# Module 3 : Differentiation and Mean Value Theorems

## Lecture 8 : Chain Rule [Section 8.1]

### Objectives

In this section you will learn the following :

- Differentiability of composite of functions, the chain rule .
- Applications of the chain rule.
- Successive differentiability of a function.

### 8.1 Chain Rule

In the previous section we analysed the differentiability of algebraic combinations of differentiable functions. In this section we analyse the differentiability of the composition of differentiable functions.

**8.1.1 Theorem (Chain Rule):**

Let $f$ and $g$ be functions such that $gof$ is defined. If $f$ is differentiable at $c$ and $g$ is differentiable at $f(c)$, then $gof$ is differentiable at $c$ and

$$(gof)'(c) = g'(f(c))f'(c)$$

Alternatively, if

$y = g(u)$ *and* $u = f(x)$,
then,

$$\left.\frac{dy}{dx}\right|_{x=c} = \left.\frac{dy}{du}\right|_{u=f(c)} \left.\frac{du}{dx}\right|_{x=c}.$$

PROOF

Click here to View the Interactive animation : Applet 8.1

**8.1.2 Example:**

Consider the function $f(x)=\sin(x^4)+\sin^4 x,\ x\in\mathbb{R}$. Then

$f(x)=(g\circ h)(x)+(h\circ g)(x)$

where

$g(x)=\sin(x)$ and $h(x)=x^4,\ x\in\mathbb{R}$.

Since, both $g$ and $h$ are differentiable everywhere, by chain rule and theorem 7.1.8, $f$ is also is differentiable at every $c$ in $\mathbb{R}$ and

$f'(c)=4c^3\cos(c^4)+4(\sin^3 c)\cos c.$

8.1.3 **Example (Differentiation of rational powers) :**

Let $u(x),\ x\in\mathbb{R}$, be any positive differentiable function and $r\in\mathbb{R}$, Then the function $\phi(x):=(u(x))^r$ is differentiable with

$$\phi'(x)=r(u(x))^{r-1}u'(x).$$

To see this, let us consider the case when $r>0$. Let $r=\frac{p}{q}$, where $p,q$ are both positive integers .

Let $f(y)=y^{\frac{1}{q}},\ y>0$. Then

$$\phi(x)=(u(x))^{\frac{p}{q}}=f(g(u(x))),$$

where $g(u)=u^p\ \forall u$. Note that $y=g(u)=u^p$. Thus, by chain rule

$$\begin{aligned}\phi'(x) &= \left.\frac{df}{dy}\right|_{g(u(x))} \times \left.\frac{dg}{du}\right|_{u(x)} \times \left.\frac{du}{dx}\right|_{x}\\ &=\left(\frac{1}{q}y^{\frac{1}{q}-1}\right)\left(pu^{p-1}\right)\left(\frac{du}{dx}\right)\\ &=\frac{p}{q}\left(u^{p\left(\frac{1}{q}-1\right)+p-1}\right)\frac{du}{dx}\\ &=\frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx}=r\,u^{r-1}\frac{du}{dx}.\end{aligned}$$

8.1.4 **Example (Derivative of the exponential function):**

In example 3.1.6 (iii) we observed that the function $\log:(0,\infty)\to\mathbb{R}$ is a bijective function which is differentiable at every point $x$. Its inverse function is called the exponential function and is denoted by

$$\exp:\mathbb{R}\to(0,\infty).$$

Since $\log(x)$ has derivative $\frac{1}{x}\neq 0$ for every $x>0$, the exponential function is differentiable. If $\exp(y)=x$, then

$$\frac{d}{dy}(\exp y)=\frac{1}{\frac{d}{dx}(\log(x))}=\frac{1}{\frac{1}{x}}=x=\exp(y).$$

Hence, $\exp(y)$ is its own derivative for every $y \in \mathbb{R}$.

**8.1.5 Example (Derivative of general power function):**

Let $\alpha \in \mathbb{R}$ and for every $x \in \mathbb{R}, x > 0$, define

$f(x) = e^{\alpha \log x}$

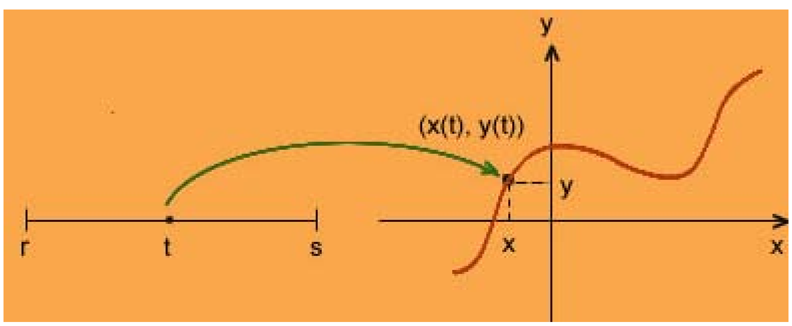
Note that $f(x) = \phi(g(\log x))$, where $\phi(u) = e^u$, $g(y) = \alpha y$. By chain rule, $f$ is differentiable at every point $x$ and

$$\begin{aligned} f'(x) &= \phi'(u)u'(y)\frac{dy}{dx} \\ &= e^u(\alpha)\frac{1}{x} \\ &= \frac{\alpha}{x}\left(e^{\alpha \log x}\right) \\ &= \alpha x^{\alpha - 1}. \end{aligned}$$

We give below some applications of the chain rule.

**8.1.6 Parametric Differentiation:**

Consider a curve $C$ in the plane $\mathbb{R}^2$ which is the graph of a function $y = f(x)$. Suppose that both the variables $x$ and $y$ are functions of another variable $t$, say, $x = x(t)$ and $y = y(t), t \in [r, s]$. Then, the function $t \mapsto (x(t), y(t))$ is called a parametric representation of the curve $C$.




Further, suppose that $f$ is differentiable as a function of $x$ and $x(t)$ is differentiable as a function of $t$. Then by the chain rule, the function $y(t) = f(x(t)), t \in [r, s]$, is differentiable as a function of $t$ and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

In case, $dx/dt \neq 0$, we can write

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

This is called parametric differentiation.

**8.1.7 Example :**

Let $x(t)=\sqrt{t}$ and $y(t)=\frac{1}{4}\left(t^2-4\right), t>0$. Then,

$$\frac{dx}{dt}=\frac{1}{2\sqrt{t}}, \ \frac{dy}{dt}=\frac{1}{4}(2t)=\frac{t}{2}.$$

Since $\frac{dx}{dt}\neq 0$ for $t>0$, we get

$$\frac{dy}{dx}=\frac{t}{2} \times 2\sqrt{t}=t^{\frac{3}{2}} = x^3, \text{ for } t>0.$$

In this case, in fact $y$ as a function of $x$ is given by $y=\frac{1}{4}\left(x^4-4\right),\ x>0$, and hence $\frac{dy}{dx}=x^3$.

**8.1.8 Note :**

In general, the graph of a curve, $t\mapsto\left(x(t),y(t)\right)$ may not arise as the graph of a function. For example

$$x=2t-\pi\sin t,\ y=2-\pi\cos t,\ t\in\mathbb{R}$$

defines a curve in the plane, called prolete cycloid. Even though at the point $t=\frac{\pi}{2}$,

$$\frac{dx}{dt}=2-\pi\cos t\neq 0,\ \frac{dy}{dx}=\pi\sin t,$$

and we have

$$\frac{dy}{dx}=\left.\frac{\pi\sin t}{2-\pi\cos t}\right|_{t=\frac{\pi}{2}}=\frac{\pi}{2},$$

however, this is not the derivative of any function $y=f(x)$.

We saw in the lecture 6 that the inverse of a one-one continuous function is also continuous. It is natural to ask the question: When is the inverse of a one one differentiable function also differentiable? The answer is the next theorem.

8.1.9 **Theorem (Derivative of the inverse function):**

Let $I$ be an open interval and $f: I\rightarrow\mathbb{R}$ be a one-one differentiable function. Let $J$ be the range of $f$ and $f^{-1}: J\rightarrow I$ be the inverse function. Then, $f^{-1}$ is differentiable at a point $f(c)$ for $c\in I$, such that, $f'(c)\neq 0$ and in that case

$$\left(f^{-1}\right)'\left(f(c)\right)=\frac{1}{f'(c)}.$$

PROOF

**8.1.9 Theorem (Derivative of the inverse function):**

Let be an open interval and $f: I\rightarrow\mathbb{R}$ be a one-one differentiable function. Let $J$ be the range of $f$ and $f^{-1}: J\rightarrow I$ be the inverse function. Then, $f^{-1}$ is differentiable at a point $f(c)$ for $c\in I$, such that $f'(c)\neq 0$ and in that case

$$\left(f^{-1}\right)'\left(f(c)\right)=\frac{1}{f'(c)}.$$

**Proof:**

Since $f$ is one-one and continuous on $I$, $f$ is either strictly increasing or strictly decreasing. Let $c \in I$. Then, $f(c) \in J$ and since $f$ is one-one, $f(x) \neq f(c)$ for $x \neq c, x \in I$. Let $y = f(x)$ and $d = f(c)$. Note that since both $f$ and $f^{-1}$ are continuous, $y \to d$ if and only if $x \to c$. Thus,

$$\lim_{y \to d}\left(\frac{f^{-1}(y)-f^{-1}(d)}{y-d}\right) = \lim_{x \to c}\left(\frac{x-c}{f(x)-f(c)}\right) = \frac{1}{f'(c)}.$$

8.1.10 **Example (differentiation of the n-th root function):**

For $n \in \mathbb{N}$, the function $f:[0,\infty) \to [0,\infty)$ defined by $f(x) := x^n$ is a one-one differentiable function with $f'(x) \neq 0$ for $x \in (0,\infty)$. Thus, the inverse function $f^{-1}(y) := y^{\frac{1}{n}}$ is differentiable for every $y \in (0,\infty)$, and its derivative at $d = f(c) = c^n$ is given by

$$\frac{1}{nc^{(n-1)}} = \frac{d^{\left(\frac{1}{n}-1\right)}}{n}.$$

Consequently (by the Chain Rule), if $r$ is any rational number, then $g(x) = x^r$ defines a differentiable function on $(0,\infty)$ and

$$g'(c) = rc^{r-1} \text{ for } c \in (0,\infty).$$

Click here to see an interactive visualization (Java) : Derivative of the inverse function : Applet 8.2

**8.1.11** **Implicit Differentiation** :

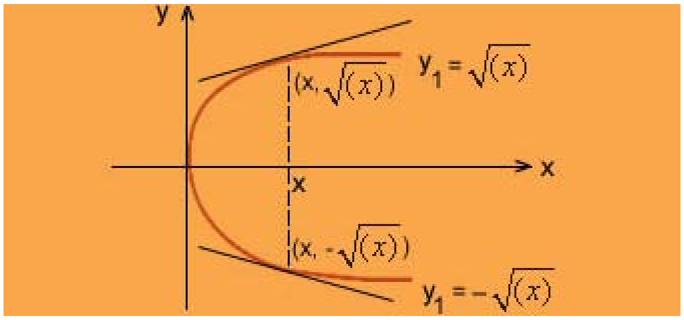
Sometimes, the relation between the independent variable $x$ and the dependent variable $y$ is not explicitly given a function $y = f(x)$, but is given as a relation $F(x,y) = 0$. For example, the relation $y = 1/x, x \neq 0$, can be written implicitly as $xy = 1$. However, the relation $x^5 + y^4 = 4$, does not allow us to represent $y$ explicitly as a function of $x$. In fact, it represents more than one function. The representation $F(x,y) = 0$ is called **implicit representation of the function** . The question one wants to answer is the following: When can we compute $\frac{dy}{dx}$ from the implicit relation $F(x,y) = 0$, without requiring to compute $y$ in terms of explicitly? To provide a complete answer to this we need a theorem from advanced calculus called "Implicit Function Theorem", which gives conditions under which an implicit equation $F(x,y) = 0$ represents an explicit function $y = f(x)$, and is differentiable. Further, it ensures that to find $\frac{dy}{dx}$ one can differentiate $F(x,y) = 0$ using the rules of differentiation and solve it for $\frac{dy}{dx}$. This theorem is stronger than the theorem on the derivatives of inverse function.

**8.1.12 Example:**

Consider the relation $y^2 = x, x > 0$, which can be written as $F(x,y) = y^2 - x = 0$. By the implicit function theorem and using chain the rule, we get

$$2y\left(\frac{dy}{dx}\right)=1, \text{ thus } \frac{dy}{dx}=\frac{1}{2y} \text{ i.e, } \frac{dy}{dx}=\frac{1}{\pm 2\sqrt{x}}.$$

Of course, the question arises: 'which is the right derivative?'. For that, we observe that $y^2 = x$ represents 2-different functions : $y=+\sqrt{x}$ and $y=-\sqrt{x}, x>0$.




## 8.1.13Example :

Consider
$$F(x,y)=x^2(x-y)^2-(x^2-y^2)=0$$

Assuming that the conditions for the implicit differentiation are satisfied, we get

$$2x(x-y)^2+x^2\left(2(x-y)\left(-\frac{dy}{dx}\right)\right)-\left(2x-2y\frac{dy}{dx}\right)=0.$$

Thus, $\left(\frac{dy}{dx}\right)\left(-2x^2(x-y)+2y\right)=2x\left[1-(x-y)^2\right]$, which gives

$$\frac{dy}{dx}=\frac{x(1-(x-y)^2)}{(-x^2(x-y)+y)}$$

This allows us to compute the derivative $\frac{dy}{dx}$ at any point $(x,y)$ which satisfies the relation $F(x,y)=0$.

For example at $(1,1), \frac{dy}{dx}=1$.

### 8.1.14Note

It may not be always possible to represent explicitly $\frac{dy}{dx}$ as a function of $x$.

**Practice Exercise 8.2 : Chain rule and applications**

1. Compute $\frac{dy}{dx}$ for the following functions wherever they are defined:

(i) $y=\sin\sqrt[3]{x}+\sqrt[3]{\sin x}$.

(ii) $y=\sqrt{x}+\sin(2x)^2$.

(iii) $y = \dfrac{(3x^2-2)^3}{(2x+3)^2}$.

2. Let $f, g$ be differentiable functions such that $f(x) = \left(g(x)\right)^3$. Given that, $f'(5) = 162$ and $g(5) = -3$, compute. $g'(5)$.

3. Let $y$ be expressible in terms of $x$ by the relation $\sin y = 2x$. Find the largest interval of the form $-a < y < a$ such that $y$ will be differentiable as a function of $x$.

**Practice Exercise 8.2 : Chain rule and applications**

Continued . .

4. Let $y$ be a differentiable function of the variable $x$ such that $\sqrt{x} + \sqrt{y} = \sqrt{a}$. Find the sum of intercepts of the tangent line $L$ to $y$ at every point is equal to $a$.

5. Let $f(x) = 5x^3 + x - 7$. Show that $f$ is a one-one differentiable function. Find $\left(f^{-1}(c)\right)'$.

6. $f(x) = x^4 + x^3 + 1,\ 0 \le x \le 2$. Show that $f$ is one-one and 3 belongs to range of $f$. Let $g = f^{-1}$ and $h(x) = f(2(g(x))$. Compute $h'(3)$.

**Recap**

In this section you have learnt the following :

- Differentiability of composite of functions, the chain rule.
- Applications of the chain rule.
- Successive differentiability of a function.

## [Section 8.2]

**Objectives**

In this section you will learn the following :

- The notion of successive differentiation.
- The Leibnitz's formula.
- The notion of related rates.

## 8.2 **Successive Differentiation**

### 8.2.1 Definition:

Let $I \subseteq \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$. We say $f$ is twice differentiable at $c \in I$ if $f$ is differentiable on $(c-\delta, c+\delta)$ for some $\delta > 0$ and the derivative function is differentiable at $c$. In that case we define the second order derivative of $f$ at $c$ to be $f': (c-\delta, c+\delta) \to \mathbb{R}$

$$f''(c) := \left(f'\right)'(c),$$

the derivative of the derivative function. It is also denoted by

$$\frac{d^2 f}{dx}\bigg|_{x=c}$$

The concept of $n$-times differentiability and the $n$th derivative of $f$ at $c$, denoted by $f^{(n)}(c)$, can be defined similarly:

$$f^{(n)}(c) := (f^{(n-1)})'(c)$$

If $f^{(n)}(c)$ exists for every $n \in \mathbb{N}$, we say $f$ is infinitely differentiable at $c$.

### 8.2.2 Examples:

(i) Consider $f(x) = x^k,\ x \in \mathbb{R}$

Then $f$ is $n$-times differentiable for every and $n \geq 1$ and

$$f^{(n)}(x) = \begin{cases} k(k-1)\,..\,(k-n)\ x^{k-n} & \text{for } n \leq k, \\ 0 & \text{for } n > k, \end{cases}$$

(ii) Let $f(x) = Sin\ x,\ x \in \mathbb{R}$. Then $f$ is also $n$-times differentiable for every $n$. It is easy to show that

$$f^n(x) = sin(x + n\frac{\pi}{2})$$

(iii) Let

$$f(x) = \begin{cases} x^2 sin\left(\dfrac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Then $f$ is differentiable at every $x$ with

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

For $f, f''(0)$ does not exist. In fact, $f'(x)$ is not even continuous at $x = 0$.

**Click here to see a visualization(Java) : Applet 8.3**

The product rule for differentiation: for differentiable functions $f$ and $g$,

$$(fg)' = f'g + fg'$$

can be extended to higher derivatives as follows.

### 8.2.3 Theorem (Leibnitz's Rule):

Then $f$ and $g$ is $n$-times differentiable at a point $c$ and both have all derivatives of orders up to $(n-1)$ in a neighborhood of $c$. Then, $fg$ is differentiable at $c$ with

$$(fg)^{(n)}(c) = f^{(n)}(c)g(c) + \binom{n}{1} f^{(n-1)}(c)g'(c) + \ldots + \binom{n}{k} f^{(n-k)}(c)g^{(k)}(c) + \ldots + f(c)g^{(n)}(c)$$

Proof:

It is easy to prove the required statement by induction on $n$. We leave the details to the reader.

### 8.2.4 Example:

Let us use Leibnitz's rule to find the thired derivative of the function $h(x) = x^2 \sin x$

Let

$$f(x) = x^2,\ g(x) = \sin x$$

Then

$$f'(x) = 2x,\ f^{(2)}(x) = 2,\ f^{(3)}(x) = 0$$

and

$$g'(x) = \cos x,\ g^{(2)}(x) = -\sin x,\ g^{(3)}(x) = -\cos x.$$

Thus

$$\begin{aligned} h^{(3)}(x) &= 3 f^{(2)}(x)\, g'(x) + 3 f'(x)\, g^{(2)}(x) + f(x)\, g^{(3)}(x) \\ &= 6 \cos x - 6x \sin x - x^2 \cos x. \\ &= (6 - x^2) \cos x - 6x \sin x. \end{aligned}$$

We saw that the derivative of a function also represents the rate of change of the function. This interpretation along with the chain rule is useful in solving problems which involve various rates of change.

### 8.2.5 Example:

If the length of a rectangle decreases at the rate of 3 cm/sec and its width increases at the rate of 2 cm/sec, find the rate of change of the area of the rectangle when its length is 10 cms and its width is 4cms. Let $x$ denote length, $y$ denote width and $A$ denote the area of the rectangle. Then by implicit differentiation

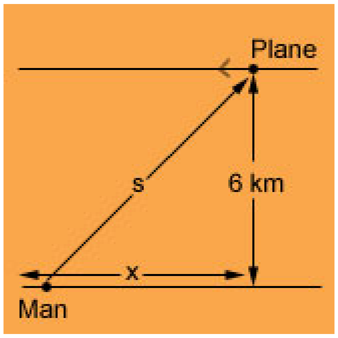
$$A = xy \Rightarrow \frac{dA}{dt} = x\frac{dy}{dt} + \frac{dx}{dt}y = 2x - 3y$$

In particular, $x = 10, y = 4 \Rightarrow \frac{dA}{dt} = 8$ cm$^2$/sec that is, the area of the rectangle increases at the rate of 8 cm$^2$/sec.

### 8.2.6 Example:

An airplane is flying in a straight path at a height of 6 Km from the ground which passes directly above a man standing on the ground. The distance $s$ of the man from the plane is decreasing at the rate of 400 km per hour when $s = 10$ km. We want to find the speed of the plane. To find this, let $x$ denote the horizontal distance of the plane from the man. We note that for $s = 10$, $x = \sqrt{(10)^2 - (6)^2} = 8$. We are given that

$$\frac{ds}{dt} = 400 \text{ when } s = 10,$$




and we have to find $\frac{dx}{dt}$. The variables $s$ and $x$ are related by $s^2 = x^2 + 6$.

Thus,

$$2\frac{ds}{dt} = 2\frac{dx}{dt}$$

Hence

$$\frac{dx}{dt} = \left(\frac{s}{x}\right)\left(\frac{ds}{dt}\right)$$

Thus, when $s = 10, x = 8, \frac{ds}{dt} = -400$,

we set

$$\frac{dx}{dt} = \frac{10}{8}(-400) = -500 \text{ km /hour}$$

Hence the plane is approaching the man with a speed of 500 km/hour.

**Practice Exercises: Successive differentiation**

**1.** Let $f(x) = (x-c)^2 g(x)$, where $c \in \mathbb{R}$ and $g(x)$ and is differentiable and $g(c) \neq 0$. Show that $f''(c) \neq 0$.

**2.** Show that for every $n \geq 1$,

**(i)** $\frac{d(\sin x)}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right)$

**(ii)** $\frac{d(\cos x)}{dx^n} = \cos\left(x + \frac{n\pi}{2}\right)$

**3.** Use Leibnitz theorem to find the third derivative of the functions

**(i)** $x^2 \sin x$

**(ii)** $x^2(x^2+1)^{-1}$

**Practice Exercises: Successive differentiation (Continued)**

**4.** Using induction on $n$, show that

**(i)** For $f(x)=\dfrac{1}{1+x},\ x\neq 1,\ f^{(n)}(x)=\dfrac{(-1)^n\, n!}{(1+x)^{n+1}}$

**(ii)** For $f(x)=\dfrac{4}{x}-\dfrac{1}{x^2},\ x\neq 0,\ f^{(n)}(x)=\dfrac{(-1)^n\, n!(3-n)}{x^{n+2}}$

**5.** The radius of the circular disc is increasing with time (think of oil pouring from a tanker in sec). Find the rate of change of the area of the disc to the radius of the disc. How fast is the area increasing when radius is 4km, if the rate of change of the radius is 5 cm/sec.

## Recap

In this section you have learnt the following :

- The notion of successive differentiation.
- The Leibnitz's formula.
- The notion of related rates.