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Module 4 : Local / Global Maximum / Minimum and Curve Sketching

Lecture 10 : Sufficient conditions for increasing / decreasing [Section 10.1]

Objectives

In this section you will learn the following :

How the knowledge about the derivatives of a function helps us to draw conclusions regarding the increasing / decreasing nature of the function.

The key result that helps us to deduce conclusions about nature of a function f from the knowledge about its derivative function f' is the following:

10.1.1 Theorem:

Let $f:(a,b)\longrightarrow\mathbb{R}$ be differentiable. Then the following hold:

(i) $f'(x)\geq 0$ for all $x\in(a,b)$ if and only if f is increasing in (a,b).

(ii) $f'(x)\leq 0$ for all $x\in(a,b)$ if and only if f is decreasing in (a,b).

(iii) if $f'(x)>0$ for all $x\in(a,b)$, then f is strictly increasing in (a,b).

(iv) if $f'(x)<0$ for all $x\in(a,b)$, then f is strictly decreasing in (a,b).

10.1.1 Theorem:

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(iii) if $f'(x)>0$ for all $x\in(a,b)$, then f is strictly increasing in (a,b).

(iv) if $f'(x)<0$ for all $x\in(a,b)$, then f is strictly decreasing in (a,b).

Proof:

To prove (i), let $f'(x)\geq 0$ for all $x\in(a,b)$. Let $x_1, x_2\in(a,b)$ with $x_1<x_2$. Then, by the mean value

theorem for $f:[x_1,x_2]\to\mathbb{R}$, we have

$f(x_2)-f(x_1)=f'(c)(x_2-x_1)$, for some $c\in(x_1,x_2)$.

Since $f'(x)\geq 0$ for all , we get

$f(x_2)-f(x_1)=f'(c)(x_2-x_1)\geq 0$.

Thus,

$f(x_2)\geq f(x_1)$ whenever $x_1,x_2\in(a,b)$, $x_1 < x_2$.

Hence, $f'(x)\geq 0$, for all $x\in(a,b)$ implies that f is increasing in (a,b) proving the only if part of (i). In fact, if $f'(x)>0$ for all $x\in(a,b)$, then the above argument also tells us that f is strictly increasing in $[a,b]$, proving (iii). To prove the converse statement in (i), suppose that f is increasing in (a,b) and $x\in(a,b)$. Then for all $y\in(x,b)$, $f(x)\leq f(y)$. Hence,

$$f'_+(x)=f'(x)=\lim_{y\to x^+}\frac{f(y)-f(x)}{y-x}\geq 0.$$

This proves (i). Proof of (ii) is similar to that of (i) and of (iv) is similar to that of (iii), and are left as exercises.

Click here to see a visualization: Applet 10.1

10.1.2 Examples:

(i) Let $f(x)=2x(1-x)$. Then $f'(x)=2-4x$. Thus

$$f'(x)\geq 0 \text{ for } 2-4x\geq 0, i.e., x\leq\frac{1}{2},$$

and

$$f'(x)\leq 0 \text{ for } 2-4x\leq 0, i.e., x\geq\frac{1}{2}.$$

Thus $f(x)$ is increasing in the interval $\left(-\infty,\frac{1}{2}\right]$ and decreasing in $\left(\frac{1}{2},\infty\right]$.

(ii) Let $f(x)=\dfrac{1}{1+x^2}-1,\ x\in\mathbb{R}$.

Then

$$f'(x)=-\frac{2x}{\left(x^2+1\right)^2}.$$

Thus,

$$f'(x)<0 \text{ for } x>0$$

and

$$f'(x)>0 \text{ for } x<0.$$

Thus f is strictly increasing in $(-\infty,0)$ and strictly decreasing in $(0,\infty)$.

(iii) Let $f(x)=\cos x+|x|,\ x\in\mathbb{R}$.

Then, f is differentiable everywhere, except $x=0$. Further,

$$f'(x)=-\sin x-1 \text{ for } x<0$$

and

$$f'(x)=-\sin x+1 \text{ for } x>0.$$

Since $|\sin x| \le 1$, $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$. Thus, $f(x)$ is strictly decreasing in $(-\infty, 0]$ and strictly increasing in $(0, +\infty]$.

10.1.3 Remarks:

(i) In part (i) of the theorem the condition $f'(x) \ge 0 \; \forall \, x \in (a,b)$ is necessary. For example $f'(c) > 0$ for some $c \in (a,b)$ need not imply f is strictly increasing in a neighborhood of c. For this, consider the function

$$f(x) = \begin{cases} x^2 + 2x, & \text{if } x \in (-1,+1) \; x \text{ irrational} \\ 0, & \text{if } x \in (-1,+1) \; x \text{ rational.} \end{cases}$$

Then, f is differentiable at $x = 0$ with $f'(0) = 2 > 0$. But, f is neither strictly increasing nor decreasing in any neighborhood of $x = 2$.

(ii) In theorem 10.1.2 we showed that $f'(x) > 0$ for all $x \in (a,b)$, implies that f is strictly increasing in $[a,b]$. However, the converse of this need not hold, i.e., $f(x)$ strictly increasing in (a,b) need not imply $f(x) > 0$. For example $f(x) = x^3$ is strictly increasing in every interval $(-a, +a)$, but $f'(0) = 0$.

10.1.4 Definitions:

Let $f : (a,b) \to \mathbb{R}$ and $c \in (a,b)$.

(i) We say f is locally order preserving if there exists some $\delta > 0$ such that

$$f(x) < f(c) < f(y) \text{ whenever } c - \delta < x < c < y < c + \delta.$$

(ii) We say f is locally order reversing if there exists some $\delta > 0$ such that

$$f(x) > f(c) > f(y) \text{ whenever } c - \delta < x < c < y < c + \delta.$$

10.1.5 Example:

Let

$$f(x) = \begin{cases} x^2 + 2x & \text{for } \; x \in (-1,1), \, x \text{ irrational} \\ 0 & \text{for } \; x \in (-1,x), \, x \text{ rational.} \end{cases}$$

Then, for $-1 < x < 0 < y < +1$, we have

$$f(0) = 0$$

$$f(x) = \begin{cases} x^2 + 2x < 0 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$$

$$f(y) = \begin{cases} y^2 + 2y > 0 & \text{if } y \text{ is irrational} \\ 0 & \text{if } y \text{ is rational.} \end{cases}$$

Thus, $f(x) < f(0) < f(y)$. Hence, f is locally order preserving.

We state next a theorem which helps us to analyze local order preserving functions. Before that we prove an important result.

Theorem (Sign of Limit):

10.1.6

Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$ be such that $\lim_{x\to c} f(x)$ exists.

(i) If $\lim_{x\to c} f(x)>0$, then there exists some $\delta>0$ such that

$$f(x)>0 \text{ whenever } 0<|x-c|<\delta.$$

(ii) If $\lim_{x\to c} f(x)<0$, then there exists some $\delta>0$ such that

$$f(x)<0 \text{ whenever } 0<|x-c|<\delta.$$

10.1.6Theorem (Sign of Limit)

Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$ be such that $\lim_{x\to c} f(x)$ exists.

(i) If $\lim_{x\to c} f(x)>0$, then there exists some $\delta>0$ such that $f(x)>0$ whenever $0<|x-c|<\delta$.

(ii) If $\lim_{x\to c} f(x)<0$, then there exists some $\delta>0$ such that $f(x)<0$ whenever $0<|x-c|<\delta$.

Proof:

We prove (i), proof of (ii) is similar.

Let $L:=\lim_{x\to c} f(x)$. Then given $\in=\frac{L}{2}$ there exists $\delta>0$ such that

$|f(x)-L|<\frac{L}{2}$ whenever $0<|x-c|<\delta$

i.e., $0<L-\frac{1}{2}<f(x)<L+\frac{L}{2}$ whenever $0<|x-c|<\delta$.

10.1.7 Theorem (Derivative test for local order preserving):

Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$ be such that f is differentiable at c.

(i) If $f'(c)>0$, then f is locally order preserving at c.

(ii) If $f'(c)<0$, then f is locally order reversing at c.

10.1.7Theorem (Derivative test for local order preserving):

Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$ be such that f is differentiable at c.

(i) If $f'(c) > 0$, then f is locally order preserving at c.

(ii) If $f'(c) < 0$, then f is locally order reversing at c.

Proof:

We prove (i), proof of (ii) is similar. Since
$f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists,
by theorem 10.1.6, there exists $\delta > 0$ such that
$0 < \dfrac{f(x) - f(c)}{x - c}$ whenever $0 < |x - c| < \delta$.
Thus
$0 < f(x) - f(c)$ whenever $0 < x - c < \delta$
and
$f(y) - f(c) < 0$ whenever $-\delta < y - c < 0$
Thus
$f(y) < f(c) < f(x)$ for $c - \delta < y < c < x < c + \delta$

Hence f is locally order preserving at c.

please refer web site for quiz

PRACTICE EXCERCISES

1. For the following functions find the intervals in which the functions are increasing, decreasing:

(i) $f(x) = -2x^2 + 4x + 3$.

(ii) $f(x) = \begin{cases} \dfrac{x^2 - 2x + 1}{x + 1}, & x \neq 1 \\ 1, & x = 0. \end{cases}$

(iii) $f(x) = |x + 2| + 1$.

(iv) $f(x) = x^{\frac{2}{3}} - 1$.

(v) $f(x) = x + \sin x$

2. Show that

(i) $\sin x > x \;\; \forall \;\; x \in (0, \pi)$.

(ii) $(1 + x)^{200} > 1 + 200x \;\; \forall \;\; x > 0$.

3. Find a polynomial function $p(x)$ which has all the following properties:

(i) Decreasing in $(-\infty, 0) \bigcup (2, +\infty)$.

(ii) Increasing in $[0, 2]$.

(iii) $p(0) = 0, \; p(2) = 2$.

4. Show that $(1 + x)^{\frac{1}{3}} < 1 + \dfrac{1}{3} x$ for every $x > 0$.

5. In the following statements, prove if you think they are true, give examples if they are false: for functions f, g on $\mathbb{R}$:

(i) If f, g are increasing, then $f+g$ is also increasing.

(ii) If f, g are increasing, than so in $f.g$.

6. Give examples of functions f, g on $\mathbb{R}$ with the following properties:

(i) f, g are both increasing but $f-g$ is decreasing.

(ii) f, g are both strictly increasing, but $f-g$ is a constant function.

(iii) f, g are both increasing and $f-g$ is also increasing.

7. Let $f:(a,b)\rightarrow\mathbb{R}$ and $c\in(a,b)$.If f is continuous, is increasing in (a,c) and f also increasing (c,b).

Show that f is increasing in (a,b).

Using this, prove the following:

$f(x)=x+\sin x$ is increasing in $(-\infty,+\infty)$.

8. State a result corresponding to exercise 7 for decreasing functions and use that to show that $f(x)=\cos x-x$ is decreasing $(-\infty,+\infty)$.

Recap

In this section you have learnt the following

- How the knowledge about the derivatives of a function helps us to draw conclusions regarding the increasing / decreasing nature of the function.

[Section 10.2]

Objectives

In this section you will learn the following :

How the knowledge about the derivatives of a function helps us to draw conclusions regarding the points of local maximum / minimum for the function.

In this section we analyze the problem of locating point of local maximum/minimum for a function.

10.2.1 Theorem (continuity test for local maximum/minimum):

Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$. Let f be continuous at c.

(i) If f is increasing in an interval $(c-\delta,c)$ and decreasing in an interval $(c,c+\delta)$, for some $\delta>0$, then f has a local maximum at c.

(ii) If f is decreasing in an interval $(c-\delta,c)$ and increasing in an interval $(c,c+\delta)$, for some $\delta>0$, then f has a local minimum at c.

10.2.1 Theorem (continuity test for local maximum/minimum):

Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$. Let f be continuous at c.

(i) If f is increasing in an interval $(c-\delta,c)$ and decreasing in an interval $(c,c+\delta)$, for some $\delta>0$, then f has a local maximum at c.

(ii) If f is decreasing in an interval $(c-\delta,c)$ and increasing in an interval $(c,c+\delta)$, for some $\delta>0$, then f has a local minimum at c.

Proof:

We prove (i), proof of (ii) is similar. Let

$$x,y\in(c-\delta,c) \text{ with } x<y$$

Then, by the given condition, $f(x)\le f(y)$. Letting $y\to c$ and using the continuity of f at c. We get

$$f(x)\le\lim_{y\to c}f(y)=f(c) \text{ for } x\in(c-\delta,c).$$

Similarly,

$$f(x)\le f(c) \text{ for } x\in(c,c+\delta).$$

Hence, f has a local maximum at c.

10.2.2 Examples:

Let

$$f(x)=\begin{cases}x^2 & \text{for } 0\le x\le 1\\ 1-x+\alpha & \text{for } x\ge 1.\end{cases}$$

We want to find α such that f has a local maximum at $x=1$. Since f is increasing in $(0,1)$ and decreasing in , by theorem **10.2.1,** it will have a local maximum at $x=1$ if it is continuous at

$(1,+\infty)$

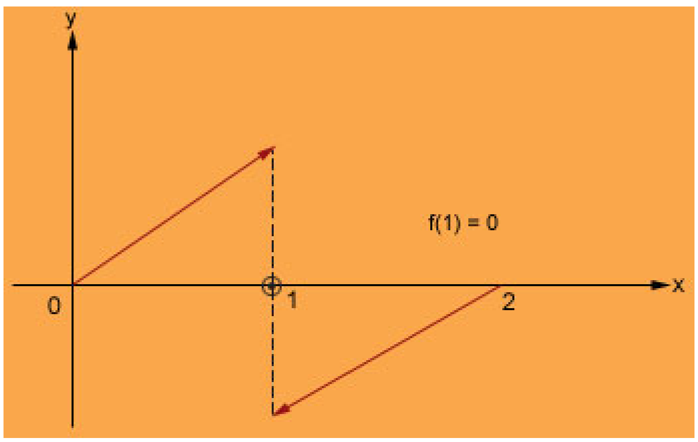
$x=1$. That will be so if $\alpha=1$.

10.2.3 Remark:

In theorem **10.2.1** , the continuity for the function f at the point $c\in(a,b)$ is necessary. For example consider the function:

$$f(x)=\begin{cases} x, & \text{if } 0\le x<1, \\ 0, & \text{if } x=1, \\ x-2, & \text{if } 1<x<2. \end{cases}$$

Then, f is not continuous at $x=1$ and has no local maximum/minimum.



A sufficient condition for the existence of local maxima/minima in terms of the first derivative is given in the next theorem.

10.2.4 Theorem (First derivative test for local maxima/minima):

Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$ and be such that f is continuous at c.

(i) If there exists some $\delta>0$, such that $f'(x)$ exists in $(c-\delta,c)\cup(c,c+\delta)$ with

$$f'(x)\ge 0 \textit{ for } x\in(c-\delta,c) \textit{ and } f'(x)\le 0 \textit{ for } x\in(c,c+\delta),$$

then, f has a local maximum at c.

10.2.7 Theorem (Second derivative test for local maximum/ minimum):

Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$.

(i) If $f'(x)$ exists in $(c-\delta,c+\delta)$ for some $\delta>0$ with $f'(c)=0$ and $f''(c)$ exists with $f''(c)<0$, then f has a local maximum at c.

(ii) If $f'(x)$ exists in $(c-\delta,c+\delta)$ for some $\delta>0$ with $f'(c)=0$ and $f''(c)$ exists with $f''(c)>0$, then f has a local minimum at c.

10.2.7 Theorem (Second derivative test for local Maximum):

Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$.

(i) If $f'(x)$ exists in $(c-\delta,c+\delta)$ for some $\delta>0$ with $f'(c)=0$ and $f''(c)$ exists with $f''(c)<0$, then f
has a local maximum at c.

(ii) If $f'(x)$ exists in $(c-\delta,c+\delta)$ for some $\delta>0$ with $f'(c)=0$ and $f''(c)$ exists with $f''(c)>0$, then f
has a local minimum at c.

Proof:

We give a proof of (i), proof of (ii) is similar. Consider $f':(c-\delta,c+\delta)\to\mathbb{R}$. Since

$$\lim_{x\to c}\left(\frac{f'(x)-f'(c)}{x-c}\right)=f''(c)<0,$$

there exists $\delta>\delta'>0$ such that

$$\left(\frac{f'(x)-f'(c)}{x-c}\right)<0 \quad \forall\, x\in\left(c-\delta',c+\delta'\right).$$

Hence,

$$f'(x)<f'(c)=0, \quad \text{if } x>c, \text{ and } f'(x)>f'(c)=0, \text{ if } x<c.$$

Now by the first derivative test, f has local maximum at $x=c$.

10.2.8 Local maximum / minimum procedure:

In view of Lemma 9.1.5, the possible points in the domain of a function $f:D\subseteq\mathbb{R}\to\mathbb{R}$, where f can have local maximum/ minimum are the following:

(i) The end points of closed intervals, if any, contained in D.

(ii) The points at which f is not differentiable.

(iii) The interior points of D such that $f'(x)=0$.

These points of (ii) and (iii) are called critical points of f. These points provide a complete list of probable points for local maximum/minimum for f. Now we can apply theorems 10.1.1, 10.2.4 and 10.2.7 to analyze for local maximum/ minimum at these points. Note that, f need not have a local maximum/minimum at a critical point.

10.2.9 Examples:

(i) Let $f(x)=\left(x^2-4\right)^{\frac{2}{3}}, x\in\mathbb{R}$.

(ii) Let $f(x)=x+\sin x,\ x\in\mathbb{R}$. The function is differentiable everywhere and

$f'(x) = 1 + \cos x.$

Thus, the critical points are $\{x \in \mathbb{R} \mid \cos x = -1\}$. However, none of these points is a point of local maximum/ minimum since $f'(x) = 1 + \cos x > 0$ for every x.

please refer web site for quiz

PRACTICE EXCERCISES

1. In the following the derivative function f' of a function f is given. Find the critical points, the intervals on which f is increasing / decreasing and the points of local maximum/minimum.

(i) $f'(x) = x^3(x^2 - 5)$.

(ii) $f'(x) = (x-1)^2 (x+2)$.

(iii) $f'(x) = x^{-\frac{1}{3}}(x-2)$.

(iv) $f'(x) = \dfrac{x^2 - 1}{x^2 + 2}$.

2. For the following functions, find the critical points and points of local maximum / minimum.

(i) $f(x) = \sin^2 x$.

(ii) $f(x) = x^4 - 8x^2 + 16$.

(iii) $f(x) = \dfrac{x^3}{3x^2 + 1}$.

(iv) $f(x) = x - 2\sin x, 0 < x < 2\pi$.

3. Find the intervals in which the function $f(x) = x^2 + bx + c$ is increasing/ decreasing. Also find points of local maximum / minimum of f.

4. Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. If $f(x)$ has three distinct real roots,
then show that $4p^3 + 27q^2 < 0$ by proving the following:

(i) $p < 0$

(ii) f has maxima at $-\sqrt{\dfrac{-p}{3}}$ and minima at $\sqrt{\dfrac{-p}{3}}$.

(iii) $f\left(-\sqrt{\dfrac{-p}{3}}\right) f \sqrt{\dfrac{-p}{3}} < 0.$

5. In each case, find a function f which satisfies all the given conditions, or else show that no such function exists:

(i) $f''(x) > 0$ for all $x \in \mathbb{R}, f'(0) = 1, f'(1) = 1.$

(ii) $f''(x) > 0$ for all $x \in \mathbb{R}, f'(0) = 1, f'(1) = 2.$

(iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}, f'(0) = 1, f(x) \leq 100$ for all $x > 0.$

(iv) $f''(x) > 0$ for all $x \in \mathbb{R}, f'(0) = 1, f(x) \le 1$ for all $x < 0$.

6. Show that a cubic polynomial can have atmost three distinct real roots.

7. Let $f : [a,b] \to \mathbb{R}$ be such that f'' exists and is continuous. If f has three distinct zeros in $[a,b]$, show that
f'' will have at least one real zero in (a,b).

Recap

In this section you have learnt the following

- How the knowledge about the derivatives of a function helps us to draw conclusions regarding the points of local maximum / minimum for the function.