Module 1 : Real Numbers, Functions and Sequences

Lecture 2 : Convergent & Bounded Sequences

[Section 2.1 : Need to consider sequences]

Objectives

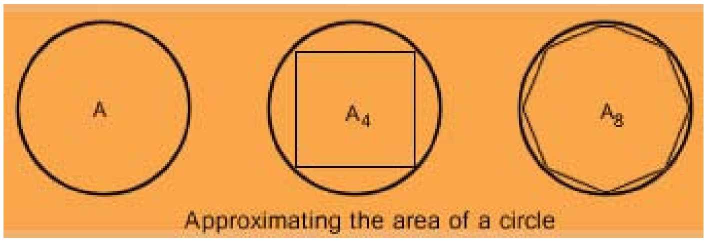
In this section you will learn the following

- The need to consider sequences.
- The concept of a sequence.

2.1 Need to consider sequences :

The aim of this lecture is to introduce the concept of a sequence. Sequences arise naturally in various fields. Any iterative process gives rise to a sequence of observations. A sequence can be thought of as a list of objects written in a definite order.

2.1.1 Example (Finding the area of the unit circle) :

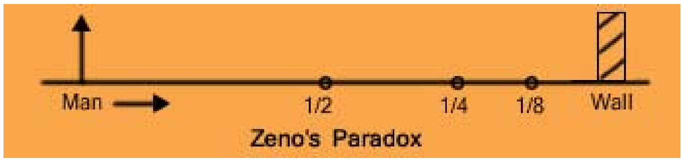


Greek mathematicians (400 B.C.) analyzed this problem by inscribing regular polygons inside the circle. If denotes the area of the n -sided polygon inscribed in the circle, then we get the sequences of numbers A_3 , A_4 ,....., A_n .

Click here to View the Interactive animation : Applet 1.3 (avaliable on WebSite).

2.1.2 Example (Zeno's paradox) :

A man standing in a room can not walk to the wall. In order to do so, he would have to go half the distance, then half the remaining distance, and then again half of what shall remains. This process can always be continued and can never be ended.



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To understand this paradox, let us assume that the man walks with a constant speed. Suppose he takes a minutes to cover the first half of the distance. The next half will be covered in $a/2$ minutes, the half of the remaining half in $a/4$ minutes, and so on. The time consumed at the n th stage will be

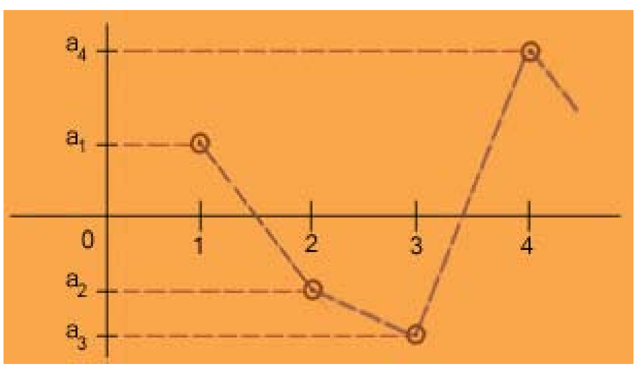
$$T_n = a + \frac{a}{2} + \frac{a}{2^2} + \ldots + \frac{a}{2^{n-1}}.$$

This gives us the sequence T_1 , T_2 , ... The paradox is that there are infinite stages and how they can covered in finite time? The paradox is resolved by proper interpretation of 'infinite'.

Let us observe that in both the examples, we are interested in finding out what happens to the sequence of observations for large n ? To analyze this problem, let us make some definitions.

2.1.3 Definition :

A sequence of elements of a set S is an ordered collection : a_1, a_2, ... , a_n . . . of elements of S . The element a_1 is called its first term, a_2 - its second term , and in general a_n as its nth term. We also write this as $\{a_n\}_{n \geq 1}$. One can also think of a sequence $\{a_n\}_{n \geq 1}$ in S as a function f defined on the set of natural numbers with values in S , i.e., $f : \mathbb{N} \to S$ with $f(n) := a_n$. Note that a sequence $\{a_n\}_{n \geq 1}$ is not the same as the set $\{a_1, a_2, \ldots\}$.



2.1.4 Examples :

1. For the sequence $\left\{\frac{1}{n}\right\}_{n \geq 1}$, $a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3}, \ldots\ldots$

2. For the sequence $\left\{(-1)^n\right\}_{n \geq 1}$, $a_1 = -1,\ a_2 = 1,\ a_3 = -1, \ldots\ldots$

3. For the sequence $\{\cos n\pi\}_{n \geq 1}$, $a_1 = -1,\ a_2 = 1,\ a_3 = -1, \ldots\ldots$

4. For the sequence $\{a_n\}_{n \geq 1,}$ where $a_1 = 1, a_2 = 2,$ and $\forall\ n \geq 3, a_n = a_{n-1} + a_{n-2}$.

Click here to View the Interactive animation : [Applet 1.5] (avaliable on WebSite).

Recap

In this section you have learnt the following

- The need to consider sequences arises from practical problems.
- The concept of a sequence.

Objectives

In this section you will learn the following

- Convergence of a sequence.

The aim of this lecture is to analyze various concepts about sequences: a sequence being bounded, monotone, and convergent.

2.2 Convergent Sequences

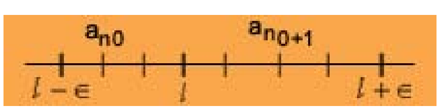
Given a sequence $\{a_n\}_{n \geq 1}$, one is interested to know: what happens to a_n as n becomes large. In the example of the area of the unit circle, we expect that for n sufficiently large, A_n will be a 'good enough' representation of π , the area of the unit circle, that is, will A_n come close to the value π , the area of the unit circle when n becomes large? How close? Will it come as close as we want? Using the concept of absolute value (which gives the notion of distance on $\mathbb{R}$) we can express it mathematically as follows:

2.2.1 Definition:

A sequence $\{a_n\}_{n \geq 1}$ is said to be convergent, if there exists $l \in \mathbb{R}$ such that given any real number $\varepsilon > 0$, we can find a natural number n_0 such that

$$l-\varepsilon < a_n < l+\varepsilon \ \forall \ n \ \geq \ n_0 , i.e., \ |a_n - l| < \varepsilon \ \forall \ n \geq n_0$$

The real number l is called a limit of $\{a_n\}_{n \geq 1}$ and we write it as $\lim\limits_{n \to \infty} a_n = l$.



If we consider the geometric visualization of the sequence $\{a_n\}_{n \geq 1}$ as a function, $f : \mathbb{N} \to \mathbb{R}, f(x) = a_n$ then saying that the number l will be the limit of the sequence means that given any horizontal strip, of width $\in$ centered at l, all but finitely many of $f(n) = a_n$ lie in this strip. Intuitively, after some stage all the elements of the sequence are close to l. Or a 'tail' of the convergent sequence lies inside any small neighborhood of l. A sequence which is not convergent is called a divergent sequence.

2.2.2 Examples:

1. The sequence $\left\{\frac{1}{n}\right\}_{n \geq 1}$ is convergent to $l = 0$. To see this, let $\varepsilon > 0$ be given. Then

$$|a_n - l| = |\frac{1}{n} - 0| = \frac{1}{n}.$$

Thus, $\frac{1}{n} < \varepsilon$ if $n > \frac{1}{\varepsilon}$. Now, if we choose $n_0 > \frac{1}{\varepsilon}$ (which is possible by the Archimedian property),

then $\forall\ n \geq n_0$,

$$\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon .$$

Hence, $\lim_{n \to \infty} a_n = 0$.

Click here to View the Interactive animation : Applet 1.6 (avaliable on WebSite).

2. The sequence $\{n\}_{n \geq 1}$ is not convergent.

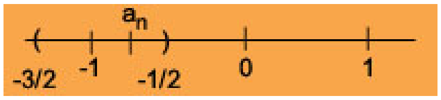
Suppose, $\lim_{n \to \infty} n = l$. Then given $\varepsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that

$$|a_n - l| = |n - l| < 1\ \forall\ n \geq n_0,$$

i.e., $l - 1 < n < l + 1\ \forall\ n \geq n_0$,

which is not true (Archimedian property). In some sense $\{n\}_{n \geq 1}$ is not convergent as it outgrows every real number. In fact, if a sequence is convergent, it can not grow arbitrarily, as we shall see in the next section.

3. Consider the sequence $\left\{(-1)^n\right\}_{n\geq1}$. Every odd term of the sequence is -1 and every even term of the sequence is $+1$. Intuitively, the elements a_n do not come closer to a single value l. We expect $\left\{(-1)^n\right\}_{n\geq1}$ to be divergent. We can write it as follows. First suppose $l \in \mathbb{R}, l = -1$. If $a_n \to l$ then given $\varepsilon = \frac{1}{2}$, there should exist some n_0 such that



$$|a_n + 1| < \frac{1}{2} \quad \forall\ n \geq n_0,$$

i.e., $-1 - \frac{1}{2} < a_n < -1 + \frac{1}{2}$,

$$\text{i.e., } -\frac{3}{2} < a_n < -\frac{1}{2} \qquad \forall\, n \geq n_0\ ,$$

which is not true for every even n, $n > n_0$. Hence, $a_n \to l = -1$ is not true. Similarly, we can show that $a_n \to l$ is not possible for any $l \in \mathbb{R}$.

Though both sequence $\{n\}_{n\geq 1}$ and $\{(-1)^n\}_{n\geq 1}$ are divergent, they are divergent for different reasons.

2.2.3 Examples:

1. Consider the sequence $\left\{\frac{n}{2n+1}\right\}_{n\geq 1}$. We show that $\lim_{n\to\infty}\left(\frac{n}{2n+1}\right) = \frac{1}{2}$. Since

$$|a_n - l| = \left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left[\frac{2n-2n-1}{2(2n+1)}\right] = \left[\frac{1}{2(2n+1)}\right] < \left[\frac{1}{4n}\right],$$

Given $\in$ > 0, we will have $|a_n - l| < \in$, if $\frac{1}{4n} < \varepsilon,\ i.e.,\ n > \frac{1}{4\varepsilon}$

So, if we choose n_0 such that $n_0 > (1/4)\varepsilon$, then $\forall\, n \geq n_0$, $|a_n - l| < \frac{1}{4n} < \frac{1}{4n_0} < \varepsilon$.

2. Consider the sequence $\left\{\frac{\cos n\pi}{\sqrt{n}}\right\}_{n\geq 1}$. Since,

$$\left|\frac{\cos n\pi}{\sqrt{n}}\right| < \frac{1}{\sqrt{n}}\ ,$$

given any $\in$ > 0, if we choose a positive integer n_0 such that $\frac{1}{n_0} < \in^2$, then for every $n > n_0$,

$$\left|\frac{\cos n\pi}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n_0}} < \in .$$

Hence, $\lim_{n\to\infty}\left(\frac{\cos n\pi}{\sqrt{n}}\right) = 0$.

At this stage it is natural to ask the question: Can a convergent sequence have two different limits? We show in the next theorem that this is not possible.

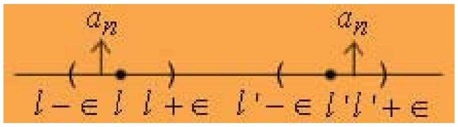
proof:

2.2.4 Theorem:

Limit a sequence is unique.

Click here to View the Interactive animation : Applet 1.7 (avaliable on WebSite).

Suppose $a_n \to l$ as well as $a_n \to l'$ with $l \neq l'$, say $l < l'$.



Take $\in = (l - l')/4$. Then by definition, there exists $n_1, n_2 \in \mathbb{N}$ such that

$l - \varepsilon < a_n < l + \varepsilon$ for all $n \geq n_1$

and

$l' - \varepsilon < a_n < l' + \varepsilon$ for all $n \geq n_2$

Thus, for $n \geq \max\{n_1, n_2\}$,

$l - \varepsilon < a_n < l + \varepsilon < l' - \varepsilon < a_n < l' + \varepsilon$

which is a contradiction. Hence, $l' - l$.

2.2.5 Note:

The technique used in the proof of the theorem is called the proof by contradiction.

For Quiz reder the WebSite.

Practice Exercises 2.2: Convergent Sequences

1. Using definition prove the following:
   - $\lim_{n \to \infty} \frac{10}{n} = 0$.
   - $\lim_{n \to \infty} \frac{5}{3n+1} = 0$.
   - $\lim_{n \to \infty} \frac{n^{2/3} \sin(n!)}{n+1} = 0$.
   - $\lim_{n \to \infty} \frac{n}{n+1} - \frac{n+1}{n} = 0$.
2. Show that the following sequences are not convergent:
   - $\frac{n^2}{n+1}$.
   - $\left\{(-1)^n \left(\frac{1}{2} - \frac{1}{n}\right)\right\}_{n \geq 1}$.

Recap

In this section you have learnt the following

- How to formulate and analyse the concept of convergence of a sequence.

Objectives

In this section you will learn the following

- The concept of a bounded sequence.

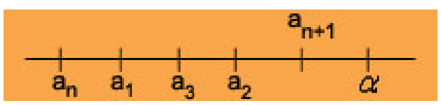
2.3 Bounded Sequences

We saw that sequence $\{n\}_{n\geq 1}$ is not convergent as its terms keep growing bigger and bigger. This motivates our next definition.

2.3.1 Definition:

1. Let $\{a_n\}_{n\geq 1}$ be a sequence.

   We say $\{a_n\}_{n\geq 1}$ is bounded above if there is some real number α such that $a_n \leq \alpha \; \forall \; n$.

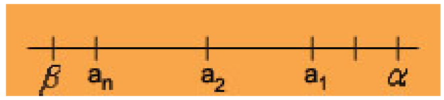
   Sequence bounded above by

2. We say $\{a_n\}_{n\geq 1}$ is bounded below if there exists a real number β such that $\beta \leq a_n \; \forall \; n$.

   Sequence bounded below by β

3. We say $\{a_n\}_{n\geq 1}$ is bounded if it is both bounded above and below.



Bounded Sequence

2.3.2 Examples:

1. Consider the sequence $\left\{\frac{(-1)^n}{n^2}\right\}_{n\geq 1}$. Since for every n , $\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$ < 1, the sequence is bounded.

2. Consider the sequence $\left\{\frac{2^n}{n}\right\}_{n\geq 1}$. It is easy to show, using induction that $2^n > n^2$ for all $n \geq 5$.

   Hence, $a_n > n$ for all large n . Hence $\left\{\frac{2^n}{n}\right\}_{n\geq 1}$ is not bounded above.

   Motivated by our remarks at the end of section 1.4, we have the following.

2.3.3 Theorem:

If $\{a_n\}_{n\geq 1}$ is convergent then it is bounded

Let $\lim_{n\to\infty} a_n = l$ **. Then given** $\epsilon > 0$**, say = 1, there exists** $n_0 \in \mathbb{N}$ **such that**

$$|a_n - l| < 1 \quad \text{for all} \quad n \geq n_0,$$
$$l - 1 < a_n < l + 1 \quad \text{for all} \quad n \geq n_0.$$

That means all a_n's, accept $a_1, a_2, \ldots, a_{n0-1}$ lie in between l - 1 and l + 1. Thus, if we define

$$\alpha := \max\{a_1, a_2, \ldots, a_{n_0-1}, l+1\}$$

and $$\beta := \min\{a_1, a_2, \ldots, a_{n_0-1}, l-1\}$$

then $$\beta \leq a_n \leq \alpha \text{ for every } n \geq 1.$$

Hence, $\{a_n\}_{n\geq 1}$ is bounded.

2.3.4 Example:

Consider the sequence $\left\{(-1)^n\right\}_{n\geq 1}$. We showed in example 2.2.2(iii) that this sequence in not convergent. Clearly, it is a bounded sequence as $|a_n| < 1$ for every n .

However, it not always easy to guess whether a sequence is convergent or not and even if it is convergent, what is its limit. We describe in next section in next section some theorems which helps us to compute limits of sequences.

For Quiz refer the WebSite.

Practice Exercises 2.3: Bounded Sequences

1. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be bounded sequences. Prove that $\{a_n \pm b_n\}_{n\geq 1}$ and $\{a_n b_n\}_{n\geq 1}$ are also bounded sequences. What can you say about the sequence $\{\alpha a_n\}_{n\geq 1}$, $\alpha \in \mathbb{R}$

2. Give examples to show that if $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are bounded sequences, $b_n \neq 0$ for every n, then $\left\{\frac{a_n}{b_n}\right\}_{n\geq 1}$ need not be a bounded sequence.

3. Let $f:\mathbb{R}\to\mathbb{R}$ be a bounded function and $\{a_n\}_{n\geq 1}$ be any sequence. Is $\{f(a_n)\}_{n\geq 1}$ a bounded sequence? Justify your claim.

Recap

In this section you have learnt the following

- A necessary condition for a sequence to be convergent is that it should be bounded.