Module 18 : Stokes's theorem and applications

Lecture 54 : Application of Stokes' theorem [Section 54.1]

Objectives

In this section you will learn the following :

- Computational applications of Strokes' theorem.
- Physical applications of Strokes' theorem.
- Sufficient conditions for a vector field to be conservative.

54.1 Applications of Stokes' theorem

Stokes' theorem gives a relation between line integrals and surface integrals. Depending upon the convenience, one integral can be computed interms of the other.

54.1.1 Example (computation of line integral):

We want to compute

$$\oint_C \mathbf{F}.d\boldsymbol{r}, \text{ where } \mathbf{F} = y\mathbf{i} + xz^3\mathbf{j} - zy^3\mathbf{k}$$

and C is the circle in the plane $z=-3, x^2+y^2=4,$ oriented anti-clockwise. To apply Stokes' theorem, let us find a convenient surface S whose boundary is $C.$ The most natural surface in this case is the circular disc

$$x^2+y^2 \leq 4, z=-3.$$

For S if we choose the normal vector $\mathbf{n}$ to be $\mathbf{k},$ then $\partial(S)=C$ will have anti-clockwise orientation, and by Stokes' s theorem we will have

$$\oint_C \mathbf{F}.d\boldsymbol{r} = \iint_S (\text{curl } \mathbf{F}\cdot\mathbf{n})\, dS.$$

Since,

$$\text{curl}(\mathbf{F})\cdot\mathbf{n} = \left[\frac{\partial}{\partial x}(xz^3) - \frac{\partial}{\partial y}(y)\right] = z^3 - 1.$$

we have

$$\oint_C (\mathbf{F}.d\mathbf{r}) = \iint_S (z^3-1)\, ds = \iint_S (-28)\, ds = -28\,(4\pi).$$

54.1.2 Example:

Let us try to apply the above technique to evaluate

$$\oint_C (\mathbf{F}.d\mathbf{r}) \text{ where } \mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{y}{x^2+y^2}\mathbf{j},$$

and C is the circle $x^2+y^2=1, z=0$ oriented clockwise. We note that for $\mathbf{F}, \text{curl}(\mathbf{F})=0,$ and hence

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS = 0,$$

whatever be the surface S, as long as $\partial(S)=C.$ We also know (see example 47.2.8)that

$$\oint (F \cdot d\mathbf{r}) = -2\pi.$$

Thus,

$$\oint_C \mathbf{F} \cdot dr \neq \iint_S (\text{curl } \mathbf{F} \cdot \mathbf{n}) dS.$$

Does this contradict Stokes' theorem? The answer is no. The reason for this is that Stokes' theorem is not applicable for the given $\mathbf{F}$ because

$$\mathbf{F}(x,y,z) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

is defined in $D := \{(x,y,z) \in \mathbb{R}^3 | (0,0,z)\}$. Thus, there does not exist any surface $S \subseteq D$ whose boundary is the unit circle $C: x^2+y^2=1$ in the xy-plane. However, we can still evaluate the integral

$$\oint_C \mathbf{F} \, . \, d\mathbf{r}$$

as follows: Consider the vector field

$$\tilde{\mathbf{F}}(x,y,z) := -\frac{y}{x^2+y^2+z^2}\mathbf{i} + \frac{x}{x^2+y^2+z^2}\mathbf{k},$$

$(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. Note that $\tilde{\mathbf{F}} = \mathbf{F}$ on $C,$ and if we consider the upper hemisphere S given by $x^2+y^2+z^2=1, z \geq 0$, then $\partial(S)=C$ and $S \cup C \mathbb{R}^3 \{(0,0,0)\}$. If we select the outward unit normal

$$\mathbf{n} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},$$

$x^2+y^2+z^2=1$ on S, then $\partial(S)=C$ gets the anti-clockwise orientation and we have by Stokes' theorem

$$\oint_C \mathbf{F} \, . \, d\mathbf{r} = \oint_C \tilde{\mathbf{F}}.d\boldsymbol{r} = \iint_S (\text{curl } \tilde{\mathbf{F}}) \cdot \mathbf{n} \, dS.$$

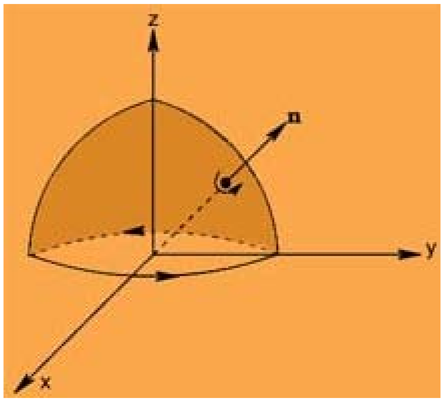

Figure: Orientation on S and $\partial(S)$

Let S be given the parameterizations :

$$\mathbf{r}\,(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (\sqrt{1-x^2-y^2})\ \mathbf{k},\,(x,y)\in D.$$

where D is the unit disc in xy-plane. Then

$$\mathbf{r}_x \times \mathbf{r}_y = -f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k},$$

for $f\,(x,y) = \sqrt{1-x^2-y^2}$. This gives

$$\mathbf{r}_x \times \mathbf{r}_y = +\frac{x}{z}\mathbf{i} + \frac{y}{z}\mathbf{k} + \mathbf{k}.$$

Also

$$\mathrm{curl}(\tilde{\mathbf{F}}) = -2\,zx\,\mathbf{i} - 2\,zy\,\mathbf{k} - z^2\mathbf{k}.$$

Hence

$$\begin{aligned}\iint_S (\mathrm{curl}\,\tilde{\mathbf{F}}\cdot\mathbf{n})\,ds &= \iint_D\left(+\frac{x}{z}\mathbf{i} + \frac{y}{z}\mathbf{k} + k\right)\cdot\left(-2\,zx\,\mathbf{i} - 2\,zy\,\mathbf{k} - z^2\,\mathbf{k}\right)dxdy\\ &= \iint_D -\left(2x^2+2y^2+2z^2\right)dxdy\\ &= -2\iint_D dxdy\\ &= -2\pi.\end{aligned}$$

54.1.3 Example (Calculation of surface integral):

In the formula

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \mathrm{curl}(\mathbf{F})\cdot\mathbf{n}dS$$

the only relation between C and S is that $\partial(S) = C$. Thus, if S_1 and S_2 are two surfaces such that $\partial(S_1) = \partial(S_2) = C$ and $\mathbf{F}$ is defined in a region including have $\iint_{S_1} \mathrm{curl}(\mathbf{F})\cdot\mathbf{n}dS = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_1} \mathrm{curl}(\mathbf{F})\cdot\mathbf{n}dS$

This is useful in computations. We consider an example. Let us evaluate

$$\iint_S (\nabla u \times \nabla v)\cdot\mathbf{n}\,dS$$

where

$$u = x^3 - y^3 + z^2, \text{ and } v = x + y + z,$$

S is the upper hemispherical sheet

$$x^2 + y^2 + z^2 = 1, z \geq 0,$$

$\mathbf{n}$ is the unit normal to S with non-negative z-component. Since

$$(\nabla u \times \nabla v) = \text{curl } (u \nabla v),$$

we have by Stokes' theorem

$$\begin{aligned}\iint_S (\nabla u \times \nabla v) . \mathbf{n}\, dS &= \oint_C (u \nabla v) \cdot d\mathbf{r} \\ &= \iint_{S_1} (\nabla u \times \nabla v) \cdot \mathbf{n}\, dS\end{aligned},$$

where we select S_1 be the circular disc $D = \left\{(x, y, z) \mid x^2 + y^2 = a^2, z = 0\right\}$.

Thus

$$\iint_S (\nabla u \times \nabla v) \cdot \mathbf{n}\, dS = \iint_D (\nabla u \times \nabla v) \cdot \mathbf{k}\, dxdy.$$

Since, $(\nabla u \times \nabla v) \cdot \mathbf{k} = 3x^2 + 3y^2$, we have

$$\begin{aligned}\iint_C (\nabla u \times \nabla v) \cdot \mathbf{n} dS &= \iint_{x^2+y^2 \leq a^2} 3\, (x^2 + y^2)\, dxdy \\ &= 3 \int_0^a \int_0^{2\pi} r^3\, dxdy \\ &= \frac{3\pi}{2}.\end{aligned}$$

54.1.4 Note (Green's theorem, a particular case of Stokes' theorem):

Consider a planer vector-field.

$$F\, (x, y) = P\, (x, y)\, \mathbf{i} + Q\, (x, y)\, \mathbf{j}.$$

Let D be a region in the xy-plane with $\partial(D) = C$ a simple closed curve. If we treat D as a flat surface, oriented $\mathbf{k}$ as the unit normal, then by Stokes' theorem, treating $\mathbf{F}$ as a vector field in z-space, $F\, (x, y) = P\, (x, y)\, \mathbf{i} + Q\, (x, y)\, \mathbf{j} + 0\, \mathbf{k}$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \text{curl}(\mathbf{F}) \cdot \mathbf{k} dxdy$$

which is Green's theorem, since

$$\text{curl}(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

We showed in section 47.2, that given points $A, B \in D \subseteq \mathbb{R}^3$ and $\mathbf{F}$, a continuously differentiable vector field on D, if the line integral

$$\int_{C(A,B)} \mathbf{F} \cdot d\mathbf{r}$$

is independent of the path C going A to B, then $\operatorname{curl}(\mathbf{F})=0$. We also stated that this condition is also sufficient if the domain D is simply connected. We prove this as an application of Stokes' theorem.

54.1.5 Theorem (Conditions for conservativeness):

Let D be a simply connected domain in $\mathbb{R}^3$ $F : D \to \mathbb{R}^3$ be a continuously differentiable vector field. Then the following are equivalent:

1. $\mathbf{F}$ *is conservative.*
2. $\int_{C(A,B)} \mathbf{F}.d\mathbf{r}$ *is independent of the path* $C(A,B)$ *joining* A *to* B.
3. $\operatorname{curl}(\mathbf{F})=0$.

PROOF

In view of the statements above, we only have to show that $(iii) \Rightarrow (ii)$. Let C be any simple closed curve in D. Since D is simply connected, we can find an orientable piecewise smooth surface $S \subseteq D$ such that $C=\partial S$. Then by Stokes' theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}(\mathbf{F})\cdot \mathbf{n}\, dr = 0.$$

Hence, by theorem $47.2.5$, (ii) words.

54.1.6 Physical interpretation of Curl:

Stokes' theorem provides a way of interpreting the curl of a vector-field $\mathbf{F}$ in the context of fluid-flows. Consider a small circular disc S_a of radius a at a point P in the domain of $\mathbf{F}$. Let $\mathbf{n}$ be the unit normal to the disc at P_0. Then by Stokes' theorem

$$
\begin{aligned}
\oint_{Ca} (\mathbf{F}\cdot \mathbf{T})\, ds &= \iint_{Ca} \mathbf{F}.dr \\
&= \iint_{Sa} (\operatorname{curl} \mathbf{F}).\mathbf{n}\, dr \\
&\cong \left(\operatorname{curl} \mathbf{F}(P_0).\mathbf{n}(P_0)\right) \iint_{Sa} dS \\
&= \left(\operatorname{curl} \mathbf{F}(P_0).\mathbf{n}\right) A(S_a)
\end{aligned}
$$

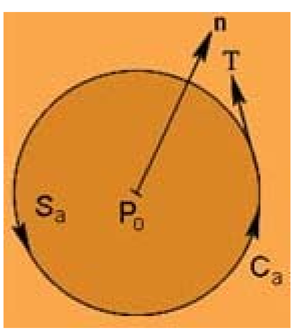

Figure: Flux along C_a

Thus,

$$\left(\text{curl } \mathbf{F}\left(P_0\right).\mathbf{n}\right)\left(P_0\right) \cong \frac{1}{A(Sa)} \oint_{Ca} (\mathbf{F}.\mathbf{T}) ds$$

Note that $\mathbf{F}.\mathbf{T}$ gives the component of F in the direction of the tangent and hence gives the rotational component of $\mathbf{F}$ along ds. Then $\oint_{C_a} (\mathbf{F}.\mathbf{T}) ds$ is called the **circulation density** of $\mathbf{F}$ around $C_\mathbf{a}$. If we let $a \to 0$ in (92), then we will have (as error in approximation goes to zero)

$$\left(\text{curl } \mathbf{F}\right)\left(P_0\right) \mathbf{n}\left(P_0\right) = \lim_{a \to 0} \frac{1}{A(Sa)} \oint_{Ca} (\mathbf{F}.\mathbf{T}) ds$$

For this reason, the normal component of $\text{curl}(\mathbf{F})$, also called the **specific circulation** of the fluid at the point P_0. Note that the specific circulation is maximum when $\text{curl}\mathbf{F}(P_0)$ and $n(P_0)$ have the same direction we can interpret Stokes' theorem.

$$\iint_S \left(\text{curl } \mathbf{F} \cdot \mathbf{n}\right) dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

as follows: the collective measure of rotational tendency is equal to the tendency of the fluid to circulate around its boundary. Thus, if $\text{curl}\boldsymbol{F} = 0$ in S, then

$$\iint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

i.e., there is no circulation tendency, or one says the fluid is **irrotational** .

Practice Exercises

1. Using Stokes' theorem, evaluate the integral

   $$\int_C \mathbf{F} \cdot d\mathbf{r},$$

   where

   1. $\mathbf{F}\,(x, y, z) = 2\,y\,\mathbf{i} + 3\,z\,\mathbf{j} + x\,\mathbf{k}$, and C is the triangle with vertices $A\,(0,0,0), B\,(0,2,0), C\,(1,1,1)$ oriented for A to

      B to C to A.

   2. $\mathbf{F}\,(x, y, z) = z^2\,\mathbf{i} + y\,\mathbf{j} + x\,z\mathbf{k}$, and C is the boundary of the upper hemi-sphere $S : z = \sqrt{4 - x^2 - y^2}$, oriented counter clockwise

   3. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + 4xy^3\mathbf{j} + y^2 x\,\mathbf{k}$ and C is the rectangle going $A\,(0,0,0)$, $B\,(1,0,0)$, $C\,(1,3,2)$, $D\,(0,3,2)$,

      oriented from A to B to C to D to A.

   4. $\mathbf{F}(x, y, z) = 3y\,\mathbf{i} + 4x\,\mathbf{j} + 2y\,\mathbf{k}$, and C is the boundary if the paraboloid $z = 4 - x^2 - y^2, z \geq 0$ oriented counter

      clockwise

   Answer:

   (i) 1

   (ii) 0

(iii) -90

(iv) 16π

2. Let $\mathbf{F}(x,y,z)=(x-z)\mathbf{i}+(y-x)\mathbf{j}+(z-xy)\mathbf{k}$, and C be the boundary of the triangle S with vertices $A(1,0,0),B(0,2,0),C(0,0,1)$. Find the following.

1. Circulation of $\mathbf{F}$ around the triangle when C is oriented counterclockwise.
2. Circulation density of $\mathbf{F}$ at $(0,0,0)$ in the direction $\mathbf{k}$.
3. Find the direction of $\mathbf{n}$ along which the circulation density of $\mathbf{F}$ at $(0,0,0)$ is maximum.

Answer:

(i) $3/2$

(ii) -1

(iii) $-\frac{1}{\sqrt{2}}\mathbf{j}-\frac{1}{\sqrt{2}}\mathbf{k}$

3. Let S be the upper half of the ellipsoid $x^2+y^2+\dfrac{z^2}{9}=1,$ oriented so that n is upward. For $\mathbf{F}(x,y,z)=x^2\mathbf{i}+y^4\mathbf{j}+z^2\sin\ xy\ \mathbf{k}$

Evaluate

$$\iint_S(\text{curl }\mathbf{F})\cdot\mathbf{n}dS,$$

Replacing S by a suitable simpler surface with the same boundary as that of S.

Answer: 0

Recap

In this section you have learnt the following

- Computational applications of Strokes' theorem.
- Physical applications of Strokes' theorem.
- Sufficient conditions for a vector field to be conservative