Module 11 : Partial derivatives, Chain rules, Implicit differentiation, Gradient, Directional derivatives

Lecture 33 : Implicit differentiation [Section 33.1]

Objectives

In this section you will learn the following :

- The concept of implicit differentiation of functions.

33 .1 Implicit differentiation

As we had observed in section 8.1, many a times a function y of an independent variable x is not given explicitly, but implicitly by a relation. In section 8.1 we had also mentioned about the implicit function theorem. We state it precisely now, without proof.

33.1.1 Implicit Function Theorem (IFT):

Let $(x_0, y_0) \in \mathbb{R}^2$ and $g : B_r(x_0, y_0) \to \mathbb{R}$ be such that the following holds:

(i) *Both the partial derivatives g_x and g_y exist and are continuous in $B_r(x_0, y_0)$.*

(ii) $g(x_0, y_0) = 0$ *and* $g_y(x_0, y_0) \neq 0$.

Then there exists some $\delta > 0$ and a function

$$\phi : [x_0 - \delta, x_0 + \delta] \to \mathbb{R}$$

such that ϕ is differentiable, its derivative ϕ' is continuous with $\phi(x_0) = y_0$ and

$$g(x, \phi(x)) = 0 \text{ for all } x \in [x_0 - \delta, x_0 + \delta].$$

33.1.2 Remark:

We have a corresponding version of the IFT for solving x in terms of y. Here, the hypothesis would be $g_x(x_0, y_0) \neq 0$.

33.1.3 Example:

Let

$$g\,(x,y) = x^2 + (y-1)^2 - 4$$

we want to know, when does the implicit expression

$$g\,(x,y) = 0$$

defines y explicitly as a function of x. We note that

$$g_x(x,y) = 2x, g_y(x,y) = 2\,(y-1),$$

and are both continuous. Since for the points $(x,y) = (\pm 2, 1)$, $g(x,y) = 0$ and

$$g_y(x,y) = 2\,(y-1) = 0 \text{ for } y = 1,$$

the implicit function theorem is not applicable. For $-2 < x < 2$, and $-1 < y < 1$, the equation $g(x,y) = 0$ defines the explicit function

$$y = 1 - \sqrt{(4-x^2)}\,,$$

and for $-2 < x < 2, 1 < y < 3$, the equation $g(x,y) = 0$ defines the explicit function

$$y = 1 + \sqrt{(4-x^2)}\,,$$

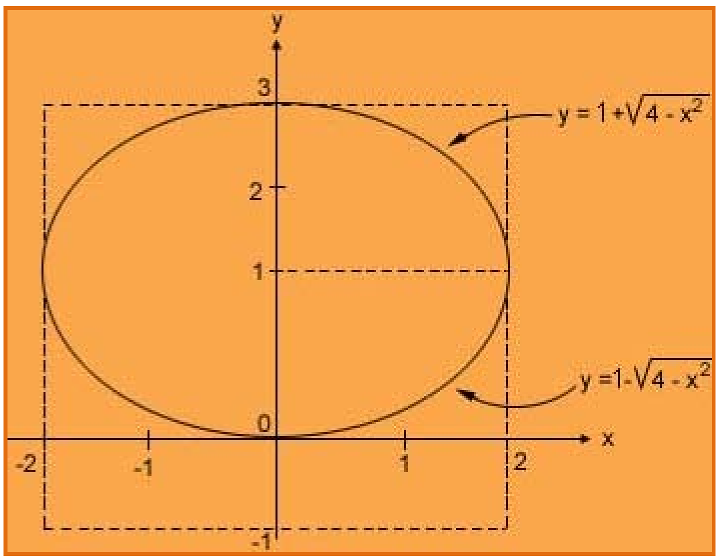

Figure 1. y is a function of x.

A result similar to that of theorem 33.1.1 holds for function of three variables, as stated next.

33.1.4 Theorem :

Let $(x_0, y_0, z_0) \in \mathbb{R}^3$ *and* $g : B_r(x_0, y_0, z_0) \to \mathbb{R}$ *be such that*

(i) g_x, g_y, g_z *exist and are continuous at* (x_0, y_0, z_0).

(ii) $g(x_0, y_0, z_0) = 0$ *and* $g_z(x_0, y_0, z_0) \neq 0$. *Then the equation* $g(x, y, z) = 0$ *determines a unique function*

$z = f\,(x,y)$ *in the neighborhood* N *of* (x_0, y_0) *such that for* $(x,y) \in N$,

$$g\,(x, y, f\,(x,y)) = 0$$

$g_z\ (x, y, f\ (x, y)) \neq 0$

and

$$f_x\ (x_0\ , y_0) = -\frac{g_x\ (x, y, f\ (x, y))}{g_z\ (x, y, f\ (x, y))},$$

$$f_y\ (x_0\ , y_0) = -\frac{g_y\ (x, y, f\ (x, y))}{g_z\ (x, y, f\ (x, y))}.$$

Practice Exercises :

(1) Show that the following functions satisfy conditions of the implicit function theorem in the neighborhood of the

indicated point.

(i) $g(x, y) = e^{x-y} + x^2 - y - 1 = 0, (x_0, y_0) = (0, 0)$.

(ii) $g(x, y) = \cos xy + x - 1 = 0, (x_0, y_0) = (1, \pi/2)$

(iii) $g(x, y) = (x^2 + y^2)^2 - 8(y^2 - x^2) = 0, (x_0, y_0) = (1, \sqrt{3})$

(iv) $g(x, y, z) = x^2 + y^2 + \sin z - 1 = 0, (x_0, y_0, z_0) = (1, 0, 0)$

(2) Find the points such that in some neighborhood of which the following function define implicitly a function

$y = f(x)\ (z = f(x, y))$

(i) $g(x, y) = x^3 + 3y^2 + 8xz^3y$

(ii) $x^2 - y^2 - 7$

Answers

(3) Let

$$g\ (x, y, z) = x^2 + 3y^2 + 5z^2 + 2xy - 4yz - 4xz - 9$$

Show that the equation $g(x, y, z) = 0$ uniquely represents y as a function of x in a neighborhood of $(-4, 2, -1)$ where as, the same cannot be said about x as a function of y, z .

Recap

In this section you have learnt the following

The concept of implicit differentiation of functions.

[Section 33.2]

Objectives

In this section you will learn the following :

- The notion of directional derivative.

33 .2 Directional Derivatives

For a function of two variables, the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ of f determine the rate of change of f along the two axes at (x_0, y_0). In fact, we can analyze the rate of change of f in any particular direction.

33.2.1Definition:

Let $P(x_0, y_0) \in D \subset \mathbb{R}^2$. Let us fix a direction in $\mathbb{R}^2$, by choosing a unit vector

$$\mathbf{u} = (u_1, u_2) \text{ with } u_1^2 + u_2^2 = 1.$$

Then the line L in D through $P(x_0, y_0)$ parallel to the vector u has parametric equation

$$x = x_0 + tu_1,\ y = y_0 + tu_2, \text{where } t \in \mathbb{R}.$$

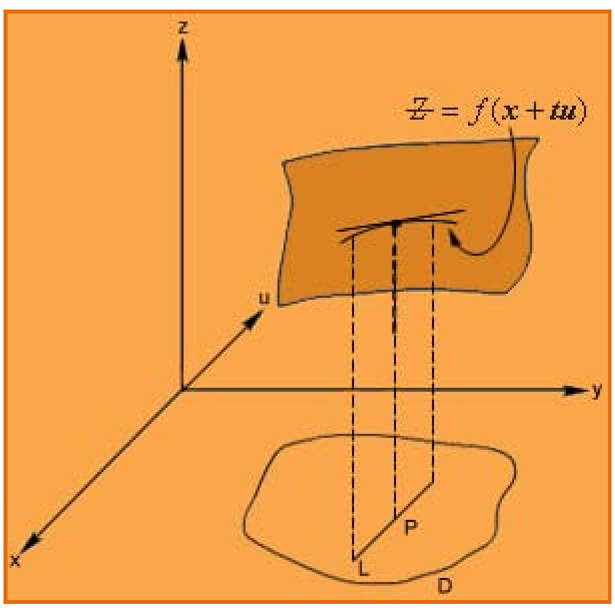

Figure 1. Derivative in the direction *u*

Whenever the limit

$$\lim_{t\to 0}\frac{f(x_0+tu_1, y_0+tu_2)-f(x_0,y_0)}{t},$$

exists is called the **directional derivative** of f at (x_0, y_0) in the direction of $\mathbf{u}$. It is denoted by $D_{\mathbf{u}} f(x_0, y_0)$.

33.2.2 Note:

The directional derivative represents the rate of change of f at the point $P(x_0, y_0)$ in the direction of the vector $\mathbf{u}$. Note that for a unit vector $\mathbf{u}=(u_1,u_2)$ if the directional derivative $(D_{\mathbf{u}}f)(P)$ exists, then $(D_{-\mathbf{u}}f)(P)$ also exists and

$(D_{\mathbf{u}}f)(P) = -(D_{-\mathbf{u}}f)(P)$.

33.2.3 Examples:

(i) Let $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x,y)=e^x\cos y$$

and $\mathbf{u}=(u_1,u_2)$, and any unit vector. Then, for $(x_0,y_0)\in\mathbb{R}^2$, we have

$$\lim_{t\to 0}\left(\frac{f(x_0+tu_1,y_0+tu_2)-f(x_0,y_0)}{t}\right)$$

$$=\lim_{t\to 0}\left(\frac{e^{x_0}e^{tu_1}\cos(y_0+tu_2)-e^{x_0}\cos y_0}{t}\right)$$

$$=e^{x_0}\cos y_0\left[\lim_{t\to 0}\left(\frac{e^{tu_1}\cos tu_2-1}{t}\right)\right]-e^{x_0}\sin y_0\left[\lim_{t\to 0}\left(\frac{e^{tu_1}\sin tu_2}{t}\right)\right]$$

$$=e^{x_0}\cos y_0\left[\lim_{t\to 0}\left(u_1e^{tu_1}\cos tu_2-u_2e^{tu_1}\sin tu_2\right)\right]-\left(e^{x_0}\sin y_0\right)u_2$$

$$=(e^{x_0}\cos y_0)u_1-(e^{x_0}\sin y_0)u_2.$$

Thus, $D_{\mathbf{u}} f(x_0,y_0)$ exists and

$$D_{\mathbf{u}} f(x_0,y_0)=(e^{x_0}\cos y_0)u_1-(e^{x_0}\sin y_0)u_2.$$

(ii) Let

$$f(x,y)=|x|+|y|, \text{ for } (x,y)\in\mathbb{R}^2.$$

Then, $D_{\mathbf{u}}f(0,0)$ exists for each of the unit vectors $\mathbf{u}=-\mathbf{i},\mathbf{j},-\mathbf{j}$ and for each of them $D_{\mathbf{u}}(0,0)=1$. Note that both $f_x(0,0)$ and $f_y(0,0)$ do not exist.

33.2.4 Theorem:

Let $f: D\subseteq\mathbb{R}^2\to\mathbb{R}$ be differentiable at $(x_0,y_0)\in D$. Then for every unit vector $\mathbf{u}=(u_1,u_2)$, the directional derivative $(D_{\mathbf{u}}f)(x_0,y_0)$ exists and

$$(D_{\underline{u}}f)(x_0,y_0)=f_x(x_0,y_0)u_1+f_y(x_0,y_0)u_2.$$

33.2.4Theorem:

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ *be differentiable at* $(x_0, y_0) \in D$. *Then for every unit vector* $\mathbf{u} = (u_1, u_2)$ *, the directional derivative* $(D_{\mathbf{u}} f)(x_0, y_0)$ *exists and*
$(D_{\underline{u}} f)(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$

Proof

By the differentiability of f at the point (x_0, y_0), we have for all t near t = *0*,

$$f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)$$
$$= f_x(x_0, y_0)tu_1 + f_y(x_0, y_0)tu_2 + tu_1\ \varepsilon_1(tu_1, tu_2) + tu_2\ \varepsilon_2(tu_1, tu_2),$$

where

$$\varepsilon_1(tu_1, tu_2), \varepsilon_2(tu_1, tu_2) \to 0 \text{ and } g_1(tu_1, tu_2) \to (0,0) \text{ as } t \to 0.$$

Now dividing both sides of the above equation by $t \neq 0$, and taking limit as $t \to 0$, we see that $(D_{\mathbf{u}} f)(x_0, y_0)$ exists and equals

$$(D_{\mathbf{u}} f)(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

33.2.5Example:

Let
$$f(x, y) = e^x \cos y, (x, y) \in \mathbb{R}^2.$$
Then, both the partial derivatives of f exist at all points,
$$f_x = e^x \cos y, f_y = e^{-x} \sin y,$$
and are continuous. Thus, f is differentiable everywhere and for any unit vector $\mathbf{u} = (u_1, u_2)$, we can directly compute
$$(D_{\mathbf{u}}\ f)(x_0, y_0) = (e^{x_0} \cos y_0)u_1 + (-e^{x_0} \sin y_0)u_2.$$
Compare this with the calculation in example $33.2.3(i)$

Practice Exercises

(1) Examine the following functions for the existence of directional derivatives at $(0,0)$ where the expressions below give

the value at $(x, y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero.

(i) $xy \dfrac{x^2 - y^2}{x^2 + y^2}$.

(ii) $\dfrac{x^3}{x^2 + y^2}$.

(iii) $\left(x^2 + y^2\right) \sin\left(\dfrac{1}{x^2 + y^2}\right)$.

(2) Let

$$f(x,y) = 0 \text{ if } y = 0$$

and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2} \text{ if } y \neq 0.$$

Show that f is continuous at $(0,0)$, both $f_x(0,0)$ and $f_y(0,0)$ exist, $D_{\underline{u}} f(0,0)$ exists for every unit vector u, but f is not differentiable at $(0,0)$.

(3) Assume that f_x and f_y exist in $B_r(1,2)$ for some $r > 0$ and are continuous at $(1,2)$. If the directional derivative of f at $(1,2)$ in the direction towards the vector $(2,3)$ is $2\sqrt{2}$ and in the direction towards $(1,0)$ is -3, then find $f_x(1,2), f_y(1,2)$ and the directional derivative of f at $(1,2)$ in the direction towards $(4,6)$.

Recap

In this section you have learnt the following

- The notion of directional derivative.