Module 11 : Partial derivatives, Chain rules, Implicit differentiation, Gradient, Directional derivatives

Lecture 32 : Chain rules [Section 32.1]

Objectives

In this section you will learn the following :

- Chain rules which help to compute partial derivatives of composite functions.

32.1 Chain rules

Recall that for $w = f(x)$, a differentiable function of one-variable, if $x = g(t)$ is also a differentiable function of t, then the composite function $w(t) = (f \circ g)(t)$ is also a differentiable function of t and

$$\frac{dw}{dt} = \frac{dt}{dx} \cdot \frac{dx}{dt}.$$

Similar results hold for functions of several variables.

We state next another such rule, the proof of which is similar to the Chain rule-I, and is left as an exercise.

32.1.1Theorem (Chain rule-I:)

Let $(x_0, y_0) \in \mathbb{R}^2$ *and* $f : B_r(x_0, y_0) \to \mathbb{R}$ *be differentiable at* (x_0, y_0). *Let* $t_0 \in \mathbb{R}$ *and*

$$x, y : (t_0 - \delta, t_0 + \delta) \longrightarrow \mathbb{R}$$

be functions such that

$$(x(t_0), y(t_0)) = (x_0, y_0) \text{ and } (x(t), y(t)) \in B_r(x_0, y_0) \text{ for all } t \in (t_0 - \delta, t_0 + \delta).$$

If x, y are both differentiable at t_0, *then the composite function*

$w : (t_0 - \delta, t_0 + \delta) \to \mathbb{R}$ *given by*

is differentiable at

$w(t) := f(x(t), y(t)),\ t \in (t_0 - \delta, t_0 + \delta)$ $\qquad t_0$

and

$$w'(t_0) = f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0).$$

Functionally, this is also written as

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

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be functions such that

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$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

Proof

Let

$$h(t) := x(t) - x_0 \text{ and } k(t) := y(t) - y_0 \text{ for } t \in (t_0 - \delta, t_0 + \delta).$$

Then, by differentiability of f, we have for $t \neq t_0$,

$$\begin{aligned}
& w(t) - w(t_0) \\
&= f(x_0 + h(t), y_0 + k(t)) - f(x_0, y_0) \\
&= f_x(x_0, y_0)[x(t) - x(t_0)] + f_y(x_0, y_0)[y(t) - y(t_0)] \\
&+ [x(t) - x(t_0)]\varepsilon_1(h(t), k(t)) + [y(t) - y(t_0)]\varepsilon_2(h(t), k(t)).
\end{aligned}$$

Thus for $t \neq t_0$

$$\frac{w(t) - w(t_0)}{t - t_0} = f_x(x_0, y_0)\left[\frac{x(t) - x(t_0)}{t - t_0}\right] + f_y(x_0, y_0)\left[\frac{y(t) - y(t_0)}{t - t_0}\right]$$

$$+\left(\frac{x(t) - x(t_0)}{t - t_0}\right)\varepsilon_1(h(t), k(t)) + \left(\frac{y(t) - y(t_0)}{t - t_0}\right)\varepsilon_2(h(t), k(t)) \qquad \text{--------(28)}$$

Since $x(t), y(t)$ are continuous, $(h(t), k(t)) \longrightarrow (0, 0)$ as $t \longrightarrow t_0$. Hence, as $t \longrightarrow t_0$, it follows from (28) that $w(t)$ is differentiable and

$$\frac{dw}{dt}(t_0) = f_x(x_0, y_0)\frac{dx}{dt}(t_0) + f_y(x_0, y_0)\frac{dy}{dt}(t_0).$$

32.1.2Theorem (Chain rule-II):

Let $(x_0, y_0) \in \mathbb{R}^2$ *and* $f : B_r(x_0, y_0) \to \mathbb{R}$ *be differentiable at* (x_0, y_0). *Let* $(s_0, t_0) \in \mathbb{R}^2$ *and*

$x, y:\ \mathrm{B}_\delta(s_0, t_0) \to \mathbb{R}^2$

be functions such that

$(x(s_0,t_0), y(s_0,t_0)) = (x_0, y_0)$ and $(x(s,t), y(s,t)) \in B_r(x_0, y_0)$ for all $(s,t) \in B_\delta(s_0, t_0)$.

If x_s, x_t, y_s, y_t *exist at* (s_0, t_0), *then the composite function*

$g : B_\delta(s_0, t_0) \to \mathbb{R},\ g(s,t) = f(x(s,t), y(s,t))$

has partial derivatives $g_s(s_0, t_0)$ *and* $g_t(s_0, t_0)$, *given by*

$$g_s(s_0,t_0) = f_x(x_0,y_0)x_s(s_0,t_0) + f_y(x_0,y_0)y_s(s_0,t_0),$$
$$g_t(s_0,t_0) = f_x(x_0,y_0)x_t(s_0,t_0) + f_y(x_0,y_0)y_t(s_0,t_0).$$

Symbolically,

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}, \quad \frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}.$$

32.1.3Examples:

(i) Let

$$f(x,y) = x^2 + y^2, \text{for } (x,y) \in \mathbb{R}^2$$

and

$$x(t) = e^t, y(t) = t, t \in \mathbb{R}.$$

Let

$$w(t) := f(x(t), y(t)), t \in \mathbb{R}.$$

Then, by chain rule, $w(t)$ is differentiable for all $t \in \mathbb{R}$, and we have

$$w'(t) = 2(e^t)e^t + 2t = 2(e^{2t} + t).$$

(ii) Let

$$f(x,y) = x^2 + y^2, \text{for } (x,y) \in \mathbb{R}^2$$

and

$$x(s,t) = s^2 - t^2, y(s,t) = 2st, \text{ for all } (s,t) \in \mathbb{R}^2.$$

For

$$g(s,t) := f(x(s,t), y(s,t)), (s,t) \in \mathbb{R}^2,$$

we have by chain rule,

$$\frac{\partial g}{\partial s} = 2(s^2 - t^2)\cdot(2s) + 2(2st)(2t) = 4s(s^2 + t^2)$$

,

$$\frac{\partial g}{\partial t} = 2(s^2 - t^2)(-2t) + 2(2st)(2s) = 4t(s^2 + t^2).$$

(iii) Let

$$f(x, y) = \frac{xy}{x^2 + y^2} \text{ for } (x, y) \neq (0,0) \text{ and } f(0,0) = 0.$$

Then, f_x, f_y exist everywhere and

$$f_x(0,0) = 0 = f_y(0,0).$$

Hence, if

$$x(t) := t \text{ and } y(t) := t^2, \text{for } t \in \mathbb{R},$$

then

$$f_x(0,0)x'(0) + f_y(0,0)y'(0) = 0.$$

However, since

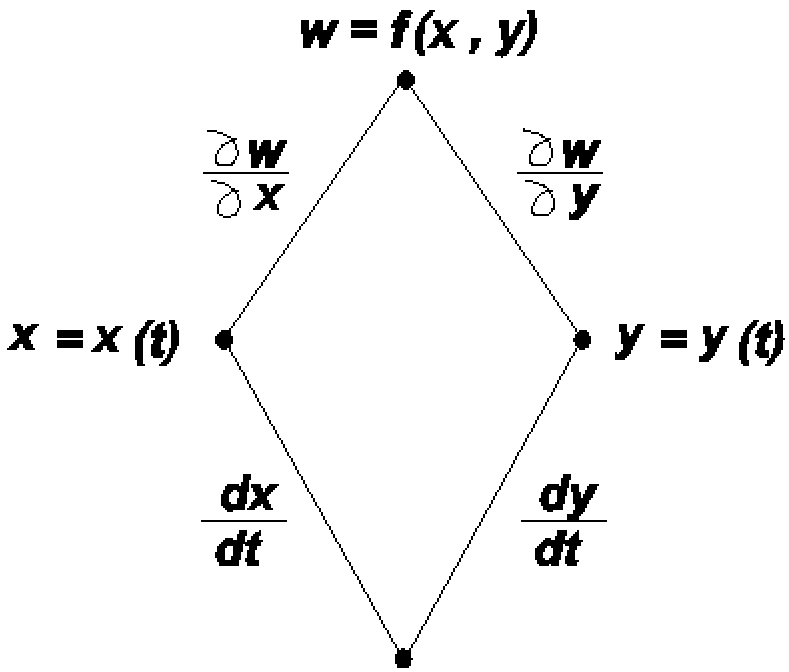
$$w(t) := f(x(t), y(t)) = t/(1+t^2), \ t \in \mathbb{R},$$

we have $w'(0) = 1.$ This does not contradict Chain Rule as f is not differentiable (in fact not even continuous) at $(0,0).$

32.1.4Remark:

Chain rule extends to functions of three or more variables as above for functions of two variables.

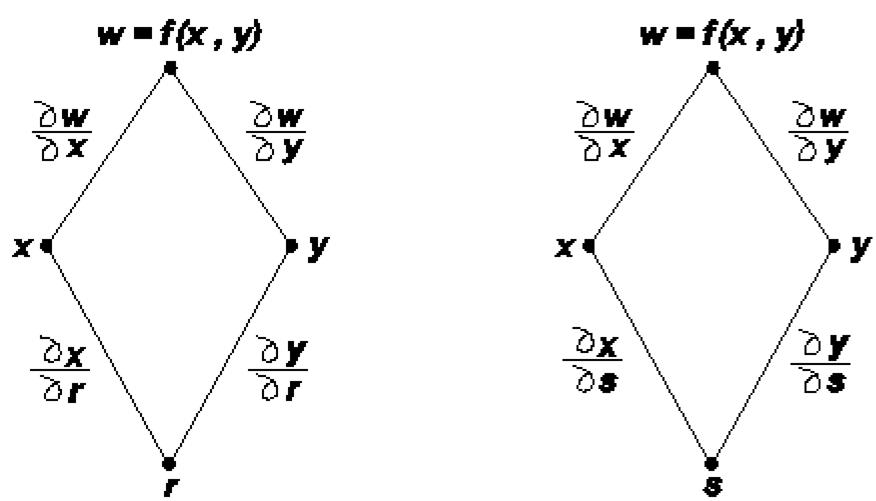
$$w = f(x, y), \; x = x(t), \; y = y(t)$$



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Figure 1. Chain rule-I

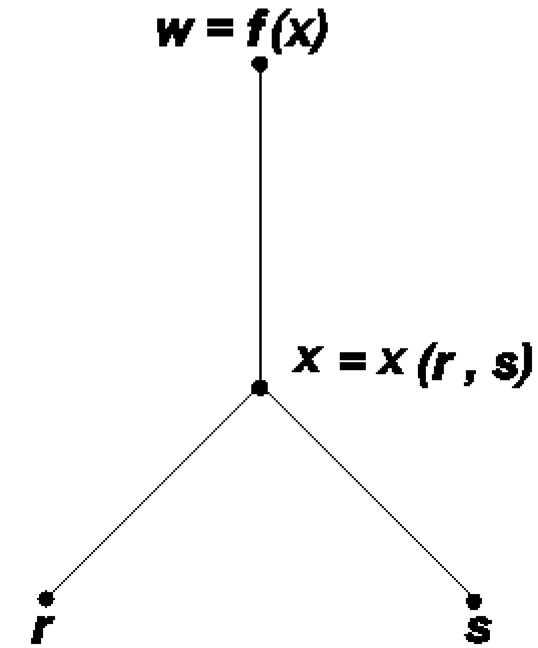
$$w = f(x, y), \; x = x(r, s), \; y = y(r, s)$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \qquad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

Figure 2. Chain rule-II

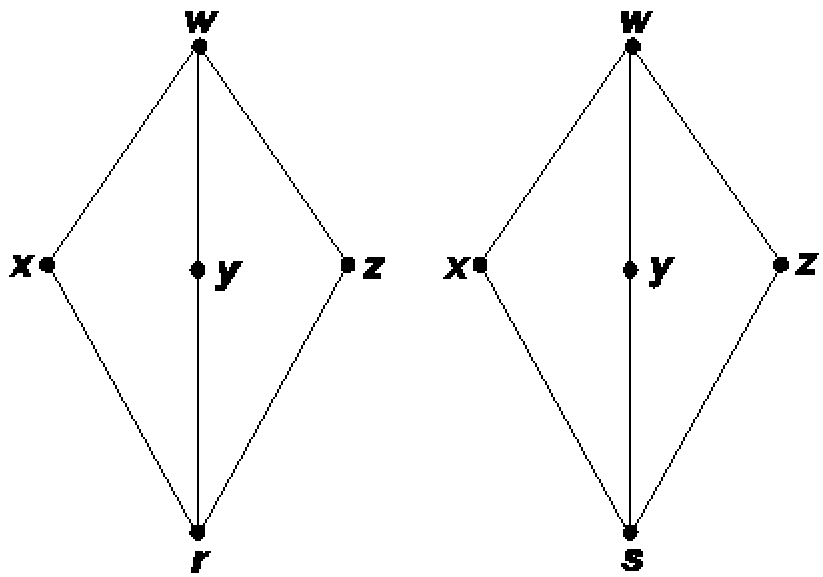
$$w = f(x)\ ,\ x = x\,(r\,,\,s)$$



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\,\frac{\partial x}{\partial r}\ ,\ \frac{\partial w}{\partial s} = \frac{dw}{dx}\,\frac{\partial x}{\partial s}$$

Figure 3. Chain rule-III

$$w = f(x, y, z),\ x = x(r, s),\ y = y(r, s),\ z = z(r, s)$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

Figure 4. Chain rule-IV

QUIZ

Practice Exercises

1. Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where

(i) $z = u \log v,\ u = x^2, v = \frac{1}{1+y}.$

(ii) $z = g(x^2 + y^2, e^{x-y}).$

[Answers]

2. If $f(u, v, w)$ is differentiable and

$u = x - y, v = y - z, w = z - x,$

show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

3. Let $f(x,y)$ be such that for some $n \geq 1$ and all $t > 0$,

$$f(tx,ty) = t^n f(x,y).$$

Show that if f is differentiable, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y).$$

4. Let a, b be constants and $w = f(ax+by)$, where f is differentiable. Show that

$$a\frac{\partial w}{\partial y} = b\frac{\partial w}{\partial x}$$

5. Let

$$w = f\left(\frac{y-x}{xy}, \frac{z-y}{yz}\right),$$

where f a differentiable function. Show that

$$x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial z} = 0.$$

Recap

In this section you have learnt the following

- Chain rules which help to compute partial derivatives of composite functions.