Module 15 : Vector fields, Gradient, Divergence and Curl

Lecture 43 : Vector fields and their properties [Section 43.1]

Objectives

In this section you will learn the following :

- Concept of Vector field.
- Various properties of vector fields.
- Continuity and differentiability of vector fields.
- Gradient vector field.

43.1 **Vector Fields and their properties**

Recall that, in module 11 we looked at functions

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R}, n = 1, 2, 3,$$

where D is an open subset of $\mathbb{R}^n$. Since these functions take scalar values, they are called *scalar fields* . For scalar fields, we also analyzed various concepts such as limit, continuity and differentiability. In this section we shall analyze functions

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m, \text{ where } m = 2, 3.$$

Such functions are called *vector fields.* Vector fields arise in mathematical representations of physical concepts, such as velocity, acceleration, and force in mechanics.

43.1.1 Definition:

A function

$$\boldsymbol{f} : D \subseteq \mathbb{R}^m \to \mathbb{R}^n, \text{ where } n = 1, 2, 3 \text{ and } m = 2, 3,$$

and D is an open subset of $\mathbb{R}^n$, is called a **vector-field** . Since for every $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^m$, we can write

$$\boldsymbol{f}(\boldsymbol{x}) = f_1(\boldsymbol{x})i + f_2(\boldsymbol{x})j + f_3(\boldsymbol{x})k,$$

where i, j, k are the unit vectors along the coordinate axes. The functions

$$f_i : D \subseteq \mathbb{R}^n \to \mathbb{R}, i = 1, 2, 3,$$

are scalar fields, and are called the *component scalar fields* of $\boldsymbol{f}$.

43.1.2 Examples:

1. **Gravitational field:**

   Let a particle A of mass M be fixed at a point P_0 and let a particle B of mass m be free to take any position P in space. By Newton 's law of gravitation, B is attracted towards A and this force of attraction, denoted by $\boldsymbol{F}(P)$, at P, is directed from P to P_0 with magnitude proportional to mM/r^2, r being the distance between the points P and P_0. Let

   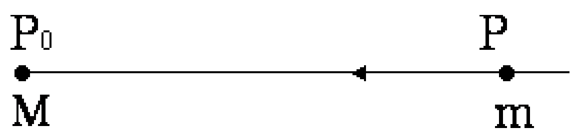

   Figure 158. Gravitational force field

   $$\|\boldsymbol{F}(P)\| = G\frac{Mm}{r^2},$$

   where G is the constant of proportionality, called the *gravitational constant* . If we introduce a coordinate system with respect to which P_0 has coordinates (x_0, y_0, z_0) and P has coordinates (x, y, z), then

   $$r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}.$$

   If we write

   $$\boldsymbol{r}(P) := (x-x_0)\,\boldsymbol{i}+(y-y_0)\,\boldsymbol{j}+(z-z_0)\,\boldsymbol{k},$$

   then

   $$\|\boldsymbol{r}(P)\| = r, \text{ and for } r>0, -\frac{\boldsymbol{r}}{|\boldsymbol{r}|} \text{ is a unit vector in the direction of the force of gravitation.}$$

   Thus,

   $$\begin{aligned}\boldsymbol{F}(P) &= \left(-\frac{\boldsymbol{r}(P)}{|\boldsymbol{r}(P)|}\right)\|\boldsymbol{F}(P)\| \\ &= -\|\boldsymbol{F}(P)\|\boldsymbol{u}(P) \\ &= \left(-\frac{GMm}{r^2}\right)u(P) \\ &= \left(-\frac{k}{\|\boldsymbol{r}(P)\|^2}\right)\boldsymbol{u}(P),\end{aligned}$$

   where

   $$\boldsymbol{u}(P) = \left(\frac{\boldsymbol{r}(P)}{\|\boldsymbol{r}(P)\|}\right) \text{ and } k := GmM$$

   is a constant. Thus, $\boldsymbol{F}(P)$ is a vector field, called the *gravitational force field.* The vector-field $\boldsymbol{F}$ is example of an *inverse square* field. Another example of inverse square field is the electrostatic force field.

2. **Velocity of a rotating body:**

Consider a rigid body B rotating about a fixed axis passing through it. Let P be any point on B. Let

(i) $\boldsymbol{v}(P)$ denote the velocity at P,

(ii) $w > 0$, denote the magnitude of the angular speed of the body,

(iii) and $\boldsymbol{a}$ denote a vector along the axis of rotation so that the rotation is clockwise when looked from the initial point of $\boldsymbol{a}$ towards the fixed point of $\boldsymbol{a}$ such that $|\boldsymbol{a}| = w$.

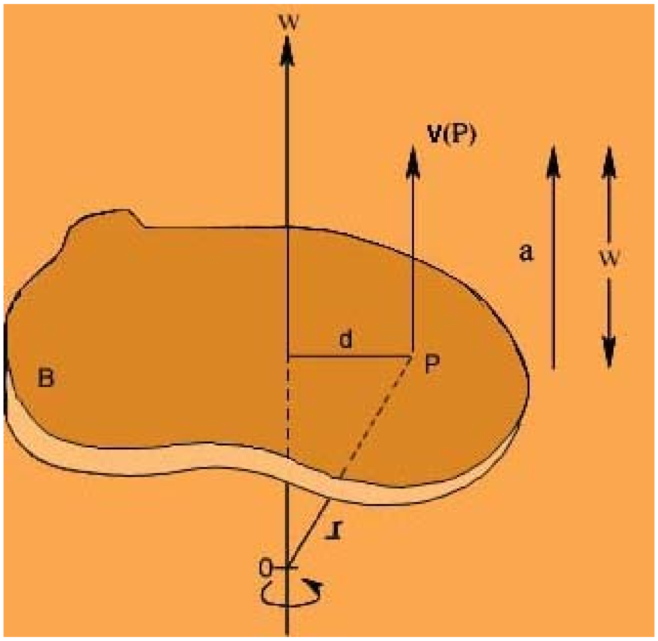

Figure 159. Velocity of a rotating body

Recall that,

$$w = \|\boldsymbol{a}\| = \frac{\text{Tangential speed at } P}{\text{distance of } P \text{ from axis}}.$$

Thus, if we chose a coordinate axis along the axis of rotation, with origin at O, and if the position vector of P is $\boldsymbol{r}$, then

$$\|\boldsymbol{v}(P)\| = wd = \|\boldsymbol{a}\|\,\|\boldsymbol{r}\|\sin\alpha,$$

where d is the distance of P from the axis of rotation and α is the angle between $\boldsymbol{a}$ and $\boldsymbol{r}$. Thus,

$$\|\boldsymbol{v}(P)\| = \|\boldsymbol{a}\| \times \boldsymbol{r},$$

and by the fact that $\|\boldsymbol{v}(P)\|$ is tangential, i.e., perpendicular to to both a and r, we have

$$\boldsymbol{v}(P) = \boldsymbol{a} \times \boldsymbol{r}.$$

This gives the velocity as a vector field. In the particular choice of axis of rotation to be $z-$axis, with $\boldsymbol{a} = w\boldsymbol{r}$, we have

$$\boldsymbol{v}(z,y,z) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & w \\ x & y & z \end{vmatrix} = -w(y\,\boldsymbol{i} - x\,\boldsymbol{j}).$$

Since, for a vector field $\boldsymbol{F}(x,y,z)$ is a vector, it is possible to represent it geometrically. Let us look at some examples.

43.1.3 Example

(i) We saw above that the velocity vector field of a body rotating about $z-$axis is given by

$\boldsymbol{v}(x,y,z) = -w(y\boldsymbol{i} + x\boldsymbol{j})$

We note that

$\|\boldsymbol{v}\|^2 = w^2(x^2 + y^2)$.

Thus,

$\|\boldsymbol{v}\|^2 = C^2 w_,^2 \text{ if } x^2 + y^2 = C^2$.

Hence, $\boldsymbol{v}$ has constant length Cw along circle of radius C. The direction is always tangential. Thus, for a right handed coordinate system the visual representation of $\boldsymbol{v}$ is as follows:

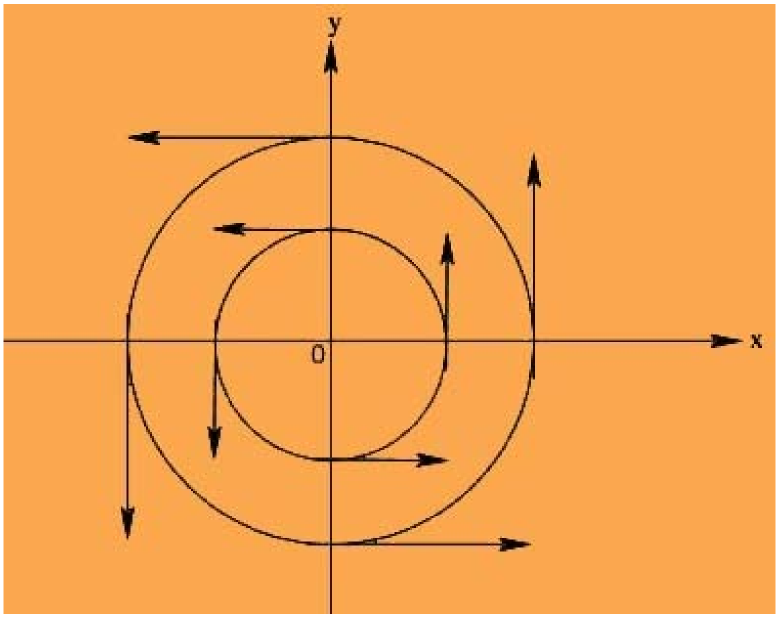

Figure 160. Velocity vector field of a rotating body

(ii) Consider the gravitational force field between two bodies of masses m_1 and m_2 respectively:

$$\boldsymbol{F}(x,y,z) = -\left(\frac{Gm_1m_2}{\|r(x,y,z)\|^2}\right)\left(\frac{\boldsymbol{r}(x,y,z)}{\|\boldsymbol{r}(x,y,z)\|}\right).$$

Thus, for m_1 at the origin and m_2 at (x,y,z), the direction of $\boldsymbol{F}(x,y,z)$ is directed towards the origin. Its magnitude is

$$\|\boldsymbol{F}(x,y,z)\| = \frac{|Gm_1m_2|}{\|r(x,y,z)\|^2}$$

For points at the same distance, the value of $\boldsymbol{F}$ is same, becoming smaller as we approach to origin. Thus a sketch is as follows:

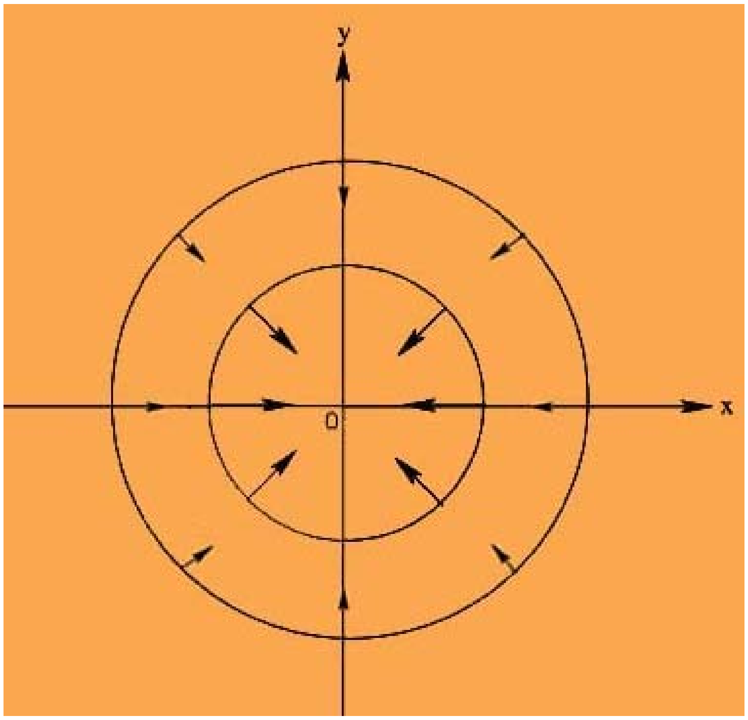

Figure 161. Gravitation force field

Some of the geometric concepts, like curves are also described by vector fields, as we shall see later.

The notion of limit for a vector-field can be defined in a manner similar to that of functions of several variables.

43.1.4 Definition:

Let $\boldsymbol{f}: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a vector field, where $n = 1, 2,$ or 3 and $m = 2$ or 3, and D is an open subset of $\mathbb{R}^n$. For $\boldsymbol{a} \in D$, we say $\boldsymbol{f}$ has a limit $\boldsymbol{L} \in \mathbb{R}^m$ at $\boldsymbol{a}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < \| \boldsymbol{x} - \boldsymbol{a} \| < \delta \text{ implies } \| \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{L} \| < \varepsilon,$$

and in that case we write

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{L}.$$

43.1.5 Remark:

It is not difficult to show that for $\boldsymbol{f} = (f_1, f_2, f_3)$, it will have limit $L = (l_1, l_2, l_3)$ at $\boldsymbol{a}$ if and only if

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} f_i(x) = \ell_i, \quad \text{for every } i = 1, 2, 3.$$

Thus, all the concepts involving limits of vector fields can be analysed in terms of its component scalar fields.

43.1.6 Definition:

Let
$\boldsymbol{f}: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$
be a vector field. We say $\boldsymbol{f}$ is *continuous* at $\boldsymbol{a} \in D$ if

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a}).$$

Limit and continuity of vector-fields behave in a fashion similar to that of scalar fields (see theorem 29.2.1). We shall assume (without proof) that vector- fields also have similar properties. We can also define differentiability and existence of partial derivatives for vector fields via their component scalar-fields.

43.1.7 Definition:

Let

$$\boldsymbol{f} : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$$

be a vector field with component functions $f_i, 1 \leq i \leq \mathrm{m}$.

(i) We say $\boldsymbol{f}$ is **differentiable** at $\boldsymbol{a} \in D$ if each f_i is differentiable at $\boldsymbol{a} \in D$.

(ii) We say $\boldsymbol{f}$ has $i\, j^{th}$ **partial derivative** , $1 \leq i \leq m, 1 \leq j \leq n,$ at a point $\boldsymbol{a} \in D$ if $\partial f_i / \partial x_j$ exists at $\boldsymbol{a}$.

(iii) We say $\boldsymbol{f}$ is continuously-differentiable if every component function is continuously differentiable.

Like functions of several variables, see section 32.2, differentiable vector-fields have nice properties. We state some particular cases.

48.1.8 Examples:

(i) Let

$$\boldsymbol{f} : D \subseteq \mathbb{R} \rightarrow \mathbb{R}^m \text{ and } g : O \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$$

be such that $(g \circ \boldsymbol{f})(t)$ is defined for every $t \in D$. If $\boldsymbol{f}$ is differentiable at $t_0 \in D \subseteq \mathbb{R}$ and g is differentiable at $\boldsymbol{f}(t_0)$, then $(g \circ \boldsymbol{f})$ is differentiable at t_0. Further,

$$(g \circ \boldsymbol{f})'(t_0) = \sum_{j=1}^{m} \frac{\partial g}{\partial x_j}(\mathbf{f}(t_0)) \frac{df_j}{dt}(t_0),$$

(ii) Let

$$\boldsymbol{f} : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ and } g : O \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^k$$

be such that $\boldsymbol{g} \circ \boldsymbol{f}$ is defined. Then, $(\boldsymbol{g} \circ \boldsymbol{f})$ has $\boldsymbol{k}$ component functions, the i^{th} component is

$$(\boldsymbol{g} \circ \boldsymbol{f})_i(x) = ((\boldsymbol{g})_i \circ \boldsymbol{f})(\boldsymbol{x}) \text{ for every } \boldsymbol{x} \in D.$$

Thus, for every $i, j,$

$$\frac{\partial(\boldsymbol{g} \circ \boldsymbol{f})_i}{\partial x_j} = \frac{\partial((\boldsymbol{g})_i \circ \boldsymbol{f})}{\partial x_j},$$

whenever f and g are differentiable.

43.1.9 Definition:

Let

$$\phi : D \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$$

be a scalar-field such that all its partial derivatives exist. The **gradient** of ϕ, denoted by $\nabla(\phi)$, is the vector-field defined by

$$(\nabla(\phi))(P) := \frac{\partial \phi}{\partial x}(P)\, \boldsymbol{i} + \frac{\partial \phi}{\partial y}(P)\, \boldsymbol{j} + \frac{\partial \phi}{\partial z}(P)\, \boldsymbol{k}, \text{for } P \in D.$$

We saw in section 35.1, that $\nabla(\phi)$ helps us to represent directional derivatives and the concept of normal to surfaces given by $f(x, y, z) = c$. Vector-fields, which arise as gradients of some scalar fields, play an important role in applications.

43.1.10 Definition:

A vector field

$$\boldsymbol{F} : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^3$$

is said to have a **potential** f, where

$$f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$$

is a differentiable scalar field, if

$$\boldsymbol{F}(P) = (\nabla f)(P) \text{ for every } P \in D.$$

In this case, F is said to be a **conservative** vector-field with potential f.

A necessary condition for a vector-field F to be conservative is given by the following:

43.1.11 Theorem (Necessary condition for a vector field to be conservative):

Let

$$\boldsymbol{F}(x,y,z) = F_1(x,y,z)\,\boldsymbol{i} + F_2(x,y,z)\,\boldsymbol{j} + F_3(x,y,z)\,\boldsymbol{k}$$

be a conservative vector-field, with a potential function $f(x,y,z)$. *If* f *is twice continuously differentiable then*

$$\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}, \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \text{ in } D.$$

PROOF

Since f is a potential of $\boldsymbol{F}$, we have

$$F_1\,\boldsymbol{i} + F_2\,\boldsymbol{j} + F_3\,\boldsymbol{k} = F = \nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k} \text{ in } D.$$

Hence,

$$\frac{\partial f}{\partial x} = F_1,\ \frac{\partial f}{\partial y} = F_2,\ \frac{\partial f}{\partial z} = F_3.$$

Thus

$$\frac{\partial F_1}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial F_2}{\partial x},$$

as f is twice continuously differentiable. Similarly

$$\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \text{ and } \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

43.1.12 Remark:

The above theorem is useful in checking that a vector-field is not conservative. We shall show later that the conditions of above theorem are also sufficient for a vector-field to be conservative when extra condition of 'simple connectedness' is imposed upon the domain.

43.1.13 Example

Consider the 2-dimensional field

$$\boldsymbol{F}(x,y) = x^2 y\,\boldsymbol{i} + xy\,\boldsymbol{j}.$$

Since

$$\frac{\partial}{\partial y}(x^2 y) = x^2 \neq y = \frac{\partial}{\partial x}(xy),$$
$\boldsymbol{F}(x,y)$ is not conservative.

43.1.14 Note:

In many calculations, the following notation is useful:

$$\nabla := \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k},$$

called the **gradient operator** . This can be applied to any scalar-field f as follows:

$$\begin{aligned}\nabla(f) &:= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right)(f) \\ &= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}.\end{aligned}$$

That the gradient operator behaves like a differentiation operator, is illustrated by the next theorem.

43.1.15 Theorem:

Let $f, g : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ *be differentiable scalar fields. Then the following holds: iii*

(i) $\nabla(f+g) = \nabla(f) + \nabla(g).$

(ii) $\nabla(fg) = f\nabla(g) + g\nabla(f).$

(iii) $\nabla(f/g) = \dfrac{g\nabla f - f\nabla g}{g^2}$, *whenever* $g \neq 0.$

All the proofs are easy and over left as exercises.

Applet 43.1: Vector fields

For Quiz refer the WebSite

Practice Exercises

1. Let $\boldsymbol{F}(x,y) = x^2\,\boldsymbol{i} - y\,\boldsymbol{j}, (x,y) \in \mathbb{R}^2.$

    1. Analyze $\lim_{(x,y)\to(0,0)} \|\boldsymbol{F}(x,y)\|$.
    2. If $(0,y)$ is such that $y > 0,$ what can you say about the direction of $\boldsymbol{F}(0,y)$.
    3. For (x,y) in the first quadrant, analyze the direction of $\boldsymbol{F}(x,y)$.

    Answer:

    (i) 0

    (ii) negative y -axis

    (iii) downward to the right

2. Let $F(x,y) = \dfrac{x}{(x^2+y^2)^{1/2}}\,\boldsymbol{i} + \dfrac{y}{(x^2+y^2)^{1/2}}\,\boldsymbol{j},$

   1. Analyze $\lim_{(x,y)\to(0,0)} \| \boldsymbol{F}(x,y) \|$.
   2. Analyze the directions of $\boldsymbol{F}(x,y)$ for $(x,0), x>0$ and $(0,y), y>0$.

Answer:

(i) 1.

(ii) unit vectors along five x-axis unit vectors along positive y-axis

3. Let $\boldsymbol{f} : D \subseteq \mathbb{R} \to \mathbb{R}^3$ and $\boldsymbol{g} : D \subseteq \mathbb{R} \to \mathbb{R}^3$ be differentiable vector fields. Prove the following:

   1. $\dfrac{d}{dt}(a\boldsymbol{f} + b\boldsymbol{g}) = a\dfrac{d}{dt}(\boldsymbol{f}) + b\dfrac{d}{dt}(\boldsymbol{g}), a, b \in \mathbb{R}.$
   2. $\dfrac{d}{dt}(\boldsymbol{f}\cdot\boldsymbol{g}) = \boldsymbol{f}\cdot\dfrac{d}{dt}(\boldsymbol{g}) + \boldsymbol{g}\cdot\dfrac{d}{dt}(\boldsymbol{f}).$
   3. $\dfrac{d}{dt}(\boldsymbol{f}\times\boldsymbol{g}) = \boldsymbol{f}\times\dfrac{d}{dt}(\boldsymbol{g}) + \boldsymbol{g}\times\dfrac{d}{dt}(\boldsymbol{f}).$

4. Let $\boldsymbol{f} : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{g} : O \subseteq \mathbb{R}^m \to \mathbb{R}$ be differentiable. Show that $\dfrac{\partial}{\partial x_i}(g \circ \boldsymbol{f})(\boldsymbol{a}) = \sum_{j=1}^{n} \dfrac{\partial g}{\partial x_j}(\boldsymbol{f}(\boldsymbol{a}))\dfrac{\partial(\boldsymbol{f}_j)(\boldsymbol{a})}{\partial x_i}$.

5. Show that the following vector-fields are not conservative:

   1. $F(x,y) = 5y^2(3y\,\boldsymbol{i} - x\,\boldsymbol{j}).$
   2. $F(x,y) = e^x(\cos y\, i + \sin y\,\boldsymbol{j}).$

6. Show that ϕ is a potential for the vector field $\boldsymbol{F}$ in some domain:

   1. $\phi(x,y) = \tan^{-1}(xy),\quad \boldsymbol{F}(x,y) = \dfrac{y}{1+x^2y^2}\,\boldsymbol{i} + \dfrac{x}{1+x^2y^2}\,\boldsymbol{j}.$
   2. $\phi(x,y) = x^2y,\quad \boldsymbol{F}(x,y) = 2xy\,\boldsymbol{i} + x^2\,\boldsymbol{j}$
   3. $\phi(x,y) = \dfrac{1}{2}\ln(x^2+y^2),\quad \boldsymbol{F}(x,y) = \dfrac{x\boldsymbol{i}+y\boldsymbol{j}}{x^2+y^2}.$

7. Find the vector-field for which the potential function is:

   1. $\phi(x,y) = 5x^2 + 3xy + 10y^2$
   2. $\phi(x,y,z) = xy\ln(x+y).$
   3. $\phi(x,y,z) = z - ye^{x^2}$

Answer:

(i) $(10x+3y)\mathbf{i}+(3x+20y)\mathbf{j}$

(ii) $\left(y\ln(x+y)+\dfrac{xy}{x+y}\right)\mathbf{i}+\left(x\ln(x+y)+\dfrac{xy}{x+y}\right)\mathbf{j}$

(iii) $\left[-2xye^{x^2}\right]\mathbf{i}+\left[-e^{x^2}\right]\mathbf{j}+\mathbf{k}$

Recap

In this section you have learnt the following

- Concept of Vector field.
- Various properties of vector fields.
- Continuity and differentiability of vector fields.
- Gradient vector field.