Module 17 : Surfaces, Surface Area, Surface integrals, Divergence Theorem and applications

Lecture 51 : Divergence theorem [Section 51.1]

Objectives

In this section you will learn the following :

- Divergence theorem, which relates line integral with a double integral.

51.1 Divergence theorem

We saw in lecture 48 (module 16) that the Green's theorem relates the line integral to double integral:

$$\iint_R \operatorname{div}(\mathbf{F})\,dxdy = \oint_C (\mathbf{F}\cdot\mathbf{n})ds$$

An extension of this result holds in $\mathbb{R}^3$ for surface integrals, which helps to represent flux across a closed surface as a triple integral.

51.1.1 Theorem (Divergence theorem):

Let G be a closed bounded region in $\mathbb{R}^3$ whose boundary is an orientable surface S. Let

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

be a continuously differentiable vector-field in an open set containing the region D. Then

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = \iiint_G (\operatorname{div}\ \mathbf{F})\,dxdydz,$$

where $\mathbf{n}$ is the outward normal to the surface S.

PROOF

(For Simple regions G)

We shall assume that the region G has the property that any straight line parallel to any one of the coordinate axes intersects G at most in one line segment or a single point. For such a region G, we have to show that

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \iiint_G (\mathbf{div}\,\mathbf{F})\, dxdydz. \qquad \text{----------(70)}$$

Let the outward normal $\mathbf{n}$ at any point on S have direction cosines $\cos\alpha, \cos\beta$ and $\cos\gamma$ i.e., let

$\mathbf{n} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$

than (26) is same as proving:

$$\iint_S \left(P\cos\alpha + Q\cos\beta + R\cos\gamma\right) dS = \iiint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} + \right) dxdydz.$$

To prove this, we shall prove the following:

$$\iint_S P\cos\alpha\, dS = \iiint_G \frac{\partial P}{\partial x}\, dxdydz, \qquad \text{----------(71)}$$

$$\iint_S Q\cos\beta\, dS = \iiint_G \frac{\partial Q}{\partial y}\, dxdydz, \qquad \text{----------(72)}$$

$$\iint_S R\cos\gamma\, dS = \iiint_G \frac{\partial R}{\partial z}\, dxdydz. \qquad \text{----------(73)}$$

Because of the special assumption on G, it can be written as

$$G = \left\{(x,y,z) \mid (x,y) \in D \subseteq \mathbb{R}^2,\ g(x,y) \le z \le h(x,y)\right\},$$

In the above D is the projection of S onto the xy-plane. Note that, for any $(x,y) \in \mathbb{R}^2$, a point $(x,y,z) \in G$ provided z lies between the surfaces $z = g(x,y)$ and $z = h(x,y)$. Thus the boundary S of G consists of an upper part S_1, the surface $z = h(x,y)$; a lower part S_2 the surface $z = g(x,y)$;, and possible the lateral part: S_3 a cylinder with base D and axis parallel to z-axis. Thus

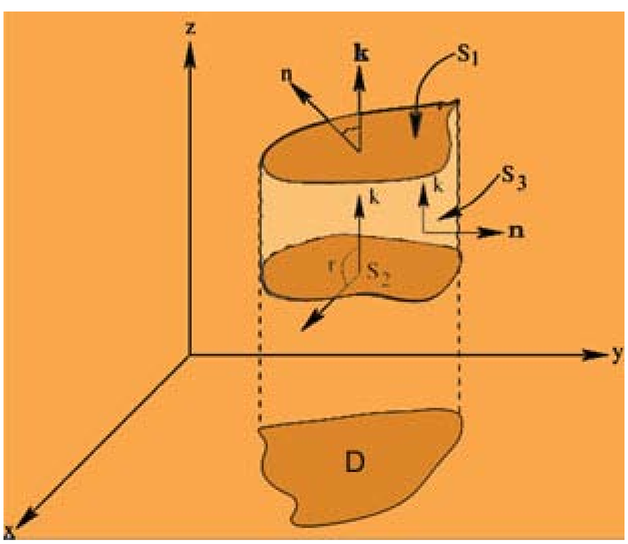


Figure 222. Caption text.

$$\iint_S R\cos\gamma\, dS = \iint_{S_1} R\cos\gamma\, dS + \iint_{S_2} R\cos\gamma\, dS + \iint_{S_3} R\cos\gamma\, dS.$$

Note that on the surface S_1 the and γ that the outward normal $\mathbf{n}$ makes with $\mathbf{k}$ is acute, on S_2 it is obtuse and on S_3 it is $\pi/2$. Hence, above becomes

$$\begin{aligned}\iint_S R\cos\gamma\, dS &= \iint_D R\left(x,y,h(x,y)\right)dxdy - \iint_D R\left(x,y;g(x,y)\right)dxdy \\ &= \iint_R\left[R\left(x,y,h(x,y)\right) - R\left(x,y,g(x,y)\right)\right]dxdy \\ &= \iint_R\left(\int_{z=g(x,y)}^{z=h(x,y)} \frac{\partial R}{\partial z}\,dz\right)dxdy \\ &= \iiint_G \frac{\partial R}{\partial z}\,dxdydx.\end{aligned}$$

This proves (71). Similarly, using the special nature of G and projecting it on yz-plane and zx-plane, respectively, equations (72) and (73) can be proved. This proves the divergence theorem for special regions.

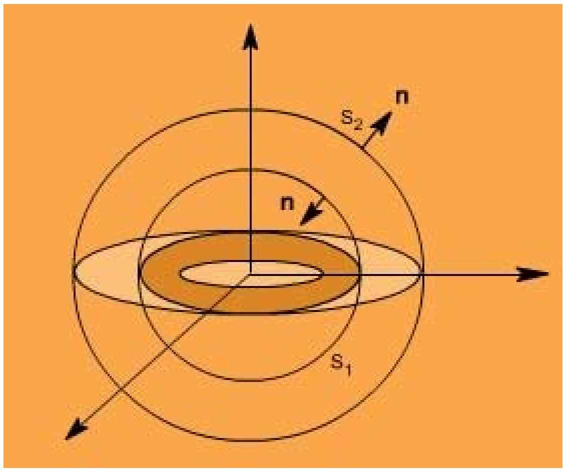
51.1.2 Note :

Divergence theorem can be extended to regions G which can be divided into finite number of simple regions. Essentially, the idea is to add the corresponding results over such regions, observing that the surface integrals over common-surface will cancel other (normals being outward).

51.1.3 Example:

Consider the solid G enclosed by two concentric spheres, say

$G=\left\{(x,y,z)\,|\,4\le x^2+y^2+z^2\le 9\right\}$



Figure

Let

$$S_1 := \left\{(x,y,z)\,|\,x^2+y^2+z^2=4\right\}$$
$$S_2 := \left\{(x,y,z)\,|\,x^2+y^2+z^2=9\right\}$$

Then G has boundary $S=S_1\cup S_2$, which is orientable, but G is not simple solid. However, we can write $G=G_1\cup G_2$

where

$$G_1=\left\{(x,y,z)\,|\,4\le x^2+y^2+z^2=9\,,\,z\ge 0\right\}$$

$$G_2=\left\{(x,y,z)\,|\,4\le x^2+y^2+z^2\le 9\,,\,z\le 0\right\}$$

Then G_1 and G_2 are both simple solids, G_1 is bounded by piecewise smooth surfaces S_2^+ upper hemisphere of S_2 the surface S_1^+ upper hemisphere of S_1 and the annulus surface S_3 in the xy-plane given by

$$\left\{(x,y,0) \mid 4 \le x^2+y^2 \le a\right\}.$$

Similarly, G_2 is bounded by S_2^-, the lower hemisphere of S_2, the surface S_1, the lower hemisphere of S_1 and S_3. Note that the outward normal on S_3 as boundary of G_2 is negative of the outward normal of S_3 as boundary of G_1. The divergence theorem is applicable to both G_1 and G_2, and we set

$$\begin{aligned}
\iiint_G (\text{ div } \mathbf{F})\, dV &= \iiint_{G_1} (\text{ div } \mathbf{F})\, dV + \iiint_{G_2} (\text{ div } \mathbf{F})\, dV \\
&= \iint_{S_2^+} (\mathbf{F}\cdot\mathbf{n})\, dS + \iint_{S_1^+} (\mathbf{F}\cdot\mathbf{n})\, dS + \iint_{S_3^+} (\mathbf{F}\cdot\mathbf{n})\, dS \\
&\quad + \iint_{S_2^-} (\mathbf{F}\cdot\mathbf{n})\, dS + \iint_{S_1^-} (\mathbf{F}\cdot\mathbf{n})\, dS - \iint_{S_3^+} (\mathbf{F}\cdot\mathbf{n})\, dS \\
&= \iint_{S_1} (\mathbf{F}\cdot\mathbf{n})\, dS + \iint_{S_2} (\mathbf{F}\cdot\mathbf{n})\, dS \\
&= \iint_{S_1} (\mathbf{F}\cdot\mathbf{n})\, dS,
\end{aligned}$$

where, the normal to S_1 is directed towards origin, while the normal to S_2 is directed outward, away from origin.

51.1.4 Example :

Similarly, consider the region G bounded by the surface S obtained by revolving a circle of redius b with center at $(0,a,0)$ about z-axis , $a > b$.

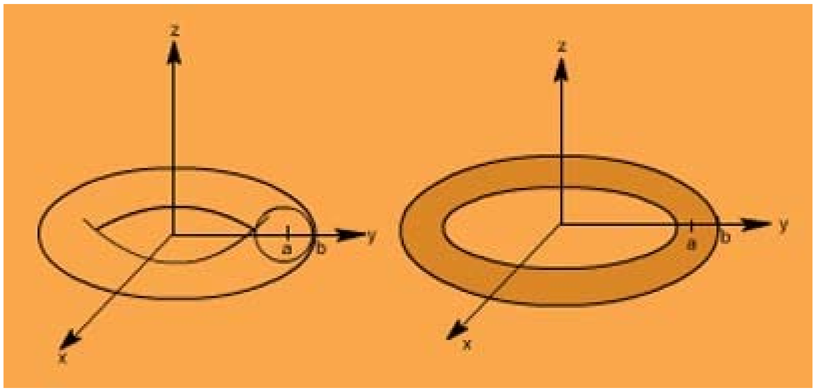


Figure: Torus

with axis being the z-axis. Then it is a simple xy- solid, its projection on xy-plane being the annulus region, as shown in figure. However, it is not a simple yz-solid or a simple xz-solid. We can divide G into four region by G_1, G_2, G_3 and G_4 by planes parsing through z-axis and parallel to xz and yz planes.

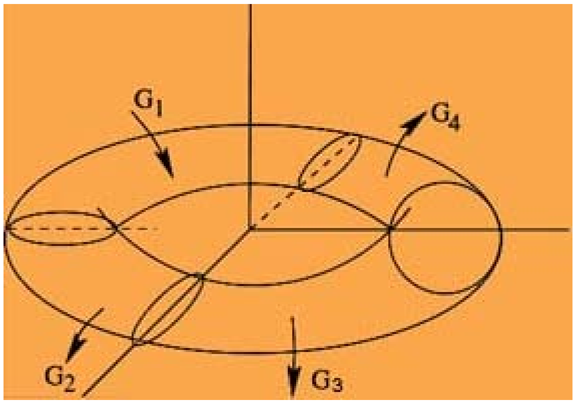


Figure 4. Forms as a unioin of simple surfaces

51.1.5 Example :

Let us verify Divergence theorem for the solid G bounded by the paraboloid $z=4-x^2-y^2$ and the xy-plane, the vector field $\mathbf{F}(x,y,z)=2z\mathbf{i}+x\mathbf{j}+y^2\mathbf{k}$. The surface binding the region G is S_1, the paraboloid $z=4-x^2-y^2$ and the surface S_2, the xy-plane. For S_2, the outward unit normal is $\mathbf{n}_2=-\mathbf{k}$. For S_1, the outward unit normal is

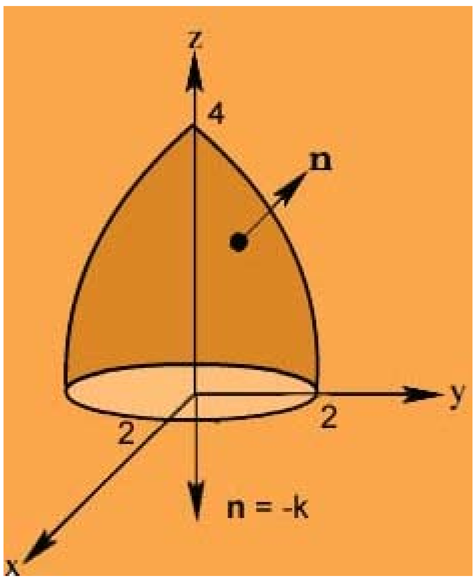


Figure 226. The Paraboloid

$$\mathbf{n}_1=\frac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{4\,x^2+4\,y^2+1}}$$

Thus

$$\begin{aligned}\iint_S(\mathbf{F}\cdot\mathbf{n})\,dS &= \iint_{S_1}(\mathbf{F}\cdot\mathbf{n}_1)\,dS+\iint_{S_2}(\mathbf{F}\cdot\mathbf{n}_2)\,dS\\ &= \iint_D(\mathbf{F}\cdot(2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}))\,dxdy+\iint_D(\mathbf{F}\cdot(-\mathbf{k}))\,dxdy,\end{aligned}$$

where $D=\left\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 4\right\}$. Hence

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \int_{-2}^{+2}\int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} (-y^2)\, dxdy + \int_{-2}^{+2}\int_{-\sqrt{4-y^2}}^{+\sqrt{4-y^2}} (4xz+2xy+y^2)\, dxdy$$

$$= \int_{-2}^{+2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4xz+2xy)\, dxdy$$

$$= \int_{-2}^{+2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left[4x\,(4-x^2-y^2)+2xy\right] dxdy$$

$$= \int_{-2}^{+2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left[16x-4x^3-4xy^2+2xy\right] dxdy$$

$$= \int_{-2}^{+2}\left[8x^2-x^4-2x^2y^2+x^2y\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$$

$$= \int_{-2}^{+2} 0\, dy = 0.$$

On the other hand, it is easy to check that $\operatorname{div}(\mathbf{F}) = 0.$ Thus

$$\iiint_G (\operatorname{div}\mathbf{F})\, dv = \iiint_G 0\, dv = 0.$$

This verifies divergence theorem.

Practice Exercises

1. Verify divergence theorem for the following:

   $\mathbf{F}(x,y,z) = x\mathbf{i}+y\mathbf{j}+z\mathbf{k},$

   G is the solid bounded by the three coordinate planes and the plane

   $2x+2y+2z=6.$

   Answer: $\dfrac{63}{2}$

2. Let G be the solid bounded by the cylinder $x^2+y^2=4,$ the plane $x+z=6$ and the plane $z=0.$ Verify divergence theorem for this solid where

   $\mathbf{F}(x,y,z) = (x^2+\sin z)\mathbf{i}+(xy+\cos z)\mathbf{j}+e^y\mathbf{k}.$

   Answer: -12π

3. Verify divergence theorem for the region G enclosed by the cylinder $x^2+y^2=9,$ the planes $z=0, z=2$ and $\mathbf{F}(x,y,z) = x^3\mathbf{i}+y^3\mathbf{j}+z^3\mathbf{k}.$

   Answer: 279π

Recap

In this section you have learnt the following

- Divergence theorem, which relates line integral with a double integral.

[Section 51.2]

Objectives

In this section you will learn the following :

- Some applications of the divergence theorem.

51.2.1 Example (Computation of surface integrals):

Consider the solid G bounded by the there coordinate planes and the plane $2x+2y+z=6$. Let S be the surface bounding this region. S is a peicewise smoth surface being the union of simple surfaces.

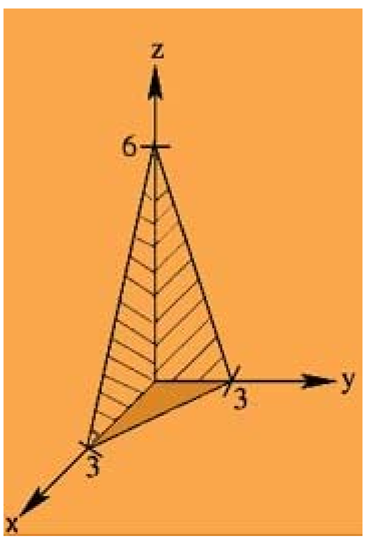


Figure: The surface S

For a given vector field $\mathbf{F}$, computing the surface integral

$$\iint_S (\mathbf{F}\cdot\mathbf{n})dS$$

is complex as the surface S is made up of four subsurface. However, this can be easily computed by computing a single triple integral. For example, if

$$\mathbf{F}(x,y,z) = x\mathbf{i}+y^2\mathbf{j}+\mathbf{k},$$

then by divergence theorem

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \iiint_G (\operatorname{div}\ \mathbf{F})\, dxdydz$$

$$= \int_0^3 \left(\int_0^3 \left(\int_0^{6-2x-2y} (2+2y) \right) dx \right) dy$$

$$= \int_0^3 \left(\int_0^3 \left[2z + 2zy \right]_0^{6-2x-2y} dx \right) dy$$

$$= \int_0^3 \left(\int_0^3 \left(12 - 4x + 8y - 4xy - 4y^2 \right) dx \right) dy$$

$$= \int_0^3 \left(18 + 6y - 10y^2 + 2y^3 \right) dy$$

$$= \left[18y + 3y^2 - \frac{10y^3}{3}\ \frac{y^4}{2} \right]_0^3$$

$$= 63/2.$$

51.2.2 Example:

Let G be a region in $\mathbb{R}^3$ enclosed between two non intersecting surfaces S_1 and S_2. Suppose both S_1 and S_2 are orientable (for example S_1 and S_2 are concentric spheres). Let S_1 be the inner-surface of G and S_2 be the outer-surface of G. Then

$$\iint_{S_1} (\mathbf{F}\cdot\mathbf{n})\, dS + \iint_{S_2} (\mathbf{F}\cdot\mathbf{n})\, dS = \iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \iiint_{\mathrm{G}} \operatorname{div}\ (\mathbf{F})\, dV.$$

If $\mathbf{F}$ is such that $\operatorname{div}(\mathbf{F}) = 0$ on G, then we have

$$\iint_{S_1} (\mathbf{F}\cdot\mathbf{n})\, ds + \iint_{S_2} (\mathbf{F}\cdot\mathbf{n})\, ds = 0,$$

where $\mathbf{n}$ is the unit outward normal to $S = S_1 \cup S_2$. This helps us to compute either of the above flux integrals in terms of the other. For example, let

$$\mathbf{F}(x,y,z) = -\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\left(x^2+y^2+z^2\right)^{3/2}},$$

and $S = S_1 \cup S_2$, where S_1 is a sphere of radius $a > 0$ and S_2 is a closed surface including the region $\left\{(x,y,z) \mid x^2+y^2+z^2 \le a^2\right\}$. Then, as $\operatorname{div}(F) = 0,$ by divergence theorem

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS + \iint_{S_a} (\mathbf{F}\cdot\mathbf{n}')\, dS = 0,$$

where S_a is the sphere centered at origin and of radius a. Note that $\mathbf{n}$ in the first integral is the outward normal, while in S_a, $\mathbf{n}'$ is the normal pointing towards origin. Thus,

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS + \iint_{S_a} (\mathbf{F}\cdot\mathbf{n})\, dS = 0$$

where, in both integrals, $\mathbf{n}$ is the outward pointing normal.

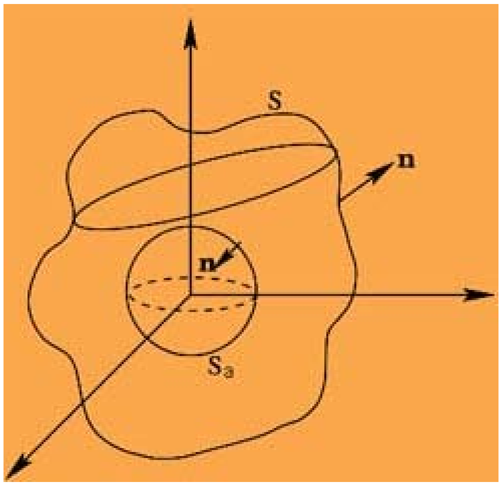


Figure:The region $S \cup S_a$

For
$S_a = \{(x,y,z) \,|\, \phi(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0\}$,

$$\mathbf{n} = \frac{2\,x\,\mathbf{i} + 2\,y\,\mathbf{j} + 2\,z\,\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{1}{a}\{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\},$$

the normalized position vector. Hence,

$$\mathbf{F}\cdot\mathbf{n} = \frac{x^2+y^2+z^2}{a\,(x^2+y^2+z^2)^{3/2}} = \frac{1}{a\sqrt{x^2+y^2+z^2}},$$

and we have

$$\begin{aligned}\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS &= \iint_{S_a} (\mathbf{F}\cdot\mathbf{n})\,dS \\ &= \iint_{S_a} \frac{1}{a^2}\,dS \\ &= \frac{4\,\pi\,a^2}{a^2} \\ &= 4\,\pi.\end{aligned}$$

51.2.3 Green's Identity and properties of Harmonic functions:

Let f, g be two scalar-fields which are twice continuously differentiable in a region which includes a solid G and its boundary surface S. Let $\mathbf{F} := f(\nabla g)$. Then,

$$\begin{aligned}\operatorname{div}(\mathbf{F}) = \nabla\cdot\mathbf{F} &= \nabla\cdot(f\,\nabla\,g) \\ &= (\nabla\,f)\cdot(\nabla\,g) + f(\nabla^2\,g),\end{aligned}$$

where

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right),$$

called the **Laplacian operator** . Thus, by the divergence theorem applied to $\mathbf{F} = f(\nabla g)$ over G, we get

$$\iiint_G \operatorname{\mathbf{div}}(\mathbf{F})\,dxdydz = \iint_S (\mathbf{F}\cdot\mathbf{n})\,dS,$$

i.e.,

$$\iiint_G \left(\nabla f \cdot \nabla g + f\,\nabla^2 g\right) dV = \iint_S \left(F\,(\nabla g \cdot \mathbf{n})\right) dS$$

$$= \iint_S f \left(\frac{\partial g}{\partial \mathbf{n}}\right) dS, \qquad \text{----------(74)}$$

where $\partial g/\partial \mathbf{n}$ is the directional derivative of g in the direction of $\mathbf{n}$. The equation (74) is called **Green's first identity** . Interchanging f and g in the above equation, we get

$$\iiint_G \left(g\,\nabla^2 f + \nabla f \cdot \nabla g\right) dV = \iint_S g \left(\frac{\partial f}{\partial \mathbf{n}}\right) dS \qquad \text{----------(75)}$$

Subtracting (75) from (74), we get

$$\iiint_G \left(f\,\nabla^2 g - g\,\nabla^2 f\right) dV = \iint_S \left(f\,\frac{\partial g}{\partial \mathbf{n}} - g\,\frac{\partial f}{\partial \mathbf{n}}\right) dS. \qquad \text{----------(76)}$$

This is called **Green's second identity** . Some of the particular cases of these identities give us the following consequences:

51.2.4 Special cases of Green's Identity :

1. Let $f \equiv 1$ in (76) . Then, as $\nabla f = 0$, we have

   $$\iiint_G \nabla^2 g\,dV = \iint_S \frac{\partial g}{\partial \mathbf{n}}\,dS.$$

   Thus, if $\nabla^2 g = 0$, (in which case the scalar field g is called **harmonic**), we have

   $$\iint_S \frac{\partial g}{\partial \mathbf{n}}\,dS = 0.$$

   The integral is the average of the rate of change of g along the normal on S . Thus, for a harmonic function on G , average of its rate of change on S is zero. This is called the **Laplace theorem** .

2. Let $f = g$ in (75). Then,

   $$\iiint_G \left(f\,\nabla^2 f + |\nabla f|^2\right) dV = \iint_S \left(f\,\frac{\partial f}{\partial \mathbf{n}}\right) ds.$$

   Suppose, either $f \equiv 0$ or $\partial f/\partial \mathbf{n} = 0$ on S. Then,

   $$\iiint_G \left(f\,\nabla^2 f + |\nabla f|^2\right) dV = 0.$$

   Further, if f is harmonic, i.e., $\nabla^2 f = 0$, we have

   $$\iiint_G |\nabla f|^2\,dV = 0,$$

   which implies that $\nabla f = 0$ in G and hence $f \equiv C$ in G . Thus, for a harmonic function in G , if either

$$\frac{\partial f}{\partial \mathbf{n}} = 0 \text{ on } S \text{ or } f = 0 \text{ on } S, \text{ then } f \equiv C \text{ in } G.$$

In particular, if as f is continuous, then

$$\left.\begin{array}{lll} f \equiv 0 & \text{on} & S, \\ \text{or} & & \\ \dfrac{\partial f}{\partial \mathbf{n}} = 0 & \text{on} & S \\ \text{and} & & \\ \nabla^2 f = 0 & \text{in} & G, \end{array}\right\} \qquad \text{----------(77)}$$

then, $f \equiv 0$ in G also. As a particular case, if f_1, f_2 are two harmonic function in G such that $f_1 = f_2$ on S, then $(f_1 - f_2)$ satisfies equations (77), and hence $f - g = 0$ in G, i.e., $f = g$ in G.

Thus, a harmonic function in G uniquely determined by its values on the boundary of G. We close this section by giving some examples of harmonic functions.

51.2.5 Examples of harmonic functions:

1. **The flow of heat in a body** : The equation governing the heat flow is

$$\frac{\partial U}{\partial t} = c^2 \nabla^2 U,$$

when C is a constant and $U(x, y, z, t)$ represents the temperature of the body at a point (x, y, z) at time t. If the flow of heat is 'steady', i.e., $U(x, y, z, t)$ does not depend upon temperature, then $\nabla^2 U = 0$, i.e., the temperature of steady heat flow is a harmonic function.

2. Consider the gravitational force on a particle B of mass m at any point (x, y, z) due to a mass M at a fixed point $A(x_0, y_0, z_0)$. The gravitation force is

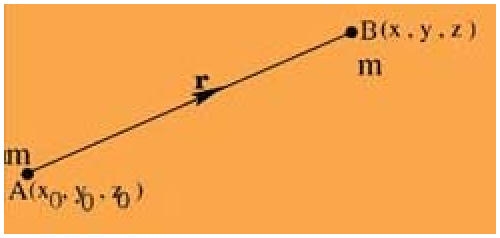


Figure: Force of gravitation between point masses

$$\mathbf{F}\,(x, y, z) = \frac{-C\,\mathbf{r}}{\left[\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}\right]^3} = \frac{-C\,\mathbf{r}}{r^3},$$

where

$$\mathbf{r} = (x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j} + (z + z_0)\,\mathbf{k},$$

$$C = G\,Mm, \text{ and } r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

We also saw that, $\operatorname{div}(\mathbf{F}) = 0$. Further, if

$$f(x,y,z) = \frac{C}{r}, \text{ then } \nabla f = \mathbf{F},$$

and such a scalar field f is called the 'potential' of the force field $\mathbf{F}$. Then, in this case,

$$\nabla^2 f = -C\,\nabla^2\left(\frac{1}{r}\right) = 0.$$

If a mass is distributed in a region R in space with density $\rho(x_0, y_0, z_0), (x_0, y_0, z_0) \in R,$ then the corresponding potential of the force field at a point (x, y, z) not occupied by the mass will be given by

$$f\ (x,y,z) = m\,G\ \iiint_G \left(\frac{\rho\ (x_0, y_0, z_0)}{r}\right) dx_0 dy_0 dz_0.$$

Hence,

$$\nabla^2 f = m\,G\left(\ \iiint_G \rho\ (x_0, y_0, z_0)\ \nabla^2\left(\frac{1}{r}\right) dx_0 dy_0 dz_0\right) = 0.$$

Thus, the potential of the gravitational force field is a harmonic function at every point which is not occupied by matter.

51.2.6 Independence of divergence of the coordinate system:

By the mean value theorem for triple integrals,

$$\iiint_R (\ \operatorname{div}(\mathbf{u}))\,dV = V(R)\ (\ \operatorname{div}\ \mathbf{u})(P_0),$$

for some point P_0 in the closed bounded region R, where $\mathbf{u}$ is a smooth vector field in a domain that includes R along with its boundary and $V(R)$ is the volume of the region R. Then, by the divergence theorem, if S is the surface bounding the region R, and is orientable , then

$$\operatorname{div}(\mathbf{u})(P) = \frac{1}{V(R)}\ \iiint_{\mathrm{R}}\ \operatorname{div}(\mathbf{u})\,dV = \frac{1}{V(R)}\ \iint_{S\,=\,\partial R} u_n\,dS.$$

Let P be a fixed point in the region R and we apply the above discussion to the region $B(P,r)$, a small sphere centered at the point P of radius r. Then, there exists a point $P_0 \in B(P,r)$, such that

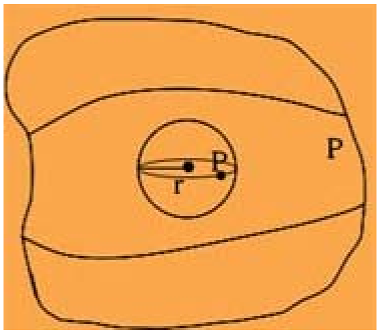


Figure: Sphere at P_0 inside R

$$\operatorname{div}(\mathbf{u})(P_0) = \frac{1}{V(B(P,r))} \iint_S \mathbf{u}_n \, dS,$$

where $V(B(P,r))$ is the volume of the sphere $B(P,r)$. If we let $r \to 0$ in the above equation, as $(P_0 \to P)$, we have

$$\operatorname{div}(\mathbf{u})(P) = \lim_{r \to 0} \left(\frac{1}{V(B(P,r))} \iint_{S = \partial(B(P,r))} \mathbf{u}_n \, dS \right). \quad (79)$$

Note that, since R and S are independent of the coordinate system, and the surface integral is a limit of approximating sums, $\operatorname{div}(u)$ is independent of the coordinate system.

51.2.7 Physical interpretation of divergence:

Recall that, the integral

$$\iint_S \mathbf{u}_n dS$$

gives the total mass of the fluid that flows across a surface S per unit time, where $\mathbf{u} = P(x,y,z)\mathbf{v}$, P being the density and $\mathbf{v}$ the velocity of the fluid. We can also interpret it as the total mass of the fluid that flows from inside of R to outside R, if $\mathbf{n}$ is the outward unit normal. Thus

$$\frac{1}{V(R)} \iint_{S = \partial(R)} \mathbf{u}_n \, dS$$

is the **average flow out of** R per unit time. Thus, equation (79) tells us that if we want to find the flow of the mass per unit volume, per unit time at a point, then this is given by the right hand side of (79), i.e., by $\operatorname{div}(\mathbf{u})(P)$. Further, if the fluid flow is steady, the fluid is incompressible, and there are no source or sink, then clearly the rate of fluid flow across a point must be zero, i.e., $\operatorname{div}(\mathbf{u}) = 0$. Conversely, if $\operatorname{div}(\mathbf{u})(P) \neq 0$, then the rate of flow across a P is not zero, hence either fluid is being produced at P or is being absorbed at P. Hence, for a steady flow of an incompressible fluid flow through R, there are no sources or sinks iff $\operatorname{div}(\mathbf{u})(P)$ =0. Note that incompressible is same as saying the density ρ is constant. Thus, $\operatorname{div}(\mathbf{u}) = 0$ iff $\operatorname{div}(\mathbf{v}) = 0$, where $\mathbf{v}$ is the velocity vector field.

Practice Exercise

Let f, g be harmonic functions in G such that $\frac{\partial f}{\partial \mathbf{n}} = \frac{\partial g}{\partial \mathbf{n}}$ on S, the boundary of G. Show that $f \equiv g + C$ on G.

1. Using divergence theorem compute the integral

   $$\iint_S \mathbf{F} \cdot \mathbf{n} dS,$$

   where S is the surface of the unit cube in $\mathbb{R}^3$ bounded by the there coordinate planes and the planes $x = 1, y = 1, z = 1,$ and

   $$\mathbf{F}(x,y,z) = 2x\,\mathbf{i} + 3y\,\mathbf{j} + z^2\,\mathbf{k}.$$

   Answer: 6

2. Find the flux of the field

   $$\mathbf{F}(x,y,z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k},$$

across the surface S consisting of the hemisphere $x^2+y^2+z^2=1, z\geq 0$ $\mathbf{F}(x,y,z)=2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}$ with base $x^2+y^2\leq 1, z=0.$

Answer: 4π

3. Use divergence theorem to verify that the volume of a solid G bounced by a closed surface S is given by either of

   the following:

   $\iint_S x\,dydz, \iint_S y\,dzdz, \iint_S z\,dxdy$.

Recap

In this section you have learnt the following

- Some applications of the divergence theorem.