Module 2 : Limits and Continuity of Functions

Lecture 6 : Properties of Continuous Functions [Section 6.1 : Discontinuities]

Objectives

In this section you will learn the following

- The discontinuity of a function at a point.
- Various types of discontinuities.

6.1 Discontinuities of a function

6.1.1 Definitions:

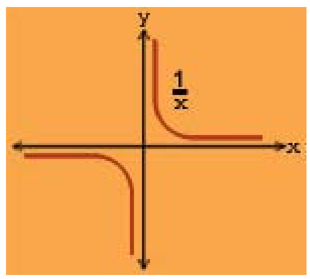
Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$ and $c \in A$.

(1) We say f is discontinuous at a point c if it is not continuous at c. This can happen in any of the following
ways:

(i) $\lim_{x\to c} f(x)$ does not exist. The nonexistence of $\lim_{x\to c} f(x)$ can happen in two ways :

(a) Essential discontinuity:

Either $\lim_{x\to c^+} f(x)$, or $\lim_{x\to c^-} f(x)$, or both does not exist. Such a point of discontinuity is called essential discontinuity of f. For example, for $f(x) = \dfrac{1}{x}$ at $x \neq 0$, both the left and the right hand limit of f at $c = 0$ do not exist. Thus, no value for $f(0)$ will make f continuous at $c = 0$.



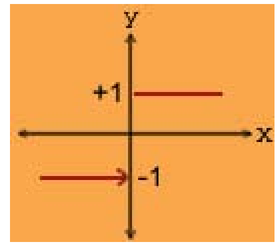
(b) Jump discontinuity:

If both $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exist, but are not equal,

then such a point of discontinuity is called jump discontinuity of f.
For example

$$f(x) = \begin{cases} +1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0, \end{cases}$$

has jump discontinuity at $x = 0$.



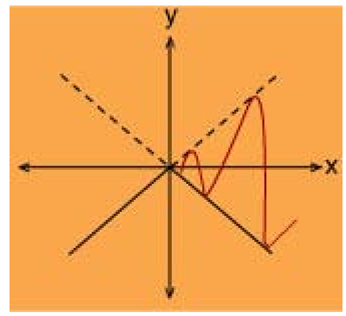
(ii) Removable discontinuity :

If $\lim_{x \to c} f(x) = l$ exists but is not equal to $f(c)$, then such a discontinuity is called a removable discontinuity, for the function can be redefined as $f(c) = l$ to make it continuous.

For example, the function

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ \alpha & \text{if } x = 0, \end{cases}$$

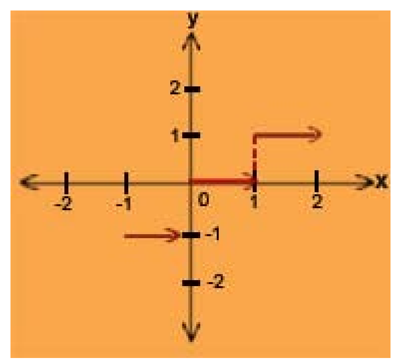
as shown in the figure has removable discontinuity at $x = 0$ $\alpha = 0$ with $f(0) = 0$.



Graph of $f(x)$

6.1.2 **Examples**:

(i) The function $f(x) = [x]$, is not continuous at *c*, if *c* is an integer, since the left limit is not equal to the right limit. The function has jump discontinuities at such points.



(ii) Consider the function $f(x) = 1$ if x is rational, and $f(x) = 0$ if x is irrational. It is discontinuous at every point. It has essential discontinuity at every point since limit at every point does not exist.

(iii) The function

$$f(x) = \sin \frac{1}{x} \text{ for } x \neq 0, f(0) = 0$$

is discontinuous at $x = 0$ .Once again it is a point of essential discontinuity $x = 0$.

Practice Exercises 6.1: Discontinuities of a function

1. Analyse the points of discontinuity and the types of discontinuity for the following functions:

$$f(x) = [x^2], x \in \mathbb{R}; \ g(x) = \sqrt{[x]}, x \geq 0$$

2. Find the type of discontinuity has at :

f $x = 0$

(i) $f(x) = \dfrac{x^3 - 3x + 2}{x^4 - 4x + 3}$.

(ii) $f(x) = \dfrac{5x^2 - 3x}{2x}$.

3. Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$

Show that f is discontinuous at every $c \in \mathbb{R}$. Analyse the type of discontinuity also.

4. Let $g(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 1 - x, & \text{if } x \text{ is irrational.} \end{cases}$

Show that g is continuous only at $c = 1/2$. Analyse the type of discontinuities at other points.

5. Let f have essential discontinuity at $x = c$. What can you say about $|f|$.

Optional Exercise (Discontinuities of a monotone function):

Try to draw the graph of a monotone function $f : [a, b] \to \mathbb{R}$ with a discontinuity at a point $c \in \mathbb{R}$. .

You will realize that such function can have only jump discontinuities, and this is not difficult to prove. In fact, one can show that monotone functions can have at most 'countable' number of discontinuities.

Recap

In this section you have learnt the following

- Jump Discontinuity
- Removable Discontinuity
- Essential Discontinuity.

[Section 6.2 : Basic properties of Continuous Functions]

Objectives

In this section you will learn the following

- Intermediate value property for continuous functions
- Maxima and minima theorem for continuous functions.
- Continuity of the inverse function.

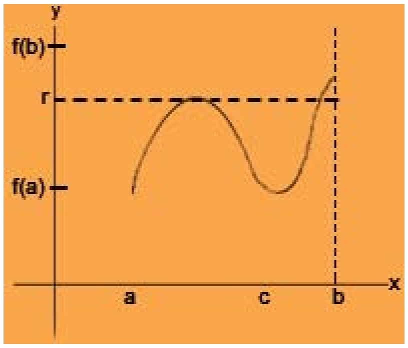
- The nth root function.

6.2 Some Basic Property of continuous functions

In this section we shall see how continuity of a function helps us to get a better picture of the function. We shall only state the results. For proofs, one may refer any book on Real Analysis.

6.2.1 Theorem (Intermediate Value Property) :

Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Let r be any real number between $f(a)$ and $f(b)$. Then there exists some $c\in[a,b]$ such that $f(c)=r$



Suppose $f(a)<f(b)$ and r is a real number between $f(a)$ and $f(b)$. Let us try to draw a picture of this.

Let $r\in\mathbb{R}$ be such that $f(a)<r<f(b)$. Saying that f is continuous implies then its graph will cut the horizontal line $y=r$ at least at one point $c\in(a,b)$.

CLICK HERE TO SEE A VISUALIZATION

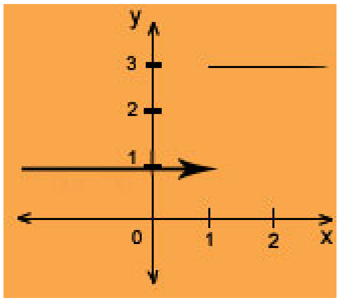
6.2.2 Corollary :

Let $f:D\subseteq\mathbb{R}\to\mathbb{R}$ be a continuous function. Then, for every interval $I\subseteq D$, the image set $f(I)$ is also an interval. Thus, a continuous function maps intervals to intervals.

6.2.3 Example :

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}3 & \text{for } x\ge 1\\ 1 & \text{for } x<1.\end{cases}$$



The function is not continuous at x =1. Observe that

$$1=f(0)<2<3=f(1),$$

however there is no value c for which $f(c)=2$.

This shows that the continuity condition in theorem 6.2.1 cannot be removed.

6.2.4 Corollary (Fixed point theorem) :

Let $f:[a,b]\to[a,b]$ be a continuous function. Then there exists some $c\in[a,b]$ such that $f(c)=c$. Such a point is called a fixed point for f .

6.2.4 Corollary (Fixed point theorem) :

Let $f:[a,b]\to[a,b]$ be a continuous function. Then there exists some $c\in[a,b]$ such that $f(c)=c$. Such a point is called a fixed point for f
.

Proof:

Consider the function $g(x)=f(x)-x,\ x\in[a,b]$.

Then, since $f(a),\ f(b)\in[a,b]$,

$$g(b)=f(b)-b<0$$

and

$$g(a)=f(a)-a>0$$

Thus, by the theorem 6.2.3, there exists $c\in[a,b]$ such that $g(c)=0$, i.e. $f(c)=c$.

6.2.5 Example (locating zeros) :

Consider the function

$$f(x)=7x^5-9x^3-1\cdot$$

It is not easy to find solutions of the equation $7x^5-5x^3-1=0$. However, we observe that

$$f(-1)=1>0>-1=f(0),$$

and f is a continuous function. Hence by theorem 2.5.1, there exists $c\in(-1,0)$ such that $f(c)=0$. Thus we are able to locate a zero of f in the interval (-1, 0).

6.2.6 **Example :**

(i) Let $f:(0,+\infty)\to\mathbb{R}$ be defined by $f(x)=\dfrac{1}{x}$. Then, f is continuous, but f is not bounded on say $(0,1)$.

(ii) Let $g:(0,1)\to\mathbb{R}$ be defined by $g(x)=x$. Then g is continuous, g is bounded but g does not attain a maximum value or a minimum value on $(0,1)$.

6.2.7 **Theorem (maximum and minimum) :**

Let $f:[a,b]\to\mathbb{R}$ be continuous. Then the following hold:

(i) f is bounded.

(ii) f attains its maximum and minimum, that is, there exist c_1 and c_2 in $[a,b]$ such that $f(c_1)\geq f(x)$, for every $x\in[a,b]$ and $f(c_2)\leq f(x)$, for every $x\in[a,b]$. Further range of $f=[m,M]$ where $m=f(c_1)$ and $M=f(c_2)$.

(iii) $f([a,b])=[m,M]$, i.e., f maps closed bounded intervals to closed bounded intervals.

We state next another important theorem.

6.2.8 Theorem (continuity of the inverse function) :

Let $f:[a,b]\to\mathbb{R}$ be continuous. If f is one-one, then f is either strictly increasing, i.e., $f(x_1)<f(x_2)$ whenever $x_1<x_2$ or it is strictly decreasing, that is, $f(x_1)>f(x_2)$ whenever $x_1<x_2$. Further, if I is the range of f, then $f^{-1}:I\to[a,b]$ is one-one and continuous.

6.2.9 Theorem (Existence of the nth-root function) :

For every positive integer n, the function $f:[0,+\infty)\to\infty, f(x)=x^n$ is a one-one onto function. The inverse function $f^{-1}:\mathbb{R}\to[0,+\infty)$, called the nth-root function, is also continuous, and is denoted by $f^{-1}(x)=\sqrt[n]{x}$.

QUIZ

Practice Excercises :

1. Show that a polynomial of odd degree has atleast one real root.
2. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\lim\limits_{x\to+\infty} f(x)=0$ and $\lim\limits_{x\to-\infty} f(x)=0$. Use theorem 6.2.7 suitably to show that there exist a point $a\in\mathbb{R}$ such that either $f(a)\geq f(x)$ or $f(x)\geq f(a)$ for every $x\in\mathbb{R}$.
3. Show that the polynomial x^4+6x^3-8 has at least two real roots.
4. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function which takes only rational values. Show that f is constant function.
5. Give the examples of continuous functions with the following properties:

(i) It maps a finite interval onto an infinite interval.

(ii) It maps an infinite interval onto a finite interval.

(iii) It maps an open interval onto a closed interval.

Recap

In this section you have learnt the following

- Important properties of continuous functions, such as the intermediate value property and the maxima/ minima property.
- Existence of the nth root function.
- Continuity of the inverse function.

Congratulations ! You have finished Lecture - 6. To view the next Lecture select it from the left hand side of the page.