Objectives

In this section you will learn the following :

- Mean value theorem for functions of several variables
- Linear approximations for functions of several variables

36 .1 Mean value theorem and Linearization

36.1.1Definition:

Let $D \subset \mathbb{R}^2$ and $P(x_0, y_0), Q(x_1, y_1) \in D$. Let

$h := x_1 - x_0$, and $k := y_1 - y_0$.

Then,

$$L(P,Q) := \{(x_0 + th, y_0 + tk) \mid 0 < t < 1\}$$

and

$$\overline{L(P,Q)} := \{(x_0 + th, y_0 + tk) \mid 0 \leq t \leq 1\}$$

are the open and the closed straight line segments joining (x_0, y_0) and (x_1, y_1).

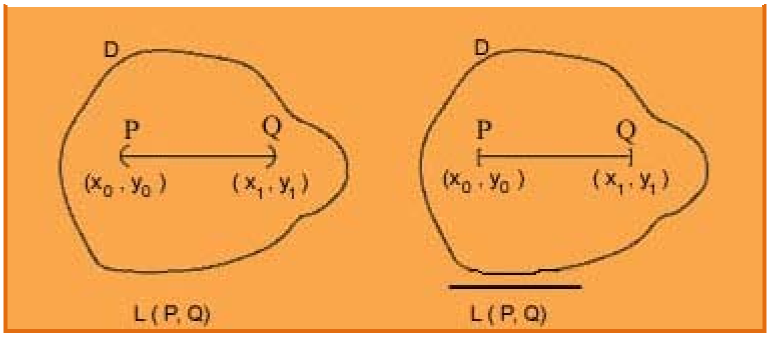

Figure 1. Open and closed line segments

36.1.2 Mean Value Theorem:

Let $D \subset \mathbb{R}^2$ and $P(x_0, y_0), Q(x_1, y_1) \in D$ be such that every point of $L(P,Q)$ is an interior point of D. Let $f : D \to \mathbb{R}$ is such that the following hold:

(i) *Both f_x and f_y exist at every interior point of D, and are continuous at every point of $L(P,Q)$.*

(ii) *f is continuous at both $P(x_0, y_0)$ and $Q(x_1, y_1)$.*
Then there exists some $\theta \in (0,1)$ such that

$$f(x_1, y_1) - f(x_0, y_0) = h\ f_x(x_0 + \theta h, y_0 + \theta k) + k\ f_y(x_0 + \theta h, y_0 + \theta k).$$

PROOF

36.1.2 Mean Value Theorem:

Let $D \subset \mathbb{R}^2$ and $P(x_0, y_0), Q(x_1, y_1) \in D$ be such that every point of $L(P,Q)$ is an interior point of D. Let $f : D \to \mathbb{R}$ is such that the following hold:

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Then there exists some $\theta \in (0,1)$ such that

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Proof

Define $\phi : [0,1] \to \mathbb{R}$ by

$$\phi(t) := f(x_0 + th, y_0 + tk),\ t \in [0,1].$$

Then, ϕ is continuous on $[0,1]$. Also, by the chain rule, ϕ is differentiable at every $t \in (0,1)$ with

$$\phi'(t) = hf_x(x_0 + th, y_0 + tk) + kf_y(x_0 + th, y_0 + tk).$$

Hence by the men value theorem for functions of one variable, there exists some $\theta \in (0,1)$ such that

$$\phi(1) - \phi(0) = \phi'(\theta).$$

Thus,

$$f(x_1, y_1) - f(x_0, y_0) = hf_x(x_0 + \theta h, y_0 + \theta k) + kf_y(x_0 + \theta h, y_0 + \theta k).$$

36.1.3 Remark:

(i) The mean value theorem for functions of two variables has consequences analogous to those of the mean value

theorem for functions of one variable. For example, if every point of D is an interior point of D and any two points of D can be joined by a finite number of straight line segments which lie in D, then any function $f : D \to \mathbb{R}$ such that $f_x = f_y = 0$ at every point of D, f must be a constant function.

(ii) As in the case of functions of a single variable, Taylor's theorem holds for functions of several variables also.
The interested reader may consult an advanced book on calculus of several variables.

36.1.4 Definition:

Let $(x_0, y_0) \in \mathbb{R}^2$ and $f : B_r(x_0, y_0) \to \mathbb{R}$ be such that f_x, f_y exist on $B_r(x_0, y_0)$ and are continuous at (x_0, y_0). Then,

$$T(x,y) := f(x_0, y_0) + (x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0)$$

is a linear function called the **linear** (or **tangent plane**) approximation of f for (x, y) near the point (x_0, y_0).

36.1.5 Note (Error estimate):

(i) If

$$h = x - x_0 \text{ and } k = y - y_0,$$

then using theorem 36.1.2, we can find $\theta \in (0,1)$ such that at points (x, y) close to (x_0, y_0),

$$f(x,y) - f(x_0, y_0) = h\, f_x(x_0 + \theta h, y_0 + \theta k) + k\, f_y(x_0 + \theta h, y_0 + \theta k).$$

Thus, the error in using linear approximation is given by

$$\begin{aligned} e_1(x,y) &:= f(x,y) - T(x,y) \\ &= f(x,y) - f(x_0, y_0) - h f_x(x_0, y_0) - k f_y(x_0, y_0) \\ &= h\, f_x(x_0 + \theta h, y_0 + \theta k) + k\, f_y(x_0 + \theta h, y_0 + \theta k) - h f_x(x_0, y_0) - k f_y(x_0, y_0) \\ &= h[f_x(x_0 + \theta h, y_0 + \theta k) - f_x(x_0, y_0)] + k[f_y(x_0 + \theta h, y_0 + \theta k) - f_y(x_0, y_0)]. \end{aligned}$$

It follows from the continuity of the partial derivatives that the error $e_1(x, y)$ tends to zero as $(x,y) \to (x_0, y_0)$, i.e., as (h,k) tends to zero. In fact, it tends to zero ‘faster' than as (h,k) tends to zero, since

$$\left| \frac{e_1(x,y)}{\sqrt{h^2 + k^2}} \right| \leq | f_x(x_0 + \theta h, y_0 - \theta k) - f_x(x_0, y_0) | + | f_y(x_0 + \theta h, y_0 + \theta k) - f_y(x_0, y_0) |$$

$$\longrightarrow 0 \text{ as } (h,k) \to 0.$$

(ii) If the second order partial derivatives of f exist and are continuous in some open ball $B_r(x_0, y_0)$ centered at

$P(x_0, y_0)$, then for $Q(x,y) \in B_r(x_0, y_0)$, it can be shown that the error in linear approximation can be estimated as follows:

$$|e_1(x,y)| \leq \frac{M_2(x,y)}{2}(|x-x_0|+|y-y_0|)^2,$$

where $M_2(x,y)$ is a real number such that

$$M_2(x,y) \geq \sup\{|f_{xx}(a,b)|, |f_{yy}(a,b)|, |f_{xy}(a,b)| \mid (a,b) \in \overline{L(P,Q)}\}.$$

36.1.6Example:

Let

$$f(x,y) = \frac{1}{(1-x-y)} \text{ for } (x,y) \in \mathbb{R}^2 \text{ with } x+y \neq 1.$$

Then,

$$f_x(0,0) = 1 = f_y(0,0).$$

Thus, the linear approximation to f for (x,y) near $(0,0)$ is

$$T(x,y) = 1+x+y.$$

To estimate the error, we note that

$$f_{xx}(a,b) = f_{xy}(a,b) = f_{yy}(a,b) = \frac{2}{(1-a-b)^3}.$$

Thus, we can take

$$M_2(x,y) = \frac{2}{(1-x-y)^3} \text{ if } 0 < x+y < 1$$

and

$$M_2(x,y) = 2 \text{ if } x+y \leq 0.$$

For example, if both $|x|,|y| < 0.1$, then

$$|e_1(x,y)| \leq \left(\frac{2}{(1-0.1-0.1)^3}\right)\left(\frac{((0-0.1)+(0-0.1))^2}{2}\right) = 0.7813 \text{ if } x+y > 0$$

and

$$|e_1(x,y)| \leq \left(\frac{2}{2}\right)((0-0.1)+(0-0.1))^2 = 0.04 \text{ if } x+y \leq 0.$$

36.1.7Example :

Consider a rectangular box of length $x = 50cms$, width $y = 20cms$ and hight $z = 15cms$. We want to find the percentage error made in measuring the volume if an error of $\pm 0.1mm$ is made in measuring each dimension of the box. Since the volume V is given by

$V(x,y,z) = xyz,$

with

$(x_0, y_0, z_0) = (50, 20, 15)$ and $(x_1, y_1, z_1) = (x_0 \pm 0.01, y_0 \pm 0.01, z_0 \pm 0.01),$

the error is given by

$e_1(x,y,z) = (\pm 0.01)\left[f_x(50,20,15) + f_y(50,20,15) + f_z(50,20,15)\right]$

$= (\pm 0.01)\left[(20)(15) + (50)(15) + (50)(20)\right]$

$= (\pm 0.01)(20\,50)$

$= \pm 20.5 cm^3$

The actual volume is

$V(50,20,15) = 15{,}000 cm^3.$

Thus, the percentage error is

$$\frac{e_1(x,y,z)}{V(x,y,z)} \times 100 = \frac{20.5}{15{,}000} \times 100 = 0.14\,\%$$

Practice Exercises

(1) For the following functions, find the linear approximation at the a bound for the error is approximation in the

specified region:

(i) $f(x,y) = 2x^2 - 2xy + y^2 + 6,\ P = (3,2),$ and

$|x-3| < 0.1, |y-2| < 0.1.$

(ii) $f(x,y,z) = xy + 2yz - 3xz,\ P = (1,1,0),$ and

$|x-1| < 0.01, |y-1| < 0.01, |z-1| < 0.01.$

Answer

(2) Let $f(x,y) = x^2 + y^2$. Find the linear approximation to f at $P = (0,0)$. Further find the points $(x,y) \neq (0,0)$

such that the error $e_1(x,y)$ has the property

$$\left|\frac{e_1(x,y)}{\sqrt{x^2+y^2}}\right| < 10^{-3}.$$

Answer

Recap

In this section you have learnt the following

- Mean value theorem for functions of several variables
- Linear approximations for functions of several variables