Module 13 : Maxima, Minima and Saddle Points, Constrained maxima and minima

Lecture 37 : Maxima and Minima [Section 37.1]

Objectives

In this section you will learn the following :

- The notions local maximum and local minimum of a function of several variables.

37 .1 Maxima and minima

37.1.1Definition:

Let $f : D \rightarrow \mathbb{R}$, where $D \subset \mathbb{R}^2$. Let $(x_0, y_0) \in D$.

(i) We say f has a **point of local maximum** at (x_0, y_0) if there is some $\delta > 0$ such that

$$(x,y) \in D \cap B_\delta(x_0, y_0) \text{ implies } f(x,y) \leq f(x_0, y_0).$$

In this case the value $f(x_0, y_0)$ is called a **local maximum** of f.

(ii) We say f has a **point of local minimum** at (x_0, y_0) if there is some $\delta > 0$ such that

$$(x,y) \in D \cap B_\delta(x_0, y_0) \text{ implies } f(x,y) \geq f(x_0, y_0).$$

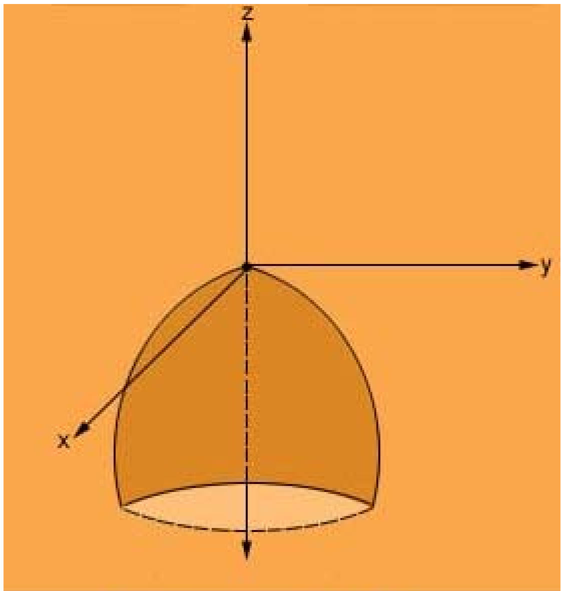
In this case the value $f(x_0, y_0)$ is called a **local minimum** of f.

37.1.2 Examples:

(i) If $D = \mathbb{R}^2$ and

$$f(x,y) = -(x^2 + y^2) \ \text{ for } (x,y) \in \mathbb{R}^2,$$

then f has a local maximum at $(0,0)$ and the local maximum is $f(0,0)=0$.



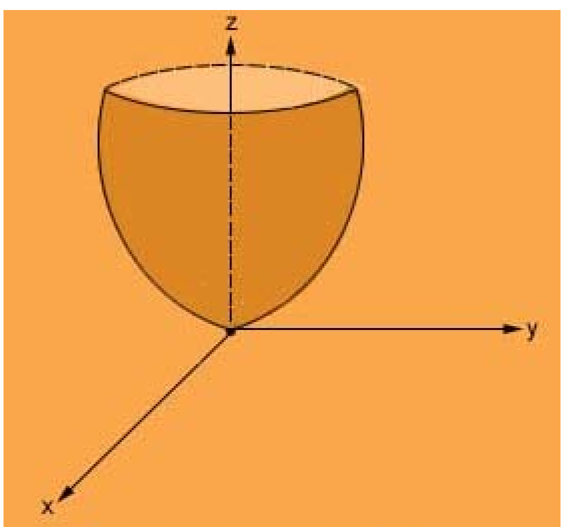
Local maximum at (0, 0, 0)

Figure 1. Local maximum at $(0,0,0)$.

(ii) If $D=\mathbb{R}^2$ and

$$f(x,y)=x^2+y^2 \text{ for } (x,y)\in\mathbb{R}^2,$$

then f has a local minimum at $(0,0)$ and the local maximum is $f(0,0)=0$.



Local Minimum at (0, 0, 0)

Figure 2. Locale minimum at $(0,0,0)$.

For functions of three or more variables, the concepts of local maximum/minimum can be defined similarly. Further, similar to function of one variable, we have the following sufficient condition for a point to be a point of local maximum/minimum.

37.1.3Theorem (Necessary conditions for local extremum or point):

Let $\boldsymbol{u} = (u_1, u_2)$ *be any unit vector.*

(i) *If* f *has a local maximum/local minimum at* (x_0, y_0) *and* $(D_{\boldsymbol{u}} f)(x_0, y_0)$ *exists,*

then

$(D_{\boldsymbol{u}} f)\,(x_0, y_0) = 0.$

(ii) *If both* $f_x(x_0, y_0)$ *and* $f_y(x_0, y_0)$ *exist and* f *has a local maximum or a local*

minimum at (x_0, y_0)*, then*

$f_x(x_0, y_0) = 0 = f_y(x_0, y_0).$

37.1.3Theorem (Necessary conditions for local extremum or saddle point):

Let $\boldsymbol{u} = (u_1, u_2)$ *be any unit vector.*

(i) *If* f *has a local maximum/local minimum at* (x_0, y_0) *and* $(D_{\boldsymbol{u}} f)(x_0, y_0)$ *exists,*

then
$(D_{\boldsymbol{u}} f)\,(x_0, y_0) = 0.$

(ii) *If both* $f_x(x_0, y_0)$ *and* $f_y(x_0, y_0)$ *exist and* f *has a local maximum or a local*

minimum at (x_0, y_0)*, then*
$f_x(x_0, y_0) = 0 = f_y(x_0, y_0).$

Proof

Suppose f has a local maximum at (x_0, y_0). Then, we can find some $\delta > 0$ such that
$f(x_0 + tu_1, y_0 + tu_2) \le f(x_0, y_0)$, whenever $0 \le t \le \delta$.

Hence, if $(D_{\boldsymbol{u}} f)(x_0, y_0)$ exists, then

$$(D_{\boldsymbol{u}} f)(x_0, y_0) = \lim_{t \to 0} \left(\frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t} \right) \le 0$$

Similarly, $(D_{\boldsymbol{u}})\, f(x, y) \ge 0$. Hence, $(D_{\boldsymbol{u}})\, f(x_0, y_0) = 0.$

The case of a local minimum is similar. This proves (i). To prove (ii), we note that if $f_x(x_0, y_0)$ exists and if $u = (1, 0)$, then
$(D_{\boldsymbol{u}})\, f(x_0, y_0) = f_x(x_0, y_0) = 0.$
Considering $u = (0, 1)$, we similarly obtain $f_y(x_0, y_0) = 0$.

As in the case of one variable we make the following definition.

37.1.4 Definition:

An interior point $(x_0, y_0) \in D$ is called a **critical point** of f if

(i) Either, both f_x and f_y exist with

$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$

(ii) Or, one or both of $f_x(x_0, y_0), f_y(x_0, y_0)$ do not exist.

As a consequence of theorem 37.1.3, we have the following:

37.1.5 Corollary:

For a function local maxima/minima can occur only at critical point or boundary points of D.

37.1.6 Example:

Let

$$f(x, y) = \sqrt{x^2 + y^2},\ (x, y) \in \mathbb{R}^2.$$

Then, $(0,0)$ is the only critical point of f, as both f_x and f_y do not exist at $(0,0)$, and at every other point both exist are non-zero. The point $(0,0)$ is obviously a point of local minimum, with local minimum being $f(0,0) = 0$. Thus, the condition in theorem $37.1.3$ is only necessary, not sufficient.

37.1.7 Example:

Consider the function

$f\ (x, y) = x^3 + y^4\ , (x, y) \in \mathbb{R}^2.$

Then

$f_x\ (x, y) = 3x^2\ ,\ f_y\ (x, y) = 4y^3.$

Thus

$f_x\ (x, y) = 0 = f_y\ (x, y)$

gives $(0,0)$ as the only critical point of f. However, at $(0,0)$, f has neither a local maximum, nor local minimum. For example, for any $\delta > 0$,

$$(-\delta/2, 0) \in B_\delta\ (0,0) \text{ and } f\ (-\delta/2, 0) = -\frac{\delta^3}{8} < 0.$$

and

$$(0, \delta/2) \in B_\delta\ (0,0) \text{ and } f\ (0, \delta/2) = \frac{\delta^4}{16} > 0.$$

Practice Exercises

(1) Show that the following functions have local maximum at the point $(0,0)$:

(i) $f\ (x, y) = \cos x \cos y \exp\left(-\sqrt{x^2 + y^2}\right),\ (x, y) \in \mathbb{R}^2.$

(ii) $f\ (x, y) = \dfrac{1}{2}\big(\left(|x| - |y|\right) - |x| - |y|\big),\ (x, y) \in \mathbb{R}^2.$

(2) Show that the following functions have local minimum at the indicated point:

(i) .

$(x^2+y^2)\exp(-x^2-y^2)$, $(x,y)\in\mathbb{R}^2$

(ii) $\sin(x^2+y^2)$.

(3) Show that the following functions have neither local maximum, nor local minimum at $(0,0)$

(i) $f(x,y)=y^2-x^2$.

(ii) $f(x,y)=xy$.

Recap

In this section you have learnt the following

- The notions local maximum and local minimum of a function of several variables.

[Section 37.2]

Objectives

In this section you will learn the following :

The notions saddle points of a function of several variables.

37.2 Saddle points

Recall, we proved that a function f can have local maximum/minimum at only critical points. However, not every critical poling is a point of local maximum/minimum.

37.2.1Example

Let

$z=f(x,y)=x^2-y^2$, $(x,y)\in\mathbb{R}^2$.

Then, both f_x and f_y exist at every point (x,y) and

$f_x=2x=0$, $f_y=2y=0$,

imply that $(0,0)$ is the only cortical point. But at $(0,0)$, f has no local maximum for

$$f(0,0)=0<f(x,0)=x^2, \text{ for all } x\in \mathbb{R}.$$

Also, f does not have a local at minimum at $(0,0)$ since

$$f(0,0)=0>f(0,y)=-y^2, \text{for all } y\in \mathbb{R}.$$

Thus, in other words, in every neighborhood of $(0,0)$, there is a curve in the domain at every point of which f takes value bigger than $f(0,0)$, and also there exists another curve in the domain of f at every point of which f takes value less than $f(0,0)$.

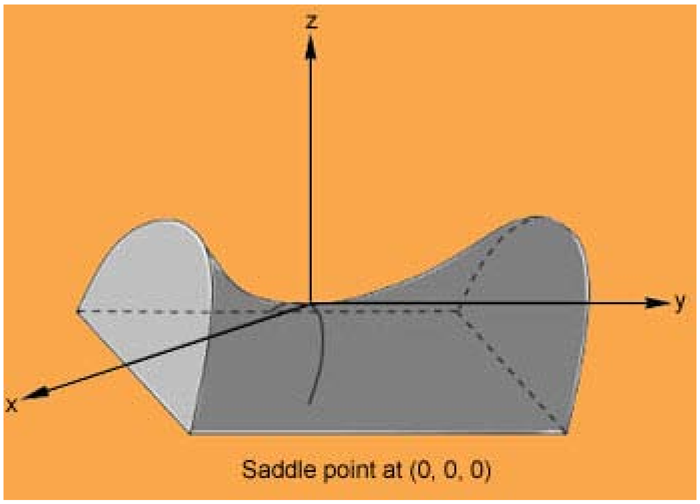


Figure 1. Saddle point at $(0,0,0)$

The above example motivates the following definition.

37.2.2 Definition:

Let $f: D\subseteq \mathbb{R}^2 \to \mathbb{R}$ and $P\in D$ be a critical point. We call a point P in D to be a **saddle point** of f if in every open ball B centered at P, there exist points Q and R in $B\cap D$ such that

$$f(Q)>f(P)>f(R).$$

37.2.3 Example:

(i) Let

$$f(x,y)=xy, (x,y)\in \mathbb{R}.$$

Since

$$f_x(x,y)=y, f_y(x,y)=x,$$

the function has only one critical point, namely, $(0,0)$. But it is neither a point of local maximum, nor a point of local minimum. For example for every $\delta>0$,

$$f(\delta/2, \delta/2) = \frac{\delta^2}{4} > 0 = f(0,0)$$

and

$$f(-\delta/2, \delta/2) = -\frac{\delta^2}{4} < 0 = f(0,0).$$

Thus, $(0,0)$ is in fact a saddle point for $f(x,y) = xy$.

(ii) Let

$$f(x,y) = 2x^2 y + y x^2, (x,y) \in \mathbb{R}^2.$$

It is easy to check that $(0,0)$ is a critical point for f. Along the line $y = x$, which passes through $(0,0)$,

$$f(x,x) = 3x^3, x \in \mathbb{R},$$

and hence f takes both positive and negative values at points as close to $(0,0)$ as we want. Since $f(0,0) = 0$, the point $(0,0)$ is a saddle point for f.

Practice Exercises

For the following function, check whether the indicated point is a point of local maximum/local minimum/saddle point for f:

(1) $f(x,y) = 4xy - x^2 - y^2, P = (0,0)$.

Answer

(2) $f(x,y) = e^x \sin y, P = (0,0)$.

Answer

(3) Let $f(x,y) = x^2 + y - e^y, (x,y) \in \mathbb{R}^2$. Show that $(0,0)$ is a critical point of f. Find suitable curves in $\mathbb{R}^2$ to deduce that f, has a saddle point at $(0,0)$.

Answer

Recap

In this section you have learnt the following

- The notions saddle points of a function of several variables.