Module 16 : Line Integrals, Conservative fields Green's Theorem and applications

Lecture 48 : Green's Theorem [Section 48.1]

Objectives

In this section you will learn the following :

- Green's theorem which connects the line integral with the double integral.

48.1 Green's Theorem for simple domains :

We analyze next the relation between the line integral and the double integral.

48.1.1 Definition:

Consider a region R in the plane defined by

$$R := \left\{ (x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, f(x) \leq y \leq g(x) \right\},$$

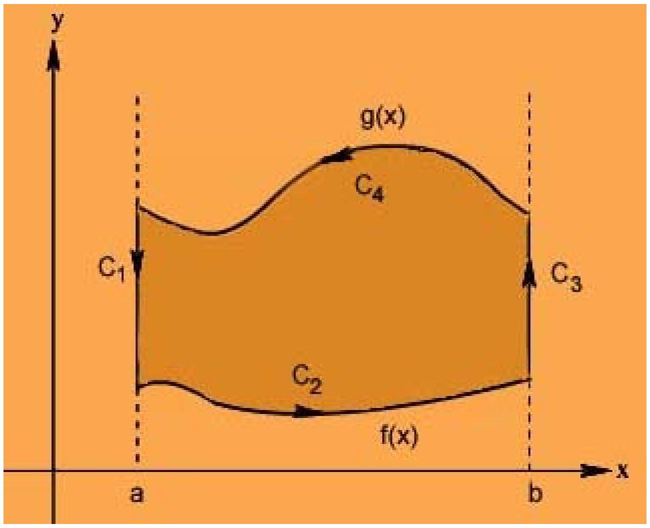

Figure: Caption text.

where $f(x), g(x)$ are continuously differentiable functions. Such a region is called a **vertically simple** region or a **type-I** region in $\mathbb{R}^2$. Let C denote the boundary of this domain, C being traversed in the counter-clockwise direction (i.e., as you move along the boundary, the domain R lies to the left). Then the boundary C can be broken into parts

$$C = C_1 \cup C_2 \cup C_3 \cup C_4,$$

where C_1 and C_3 are the vertical line segments, C_2 is the graph

$$\{(x, f(x)) \mid a \le x \le b\}$$

and C_4 is the graph

$$\{(x, g(x)) \mid b \le x \le a\}.$$

48.1.2 Theorem (Green's theorem for simple domains):

Let $P, Q : U \to \mathbb{R}$ be continuously-differentiable scalar fields, where $U \subset \mathbb{R}^2$. Let R be a vertically simple region in $\mathbb{R}^2$ which can be represented as

$$R = \{(x, y) \mid a \le x \le b, f(x) \le y \le g(x)\}$$

where both the function f, g are continuously differentiable. Further, let U be such that R and its boundary C are inside U. Then

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

where C is given the counter-clockwise orientation.

PROOF

Let us write

$$C = C_1 \cup C_2 \cup C_3 \cup C_4,$$

where the curves C_1, C_2, C_3 and C_4 have, respectively, the following parameterizations

$$\begin{aligned}
&C_1 \text{ is } \mathbf{r}_1(y) := a\,\mathbf{i} + y\,\mathbf{j},\ g(a) \le y \le f(a),\\
&C_2 \text{ is } \mathbf{r}_2(x) := x\,\mathbf{i} + f(x)\,\mathbf{j},\ a \le x \le b,\\
&C_3 \text{ is } \mathbf{r}_3(y) := b\,\mathbf{i} + y\,\mathbf{j},\ f(b) \le y \le g(b),\\
&C_4 \text{ is } \mathbf{r}_4(x) := x\,\mathbf{i} + g(x)\,\mathbf{j},\ b \le x \le a,
\end{aligned}$$

Then

$$\oint_C P dx = \int_{C_1} P dx + \int_{C_2} P dx + \int_{C_3} P dx + \int_{C_4} P dx$$

$$= 0 + \int_{x=a}^{x=b} P(x_1 f(x))\, dx + 0 + \int_{x=b}^{x=a} P(x_1 g(x))\, dx$$

$$= \int_{x=a}^{x=b} \left[P(x, f(x)) - P(x, y(x)) \right] dx$$

$$= \int_{x=a}^{x=b} \left[\int_{y=g(x)}^{f(x)} \frac{\partial P}{\partial y}(x, y)\, dy \right] dx$$

$$= -\iint_R \frac{\partial P}{\partial y}(x, y)\, dx\, dy \qquad \text{----------(47)}$$

A similar calculation will give us

$$\oint_C Q dy = \iint_R \frac{\partial Q}{\partial x}(x,y)\, dx\, dy \qquad \text{----------(48)}$$

From (47) and (48) we have

$$\oint_C P dx + Q dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\, dy.$$

48.1.3 Note :

Arguments similar to the above theorem will tell us that conclusion of Green's theorem also holds for regions R of the type :

$$R = \left\{(x,y) \in \mathbb{R}^2 \mid C \le y \le d, f(y) \le x \le g(y)\right\},$$

where f, g are continuously differentiable. Such regions are called **horizontally simple** regions or **type-II regions** in $\mathbb{R}^2$.

48.1.4 Example :

Let us verify Green's theorem for scalar field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, where

$$P(x,y) := -x^2 y, Q(x,y) := xy^2,$$

and the region R is given by

$$R := \left\{(x,y) \mid x^2 + y^2 \le a\right\}.$$

The boundary C of R is the unit circle.

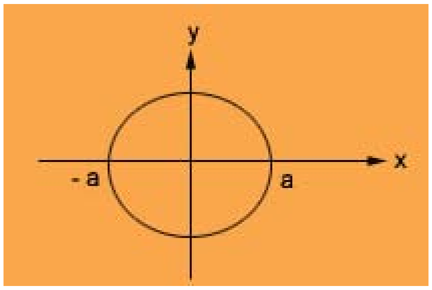

Figure 196. R **with boundary circle**

A parameterization of C is

$\mathbf{r}(t) = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j}, 0 \le t \le 2\pi.$

Thus,

$$\oint_C Pdx+Qdy=\int_0^{2\pi}\left(-a^3\cos^2 t\ \sin t\right)\left(-a\ \sin t\right)$$

$$\int_0^{2\pi}\left(-a^3\cos t\ \sin^2 t\right)\left(-a\ \cos t\right)$$

$$=a^4\int_0^{2\pi}\left[+\cos^2 t\ \sin^2 t+\cos^2 t\ \sin^2 t\right]dt$$

$$=\frac{2a^4}{4}\int_0^{2\pi}\sin^2 2t dt$$

$$=\frac{a^4}{4}\left[\int_0^{2\pi}\left(1-\cos\ 4t\right)dt\right.$$

$$=\frac{\pi a^4}{2}-\frac{a^4}{4}\int_0^{2\pi}\cos\ 4t dt$$

$$=\frac{\pi a^4}{2}.$$

And

$$\iint_R\left[\frac{\partial}{\partial x}(xy^2)-\frac{\partial}{\partial y}(-x^2 y)\right]dx\,dy=\int_{-a}^{+a}\left[\int_{\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}(y^2+x^2)\,dy\right]dx$$

$$=\int_0^{2\pi}\int_0^a r^2\ r\ dr\ d\theta \qquad \text{(in polar coordinates)}$$

$$=2\pi\left[\frac{r^4}{4}\right]_0^a$$

$$=\frac{\pi a^4}{2}.$$

Hence

$$\oint_C Pdx+Qdy=\iint_R\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]dx\,dy.$$

48.1.5 Extending Green's theorem beyond simple regions :

In theorem 48.1.1 we proved green's theorem for simple domains (horizontally simple or vertically simple). Though a complete and rigorous argument is beyond the scope of these notes, we show how Green's theorem extends to more general regions as follows:

1. Let R be the region enclosed by a simple closed curve C. Then

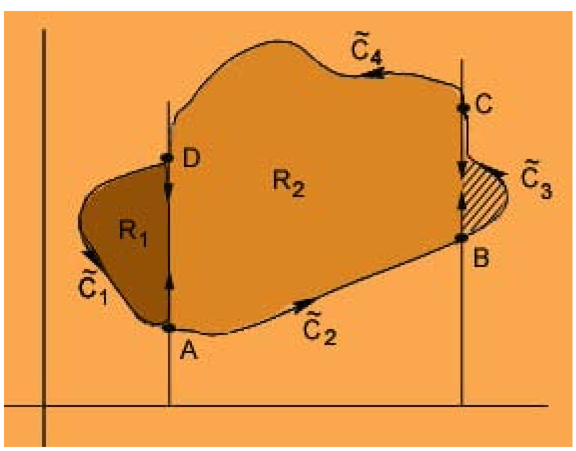

Figure: Splitting a region into simple regions

The region can be subdivided into finite number of simple regions by line segments parallel to the axes. As in

R

figure above, R is divided into three non overlapping regions $R_1, R_2,$ and R_3. For each region R_i, Green's theorem holds, and we have

$$\oint_{R_i} Pdx + Qdy = \iint_{C_i} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx\,dy, i = 1, 2, 3,$$

where C_i is the boundary of the region R_i, traversed counter clockwise direction. Since

$$C_1 = \tilde{C}_1 \cup C(A, B),$$

$$C_2 = \tilde{C}_2 \cup C(B, C) \cup \tilde{C}_4 \cup C(D, A)$$

$$C_3 = \tilde{C}_3 \cup C(B, C),$$

where $C(A, D), C(B, C), C(D, A),$ and $C(C, B)$ are the line segments joining the first point with the second point. Thus, adding the above equations for $i = 1, 2, 3$, we have

$$\begin{aligned}
\iint_R Pdx + Qdy &= \iint_{R_1} (Pdx + Qdy) + \iint_{R_2} (Pdx + Qdy) + \iint_{R_3} (Pdx + Qdy) \\
&= \oint_{\tilde{C}_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy + \oint_{\tilde{C}_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \\
&\quad + \oint_{\tilde{C}_3} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy + \oint_{\tilde{C}_4} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \\
&= \oint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy
\end{aligned}$$

This says that Green's theorem holds for the region R.

2. As a particular case of (i) we get that Green's theorems holds for regions as shown in the figure below :

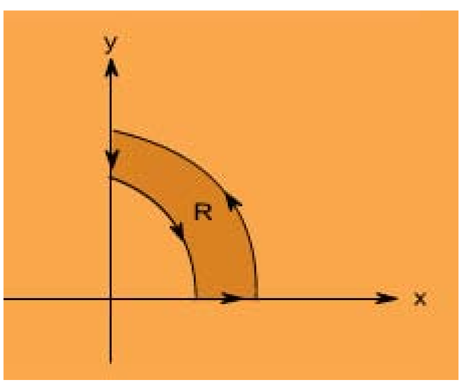

Figure: Simple region

Joining any two or more such regions, we get Green's theorem for regions of the type:

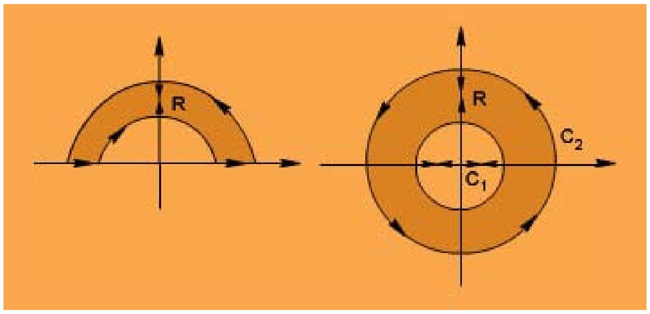

Figure: Joining simple regions

For example, for the annular region $R=\left\{(x,y)\,|\,a^2<x^2+y^2<b^2\right\}$, the Green's theorem states:

$$\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\int_{C_1}\left(P\,dx+Q\,dy\right)+\int_{C_2}P\,dx+Q\,dy$$

Note that C_2 is oriented counter-clockwise while C_1 is clockwise. Thus, Green's theorem extends to domains with finite number of holes also:

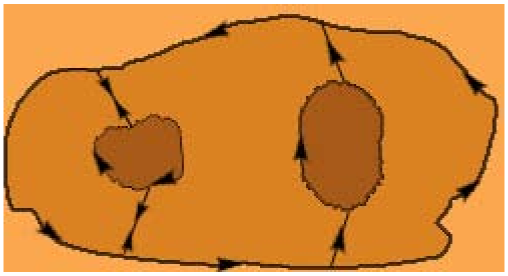

Figure: Splitting a region into simple regions

48.1.6 Example:

Consider the line integral of the vector field

$$\mathbf{F}(x,y)=\frac{-y}{x^2+y^2}\,\mathbf{i}+\frac{x}{x^2+y^2}\,\mathbf{j}\,,\ (x,y)\neq(0,0)\,,$$

over any piecewise smoothly closed curve C that does not include the origin. Let C be oriented counter-clockwise.

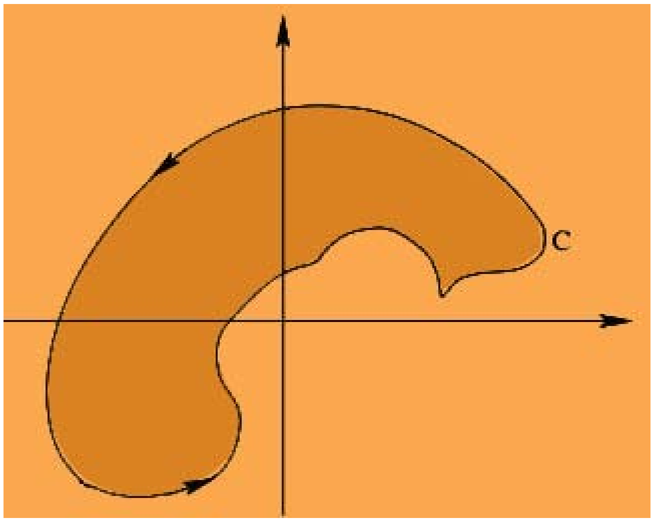

Figure: Piecewise smooth curve not including origin

Then, by Green's theorem, if R is the region enclosed,

$$\oint_C\left(-\frac{y}{x^2+y^2}\right)dx+\left(\frac{x}{x^2+y^2}\right)dy=\iint_R\left[\frac{\partial}{\partial x}\left(+\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right]dy$$
$$=\iint_R 0\,dxdy$$
$$=0.$$

Let us also compute the line integral of $\mathbf{F}$ over a piecewise smooth, simple closed curve C that includes origin, C oriented counter clockwise.

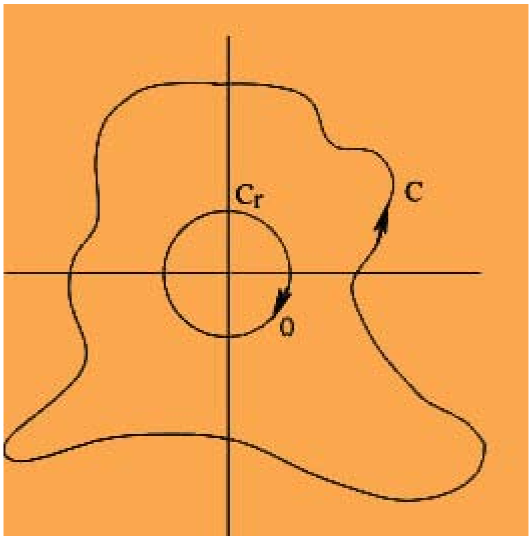

Figure: Piecewise closed curve including origin

Since C does not intersect the origins, we can find some $r>0$ such that the circle

$$C_r := \left\{(x,y)\in \mathbb{R}^2 \,|\, x^2+y^2=r^2\right\}$$

lies completely inside C. We orient C_r clockwise, and consider the region R enclosed by C and C_r. Thus, by Green's theorem, we have

$$\oint_C\left(-\frac{y}{x^2+y^2}\right)dx+\left(\frac{x}{x^2+y^2}\right)dy$$
$$+\int_{C_r}\left(-\frac{y}{x^2+y^2}\right)dx+\left(\frac{x}{x^2+y^2}\right)dy$$
$$=\iint_R\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right]$$
$$=0.$$

Hence

$$\oint_C\left(-\frac{y}{x^2+y^2}\right)dx+\left(\frac{x}{x^2+y^2}\right)dy=\oint_{-C_r}\left(-\frac{y}{x^2+y^2}\right)dx+\left(\frac{x}{x^2+y^2}\right)dy$$
$$=\int_0^{2\pi}dx$$
$$=2\pi.$$

48.1.6 Some other forms of Green's Theorem:

1. **Flux form of Green's theorem:**

   Consider a simple closed curve C oriented counter-clockwise. Let R be the region enclosed by it. Let the arclength parameterizations for C be

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}.$$

Then, the unit tangent vector to C at a point is given by

$$\mathbf{T}(s) = r'(s) = x'(s)\mathbf{i} + y'(s)\mathbf{j}$$

and the unit-normal, pointing outward to the region R, is given by

$$\mathbf{n}(s) = y'(s)\mathbf{i} - x'(s)\mathbf{j}.$$

Let

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$$

be a continuously differentiable vector-field in a domain D which includes both C and R. Then, by Green's theorem

$$\iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dxdy = \oint_C Pdx + Qdy. \qquad \text{----------(49)}$$

If we write

$$\mathbf{G} = Q\mathbf{i} - P\mathbf{j},$$

then,

$$\operatorname{div}(\mathbf{G}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \qquad \text{----------(50)}$$

and

$$\mathbf{G} \cdot \mathbf{n} = (Q\mathbf{i} - P\mathbf{j}) \cdot (y'(s)\mathbf{i} - x'(s)\mathbf{j})$$

$$= P x'(s) + Q y'(s). \qquad \text{----------(51)}$$

From (49), (50) and (51), we have

$$\iint_R \operatorname{div}(G)\, dxdy = \oint_C (G.\mathbf{n})\, ds.$$

Thus, the flux of $\mathbf{G} = \mathbf{Q}i - \mathbf{P}j$ across a simple closed curve C is the double integral of the divergence of G over R, the region enclosed by C. This is called the **flux-form of Green's theorem** .

2. **Work form of Green's Theorem**

Consider a force field

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j}.$$

and representing

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \operatorname{curl}(\mathbf{F}) \cdot \mathbf{k} \qquad \text{----------(52)}$$

Also on the other hand

$$\oint_C Pdx + Qdy = \oint_C (P\mathbf{i} + Q\mathbf{j}) \cdot \left(x'(s)\mathbf{i} + y'(s)\mathbf{j}\right) ds$$

$$= \oint_C (\mathbf{F}.\mathbf{T})\, ds. \qquad \text{----------(53)}$$

Thus, by Green's theorem, using (52), (53) we have

$$\oint_C (\mathbf{F}.\mathbf{T}).ds = \iint_R \left[\text{curl}\,(\mathbf{F}.\mathbf{k})\right] dxdy.$$

The integral on the left represents the work done in moving against a force field $\mathbf{F}$ along the curve C. This is called the **work from of the Green's theorem** . This is also called the **Curl form** of the Green's theorem

Practice Exercises

1. Verify Green's theorem in each of the following cases:
   1. $P(x,y) = -xy^2; Q(x,y) = x^2 y;\ R\left\{(x,y) \mid x \geq 0, 0 \leq y \leq 1 - x^2\right\}$
   2. $P(x,y) = 2xy; Q(x,y) = e^x + x^2;$ where, R is the region inside the triangle with vertices $(0,0),(1,0),(1,1)$.
2. Verify Green's theorem for

   $$F\,(x,y) = (x^2 + y)\,\mathbf{i} + (4\,x - \cos y)\,\mathbf{j},$$

   R the region that is inside the square with vertices $(0,0),(5,0),(5,5)$ and $(0,5)$ but is outside the rectangle with vertices $(1,1),(3,1),(3,2)$ and $(1,2)$ and C is the boundary of this region.
   Answer: Both integral are equal to 69
3. Show that Green's theorem is applicable for the region R, the inside of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ and out side the circle $x^2 + y^2 = 1$, and hence compute

   $$\int_C 2\,xydx + (x^2 + 2)\,dy$$ where C is the boundary of the region R.

   Answer: 10π
4. Let f, g be differentiable function of a single variable and C is a piecewise smooth simple closed path. Show that $\int_C f\,(x)\,dx + g\,(y)\,dy = 0.$
5. Verify the flux form and the curl form of the Green's theorem for the following:
   1. $F\,(x,y) = (x - y)\,\mathbf{i} + x\,\mathbf{j}$, and R is the region bounded by the circle $\mathbf{r}\,(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}\,, 0 \leq t \leq 2\pi$
   2. $F\,(x,y) = (x - y)\,\mathbf{i} + (y - x)\,\mathbf{j}$, and R is the region bounded by the lines $x = 0\,, x = 1\,, y = 0\,, y = 1.$

   Answer:

   (i) π

   (ii) 2

Recap:

In this section you have learnt the following

- Green's theorem which connects the line integral with the double integral.

[Section 48.2]

Objectives

In this section you will learn the following :

- Computations of line integrals and area enclosed by a curve.
- Sufficient condition for conservativeness of a vector field.
- Integral formulae for the laplacian.

48.2 Applications of Green's Theorem:

48.2.1 Evaluations of line-integrals :

Green's theorems help us to evaluate certain line integrals by evaluating the corresponding double integral. For example, we want to compute the line integral

$$I = \oint_C \left(5 - xy - y^2\right) dx - \left(2xy - x^2\right) dy$$

where C is the boundary of the square

$$R = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}.$$

Using Green's theorem, we can convert this to a double integrals. Let C, the boundary of the region R, be given anticlockwise orientation.

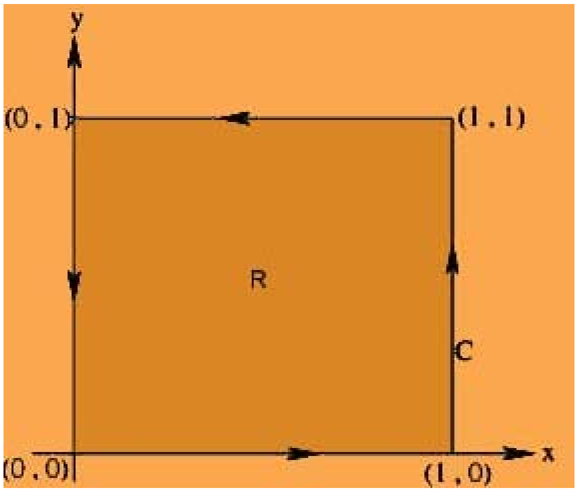

Figure: Design R, with oriented boundary

Then, by Green's theorem,

$$I=\iint_R\left[\frac{\partial}{\partial x}(-2xy+x^2)-\frac{\partial}{\partial y}(5-xy-y^2)\right]dxdy$$

$$=\int_0^1\left(\int_0^1(-2y+2x+x+2y)\right)dxdy$$

$$=\int_0^1\left(\int_0^1 3x\,dx\right)dy=\frac{3}{2}\int_0^1 dy=\frac{3}{2}.$$

48.2.2 Area enclosed by a curve :

Let C be a simple closed curve in $\mathbb{R}^2$, enclosing a region R. Consider the vector field

$$\mathbf{F}=P\mathbf{i}+Q\mathbf{j},$$

where $P(x,y)=y$ and $Q(x,y)=0$ in R. Then, by Green's theorem,

$$-\oint_C ydx=\iint_R dxdy. \qquad \text{----------(54)}$$

Similarly, if $P(x,y)=0$ and $Q(x,y)=x$ in R, we have by Green's theorem,

$$\oint_C xdy=\iint_R dxdy. \qquad \text{----------(55)}$$

From (54) and (55), we have

$$2\left(\text{Area of region R}\right)=\ 2\iint_R dxdy\ =\ \oint_C xdy-ydx.$$

Thus, the area of the region R is given by

$$A=\frac{1}{2}\left(\oint_C xdy-ydx\right).$$

In case C have parameterizations:

$$\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j},\ a\le t\le b,$$

Then, the area enclosed by C is given by

$$A=\frac{1}{2}\int_a^b\left[x(t)y'(t)-y(t)x'(t)\right]dt.$$

If C has polar representation

$$x(\theta)=r\cos\theta, y(\theta)=r\sin\theta,\ \theta_0\le\theta\le\theta_1,$$

we get

$$A=\frac{1}{2}\int_{\theta=\theta_0}^{\theta_1} r^2\left(\cos^2\theta+\sin^2\theta\right)d\theta$$

$$=\frac{1}{2}\int_{\theta_0}^{\theta_1} r^2\,d\theta.$$

48.2.3 Sufficient condition for a vector-field to be conservative :

Suppose D is a simply-connected open set in $\mathbb{R}^2$ and $\mathbf{F}:D\to\mathbb{R}^2$ is a vector field,

$$\mathbf{F}=P\mathbf{i}+Q\mathbf{j}$$

such that

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x} \text{ in } D. \qquad ----------(56)$$

In this case, we claim that there exists a function $\phi : D \to \mathbb{R}$ such that

$$\frac{\partial \phi}{\partial x}=P \text{ and } \frac{\partial \phi}{\partial y}=Q.$$

In other words, a planer-vector field in a simply connected region is conservative if (56) is satisfied. In view of theorem $47.2.8$, it is enough to show that the line integral

$$\oint_C Pdx+Qdy=0.$$

for every simple closed curve C in D. This indeed happens: , first given a simle closed curve C, as D is simply conected, we can find a region $R \subset D$ such that C is the boundary of R. Then, by Green's theorem,

$$\oint_C Pdx+Qdy=\iint_R\left(\frac{\partial \phi}{\partial x}-\frac{\partial y}{\partial y}\right)dxdy=0.$$

Hence, $\mathbf{F}$ is conservative.

48.2.4 Change of Variable formula in $\mathbb{R}^2$:

As another application of the Green's theorem we can deduce the change of variables formula, as stated in theorem 42.1.2. Let

$$T : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$$

be a transformation defined on an open simply connected subset U of $\mathbb{R}^2$ such that if

$$T(u,v)=(X(u,v),Y(u,v))=(x,y),$$

then X,Y have continuous partial derivatives, both X,Y are one-one onto and

$$J(T,(u,v))=\begin{vmatrix}\dfrac{\partial X}{\partial u} & \dfrac{\partial Y}{\partial u}\\[2ex] \dfrac{\partial X}{\partial v} & \dfrac{\partial Y}{\partial v}\end{vmatrix}\neq 0, \text{ for any } (u,v).$$

Let $R^* \subset U$ be simply connected, with $T(R^*)=R$. Then, we claim that

$$\iint_R f(x,y)\,dxdy=\iint_{R^*} f(X(u,v),Y(u,v)\,|\,J(T,(u,v))\,|)\,dudv.$$

Let us first prove it in the particular case when

$$f=1 \text{ and } J(T,(u,v))>0 \text{ for all } (u,v)\in \mathbb{R}^*.$$

Similar arguments can be given for general case. Let C be the boundary of R and C^* the boundary of $R^* := T(R)$. We observe that as $J(T,(u,v))>0$ if C is oriented anticlockwise, then $T(C)=C^*$ is also anticlockwise. Let $\alpha : [a,b] \to \mathbb{R}^*$

$$\alpha(t)=(u(t),v(t)), t\in[a,b]$$

be a parameterizations of C^* in the anticlockwise direction. Then

$$\beta : [a,b] \to \mathbb{R}$$

$$\beta(t) := \alpha(X(u(t), v(t)), Y(u(t)), (v(t)), t \in [a,b]$$

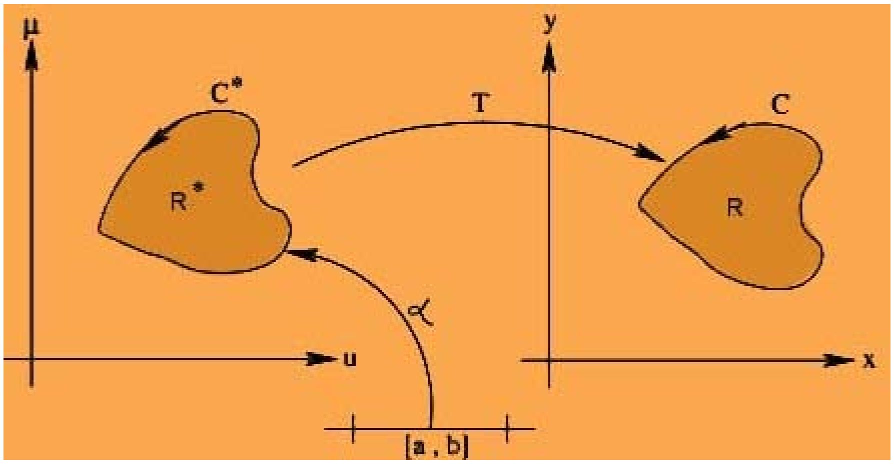

Figure: Transframation T

is a parameterizations of C in the anticlockwise direction. Now by Green's theorem

$$\begin{aligned}
\iint_R dxdy &= \oint_C xdy \\
&= \int_a^b X\big(u(t), v(t)\big)\left(\frac{\partial Y}{\partial u}\frac{du}{dt} + \frac{\partial Y}{\partial v}\frac{dv}{dt}\right)dt \\
&= \int_a^b \left[\left(X\frac{\partial Y}{\partial u}\right)\frac{du}{dt} + \left(X\frac{\partial Y}{\partial v}\right)\frac{dv}{dt}\right]dt \\
&= \oint_{C^*}\left(X\frac{\partial Y}{\partial u}du + X\frac{\partial Y}{\partial v}dv\right) \\
&= \iint_{R^*}\left[\frac{\partial}{\partial u}\left(X\frac{\partial Y}{\partial v}\right) - \frac{\partial}{\partial v}\left(X\frac{\partial Y}{\partial u}\right)\right]dudv \\
&= \iint_{R^*} J\big(T(u,c)\big)dudv.
\end{aligned}$$

48.2.5 Integral formulas for the Laplacian :

Recall that the operator ∇^2, called the **Laplacian operator,** in the plane is defined by

$$\nabla^2(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

where f is any scalar field having second order partial derivatives. Let us consider one such function $w(x,y)$ and define the vector field

$$\mathbf{F} := -\frac{\partial w}{\partial y}\mathbf{i} + \frac{\partial w}{\partial x}\mathbf{j}.$$

We apply Green's theorem to $\mathbf{F}$ to get

$$\iint_R (\nabla^2 w)\, dxdy = \iint_R \left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) \right] dxdy$$

$$= \int_C \left(-\frac{\partial w}{\partial y} dx + \frac{\partial w}{\partial x} dy \right)$$

$$= \oint_C \left(-\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{dy}{ds} \right) ds$$

$$= \oint_C \left(\frac{\partial w}{\partial x}\mathbf{i} + \frac{\partial w}{\partial y}\mathbf{j} \right)\left(\frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j} \right) ds$$

$$= \oint_C (\nabla w).\mathbf{n} ds,$$

where $\mathbf{n}$ is the unit normal to C for the given orientation. Recalling that for a differentiable function w, $(\nabla w) \cdot \mathbf{n}$ is the directional derivative of w in the direction of $\mathbf{n}$, let us write

$$\frac{\partial w}{\partial \mathbf{n}} := (\nabla w) \cdot \mathbf{n}.$$

Then, we have

$$\iint_R (\nabla^2 w)\, dxdy = \int_C \frac{\partial w}{\partial \mathbf{n}}\, ds. \qquad \text{----------(57)}$$

In the particular we have the following,

$$\text{if } \frac{\partial w}{\partial n} \equiv 0 \text{ on } C \text{ then } \iint_R \nabla^2 w\, dxdy = 0. \qquad \text{----------(58)}$$

Another such formula is obtained by considering a scalar field w as above, and observing

$$w\frac{\partial w}{\partial n} = w(\nabla w \,.\, \mathbf{n}),$$

by Green's theorem we get

$$\oint_C w\frac{\partial w}{\partial n}\, ds = \oint_C w\left(\nabla w.\, \mathbf{n}\right)\, ds$$

$$= \frac{1}{2}\oint_C \left(\nabla\left(w^2\right).\, \mathbf{n}\right)\, ds$$

$$= \frac{1}{2}\iint_R \operatorname{div}\ (\nabla (w^2))\, dxdy$$

$$= \iint_R \operatorname{div}\, (w \nabla w)\, dxdy$$

$$= \iint_R \left[(\nabla w \,.\, \nabla w + w)(\nabla^2 w) \right] dxdy$$

$$= \iint_R \left[\, |\nabla (w)|^2 + w\, (\nabla^2 w) \right] dxdy. \qquad \text{----------(59)}$$

Equations (57), (58) and (59) are called **Green's identities** . Scalar fields $w(x,y)$ which satisfy the relation $\nabla^2 w = 0$, are called **harmonic functions** . For a harmonic function w, if $\frac{\partial w}{\partial n} = 0$ on C, then equation (59) gives

$$0 = \iint_R |\nabla w|^2 \, dxdy.$$

Hence

$$|\nabla w|^2 = 0, \ \text{ i.e., } \ \frac{\partial w}{\partial x}, \frac{\partial y}{\partial y} = 0, \text{ in } R.$$

In case R is such that any two points can be connected by a path parallel to axes, this implies that w is a constant scalar field. Thus,

for a harmonic function w, if its rate of change along the unit tangent direction to C in zero, then w is constant in the region enclosed by C.

Practice Exercises

1. Use Green's theorem to evaluate the integral $\oint_C y^2\,dx + x\,dy$ where C is the boundary of the region R given by

    1. R is the square with vertices $(0,0),(2,0),(2,2),(0,2)$.
    2. R is the square with vertices $(\pm 1, \pm 1)$.
    3. R is the disc of radius 2 and center $(0,0)$.

    Answers:

    (i) -4,

    (ii) 4,

    (iii) 4π

2. Find the area of the following regions using Green's theorem:

    1. The area lying in the first quadrant of the cardioid $r = a(1-\cos\theta)$.
    2. The region under one arch of the cycloid $r = a(t-\sin t)i + a(1-\cos t)\underline{0}\,j,\ 0 \le t \le 2\pi$.
    3. The region bounded by the limaçon $r = 1-2\cos\theta,\ 0 \le \theta \le \pi/2$ and the two axes.

    Answer:

    (i) $\frac{a^2}{8}(3\pi - 8)$.

    (ii) $2\pi a^2$

    (iii) $\frac{3\pi - 8}{4}$

3. Compute the area enclosed by

    1. the cardioid: r= a(1- $\cos\theta), 0 \le \theta \le 2\pi$,
    2. the lemniscate: $r^2 = a^2 \cos 2\theta; -\pi/4 \le \theta \le \pi/4$.

    Answer:

(i) $A=\frac{3\pi a^2}{2}$.

(ii) $A=a^2/2$.

4. Let C be a simple closed curve in the xy-plane enclosing a region R. The polar moment of inertia of R is defined to be

$$I_0 := \iint_R (x^2+y^2)\,dxdy.$$

Show that

$$I_0 := \frac{1}{3}\oint_C x^3dy - y^3dx.$$

5. Consider $a=a(x,y), b=b(x,y)$ having continuous partial derivatives on the unit disc $D=\{(x,y) \mid x^2+y^2\le 1\}$. If $a(x,y)\equiv 1, b(x,y)\equiv y$ on circle C, the boundary of D. Let

$\boldsymbol{u} := a\boldsymbol{i}+b\boldsymbol{j}; v := (a_x-a_y)\mathbf{i}+(b_x-b_y)\,\boldsymbol{j}, \mathbf{w}=(b_x-b_y)\mathbf{i}+(a_x-a_y)\boldsymbol{j}$.

Using Green's theorem show that $\iint_D \boldsymbol{u}\cdot\boldsymbol{u}\,dxdy=0$ and $\iint_D \boldsymbol{u}\cdot\mathbf{w}\,dxdy=-\pi$.

6. Let C be any closed curve in the plane. Show that $\oint_C \nabla(x^2-y^2)\cdot\mathbf{n}ds=0$.

7. Using Green's identities , compute

$$\oint_C \frac{\partial w}{\partial n}ds$$

for the following:

1. $w(x,y)=e^x+e^y$, $R=\{(x,y) \mid 0\le x\le 2, 0\le y\le 1\}$.
2. $w(x,y)=e^x\sin y$, and R is the triangle with vertices $(0,0),(4,2),(0,2)$.

Answer:

(i) e^2+2e-3.

(ii) 0

Recap

In this section you have learnt the following

- Computations of line integrals and area enclosed by a curve.
- Sufficient condition for conservativeness of a vector field.
- Integral formulae for the laplacian.