Module 5 : Linear and Quadratic Approximations, Error Estimates, Taylor's Theorem, Newton and Picard Methods

Lecture 14 : Taylor's Theorem [Section 14.1]

Objectives

In this section you will learn the following :

Taylor's theorem and its applications

14.1 Taylor 's Theorem and its applications

In previous section we used $L(x; a)$, the tangent line, a polynomial of degree one in x to approximate a given function $f(x)$ for x near $x = a$. One can try to approximate the function f by a higher degree polynomial, hoping that the polynomial of higher degree will give a better approximation to f for x near a. To analyze this, we need a generalization of the extended mean value theorem:

14.1.1Theorem (Taylor's Theorem):

Let $\delta > 0$ *and* $f : [a-\delta, a+\delta] \to \mathbb{R}$ *be such that*

(i) $f, f', \ldots, f^{(n)}$ *exist and are are continuous on* $[a-\delta, a+\delta]$.

(ii) $f^{(n+1)}$ *exist on* $[a-\delta, a+\delta]$.

Then,

$$f(x) = f(a) + f'(a)(x-a) + \ldots + \left(\frac{f^{(n)}(a)}{n!}\right)(x-a)^n + \left(\frac{f^{(n+1)}(c)}{(n+1)!}\right)(x-a)^{n+1}.$$

for some

$c \in (a-\delta, a+\delta)$

The above expression is also known as the Taylor 's formula for f around a.

Proof:

We assume the proof. The interested reader may refer a book on advanced calculus.

14.1.2 Definition:

Let f be as in theorem 14.1.1.

(i) The polynomial

$$T_n(x;a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

is called the nth degree Taylor polynomial of f around a.

(ii) The term

$$R_n(x;a) = f(x) - T_n(x;a) = \left(\frac{f^{(n+1)}(c)}{(n+1)!}\right)(x-a)^{n+1}$$

is called the nth-Remainder term of the Taylor 's formula for f around a.

(iii) In case

$$\lim_{n\to\infty} |R_n(x;a)| = 0,$$

we say that f has Taylor series around a and write it as

$$f(x) = \sum_{n=0}^{+\infty}\left(\frac{f^{(n)}(a)}{n!}\right)(x-a)^n.$$

14.1.3 Example:

(i) Let

$$f(x) = \frac{1}{x},\ x \neq 0.$$

Then

$$f^{(k))}(x) = \frac{(-1)^k n!}{x(k+1)},\ k \geq 1.$$

Thus for any $a > 0$,

$$T_n(x;a) = f(a) - \frac{(x-a)}{a^2} + \frac{(x-a)^2}{a^3} - \dots + \frac{(-1)^n (x-a)^n}{a^{n+1}}.$$

(ii) Let

$$f(x) = \sin x,\ x \in \mathbb{R}$$

Then, for $k \in \mathbb{N}$,

$$f^{(k)}(x) = \begin{cases} (-1)^{k/2} \sin x, & \text{if } k \text{ is even} \\ (-1)^{(k-1)/2} \cos x, & \text{if } k \text{ is odd} \end{cases}$$

Thus, for nth Taylor polynomial of $f(x)$ around $a = 0$ is given by

$$T_n(x;a) = \begin{cases} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{(n-1)/2} \frac{x^n}{n!}, & \text{for } n \text{ odd} \\ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{(n-2)/2} \frac{x^{n-1}}{(n-1)!}, & \text{for } n \text{ even} \end{cases}$$

(iii) Let $f(x)$ be a polynomial of degree n:

$$f(x) := a_0 + a_1 x + \dots\dots + a_n x^n,\ a_n \neq 0.$$

Then

$$f^{(k)}(x) = \begin{cases} n(n-1)\dots\dots(n-k)\, a_n x^{n-k} & \text{for } 1 \le \text{k} \le \text{n} \\ 0 & \text{for k} > \text{n} \end{cases}$$

Thus, for f,

$$T_n(f;a) = T_{n+m}(f;a) \quad \text{for all } m \geq n.$$

For example, for

$$f(x) = x^3 - 2x + 4,$$

$$f'(x) = 3x^2 - 2,\ f''(x) = 6x,\ f'''(x) = 6 \text{ and } f^{(iv)}(x) \equiv 0$$

Thus, for $a = 2$

$$T_3(x;2) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$
$$= 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$$

14.1.4Note:

For $n = 1$, the Taylor polynomial for f is nothing but the linear approximation. For a function f, which is $n+1$ times differentiable in (a,b), we can use the nth degree Taylor 's polynomial to approximate f in a small interval around a. We analyze this for $n = 2$ in the next section.

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14.1.4 Note:

For $n = 1$, the Taylor polynomial for f is nothing but the linear approximation. For a function f, which is $n+1$ times differentiable in (a,b), we can use the nth degree Taylor 's polynomial to approximate f in a small interval around a. We analyze this for $n = 2$ in the next section.

PRACTICE EXERCISES

1. Find the nth Taylor polynomial of f around a, that is,

$$f_n(x) = f(a) + f'(a)(x-a) + \dots + \frac{f_n(a)}{n!}(x-a)^n,\ x \in \mathbb{R},$$

when $a = 0$ and $f(x)$ is as below:

(i) $x^3 + 3x^2 + 5$.

(ii) $\tan^{-1} x$.

(iii) $\sin x$.

(iv) $\cos x$.

(v) $\exp(x)$.

(vi) $\exp(-x)$.

2. Show that for all $x \in \mathbb{R}$,

(i) $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$.

(ii) $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$.

(iii) $\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

3. Let $T_n(f;a)$ denote the Taylor 's polynomial of order n for f at $x = a$. Prove the following:

(i) $T_n(\alpha f;a) = \alpha T_n(f;a)$ for every $\alpha \in \mathbb{R}$.

(ii) $T_n(f+g;a) = T_n(f;a) + T_n(g;a)$.

(iii) .

$$\left(T_n(f\,;a)\right)' = T_{n-1}(f'\,;a)$$

PRACTICE EXERCISES

4. Let f, g be functions having derivatives of order n at $x = a$ such that

$$f(x) = P_n(x) + x^n g(x),$$

where $P_n(x)$ is a polynomial of degree $n \geq 1$ If $g(x) \to 0$ as $x \to 0$, show that

$$P_n(x) = T_n(x;0).$$

5. Using exercises (3) and (4) above find $T_n(f,0)$ for the following :

(i) $f(x) = \dfrac{1}{1-x}$ $\quad \left(\text{Hint: } \dfrac{1}{1-x} = 1 + x + \ldots + x^n + \dfrac{x^{n+1}}{1-x}\right)$

(ii) $f(x) = \dfrac{1}{1+x}$.

(iii) $f(x) = \dfrac{1}{1-x^2}$ $\quad \left(\text{Hint: } \dfrac{2}{1-x^2} = \dfrac{1}{1-x} + \dfrac{1}{1+x}\right)$

(iv) $f(x) = \dfrac{1}{1+x^2}$.

(v) $f(x) = \dfrac{x}{1-x^2}$.

6. Using exercise (3) and 5(iv), show that for $f(x) = \tan^{-1}(x)$,

$$T_4(f,0) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}.$$

Recap

In this section you have learnt the following

- Taylor's theorem and its applications

[Section 14.2]

Objectives

In this section you will learn the following :

- How to estimate a function by a quadratic function and how to estimate the error.

14.2 Quadratic approximations

14.2.1 Definitions:

For $n = 2$, the Taylor 's polynomial

$$T_2(x;\, a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 ,$$

is called the quadratic approximation of f for x near a, and is also denoted by $Q(x\,;\, a)$.

14.2.2 Examples:

(i) Consider the function

$$f(x) = \frac{1}{1-x} \quad \text{for} \quad x \neq 1 .$$

Then,

$$f(x) \;=\; \frac{1}{\left(1-x^2\right)} \quad \text{and} \quad f''(x) \;=\; \frac{-2}{(1-x)^3} \quad \text{for} \quad x \neq 1$$

Thus, for x near $a \neq 1$,

$$Q(x;a) \;=\; \frac{1}{(1-a)} \;+ \frac{1}{(1-a)^2}\,(x-a) \;+\; \frac{-2(x-a)^2}{2(1-a)^3} \;.$$

For example for $a = 0$,

$$Q(x;0) \;=\; 1 + x + x^2 .$$

(ii) Consider the function

$$f(x) = \exp(x), \quad x \in \mathbb{R} .$$

Then

$$f(0) = f'(0) = f''(0) = \exp(0) = 1 .$$

Thus, near $a = 0$, $f(x)$ has quadratic approximation

.

$$Q(x;\,0) = 1 + x + \frac{x^2}{2}$$

(iii) Let

$$f(x) = \sqrt{x},\ \ x \in \mathbb{R},\ x > 0$$

Then,

$$f'(x) = \frac{1}{2\sqrt{x}}\ ,\ f''(x) = -\frac{1}{4(x)^{3/2}}$$

Thus, near $x = 1$, $f(x)$ has quadratic approximation:

$$\begin{aligned} Q(x,1) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 \\ &= 1 + \frac{(x-1)}{2} - \frac{(x-1)^2}{8}\,. \end{aligned}$$

For example, this given

$$\begin{aligned} \sqrt{0.5} &\simeq 1 - \left(\frac{0.5}{2}\right) - \frac{(0.5)^2}{8} \\ &= 0.719. \end{aligned}$$

Like for linear approximations, it is natural to ask the question:

How well does $Q(x;a)$ approximate $f(x)$ for near a?

An answer to this question is the following:

14.2.3Corollary:

Let $a, x \in \mathbb{R}$ with $a \neq x$ and I be the closed interval with end points a and x. Let $f : I \to \mathbb{R}$ be such that

(i) The functions f, f', f'' are all continuous.

(ii) For every c between a and x,

$f'''(c)$ exists and $|f'''(c)| \leq M_3(x)$,

then $e_2(x;a) = f(x) - Q(x;a)$ satisfies the following:

$$|e_2(x;a)| \leq \left(\frac{M_3(x)}{3!}\right)|x-a|^3.$$

PROOF

14.2.3 Corollary:

Let $a, x \in \mathbb{R}$ with $a \neq x$ and I be the closed interval joining a and x. Let $f : I \to \mathbb{R}$ be such that

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then $e_2(x;a) = f(x) - Q(x;a)$ satisfies the following:

$$|e_2(x;a)| \leq \left(\frac{M_3(x)}{3!}\right)|x-a|^3.$$

Proof:

Follows trivially from Taylor 's Theorem for $n = 3$.

14.2.4 Example:

Consider the function

$$f(x) = 1/(1-x) \text{ for } x \neq 1.$$

Then

$$f'''(c) = 6/(1-c)^4 \text{ for } c \neq 1.$$

We saw in example 14.2.1, that the quadratic approximation for f near the point $a = 0$ is given by

$$Q(x;0) = 1 + x + x^2.$$

Let us estimate the error

$$e_2(x;0) = f(x) - Q(x;0) \text{ for } x < 1.$$

Let us fix a $x < 1$. We have to consider two cases.

Case (i): $0 < x < 1.$

In this case, for $0 < c < x$, we have

$$\left|f'''(c)\right| = \left|\frac{6}{(1-c)^4}\right| \leq \left|\frac{1}{(1-x)^4}\right| := M_3(x).$$

Thus

$$\left|e_2(x,0)\right| \leq \left(\frac{|x|^3}{(1-x)^4}\right).$$

For example,

for all $0 < x < 0.1, \left|e_2(x,0)\right| \leq |x|^3 \leq (0.1)^3 = 0.001.$

Thus, for all $x \in [0, 0.1]$,

$Q(x,0) = 1 + x + x^2$ differs from $f(x) = \dfrac{1}{1-x}$ at most by 0.001

Case (ii): $x < 0$.

In this case, for $x < c < 0$, we have

$$\left|f'''(c)\right| = \frac{6}{(1-c)^4} < 6 := M_3(x).$$

Thus,

$$\left|e_2(x,0)\right| \leq |x|^3.$$

For example,

for $x = -.01 < 0, \left|\in_2(x,0)\right| \leq (.01)^3 = 0.001.$

14.2.5 Note (Rate of convergence for the error):

Since

$$f(x) = Q(x;a) + e_2(x;a),$$

clearly, $e_2(x;a) \to 0$ as $x \to a$. In fact, the error in the quadratic approximation tends to zero at a rate faster than $(x-a)^2$. To see this, note that

$$e_2(x;a) := \frac{(x-a)^3}{3!} f'''(c),$$

and hence

$$\lim_{x \to a}\left(\frac{e_2(x;a)}{(x-a)^2}\right) = \lim_{x \to a}\left(\frac{f'''(c)(x-a)}{3!}\right) = 0.$$

CLICK HERE TO SEE AN INTERACTIVE VISUALIZATION : Applet 14.1

PRACTICE EXERCISES

1. Let $a \in \mathbb{R}$ and $F(x) = c_0 + c_1(x-a) + c_2(x-a)^2$ for $x \in \mathbb{R}$. If f is twice differentiable at a, show that F

is the quadratic approximation of f near a if and only if

$f(a) = F(a) = c_0$, $f'(a) = F'(a) = c_1$ and $f''(a) = F''(a) = c_2$.

2. Let

$$f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} \text{ for } x > 0.$$

Write down the linear and the quadratic approximations $L(x)$ and $Q(x)$ of $f(x)$ near the point $x = 4$. Estimate the errors.

3. For the following functions find the quadratic approximations near the point $a = 0$, and also find the error

bounds valid for $|x| \le 0.03$:

(i) $f(x) = \sin(x^2)$.

(ii) $f(x) = \sqrt[3]{1+x}$.

(iii) $f(x) = \dfrac{1}{\sqrt{1+x}}$.

(iv) $f(x) = \dfrac{x}{1+x^2}$.

Recap

In this section you have learnt the following

How to estimate a function by a quadratic function and how to estimate the error.