Module 3 : Differentiation and Mean Value Theorems

Lecture 7 : Differentiation

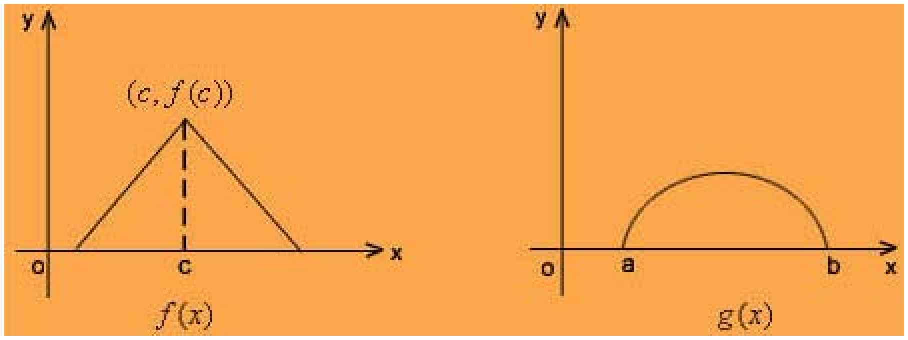
Objectives

In this section you will learn the following :

- The concept of derivative.
- Various interpretations of the derivatives.

7.1 Differentiation

We saw in the previous module that the concept of continuity helps us to understand a function better: the graph of a continuous function does not have any breaks. Next, we look at a property which helps us to analyze 'smoothness' of the graph. Let us look at the graphs of the following functions f and g :

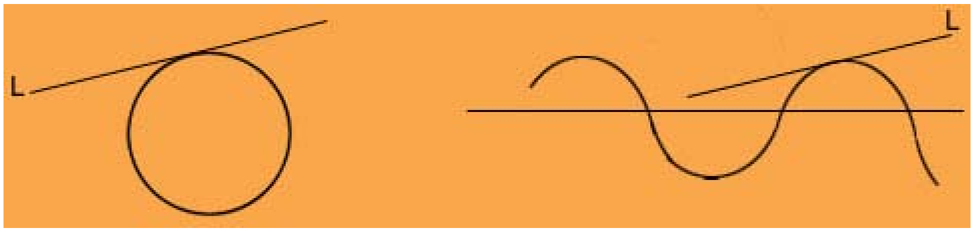


Though both the functions are continuous functions, their graphs do not have any breaks, the graph of f is 'qualitatively' different from the graph of g. We shall make this more precise. The graph of f has an 'edge' or a 'corner' at a point $x = c$, where as, the graph of g is 'smooth'. Geometrically, we can draw the tangent line to the graph of g at every point, where as this is not the case for the graph of the function f at the point $(c, f(c))$. Note that, to draw the tangent line at a point, we only need to know its slope. But before we try to do so, we have to decide what we mean by the 'tangent line'? This is not as simple to answer as it seems. For nice graphs, like that of a circle, it is easy: it is a line that intersects the circle only at one point. However, if we consider graph, like that of the trigonometric function $\cos x$, then the line L seems a tangent, is not acceptable by the above definition.

The line will be an acceptable tangent if we modify our definition that the line should intersect the graph

L

only at the point under consideration in some neighborhood of it. Now in order to find the slop of the tangent at a point P, consider a nearby point Q on the graph and consider the secant line, the line through P and Q.



The tangent line at P now can be thought as the line obtained as the point Q moves on the graph and comes closer to the point P, eventually merging with P. If the graph is that of a function $y = f(x)$, P is $(c, f(c))$ and Q is $(x, f(x))$, then

the slop of the line $PQ = \dfrac{f(c) - f(x)}{c - x}$,

and the slop of the tangent line at P should be the limiting case of the above slop as x approaches c.

Click here to View the Interactive animation : Applet 7.1

This motivates our next definition.

7.1.1 Definition:

Let $f : (a,b) \to \mathbb{R}$ and $c \in (a,b)$.

(i) We say that f is **differentiable** at the point c if

$$\lim_{x \to c} \frac{f(c) - f(x)}{c - x}$$

exists, and in that case

$$f'(c) := \lim_{x \to c} \frac{f(c) - f(x)}{c - x}$$

is called the **derivative** of f at the point c.

(ii) We say that f is **differentiable** on (a,b) if $f'(c)$ exists for each $c \in (a,b)$. The function f' is called the **derivative** of f.

(iii) The limit,

$$f'_-(c) := \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

is called the **left-hand derivative** of f at c

(iv) The limit

$$f'_+(c) := \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

is called the **right-hand derivative** of f at c.

Note that exists iff both both and exist and are equal.

$f'(c)$ $f'_-(c)$ $f'_+(c)$

7.1.2 Examples:

(i) Let $f:(a,b)\to\mathbb{R}$ be any constant function. Then, for any $c\in(a,b)$, since $f(x)-f(c)=0$, we have $f'(c)=0$ for all c.

(ii) Consider the function $f:\mathbb{R}\to\mathbb{R}, f(x):=x^n, n\in\mathbb{N}$. For any $c\in\mathbb{R}, f'(c)=nc^{n-1}$. To see this, note that

$$\lim_{h\to 0}\left(\frac{(c+h)^n-c^n}{h}\right)=\lim_{h\to 0}\left\{nc^{n-1}+h\left[\binom{n}{2}c^{n-2}+\ldots+h^{n-1}\right]\right\}=nc^{n-1}$$

Hence,

$$f'(c)=nc^{n-1}$$

(iii) The function $f:\mathbb{R}\to\mathbb{R}, f(x)=\sin x$, is differentiable at every $c\in\mathbb{R}$. To see this, observe that

$$\lim_{h\to 0}\frac{\cos h-1}{h}=\lim_{h\to 0}\left(\frac{cos^2 h-1}{h(1+\cos h)}\right)=\lim_{h\to 0}\left(-\left(\frac{\sin h}{1+\cos h}\right)\right)\left(\frac{\sin h}{h}\right)=0$$

Hence, using the formula

$$\lim_{h\to 0}\frac{\sin(c+h)-\sin c}{h}=\lim_{h\to 0}\left[\sin c\left(\frac{\cos h-1}{h}\right)+\cos c\left(\frac{\sin h}{h}\right)\right].$$

$$=\cos c$$

Similarly, the function $\cos x$ is differentiable at every $c\in\mathbb{R}$ and its derivative at $x=c$ is $-\sin c$.

(iv) The function $f:\mathbb{R}\to\mathbb{R}, f(x)=|x|$ is not differentiable at $c=0$ since

$$f'_-(0)=-1\neq 1=f'_+(0).$$

(v) Consider the function $f(x)=x^{1/3}, x\in\mathbb{R}$. Then

$$\frac{f(x)-f(0)}{x}=x^{-2/3}.$$

Thus, $\lim_{x\to 0}\left[\frac{f(x)-f(0)}{x}\right]=+\infty$.

Hence, $f(x)=x^{1/3}$ is not differentiable at $x=0$.

Click here to View the Interactive animation : Applet 7.2

Another equivalent way of saying that f is differentiable at $x=c$ is the following:

7.1.3 Proposition

Let $f:(a,b)\to\mathbb{R}$ and $c\in(a,b)$. Then, f is differentiable at c if and only if there exists a real number $\alpha\in\mathbb{R}$ and a function $\varepsilon(h):(-\delta,+\delta)\to\mathbb{R}$, for some $\delta>0$ such that

$$\lim_{h\to 0}\varepsilon(h)=0 \text{ and } f(c+h)=f(c)+h[\varepsilon(h)+\alpha].$$

And, in this case $\alpha=f'(c)$.

Proof

7.1.3 Proposition

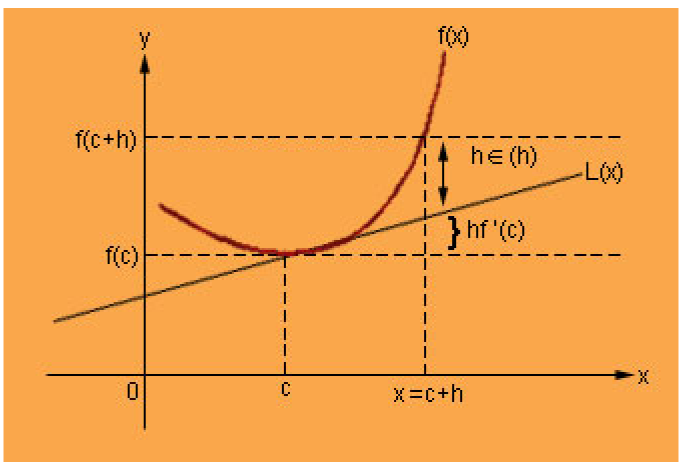
Let $f:(a,b)\to\mathbb{R}$ *and* $c\in(a,b)$. *Then,* f *is differentiable at* c *if and only if there exists a real number* $\alpha\in\mathbb{R}$ *and a function* $\varepsilon(h):(-\delta,+\delta)\to\mathbb{R}$, *for some* $\delta>0$ *such that*
$\lim_{h\to 0}\varepsilon(h)=0$ *and* $f(c+h)=f(c)+h[\varepsilon(h)+\alpha]$.
And, in this case $\alpha=f'(c)$.

Proof:

Suppose f is differentiable at c. Let δ be such that $(c-\delta,c+\delta)\subseteq(a,b)$. Define

$$\in(h):=\begin{cases}\dfrac{f(c+h)-f(c)}{h}-f'(c), & 0<|h|<\delta\\ 0 & \text{for } h=0\end{cases}$$

Then, f being differentiable at $x=c$ implies that $\lim_{h\to 0}\in(h)=0$, and the required claim follows with $\alpha:=f'(c)$.



Conversely, if (i) and (ii) are satisfied for f, then clearly

$$\lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}-\alpha\right)\le\lim_{h\to 0}|\in(h)|=0.$$

Hence, f is differentiable at $x=c$ with $\alpha=f'(c)$.

7.1.4 Note:

Note that, for a function f differentiable at $x=c$, the function $\in(h)$ as defined (in the proof of above proposition):

$$\in(h) := \begin{cases} \dfrac{f(c+h)-f(c)}{h} - f'(c), & 0<|h|<\delta \\ 0 & \text{for } h=0 \end{cases}$$

where δ is such that $(c-\delta, c+\delta) \subseteq (a,b)$ is the error made in measuring the slop of the tangent to the graph of f at $x=c$ by the slop of the secant joining $(c, f(c))$ and $(c+h, f(c+h))$. Further, if $\delta > 0$ is such that $(-\delta, +\delta) \subseteq (a,b)$ then for $0<|h|<\delta$.

$$f(c+h) = [f(c) + hf'(c)] + h \in (h).$$

Thus, $f(c+h) - f(c)$, the change in the values of f from $x=c$ to $x=c+h$, is approximately given by $hf'(c)$ the error being $h\in(h)$. Hence, in a neighborhood of c, the function f can be approximated by a linear function, $L(x) := f(c) + (x-c)f'(c)$. The quantity $df(c;h) = hf'(c)$, is called the **differential** of f at c. This aspect of differentiation plays an important role in many applications.

We show next that the property of a function being differentiable is stronger than that of continuity.

7.1.6 Examples

(i) Let

$$f(x) := \begin{cases} \dfrac{1}{x}, & x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

The function f is not differentiable at $x=0$, since it is not continuous at $x=0$.

(ii) Let $f(x) = x^{\frac{1}{n}}, x > 0$, where $n \geq 2$ is a fixed positive integer. We saw, in example 2.3.2(iv), that f is a continuous function. Let $g(x) := x^n, x > 0$, and $c > 0$ be a fixed point. Define for $x > 0$

$$f^*(x) = \begin{cases} \dfrac{g(x)-c}{x-c^{\frac{1}{n}}} & \text{for } x \neq c^{\frac{1}{n}} \\ \left.\dfrac{dg}{dx}\right|_{x=c^{\frac{1}{n}}} & \text{for } x = c^{\frac{1}{n}}. \end{cases}$$

Then, f^* is a continuous function, and f being also continuous, $f^* o f$ is continuous with

$$\left(f^* o f\right)(c) = \left.\frac{dg}{dx}\right|_{x=c^{\frac{1}{n}}} = n\left(c^{\frac{1}{n}}\right)^{n-1}.$$

Thus,

$$\lim_{x\to c}\left(\frac{x^{\frac{1}{n}}-c^{\frac{1}{n}}}{x-c}\right) = \lim_{x\to c}\left(\frac{1}{f^*\left(x^{\frac{1}{n}}\right)}\right)$$

$$= \lim_{x\to c}\left(\frac{1}{\left(f^* of\right)(x)}\right)$$

$$= \frac{1}{\left(f^* of\right)(c)}$$

$$= \frac{1}{n\left(c^{\frac{1}{n}}\right)^{n-1}}$$

$$= \frac{1}{n}c^{\frac{1}{n}-1}.$$

(iii) For the sake of giving example and illustrations, we shall assume the existence of the logrithmic function
(which we shall define later in lecture):

$$\log : (0,\infty)\to\mathbb{R}$$

is a bijective function with the following properties :

(i) $\log(xy) = \log(x) + \log(y)$ for every $x, y > 0$ ·

(ii) $\log\left(\frac{1}{x}\right) = -\log(x)$ for every $x > 0$ ·

(iii) $\log(1) = 0.$

(iv) $\log(x)$ is differentiable everywhere with

$$\frac{d}{dx}(\log x) = \frac{1}{x}, \qquad x > 0.$$

7.1.7 **Note:**

(i) The converse of the theorem 7.1.5 is false. For example, consider the function

$f(x) = |x|, x\in\mathbb{R}$. It is continuous everywhere, but it is not differentiable at $c = 0$.

(ii) Saying that a function f is not differentiable at $x = c$ means that either of the following happens:
Case (i) : The function is not continuous at $x = c$.

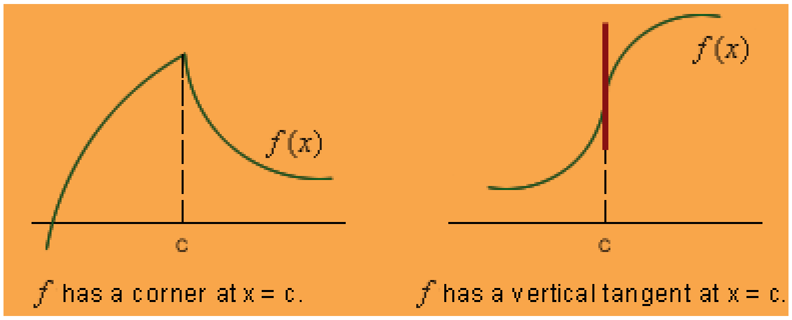
Case(ii): The function is continuous at $x = c$, but

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \text{ does not exist}.$$

For example, both the left and the right hand limits

$$\lim_{\substack{h\to 0\\ h<0}} \frac{f(c+h)-f(c)}{h} \text{ and } \lim_{\substack{h\to 0\\ h>0}} \frac{f(c+h)-f(c)}{h}$$

may exist, but are not equal. In that case we say that f has a corner at $x=c$.



f has a corner at x = c. f has a vertical tangent at x = c.

Another possibility is that

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = +\infty \text{ or } -\infty$$

In that case, we say that f has a vertical tangent at $x=c$.

Some results that help us in computing the derivatives are given in the next theorem.

7.1.8 Theorem:

Let $f,g:(a,b)\to\mathbb{R}$ and $c\in(a,b)$. If f,g are differentiable at c then the following hold:

(i) The functions $(f\pm g)$ are differentiable at $x=c$ and

$$(f\pm g)'(c) = f'(c) \pm g'(c).$$

(ii) For every $r\in\mathbb{R}$, the function rf is differentiable at c and

$$(rf)'(c) = rf'(c).$$

(iii) The function fg is differentiable at c and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

(iv) If $g(c)\neq 0$, then the function f/g is also differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$$

.

PROOF

7.1.9 Examples:

(i) It follows as a consequence of the above theorem that every polynomial function

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

is differentiable. Similarly, every rational function

$$r(x) = \frac{p(x)}{q(x)}, \; q(x) \neq 0,$$

is differentiable, whenever it is defined. Its derivatives can be computed using the examples and theorem 7.1.8 given above.

(ii) Let $f(x) = x^n, x \in \mathbb{R}$, where n is any negative integer. Then, using theorem 7.1.8(iii), f is differentiable at every $x \neq 0$ with

$$f'(x) = -\frac{nx^{-n-1}}{\left(x^{-n}\right)^2} = -nx^{n-1}.$$

7.1.10 Other notations for the derivative:

We have already used the notation $f'(c)$ for the derivative of f at the point c. Some other notations are as follows:

$$\left.\frac{df}{dx}\right|_{x=c} ; \; Df(c).$$

Sometimes, the notation $\frac{dy}{dx}$ is used for the derivatives of $y = f(x)$ at the point x. In this notation the use of the variable x is ambiguous , as it is used both for point where the derivative is being evaluated as well as the variable of differentiation. Any case, the symbol $\frac{df}{dx}$ or $\frac{dy}{dx}$, for $y = f(x)$, is just a notation for the value $f'(x)$, and it should not be regarded as a fraction of df and dx, dy and dx.

7.1.11 Extension of the derivative concept:

Let $f : [a,b] \to \mathbb{R}$. We say f is differentiable at the end point a if $f'_+(a)$ exists. Similarly, we say f is differentiable at the end point b as well as $f'_-(b)$ exist.

7.1.12 Example:

Let us find the tangent line to $f(x) = x^2$ at a point, which passes through the point (3,5). The equation of the tangent line to the graph of f at the point (c, c^2) is

$$y = f(c) + \left(x - c\right) f'(c) = c^2 + 2(x-c)c.$$

To pass through (3,5), we must have

$$5 = c^2 + 2c(3-c), \; i.e., \; c = 1 \text{ or } 5.$$

Hence, the points on the graph of f are $(1,1)$ and $(5,25)$ the corresponding tangent lines are

$$y = 2x - 1, y = 10x - 25$$

7.1.13 Interpretation of $f'(c)$:

(i) Geometric interpretation:

That f is differentiable at c means that the graph of $y = f(x)$ is 'smooth' at c, i.e., it is possible to define analytically the notion of unique tangent line to f at c as follows:

$$y = L(x) := (x - c) f'(c) + f(c).$$

If $f'(c) \neq 0$,then

$$y = n(x) := \frac{-(x - c)}{f'(c)} + f(c)$$

is called the normal to f at $x = c$, and $x = c$ is the equation of the normal when $f'(c) = 0$.

(ii) Rate of change:

For a function $f : (a,b) \to \mathbb{R}$, and $c \in (a,b)$, the ratio $(f(c) - f(x))/(c - x)$ can be thought of the average change in the values of f when its argument changes from c to a near by point x. Thus, if f is differentiable at c, then

$$f'(c) = \lim_{x \to c} \frac{f(c) - f(x)}{x - c}$$

can be taken as the rate of change of f at c.

(iii) Physical Interpretation of $f'(c)$:

Let $f(t)$ denote the distance traveled up to time t by a body in linear motion. Then, $f'(t)$ represents the rate of change of distance at time t, is called the instantaneous velocity of the body at time t.

7.1.14 Examples:

(i) Let

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{for } x \in (0,1] \\ 0 & \text{for } x = 0 \end{cases}$$

Then, f is differentiable at every point $x \in (0,1]$, but not at $x = 0$.

(ii) Let $f(x) = x^{\frac{1}{3}}, x \in [-1, 0]$. Then f is differentiable on $[-1, 0)$ but not at $x = 0$. In fact f has vertical tangent at $x = 0$.

Practice Excercises: Derivative

1. Using the definition evaluate $f'(x)$ for the following

(i) $f(x) = x^3$ at $x = 0$.

(ii) $f(x) = \sqrt{x}$ at $x = 0$.

(iii) $f(x) = -\dfrac{1}{x^2}, x \neq 0$, for any point $x = c$.

(iv) $f(x) = \dfrac{1}{\sqrt{x}}, x \neq 0$, at $x = 1$.

2. Show that $f(x) = x^{\frac{1}{3}}, x \in \mathbb{R}$ is not differentiable at $x = 0$. Does the graph of f have a tangent at $(0,0)$?

3. Give example of a continuous function which is differentiable everywhere except at point $x \neq \pm 1$.

4. Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x = a$. Hence, $|f(x)|$ is differentiable at $x = a$ if and only if $|f(a)| = 0$.

5. Let P be any polynomial of degree n such that $p(c) = 0$ and $p'(c) = 0$. Show that $P(x) = (x-c)^2\, \mathbb{Q}(x)$, where $\mathbb{Q}(x)$ is a polynomial of degree $(n-2)$.

6. Find the equation of tangent to the curve $y = x^2 - 4$ at the point when its graph intersects the two axes.

7. Find the angle of intersection of the graphs of function

(i) $f(x) = \dfrac{x^2}{2}, g(x) = \dfrac{x^3}{2}$.

(ii) $f(x) = \dfrac{x^2}{2}$, $g(x) = x^2(2-x)$.

8. Let $\phi = f_1 f_2 \dots f_n$ where each f_i is differentiable at $x = c$. Show that if $\phi(c) \neq 0$,

Then, $\dfrac{\phi'(c)}{\phi(c)} = \dfrac{f_1'(c)}{f_1(c)} + \dots\dots + \dfrac{f_n'(c)}{f_n(c)}$. \

9. Let $f(x) = |x+1| + |x-1|$. Find the points $x \in \mathbb{R}$ where f is not differentiable. Can a tangent be defined to the graph of f at these points.

10. Find the values of $b = c$ such that for $f(x) = x^2 + bx + c$, the line $y = x$ is a tangent to the graph of f at $(2, f(2))$.

11. Give an example of a function $f:(a,b)\to\mathbb{R}, c\in(a,b)$ such that $\frac{f(c+h)-f(c-h)}{2h}$ exists but f is not differentiable at $x=c$.

(Note that the converse holds)

12. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by, for $k\geq 0$ an integer,

$$f(x)=\begin{cases} x^k\sin\left(\frac{1}{x}\right) & \text{for } x\neq 0, \\ 0 & \text{if } x=0. \end{cases}$$

Prove the following statements

(i) For $k=0$, f is not continuous at $x=0$.

(ii) For $k=1$, f is continuous at $x=0$.

(iii) For $k=2$, f is differentiable at $x=0$, but $f'(x)$ is not continuous at $x=0$.

(iv) For $k=3$, f is differentiable at $x=0$, and $f'(x)$ is continuous everywhere.

13. Let

$f:[-1,1]\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} x^2\sin\left(\frac{1}{x^2}\right) & \text{for } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

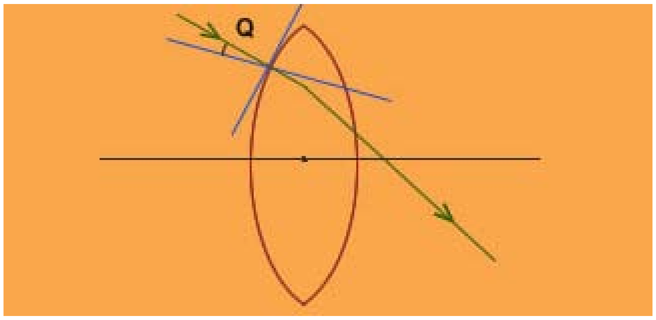
Show that f is differentiable everywhere on $\left[-1,1\right]$ but $f'(x)$ is not bounded on $\left[-1,1\right]$, (hence is also not continuous on $\left[-1,1\right]$).

Historical comments:

The notion of derivative of a function at a point evolved out of the efforts of the mathematicians during the seventeenth century to solve the following problems:

(i) The tangent line problem:

The problem is: how to define tangent line to curve at a point? This problem arose in the study of passage of light through a lens. It was important to know the angle at which a ray of light strikes the surface of the lens.



(ii) Problems in mechanics:

How to represent intantaneous velocity and acceleration of a moving body?

(iii) Maxima / minima problems:

How to find the maximum and minimum of a function? For example, to find the angle at which a missile should be fired so that it has maximum range. In astronomy, it is of interest to know when will a particular planet be at a maximum/ minimum distance from earth.

Mathematicians who contributed partially to solve these problems were Pierre de Fermat (1601-1665), Rene Descartes (1596-1650), Christian Huygens (1629-1695), and Issac Barrow (1630-1677). However it was the work of Issac Newton (1642-1717) and Gottfried Wilhelm von Leibniz (1646-1716) which laid the foundation for calculus. For detailed biographies of these mathematicians visit: http://www.gap.des.st-and.ac.uk/history/mathematics.

Recap

In this section you have learnt the following :

- The concept of derivative.
- Various interpretations of the derivatives.