Module 16 : Line Integrals, Conservative fields Green's Theorem and applications

Lecture 47 : Fundamental Theorems of Calculus for Line integrals [Section 47.1]

Objectives

In this section you will learn the following :

- Fundamental theorem of calculus for line integrals.
- Physical applications of this theorem.

47.1 Fundamental Theorems of Calculus for Line integrals

The fundamental theorem of calculus for definite integration helped us to compute $\int_a^b f(x)dx.$ If f has an anti-derivative F, then

$$\int_a^b f(x)\,dx = F(b) - F(b). \qquad \text{----------(38)}$$

In fact, if f is continuous on (a,b), then f and F are related by $f(x) = F'(x)$ for every $x \in (a,b)$, i.e., an antiderivative F of f is given by

$$F(x) := \int_a^x f(t)dt, x \in [a,b]. \qquad \text{----------(39)}$$

We shall extend both these parts of the fundamental theorem of calculus for line integrals.

47.1.1 Theorem (Fundamental theorem for line integrals):

1. Let D be an open set in $\mathbb{R}^3$ and $\phi : D \subset \mathbb{R}^3 \to \mathbb{R}$ be a continuously differentiable scalar field. Let

   $A, B \in D$ and let $C, \mathbf{r} : [a,b] \to D$, be any smooth curve in D such that initial point of C is A and final point of C is B. Then

   $$\int_C (\nabla \phi) \cdot d\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)).$$

2. Let $\mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}$ be a continuously differentiable vector-field such that F is conservative, i.e., $\mathbf{F} = \nabla \phi$ for some

   continuously differentiable scalar field ϕ on D. Then, for $A, B \in D$ and for any smooth curve

   $$\mathbf{r} : [a,b] \to D, \text{ with } \mathbf{r}(a) = A \text{ and } \mathbf{r}(b) = B,$$

   We have

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)).$$

   PROOF

   1. Consider the function $g : [a,b] \to \mathbb{R}$,

      $$g(t) := (\phi \circ \mathbf{r})(t), t \in [a,b].$$

   2. Then g is a continuously differentiable function with

$$g'(t) = (\nabla\phi)(\mathbf{r}(t)) \cdot \mathbf{r}'(t), t \in (a,b).$$

Hence, by fundamental theorem of calculus for a single variable,

$$\begin{aligned}\int_C (\nabla\phi) \cdot \mathbf{r} &= \int_a^b (\nabla\phi)(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_a^b g'(t)\, dt \\ &= g(b) - g(a) \\ &= \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)).\end{aligned}$$

(ii) This follows from (i).

We give next some applications of this theorem.

47.1.2 Applications :

1. **Independence of work done:**

   The above theorem says that if a vector-field is conservative, then the work done in moving from one point to another does not depend upon the path taken.

2. **Principle of energy conservation:**

   Let $\mathbf{F}$ be a conservative force field in a domain D with

   $$\mathbf{F}(x,y,z) = (\nabla\phi)(x,y,z)\ (x,y,z) \in D.$$

   The scalar field ϕ is called a **potential function** for the vector field $\mathbf{F}$, and the scalar field

   $$V(x,y,z) = -\phi(x,y,z), (x,y,z) \in D$$

   is called the **potential energy** of the field at the point (x,y,z). The above theorem tells us that W, the work done by $\mathbf{F}$ on a particle that moves along any path C from a point (x_0,y_0,z_0) to a point (x_1,y_1,z_1) is related to the potential energy of the body by the equation

   $$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \phi(x_1,y_1,z_1) - \phi(x_0,y_0,z_0)$$

   $$= -\left[V(x_1,y_1,z_1) - V(x_0,y_0,z_0)\right],$$

   i.e., the work done by the conservative force field is equal to the negative of the change in potential energy. Thus, in particular if C is a closed curve, then there is no change in the potential energy, and hence the work done is zero. Further, suppose the particle being moved has mass m, velocity v_0 at the initial point (x_0,y_0,z_0) and v_1 at the point (x_1,y_1,z_1). Then, the work energy relationship says that the work done is also equal to the change in kinetic-energy. Hence, for a conservative force field,

   $$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = \int_C \mathbf{F} \cdot d\mathbf{r} = -(V_1 - V_0),$$

   where

   $$V_1 = V(x_1,y_1,z_1) \text{ and } V_0 = V(x_0,y_0,z_0).$$

   Thus,

   $$\frac{1}{2}mv_1^2 + V_1 = \frac{1}{2}mv_0^2 + V_0.$$

   This equation stats that the total energy, i.e., the kinetic energy plus the potential energy, of the particle does not change if it is moved from one point to another in a conservative vector field. This is called the **principle of energy**

conservation . This is the reason that for $\mathbf{F} = \nabla\phi$, we say that $\mathbf{F}$ is conservative.

Mathematically, if a vector field $\mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ is conservative and $A, B \in D$, then the integral

$\int_{C(A,B)} \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C

joining the points A and B . It is natural to ask the question:

"Is the converse true, i.e., if $\mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ is such that for given points $A, B \in D$ the integral $\int_{C(A,B)} \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C joining A to B , is $\mathbf{F}$ conservative?"
Thus, given a vector-field with the above property, one would like to construct a potential function for it, i.e., try to extend equation (39) to line integrals. Recall that, for a function f of one variable, to construct an antiderivative in $[a,b]$, we simply defined it to be

$$\int_a^x f(t)dt.$$

In the present situation, given a point $A(x_0, y_0, z_0) \in D \subset \mathbb{R}^3$, we would like to define

$$\phi(x,y,z) := \int_{C(A,B)} \mathbf{F} \cdot d\mathbf{r}, \qquad \text{---------(40)}$$

where C is a curve in D joining $A(x_0, y_0, z_0)$ to any arbitrary point $B(x,y,z) \in D$.

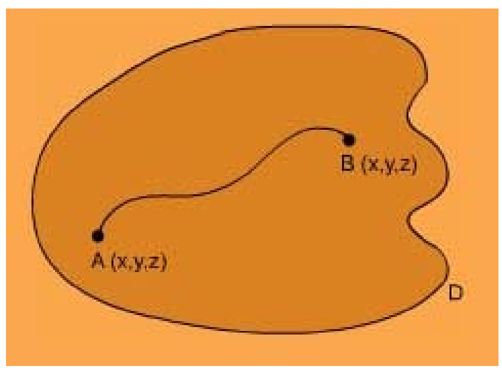

Figure: Definition of $\phi(x,y,z)$.

We observe that the given property of the vector field $\mathbf{F}$ tells us that $\phi(x,y,z)$, as given by equation (40) , is well-defined since the line-integral does not depend upon the choice of the curve C joining A to B . However, there is one problem:
How to ensure that given a point $A(x_0, y_0, z_0) \in D$, there will be at least one path $C(A,B)$,
completely in domain D, joining A to any arbitrary point $B(x,y,z) \in D$?
That this may not be always possible for some domains is illustrated in the next example.

47.1.3 Example :

Consider the domain $D \subseteq \mathbb{R}^2$, the plane $\mathbb{R}^2$ minus the shaded annulus region, i.e.,

$$D := \left\{(x,y) \in \mathbb{R}^2 \mid 3 < x^2 + y^2 < 2\right\}$$

Let

$A \in \{(x,y) \mid x^2 + y^2 < 2\}$ and $B \in \{(x,y) \mid x^2 + y^2 > 3\}$.

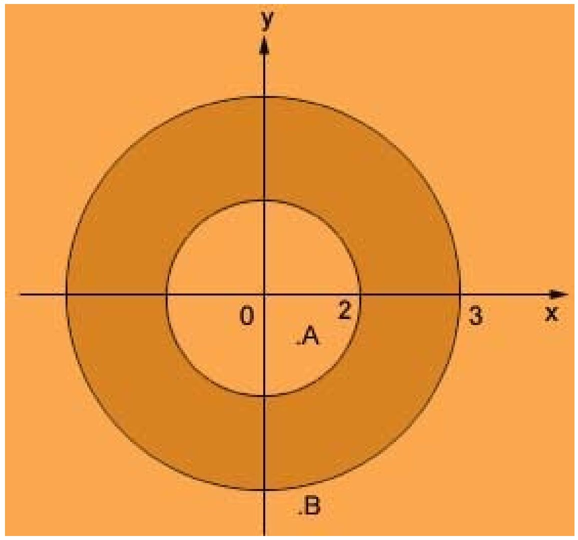

Figure: The region D

Then, the points $A, B \in D$ cannot be joined by any continuous path C completely lying inside D. To go from the point A to the point B, one has to cross the annulus region.

Practice Exercises

1. For the following show that $\mathbf{F} = \nabla \phi$, for sum ϕ. Use this to compute $\int_c \mathbf{F} \cdot d\mathbf{r}$ for the given C:

    1. $\mathbf{F} = 2\,xy^3\,\mathbf{i} + (1+3x^2\,y^2)\,\mathbf{j}$, C is the line segment joining $(0,0,0)$ with $(1,1,0)$.
    2. $\mathbf{F} = e^y\,\mathbf{i}$, C is the semi-circular path in the upper half xy-plane joining $(-1,0)$ to $(1,0)$

    Answer:

    (i) $\phi(x,y) = x^2y^3 + y$; $\int_c \mathbf{F}\, d\,\mathbf{r} = 3.$

    (ii) $\phi(x,y) = x\,e^y$, $\int_c \mathbf{F} \cdot d\mathbf{r} = -2.$

2. Using fundamental theorem of calculus, Evaluate $\oint_C xe^{-y^2}dx + [-x^2ye^{-y^2} + 1/(x^2+y^2)]dy$ around the square determined by $|x| \leq a, |y| \leq a$ traced in the counter clockwise direction.

    (Hint: $xe^{-y^2}dx + (-x^2ye^{-y^2})dy = d(x^2e^{-y^2})$.)

    Answer: $0.$

3. Find a non-zero function $g(x)$ such that the vector-field $\mathbf{F}(x,y) = h(x)\left[\left(x\sin y + y\cos y\right)\mathbf{i}\left(x\cos y - y\sin y\right)\mathbf{j}\right]$ has a potential function.

    Answer: $g(x) = e^x$

Recap

In this section you have learnt the following

- Fundamental theorem of calculus for line integrals.
- Physical applications of this theorem.

[Section 47.2]

Objectives

In this section you will learn the following :

- Necessary and Sufficient conditions on the domain and the vector field to be conservative.
- The motion of simple connectedness of a domain.

47.2 Conservative Vector Fields

The independence of the line integral $\int_C fds$ over the path C joining two points A and B can also be described in terms of closed paths as follows:

47.2.1 Theorem:

Let $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ be any scalar-field. Then the following are equivalent:

1. *For any two points $A, B \in D$,the line integral $\int_{C(A,B)} f\, ds$ does not depend upon the path C joining A and B*
2. $\oint_C f\, ds = 0$ *for every closed path C in D.*

PROOF

$(i) \Rightarrow (ii)$: Let C be any closed path in D with parameterizations

$\mathbf{r} : [a,b] \to \mathbb{R}^3, \mathbf{r}(a) = \mathbf{r}(b).$

Choose $c \in (a,b)$ and consider the curves

$C_1,\ \mathbf{r}_1(t) := \mathbf{r}(t), a \leq t \leq c, \text{ and } C_2\ \ \mathbf{r}_2(t) := \mathbf{r}.(t), c \leq t \leq b.$

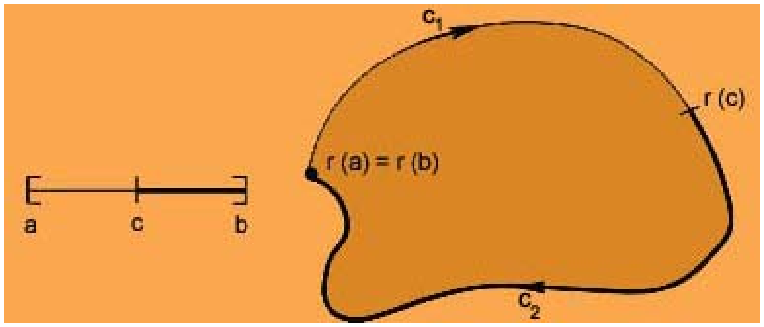

Figure 189. Caption text.

Then, C is the union of the two curves C_1 and C_2. Further, $-C_2$ is the curve with initial point $\mathbf{r}(b)=\mathbf{r}(a)$ and final point $\mathbf{r}(c)$. Thus, by (i)

$$\int_{C_1} f\, ds = \int_{-C_2} f\, ds = -\int_{C_2} f\, ds$$

Hence,

$$\oint_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds = 0$$

$(ii) \Rightarrow (i)$: Let C_1 and C_2 be any two curves in D such that both have same initial and final points. Then,

$$C = C_1 \cup (-C_2)$$

is a closed curve, and by the given property

$$0 = \oint_C f\, ds = \int_{C_1} f\, ds + \int_{-C_2} f\, ds = \int_{C_1} f\, ds - \int_{C_2} f\, ds.$$

Hence

$$\int_{C_1} f\, ds = \int_{C_2} f\, ds$$

We saw in example 47.1.3 that the existence of potential for a vector field $\mathbf{F}$ depends upon the nature of the domain of $\mathbf{F}$. This motivates our next definition.

47.2.2 Definition :

Let D be a region in $\mathbb{R}^3$. We say D is **connected** if any two points in D can be joined by a piecewise smooth curve completely in D.

47.2.3 Examples :

1. In $\mathbb{R}$, the only connected sets are intervals.

2. In $\mathbb{R}^2$, examples of connected sets are:

$$\text{open balls } B_r(x,y) = \left\{(\alpha,\beta) \in \mathbb{R}^2 \mid \| (x,y) - (\alpha,\beta) \| < r\right\},$$

$$\text{annulus regions } \left\{(x,y) \in \mathbb{R}^2 \mid r < x^2 + y^2 < s\right\} \text{ for } r,s \in \mathbb{R} \text{ with } 0 < r < s,$$

$$\text{open rectangles } (a,b) \times (c,d).$$

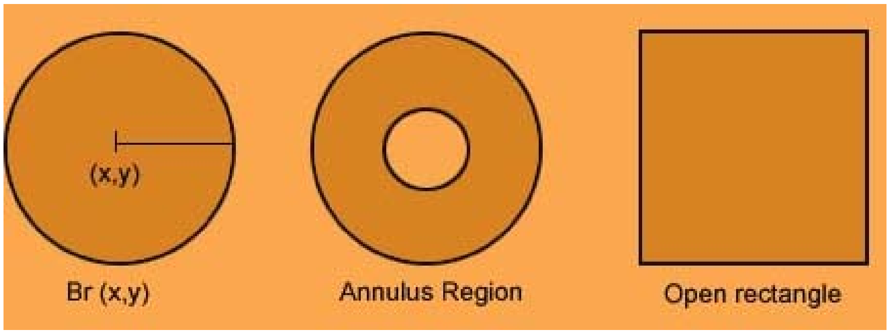

Figure: Connected subsets in $\mathbb{R}^3$

3. It is easy to see that in $\mathbb{R}^3$, every convex set is connected. In fact, by definition, any two points in a convex set can be joined by a line segment.

   For connected regions, we can answer our question: which vector fields have a potentials?

47.2.4 Theorem (Existence of potential):

Let $\mathbf{F}: D \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous vector-field, where D is an open connected set. If for any curve C in D, the line-integral $\int_C \mathbf{F}\, d\mathbf{r}$, depends only upon the initial and final point of C, then there exists a scalar field $\phi: D \subset \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F} = \nabla \phi$.

PROOF

Let us fix any point $P(x_0, y_0, z_0) \in D$. For any point $Q(x,y,z) \in D$, let $C(P,Q)$ be any smooth curve with initial point P and final point Q, at least one such curve exists as D is connected. Define

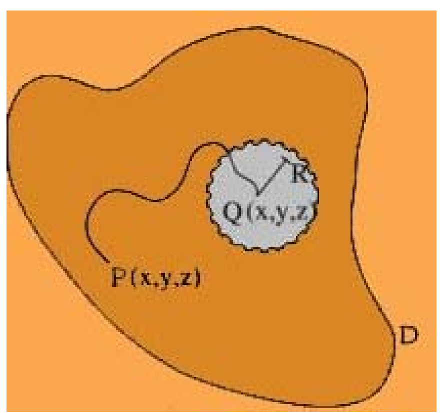

Figure 191. The path $C(P,Q)$ in D

$$\phi(x,y,z) = \int_{C(P,Q)} \mathbf{F} \cdot d\mathbf{r},\ Q(x,y,z) \in D.$$

Further, the value $\phi(x,y,z)$ does not depend upon the curve joining P to Q. Thus, the function $(x,y,z) \mapsto \phi(x,y,z)$ is well-defined. We show that ϕ is the required scalar-field. For $(x,y,z) \in D$, since D is open, we can select $r > 0$ such that

$$B_r(x,y,z) \subseteq D,$$

where $B_r(x,y,z)$ is the open ball in $\mathbb{R}^3$ with center (x,y,z) and radius r. Let $0 \le h < r$ and $\mathbf{u} = (u_1, u_2, u_3)$ be any given unit vector in such that

$\mathbb{R}^3$

$\mathbf{x}+h\mathbf{u} \in B_r(x,y,z), 0 \le h < 1.$

Let R be the point $(x+hu_1, y+hu_2, z+hu_3)$ and $L(Q,R)$ denote the line segment joining Q to R. Then,

$$\frac{\phi(\mathbf{x}+h\mathbf{u})-\phi(\mathbf{x})}{h} = \frac{1}{h}\left[\int_{C(P,R)} \mathbf{F}\cdot d\mathbf{r} + \int_{L(P,Q)} \mathbf{F}\cdot d\mathbf{r}\right].$$

Using the properties of the line integral, we have

$$\frac{\phi(\mathbf{x}+h\,\mathbf{u})-\phi(\mathbf{x})}{h} = \frac{1}{h}\int_{L(Q,R)} \mathbf{F}\cdot d\mathbf{r}.$$

Let for the line segment $L(Q,R)$ us choose the arc-length parameterization

$x \mapsto \mathbf{x}+sh\,\mathbf{u}, 0 \le s \le 1.$

Then

$$\frac{\phi(\mathbf{x}+h\,\mathbf{u})-\phi(\mathbf{x})}{h} = \frac{1}{h}\int_{L(Q,R)} \mathbf{F}\cdot d\,\mathbf{r} = \frac{1}{h}\int_0^1 \left[F(\mathbf{x}+s\,h\mathbf{u})\cdot(h\mathbf{u})\right]ds.$$

Selecting $\mathbf{u}=\mathbf{i}$, we have

$$\begin{aligned}\frac{\phi(\mathbf{x}+h\,\mathbf{i})-\phi(\mathbf{x})}{h} &= \frac{1}{h}\left[\int_0^1 \mathbf{F}(\mathbf{x}+s\,h\,\mathbf{i})\cdot(h\mathbf{i})\right]ds \\ &= \int_0^1 F_1(\mathbf{x}+s\,h\,\mathbf{i})\,ds \\ &= \frac{1}{h}\int_0^h F_1(\mathbf{x}+s\,\mathbf{i})\,ds.\end{aligned}$$

If we write

$$g(t) := \int_0^t F_1(\mathbf{x}+s\,\mathbf{i})\,ds, t \in \mathbb{R},$$

then $g(t)$ is differentiable at $t=0$, with $g'(0) = F_1(\mathbf{x})$. Hence,

$$\lim_{h\to 0}\left[\frac{\phi(\mathbf{x}+h\,\mathbf{i})-\phi(\mathbf{x})}{h}\right] = \lim_{h\to 0}\left[\frac{g(h)-g(0)}{h}\right] = g'\ (0) = F_1(\mathbf{x}).$$

Thus, ϕ is differentiable with respect to x and

$$\frac{\partial\phi}{\partial x}(\mathbf{x}) = F_1(\mathbf{x}).$$

Similarly, by choosing $\mathbf{u}=\mathbf{j}$ and $\mathbf{u}=\mathbf{k}$, we will get $\nabla\phi = \mathbf{F}$.

Theorems 47.2.1 and 47.2.4 give us the following:

47.2.5 Theorem :

Let $D \subset \mathbb{R}^3$ be an open connected subset of $\mathbb{R}^3$ and $\mathbf{F}: D \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous vector-field. Then the following statements are equivalent :

1. *There exists a scalar-field $\phi: D \subset \mathbb{R}^3 \to \mathbb{R}$ such that*

$\mathbf{F} = \nabla \phi,$

i.e., $\mathbf{F}$ is conservative.

2. *For any two points $P, Q \in D$, the line integral $\int_{C(P,Q)} \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C joining P and Q.*

3. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$, *for every closed smooth curve C in D.*

PROOF

$(i) \Rightarrow (ii)$:

If $\mathbf{F} = \nabla \phi$, then by theorem $47.1.1$ for any curve C joining $P, Q \in D$,

$$\int_{C(P,Q)} \mathbf{F} \cdot d\mathbf{r} = \phi(Q) - \phi(P).$$

Hence (ii) holds.

$(ii) \Rightarrow (i)$

Given that (ii) holds, by theorem $47.2.3$, there exists a scalar field $\phi : D \to \mathbb{R}$ such that $\mathbf{F} = \nabla \phi$. Hence (i) holds.

$(ii) \Longleftrightarrow (iii)$:

We have already proved this in theorem $47.2.3$

In order to be able to use theorem $47.2.3$ effectively, i.e., given a vector-field $\mathbf{F}$ to be able to check whether it is conservative or not, one has to verify the condition that for all points $P, Q \in D$ and all curves $C(P,Q)$ joining P to Q, the line integral $\int_{C(P,Q)} \mathbf{F} \cdot d\mathbf{r}$ is independent of the curve C. This condition seems difficult to check. One aim is to find some verifiable necessary and sufficient conditions for $\mathbf{F}$ to be conservative. A simple necessary condition for $\mathbf{F}$ to be conservative is given by our next theorem.

47.2.6 Theorem (Necessary Condition for $\mathbf{F}$ to be conservative):

Let $D \subset \mathbb{R}^3$ be an open connected set and $\mathbf{F} : D \to \mathbb{R}^3$ be a continuously differentiable vector-field. If

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

is conservative, then

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}, 1 \le i \le 3, 1 \le j \le 3, \text{ i.e., } \operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = 0.$$

PROOF

Since $\mathbf{F}$ is conservative,

$\mathbf{F} = \nabla\phi$ for some scalar field ϕ.

Since $\mathbf{F}$ is continuously differentiable, ϕ is twice-continuously differentiable, and we have

$$F_i = \frac{\partial\phi}{\partial x_i}, 1 \le i \le 3.$$

Since ϕ is twice continuously differentiable, we get

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial^2\phi}{\partial x_j \partial x_i} = \frac{\partial^2\phi}{\partial x_i \partial x_j} = \frac{\partial F_j}{\partial x_i}, 1 \le i, j \le 3,$$

proving the required claim.

The above theorem is useful in verifying that a scalar field $\mathbf{F}$ is not conservative.

47.2.7 Example:

Let
$\mathbf{F} = x\mathbf{i} + xy\mathbf{j}, (x, y, z) \in \mathbb{R}^3$.

Then,

$$F_1(x,y,z) = x, F_2(x,y,z) = xy \text{ and } F_3(x,y,z) = 0.$$

Since

$$\frac{\partial F_1}{\partial y} = 0 \neq y = \frac{\partial F_2}{\partial x},$$

the vector field $\mathbf{F}$ is not conservative.

We give next an example to show that the condition of theorem 47.2.6, i.e., $\operatorname{curl}((\mathbf{F})) = 0$ is only necessary, and not sufficient.

47.2.8 Example:

Let
$$\mathbf{F}(x,y,z) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}, (x,y) \neq (0,0).$$

Note that,

$$\left\{(x,y,z) \in \mathbb{R}^3 | (x,y) = (0,0)\right\}$$

is the set $(\{0\} \times \{0\} \times \mathbb{R}) \subset \mathbb{R}^3$. Thus, $\mathbf{F} : D \to \mathbb{R}$ is a continuously differentiable vector field on an open connected set $D := \mathbb{R}^3 \setminus \left\{(0,0,z) | z \in \mathbb{R}\right\}$ with

$$F_1(x,y,z) = -\frac{y}{x^2 + y^2}, F_2(x,y,z) = \frac{y}{x^2 + y^2}.$$

Thus,

$$\frac{\partial F_1}{\partial y} = -\frac{\left(x^2 + y^2\right) - 2y^2}{\left(x^2 + y^2\right)^2} = -\frac{x^2 - y^2}{x^2 + y^2}$$

and

$$\frac{\partial F_2}{\partial x} = -\frac{(x^2+y^2)-2x^2}{(x^2+y^2)} = -\frac{x^2-y^2}{x^2+y^2}.$$

Hence

$$\frac{\partial F_1}{\partial y}(x,y,z) = \frac{\partial F_2}{\partial x}(x,y,z), \text{ for all } (x,y,z) \in D.$$

However, if we consider C, the curve $x^2+y^2=1$, in D, with parameterization

$$x(\theta) := \cos\theta, y(\theta) := \sin\theta, 0 \le \theta < 2\pi,$$

we have

$$\oint_C \mathbf{F}.dr = \int_0^{2\pi} (-\sin\theta\,\mathbf{i}, \cos\theta\,\mathbf{j}).(-\sin\theta\,\mathbf{i}, \cos\theta\,\mathbf{j})\,d\theta = 2\pi \ne 0$$

In fact, if we take the closed curve $\tilde{C}$

$$\mathbf{r}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, 0 \le \theta \le 4\pi,$$

i.e., the curve that circles the z-axis twice, then

$$\oint_C \mathbf{F}.dr = 4\pi.$$

Thus, though $\mathbf{F}$ satisfies conditions of theorem, it is not conservative. To make the condition $\text{curl}((\mathbf{F})) = 0$ to be sufficient also for $\mathbf{F}$ to be conservative, one has to impose more conditions on the domain D of $\mathbf{F}$.

47.2.9 Definition :

1. A subset $D \subset \mathbb{R}^2$ is said to be **simply connected** if no simple closed curve in D encloses points that are not in the region D. Intuitively, in $\mathbb{R}^2$ a set without 'holes' is simply connected.

2. A region $D \subset \mathbb{R}^3$ is said to be **simply connected** if S for every simple closed curve C in D there exists a surface S in D whose boundary is C.

47.2.10 Examples:

1. For example, the region enclosed by a circle, ellipse, a rectangular path are all simply connected sets in $\mathbb{R}^2$. The region $D = \mathbb{R}^2 \setminus \{(x,y) | 2 \le x^2+y^2 \le 3\}$ is not simply connected. In particular is not simply connected. There are closed curves C that enclose points not in D, for example origin.

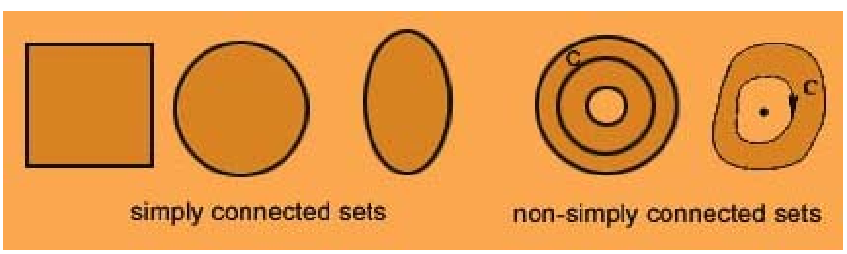

Figure: Simply connected and non- simply connected sets in $\mathbb{R}^2$

2. In the interior of a sphere is simply connected. Interior of two concentric spheres is also simply connected. Torus is

$\mathbb{R}^3$,

not simply connected. For example, in the figure below, the curve C is not the boundary of any surface in D.

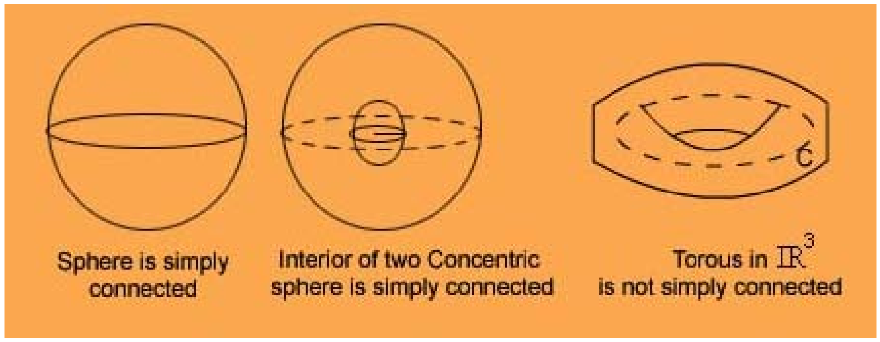

Figure: Simply connected and non- simply connected sets in $\mathbb{R}^3$

47.2.11 Note:

A more mathematically rigorous definition of simple connectedness is the following. Let $D \subset \mathbb{R}^3$ be connected. We say D is simply connected if given any two points A and B in D and any two curves $\mathbf{r}_1$ and $\mathbf{r}_2$ in D,

both having initial point A and final point B, the curve $\mathbf{r}_1$ can be continuously deformed to $\mathbf{r}_2$, i.e., there exists a continuous function

$F : [0,1]\times[0,1] \to \mathbb{R}^3$ such that $F(0,s) = \mathbf{r}_1(s)$ and $F(1,s) = \mathbf{r}_2(s)$ for all $s \in I$.

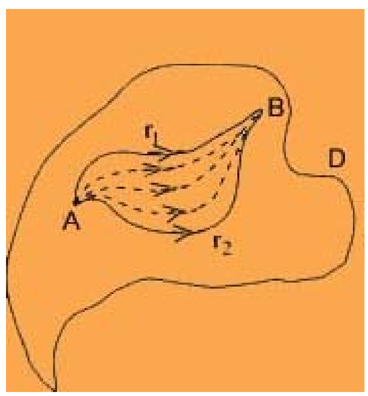

Figure: Continuous deformation of $\mathbf{r}_1$ to $\mathbf{r}_2$

We state a necessary and sufficient condition for a vector field to be conservative. We shall prove this in module 18 (theorem 54.1.5)

47.2.12 Theorem (Sufficient condition for a field to be conservative) :

Let $D \subset \mathbb{R}^3$ be a simply connected open set and $\mathbf{F} : D \to \mathbb{R}^3$ be a continuously differentiable vector-field. If curl $(\mathbf{F}) = 0$, then there exists a scalar-field $\phi : D \to \mathbb{R}$ such that $\mathbf{F} = \nabla\phi$, i.e., F is conservative.

47.2.13 Example (Calculation of potential function):

Let us consider the vector field

$$\mathbf{F}(x,y,z) = (y^2z^2 \cos x - 4x^3z)\mathbf{i} + (2z^3y \sin x)\mathbf{j} + (3y^2z^2 \sin x - x^4)\mathbf{k},\ (x,y,z) \in \mathbb{R}^3.$$

In case $\mathbf{F}$ is to be conservative with potential $\phi(x,y,z)$, we should have

$$\frac{\partial \phi}{\partial x} = y^2 z^3 \cos x - 4x^3 z,$$

$$\frac{\partial \phi}{\partial y} = 2z^3 y \sin x, \qquad \text{----------}(42)$$

and

$$\frac{\partial \phi}{\partial z} = 3y^2 z^2 \sin x - x^4. \qquad \text{----------}(43)$$

A general function $\phi(x,y,z)$ which satisfies (41) can be obtained as follows. First we integrate (41) with respect to x. This gives us

$$\phi(x,y,z) = y^2 z^3 \sin x - x^4 z + \alpha(y,z), \qquad \text{----------}(44)$$

where $\alpha(y,z)$ is a continuously-differentiable function of y,z -variables. But then, differentiating (44) with respect to y and using (42), we have

$$\frac{\partial \phi}{\partial y} = 2yz^2 \sin x + \frac{\partial \alpha}{\partial y} = 2yz^3 \sin x .$$

Hence,

$$\frac{\partial \alpha}{\partial y}(y,z) = 0 .$$

Thus $\alpha(y,z)$ depends upon z alone. Let us take $\alpha(y,z) = \beta(z)$. Then, from (44) we have

$$\phi(x,y,z) = y^2 z^3 \sin x - x^4 z + \beta(z). \qquad \text{---------}(45)$$

Once again differentiating (45) and using (43), we have

$$\frac{\partial \phi}{\partial z} = 2y^2 z^2 \sin x - x^4 - \beta'(z) = 3y^2 z^2 \sin x - x^4.$$

Hence $\beta'(z) = 0 \ \forall\, z$, implying that $\beta(z) \equiv C$, a constant. Hence, (45) gives

$$\phi(x,y,z) = y^2 z^3 \sin x - x^4 z + C$$

Now we can check that $\nabla \phi = \mathbf{F}$. Hence, $\mathbf{F}$ is conservative with potential ϕ.

47.2.14 Note :

Recall that, in example 47.2.8 , we showed that the vector-field

$$F(x,y,z) = -\frac{y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}, (x,y,z) \in \mathbb{R}^3 \setminus z-\text{axis}$$

satisfies the conditions that $$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2},$$

however, $\mathbf{F}$ is not conservative. The natural question arises: why can we not apply the process of the previous example? The reason is that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of the path, and hence (as in example 47.2.13) we cannot always select path parallel to the axes to integrals. Let us analyze this in detail. Let , for some in

$\mathbf{F} = \nabla\phi$ ϕ

$D := \left\{ (x,y,z) \in \mathbb{R}^3 \mid (x,y) \neq (0,0) \right\}$.

Then

$$\frac{\partial \phi}{\partial x}(x,y,z) = -\frac{y}{x^2+y^2}. \qquad \text{----------(46)}$$

Hence, if we integrate, we get

$$\phi(x,y,z) = -y\int\left(\frac{1}{y^2+x^2}\right)dx + \alpha(y).$$

This will give us

$$\phi(x,y,z) = -\left(\tan^{-1}\frac{x}{y}\right) + \alpha(y).$$

This is where the problem arises. The function

$$\tan^{-1}\left(\frac{x}{y}\right), (x,y,z) \in D$$

is not single valued, since $\tan : \mathbb{R} \to \mathbb{R}$ is not a one-one-function. In fact, if we consider

$$\tan : (-\pi/2, \pi/2) \to \mathbb{R}, \qquad \theta \mapsto \tan\theta,$$

then it is one-one, onto and the corresponding inverse function $\tan^{-1}$is called the **principle branch** of $\tan^{-1}\theta$. Thus, in order to make the above calculations possible, we can decide to select the principle branch of $\tan^{-1} x$. But then, this function is not everywhere continuous. In order to set a valid solution, we can modify our domain, as shown in the next example.

47.2.15 Example

Consider

$$D := \left\{ (x,y,z) \in \mathbb{R}^3 \mid x \neq 0, y > 0 \right\},$$

i.e., D consists of the space $\mathbb{R}^3$ minus the plane consisting of negative part of the x -axis (including 0) and the z -axis. Define

$$\phi(x,y,z) := \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & if \quad x < 0 \\ \pi/2 & if \quad x = 0, y = 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & if \quad x < 0, y > 0 \\ \tan^{-1}\left(\frac{y}{x}\right) - \pi & if \quad x < 0, y < 0 \end{cases}$$

Then ϕ is the required potential for

$$\mathbf{F}(x,y,z) = -\frac{y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}, (x,y,z) \in D.$$

Practice Exercises

1. Show that

$$\mathbf{F}\,(x,y,z) = y\,\mathbf{i} + (z\cos yz + x)\,\mathbf{j} + (y\cos yz)\,\mathbf{k}$$

is a conservative vector field by finding a function ϕ such that $\mathbf{F} = \nabla \phi$.

Answer: $\phi(x,y,z) = xy + \sin y z$

2. Show that $\mathbf{F} = \nabla \phi$, for some ϕ, where

   $\mathbf{F}(x,y,z) = (2xyz + \sin x)\,\mathbf{i} + x^2 z\,\mathbf{j} + x^2 y\,\mathbf{k}$,
   and hence compute $\int_c F \cdot ds$, where C is $r(t) = \cos^5 t\,\mathbf{i} + \sin^3 t\,\mathbf{j} + t^4\,\mathbf{k}$, $0 \le t \le \pi$

   Answer:

   $\phi(x,y,z) = x^2yz - \cos x$
   $\int_C F \cdot ds = -2$

3. Show that the line integral

   $$\int_C y\,z\,dx + x\,z\,dy + y\,x^2\,dz$$

   is not independent of the path C.

4. Show that the vector field

   $$\mathbf{F}(x,y) = \frac{y}{x^2+y^2}\,\mathbf{i} - \frac{x}{x^2+y^2}\,\mathbf{j}$$
   is not conservative by evaluating the integrals

   $$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \text{ and } \int_{C_2} \mathbf{F} \cdot d\mathbf{r},$$
   where C_1 is the curve $r_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \le t \le \pi$ and C_2 is the curve

   $r_2(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j}$, $0 \le t \le \pi$

Recap

In this section you have learnt the following

- Necessary and Sufficient conditions on the domain and the vector field to be conservative.
- The motion of simple connectedness of a domain.