Module 1 : Real Numbers, Functions and Sequences

Lecture 3 : Monotone Sequence and Limit theorem

[Section 3.1 : Monotone Sequences]

Objectives

In this section you will learn the following

- The concept of a sequence to be monotonically increasing/ decreasing.
- Convergence of monotone sequences.
- Completeness axiom of real numbers.

3.1 Monotone Sequences

3.1.1 Definition:

1. A sequence $\{a_n\}_{n\geq 1}$ said to be monotonically increasing, if $a_{n+1} \geq a_n \;\; \forall \; n \geq 1$

2. A sequence $\{a_n\}_{n\geq 1}$ said to be monotonically decreasing, if $a_{n+1} \geq a_n \;\; \forall \; n \geq 1$

   We can describe now the completeness property of the real numbers.

3.1.2 Completeness property

Every monotonically increasing sequence which is bounded above is convergent.

3.1.3 Theorem:

If $\{a_n\}_{n\geq 1}$ is monotonically decreasing and is bounded below, it is convergent.

PROOF -------------

Follows from the following facts.

1. $\{a_n\}_{n\geq 1}$ is monotonically decreasing if and only if $\{-a_n\}_{n\geq 1}$ monotonically increasing.
2. $\{a_n\}_{n\geq 1}$ is bounded below if and only if $\{-a_n\}_{n\geq 1}$ is bounded above.

3. $\{a_n\}_{n\geq 1}$ is convergent if and only if $\{-a_n\}_{n\geq 1}$ is convergent .

3.1.4 Examples:

1. Sequence $\{2n\}_{n\geq 1}$ is monotonically increasing and is not bounded above.

2. Sequence $\left\{\frac{1}{n}\right\}_{n\geq 1}$ is monotonically decreasing and is bounded below,say by 0 .

3. Sequence $\left\{(-1)^n\right\}_{n\geq 1}$ is neither monotonically increasing nor decreasing.

4. Let $a_1 = 1, a_n := \frac{3a_{n-1}+2}{6} \; \forall \; n \geq 2.$ As shown in problem 2.1(Lecture 1) , $a_n \geq \frac{2}{3} \; \forall \; n$ and $a_n \geq a_{n+1} \; \forall \; n$. Hence, $\{a_n\}_{n\geq 1}$ is convergent. Let $l = \lim\limits_{n\to\infty} a_n$. Then,
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3a_{n-1}+2}{6}.$$
Hence, $l = \frac{3l+2}{6}$. This implies that $l = 2/3$. Hence, $\lim\limits_{n\to\infty} a_n = 2/3$

For Quiz refer the WebSite

Practice Exercises 3.1: Monotone Sequences

1. Determine whether the sequences are increasing or decreasing:

$$\text{(i)} \left\{\frac{n}{n^2+1}\right\}_{n\geq 1} \quad \text{(ii)} \left\{\frac{2^n 3^n}{5^{n+1}}\right\} \quad \text{(iii)} \left\{\frac{1-n}{n^2}\right\}_{n\geq 1}.$$

2. Show that the following sequences are convergent by showing that they are monotone and bounded. Find their limits also:
   - $a_0 = 1, a_{n+1} := \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \forall \; n \geq 1.$
   - $a_1 = \sqrt{2}, a_{n+1} := \sqrt{2+a_n} \;\; \forall \; n \to 1.$
   - $a_1 = 2, a_{n+1} := \mathbf{3} + \frac{a_n}{2}.$

3. Let $a_1 > b_1 > 0$ and $a_{n+1} := \frac{a_n + b_n}{2}, b_{n+1} = \frac{2a_nb_n}{a_n+b_n}$.

   Show that $\{a_n\}_{n\geq 1}$ is a decreasing sequence, $\{b_n\}_{n\geq 1}$ is an increasing sequence and $a_n > a_{n+1} > b_{n+1} > b_n$ for every n . .

Recap

In this section you have learnt the following

- The concept of a sequence to be monotone sequence.

- Monotonically increasing/ decreasing sequences converse if they are bounded above/ below.

Objectives

In this section you will learn the following

- Techniques of computing limits of sums, differences, products and quotients of sequences.
- The Sandwich Theorem.

3.2 Limit Theorems on sequences :

Some more theorems which helps us in computing limits of sequences are as follows:

3.2.1 Theorems (Algebra of limits):

Let $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$ be sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then the following hold:

1. A sequence $\{x_n + y_n\}_{n\geq 1}$ is convergent and $\lim_{n\to\infty}(x_n + y_n) = x + y$.
2. A sequence $\{x_n y_n\}_{n\geq 1}$ is convergent and $\lim_{n\to\infty}(x_n y_n) = xy$.
3. If $y \neq 0$, then $\frac{x_n}{y_n}$ is defined for all $n \geq n_1$, for some $n_1 \in \mathbb{N}$ and the sequence $\left\{\frac{x_n}{y_n}\right\}_{n\geq n_1}$ is convergent with

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{x}{y} .$$

1. Let $\alpha \in > 0$ be given. Choose $n_1, n_2 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{2} \ \forall\, n \geq n_1 \text{ and } |y_n - y| < \frac{\varepsilon}{2} \ \forall\, n \geq n_2$$

Let .Then $n_0 := \max\{n_1, n_2\}$. Then $\forall\, n \ge n_0$,

$$\left|x_n + y_n - (x+y)\right| \;\le\; |x_n - x| + |y_n - y| < \varepsilon$$

This proves (i).

2. To prove (ii), first note that $\{x_n\}_{n\ge1}$ and $\{y_n\}_{n\ge1}$ being convergent, are bounded sequences by theorem 1.5.3.

Let $|x_n| < M_1$ and $|y_n| < M_2 \;\forall\, n$.

Then, $|x| \le M_1$ by exercise (ii). Suppose both x, y $\ne$ 0. Let an $\varepsilon > 0$ be given Choose n_1 *and* $n_2 \in \mathbb{N}$ such that $|x_n - x| < \dfrac{\varepsilon}{2M_2} \;\forall\; n \ge n_1$ and $|y_n - y| < \dfrac{\varepsilon}{2M_1} \;\forall\; n \ge n_2$

Note that $M_1 > 0$ and $M_2 > 0$. Then, $\forall\, n \ge n_0 := \max\{n_1, n_2\}$

$$\begin{aligned}
|x_n y_n - xy| &\le |x_n y_n - x y_n| + |x y_n - xy| \\
&\le |y_n|\,|x_n - x| + |x|\,|y_n - y| \\
&\le M_2\,|x_n - x| + M_1 |y_n - y| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Hence, $\lim\limits_{n\to\infty}(x_n y_n) = xy$. This proves (ii) when $x, y \ne 0$. The case when $x = 0$ or $y = 0$ is easy and is left as an exercise.

3. To prove (iii), we have to show that given an $\varepsilon > 0, \exists\, n_0$ and such that

$$\left|\frac{x_n}{y_n} - \frac{x}{y}\right| < \varepsilon \;\;\forall\, n \ge n_0$$

Since $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$,we have $n_1, n_2 \in \mathbb{N}$ such that

$$\left.\begin{aligned}
&|x_n - x| < \epsilon \;\forall\, n \ge n_1 \\
&\text{and} \\
&|y_n - y| < \epsilon \;\forall\, n \ge n_2
\end{aligned}\right\}.$$

Now,

$$\begin{aligned}
\left|\frac{x_n}{y_n} - \frac{x}{y}\right| &= \left|\frac{x_n y - x y_n}{y_n y}\right| \\
&= \left|\frac{x_n y - xy + xy - x y_n}{y_n y}\right| \\
&\le \frac{|x_n - x|\;|y| \;+\; |x|\;|y_n - y|}{|y_n y|}
\end{aligned}$$

If we take $n_0 := \max\{n_1, n_2\}$, the above inequality will give us

$$\left|\frac{x_n}{y_n} - \frac{x}{y}\right| \le \frac{|y|\,\varepsilon + |x|\,\varepsilon}{|y_n|\,|y|} \quad \text{for } n \ge n_0 \qquad (1)$$

Appearance of y_n in the denominator on the right hand side is to be removed. That we can do if we can say $\frac{1}{|y_n|} < c$ for some constant c and for all n large, i.e., $|y_n| > \frac{1}{c}$ for all large n. This is true since $\lim_{n\to\infty} y_n = y \neq 0$ and we can choose n_3 such that $\forall\, n \ge n_3.$ $|y_n - y| < \frac{|y|}{2}$.

Thus, $\forall\, n \ge n_3,$

$$\begin{aligned} |y_n| &\ge |y| - |y_n - y| \\ &> |y| - \frac{|y|}{2} \\ &= \frac{|y|}{2} > 0 \end{aligned} \qquad (2)$$

Thus, for $n > \max\{n_1, n_2, n_3\}$, $\frac{x_n}{y_n}$ is defined and we will have from (1) and (2),

$$\left|\frac{x_n}{y_n} - \frac{x}{y}\right| < 2\left(\frac{|y|\,\varepsilon + |x|\,\varepsilon}{|y|\,|y|}\right) = 2\,\varepsilon\left(\frac{|y| + |x|}{|y|^2}\right).$$

To bring this estimate to the required form, we make some changes. We choose n_1 such that $|x_n - x| < \frac{1}{4}\,|y|\,\varepsilon,\ n \ge n_1,$ and if $x \neq 0,$ we choose n_2 such that

$$|y_n - y| < \frac{1}{4}\,\frac{|y|^2\,\varepsilon}{|x|}\ \ \forall\, n \ge n_2.$$

Then for $n \ge \max\{n_1, n_2, n_3\}$ we will have

$$\begin{aligned} \left|\frac{x_n}{y_n} - \frac{x}{y}\right| &\le 2\left(\frac{|x_n - x|\,|y| + |x|\,|y_n - y|}{|y|^2}\right) \\ &< 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon. \end{aligned}$$

In case $x = 0$, clearly

$$\left|\frac{x_n}{y_n}\right| \le \frac{2\,|x_n|}{|y|} < \epsilon \text{ for } n \ge n_1.$$

This proves (iii) .

3.2.2 Sandwich Theorem:

Let $a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$. If $a_n \to l$ and $c_n \to l$, then $b_n \to l$.

Let $\in > 0$. Using definition, find natural numbers n_1 and n_2 such that

$n \geq n_1 \Rightarrow |a_n - l| < \in$ and $n \geq n_2 \Rightarrow |c_n - l| < \in$.

Let $n_0 := \max\{n_1, n_2\}$. Then, for $n \geq n_0$ we have

$l - \in \ < a_n \ \leq b_n \ \leq c_n \ < l \ + \in,$

i.e., $|b_n - l| < \in$. Thus, $b_n \to l$ as $n \to \infty$

3.2.3 Examples (Some important limits):

1. Let $x \in \mathbb{R}$ with $|x| < 1$. Then, $x^n \to 0$ as $n \to \infty$. To see this, note that for $x \neq 0$ we can write

$$\frac{1}{|x|} = 1 + h \quad \text{for some } h \geq 0.$$

Hence, $$\frac{1}{|x|^n} = (1+h)^n \ .$$

Using binomial theorem, we get $\dfrac{1}{|x|^n} = 1 + nh + \ldots + h^n \ > \ nh$.

Thus, $0 < |x|^n < \dfrac{1}{nh}$.

Hence, by the Sandwich Theorem, $x^n \to 0$ as $n \to \infty$.

2. For x > 0, the sequence $\left\{ x^{\frac{1}{n}} \right\}_{n \geq 1}$ is convergent and $\lim\limits_{n \to \infty} x^{\frac{1}{n}} = 1$. To see this, let us first suppose that

x > 1.Then, $x^{\frac{1}{n}} > 1$ for every $n \geq 1$. Let $x^{\frac{1}{n}} = 1 + d_n, d_n > 0$.

Once again, for every $n \geq 1$.

$$x = (1+d_n)^n = 1 + nd_n + \ldots + (d_n)^n > 1 + nd_n$$

Hence, $d_n < \dfrac{(x-1)}{n}$ and by Sandwitch theorem, $d_n \to 0$, i.e., $x^{\frac{1}{n}} \to 1$ Next, suppose $0 < x < 1$.

Then $0 < x^{\frac{1}{n}} < 1$ for every n. Let

$$x^{\frac{1}{n}} = \frac{1}{1+h_n}, h_n > 0 \ .$$

Once again, for every $n \geq 1$,

$$x = \frac{1}{(1+h_n)^n} \ \leq \ \frac{1}{1+nh_n} \ < \ \frac{1}{nh_n}.$$

Hence, $h_n < \dfrac{1}{nx}$ implying that $h_n \to 0$.

Thus $x^{\frac{1}{n}} \to 1$. For x = 1, clearly $x^{\frac{1}{n}} = 1$ for every n , and hence $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$

For Quiz refer the WebSite

Practice Exercises 3.2 : Limit Theorems

1. Show that the following limits exists and find them:
    1. $\lim_{n\to\infty}\left(\dfrac{n}{n^2+1}+\dfrac{n}{n^2+2}+.... +\dfrac{n}{n^2+n}\right)$.
    2. $\lim_{n\to\infty}\left(\dfrac{n!}{n^n}\right)$.
    3. $\lim_{n\to\infty}\left(\dfrac{n^3+3n^2+1}{n^4+8n^2+2}\right)$.
    4. $\lim_{n\to\infty}(n)^{1/n}$.
    5. $\lim_{n\to\infty}\dfrac{\cos \pi\sqrt{n}}{n^2}$.
    6. $\lim_{n\to\infty}\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$.

2. If $\lim_{n\to\infty} a_n = L$, find $\lim_{n\to\infty} a_{n+1}$, $\lim_{n\to\infty}|a_n|$.

3. If $\lim_{n\to\infty} a_n = L > 0$ show that there exists $n_0 \in \mathbb{N}$ such that $a_n \ge \dfrac{L}{2}\ \forall\, n \ge n_0$

4. If $a_n \ge 0$ and $\lim_{n\to\infty} a_n \ge 0$, show that $\lim_{n\to\infty} a_n^{1/2} = 0$.

5. For given sequences $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$, prove or disprove the following:
    1. $\{a_nb_n\}_{n\ge1}$ is convergent, if $\{a_n\}_{n\ge1}$ is convergent.
    2. $\{a_nb_n\}_{n\ge1}$ is convergent, if $\{a_n\}_{n\ge1}$ is convergent and $\{b_n\}_{n\ge1}$ is bounded.

6. A sequence $\{a_n\}_{n\ge1}$ is said to be **Cauchy** if for any $\in > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \in \forall\, m, n \ge n_0$. In other words, the elements of a Cauchy sequence come arbitrarily close to each other after some stage.

    Show that every convergent sequence is also Cauchy. (In fact, the converse is also true, i.e., every Cauchy sequence in $\mathbb{R}$ is also convergent. We shall assume this fact.)

7. Let $\{a_n\}_{n\ge1}$ be a sequence such that $|a_n| \le M$ for every n. Show that .

$$\left|\lim_{n\to\infty} a_n\right| \le M$$

8. Show that a sequence $\{a_n\}_{n\ge1}$ is convergent if and only if the subsequence $\{a_{2n}\}_{n\ge1}$ and $\{a_{2n+1}\}_{n\ge1}$ are both convergent to the same limits.

9. Is every Cauchy sequence bounded?

Additional Remarks :

1. In Practice exercise 2(i) of section 1.6, we defined $a_0 = 1, a_{n+1} := \frac{1}{2}\left(a + \frac{2}{a_n}\right) \forall\, n \ge 1.$

   The sequence $\{a_n\}_{n\ge1}$ is a monotonically decreasing sequence of rational numbers which is bounded below. However, it cannot converge to a rational (why?). This exhibits the need to enlarge the concept of numbers beyond rational numbers. The sequence $\{a_n\}_{n\ge1}$ converges to $\sqrt{2}$ and its elements a_n's are used to find rational approximation (in computing machines) of $\sqrt{2}$.

2. To prove that a sequence $\{a_n\}_{n\ge1}$ is convergent to L , one needs to find a real number L (not given by the sequences) and verify the required property. However, the concept of 'Cauchyness' of a sequence is purely an 'intrinsic' property which can be verified purely by the given sequence. Still a sequence is Cauchy if and only if it is convergent.

3. Using the completeness property we can say that the sequence $\{A_{2n}\}_{n\ge2}$ of the areas of $2n$ -sided regular polygons inside the unit circle is an increasing sequence which is bounded above. Its limit is denoted by π ,
   called pi. This gives a definition of π. It is also an irrational number.

Optional Exercises :

1. Let $\{a_n\}_{n\ge1}$ be a sequence and let $s_n := \frac{a_1 + \ldots + a_n}{n}, n \ge 1$

   1. Show that $\{s_n\}_{n\ge1}$ is convergent to l , whenever $\{a_n\}_{n\ge1}$ is convergent to l .
   2. Given an example to show that the converse of (i) need not be true.

2. Prove that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n\ge1}$ is convergent as follows:

   1. Expand $\left(1+\frac{1}{n}\right)^n$ by binomial theorem and use the fact $\binom{n}{k}\left(\frac{1}{n}\right)^k \le \frac{1}{k!} < \frac{1}{2^{k-1}}, 1 \le k \le n,$ to show that $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n\ge1}$ is bounded.
   2. Using the fact that for all $k \ge 2,$

   $$\binom{n+1}{k}\left(\frac{1}{n+1}\right)^k \ge \binom{n}{k}\left(\frac{1}{n}\right)^k ,$$

show that $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n\geq 1}$ is monotonically increasing.

3. Use completeness property to deduce $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists.

The limit, denoted by e, is called **Euler's number**. It is an irrational number and the above sequence is used to find its approximate values.

Recap

In this section you have learnt the following

- Limits of complicated sequences can be computed by expressing them as sums, differences, products and quotients of convergent sequences.
- A sequence becomes convergent if it can be sandwiched between two convergent sequences.

Objectives

In this section you will learn the following

- The concept of a sequence being convergent to $+\infty$ or $-\infty$.
- The concept of a subsequence of sequence.

3.3 Some Extensions of the Limit concept:

3.3.1 Definition:

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

(i) We say $\{a_n\}_{n\geq 1}$ **converges to** $+\infty$ if for every $\alpha\in\mathbb{R}$, $\{a_n\}_{n\geq 1}$ is ultimately bigger than α, i.e., given $\alpha\in\mathbb{R}, \exists\ n_0\in\mathbb{N}$ such that $a_n>\alpha\ \forall\ n\geq n_0$. We write this as $\lim_{n\to\infty}a_n=+\infty$.

(ii) We say $\{a_n\}_{n\geq 1}$ **converges to** $-\infty$ if for every $\alpha\in\mathbb{R}$, $\{a_n\}_{n\geq 1}$ is ultimately smaller than α, i.e., given $\alpha\in\mathbb{R}, \exists\ n_0\in\mathbb{N}$ such that $a_n<\alpha\ \forall\ n\geq n_0$. We write this as $\lim_{n\to\infty}a_n=-\infty$.

(iii) We say a sequence $\{a_n\}_{n\geq 1}$ is **divergent properly** if either it converges to $+\infty$ or it converges to $-\infty$.

3.3.2 Examples:

(i) Consider the sequence $\{2n\}_{n\geq 1}$. Given any α, by the Archimedian property, we can find positive integer

N

such that $N > \alpha/2$. Thus, for every $n \geq N, x_n = 2n \geq 2N > \alpha$. Hence, $\lim_{n\to\infty} a_n = +\infty$.

(ii) Let $\alpha \in \mathbb{R}$, $\alpha > 1$ .Consider the sequence $\{\alpha^n\}_{n\geq 1}$. We shall show that $\lim_{n\to\infty} \alpha^n = +\infty$. To see this let $\alpha = 1+\beta$, where $\beta > 0$. Using Binomial theorem,

$$\alpha^n = (1+\beta)^n = 1 + n\beta + \frac{n(n-1)}{2}\beta^2 + \ldots + \beta^n > n\beta.$$

Once again, using the Archimedian property, we can find positive integer N such that $N\beta > \alpha$. Then, by the above inequality, we get $\alpha^n > n\beta > \alpha$ for every $n \geq N$. Hence, $\lim_{n\to\infty} a_n = +\infty$. Here are some intuitively obvious results.

3.3.3 Theorem:

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

(i) If $\{a_n\}_{n\geq 1}$ converges to $+\infty$, then it is not bounded above.

(ii) If $\{a_n\}_{n\geq 1}$ converges to $-\infty$, then it is not bounded below.

(iii) If $\{a_n\}_{n\geq 1}$ is monotonically increasing and not bounded above then, $\{a_n\}_{n\geq 1}$ converges to $+\infty$.

(iv) If $\{a_n\}_{n\geq 1}$ is monotonically decreasing and not bounded below then, $\{a_n\}_{n\geq 1}$ converges to $-\infty$.

PROOF

3.3.4 Definition:

Let $\{a_n\}_{n\geq 1}$ be a sequence and let $\{n_k\}_{k\geq 1}$ be as strictly increasing sequence of natural numbers. Then $\{a_{n_k}\}_{k\geq 1}$ is called a subsequence of $\{a_{n_k}\}_{k\geq 1}$. In some sense, $\{a_{n_k}\}_{k\geq 1}$ is a part of $\{a_n\}_{n\geq 1}$ with due regard to the order of the terms.

3.3.5 Theorem:

Let $\{a_n\}_{n\geq 1}$ be a sequence. Then $\{a_n\}_{n\geq 1}$ is convergent to l iff every subsequence of $\{a_n\}_{n\geq 1}$ is convergent to l

PROOF

QUIZ

Practice Exercises 3.3 : Extension of Limit Concept

(1) Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be two sequences of positive real numbers such that $\lim_{n\to\infty} \frac{x_n}{y_n} = L$ exists and $L > 0$ Show that if $\{x_n\}_{n\geq 1}$ is convergent to $+\infty$, then $\lim_{n\to\infty} y_n = +\infty$.

(2) Give an example to show that conclusion of (1) need not hold for the cases when $L = 0$ or $L = +\infty$

(3) If , show that .

$$\lim_{n\to\infty}\left(\frac{a_n}{n}\right)=L>0 \qquad \lim_{n\to\infty} a_n=+\infty$$

(4) Let $\{x_n\}_{n\geq 1}$ be an unbounded sequence. Show that there exists a subsequence of $\{x_n\}_{n\geq 1}$ which is convergent to $+\infty$ or $-\infty$.

Recap

In this section you have learnt the following

- How to extend the notion of a sequence to be convergent to $+\infty$ / $-\infty$.
- The notion of a subsequence of a sequence.
- A sequence is convergent if every subsequence is convergent to the same limit.