Module 12 : Total differential, Tangent planes and normals

Lecture 35 : Tangent plane and normal [Section 35.1]

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Objectives

In this section you will learn the following :

- The notion tangent plane to a surface.
- The notion of normal line to a surface.

35.1 Tangent Planes and Normals to Surfaces

In the case of function of a single variable, the derivative helped us to formalize the concept of tangent and normal to the graph of the function. Similar results hold for functions of several variables, the role of the derivative being played by the gradient.

35.1.1Theorem:

Let $D \subset \mathbb{R}^3$ *and* $F : D \to \mathbb{R}$ *be a differentiable function. Let Then* S *be the level surface given by* F,

$$S = \{(x, y, z) \in D \mid F(x, y, z) = 0\}.$$

Let $P = (x_0, y_0, z_0)$ *be a point on* S *and* C *be any smooth curve lying on* S *and passing through* P. *Then,*

$$\nabla F(P) \cdot \underline{\boldsymbol{t}} = 0, \text{ where } \underline{\boldsymbol{t}} \text{ is the vector tangent to } C \text{ at } P.$$

35.1.1Theorem:

Let $D \subset \mathbb{R}^3$ *and* $F : D \to \mathbb{R}$ *be a differentiable function. Let Then* S *be the level surface given by* F,
$S = \{(x, y, z) \in D \mid F(x, y, z) = 0\}.$
Let $P = (x_0, y_0, z_0)$ *be a point on* S *and* C *be any smooth curve lying on* S *and passing through* P.

Then,
$\nabla F(P) \cdot \underline{t} = 0$, where $\underline{t}$ is the vector tangent to C at P.

Proof

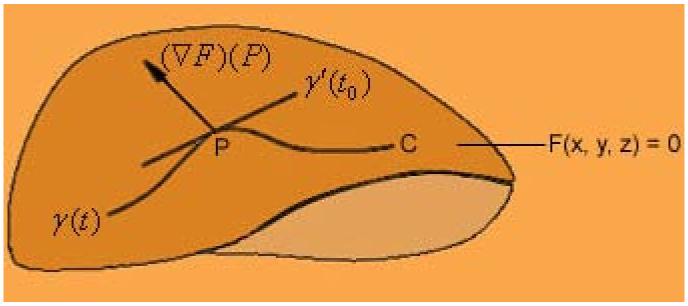

Figure 1. $\nabla F(P)$ **is normal to** S

Let C have the parameterization
$\underline{\gamma}(t) = (x(t), y(t), z(t)), \alpha \le t \le \beta$, with $\underline{\gamma}(t_0) = (x_0, y_0, z_0)$.
Then,
$F(x(t), y(t), z(t)) = 0$ for all $t \in [\alpha, \beta]$,
and $P = (x(t_0), y(t_0), z(t_0))$ for some $t_0 \in (\alpha, \beta)$. Further
$\underline{\gamma}'(t_0) = (x'(t_0), y'(t_0), z'(t_0))$
is the **tangent vector** to C at P, provided $\gamma'(t_0) \ne (0,0,0)$. Now
$G(t) := F(x(t), y(t), z(t)) = 0$ for all $t \in [\alpha, \beta]$.
and hence, by the chain rule , we have
$0 = G'(t_0) = F_x(P)x'(t_0) + F_y(P)y'(t_0) + F_z(P)z'(t_0) = \nabla F(P) \cdot \underline{t}$,

where $\underline{t} = \underline{\gamma}'(t_0)$.

In view of the above theorem, if the vector

$$\nabla F(P) = \left(F_x(P), F_y(P), F_z(P)\right) \ne 0,$$

is perpendicular to the tangent vector to every smooth curve C on S through P. Then, all these tangent vectors will lie in the same plane which is perpendicular to $\nabla F(P)$. This motivates our next definition.

35.1.2 Definition:

Let a surface S be given by
$S = \{(x, y, z) \in D \mid F(x, y, z) = 0\}$,
where $F : D \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function. Let $P = (x_0, y_0, z_0)$ be a point on S such that $\nabla F(P) \ne (0,0,0)$. Then, $\nabla F(P)$ is called the **normal** to S at the point P. The plane through the point P with normal $\nabla F(P)$ is called the **tangent plane** to the surface S at P and is given by

$$F_x(P)(x - x_0) + F_y(P)(y - y_0) + F_z(P)(z - z_0) = 0.$$

The line, with parametric equation
$x = x_0 + F_x(P)t,. \; y = y_0 + F_y(P)t, \; z = z_0 + F_z(P)t,$

is called the **normal line** to S through P. In case $F_x(P), F_y(P), F_z(P)$ are all nonzero, these equations can

be written as

$$\frac{x - x_0}{F_x(P)} = \frac{y - y_0}{F_y(P)} = \frac{z - z_0}{F_z(P)}.$$

35.1.3Example:

Given $D \subset \mathbb{R}^2$ and a differentiable function $f : D \to \mathbb{R}$, consider the surface given by

$$S = \{(x, y, z) \mid z = f(x, y), (x, y) \in D\}.$$

This is a special case of the earlier discussion, for if we let

$$F(x,y,z) = f(x,y) - z, \ (x, y) \in D, z \in \mathbb{R} \ ,$$

then

$$S = \{(x,y,z) \mid F(x,y,z) = 0\}.$$

Thus, for any point $(x_0, y_0) \in D$, if $z_0 = f(x_0, y_0)$, then for $P = (x_0, y_0, z_0)$

$$\nabla F(P) = (f_x(x_0, y_0), f_y(x_0, y_0, -1) \neq 0,$$

and the tangent plane to S is given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = z - f(x_0, y_0).$$

The normal line to the surface at (x_0, y_0) is given by

$$x = x_0 + f_x(x_0, y_0)t, \ \ y = y_0 + f_y(x_0, y_0)t, \ \ z = f(x_0, y_0) - t.$$

35.1.4 Example:

(i) Consider the hyperboloid

$$z^2 = 2x^2 + 2y^2 + 4.$$

We re-write this as

$$F(x, y, z) = 2x^2 + 2y^2 - z^2 + 4 = 0$$

Then

$$F_x(x,y,z) = 4x \ , \ F_y(x,y,z) = 4y \ , \ F_z(x,y,z) = -2z.$$

At the point $(1, 1, 2)$

$$F_x(1,1,2) = 4 \ , \ F_y(1,1,2) = 4 \ , \text{and } F_z(1,1,2) = -4.$$

Thus, the equation of the normal line to the hyperboloid at $(1, 1, 2)$ is

$$\frac{x-1}{4} = \frac{y-1}{4} = \frac{z-2}{-4},$$

and the equation of the tangent plane to the hyperboloid at $(1, 1, 2)$ is given by

$$4(x-1) + 4(y-1) - 4(z-2) = 0,$$

i.e.,

$$4x + 4y - 4z = 0.$$

(ii) Consider the curve of intersection of the sphere $x^2 + y^2 + z^2 = 6$ and the plane $x - y - z = 0$. The point

$P = (2, 1, 1)$ lies on this curve. We want to find tangent line to this curve C at P. Let

$$F(x, y, z) = x^2 + y^2 + z^2 - 6$$

and

$G(x,y,z) = x - y - z$.

Then, the tangent line to C at P is the line orthogonal to both $\nabla F(P)$ and $\nabla G(P)$. Thus, the required line is parallel to the vector is $\nabla F(P) \times \nabla G(P)$. In our case

$$\nabla F(x,y,z) = (2x, 2y, 2z),$$

$$\nabla G(x,y,z) = (1,-1,-1).$$

Thus

$$(\nabla F)(P) \times \nabla G(P) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 2 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= 2\,\mathbf{j} - 2\,\mathbf{k} = 2\,(\mathbf{j} - \mathbf{k}).$$

Thus, the required tangent line is the line through $(2,1,1)$ parallel to the vector $(\mathbf{j} - \mathbf{k})$, i.e.,

$$\mathbf{r}(t) = (2\,\mathbf{i} + \mathbf{j} + k) + t\,(2\,\mathbf{j} - 2\,\mathbf{k})$$
$$= 2\,\mathbf{i} + (1+t)\,\mathbf{j} + (1-t)\,\mathbf{k},\ t \in \mathbb{R}.$$

35.1.5 Note (Estimate of change of a given direction) :

The motion of directional derivative can be used to estimate the change in the values of a function f near a point P in a given direction. If P is (x_0, y_0, z_0) and the direction is given by the unit vector $\mathbf{u}$, then the approximation to this change is given by

$$f(P + t\,\mathbf{u}) - f(P) \cong \big((\nabla f)(P) \,.\, \mathbf{u}\big)\,|t|$$

35.1.6 Example:

For the function

$$f(x,y,z) = x + x\cos z - y\sin z + y,$$

$$(\nabla f)(x,y,z) = \big(1 + \cos z,\ 1 - \sin z,\ x - y\big).$$

Thus, for $P = (2,-1,0)$,

$$(\nabla f)(P) = (3,1,3).$$

The approximate change in the value of f in moving from P towards the point $Q = (0,1,2)$ by a distance of 0.2 units is given by

$$f(3.02, 1.02, 3.02) - f(3,1,3) \cong \big((\nabla f)(P) \,.\, \mathbf{u}\big)\,(.02),$$

where

$$\mathbf{u} = \frac{1}{\sqrt{10}}(3,0,1).$$

Thus, the approximate change is

$$\frac{02}{\sqrt{10}}[(3,1,3)\,.\,(3,0,1)] = \frac{24}{\sqrt{10}}.$$

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Practice Exercises

(1) Find the equation of the tangent plane and the normal line to the surface at the indicated point:

(i) $xy - z = 0,\ P = (-2,-3,6).$

(ii) $xy^2 + yz^2 + zx^2 = 25,\ P = (1,2,3).$

(iii) $z = e^y \cos x,\ P(0,0,1).$

Answer

(2) Find the smallest positive angle between the normals to the surfaces

$z = e^{xy} - 1$ and $z = \ln\sqrt{x^2 + y^2}$
at the point $(0,1,0)$.

Answer

(3) Compute $(D_u f)(2,2,1)$, where

$f(x,y,z) = 3x + 5y + 2z$
and
$\boldsymbol{u}$ is the direction of the outward normal to sphere $x^2 + y^2 + z^2 = 9.$

Answer

(4) Consider the cone

$z^2 = 4(x^2 + y^2)$

Let $P(x_0, y_0, z_0)$ be a point on the cone, other than the vertex. Show that the tangent plane to the cone at P always parses through the vertex.

(5) Find the parametric equations of the tangent line to the curve of intersection of the surface

$z = x^2 + y^2$ and $x^2 + 4y^2 + z^2 = 9$

at the point $P = (4,3,5).$

Answer

(6) For the given functions, find the direction in which it has maximal directional derivative and find its value also:

(i) $f(x,y)=x^2e^y$ at $P=(-2,0)$.

(ii) $f(x,y,z)=(\sin xy)e^{-z^2}$ at $P(1,\pi,0)$.

Answer

Recap

In this section you have learnt the following

- The notion tangent plane to a surface.
- The notion of normal line to a surface.

[Section 35.2]

Objectives

In this section you will learn the following :

- The notion of higher order partial derivatives
- The conditions that ensure the equality of the mixed partial derivatives

35.2 Higher Order Partial Derivatives

35.2.1Definition:

Let $(x_0,y_0)\in\mathbb{R}^2$ and $f:B_r(x_0,y_0)\to\mathbb{R}$ be such that both f_x and f_y exist at every point of $B_r(x_0,y_0)$. This gives us functions

$$f_x:B_r(x_0,y_0)\to\mathbb{R}\ \text{ and }\ f_y:B_r(x_0,y_0)\to\mathbb{R}.$$

We can analyze the existence of the partial derivatives of the functions f_x and f_y with respect to the variable x and y, namely

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right),$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yx} = \frac{\partial^2 f}{\partial x \partial y} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

These functions, whenever exist, are called the **second order partial derivatives** of f. Higher order partial derivatives for functions of three variables can be defined similarly.

35.2.2 Remark:

Higher order partial derivatives are useful in studying physical phenomenon. For example, if $u(x,t)$ represents the temperature (of a uniform rod) at position x and time t, then it satisfies the (one-dimensional) **heat equation**

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t},$$

where c is a constant (determined by the rod). Also, if $v(x,y)$ represents the (electrostatic or gravitational) potential at a point (x,y) on a thin plate, then it satisfies the (two-dimensional) **Laplace equation** :

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Lastly, if $w(x,t)$ represents the height of the wave at distance x and time t, then it satisfies the (one-dimensional) **wave equation** :

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2},$$

where c is the velocity with which the wave is propagated.

35.2.3 Examples:

Let

$$f : \mathbb{R}^2 \to \mathbb{R} \quad f(x,y) = e^x \cos y, \ (x,y) \in R^2.$$

All higher order partial derivatives of f exist at every point $(x_0,y_0) \in \mathbb{R}^2$. Since

$$f_x(x_0,y_0) = e^{x_0} \cos y_0 \text{ and } f_y(x_0,y_0) = -e^{x_0} \sin y_0,$$

we have, for example,

$$f_{xx}(x_0,y_0) = e^{x_0} \cos y_0, \ f_{yy}(x_0,y_0) = -e^{x_0} \cos y_0$$

and

$$f_{xy}(x_0,y_0) = -e^{x_0} \sin y_0 = f_{yx}(x_0,y_0).$$

We prove next an important theorem for mixed partial derivatives f_{xy} and f_{yx}.

35.2.4 Theorem (Equality of mixed Derivatives:

Let $(x_0,y_0) \in \mathbb{R}^2$ *and* $f : B_r(x_0,y_0) \to \mathbb{R}$ *be such that* f_x, f_y, f_{xy} *and* f_{yx} *exist on* $B_r(x_0,y_0)$ *and are* *continuous at* (x_0, y_0). *Then*

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

35.2.4Theorem (Equality of mixed Derivatives):

Let $(x_0, y_0) \in \mathbb{R}^2$ ***and*** $f : B_r(x_0, y_0) \to \mathbb{R}$ ***be such that*** f_x, f_y, f_{xy} ***and*** f_{yx} ***exist on*** $B_r(x_0, y_0)$ ***and are continuous at*** (x_0, y_0). ***Then***
$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$

Proof

Let $\varepsilon > 0$ be given. By the continuity of f_{xy} and f_{yx} at (x_0, y_0), we can find $\delta > 0$ such that $\delta \le r$ and for any $(x_1, y_1) \in B_\delta(x_0, y_0)$, we have

$$|f_{xy}(x_1, y_1) - f_{xy}(x_0, y_0)| < \frac{\varepsilon}{2}, \text{ and } |f_{yx}(x_1, y_1) - f_{yx}(x_0, y_0)| < \frac{\varepsilon}{2}.$$

For $(h, k) \in B_\delta(0, 0), h \ne 0, k \ne 0,$ fixed, define

$$F : (x_0 - \delta, x_0 + \delta) \longrightarrow \mathbb{R} \text{ and } G : (y_0 - \delta, y_0 + \delta) \longrightarrow \mathbb{R}$$

as follows

$$F(x) := f(x, y_0 + k) - f(x, y_0) \text{ and } G(y) := f(x_0 + h, y) - f(x_0, y).$$

Then, F and G are differentiable, and

$$F(x_0 + h) - F(x_0) = G(y_0 + k) - G(y_0).$$

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Applying Lagrange's mean value theorem to the function F, we can find some c_1 between x_0 and $x_0 + h$, such that

$$F(x_0 + h) - F(x_0) = hF'(c_1) = h[f_x(c_1, y_0 + k) - f_x(c_1, y_0)]$$

Also, by Lagrange's mean value theorem applied to the function $y \mapsto f_x(c_1, y)$, gives us some d_1 between y_0 and $y_0 + k$ such that

$$f_x(c_1, y_0 + k) - f_x(c_1, y_0) = kf_{xy}(c_1, d_1).$$

35.2.5Example:

Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $f(0,0) = 0$ and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2} \quad \text{for}(x, y) \ne (0,0).$$

We have

$$f_x(0, y_0) = -y_0 \quad \text{for any} \quad y_0 \in \mathbb{R},$$

and

$$f_y(x_0, 0) = x_0 \quad \text{for any} \quad x_0 \in \mathbb{R}.$$

Thus,

$$f_{xy}(0,0) = -1 \neq 1 = f_{yx}(0,0).$$

This shows that the continuity hypothesis in the above theorem cannot be dropped. Indeed, for $(x_0, y_0) \neq (0,0)$, we have

$$f_{xy}(x_0, y_0) = \frac{(x_0^2 - y_0^2)(x_0^4 + 10x_0^2 y_0^2 + y_0^4)}{(x_0^2 + y_0^2)^3} = -f_{yx}(x_0, y_0).$$

PRACTICE EXERCISES

(1) Let

$$f(x,y) = \begin{cases} -\left(\dfrac{xy}{x^2+y^2}\right) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

Is $f_{xy}(0,0) = f_{xy}(0,0)$? Justify your claim.

Answer

(2) Consider the function

$$z = xy + \left(\frac{e^y}{y^2+1}\right).$$

(3) Show that $\partial z/\partial x$ and $\partial z/\partial y$ satisfy the conditions of theorem 35.2.4. Hence, compute $\partial^2 z/\partial x \partial y$ by computing $\partial^2 z/\partial y \partial x$.

(4) Show that the following functions satisfy the Laplace equation :

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

(i) $f(x,y,z) = 2(x^2+y^2) - 4z^2.$

(ii) $f(x,y,z) = \sqrt{x^2+y^2+z^2}.$

(iii) $f(x,y,z) = e^x \sin y + e^y \sin x.$

(5) Let

$$f(x,y) = x^3y^5 - 2x^2y + x.$$

Show that

$$f_y \, xy = x \, f_{yy}.$$

(6) Let

$$f(x,y,z) = x^3y^5z^7 + xy^2 + y^3z.$$

Show that

$$z\, f_{zz} = 5\, f_{xyz}.$$

Recap

In this section you have learnt the following

- The notion of higher order partial derivatives
- The conditions that ensure the equality of the mixed partial derivatives