Module 13 : Maxima, Minima and Saddle Points, Constrained maxima and minima

Lecture 39 : Absolute maxima / minima [Section 39.1]

Objectives

In this section you will learn the following :

- The notion of absolute maxima/minima for functions of several variables.
- Method of finding absolute maxima/minima.

39 .1 Absolute maxima/minima

39.1.1Definition:

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$. If there exists a point $\boldsymbol{a} \in D$ such that

$$f(x, y) \leq f(\boldsymbol{a}), \text{for all } (x, y) \in D,$$

then the number $f(a)$ is called the **absolute maxima** of f in D. Similarly, if there exists a point $\boldsymbol{b} \in D$ such that

$$f(x, y) \geq f(\boldsymbol{b}), \text{for all } (x, y) \in D,$$

then the number $f(\boldsymbol{b})$ is called the **absolute minima** of f in D.

39.1.2**Note:**

Recall that If D is closed and bounded, and f is continuous, then by theorem 30.2.4, both absolute maxima and absolute minimum exist.

39.1.3 Theorem:

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$.

(i) *Let f assume its absolute maximum at a point $a \in D$. Then, either at a is a boundary point of D, or is a critical point of f in D.*

(ii) *Let assume f its absolute minimum at a point $b \in D$. Then, either b is a boundary point of D, or is a critical point of f in D.*

39.1.3Theorem:

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$.

(i) *Let f assume its absolute maximum at a point $a \in D$. Then, either at a is a boundary point of D, or is a critical point of f in D.*

(ii) *Let assume f its absolute minimum at a point $b \in D$. Then, either b is a boundary point of D, or is a critical point of f in D.*

Proof

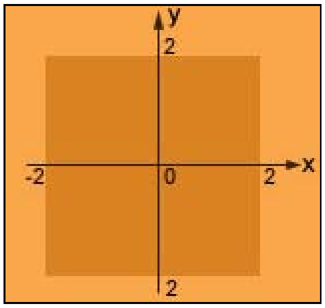
Suppose a is not a boundary point of D. Then, f must assume its maximum at some interior point of D. Thus, $a \in D$ is an interior point of D. If both $f_x(a)$ and $f_y(a)$ exist, then $\nabla f(a) = 0$ by theorem 37.1.3. In any case, a is a critical point of f. Similar arguments hold for b.

39.1.4Note:

To find the absolute maximum M and the absolute minimum m of a function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ on D, we compare the values of f at the critical points of f in D and the absolute maximum and the absolute minimum of the restriction of f to the boundary of D. The latter can often be found by reducing it to a one variable problem.

39.1.5Examples:

(i) Suppose



$$D = \{(x,y) \in \mathbb{R}^2 \mid |x| \le 2, |y| \le 2\}$$

and $f : D \to \mathbb{R}$. is given by

$$f(x,y) = 4xy - 2x^2 - y^4.$$

Since D is a closed bounded set and f is a continuous function, it has both, absolute maximum and absolute minimum in D For f both the partial derivatives exist everywhere and

$$f_x(x_0,y_0) = 4y_0 - 4x_0,\ f_y(x_0,y_0) = 4x_0 - 4y_0^3.$$

Further,

$$\nabla f(x_0, y_0) = 0 \text{ for } (x_0, y_0) = (0,0), (1,1) \text{ and} (-1,-1).$$

Also, $(x_0, y_0) \in D$ is a boundary point if

$$x_0 = 2 \text{ or } x_0 = -2 \text{ or } y_0 = 2 \text{ or } y_0 = -2.$$

Due to symmetry of the domain, $f(-x,-y) = f(x,y)$. Thus, we need only determine the absolute maximum and minimum of the functions

$$f(2,y) = 8y - 8 - y^4, \text{ for } -2 \le y \le 2$$

and

$$f(x,2) = 8x - 2x^2 - 16, \text{ for } -2 \le x \le 2.$$

It is easy to check that the function

$$f(2,y) \text{ has absolute maximum at } y = \sqrt[3]{2}$$

and

$$\text{absolute minimum at } y = -2.$$

Also,

$$f(x,2), \text{ absolute maximum at } x = 2 \text{ and absolute minimum at } x = -2.$$

Finally, we compare these values of f

$$f(0,0) = 0, f(1,1) = 1, f(2,\sqrt[3]{2}) = 6\sqrt[3]{2} - 8, f(2,-2) = -40, f(2,2) = -8,$$

here we have ignored the points $(-1,-1)$ and $(-2,2)$ due to symmetry. Thus,

$$\text{the absolute maximum of } f \text{ is } 1,$$

which is attained at the points $(1,1)$ as well as at $(-1,-1)$, and

$$\text{the absolute minimum of } f \text{ is } -40,$$

which is attained at $(2,-2)$ as well as at $(-2,2)$.

(ii) Let us find the triangle for which the product of the sines of the three angles is the largest. If we denote two

angles by x and y, then the required function to be maximized is

$$f(x,y) = \sin x \sin y \sin(x+y), \text{ where } \quad 0 \le x, y, x+y \le \pi.$$

It is obvious from the nature of the problem that the function will have absolute maximum. Note that

$$f(x,y) = 0 \text{ if } x, y \text{ or } x+y = 0, \text{ or } \pi.$$

Thus, f vanishes at each boundary point. At other points, i.e., for $0 < x, y, x+y < \pi$, the equations $f_x(x,y) = 0 = f_y(x,y)$ are given by

$$\cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y) = 0,$$
$$\sin x \cos y \sin(x+y) + \sin x \sin y \cos(x+y) = 0.$$

Since

$$\sin x \neq 0 \text{ and } \sin y \neq 0 \text{ for } 0 < x, y, < \pi,$$

above equations give us

$$\sin(2x+y) = 0 = \sin(2y+x).$$

As $0 < x+y < \pi$, we have $0 < 2x+y, 2y+x < 2\pi$, and hence, the critical points of f are given by

$$2x+y = 0 = 2y+x, \text{i.e.}, x = y = \frac{\pi}{3}.$$

Since $f(\pi/3, \pi/3) > 0$, follows that

f has an absolute maximum at $(\pi/3, \pi/3)$.

Thus, the desired triangle is equilateral.

Practice Exercises

(1) Find the absolute minimum and the absolute maximum of the function

$$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$$

on the closed triangular plate bounded by the lines

$$x = 0, y = 2 \text{ and } y = 2x.$$

Answer

(2) Find the absolute maximum and the absolute minimum of

$$f(x,y) = (x^2 - 4x)\cos y$$

over the region

$$R = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x \le 3, -\pi/4 \le y \le \pi/4\}.$$

Answer

Recap

In this section you have learnt the following

- The notion of absolute maxima/minima for functions of several variables.
- Method of finding absolute maxima/minima.

[Section 39.2]

Objectives

In this section you will learn the following :

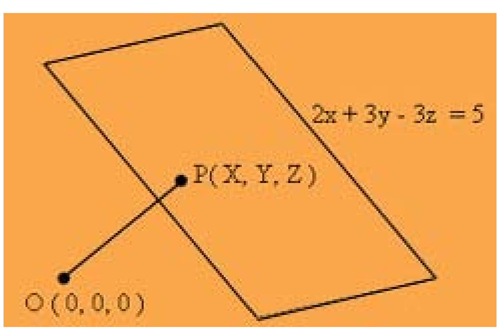
- The notion of constrained maxima/minima for functions of several variables.
- Lagrange's Method of finding constrained maxima/minima.

39.2 Constrained maxima/minima

In many practical problems, one has to find extreme values of a function whose domain is constrained to lie on a particular region in space. Let us look at some examples.

39.2.1 Examples:

(i) Find the point $P(x,y,z)$ closest to the origin on the plane $2x+y-3z=5$.



Here we want to find (x,y,z) in the plane $2x+y-3z=5$ such that the value

$$f(x,y,z) := \sqrt{x^2+y^2+z^2}$$

is the smallest.

(ii) A space satellite in the shape of an ellipsoid $4x^2+y^2+4z^2=16$ enters earth's atmosphere and its surface begins to heat. After an hour, the temperature at the point (x,y,z) on the surface of the probe is

$$T(x,y,z)=8x^2+xyz-16z+600.$$

One would like to find the hottest point on the surface of the probe. That leads to the problem of finding absolute maximum T for (x,y,z) on the ellipsoid.

Mathematically, the problem is to find the absolute maximum/minimum of a function

$f: D \to \mathbb{R}$, where (x,y) is constrained to satisfy $g(x,y)=0$.

In case we can solve $g(x,y)=0$ for one of the variables in terms of the other, the problem can be reduced to a problem of one variable. But, often this is difficult. A method to handle such problems, without having to solve the constraint equation and giving preference to one of the variables. This method is based on the following theorem:

39.2.2Theorem (Lagrange multiplier theorem):

Let $(x_0,y_0)\in \mathbb{R}^2$ *and*

$$f,g: B_r(x_0,y_0)\to \mathbb{R}$$

be such that the the following holds:

(i) *Both the partial derivatives of* f *and* g *exist in* $B_r(x_0,y_0)$ *and are continuous at* (x_0,y_0).

(ii) $g(x_0,y_0)=0$ *and* $\nabla g(x_0,y_0)\neq (0,0)$.

(iii) *The function* f *has a local extremum at* (x_0,y_0), *when restricted to* C, *the level curve*

$$C=\{(x,y)\in \mathbb{R}^2 \mid g(x,y)=0\}.$$

Then,

$$\nabla f(x_0,y_0)=\lambda \nabla g(x_0,y_0) \text{ for some } \lambda\in \mathbb{R}.$$

PROOF

39.2.2Theorem (Lagrange multiplier theorem):

Let $(x_0,y_0)\in \mathbb{R}^2$ *and*
$f,g: B_r(x_0,y_0)\to \mathbb{R}$
be such that the the following holds:

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(ii) $g(x_0,y_0)=0$ *and* $\nabla g(x_0,y_0)\neq (0,0)$.

(iii) *The function* f *has a local extremum at* (x_0,y_0), *when restricted to* C, *the level curve*

$C = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 0\}$.

Then,

$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some $\lambda \in \mathbb{R}$.

Proof

Since $\nabla g(x_0, y_0) \neq (0,0)$, we have

$g_x(x_0, y_0) \neq (0,0)$ or $g_y(x_0, y_0) \neq (0,0)$.

Suppose, $g_y(x_0, y_0) \neq (0,0)$. Then, using implicit function theorem, we can find a function $y : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$ such that

$g(x, y(x)) = 0$ for all $x \in [x_0 - \delta, x_0 + \delta]$.

Hence, by chain rule,

$g_x(x_0, y_0) + g_y(x_0, y_0)\; y'(x_0) = 0.$ ----------(32)

Also, since f has a local extremum at the point (x_0, y_0) when restricted to C, if we define

$\phi : [x_0 - \delta, x_0 + \delta] \to \mathbb{R},\ \ \phi(x) := f(x, y(x)),$

then ϕ has a local extremum at x_0. Therefore,

$\phi'(x_0) = f_x(x_0, y_0) + f_y(x_0, y_0) y'(x_0) = 0.$ ----------(33)

It follows from the equations (32) and (33),

$f_y(x_0, y_0)\, g_x(x_0, y_0) = f_x(x_0, y_0)\, g_y(x_0, y_0),$

and hence

$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0),$

where

$\lambda = f_y(x_0, y_0) / g_y(x_0, y_0).$

39.2.3Note(Lagrange's multiplier method):

In view of the above theorem, , to determine the absolute maximum/ minimum of a function $f(x,y)$ subject to the constraint $g(x,y) = 0$, we follow the following steps:

Step (i): Solve the equations

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{and} \quad g(x,y) = 0. \qquad \text{---------(34)}$$

Let

$$S_1 := \{(x,y) \mid \text{equation (34) is satisfied}\}.$$

Step (ii): Let

$$S_2 := \{(x,y) S \mid g(x,y) = 0\ f_x(x,y) \ \text{ or } f_y(x,y) \ \text{ does not exist, or } \nabla g(x_0, y_0) = (0,0).\}$$

Step (iii): Evaluate f at each of the points in $S_1 \cup S_2$. Find M, the largest of these values and m, the smallest of these values.

Step (iv): Ensure that M and m are the required absolute maximum and absolute minimum of f respectively for the given constraints.

39.2.4Examples:

(i) Let us find the maximum and the minimum of $f(x,y) = xy$ on the unit circle, that is, subject the the constraint

$g(x,y) = x^2 + y^2 - 1 = 0$. Since the conditions of the theorem 39.2.1 are satisfied, we consider the equations

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{and} \quad g(x,y) = 0,$$

i.e.,

$$y = 2\lambda x, \quad x = 2\lambda y \quad \text{and} \quad x^2 + y^2 - 1 = 0.$$

It is easy to check that the points (x, y) that satisfy these equations are

$$(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}).$$

Since the unit circle is closed and bounded and f is continuous on it, f attains its absolute maximum/minimum on it, and are the largest/smallest of the values

$$f(1/\sqrt{2}, 1/\sqrt{2}) = f(-1/\sqrt{2}, -1/\sqrt{2}) = \frac{1}{2}$$

and

$$f(1/\sqrt{2}, -1/\sqrt{2}) = f(-1/\sqrt{2}, 1/\sqrt{2}) = \frac{-1}{2}.$$

Thus, f has absolute maximum $1/2$, absolute minimum $-1/2$.

(ii) Let us find the minimum of the function

$$f(x, y) = x^2 + y^2 \ \text{ subject to the constraint } \ g(x, y) = (x-1)^3 - y^2 = 0.$$

The equations

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = 0,$$

we have

$$2x = 3\lambda(x-1)^2, 2y = -2\lambda y, (x-1)^3 - y^2 = 0.$$

have no solutions for $y \neq 0$. For $y = 0$, $g(x, y) = 0$, gives have $x = 1$. But then

$$\nabla g(1, 0) = 0 \ \text{ and } \ \nabla f(1, 0) = (2, 0).$$

Thus, the equation

$$\nabla f(1, 0) = \lambda \nabla g(1, 0) \text{ is not satisfied for any } \lambda \in \mathbb{R}.$$

Hence, the condition $\nabla g(x_0, y_0) \neq (0, 0)$ can not be dropped in theorem 39.2.1. However, $f(x, y)$ is the distance between origin and a point on the surface $(x-1)^3 = y^2$. Geometrically it is obvious that the minimum of f is 1 and this is attained at $(1, 0)$.

39.2.5 Constrained extremum for three Variables:

There is a result analogous to the two variable, to solve the problem of constrained maxima / minima for functions of three variables. We solve the equations

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), g(x, y, z) = 0,$$

in the unknowns x, y, z and λ at which $\nabla g(x, y, z) \neq (0, 0, 0)$ and compare the values of f at these points to locate the constrained maxima/minima of f.

39.2.6 Examples:

Let us find the points on the surface $z^2 = xy+4$ closest to the origin. This is same as minimizing the function

$$f(x,y,z) = x^2+y^2+z^2 \text{ with constraint } g(x,y,z) = xy+4-z^2.$$

Note that although the set

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid g(x,y,z) = 0\}$$

is not bounded, the set

$$S_1 = \{(x,y,z) \in E : x^2+y^2+z^2 \leq r\}$$

is closed and bounded, where $r = x_0^2+y_0^2+z_0^2$ for some $(x_0, y_0, z_0) \in E$. Further, the minimum of f on S_1 equals the minimum of f on S, which exists as f is continuous. To find this, we solve the equations

$$\nabla f = \lambda \nabla g, \text{i.e.}, 2x = \lambda y, 2y = \lambda x, 2z = -2\lambda z.$$

Since

$$\lambda = 0 \text{ implies } (x,y,z) = (0,0,0) \text{ and } g(0,0,0) \neq 0,$$

we may assume that $\lambda \neq 0$. Then, it is easy to see that the only common solutions of

$$\nabla f = \lambda \nabla g \text{ and } g = 0$$

are the points

$$(0,0,2), (0,0,-2), (2,-2,0) \text{ and } (-2,2,0).$$

Further,

$$f(0,0,\pm 2) = 4 \text{ and } f(\pm 2, \mp 2, 0) = 8.$$

Thus, $(0,0,\pm 2)$ are the points on the surface $z^2 = xy+4$ closest to the origin.

39.2.7 Remark:

The method of Lagrange's multipliers extents when we have more than one constraint. Suppose we want to find extremum a function $f(x,y,z)$ with constraints

$$g(x,y,z)=0 \text{ and } h(x,y,z)=0,$$

where g and h have continuous partial derivatives in a neighborhood of (x_0, y_0, z_0). These can be found by comparing the values of f at points which satisfy the simultaneous equations

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z),$$

$$g(x,y,z) = 0 = h(x,y,z),$$

and for which

$$\nabla g(x,y,z) \neq (0,0,0),\ \nabla h(x,y,z) \neq (0,0,0)$$

and

$$\nabla g(x,y,z) \text{ is not parallel to } \nabla h(x,y,z).$$

39.2.8 Example:

Let us analyze the problem of finding find the points on the intersection of the planes

$x+y+z=1$ and $3x+2y+z=6$

that are closest to the origin. This is same as finding the minimum value of

$f(x,y,z) = x^2+y^2+z^2,$

with constraints,

$g(x,y,z) = x+y+z-1 = 0$ and $h(x,y,z) = 3x+2y+z-6 = 0.$

The equations to be solved are

$x = \dfrac{\lambda+3\mu}{2},\ y = \dfrac{\lambda+2\mu}{2},\ z = \dfrac{\lambda+\mu}{2},$

$x+y+z-1=0,\ 3x+2y+z-6=0$

Substituting the values of $x\ y, z$ from the first three in the last two equations gives

$3\lambda+6\mu=2 \quad \text{and} \quad 3\lambda+7\mu=6.$

This gives

$\mu_0 = 4$ and $\lambda_0 = -22/3,$ and hence $(x_0, y_0, z_0) = (7/3, 1/3, -5/3).$

That this is the required point, can be justified as in the previous example.

Practice Exercises

(1) The temperature at a point (x,y,z) in 3-space is given by $T(x,y,z) = 400xyz^2$. Find the highest temperature on the unit sphere $x^2+y^2+z^2=1.$

[Answer]

(2) Find the point nearest to the origin on the surface defined by the equation $z = xy+1$

[Answer]

(3) Using the Lagrange's method of multiplier, show that the minimum value of f is 0 and is attained at

$(0,0,1)$

and the maximum value of $f(x,y,z) = x^2y^2z^2$ subject to the constraint that (x,y,z) lies on the unit sphere is $1/27$ and it is attained at $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Using these, deduce the A.M.-G.M. inequality: for three nonnegative real numbers $x,y,z \in \mathbb{R}$,

$$\left(x^2y^2z^2\right)^{1/3} \le \frac{(x^2+y^2+z^2)}{3}.$$

(4) A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x,y,z) on the surface of the probe is given by

$$T(x,y,z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the surface of the probe.

Answer

(5) Maximize the function

$$f(x,y,z) = xyz$$

subject to the constraints

$$x+y+z = 40 \text{ and } x+y = z.$$

Answer

(6) Minimize the quantity

$$f(x,y,z) = x^2+y^2+z^2$$

subject to the constraints

$$x+2y+3z = 6 \ \text{ and } \ x+3y+9z = 9.$$

Answer

Recap

In this section you have learnt the following

- The notion of constrained maxima/minima for functions of several variables.
- Lagrange's Method of finding constrained maxima/minima.