Module 7 : Applications of Integration - I

Lecture 21 : Relative rate of growth of functions [Section 21.1]

Objectives

In this section you will learn the following :

- How to compare the rate of growth of two functions.

21.1 Relative Growth Rates of functions

In many applications, it becomes important to know how fast a function is increasing. This is often analysed by comparing it with other functions. Here are some examples:

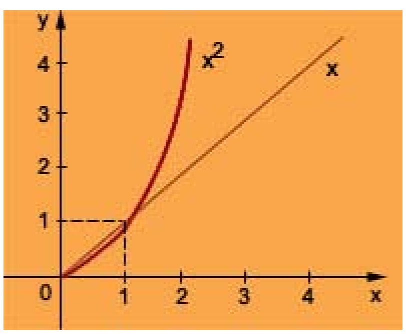
21.1.1Examples:

(i) Consider the functions $f(x) = x^2$ and $g(x) = x$. Both these functions increase as x increases. However, $f(x)$ increases at a rate much faster than $g(x)$. For example

$$f(1) = g(1) = 1,$$

where as

$$f(10^8) = (10^8)^2 = 10^{16}, g(10^8) = 10^8 \cdot$$



(ii) Let $f(x) = 10^x$ and $g(x) = x^{10}$, $x \in \mathbb{R}$. Then for $x = 100$

$f(100) = 10^{100}, g(x) = (100)^{10} = 10^{20}.$

The comparison is even more striking for $x = 1000$:

$f(1000) = 10^{1000}, g(x) = (1000)^{10} = 10^{30}.$

Thus, one can say that $f(x)$ grows much faster than $g(x)$ as x increases.

To make these ideas more precise, we have the following:

21.1.2 Definition:

Let $\alpha > 0$ and $f, g : (\alpha, \infty) \to (0, \infty)$. We say that

(i) f and g grow at the same rate as $x \to \infty$ if

$$\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right) = l \in \mathbb{R} \text{ and } l \neq 0.$$

(ii) f grows slower than g as $x \to \infty$ if

$$\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right) = 0.$$

(iii) f grows faster than g as $x \to \infty$ if

$$\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right) = \infty.$$

21.1.3 Examples:

(i) Let

$$f(x) = 2x^2 \text{ and } g(x) = (x+1)^2 \text{ for } x > 0.$$

Then

$$\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to\infty} 2\left(\frac{x}{x+1}\right)^2 = \lim_{x\to\infty} 2\left(\frac{1}{1+\frac{1}{x}}\right)^2 = 2.$$

Thus, f and g grow at the same rate as $x \to \infty$.

(ii) Let

$f(x) = x$ and $g(x) = \ln(x), x > 0.$

Then

$$\lim_{x\to+\infty}\left(\frac{x}{\ln x}\right) = \lim_{x\to\infty}\left(\frac{x}{\frac{1}{x}}\right) \qquad \left(\frac{\infty}{\infty}\text{form}\right)$$

$$= \infty.$$

Thus, $f(x) = x$ grows faster than $g(x) = \ln(x)$ as $x \to \infty$.

(iii) Let $f(x) = x$ and $g(x) = e^x, x \in \mathbb{R}$

Then

$$\lim_{x \to +\infty}\left(\frac{x}{e^x}\right) = \lim_{x \to +\infty}\left(\frac{1}{e^x}\right) \qquad \frac{\infty}{\infty}$$

$$= 0$$

Thus, $f(x) = x$ grows slower than $g(x) = e^x$ as $x \to \infty$.

Using this and (ii) above, it follows that e^x grows faster than $\ln x$ as $x \to \infty$.

(iv) Let $f(x) = e^x$ and $g(x) = 2^x$, $x \in \mathbb{R}$. Since

$$\lim_{x \to +\infty}\left(\frac{2^x}{e^x}\right) = \lim_{x \to +\infty}\left(\frac{\ln 2.2^x}{e^x}\right)$$

$$= \ln(2)\left(\lim_{x \to \infty}\left(\frac{2}{e}\right)^x\right) \qquad \left(\frac{\infty}{\infty}\text{form}\right)$$

$$= 0,$$

the last equality follows from the fact that $2 < e$ and $\lim_{x \to \infty} a^x = 0$ for $0 < a < 1$. Hence $f(x) = e^x$ grows faster than $g(x) = 2^x$.

(v) Let $a, b > 0$ be real numbers and $f(x) = \log_a(x)$, $g(x) = \log_b(x)$, $x > 0$. Then

$$\lim_{x \to +\infty}\left(\frac{\log_a(x)}{\log_b(x)}\right) = \lim_{x \to \infty}\left(\frac{\ln x}{\ln a} \times \frac{\ln(b)}{\ln(x)}\right)$$

$$= \frac{\ln(b)}{\ln(a)} \neq 0,$$

Thus, $f(x) = \log_a(x)$ and $g(x) = \log_b(x)$ grow at the same rate.

PRACTICE EXERCISES :

1. Compare the rate of growth, as $x \to \infty$, of the function x^2 with the following:

(i) $x^2 + \sqrt{x}$.

(ii) $\sqrt{x^4 - x^3 + 3}$.

(iii) 2^x.

(iv) $x\ln(x)$.

(a) $x^2\exp(-x)$.

(v) x^x.

2. Prove the following:

(i) For any two positive real numbers α, β, the function $(\ln(\ln x))^\alpha$ grows slower that $(\ln x)^\beta$ as $x \to \infty$.

(ii) For any two natural numbers n, m with $n < m$ the function x^n grows slower than x^m as $x \to \infty$.

(iii) For any two real number $b > a > 1$ the function a^x grows faster than b^x as $x \to \infty$.

(iv) For any real number $\beta > 0$ and $n \in \mathbb{N}$ and the function x^n grows faster than $(\ln x)^\beta$ as $x \to \infty$.

(v) For any real numbers $a > 1$ and $n \in \mathbb{N}$ and the function a^x grows faster than x^n as $x \to \infty$.

3. Using exercise(1) arrange the following functions in the descending rate of growth, that is, from the fastest growing to the slowest growing, as $x \to \infty$:

$$2^x, e^x, x^x, (\ln x)^x, e^{x/2}, x^{1/2}, \log_2 x, \ln(\ln x), (\ln x)^2, x^e, x^2, \ln x, (2x)^x, x^{2x}$$

4. Let f and g be functions such that $f(x)$ grow faster than $g(x)$ as $x \to \infty$. Prove that the functions $f(x)$ and $f(x) \pm g(x)$ grow at the same rate as $x \to \infty$.

5. Let $r \in (0, \infty)$ is fixed let

$f(x) = e^x$, $g(x) = \ln x$ and $h_r(x) = x^r$, for $x \in (0, \infty)$.
Show that, as $x \to \infty$, g grows slower than h_r but f grows faster than h_r.

Recap

In this section you have learnt the following

- How to compare the rate of growth of two functions.

[Section 21.2]

Objectives

In this section you will learn the following :

- How to compute the area enclosed between the graphs of two functions.

21.2 Area between two Curves

In the previous module, we defined the area under a curve $y = f(x)$, where $f:[a,b] \to \mathbb{R}$ is a non-negative Riemann integrable function. We can view this as the area between the curves $y = f(x)$ and $y = g(x) = 0$ and the lines $x = a,\ x = b$. This can be extended to the area between any the two curves.

21.2.1 Definition:

Let

$$y = f(x) \text{ and } y = g(x) \text{ for } x \in [a,b],$$

be two curves, where $f, g:[a,b] \to \mathbb{R}$ are Riemann integrable and

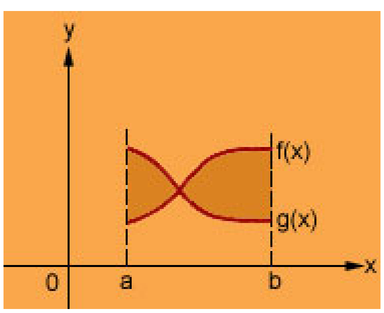
$$f(x) \le g(x) \text{ for all } x \in [a,b]. \qquad \text{----------} \ (9)$$

We define the area bounded by $f(x), g(x)$ and the lines $x = a,\ x = b$ by the formula

$$A := \int_a^b \left[g(x) - f(x)\right] dx \cdot$$

In general, where (9) need not hold, the area bounded by $f(x), g(x)$ and the lines $x = a, x = b$ is defined by the formula

$$A := \int_a^b \left|g(x) - f(x)\right| dx \cdot$$



21.2.2 Note:

To set up the integral for computing the area of a region, the following steps are useful:

Step 1: Sketch the region whose area is to be determined.

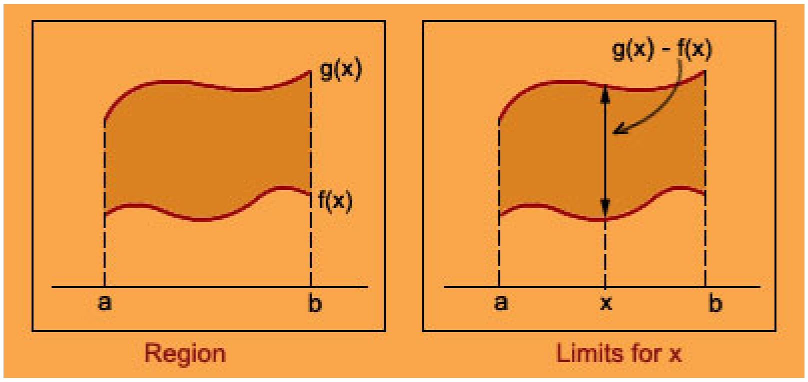
Step 2: At an arbitrary point x draw a vertical line through the region from bottom to the top boundaries of the

region. This line segment is given by , if is the top and is the bottom

$g(x)-f(x)$ $g(x)$ $f(x)$

boundary.

Step 3: Imagine moving the line segment to left, and to right. The leftmost point till which the line segment remains inside, say $x=a$, gives the lower limit of integration. Similarly, the rightmost point till which the line segment remains inside, say $x=b$, gives the upper limit of integration.



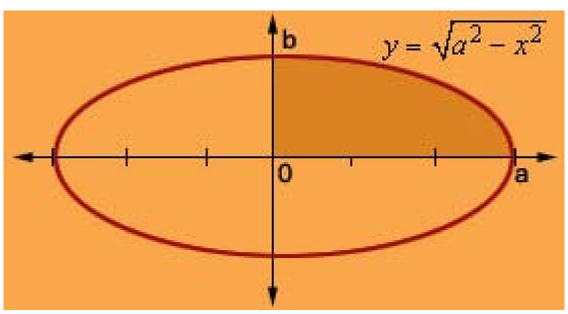
21.2.3 Examples:

(i) Let us compute the area A of the region enclosed by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

By symmetry, the required area is four times the area bounded by the curve

$$y=\frac{b}{a}\sqrt{a^2-x^2}, \text{ for } 0\le x\le a.$$



Hence,

$$A=\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx.$$

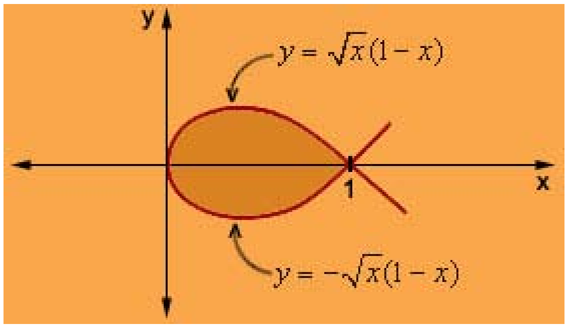
Making the substitution, $x=\sin\theta$, we get

$$A==\frac{4b}{a}a^2\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}=\pi ab.$$

(ii) Let us compute the area A of the region enclosed by the loop of the curve

$$y^2=x(1-x)^2$$

The required area A can be thought of as the area between the two curves



$y=\sqrt{x}(1-x)$ and $y=-\sqrt{x}(1-x)$ for $0\le x\le 1$

Hence,

$$A=2\int_0^1\sqrt{x}\,(1-x)\,dx=2\int_0^1\left(x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)dx=\frac{8}{15}.$$

21.2.4Note:

A variation of the definition 21.2.1 is the following: Let

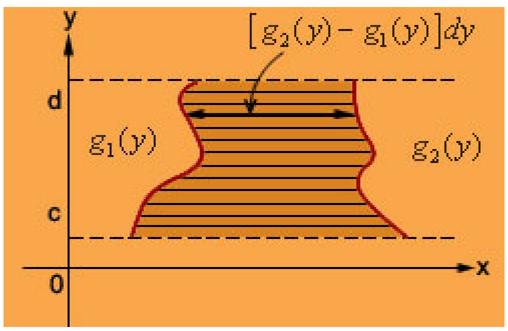
$$g_1,g_2:[c,d]\to\mathbb{R}$$

be Riemann integrable with

$$g_1(y)\le g_2(y) \text{ for all } y\in[c,d].$$

Then the area bounded by the graphs of $x=g_1(y), x=g_2(y)$ between the lines $y=c$ and $y=d$ is given by

$$A=\int_c^d[g_2(y)-g_1(y)]\,dy.$$



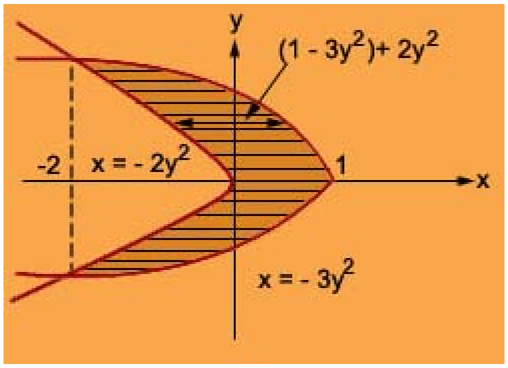
21.2.5Examples:

(i) Let us determine the area of the plane region bounded by the parabolas

$$x=-2y^2 \text{ and } x=1-3y^2.$$

For this, we first find the points of intersections of the two curves, which are given by

$$-2y^2=1-3y^2, \text{ i.e., } y=1 \text{ or } y=-1,$$



Since

$$1-3y^2 \geq -2y^2 \text{ for } -1 \leq y \leq 1,$$

the required area is

$$A = \int_{-1}^{1}\left[\left(1-3y^2\right)-\left(-2y^2\right)\right] dy = \int_{-1}^{1}\left(1-y^2\right) dy = \frac{4}{3}.$$

(ii) Area of a sector:

Compute the area A of the sector of a circle of radius r marked by the points $(0,0), (r,0)$ and $(r\cos\theta, r\sin\theta)$ where $0 < \theta \leq \pi/2$ is fixed. This area can be regarded as the area of the region bounded by the curves

$$x = y(\cot\theta), \text{ and } x = \sqrt{r^2 - y^2}, 0 \leq y \leq r\sin\theta.$$

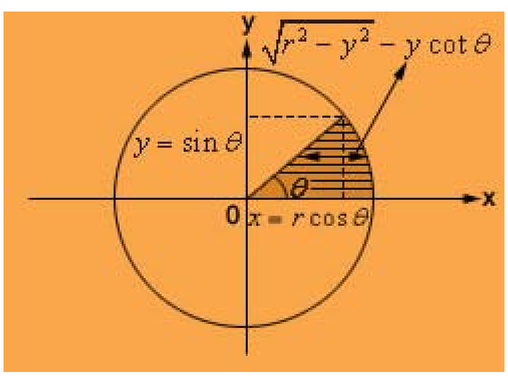
Hence

$$A = \int_0^{r\sin\theta}\left[\sqrt{r^2-y^2} - y(\cot\theta)\right] dy$$

$$= \int_0^{\theta} r^2\cos^2 u \, du - \cot\theta\left(\frac{r^2\sin^2\theta}{2}\right)$$

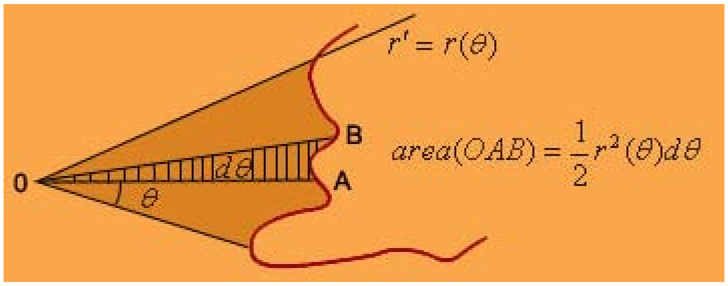
$$= \frac{r^2\theta}{2}.$$

By symmetry, this formula also holds when $(\pi/2) < \theta \leq 2\pi$. In particular, the area of a disc of radius r is πr^2



21.2.6 Area in polar coordinates:

Consider a curve $r = r(\theta), \theta \in [\alpha, \beta]$. Let us find the area of the sector bounded by this curve and the rays $\theta = \alpha, \theta = \beta$, with $0 < \beta - \alpha \leq 2\pi$.



In view of the formula in example above, we can define

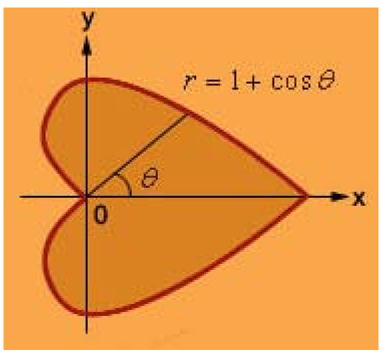
$$Area(AOB) := \lim_{|P|\to 0}\left(\sum_{i=1}^{n}\left(\frac{r^2(\gamma_i)}{2}(\theta_i - \theta_{i-1})\right)\right) = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\theta)d\theta,$$

provided the function $r : [\alpha, \beta] \to \mathbb{R}$ is Riemann integrable.

21.2.7Examples:

(i) Consider the region enclosed by the cardiod

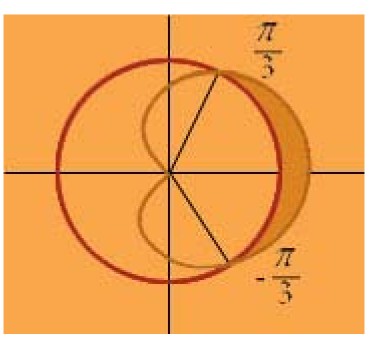
$r = 1 + \cos\theta, 0 \le \theta \le 2\pi.$



The area of this region is

$$A = \frac{1}{2}\int_0^{2\pi}(1+\cos\theta)^2\, d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\left[1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right]d\theta$$
$$= \frac{1}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{2}\right]_{\theta}^{2}\pi$$
$$= 3\pi$$

(ii) Find the area outside the circle $r = 6$, and inside the cardiod $r = 4(1+\cos\theta)$



Since $r=6$ and $r=4(1+\cos\vartheta)$ intersect for ϑ given by

$$6=4+4\cos\vartheta,$$

$$i.e., \vartheta=\frac{\pi}{3},\frac{5\pi}{3},$$

We can take limits of integration being $\vartheta=-\frac{\pi}{3}$ to $\frac{\pi}{3}$. Thus the required area is given by

$$A=\int_{-\pi/3}^{+\pi/3}\frac{1}{2}\left[4\left(1+\cos\vartheta\right)\right]^2 d\vartheta-\int_{-\pi/3}^{+\pi/3}\frac{1}{2}(6)^2 d\vartheta$$

$$=\int_{-\pi/3}^{+\pi/3}\left(16\cos\vartheta+8\cos^2\vartheta-10\right)d\vartheta$$

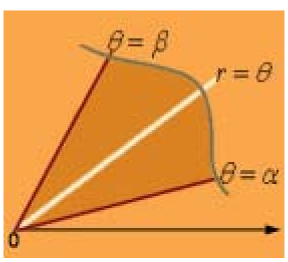
$$=\left[16\sin\vartheta+4\vartheta+2Sin2\vartheta-10\vartheta\right]_{-\pi/3}^{+\pi/3}$$

$$=18\sqrt{3}-4\boldsymbol{\pi}.$$

21.2.8 Remark:

(i) As in Cartesian coordinates, in most of the area computation problems determining the limits of integration is
the hardest part. The following steps are useful for this:

Step 1: Sketch the region whose area is to be determined.
Step 2: Draw a radial line from the pole passing through the region. Find the length of the segment inside this region.
Step 3: To find the limits of integration, rotate the radial line around the pole in anticlockwise direction to find the
lower limit $\theta=\alpha$, where the radial line starts intersecting, till $\theta=\beta$, when it stops intersecting the region.



(ii) The area of a more general planer regions can be defined using the notion of double integrals in such a way
that is consistent with the formulae given above. This will be done in a later module no 14.

PRACTICE EXERCISES

1. Find the area of the region bounded by the given curves in each of the following case:

(i) $\sqrt{x}+\sqrt{y}=1, x=0$ and $y=0$.

(ii) $y=x^4-2x^2$ and $y=2x^2$.

(iii) $x=3y-y^2$ and $x+y=3$.

2. Let $f(x)=x-x^2$ and $g(x)=ax$. Determine a so that the region above the graph of g and below the

graph

of f has area 4.5.

3. Find the area of the region bounded by the given curves in each of the following case:

(i) The region enclosed by the curve

$r = \cos 2\theta, 0 \le \theta \le 2\pi$.

(ii) Inside the circle $r = 6a\cos\theta$ and outside the cardioid $r = 2a(1+\cos\theta)$.

(iii) The region bounded by the curve $r = \sqrt{\cos 2\theta}, r = 2\cos\theta$ and the line $\theta = 0$

(iv) The region inside the circle $r = 1$ and outside the cardioid $r = 1-\cos\theta$

Consider the curve (called Lemniscate)

$r^2 = a^2\cos 2\theta, 0 \le \theta \le 2\pi$.

Show that the area bounded by this curve is not given by the following integral:

$\int_0^2 \pi \frac{1}{2} r^2 d\theta = \frac{a^2}{2}\int_0^2 \pi \cos^2\theta d\theta.$

Sketch the curve and find the correct area.

Recap

In this section you have learnt the following

- How to compute the area enclosed between the graphs of two functions.