Module 10 :   Scaler fields, Limit and   Continuity

Lecture 28 : Series of functions [Section 28.1]

Objectives

In this section you will learn the following :

- The notion of distance in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Notions of neighborhoods of points.
- Notion of convergence of sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$.

28 .1 Spaces $\mathbb{R}^2$ and $\mathbb{R}^3$

In these notes we shall develop many concepts that helps us understanding functions of several variables: limits, continuity, differentiability, problems of maxima minima, and integration. We shall see how these concepts resemble, and are different, that of a single variable. Since most of the theorems that can be proved for functions of two variables can be extended to functions of three or more variables, we confine our analysis to two/three variables only.

28.1.1 The spaces $\mathbb{R}^2$ and $\mathbb{R}^3$

Recall that for the function of one variable, the underlying set was $\mathbb{R}$, the set of real numbers. For two variables, corresponding role is played by the set $\mathbb{R}^2$, the Cartesian product of $\mathbb{R}$ with itself:

$$\mathbb{R}^2 := \{(x,y) \mid x,y \in \mathbb{R}\},$$

the set of all ordered pairs (x,y) , where $x,y \in \mathbb{R}$. Geometrically, $\mathbb{R}^2$ is represented by points in the plane. For $(x,y) \in \mathbb{R}^2$, x,y represents the Cartesian coordinates of the point P in plane with respect to an orthogonal system of coordinate axes. Similarly, $\mathbb{R}^3$ space can be represented by

$$\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\},$$

where x,y,z called the Cartesian coordinates or components of a point.

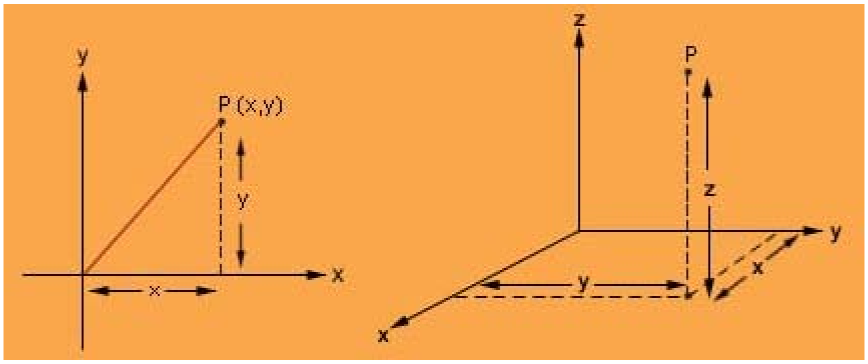

Figure 1. Coordinates of a point in $\mathbb{R}^2$ and $\mathbb{R}^3$

In the language of vectors, a point P in $\mathbb{R}^2/\mathbb{R}^3$ represents the vector $\boldsymbol{u} = OP$, with initial point O and final point P. Vectors can be added, scalar multiplied component wise. For example for vectors $\boldsymbol{u} = (x_1, y_1, z_1)$, $\boldsymbol{v} = (x_2, y_2, z_2)$, the vector sum is

$$\boldsymbol{u} + \boldsymbol{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

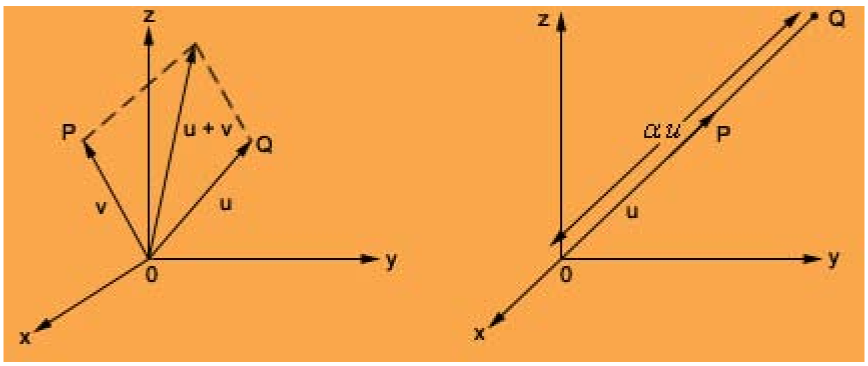

Figure 2. Scalar multiplication and addition of vectors

and the multiplication of $\boldsymbol{u} = (x_1, y_1, z_1)$ by a scalar α is

$$\alpha\boldsymbol{u} = (\alpha x_1, \alpha y_1, \alpha z_1).$$

The origin represents the vector

$$\mathbf{0} = (0, 0, 0).$$

Various properties of these operations are given in Exercise 1. We identify points in $\mathbb{R}^3/\mathbb{R}^3$ with the vectors they represent. A vector $\mathbf{u} = (x, y, z)$ is also represented as $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are vectors of unit length along the respective coordinate axes.

The notion of 'closeness' between points on $\mathbb{R}^2/\mathbb{R}^3$ can be described in terms of the magnitudes of vectors. We

do this for points in $\mathbb{R}^2$. For points in $\mathbb{R}^3$, these can be defined by adding suitable terms corresponding to the third component.

Similar to the situation in $\mathbb{R}$, we use the above notion of distance to define the neighborhoods in $\mathbb{R}^2/\mathbb{R}^3$. For $x \in \mathbb{R}, r > 0$, the r-neighborhood of a point in $\mathbb{R}$ was the open interval $(x-r, x+r)$, that is the set of all points $y \in \mathbb{R}$ which are at most at a distance r from x. This motivates the next definition.

28.1.2 Definition:

(i) For a point P in $\mathbb{R}^2$ with $P = (x, y)$,

$$\| P \| := \sqrt{x^2 + y^2}$$

is called the **norm or magnitude** of the vector.

(ii) For $P = (x_1, y_1), Q(x_2, y_2) \in \mathbb{R}^2$, the distance between then, denoted by $\| P - Q \|$,

is defined to be

$$\| P - Q \| := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

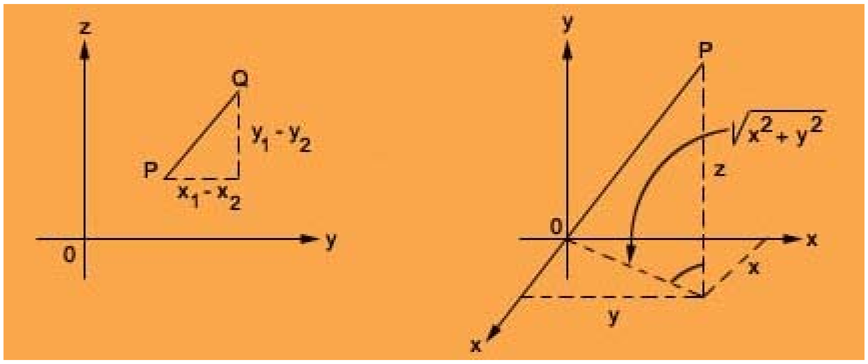

Figure 3. Norm of a vector

(iii) For vectors $\boldsymbol{u} = P = (x_1, y_1)$ and $\boldsymbol{v} = Q = (x_2, y_2)$, the **dot product** of these vectors is defined to be

$\boldsymbol{u} \cdot \boldsymbol{v} = x_1 x_2 + y_1 y_2.$

28.1.3 Theorem:

For vectors $\boldsymbol{u} = (x_1, y_1), \boldsymbol{v} = (x_2, y_2), \boldsymbol{w} = (x_3, y_3),$ and scalars $\alpha, \beta \in \mathbb{R},$ the following holds:

(i) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}.$

(ii) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{v} + (\boldsymbol{u} + \boldsymbol{w}).$

(iii) $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}.$

(iv) $\boldsymbol{u} + (-1)\boldsymbol{u} = \mathbf{0}.$

(v) $(\alpha + \beta)\boldsymbol{u} = \alpha\boldsymbol{u} + \beta\boldsymbol{u}.$

(vi) $\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{u} + \alpha\boldsymbol{v}.$

(vii) $(\alpha\beta)\boldsymbol{u} = \alpha(\beta\boldsymbol{u}).$

(viii) $1(\boldsymbol{u}) = \boldsymbol{u}$ and $0(\boldsymbol{u}) = \mathbf{0}.$

(ix) $\boldsymbol{u} \cdot \boldsymbol{v} = \| \boldsymbol{u} \| \; \| \boldsymbol{v} \| \cos\theta,$ where $0 \le \theta \le \pi$ is the angle between the vectors OP and $OQ.$

(x) $| \boldsymbol{u} \cdot \boldsymbol{v} | \le \| \boldsymbol{u} \| \| \boldsymbol{v} \|$. This is called **Cauchy- Schwarz** inequality.

(xi) $\| \boldsymbol{u} \|^2 = \boldsymbol{u} \cdot \boldsymbol{u}.$

(xii) $\| \boldsymbol{u} \| \ge 0$ *and* $\| \boldsymbol{u} \| = 0$ if and only if $\boldsymbol{u} = 0.$

(xiii) $\| \boldsymbol{u} + \boldsymbol{v} \| \le \| \boldsymbol{u} \| + \| \boldsymbol{v} \|$ .This is called **triangle- inequality**.

Proof:
All the statements are easy to prove.

28.1.4 Definition:

For $P=(x_0,y_0)\in\mathbb{R}^2$, the r**-circular neighborhood** of the point (x_0,y_0) is the set of all points $Q(x,y)\in\mathbb{R}^2$, which are at a distance at most r from P, i.e., the set

$$B(P,r)=B((x_0,y_0),r)=\{\,Q(x,y)\in R^2\,|\,\|P-Q\|<r\}.$$

This is also called an **open ball** at $P(x_0,y_0)$ with radius r. The open ball $B((x_0,y_0),r)$ without the center (x_0,y_0) is called a **deleted open ball** and will be denoted by $B_0((x_0,y_0),r)$. Thus,

$$B_0((x_0,y_0),r)=\{(x,y)\in\mathbb{R}^2\,|\,0<\|(x,y)-(x_0,y_0)\|<r\}$$
$$=\{(x,y)\in\mathbb{R}^2\,|\,0<\sqrt{(x-x_0)^2+(y-y_0)^2}<r\}.$$

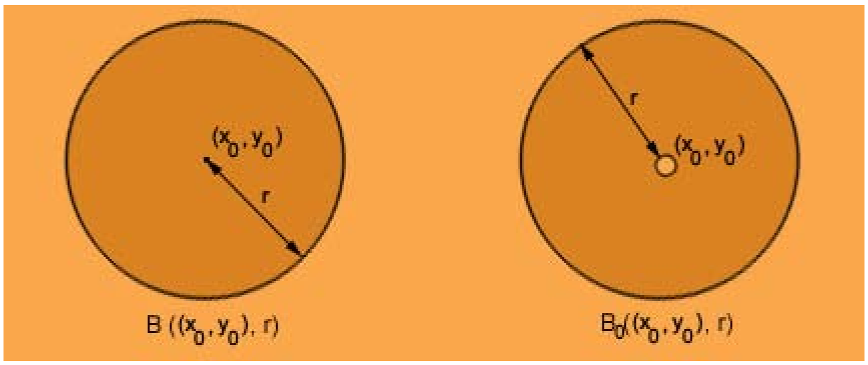

Figure 4. Open ball and deleted open ball.

For points $P(x_1,y_1,z_1),Q(x_2,y_2,z_2)\in\mathbb{R}^3$,

$$\|P-Q\|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2},$$

and

$$B_0((x_0,y_0,z_0),r)=\{(x,y,z)\in\mathbb{R}^3\,|\,0<\|(x,y,z)-(x_0,y_0,z_0)\|<r\}.$$

Recall that in $\mathbb{R}$, the notion of an sequence helped us to understand the various properties of functions of a single variable. We can do the same in $\mathbb{R}^2$ and $\mathbb{R}^3$.

28.1.5Definition:

An sequence in $\mathbb{R}^2$ is an ordered collection of points $P_1,P_2,P_3,\ldots$. We write it as $\{P_n\}_{n\geq 1}$. In analogy with sequence in $\mathbb{R}$, we say sequence $\{P_n\}_{n\geq 1}$ in $\mathbb{R}^2$ **converges** to a point $Q\in\mathbb{R}^2$ if the distance between P_n

and Q goes to zero, that is

$$\lim_{n\to\infty}\|P_n-Q\|=0.$$

We can write it in terms of ε- N definition. This means that given $\varepsilon>0$, there exists an $N\in\mathbb{N}$, such that

$$\|P_n-Q\|<\varepsilon \text{ for all } n\geq N.$$

We write this as

$$P_n\to Q \quad \text{as} \quad n\to\infty \quad \text{or} \lim_{n\to\infty}P_n=Q.$$

An equivalent criterion is given by the following theorem.

28.1.6Theorem:

Let $P_n(x_n, y_n)$ and $Q(x, y) \in D \ \forall\ n \geq 1$. *Then, the following are equivalent:*

(i) $\lim_{n\to\infty} P_n = Q$.

(ii) $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$.

Practice Exercises

(1) Prove theorem 28.1.3.

(2) For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, and $\alpha \in \mathbb{R}$, prove the following:

(i) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{w}$.

(ii) $(\mathbf{u}\cdot\mathbf{v}) = (\mathbf{v}\cdot\mathbf{u})$

(iii) $(\alpha\mathbf{u})\cdot\mathbf{v} = \alpha(\mathbf{u}\cdot\mathbf{v})$.

(iv) $<\mathbf{u}, \mathbf{v}>= 0$ for every $\mathbf{v} \in \mathbb{R}^3$ implies $\mathbf{u} = \mathbf{0}$.

(v) $||\mathbf{u}-\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2(\mathbf{u}.\mathbf{v})$.

(3) Give examples to show that for $\mathbf{u} \neq \mathbf{0}$

$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$

need not imply $\mathbf{v} = \mathbf{w}$.

(4) Cross product of vectors

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j} + u_3\,\mathbf{k}\ ,\ \mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$$

The cross product of $\mathbf{u}$ with $\mathbf{v}$ is the vector, $\mathbf{u}\times\mathbf{v}$, given by

$$\mathbf{u}\times\mathbf{v} = (u_2\,v_3 - u_3\,v_2)\,\mathbf{i} - (u_1\,v_3 - u_3\,v_1)\,\mathbf{j} + (u_1\,v_2 - u_2\,v_1)\,\mathbf{k}.$$

(A convenient way to remember this is to write $\mathbf{u}\times\mathbf{v}$ as a determinant:)

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Prove the following statements for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$:

(i) $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$.

(ii) $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$.

(iii) $\alpha(\mathbf{u}\times\mathbf{v}) = (\alpha\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(\alpha\mathbf{v})$.

(iv) $\mathbf{u}\times\mathbf{0} = \mathbf{0}$.

(v) $\mathbf{u}\times\mathbf{v} = \mathbf{0}$

(vi) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$.

(vii) $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{0} = v\cdot(\mathbf{u}\times\mathbf{v})$.

(viii) $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\varphi$, where φ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

(ix) $\mathbf{u}\times\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u} = \alpha\mathbf{v}$ for some $\alpha \in \mathbb{R}$.

(5) For $\varepsilon > 0$, let

$$R((x_0, y_0), \varepsilon) = \{(x, y) \in \mathbb{R}^2 \,|\, |x - x_0| < \varepsilon \text{ and } |y - y_0| < \varepsilon\}.$$

This is called an ε**-rectangular neighborhood** at (x_0, y_0). Show that every circular neighborhood of a point (x_0, y_0) includes a circular neighborhood and vice-versa.

(6) A sequence $\{P_n\}_{n\geq 1}$ in $\mathbb{R}^2$ is said to be **Cauchy** if for all $\varepsilon > 0$, there exists N such that $\|P_n - P_m\| < \varepsilon$ for all $n, m \geq N$. Prove the equivalence of the following:

(i) $\{P_n\}_{n\geq 1}$ is convergent

(ii) Both , $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$, are Cauchy sequence in $\mathbb{R}$.

(iii) $\{P_n\}_{n\geq 1}$ is Cauchy in $\mathbb{R}^2$.

Recap

In this section you have learnt the following

- The notion of distance in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Notions of neighborhoods of points.
- Notion of convergence of sequences in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Continuity

Lecture 28 : Series of functions [Section 28.2]

Objectives

In this section you will learn the following :

- The notion of functions with domains in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Notions of contour lines and level curves.

28 .2 Functions of two or three variables:

Functions of two or more variables arise in almost every field. For example, in topography, the height of the land depends on the two coordinates that gives its location. The reaction rate of two chemicals A and B depends upon their concentrations A and B, and the temperature t. The pressure at a place can be regarded as a function of time, temperature and the location of the place. The amount of food grown in India depends upon the amount of rain and the amount of fertilizer used. The strength of gravitational attraction between two bodies depends on their masses and the distance between them.

28.2.1 Definition:

For $D \subseteq \mathbb{R}^2 (\text{ or } \mathbb{R}^3)$, a function

$f : D \longrightarrow \mathbb{R}$

is called a function of two (or three) variables. The set D is called its domain and the set

$f(D) = \{f(x) \mid x \in D\}$

is called the **range** of the functions . If f is given by a formula, then the set of all those points $x \in \mathbb{R}^2 (\text{ or } \mathbb{R}^3)$ for which $f(x)$ is defined is called the natural domain of f .

28.2.2 Examples:

(i) Let

$$f(x, y) = \sqrt{16 - x^2 - y^2}.$$

Then f , has natural domain the set

$$\{(x, y) \in R^2 \mid x^2 + y^2 \leq 16\},$$

and its range is the interval $[0, 4]$.

(ii) Let

$$f(x, y) = \frac{x + y}{x - y}.$$

Then, the natural domain of f is the set

$$\{(x, y) \in R^2 \mid x \neq y\},$$

and its range its range is the set $\mathbb{R}$.

Recall that, a function of one variable can be represented geometrically as a curve in the plane. We may try to represent a function of two variables by a 'surface' in $\mathbb{R}^3$.

28.2.3 Definition:

Let $f : D \longrightarrow \mathbb{R}$ be a function of two variables. Let

$$G(f) = \{(x, y, z) \in R^3 \mid z = f(x, y), (x, y) \in D\} \subseteq R^3.$$

The set $G(f)$ is called the **graph** of the function f.

One can get an idea of $G(f)$, the graph of a function f, as follows:

28.2.4 Definition:

For a function f of two variables, for each $c \in \mathbb{R}$, the set

$$\{(x, y) \in D \mid f(x, y) = c\}$$

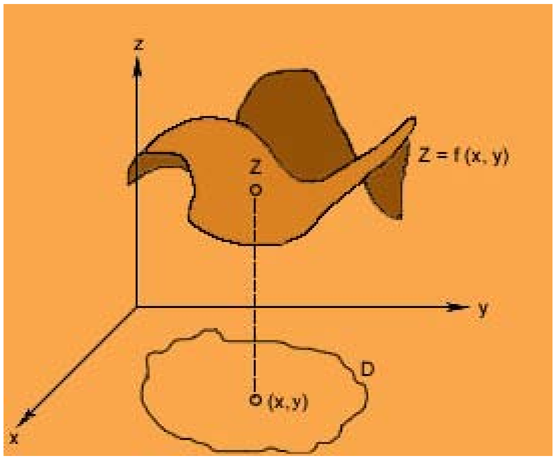

Figure 1. Graph of a unction of two variables

gives a curve in the xy-plane. This is called a level curve of the function $f(x, y)$ at $z = c$. This is the set of points in the domain of f where f takes a constant value c.

28.2.5 Examples:

(i) For

$$f(x, y) = \frac{x+y}{x-y}, \quad x \neq y \text{ and } c \in \mathbb{R}$$

the level curves of f are

$$\left\{(x, y) \;\; \frac{x+y}{x-y} = c\right\} = \left\{(x, y) \;\; x \neq y, y = \left(\frac{c-1}{c+1}\right) x\right\}.$$

These are the lines through the origin with slope $(c-1)/(c+1)$.

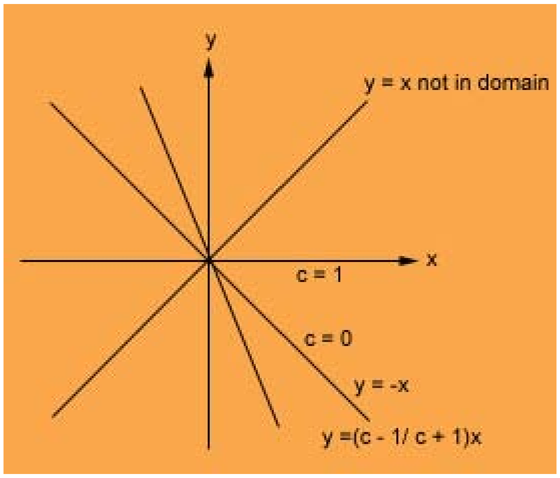

Figure 2. Level curves for $f(x,y)=\frac{x+y}{x-y}, y \neq x$.

(ii) For

$$f(x,y)=100-x^2-y^2,$$

the level curves are

$$\{(x,y)\in \mathbb{R}^2 \mid 100-x^2-y^2=c\}.$$

These are circles centered at origin with radius $\sqrt{100-c}$. Thus, they are defined only for $100 \geq c$.

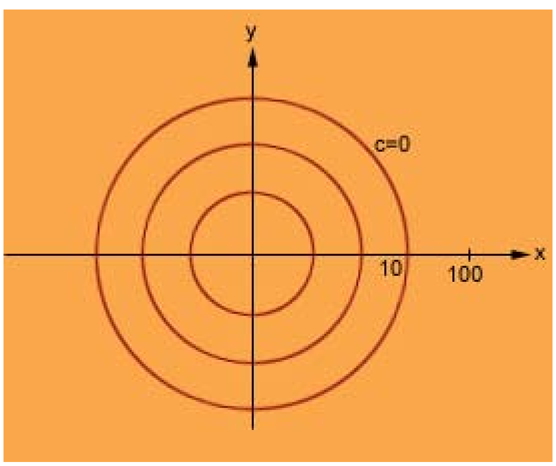

Figure 3. Level curves for $f(x,y)=100-x^2-y^2$.

Another concept that helps us to visualize $G(f)$, the graph of a function, is as follows:

28.2.6 Definition:

For every $c \in \mathbb{R}$, the **contour line** for a function

$$f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$$

is the curve in 3-space given by

$\{(x,y,z)\in \mathbb{R}^3 \mid (x,y)\in D, z=f(x,y)=c\}.$

Contour line indicates the points on the surface $z=f(x,y)$ that are at a given height $z=c$ or it is the section of the surface $z=f(x,y)$ by the plane $z=c$.

28.2.7Examples:

(i) For

$$f(x,y)=\frac{x+y}{x-y}, x\neq y,$$

the contour line for $c\in\mathbb{R}$ is

$$\left\{(x,y,z)\ \frac{x+y}{x-y}=z=c\right\},$$

that is

$$\left\{\left(x,\frac{c-1}{c+1}x,c\right)\ x\in\mathbb{R}\right\}.$$

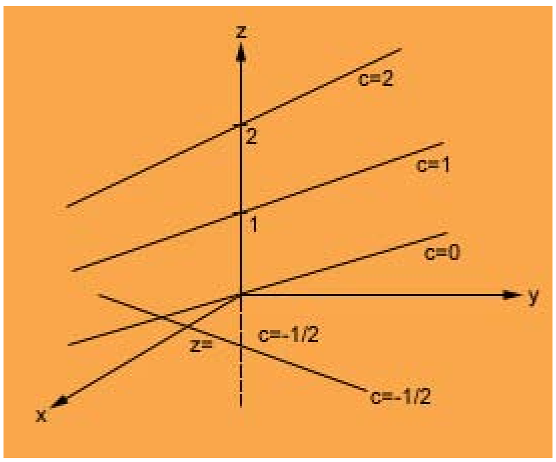

Figure 4. Contour lines for $f(x,y)=\frac{x+y}{x-y}, y\neq x.$

(ii) For

$$f(x,y)=100-x^2-y^2,$$

the contour line for $c\in\mathbb{R}$ is the circle

$$x^2+y^2=100-c \text{ in the plane } z=c.$$

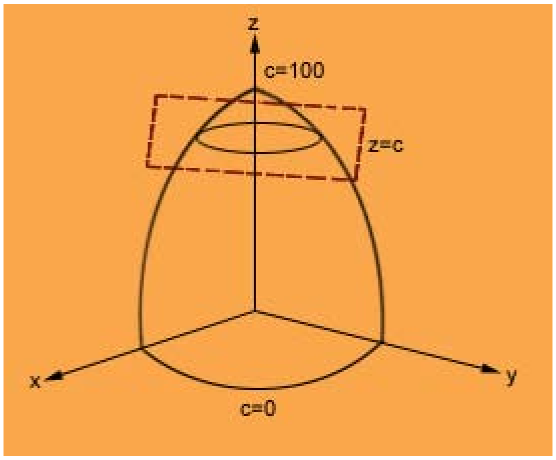

Figure 5. Contour lines for $f(x,y)=100-x^2-y^2$.

28.2.8 Note:

(i) Like taking sections of a surface by planes perpendicular to z-axis, one can also consider sections of the surface by planes perpendicular to x-axis or y-axis. These sections are care x**-traces** (y**-traces**) of the surface.

(ii) For functions of three variables, contour lines generalize to contour surface (or also called **level surfaces**).
For example for

$$f(x,y,z)=x^2+y^2+z^2,$$

the level surfaces are

$$x^2+y^2+z^2=k,$$

spheres concentric about the origin in the x,y,z coordinate system.

QUIZ

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Practice Exercises

(1) For the following functions find the natural domain and range:

(i) $z=\sqrt{x+y}$.

(ii) $z=\sqrt{x^2-y^2}$.

(iii) $z=\log(x+5y)$.

(iv) $z=\tan^{-1}\left(\dfrac{x^2}{x^2+y^2}\right)$.

Answers

(2) For the following functions, sketch the contour lines corresponding to $z=-2,-1,0,1,2,3$.

(i) $z=y^2$.

(ii) $z=x^2-y^2$.

(iii) $z=x^2y$.

(3) For the following, sketch the level curves for the following:

(i) $z = \cos(y)$.

(ii) $z = \cos(xy)$.

(iii) $z = | xy |$.

(iv) $z = \sin\left(\sqrt{x^2 + y^2}\right)$.

Answers

Recap

In this section you have learnt the following

- The notion of functions with domains in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- Notions of contour lines and level curves.