Module 6 : Definition of Integral

Lecture 17 : Fundamental theorem of calculus [Section 17.1]

Objectives

In this section you will learn the following :

- Fundamental theorem of calculus, which relates integration with differentiation.

17.1 Fundamental Theorem of Calculus

In this lecture, we describe an important theorem which connects integration with differentiation. We first make a simple observation:

17.1.1Proposition:

Let $f:[a,b] \to \mathbb{R}$ be an integral function. If $A \in \mathbb{R}$ is such that

$$L(P,f) \leq A \leq U(P,f)$$

for every partition P of $[a,b]$, then f is integrable and

$$A = \int_a^b f(x)dx \cdot$$

PROOF

17.1.1Proposition:

Let $f:[a,b] \to \mathbb{R}$ be an integral function. If $A \in \mathbb{R}$ is such that
$L(P,f) \leq A \leq U(P,f)$
for every partition P of $[a,b]$, then f is integrable and
$A = \int_a^b f(x)dx \cdot$

Proof:

Since f is integrable, there exists a sequence $\{P_n\}_{n\geq 1}$ of refinement partitions of $[a,b]$ such that

$$\lim_{n\to\infty}\left[U(P_n,f)-L(P_n,f)\right]=0.$$

By the given hypothesis,

$$L(P_n,f) \leq A \leq U(P_n,f), \text{for every } n\geq 1.$$

Hence,

$$\lim_{n\to\infty}U(P_n,f)=A=\lim_{n\to\infty}L(P_n,f).$$

Thus, by definition

$$A=\int_a^b f(x)dx.$$

17.1.2Fundamental Theorem of Calculus - I (FTC-I):

Let $f,F:[a,b]\to\mathbb{R}$ be functions with the following properties:

(i) f is integrable on $[a,b]$.

(ii) F is continuous on $[a,b]$.

(iii) F is differentiable on (a,b) with $F'(x)=f(x)$ for all $x\in(a,b)$.

Then,

$$\int_a^b f(t)dt = F(b)-F(a).$$

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(iii) F is differentiable on (a,b) with $F'(x)=f(x)$ for all $x\in(a,b)$.
Then, $\int_a^b f(t)dt = F(b)-F(a)$.

Proof:

Let $P=\{a=x_0, x_1,\ldots, x_n=b\}$ be any partition of $[a,b]$. Then

$$F(b)-F(a)=F(x_n)-F(x_0)=\sum_{k=1}^{n}\left[F(x_k)-F(x_{k-1})\right].$$

By the mean value theorem for F on $[x_{k-1},x_k]$, there exists $c_k\in(x_{k-1},x_k)$ such that

$$F(x_k)-F(x_{k-1}) = F'(c_k)(x_k-x_{k-1}).$$

Since, $F'(x)=f(x)$, for all $x\in(a,b)$, we have

$$F(x_k)-F(x_{k-1}) = f(c_k)(x_k-x_{k-1}).$$

From equations (2) and (3), we get

$$F(b)-F(a) = \sum_{k=1}^{n} f(c_k)(x_k-x_{k-1}).$$

Thus, for every partition P of $[a,b]$.

$$L(P,f) \leq F(b)-F(a) \leq U(P,f).$$

Hence, $F(b) - F(a) = \int_a^b f(x)\, dx$.

Back

17.1.3 Examples:

(i) Since, for every $n \geq 1$,

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1},$$

for every interval $[a, b]$,

$$\int_a^b nx^{n-1}\, dx = b^n - a^n,$$

i.e.,

$$\int_a^b x^{n-1}\, dx = \frac{b^n - a^n}{n}.$$

(ii) Since

$$\frac{d}{dx}(\sin x) = \cos x,$$

for $a, b \in \mathbb{R}$ with $a < b$,

$$\int_a^b \cos dx = \sin(b) - \sin(a).$$

(iii) For the function $f(x) = \exp(x), x \in \mathbb{R}$

$$\frac{d}{dx}\left(\exp(x)\right) = \exp(x).$$

Hence, for $a, b \in \mathbb{R}$ with $a < b$,

$$\int_a^b e^x dx = e^b - e^a.$$

17.1.4 Definition :

Let $f, F : [a, b] \to \mathbb{R}$ be functions such that F is differentiable and

$$F'(x) = f(x) \text{ for all } x \in (a, b).$$

Then, F is called an antiderivative of f on $[a, b]$.

17.1.5 Examples :

(i) Let $F(x) = x^2, x \in \mathbb{R}$. Since $f'(x) = 2x$, which is a continuous function, an antiderivative of $f(x) = 2x$ is

. Infact, for any $n \neq -1$, since

$F(x) = x^2$

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

we deduce that the function $f(x) = x^n$ has antiderivative

$$F(x) = \frac{x^{n+1}}{n+1},\ x \in \mathbb{R},\ n \neq -1$$

(ii) For $F(x) = \cos x,\ x \in \mathbb{R},\ F'(x) = -\sin x$, implies that $f(x) = \sin x$ has an antiderivative, namely

$$f(x) = \cos x.$$

17.1.6 Remark:

If $F(x)$ is an antiderivative of $f(x)$, then clearly

$G(x) := F(x) + c$, c a fixed constant, is also an antiderivative of f. Thus antiderivative of a function $f(x)$ is not unique. Any two antiderivatives differ by a constant.

17.1.7 Definition:

Let $f : [a, b] \to \mathbb{R}$. The set of all the antiderivatives of f is denoted by

$$\int f(x)\,dx,$$

and is called the indefinite integral or just integral of f. Since any of two elements of this set differ only by a constant, we also write

$$\int f(x)\,dx = F(x) + c,$$

where F is some antiderivative of f.

17.1.8 Examples:

In view of examples 17.2.5, we can write

$$\int x^n = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$$

and

$$\int \cos x\,dx = \sin x + c.$$

In view of theorem 17.1.1, since the knowledge about the antiderivative of a function is useful in calculating the integral of the function, it is natural to ask the question:

Given a function $f : [a, b] \to \mathbb{R}$, can we always find an antiderivative of f?

The answer to this is given by the following:

17.1.9 Fundamental Theorem of Calculus - II (FTC - II):

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then

$$F(x) = \int_a^x f(t)\,dt,\ x \in [a, b],$$

is differentiable with , i.e., has an anti-derivative, namely F.

$$f(x) = F'(x) \qquad f$$

17.1.10 Remark:

Though the above theorem tells us that every continuous function has an anti-derivative, it may not be always possible to find it explicitly. Some methods that help us to do this, are discussed in the next section.

PRACTICE EXERCISES

1. Let f have an antiderivative F and g have an antiderivative G. Find an antiderivative of the following in terms

of F and G:

(i) $\alpha f, \alpha \in \mathbb{R}$.

(ii) $f + g$.

2. Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\beta \in \mathbb{R}$ is given, then there is a unique antiderivative F of f

such that $F(\alpha) = \beta$ for a given $\alpha \in [a, b]$.

3. For the following f, find unique antiderivative F with the specified values at a specified point:

(i) $f(x) = 3x^2, \; F(2) = 10$.

(ii) $f(x) = x^2 + x^3 + x^4, \; F(1) = 0$.

(iii) $f(x) = x^{\frac{1}{3}}, \; F(1) = 0$.

4. Find the average values of the following functions over the indicated intervals:

(i) $f(x) 3x^2 - 2x, [0, 2]$.

(ii) $f(x) = 4 - x^2, [-1, 1]$.

(iii) $f(x) = \cos x, [0, 3\pi / 2]$.

Recap

In this section you have learnt the following

- Fundamental theorem of calculus,which relates integration with differentiation.

(Section 17.2)

Objectives

In this section you will learn the following :

- Integration by parts formula

- Integration by substitution

- Leibnitz's formula for differentiating integral with variable limits

17.2 Applications of fundamental theorem of calculus

17.2.1Theorem (Integration by Parts):

Let $F, G : [a,b] \to \mathbb{R}$ be differentiable functions such that both F', G' are Riemann integrable on $[a,b]$. Then

$$\int_a^b F(x)G'(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x)\,dx$$.

17.2.1Theorem (Integration by Parts):

Let $F, G : [a,b] \to \mathbb{R}$ be differentiable functions such that both F', G' are Riemann integrable on $[a,b]$. Then

$$\int_a^b F(x)G'(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b F'(x)G(x)\,dx \cdot$$

Proof:

Note that, by the product rate for differentiation

$$(FG)' = F'G + G'F \cdot$$

Since both $F'G$ and FG' are integrable, by FTC-I, we have

$$\begin{aligned} F(b)G(b) - F(a)G(a) &= \int_a^b (FG)'(x)\,dx \\ &= \int_a^b \Big(F'(x)G(x) + F(x)G'(x)\Big)dx \\ &= \int_a^b F'(x)G(x)dx + \int_a^b F(x)G'(x)\,dx. \end{aligned}$$

17.2.2Theorem (Integration by direct Substitution):

Let $f:[a,b]\to\mathbb{R}$ $g:[c,d]\to\mathbb{R}$ be functions such that

(i) f is continuous on $[a,b]$.

(ii) g is differentiable on $[c,d]$ with $g(c)=a$ and $g(d)=b$.

(iii) g' Riemann integrable on $[c,d]$.

Then

$$\int_c^d f(g(t))g'(t)\,dt=\int_a^b f(x)\,dx\cdot$$

17.2.2 Theorem (Integration by direct Substitution):

Proof:

Since f is continuous, by FTC-I, f has an antiderivative, say F. Then
$F'(x)=f(x)$ for all x.
Also by the chain rule,
$(Fog)'(x)=F'(g(x))g'(x)=f(g)(x)g'(x)$.
Thus, by FTC-I

$$(Fog)(d)-(Fog)(c)=\int_c^d (Fog)'(x)dx$$

$$=\int_c^d f(g(x))g'(x))dx.\qquad\text{------------------------------------(4)}$$

Also, again by FTC-I,

$$(Fog)(d)-(Fog)(c)=F(g(d))-F(g(c))$$

$$=F(b)-F(a)$$

$$=\int_a^b f(x)\,dx.\qquad\text{----------------------------------(5)}$$

Proof is complete from (4) and (5).

Theorems 17.2.1 and 17.2.2 give us techniques to evaluate definite integrals.

17.2.3Examples:

(i) To evaluate

$$\int xe^{\alpha x}dx,\alpha\neq 0\,,$$

we write

$$F(x)=x,G(x)=e^{\alpha x}\,.$$

Then

$$F'(x)=1 \text{ and } G'(x)=\alpha e^{\alpha x}.$$

Thus, by theorem 17.2.1,

$$\int_a^b x\,e^{\alpha x}\,dx = \frac{1}{\alpha}\int_a^b x\left(\alpha e^{\alpha x}\right)dx$$
$$= \frac{1}{\alpha}\left[\left\{b\left(\alpha e^{\alpha b}\right) - a\left(\alpha e^{\alpha x}\right)\right\} - \int_a^b e^{\alpha x}dx\right]$$
$$= \frac{1}{\alpha}\left(b\alpha e^{\alpha b} - a\alpha e^{\alpha b}\right) - \left[\frac{e^{\alpha x}}{\alpha}\int_a^b\right]$$
$$= (be^{\alpha b} - ae^{\alpha a}) - \frac{e^{\alpha b} - e^{\alpha a}}{\alpha}.$$

(ii) To compute $I = \int_0^1 2x(x^2+1)^{\frac{1}{2}}dx$, let us write

$$f(u) := u^{\frac{1}{2}}, u := g(x) = x^2 + 1.$$

Then by theorem 17.2.2,

$$I = \int_{c=0}^{d=1} f(g(x))\, g'(x)\, dx = \int_a^b u^{\frac{1}{2}}\, du,$$

where $a = g(c) = 1$ and $b = g(d) = 2$

Hence

$$I = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=1}^{u=2} = \frac{2}{3}\left[2^{\frac{3}{2}} - 1\right].$$

17.2.4Theorem (Leibnitz Rule):

Let $f:[a,b] \to \mathbb{R}$ be continuous and $u, v:[\alpha, \beta] \to [a,b]$ be differentiable. Then $\forall \gamma \in [\alpha, \beta]$

$$\frac{d}{dx}\left(\int_{u(x)}^{v(x)} f(t)\,dt\right)\Bigg|_{x=\gamma} = f\left(v(\gamma)\right)v'(\gamma) - f\left(u(\gamma)\right)u'(\gamma).$$

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Proof:

Since for $x \in [a,b]$,
$F(x) := \int_a^x f(t)\,dt$ is differentiable with $F'(x) = f(x)$,
by chain rule, for all $\gamma \in [\alpha, \beta]$, we have

$$\frac{d}{dx}(Fou)\Bigg|_{x=\gamma} = F'\left(u(\gamma)\right)u'(\gamma) = f\left(u(\gamma)\right)u'(\gamma), \qquad (6)$$

and

$$\frac{d}{dx}(Fov)\Big|_{x=\gamma} = F'\big(v(\gamma)\big)v'(\gamma) = f\big(v(\gamma)\big)v'(\gamma). \qquad (7)$$

Also by FTC-I,

$$F(v(x)) - F(u(x)) = \int_{u(x)}^{v(x)} F'(t)\,dt = \int_{u(x)}^{v(x)} f(t)\,dt. \qquad (8)$$

Hence, by (6), (7) and (8), we have

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt\Big|_{x=\gamma} = f\big(v(\gamma)\big)v'(\gamma) - f\big(u(\gamma)\big)u'(\gamma).$$

17.2.5Example:

Let

$$F(x) = \int_{\frac{1}{x}}^{x} \frac{1}{t}\,dt, x > 0.$$

Then, by theorem 17.2.4, $F'(x)$ exists and for x > 0,

$$F'(x) = \frac{1}{x} - \left(\frac{1}{\frac{1}{x}}\right)\left(-\frac{1}{x^2}\right) = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}.$$

PRACTICE EXERCISES

1. Using Leibnitz's Rule, compute the following:

(a) $\frac{d^2y}{dx^2}$, if $y = \int_0^x \frac{dt}{\sqrt{1+t^2}}$.

(b) $\frac{dF}{dx}$, if for $x \in \mathbb{R}$

(i) $F(x) = \int_1^{2x} \cos(t^2)\,dt$.

(ii) $F(x) = \int_0^{x^2} \cos(t)\,dt$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R}, \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)\,dt.$$

Show that

$$g(0) = 0 = g'(0)$$

and g satisfies the following:

$$g''(x) + \lambda^2 g(x) = f(x) \text{ for all } x \in \mathbb{R}.$$

3. Let p be a real number and let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that

$$f(x+p)=f(x) \text{ for all } x\in\mathbb{R}.$$

Let

$$\phi(x):=\int_x^{x+p} f(t)dt \quad x\in\mathbb{R}\cdot$$

Show that ϕ is a constant function, independent of p.

4. Let $f:[0,\infty)\to(0,\infty)$ a continuous function. For any $b>0$, let $G(b)$ denote the area bounded by the x-axis, the lines $x=0$, $x=b$ and the curve $y=f(x)$. If, is given by

$$G(b):=\sqrt{b^2+1}-1 \text{ for each } b>0,$$

determine the function f.

5. Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Show that for every $x\in[a,b]$,

$$\int_a^x\left[\int_a^u f(t)\,dt\right]du=\int_a^x (x-u)\,f(u)du.$$

6. Integration by inverse substitution:

Let $f:[a,b]\mapsto\mathbb{R}$ and $\phi:[c,d]\mapsto[a,b]$ be such that the following are satisfied:

(i) f is continuous.

(ii) ϕ is onto.

(iii) ϕ' exists, is continuous on $[c,d]$ and $\phi'(y)\neq 0$ for all $y\in[c,d]$.

Show that ϕ is one-one, and hence ϕ^{-1} exists. Using direct substitution for ϕ, show that

$$\int_a^b f(x)\,dx=\int_\alpha^\beta f(\phi(y))\,\phi'(y)dy,$$

where $\alpha=\phi^{-1}(a)$ and $\beta=\phi^{-1}(b)$.

7. Using direct/indirect substitution, compute the following:

(i) $\displaystyle\int\frac{2x}{\sqrt{9+x^2}}\,dx,$

(use $u=9+x^2$).

(ii) $\displaystyle\int\frac{dx}{\sqrt{9+x^2}}$,

(use inverse substitution).

$$x = 3\tan\phi$$

Recap

In this section you have learnt the following

- Integration by parts formula

- Integration by substitution

- Leibnitz's formula for differentiating integral with variable limits