

## Module 5 : Linear and Quadratic Approximations, Error Estimates, Taylor's Theorem, Newton and Picard Methods

### Lecture 15 : Newton's method [Section 15.1]

#### Objectives

In this section you will learn the following :

- Newton's Method for locating zeros of functions.

#### 15.1 Newton 's Method for locating zeros of a function

In many physical problems, it is required to answer the following:

- At what points  $x, f(x) = 0$ ?

Let us consider the particular case when  $f(x)$  is a polynomial. The problem is simple to solve when  $f$  is linear, i.e.,

$$f(x) = ax + b = 0, \text{ for } a \neq 0, \text{ if and only if } x = -b/a.$$

For a quadratic function

$$f(x) = ax^2 + bx + c = 0, a \neq 0, \text{ if and only if } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

One can use special methods to find the roots of a cubic and a biquadratic polynomial. However, there exist no simple method of finding the roots of polynomials of degrees  $\geq 5$ . The existence of roots, in general, is given by the Fundamental Theorem of Algebra, which says that every polynomial of degree  $n$  has  $n$  roots (may be complex). However, there is no general procedure of ensuring when it has real roots, and how to find those roots. The problem of finding zeros of general functions is even more complicated. For example, for

$$f(x) = x - \sin x, x \in \mathbb{R},$$

the only root is 0, but it is not easy to guess a root of

$$g(x) = x - \cos x, x \in \mathbb{R}.$$

Since the problem of finding a root can be difficult, one attempts to find approximations to a guessed root. One such method is to approximate the function itself by its tangent line approximation, i.e., replace the graph of  $y = f(x)$  by the tangent line near the points where its expected that  $f(x) = 0$ . The method is as follows:

### 15.1.1 Newton-Raphson Method:

Let  $f: A \rightarrow \mathbb{R}$ . To estimate a solution of  $f(x) = 0$ , we use the following iterative process.

- Guess a point  $x_0$  near a root of  $f$  such that  $f'(x_0) \neq 0$ . The point  $x_0$  is called the initial point.
- Replace  $f(x)$  by the tangent line approximation of  $f$  for  $x$  near  $x_0$ , i.e, by

$$L(x, x_0) = f(x_0) + f'(x_0)(x - x_0), x \in \mathbb{R}.$$

Solve  $L(x, x_0) = 0$  to get

$$x_1 := x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This is the point where the tangent to the graph of  $f(x)$  at  $(x_0, f(x_0))$  cuts the x-axis.

- Replace  $x_0$  by  $x_1$  and if  $f'(x_1) \neq 0$  carry out the procedure given above to find  $x_2$  i.e., find  $x_2$  by

$$x_2 := x_1 - \frac{f(x_1)}{f'(x_1)}.$$

This is the point where the tangent to the graph of  $f(x)$  at  $(x_1, f(x_1))$  cuts the x-axis.

- Continue the above procedure of this process: having obtained the next point  $x_{n+1}$  is obtained by

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, \dots$$

The above process will give a sequence  $\{x_n\}_{n \geq 0}$  of points, where  $x_0$  is the initial point. This sequence is called Newton sequence for  $f$  with the starting point  $x_0$ .

Geometrically, it looks that the sequence  $\{x_n\}_{n \geq 1}$  will converge to a root of  $f$ . Let us look at some examples.

### 15.1.2 Example:

Let

$$f(x) = x^3 - 3.$$

We note that  $f$  is a continuous function. To make a guess for a point  $x$  such that  $f(x) = 0$ , we try to find points  $a$  and  $b$  such that  $f$  changes sign at these points. We note that

$$f(5/4) = (125/64) - 3 < 0 \text{ and } f(3/2) = (27/8) - 3 > 0.$$

Thus, by the intermediate value property for continuous functions,  $f$  has a root in  $[5/4, 3/2]$ . Let us find the Newton sequence for this function with the starting point

$$x_0 = 5/4 = 1.25$$

We have

$$x_{n+1} = x_n - \frac{x_n^3 - 3}{3x_n^2} = \frac{2}{3}x_n + \frac{1}{x_n^2}, \text{ provided } x_n \neq 0.$$

This gives us

$$x_0 = 5/4 = 1.25, x_1 = 1.473333\dots, x_2 = 1.442900\dots, x_3 = 1.442249\dots$$

If we take  $x_0 = 3/2 = 1.5$ , then we get

$$x_1 = 1.444444\dots, x_2 = 1.442252\dots, x_3 = 1.442249\dots$$

Apparently, both the Newton sequences are coming closer close to the value  $x = 1.442249$ , but we cannot be sure.

This leads to the following questions:

- When does the Newton sequence  $\{x_n\}_{n \geq 1}$  converge?
- If the Newton sequence  $\{x_n\}_{n \geq 1}$  is convergent, does it converge to a root of  $f(x)$ ?

Some answers are provided in the next proposition.

### 15.1.3 Proposition:

- (i) In general, the Newton sequence need not converge.
- (ii) The Newton sequence can diverge.
- (iii) If for a Newton sequence,

$$x_{n_0+1} = x_{n_0} \text{ for some } n_0, \text{ then } f(x_{n_0}) = 0,$$

i.e.,  $x_{n_0}$  is a root of  $f$ .

- (iv) Let the Newton sequence  $\{x_n\}_{n \geq 1}$  of  $f$  be convergent to a point  $c$  in the domain of  $f$ . If  $f$  is continuous at  $c$  and  $\{f'(x_n)\}_{n \geq 1}$  is bounded, then  $c$  is a zero of  $f$ .
- (v) The Newton method is very sensitive to the starting point.



### 15.1.3 Proposition:

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- (ii) The Newton sequence can diverge.
- (iii) If for a Newton sequence,

$x_{n_0+1} = x_{n_0}$  for some  $n_0$ , then  $f'(x_{n_0}) = 0$ ,  
i.e.  $x_{n_0}$  is a root of  $f$ .

- (iv) Let the Newton sequence  $\{x_n\}_{n \geq 1}$  of  $f$  be convergent to a point  $c$  in the domain of  $f$ . If  $f$  is continuous at  $c$  and  $\{f'(x_n)\}_{n \geq 1}$  is bounded, then  $c$  is a zero of  $f$ .
- (v) The Newton method is very sensitive to the starting point guess.

Proof:

(i) To show that the Newton sequence need not converge at all, consider the function

$$f(x) = \begin{cases} \sqrt{x-1}, & \text{if } x \geq 1, \\ -\sqrt{1-x}, & \text{if } x < 1. \end{cases}$$

Then

$$f'(x) = \begin{cases} 1/2\sqrt{x-1}, & \text{if } x > 1, \\ 1/2\sqrt{1-x}, & \text{if } x < 1. \end{cases}$$

The Newton sequence for  $f$  with any starting point  $x_0 \neq 1$ , is given by

$$x_{n+1} = x_n - 2(x_n - 1) = -x_n + 2.$$

In the next section we shall answer the question:

- Under what conditions on  $f$ , does the Newton sequence  $\{x_n\}$  converge to a root of  $f$ ?

We close this section by analyzing the rate of convergence of a Newton sequence.

#### 15.1.4 Note (Rate of convergence in Newton's method):

Let  $f : [a, b] \rightarrow \mathbb{R}$ , be such that

- (i)  $f, f'$  and  $f''$  for all continuous on  $[a, b]$ .
- (ii)  $f'(x) \neq 0$  for all  $x \in [a, b]$ .
- (iii)  $f(r) = 0$  for some  $r \in [a, b]$ .

Then, it can be shown that for any two consecutive iterates  $x_n$  and  $x_{n+1}$  of Newton's sequence satisfy the following:

$$|x_{n+1} - r| \leq \alpha |x_n - r|^2,$$

where  $\alpha$  is some constant. Thus if we write  $E_n := x_n - r$ , then

$$|\alpha E_{n+1}| \leq |\alpha E_n|^2.$$

One says that the in the Newton's method the rate of convergence is quadratic. For example,

if  $|\alpha E_n| < 0.01$ , then at the next stage  $|\alpha E_{n+1}| < (0.01)^2$ .

**CLICK HERE TO SEE AN INTERACTIVE VISUALIZATION : [Applet 15.1](#)**

#### PRACTICE EXERCISES

1. For each of the following functions, show that equation  $f(x) = 0$  has a unique root in the interval  $I$  mentioned against it. Use Newton's method starting with the given initial point  $x_0$  to find an approximate value of this root.
  - (i)  $f(x) = x^3 - 3, I = [1.25, 1.5], x_0 = 1.25$  or  $x_0 = 1.5$ .
  - (ii)  $f(x) = x - 1 - \frac{\sin x}{2}, I = [0, 2], x_0 = 1.5$ .  
(You may have to use a calculator.)
2. Consider the function
 
$$f(x) = (x-1)^{\frac{1}{3}} \text{ for } x \in \mathbb{R}.$$
 If  $x_0 \neq 1$ , show that the Newton sequence for  $f$  is unbounded.
3. The function
 
$$f(x) = x^2 - 2x - 3 \text{ for } x \in \mathbb{R}$$
 has two roots:  $x = -1$  and  $x = 3$ . Show that if the initial point  $x_0 < 1$ , then the Newton sequence converges to the root  $-1$  and if  $x_0 > 1$ , then the Newton sequence for  $f$  converges to the other root  $x = 3$ .

4. Show that the function

$$f(x) = x^3 - 2x + 5$$

has only real zeros. Find its approximate value by Newton's method with a suitable choice of the initial point.

5. Show that the function

$$f(x) = x^3 - x - 1$$

has three real zeros. Use them to locate these zeros by selecting initial points as  $-2, 0, 2$ .

Recap

In this section you have learnt the following

- Newton's Method for locating zeros of functions.

## Module 5 : Linear and Quadratic Approximations, Error Estimates, Taylor's Theorem, Newton and Picard Methods

### Lecture 15 : Picard's method [Section 15.2]

#### Objectives

In this section you will learn the following :

- Picard's Method for finding fixed points of functions.
- Sufficient conditions for the convergence of a Newton sequence.

#### 15.2 Picard's Method for locating fixed points of a function

To answer the question about the convergence of a Newton sequence for a function  $f$ , assume that  $f'(x) \neq 0$ . Define

$$g(x) := x - \frac{f(x)}{f'(x)}.$$

Then, a point  $c$  is a zero of  $f$  if and only if  $g(c) = c$ . Further, for the starting point  $x_0$ , the Newton sequence can be rewritten as:

$$x_{n+1} := g(x_n), n \geq 0,$$

provided for the function  $g$ , its range is a subset of its domain. Thus, the question of convergence of the Newton sequence for  $f$  is equivalent to the convergence of the sequence  $\{x_n\}_{n \geq 1}$  generated by  $g$ .

This motivates our next definition.

##### 15.2.1 Definition:

Let  $A \subseteq \mathbb{R}$  and  $g : A \rightarrow \mathbb{R}$  be such that  $g(A) \subseteq A$ .

- (i) We say a sequence  $\{x_n\}_{n \geq 1}$  in  $A$  is a Picard sequence for  $g$  with initial point  $x_0$  if

$$x_{n+1} := g(x_n), n \geq 0.$$

- (ii) A point  $c \in A$  is called a fixed point of  $g$  if  $g(c) = c$ .

Thus, to locate a root of a function  $f$ , it is enough to find a fixed point of the function

$$g(x) := x - (f(x)/f'(x)).$$

That is, the Newton sequence for  $f$  is the corresponding Picard sequence for  $g$  as defined above. Let us observe the following:

### 15.2.2 Theorem:

Let  $A \subseteq \mathbb{R}$  and  $g: A \rightarrow \mathbb{R}$  be such that  $g(A) \subseteq A$ . If the Picard sequence  $\{x_n\}_{n \geq 1}$  is convergent to some point  $c \in A$  for and  $g$  is continuous at  $c$ , then  $c$  is a fixed point of  $g$ .



### 15.2.2 Theorem:

Let  $A \subseteq \mathbb{R}$  and  $g: A \rightarrow \mathbb{R}$  be such that  $g(A) \subseteq A$ . If the Picard sequence  $\{x_n\}_{n \geq 1}$  is convergent to some point  $c \in A$  for and  $g$  is continuous at  $c$ , then  $c$  is a fixed point of  $g$ .

Proof:

We note that

Thus,  $c \in A$  is such that  $g(c) = c$ .

In view of this theorem, to locate the zero of a function  $f$ , it is equivalent to locate fixed point the corresponding function  $g(x) = x - (f(x)/f'(x))$ . The above theorem says that a fixed point  $g$  of can be located if the Picard sequence converges, and  $g$  is continuous.

This raises the natural questions:

- When does a function  $g$  have a fixed point?
- When does the Picard sequence converge?

The answers are given by the next theorem.

### 15.2.3 Theorem (Picard's Convergence):

Let  $g: [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then, the following hold:

- (i) If  $g([a, b]) \subset [a, b]$ , then  $g$  has a fixed point in  $[a, b]$ .
- (ii) Let  $g$  has a fixed point  $x^* \in [a, b]$  and  $\{x_n\}_{n \geq 1}$  is a Picard sequence for  $g$  with any starting point

$x_0 \in [a, b]$ . Suppose,  $g'(x)$  exists for all  $x \in (a, b)$  with

$$|g'(x)| \leq \alpha \text{ for all } x \in (a, b),$$

for some constant  $\alpha < 1$ . Then

- (1)  $\{x_n\}_{n \geq 1}$  satisfies the relation:

$$|x_{n+1} - x^*| \leq \alpha^{n+1} |x_0 - x^*|, n = 0, 1, 2, \dots$$

- (2) The function  $g$  has only one fixed point and  $\{x_n\}_{n \geq 1}$  converges to that fixed point.



### 15.2.3 Theorem (Picard's Convergence):

Proof:

To prove (i), consider the function

$$h(x) := g(x) - x \text{ for } x \in [a, b]$$

Then,  $h$  is a continuous function. Since  $g(a), g(b) \in [a, b]$ ,

$$h(a) = g(a) - a \geq 0 \text{ and } h(b) = g(b) - b \leq 0.$$

Thus, by the Intermediate Value Property for continuous function, there is exists

$x^* \in [a, b]$  such that

$h(x^*) = 0$ , that is  $g(x^*) = x^*$ . To prove (2) of (ii), suppose that  $x^{**}$  is another fixed point

of  $g$  in  $[a, b]$ . Then, by the mean value theorem for  $g$  in the interval joining  $x^*$  and  $x^{**}$ ,

we have

$$|x^{**} - x^*| = |g(x^{**}) - g(x^*)| = |g'(c)| |x^{**} - x^*|,$$

for some  $c$  between  $x^*$  and  $x^{**}$ . But this is impossible, since  $|g'(c)| < 1$ . Thus,  $x^*$  is the

unique fixed point of  $g$  in  $[a, b]$ . This proves part of (2) in (ii). We next prove (1) in

(ii). Let us fix any  $x_0 \in [a, b]$ . Then

$$x_{n+1} := g(x_n), n = 0, 1, 2, \dots$$

Thus, by the mean value theorem for  $g$  in the interval joining  $x^*$  and  $x_n$  for every  $n \geq 1$ ,

there exists some  $c_n$  between  $x_n$  and  $x^*$  such that

$$|x_{n+1} - x^*| = |g(x_n) - g(x^*)| = |g'(c_n)| |x_n - x^*|.$$

Since  $|g'(c_n)| \leq \alpha$ , we have

$$|x_{n+1} - x^*| \leq \alpha |x_n - x^*| \leq \alpha^2 |x_{n-1} - x^*| \leq \dots \leq \alpha^{n+1} |x_0 - x^*|.$$

This proves (iii). Finally,

$0 \leq \alpha < 1$  implies that  $\alpha^{n+1} \rightarrow 0$  as  $n \rightarrow \infty$ .

Hence,  $x_{n+1} \rightarrow x^*$ . This proves the theorem completely.

### 15.2.4 Example:

Consider the function

$$g(x) = \cos x, \text{ for } x \in [0, 1].$$

Then

$$g([0, 1]) \subset [0, 1].$$

Further, for  $x \in (0, 1)$ ,

$$|g'(x)| = |-\sin x| = \sin x \leq \alpha := \sin 1 < 1.$$

Hence, there is a unique  $x^* \in [0, 1]$  such that  $\cos x^* = x^*$ . In fact, if  $x_0 \in [0, 1]$ , then the Picard sequence given by

$$x_0, x_1 := \cos x_0, x_2 := \cos(\cos x_0), \dots,$$

satisfies the relation

$$|x_{n+1} - x^*| \leq (\sin 1)^{n+1} |x_0 - x^*|, n = 0, 1, 2, \dots,$$

and it converges to  $x^*$ .

As a consequence of Picard's convergence theorem, we have the following:

### 15.2.5 Corollary (Convergence of Newton sequence):

Let  $f: [a, b] \rightarrow \mathbb{R}$  be such that  $f, f'$  are continuous,  $f'(x) \neq 0$  for all  $x \in [a, b]$  and  $f''(x)$  exists for all  $x \in (a, b)$ . Further suppose that

(i) For all  $x \in [a, b]$ ,

$$x - \frac{f(x)}{f'(x)} \in [a, b].$$

(ii) There exists  $\alpha < 1$  with the property

$$\left| \frac{f(x)f''(x)}{[f'(x)]^2} \right| \leq \alpha \text{ for all } x \in (a, b).$$

Then, there is a unique  $x^* \in [a, b]$  such that  $f(x^*) = 0$ . Further, the Newton sequence for  $f$  with any starting point  $x_0 \in [a, b]$  converges to  $x^*$ .



### 15.2.5 Corollary (Convergence of Newton sequence):

Proof:

Let

$$g(x) := x - \frac{f(x)}{f'(x)} \text{ for all } x \in [a, b].$$

Then by (i),  $g: [a, b] \rightarrow \mathbb{R}$  is continuous

on  $[a, b]$  and is differentiable on  $(a, b)$ . Further, by (ii), for all  $x \in (a, b)$ ,

$[a, b]$  $(a, b)$  $x \in (a, b)$ 

$$|g'(x)| = \left| 1 - \frac{(f'(x))^2 f(x) f''(x)}{f'(x)^2} \right| = \left| \frac{f(x) f''(x)}{(f'(x))^2} \right| \leq \alpha < 1.$$

Hence by the Picard's Convergence Theorem, the function  $g$  has a unique fixed point  $x^*$  in  $[a, b]$ , which is then the unique root of  $f$  in  $[a, b]$ . Further, if  $x_0 \in [a, b]$  is any starting point, then the Newton sequence for  $f$  is, in fact, the Picard sequence for  $g$ . Hence,  $x_n \rightarrow x^*$ , as  $n \rightarrow \infty$ .

### 15.2.6 Example:

Let

$$f(x) = x^3 - 3 \text{ for } x \in [5/4, 3/2].$$

Then,

$$x - [f(x)/f'(x)] = (2x^3 + 3)/3x^2.$$

we claim that

$$\frac{5}{4} \leq \frac{2x^3 + 3}{3x^2} \leq \frac{3}{2}, \text{ for all } x \in [5/4, 3/2].$$

This will be so if

$$15x^2 \leq 8x^3 + 12 \text{ and } 4x^3 + 6 \leq 9x^2,$$

i.e.,

$$8x^3 - 15x^2 + 12 \geq 0 \text{ and } 4x^3 - 9x^2 + 6 \leq 0 \text{ for all } [5/4, 3/2].$$

which is easy to check. Also, for  $x \in [5/4, 3/2]$ , we have

$$\left| \frac{f(x) f''(x)}{(f'(x))^2} \right| = \left| \frac{(x^3 - 3) 6x}{9x^4} \right| = \frac{2}{3} \left( \frac{|x^3 - 3|}{x^3} \right) \leq \frac{2}{3} < 1.$$

Hence, if  $x_0 \in [5/4, 3/2]$ , then the Newton sequence given by

$$x_{n+1} = \frac{2}{3} x_n + \frac{1}{x_n^2}, n = 0, 1, 2, \dots,$$

converges to the positive cube root of 3.

### 15.2.7 (Linear convergence of Picard's method):

Let  $g = [a, b] \rightarrow [a, b]$  be continuous. Suppose that  $g'(x)$  exists for all  $x \in (a, b)$  and there is a constant  $\alpha$  such that

$$|g'(x)| \leq \alpha < 1 \text{ for all } x \in (a, b).$$

Then, it can be shown that  $g$  has a unique fixed point  $x^*$  in  $[a, b]$  and that for any initial

point  $x_0 \in [a, b]$ , the Picard sequence  $x_n$  for  $g$  satisfies

$$|x_{n+1} - x^*| \leq \alpha |x_n - x^*|,$$

i.e., the convergence of the Picard's sequence is linear.

### Practice Exercises

1. Let

$$f(x) = (x-1)^2.$$

Show that the sufficient condition

$$\left| \frac{f(x)f''(x)}{f'(x)^2} \right| < 1$$

for the convergence of Newton's sequence is not satisfied for all  $x$  in  $[0, 2]$ , however for any starting point  $x_0 \neq 1$ , the Newton's sequence  $\{x_n\}_{n \geq 0}$  actually converges to the root  $x = 1$ .

2. Show that each of the following functions maps the interval  $I$  mentioned against it into itself and has a unique

fixed point in that interval. Also, show that for any initial point  $x_0$  in the interval, Picard sequence converges to this unique fixed point:

(i)  $g(x) = \cos x, I = [0, 1]$ .

(ii)  $g(x) = \frac{1}{1+x^2}, I = [0, 1]$ .

(iii)  $g(x) = 1 + \frac{\sin x}{2}, I = [0, 2]$ .

(Let  $x_0 = 1.5$  and compare the iterates with those of the Newton method found in Problem 1(ii) of previous section.)

3. Let  $g(x) = \sqrt{x}, x \in [0, 1]$ . Show that for the initial guess  $x_0 > 0$ , the Picard's sequence converges to the fixed point  $x = 1$ . Why does it not contradict theorem 15.2.3.

4. Let  $g: [a, b] \rightarrow [a, b]$  be a one-one function. Prove the following:

(i)  $c \in [a, b]$  is a fixed point of  $g$  if and only if it is also a fixed point for  $g^{-1}$ .

(ii) If  $|g'(x)| > \beta > 1$  for all  $x \in [a, b]$  and  $c \in [a, b]$  is a fixed point of  $g$ , then it is unique, and

$$c = \lim_{n \rightarrow \infty} (y_n),$$

where  $y_0 = g(x_0)$  for some  $x_0 \in [a, b]$  and  $y_{n+1} := g(y_n) \forall n \geq 0$ .

5. Using exercise (4), find fixed points of the following:

(i)  $f(x) = 2x + 1, x \in [-2, -1]$ .

(ii)  $f(x) = x^2 + x - 1, x \in [-2, -1]$ .

Recap

In this section you have learnt the following

- Picard's Method for finding fixed points of functions.
- Sufficient conditions for the convergence of a Newton sequence.