Module 9 : Infinite Series, Tests of Convergence, Absolute and Conditional Convergence, Taylor and Maclaurin Series

Lecture 25 : Series of numbers [Section 25.1]

Objectives

In this section you will learn the following :

- Convergence of a series of numbers.

25.1 Series of numbers

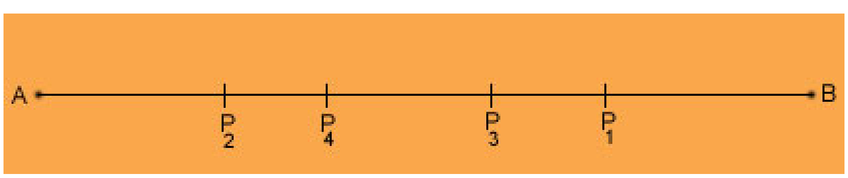
Given any finite collection of numbers

$$a_1, a_2, \ldots, a_n,$$

we know how to find their sum. For this, we take a_1, add to it a_2 to get $a_1 + a_2$, add to this sum a_3, to get $(a_1 + a_2) + a_3$, and so on till we reach a_n. In fact, the associative and the commutative properties tell us that we can add them in any order. In many practical problems one would like to add an infinite collection of numbers. Let us look at an example.

25.1.1 Example

Consider two cyclists A and B, at a distance one kilometer apart, moving towards each other, at a constant speed of 1km/hour. A man shuttles between A and B at a constant speed of 2km/hour till the two cyclists meet each other. How far away from the starting point P the man will stop shuttling and what would be the total distance covered by the man?



The answer to the first part of the question is obvious (to some one who has the knowledge that A and B are moving towards each other at a constant speed, and hence they would meet at the point Q, midway between P and Q). To answer the second question, let the starting point for the man be P and let it move towards Q. Let $P_1, P_2, P_3, \ldots, P_n, \ldots$ denote the consecutive points of the man's turning back from A and B. This gives us a sequence

$$a_1 = AP_1,\ a_2 = P_1 P_2, \ldots, a_n = P_n P_{n+1}, \ldots$$

The distance covered up to n^{th} turning is given by

$$s_n := AP_1 + P_1 P_2 + \ldots + P_n P_{n+1} = a_1 + \ldots + a_n .$$

It is easy to show that

$$AP_1 = a_1 = 2/3, P_1 P_2 = a_2 = \frac{2}{9}, \text{ and for any } n, AP_n = a_n = \frac{2}{3^{n-1}} .$$

Hence,

$$s_n = \frac{2}{3} + \frac{2}{9} + \ldots + \frac{2}{3^{n-1}} = \left[1 - \frac{1}{3^n}\right].$$

One is interested in knowing what happens as n becomes larger and larger.

We are given a sequence $\{a_n\}_{n\geq 1}$ for real numbers. What should be a method of finding the sum of all the terms of this sequence? An intuitively obvious method is to 'carry on' the ideas of finite sums. For every $n \in \mathbb{N}$, we can find

$$S_n := a_1 + a_2 + \ldots + a_n ,$$

and let this process continue, i.e, consider the convergence of the sequence $\{S_n\}_{n\geq 1}$. In view of this, let us make the following definitions:

25.1.2 Definition:

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

(i) For every $n \geq 1$, define

$$S_1 = a_1 \text{ and } S_n = a_1 + \ldots + a_n = \sum_{i=1}^{n} a_i, \text{ for all } n \geq 1$$

The pair $\left(\{a_n\}_{n\geq 1}, \{S_n\}_{n\geq 1}\right)$ is called a series of real numbers and S_n is called the n^{th}-*partial sum* of the series.

(ii) We say $\left(\{a_n\}_{n\geq 1}, \{S_n\}_{n\geq 1}\right)$ is a *convergent series* if the sequence $\{S_n\}_{n\geq 1}$ is convergent. In that case, the $\lim_{n\to\infty} S_n$ is called the sum of the series.

(iii) The series $\left(\{a_n\}_{n\geq 1}, \{S_n\}_{n\geq 1}\right)$ is said to be divergent series if it is not convergent.

25.1.3 Note:

Note that a series is determined by a sequence $\{a_n\}_{n\geq 1}$ of real numbers and the corresponding sequence of partial sums $\{S_n\}_{n\geq 1}$. In fact $\{S_n\}_{n\geq 1}$ determines the sequence $\{a_n\}_{n\geq 1}$ uniquely since

$$a_1 = S_1 \text{ and } a_{n+1} = S_{n+1} - S_n \text{ for every } n \geq 1 .$$

In view of this, the series $\left(\{a_n\}_{n\geq 1}, \{S_n\}_{n\geq 1}\right)$ is often denoted by the symbol $\sum_{n=1}^{\infty} a_n$. And if it is convergent with sum s, we write

$$\sum_{n=1}^{\infty} a_n = s$$

A word of caution: the notation $\sum_{n=1}^{\infty} a_n$ is just a convenient symbol for the more cumbersome notation $\left(\{a_n\}_{n\geq 1}, \{S_n\}_{n\geq 1}\right)$ for a series. It becomes a number only when the series is convergent.

25.1.4Example

(i) Consider the series

$$\sum_{n=1}^{\infty} a_n, \text{ where } a_n = (-1)^{n+1} \; n \geq 1.$$

Since

$S_n = 1$ if n is even and $S_n = 0$ if n is odd

the sequence $\{S_n\}_{n\geq 1}$ is not convergent. Hence the series is also not convergent.

(ii) Geometric series

Consider the series

$$\sum_{n=1}^{\infty} ar^n, \text{ where } r \in \mathbb{R} \text{ is fixed}$$

This is called *geometric series* with *common ratio* r. The convergence of this series depends upon the value of the common ratio. For $r = 1$, the n^{th} partial sum is $S_n = na$. Since,

$$\lim_{n\to\infty} S_n = \pm\infty,$$

the series is divergent for $r = 1$. Similarly, for $r = -1$, the n^{th} partial sum is

$S_n = na$ if n is odd, and $= 0$ if n is even.

Once again, $\lim_{n\to\infty} S_n$ does not exist, and hence the series is divergent for $r = -1$ also.

Finally, for $|r| \neq 1$,

$$S_n - rS_n = a - ar^{n+1}$$

Since $r \neq 1$, we have

$$S_n = \frac{a - ar^{n+1}}{1-r} = \frac{a}{1-r} - \frac{ar^{n+1}}{1-r}.$$

For $|r| < 1$, since $r^{n+1} \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}$$

Hence, the series is convergent whenever $|r| < 1$ and its sum is $1/1-r$.

(iii) As in example 25.1.1. the series $\sum_{n=1}^{\infty} a_n$ is convergent to 1, since

$$\frac{1}{3^{n-1}} \to 0$$

Hence, the man would have covered a total distance of one kilometer from start to finish. Let us also compute the total distance the cyclist A would covered till it meets B. Clearly, if they meet at the point P, then AP is the limit of the distances AP_n. To find AP_n, let us observe that for every $n \geq 1$.

$$AP_n := AP_1 - P_1P_2 + P_2P_3 - \ldots + (-1)^{n-1} \; P_{n-1}P_n.$$

If we write

$$b_1 = a_1 = AP_1 \text{ and } b_n = (-1)^{n-1} \, a_n,$$

then is the partial sum of the series , and is given by

AP_n $\quad n^{th}$ $\quad \sum_{n=1}^{\infty} b_n$

$$\begin{aligned} AP_n &= \frac{2}{3} - \frac{2}{9} + \ldots + \frac{-1}{3^{n-1}} \\ &= \frac{1}{2}\left[1 - \frac{1}{3} + \ldots + \left(\frac{-1}{3}\right)^{n-2}\right] \\ &= \frac{1}{2}\left[1 - \frac{1}{3^{n-1}}\right] \end{aligned}$$

Since

$$\frac{1}{3^{n-1}} \to 0 \text{ as } n \to \infty$$

we have

$$\sum_{n=1}^{\infty} b_n = \frac{1}{2}.$$

Hence, the cyclists will meet exactly midway.

(iv) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$. For this

$$S_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \text{ for every } n \geq 1$$

For $n = 2^k$, we have

$$\begin{aligned} S_{2^k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^{k-1}} + \frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{2^2-1} + \frac{1}{2^2}\right) + \ldots + \left(\frac{1}{2^{k-1}+1} + \ldots + \frac{1}{2^k}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \ldots + \left(\frac{1}{2^k} + \ldots + \frac{1}{2^k}\right) \\ &= 1 + \frac{k}{2}. \end{aligned}$$

This shows that $\{S_2k\}_{k\geq 1}$ is an unbounded, sequence. Thus, $\{S_n\}_{n\geq 1}$ is also unbounded and hence not convergent. Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is known as the harmonic series.

(v) Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

It is called the alternating harmonic series. Let us look at the odd and the even terms of $\{S_n\}_{n\geq 1}$, the sequence of partial sums of the series. For any k,

$$\begin{aligned} S_{2k} &= 1 - \frac{1}{2} + \frac{1}{3} + \ldots - \frac{1}{2^k} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right) \end{aligned}$$

Similarly, we can write

$$S_{2k+1} = 1-\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \ldots - \left(\frac{1}{2k} - \frac{1}{2k+1}\right)$$

Form these, it follows that $\{S_{2k}\}_{k\geq 1}$ is monotonically increasing while $\{S_{2k+1}\}_{k\geq 1}$ is monotonically decreasing. Further

$$0 < S_{2k} < S_{2k} + \frac{1}{2k+1} = S_{2k+1} \leq 1.$$

Hence, by the completeness property of real numbers, both the sequences $\{S_{2k}\}_{k\geq 1}$ and $\{S_{2k+1}\}_{k\geq 1}$ are convergent. In fact, the relation also tells us that they have the same limit. Hence, the sequence $\{S_n\}$, is also convergent convergent by exercise (8), section 1.7. Note that, we have not found its sum, we have just proved that it is convergant.

(vi) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since it is a series of positive terms, the sequence $\{S_n\}_{n\geq 1}$ of partial sums is a monotonically increasing sequence. Let

$$n_k := 2^k - 1, \ k \geq 1$$

Then, we claim that for every k,

$$0 < S_{n_k} \leq 1 + \frac{1}{2} + \ldots + \frac{1}{2^{k}\text{-}1} \qquad \text{---------(16)}$$

Clearly, $S_{n_1} = 1$ and if (16) holds for n_k, then

$$S_{n_k+1} = S_{n_k} + \left(\frac{1}{\left(2^k\right)^2} + \frac{1}{\left(2^k+1\right)^2} + \ldots + \frac{1}{\left(2^{k+1}-1\right)^2}\right)$$

$$< 1 + \frac{1}{2} + \ldots + \frac{1}{2^{k-1}} + \frac{2^k}{\left(2^k\right)^2}$$

$$= 1 + \frac{1}{2} + \ldots + \frac{1}{2^{k-1}} + \frac{1}{2^k}.$$

Thus, by induction, (16) holds for every k. Hence,

$$0 < S_{n_k} \leq \lim_{k\to\infty}\left(\sum_{j=1}^{k} \frac{1}{2^{j-1}}\right) = 2$$

Thus, the sequence $\{S_{n_k}\}_{k\geq 1}$, is bounded. Since, $\{S_n\}_{n\geq 1}$ is a monotonically increasing sequence, and for every n we can find a positive integer k such that $n \leq n_k := 2^k - 1$, we have

$$S_n \leq S_{nk} \leq 2,$$

Hence, $\{S_n\}_{n\geq 1}$ is a monotonically increasing sequence, bounded sequence. By the completeness property of $\mathbb{R}$, $\{S_n\}_{n\geq 1}$ is convergent. Hence, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

CLICK HERE TO SEE AN INTERACTIVE VISUALIZATION - APPLET

PRACTICE EXERCISES

1. For the following series compute their n^{th}-partial sum and show that the series is convergent:

(i) $\sum_{n=1}^{\infty}\left(\frac{1}{(n+2)-(n+3)}\right)$.

(ii) $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$.

(iii) $\sum_{n=1}^{\infty} 5^n\, 7^{1-n}$.

(iv) $\sum_{n=1}^{\infty}\ln\left(1-\frac{1}{(n+1)^2}\right)$.

(v) $\sum_{n=1}^{\infty}\left(1-\frac{1}{(9n^2+3n-2)}\right)$.

2. For the following series compute their n^{th}-partial sum and show that the series is not convergent:

(i) $\sum_{n=1}^{\infty}\frac{2^n-1}{4}$.

(ii) $\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right)$.

(iii) $\sum_{n=1}^{\infty} 5^{2n}\, 7^{1-n}$.

(iv) $\sum_{n=1}^{\infty}\cos(n\pi)$.

3. For a series $\sum_{n=1}^{\infty} a_n$ let S_n be its n^{th}-partial sum such that

$$S_{n+1}=\frac{S_n+1}{2} \text{ for every } n\geq 1.$$

Express S_n in terms of a_1 and show that the series is convergent with sum equal to 1.

4. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Find necessary and sufficient conditions for the series

$$\sum_{n=1}^{\infty}(a_n-a_{n+1})$$

to be convergent.

5. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Let

$$t_n := \sum_{k=n}^{\infty} a_k, \text{ for } n\geq 1.$$

Show that the sequence $\{t_n\}_{n\geq 1}$ converges.

Recap

In this section you have learnt the following

- Convergence of a series of numbers.

Series, Tests of Convergence, Absolute and Conditional Convergence, Taylor and Maclaurin Series

Lecture 25 : Simple tests of convergence [Section 25.2]

Objectives

In this section you will learn the following :

- Various ways to analyze the convergence of a series of numbers.

25.2 Simple tests of convergence

In view of the Cauchy completeness property of $\mathbb{R}$ we have the following:

25.2.1Theorem (Cauchy criterion) :

A series $\sum_{n=1}^{\infty} a_n$ *is convergent if and only if for every* $\in > 0$, *there exists* $N \in \mathbb{N}$ *such that*

$$|S_m - S_n| = |x_{n+1} + \ldots + x_m| < \in \text{ for every } m > n > N.$$

25.2.1Theorem (Cauchy criterion) :

A series $\sum_{n=1}^{\infty} a_n$ *is convergent if and only if for every* $\in > 0$, *there exists* $N \in \mathbb{N}$ *such that*

$|S_m - S_n| = |x_{n+1} + \ldots + x_m| < \in$ for every $m > n > N$.

Proof:

This is just a restatement of the fact that $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\{S_n\}_{n \geq 1}$, the sequence of partial sums is Cauchy.

25.2.2 Corollary (n^{th}- term test) :

(i) If a series $\sum_{n=1}^{\infty} a_n$, is convergent, then

$$\lim_{n \to \infty} a_n = 0.$$

(ii) If

$$\lim_{n \to \infty} a_n \neq 0,$$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If

$$\lim_{n \to \infty} a_n = 0,$$

then the series $\sum_{n=1}^{\infty} a_n$, may either converge or diverge.

The above corollary is most useful in proving that a series is not convergent.

25.2.2 Corollary (n^{th}- term test) :

(i) If a series $\sum_{n=1}^{\infty} a_n$, is convergent, then

$$\lim_{n \to \infty} a_n = 0.$$

(ii) If

$$\lim_{n \to \infty} a_n \neq 0,$$

then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If

$$\lim_{n \to \infty} a_n = 0,$$

then the series $\sum_{n=1}^{\infty} a_n$, may either converge or diverge.

Proof:

(i) Follows from theorem 25.2.1 since,

$a_n = S_{n+1} - S_n,$

Statement (ii) is just a restatement of (i). Finally, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n}.$$

For both the series, $\lim_{n \to \infty} a_n = 0$. However, the first is a convergent series, while the second is a divergent series.

25.2.3 Examples:

(i) Consider the geometric series $\sum_{n=1}^{\infty} r^n$, where $|r| > 1$. Since,

$$|r|^n \to \infty,$$

the series is not convergent.

$$\sum_{n=1}^{\infty} r^n$$

(ii) Consider the series

$$\sum_{n=1}^{\infty}\left(\frac{n^2+n+3}{2n^2+1}\right).$$

Since,

$$\lim_{n\to\infty}\left(\frac{n^2+n+3}{2n^2+1}\right)=\lim_{n\to\infty}\left(\frac{1+\frac{1}{n}+\frac{3}{n^2}}{2+\frac{1}{n^2}}\right)=\frac{1}{2}\neq 0,$$

the series is not convergent.
The limit theorems on convergent sequences imply the following results:

25.2.4Theorem:

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series and $c\in\mathbb{R}$. Then the following hold:

(i) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then the series

$$\sum_{n=1}^{\infty}(a_n+b_n),\sum_{n=1}^{\infty}(a_n-b_n),\sum_{n=1}^{\infty}(a_nb_n) \text{ and} \sum_{n=1}^{\infty}(a_n)$$

are all convergent and

$$\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n,$$

$$\sum_{n=1}^{\infty}(a_n-b_n)=\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}b_n,$$

and

$$\sum_{n=1}^{\infty}(ca_n)=c\left(\sum_{n=1}^{\infty}a_n\right).$$

(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent and $c\neq 0$ then $\sum_{n=1}^{\infty} c(a_n)$ is also divergent.

PROOF

25.2.4Theorem:

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series and $c\in\mathbb{R}$. Then the following hold:

(i) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, then

$$\sum_{n=1}^{\infty}(a_n+b_n),\sum_{n=1}^{\infty}(a_n-b_n),\sum_{n=1}^{\infty}(a_nb_n) \text{ and} \sum_{n=1}^{\infty}(a_n)$$

are all convergent and

$$\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n,$$

$$\sum_{n=1}^{\infty}(a_n-b_n)=\sum_{n=1}^{\infty}a_n-\sum_{n=1}^{\infty}b_n,$$

and

$$\sum_{n=1}^{\infty}(ca_n)=c\left(\sum_{n=1}^{\infty}a_n\right).$$

(ii) If $\sum_{n=1}^{\infty}a_n$ is divergent and $c \neq 0$ then $\sum_{n=1}^{\infty}c(a_n)$ is also divergent.

Proof:

(i) Follow from application of the limit theorems to the partial sum of the series under consideration.

The statement (ii) follows from the fact that if $\sum_{n=1}^{\infty}(ca_n)$ is convergent, then by (i)

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{1}{c}(ca_n)$$

will also be convergent.

25.2.5Example:

(i) Consider the series

$$\sum_{n=1}^{\infty}\left(\frac{2}{3^n}+\left(\frac{3}{4}\right)^n\right).$$

Since it is a sum of two convergent series:

$$\sum_{n=1}^{\infty}\frac{2}{3^n} \text{ and } \sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n,$$

it is also convergent.

(ii) Consider the series

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\left(\frac{5}{2^n}+\frac{2}{n}\right).$$

This is a divergent series, for if it were convergent, then

$$\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)=\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}\left(\frac{5}{2^n}\right)$$

would also be convergent, which is not true.

Here is another simple test that can be used for analyzing convergence of series with non-negative terms.

25.2.6Theorem (Comparison test):

Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be sequences of real numbers such that $0 \leq a_n \leq b_n$ ultimately, i.e., there exists some $N \in \mathbb{N}$, such that

$$a_n \leq b_n \text{ for every } n \geq N,$$

then, the following hold:

(i) If the series $\sum_{n=1}^{\infty}b_n$ is convergent, then so is the series $\sum_{n=1}^{\infty}a_n$.

(ii) If the series $\sum_{n=1}^{\infty}a_n$ is divergent, then so is the series $\sum_{n=1}^{\infty}b_n$.

25.2.7 Examples:

(i) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \text{ where } 0 \le p < \infty.$$

For $0 \le p \le 1$, since

$$n^p \le n \text{ for every } n \ge 1,$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is divergent for } 0 \le p \le 1.$$

For $p > 1$, since

$$n^p \ge n^2 \text{ for } n \ge 2,$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $n \ge 2$. We shall prove(in the next section) that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent to $p > 1$.

(ii) Consider the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n^2 + n + 1} \right).$$

Since

$$\frac{1}{2n^2 + n + 1} < \frac{1}{n^2} \text{ for every } n \ge 1,$$

and the sum $\sum_{\infty}^{n=1} \frac{1}{n^2}$ is convergent, the given series is also convergent.

(iii) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

For $n \ge 4$, it is easy to check (by induction) that

$$\frac{1}{n!} < \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent, the series $\sum_{n=4}^{\infty} \frac{1}{n!}$ is also convergent.

PRACTICE EXERCISES

1. Using comparison test, analyze the convergence/divergent of the following series:

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$.

(ii) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n+1}$.

(iii) $\sum_{n=1}^{\infty} \frac{\sin^2(2n)}{ln(n)}$.

2. Let $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n, \sum_{n=1}^{\infty} c_n$ be series of non-negative reals. Prove the following

(i) If

$$b_n = \frac{a_n + a_{n+1}}{2} \text{ for every } n,$$

and $\sum_{n=1}^{\infty} a_n$ is convergent, then so is $\sum_{n=1}^{\infty} b_n$.

(ii) If

$$a_n + b_n \leq c_n \text{ for every } n,$$

and $\sum_{n=1}^{\infty} c_n$ is convergent, then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent.

(iii) If

$$\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right) = 0$$

and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(iv) If

$$\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right) = +\infty$$

and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Recap

In this section you have learnt the following

Various ways to analyze the convergence of a series of numbers.