Module 11 : Partial derivatives, Chain rules, Implicit differentiation, Gradient, Directional derivatives

Lecture 31 : Partial derivatives [Section 31.1]

Objectives

In this section you will learn the following :

- The notion of partial derivatives for functions of several variables.

31.1.1 Partial derivatives

Given a function of two (or more) variables some properties of it can be deduced by assigning some fixed numerical value to all but one variables and treating it as a function of remaining variables only.

For example if $f(x,y)$ is a function of two variables, then for $x = x_0$ fixed, we get a function of a single variable

$y \mapsto f(x_0, y).$

And similarly, for $y = y_0$ fixed, we get the function of a single variable

$x \mapsto f(x, y_0).$

These functions do not give us complete information about the function f. For example, both of them may be continuous at x_0 and y_0, respectively, but f need not be continuous at (x_0, y_0). However, they can give us some useful information about f. For example if either of $x \to f(x, y_0)$ (or $y \to f(x_0, y)$) is discontinuous at x_0 (or at y_0), then clearly f cannot be continuous at (x_0, y_0). We next look at the differentiability properties of these functions.

Let $(x_0, y_0) \in D \subseteq \mathbb{R}^2$ be an interior point and $f : D \to \mathbb{R}$. To understand how does the surface $z = f(x,y)$ behave near the point (x_0, y_0), consider the curve obtained by the intersection the surface $z = f(x,y)$ with the plane $y = y_0$ at $\left(x_0, y_0, f(x_0, y_0)\right)$. This will give us a curve $x \to f(x, y_0)$ in the plane $y = y_0$. One way of understanding $z = f(x,y)$ near (x,y) is to analyze this curve. For example we can try to draw tangent to this curve at $\left(x_0, y_0, f(x_0, y_0)\right)$, i.e., analyze whether the function

$x \to f(x, y_0)$ has a derivative at x_0 or not.

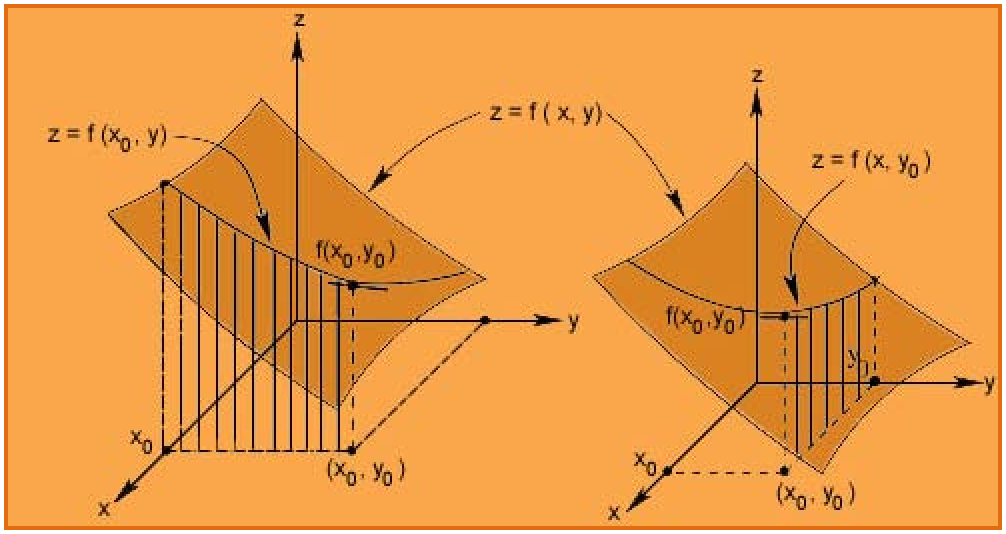

Figure 1. Partial Derivative

31.1.2 Definition:

Let $(x_0, y_0) \in D \subseteq \mathbb{R}^2$ be an interior point and $f : D \to \mathbb{R}$. The **partial derivative** of f with respect to x at (x_0, y_0), denoted by $f_x(x_0, y_0)$ or $\frac{\partial f}{\partial x}(x_0, y_0)$, is the limit

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

if it exists. Likewise, the **partial derivative** of f with respect to y at (x_0, y_0), denoted by $f_y(x_0, y_0)$ or $\frac{\partial f}{\partial y}(x_0, y_0)$, is the limit

$$\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k},$$

if it exists.

31.1.3 Examples:

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \sin xy.$$

Then, both the partial derivatives of f exists at every $(x_0, y_0) \in \mathbb{R}^2$. In fact,

$$f_x(x_0, y_0) = y_0 \cos(x_0 y_0) \text{ and } f_y(x_0, y_0) = x_0 \cos(x_0 y_0).$$

These are obtained by differentiating the functions of one variables $\sin(x y_0)$ and $\sin(x_0 y)$, respectively.

(2) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y)=\sqrt{x^2+y^2}.$$

For $(x_0,y_0)\neq(0,0)$, we have

$$f_x(x_0,y_0)=\frac{x_0}{\sqrt{x_0^2+y_0^2}} \text{ and } f_y(x_0,y_0)=\frac{y_0}{\sqrt{x_0^2+y_0^2}}$$

At $(0,0)$, since

$$\frac{f(x,0)-f(0,0)}{x}=\frac{\sqrt{x^2}}{x}=\frac{|x|}{x},$$

clearly $f_x(0,0)$ does not exist. Similarly, $f_y(0,0)$ does not exist.

(3) Let

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

Then

$$f_x(0,0)=0=f_y(0,0),$$

and at $(x_0,y_0)\neq(0,0)$,

$$f_x(x_0,y_0)=\frac{y_0(y_0^2-x_0^2)}{(x_0^2+y_0^2)^2},\quad f_y(x_0,y_0)=\frac{x_0(x_0^2-y_0^2)}{(x_0^2+y_0^2)^2}.$$

Thus, both the partial derivatives of f exist everywhere. However, f is not continuous at $(0,0)$. To see this, note that along the line $y=mx$,

$$\lim_{x\to 0} f(x,mx)=\frac{mx^2}{x^2+m^2x^2}=\frac{m}{1+m^2}.$$

Hence

$$\lim_{(x,y)\to(0,0)} f(x,y) \text{ does not exist.}$$

31.1.4Note:

Examples 31.1.2 show that the existence of both the partial derivatives at a point need not imply continuity of the function at that point. The reason being that the partial derivatives only exhibit the rate of change of $f(x,y)$ only

along two paths, namely the ones parallel to axes. A more general concept of rate of change of function, which takes into account all directions is described in the next section.

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QUIZ

Practical Exercises

(1) Examine the following functions for the existence of partial derivatives at $(0,0)$. The expressions below give

the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

(i) $xy\left(\dfrac{x^3}{x^2+y^2}\right)$.

(ii) $xy\dfrac{x^3}{x^2+y^2}$.

(iii) $\dfrac{x^2y}{x^4+y^2}$.

(iv) $xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right)$.

(v) $||x|-|y||-|x|-|y|$.

(vi) $\dfrac{\sin^2(x+y)}{|x|+|y|}$.

Answers

(2) Let

$$f(x,y) = (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) \text{for } (x,y) \neq (0,0), \text{ and } f(0,0)=0.$$

Show that f is continuous at $(0,0)$, and both the partial derivatives of f exist but are not bounded in $B_r(0,0)$ for any $r>0$.

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \left(\frac{xy}{x^2+y^2}\right) \text{for } (x,y) \neq (0,0) \text{ and, } f(0,0)=0.$$

Show that both f_x and f_y exist at every $(x_0,y_0) \in \mathbb{R}^2$, but f is not continuous at $(0,0)$.

(4) Let $f(0,0)=0$ and

$$f(x,y) = \begin{cases} x\sin(1/x)+y\sin(1/y), & \text{if } x \neq 0, y \neq 0 \\ x\sin 1/x, & \text{if } x \neq 0, y = 0 \\ y\sin 1/y, & \text{if } y \neq 0, x = 0. \end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

(5) Let $D \subset \mathbb{R}^2$, $(x_0,y_0) \in D$ and $f : D \to \mathbb{R}$ be such that both f_x and f_y exist and are bounded in

$B_r(x_0, y_0)$ for some $r > 0$. Prove that f is continuous at (x_0, y_0).

(6) Euler's Theorem: Suppose $F : \mathbb{R}^3 \to \mathbb{R}$ has the property that there exists $n \in \mathbb{N}$ such that

$$F(tx, ty, tz) = t^n F(x, y, z) \text{ for all } t \in \mathbb{R} \text{ and } (x, y, z) \in \mathbb{R}^3.$$

(Such a function is said to be homogeneous of degree n.) If the first order partial derivatives of f exist and are continuous, then show that

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = nF.$$

Recap

In this section you have learnt the following

- The notion of partial derivatives for functions of several variables.

[Section 31.2]

Objectives

In this section you will learn the following :

The notion of differentiability for functions of several variables.

31 .2 Differentiability

Recall that for a function $y = f(x)$ of one variable, the concept of differentiability at a point x_0 allowed us to approximate the function f by a linear function in the neighborhood of x_0. Analytically f differentiable at x_0 is equivalent to the fact that

$$f(x_0+h)=f(x_0)+hf'(x_0)+h\varepsilon_1(h),$$

for all h sufficiently small, where $\varepsilon_1(h)\to 0$ as $h\to 0$. The expression

$$L(x)=f(x_0)+hf'(x_0)$$

is the linear (or tangent line) approximation and $\varepsilon_1(h)$ is the error for the linear approximation. For function of two variables, the linear approximation motivates the following definition:

31.2.1 Definition:

A function
$f: D\subseteq \mathbb{R}^2\to\mathbb{R}$
is said to be **differentiable** at $(x_0,y_0)\in D$ if the following hold:

(i) Both $f_x(x_0,y_0), f_y(x_0,y_0)$ exist.

(ii) There exists $\delta>0$ such that

$$(x_0+h,y_0+k)\in D \text{ for all } |h|<\delta, |k|<\delta.$$

(For example this condition will be satisfied if (x_0,y_0) is an interior point of the domain D.)

(iii) There exist functions $\varepsilon_1(h,k)$ and $\varepsilon_2(h,k), |h|<\delta, |k|<\delta$ such that

$$\varepsilon_1(h,k)\to 0,\ \varepsilon_2(h,k)\to 0$$

and

$$f(x_0+h,y_0+k)=f(x_0,y_0)+hf_x(x_0,y_0)+kf_y(x_0,y_0)+h\varepsilon_1(h,k)+k\varepsilon_2(h,k).$$

31.2.2 Note(Necessary and Sufficient condition for differentiability):

Let f be differentiable at (x_0,y_0). Then by definition,

$$\left|\frac{f(x_0+h,y_0+k)-f(x_0,y_0)-hf_x(x_0,y_0)-kf_y(x_0,y_0)}{\sqrt{h^2+k^2}}\right|\le|\varepsilon_1(h,k)|+|\varepsilon_2(h,k)|.$$

Hence,

$$\lim_{(h,k)\to(0,0)}\left(\frac{f(x_0+h,y_0+k)-f(x_0,y_0)-hf_x(x_0,y_0)-kf_y(x_0,y_0)}{\sqrt{h^2+k^2}}\right)=0. \qquad \text{----------} \quad (25)$$

In fact, the converse also holds, i.e., if (25) holds then f is differentiable. We assume this fact.

As an application of the above equivalence, we have the following:

31.2.3 Example:

(i) Let

$$f(x,y)=\sqrt{x^2+y^2}.$$

Then f is not differentiable at $(0,0)$ as both $f_x(0,0)$ and $f_y(0,0)$ do not exit.

(ii) Let

$$f(x,y)=\begin{cases}(x^2+y^2)\sin(\frac{1}{x^2+y^2}), & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

Then $f(0,y)=0$ for all y and $f(x,0)=0$ for all x. Hence,

$f_x(0,0)=0=f_y(0,0).$

Also

$$\left|\frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}}\right|=\left|\frac{(h^2+k^2)\sin(\frac{1}{h^2+k^2})}{\sqrt{h^2+k^2}}\right|$$
$$\leq |\sqrt{h^2+k^2}|$$
$$\to 0 \text{ as } (h,k)\to(0,0).$$

Hence, f is differentiable at $(0,0)$.

As in the case of function of one variable, above notion of differentiability implies continuity of the function.

31.2.4Theorem:

Let $f:D\to\mathbb{R}$ be such that f is differentiable at $(x_0,y_0)\in D$. Then f is continuous at (x_0,y_0).

PROOF

31.2.4Theorem:

Let $f:D\to\mathbb{R}$ be such that f is differentiable at $(x_0,y_0)\in D$. Then f is continuous at (x_0,y_0).

Proof

Since f differentiable at (x_0,y_0), we have

$|f(x_0+h,y_0+k)-f(x_0,y_0)|\leq|h||\varepsilon_1(h,k)+f_x(x_0,y_0)|+|k||\varepsilon_2(h,k)+f_y(x_0,y_0)|,$

where both $\varepsilon_1(h,k),\varepsilon_2(h,k)\to 0$ as $(h,k)\to(0,0)$. Thus, we have

$\lim_{(h,k)\to(0,0)} f(x_0+h,y_0+k)=f(x_0,y_0).$

Hence, f is continuous at (x_0,y_0).

We describe next a sufficient condition for the differentiability of $f(x,y)$.

31.2.5 Increment Theorem (Sufficient condition for differentiability):

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$ be such that the following hold:

(i) *For some $r > 0$, both f_x and f_y exist at all points in $B\big((x_0, y_0), r\big)$.*

(ii) *Both f_x and f_y are continuous at the point (x_0, y_0). Then, f is differentiable at (x_0, y_0).*

31.2.5 Increment Theorem (Sufficient condition for differentiability):

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$ be such that the following hold:

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(ii) *Both f_x and f_y are continuous at the point (x_0, y_0). Then, f is differentiable at (x_0, y_0).*

Proof

Let $(h,k) \in B\big((0,0), r\big)$. Applying Mean Value Theorem to the functions
$x \mapsto f(x, y_0)$ and $y \mapsto f(x_0 + h, y)$,
we can find points c between x_0 and $x_0 + h$ and d between y_0 and $y_0 + k$ such that

$$f(x_0 + h, y_0) - f(x_0, y_0) = f_x(c, y_0)h, \qquad \text{.............................} \ (26)$$

$$f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) = f_y(x_0 + h, d)k. \qquad \text{..............................} \ (27)$$

Thus, using (26) and (27) we have

$$\begin{aligned} f(x_0 + h, y_0 + k) - f(x_0, y_0) &= hf_x(c, y_0) + kf_y(x_0 + h, d). \\ &= hf_x(x_0, y_0) + h[f_x(c, y_0) - f_x(x_0, y_0)] + kf_y(x_0, y_0) \\ &\quad + k[f_y(x_0 + h, d) - f_y(x_0, y_0)] \\ &= hf_x(x_0, y_0) + h\varepsilon_1(h,k) + kf_y(x_0, y_0) + k\varepsilon_2(h,k), \end{aligned}$$

where

$\varepsilon_1(h,k) := f_x(c, y_0) - f_x(x_0, y_0),$

$\varepsilon_2(h,k) := f_y(x_0 + h, d) - f_y(x_0, y_0).$

Since f_x, f_y are continuous at (x_0, y_0), we obtain

$$\lim_{(h,k)\to(0,0)} \varepsilon_1(h,k) = 0 \ \text{ and } \lim_{(h,k)\to(0,0)} \varepsilon_2(h,k) = 0.$$

Hence, f is differentiable at (x_0, y_0).

31.2.6 Remark:

The condition that f_x, f_y exists near (x_0, y_0) and are continuous at (x_0, y_0) are only sufficient for the differentiability of f at (x_0, y_0). These are not necessary. For example, the function

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right), & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0),\end{cases}$$

is differentiable at (x_0,y_0), as shown in example $31.2.3(ii)$. However,

$$f_x(x,y)=-\left(\frac{2x}{x^2+y^2}\right)\cos\left(\frac{1}{x^2+y^2}\right)+2x\sin\frac{1}{(x^2+y^2)},$$

which does not converge to $0=f_x(0,0)$ as $(x,y)\to(0,0)$. In fact, along the path $y=0$, the function $f_x(x,0)$ is unbounded.

Practice Exercises

(1) Examine the following functions for differentiability at $(0,0)$. The expressions below give the value of the

function at $(x,y)\neq(0,0)$. At $(0,0)$, the value should be taken as zero.

(i) $xy\dfrac{x^2-y^2}{x^2+y^2}$.

(ii) $\dfrac{x^3+2y^3}{x^2+y^2}$.

(iii) $(x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right)$.

Answers

(2) Let

$$f(x,y)=\frac{xy^4}{x^2+y^2},\ \text{if}\ \ x\neq 0, \text{and } f(0,y)=0 \text{ for all } y.$$

Show that if f is differentiable at $(0,0)$. Analyze the continuity of the partial derivatives f_x and f_y at $(0,0)$.

(3) Let f, g be both differentiable at a point (a,b). Show that the functions $f\pm g$, and fg are also differentiable at (a,b).

Recap

In this section you have learnt the following

- The notion of differentiability for functions of several variables.