Module 17 : Surfaces, Surface Area, Surface integrals, Divergence Theorem and applications

Lecture 49 : Surfaces and parameterizations [Section 49.1]

Objectives

In this section you will learn the following :

- The notion of a surface
- Parameterization of a surface

49.1.1 Definition :

Let $F : D \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}$ be a scalar field and $c \in \mathbb{R}$. Then

$$S = \{(x,y,z) \subseteq D \,|\, F(x,y,z) = c\}$$

is called a **surface** in D

49.1.2 Examples :

1. **Linear surface:**

   Let $F(x,y,z)=ax+by+cz$, be a linear function of the variables x,y and z, where $a,b,c\in\mathbb{R}$. Then, the surface

   $$IP:=\left\{(x,y,z)\in\mathbb{R}^3 \mid ax+by+cz=k\right\}$$

   is called a **plane**.

2. **Quadratic surface:**

   Let

   $$F(x,y,z)=Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+ax+Hy+Iz+J,$$

   where $(x,y,z)\in\mathbb{R}^3$ and A,B,C,D,E,F,G,H,I and J are real numbers. This is called a **second-degree equation in variables** x,y,z. The surface

   $$S:=\left\{(x,y,z)\in\mathbb{R}^3 \mid F(x,y,z)=k\right\}$$

   is called a **quadratic surface** in $\mathbb{R}^3$. Some special cases are of a quadratic surface are the following:

   1. For

      $$F(x,y,z)=x^2+y^2 \text{ with } k=a^2 \text{ for some } a>0,$$

      the quadratic surface

      $$S=\left\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2=a^2\right\},$$

      is called the **circular-cylinder**.

   2. For

      $$F(x,y,z)=x^2+y^2+z^2 \text{ with } k=a^2, a>0,$$

      the quadratic surface

      $$\left\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=a^2\right\}$$

      is the **sphere** of radius a.

   3. For

      $$F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}, k=1,$$

      the quadratic surface

      $$S=\left\{(x,y,z)\in\mathbb{R}^3 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right\}$$

      is called the **ellipsoid**.

   4. For

$$F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}\ , k=1$$

the quadratic surface

$$\left\{(x,y,z)\in \mathbb{R}^3 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}\ =1\right\}$$

is called the **hyperboloid of on sheet** .

3. **Surfaces as graph of a function:**

   Surfaces also arise as graphs of functions of two variables. Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function. Then its graph

   $$G(f):=\left\{(x,y,z)\in \mathbb{R}^3 \mid (x,y)\in D, z=f(x,y)\right\}$$

   $$=\left\{(x,y,z)\in \mathbb{R}^3 \mid F(x,y,z)=f(x,y)-z=0\ , (x,y)\in D\right\}$$

   is a surface. In the vector form we can write $G(f)$ as the image of the function $(x,y)\mapsto \mathbf{r}(x,y)$ given by

   $$\mathbf{r}(x,y)=x\,\mathbf{i}+y\,\mathbf{j}+f(x,y)\,\mathbf{k}\ , (x,y)\in D.$$

   The above example motivates the following:

49.1.3 Definition :

1. A function $\mathbf{r} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$,

   $$\mathbf{r}(u,v)=x(u,v)\,\mathbf{i}+y(u,v)\,\mathbf{j}+z(u,v)\,\mathbf{k}, (u,v)\in D$$

   is called a **parameterization** of the surface

   $$S:=\left\{(x,y,z)\in \mathbb{R}^3 \mid x=x(u,v), y=y(u,v), z=z(u,v), (u,v)\in D\right\}.$$

2. We say S is a **smooth surface** if all the functions $x(u,v)$ $y(u,v)$ $z(u,v)$ are continuous and have continuous partial derivatives in D.

49.1.4 Examples :

1. The Plane

   $$IP=\left\{(x,y,z)\in \mathbb{R}^3 \mid ax+by+cz=d\right\}$$

   has parameterizations

   $$\mathbf{r}\ (x,y)=x\,\mathbf{i}+y\,\mathbf{j}+\left[\frac{d}{c}-\left(\frac{a}{c}\,x+\frac{b}{c}\,y\right)\right]\mathbf{k}, (x,y)\in \mathbb{R}^2,$$

   if $c\neq 0$.

2. Let S be the sphere of radius a ,

$$S=\left\{(x,y,z)\in \mathbb{R}^3 \mid x^2+y^2+z^2=a^2\right\}.$$

A parameterizations of S is given by the spherical coordinates:

$$\mathbf{r}\,(\theta,\phi)=x(\theta,\phi)\,\mathbf{i}+y(\theta,\phi)\,\mathbf{j}+z(\theta,\phi)\,\mathbf{k},$$

where

$$\left.\begin{array}{lcl} x\,(\theta,\phi) & = & a\cos\theta\cos\phi \\ y\,(\theta,\phi) & = & a\cos\theta\sin\phi \\ z\,(\theta,\phi) & = & a\sin\phi \end{array}\right\},\ 0\le\theta\le 2\pi, -\pi/2\le\phi\le\pi/2.$$

3. Consider the elliptic-cylinder

$$S=\left\{(x,y,z)\in \mathbb{R}^3 \mid x^2+4y^2=4\,,\, z\in\mathbb{R}\right\}.$$

A parameterization of S is given by

$$\mathbf{r}(\theta,z)=x(\theta,z)\,\mathbf{i}+y(\theta,z)\,\mathbf{j}+z(\theta,z)\,\mathbf{k},$$

where

$$\left.\begin{array}{lcl} x\,(\theta,z) & = & 2\cos\theta \\ y\,(\theta,z) & = & 2\sin\theta \\ z\,(\theta,z) & = & z \end{array}\right\},\ 0\le\theta\le 2\pi, z\in\mathbb{R}.$$

49.1.5 Definition :

Given a surface S with parameterization $\mathbf{r}\,(u,v)$, $(u,v)\in D$, the curves $v\mapsto \mathbf{r}\,(u,v)$ for a fixed u

and

$v\mapsto \mathbf{r}(u,v)$ for any fixed v,

are called the **coordinate-curves** of the surface S.

Coordinate curves are useful in visualizing the surface.

49.1.6 Example :

Consider a surface S with parameterization:

$$\mathbf{r}\,(u,v)=u\cos v\,\mathbf{i}+u\sin v\,\mathbf{j}+u\,\mathbf{k}\,,\,(u,v)\in\mathbb{R}^2.$$

For u_0 fixed, the coordinate curve

$$v\mapsto r(u_0,v)=u_0\cos v\,\mathbf{i}+u_0\cos v\,\mathbf{j}+u_0\ \mathbf{k}$$

is a circle of radius u_0 in the plane $z=u_0$. For v_0 fixed, the coordinate curve

$$u\mapsto r(u,v_0)=u\cos v_0\ \mathbf{i}+u\sin v_0\ \mathbf{j}+u\ \mathbf{k}$$

is the line at an angle v_0 from the z-axis.

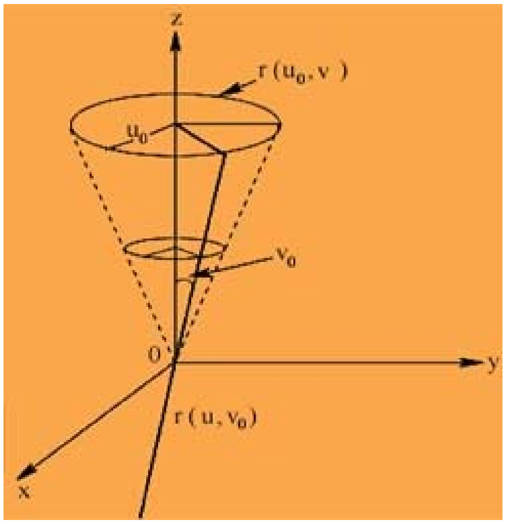

Figure: Coordinate curves of S

In section 35.1, we defined the normal vector to S, namely the vector ∇F, provided it is nonzero. We describe this in terms of a parameterization of S.

49.1.7 Theorem:

Let

$$S = \{(x,y,z) \subseteq D \mid F(x,y,z) = c\}$$

be a smooth surface with a parameterization

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \ (u,v) \in D.$$

Then,

$$\nabla \mathbf{F} = \pm(\boldsymbol{r}_u \times \boldsymbol{r}_v),$$

and the unit normal vector to S at any point is given by

$$\mathbf{n} = \pm\frac{\nabla F}{|\nabla F|} = \pm\left(\frac{\mathbf{r}_u \times \mathbf{r}_\mathbf{v}}{|\mathbf{r}_u \times \mathbf{r}_\mathbf{v}|}\right).$$

Since $(x,y,z) \in S$ satisfies $F(x,y,z) = c$, we have

$$F(x(u,v), y(u,v), z(u,v)) = 0, \ (u,v) \in D.$$

Hence,

$$0 = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u}$$

$$= (\nabla F)\cdot \mathbf{r}_u. \qquad \text{----------(60)}$$

Similarly,

$$(\nabla F)\cdot \mathbf{r}_\mathbf{v} = 0. \qquad \text{----------(61)}$$

From (60) and (61) it follows that at every point on S, the vectors ∇F and $\mathbf{r_u} \times \mathbf{r_v}$ are parallel to each other, i.e.,

$$\nabla F = \pm (\mathbf{r}_u \times \mathbf{r}_v).$$

Hence, the unit normal vector to S at any point is given by

$$\mathbf{n} = \pm \left(\frac{\mathbf{r}_u \times \mathbf{r_v}}{|\mathbf{r}_u \times \mathbf{r_v}|} \right).$$

49.1.8 Example:

Let a surface S be explicitly representable in the form

$$z = h(x,y), (x,y) \in R.$$

Then,

$$S = \{(x,y,z) | F(x,y,z) = h(x,y) - z = 0, (x,y) \in R\}.$$

Hence,

$$\nabla F = h_x \mathbf{i} + h_y \mathbf{j} - \mathbf{k}.$$

Thus, if we give S the parameterization

$$\mathbf{r}(x,y) = x \mathbf{i} + y \mathbf{j} + h(x,y) \mathbf{k},$$

then

$$\mathbf{r}_x \times \mathbf{r}_y = \pm \nabla F = \pm (h_x \mathbf{i} + h_y \mathbf{j} - \mathbf{k}).$$

Hence, the unit normal is

$$\mathbf{n} = \pm \left[\frac{-h_x \mathbf{i} - h_y \mathbf{j} + \mathbf{k}}{\sqrt{1 + h_x^2 + h_y^2}} \right].$$

Practice Exercise

1. Find a parameterizations of the surface S given below:
    1. The portion of the cone

       $$2z = \sqrt{x^2 + y^2},$$

       in the first octant between the planes $z = 0$ and $z = 3$.
    2. The portion of the sphere

       $$x^2 + y^2 + z^2 = 3,$$

       between the planes $2z = \sqrt{3}$ and $2z = -\sqrt{3}$.
    3. The portion of the plane

       $$x + y + z = 1$$

       inside the cylinder $x^2 + y^2 = 9$.

Answer:

(i) $\mathbf{r}(u,v) = (u\cos v)\mathbf{i} + (u\sin v)\mathbf{j} + \frac{u}{2}\mathbf{k}, 0 \le u \le 6, 0 \le v \le 2\pi$

(ii) $\mathbf{r}(\varphi,\phi) = \left(\sqrt{3}\sin\phi\cos\varphi\right)\mathbf{i} + \left(\sqrt{3}\sin\phi\sin\varphi\right)\mathbf{j} + \left(\sqrt{3}\cos\phi\right)\mathbf{k},$

$0 \le \varphi \le 2\pi, \pi/3 \le \phi \le 2\pi/3$

(iii) $\mathbf{r}(u,v) = (u\cos v)\mathbf{i} + (u\sin v)\mathbf{j} + \left(1 - u\cos v - 4\sin v\right)\mathbf{k} 0 \le u \le 3, 0 \le v \le 2\pi$

2. Compute the unit normal vector at a point on the following surfaces:

   1. $\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + \left(u^2 + v^2\right)\mathbf{k}, (u,v) \in \mathbb{R}^2.$
   2. $\mathbf{r}(u,v) = (2+\cos u)\cos v\mathbf{i} + (2+\cos v)\sin v\mathbf{j} + \sin u\mathbf{k}, 0 \le u \le 2\pi, 0 \le v \le 2\pi.$

   Answer:

   (i) $\dfrac{-2u\mathbf{i} - 2v\mathbf{j} + k}{\sqrt{5}}$

   (ii) $\dfrac{-\left(2+\cos v\right)\left(\cos v\cos u\mathbf{i} + \sin v\cos u\mathbf{j} + \sin u\mathbf{k}\right)}{\sqrt{2+\cos u}}$

3. A surface S is obtained by revolving the graph of the function $f(x) = 1/x, 1 \le x \le 5,$ about x-axis. Find a parameterization of S.

   Answer:

   $$\mathbf{r}(x,y) = x\mathbf{i} + \frac{1}{x}\cos v\mathbf{j} + \frac{1}{x}\sin v\mathbf{k}, 1 \le x \le 5, 0 \le v \le 2\pi$$

Recap

In this section you have learnt the following

- The notion of a surface
- Parameterization of a surface

[Section 49.2]

Objectives

In this section you will learn the following :

- How to define the integrals of a scalar field over a curve.

49.2 Area of a Surface

49.2.1 Definition:

Let S be a surface with a parameterization $\mathbf{r} : R \subseteq \mathbb{R}^2 \to \mathbb{R}^3$. We define the area of the surface S to be the

$$\text{Surface area of } \; S = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$$

49.2.2 Note:

1. The justification for the above definition is the following. Consider a small rectangular area element ΔR in the $u, v-$plane with sides parallel to the u, v axis. Let its image under $\mathbf{r}$ be the region ΔS in $\mathbb{R}^3$. We can imagine ΔS to be same as the parallelogram with sides $\mathbf{r}_u$, $\mathbf{r}_v$, and hence the area of ΔS is approximately equal to the area of the parallelogram, which is given by $|(\mathbf{r}_u \Delta v) \times \mathbf{r}_u \Delta v|$. Thus

   $$\Delta S = |\mathbf{r}_u \times \mathbf{r}_v| \, \Delta u \, \Delta v$$

   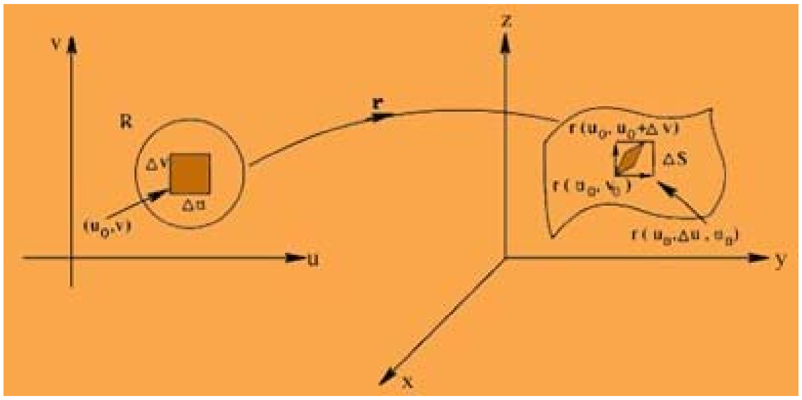

   Figure: ΔS is approximately a parallelogram

   Hence, it is natural to define the area of the surface S to be the double integral
   $\iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv.$

2. For the above definition to be meaningful, we should have the condition that the surface $\mathbf{r}(u, v)$ is such that
   $(\mathbf{r}_u \times \mathbf{r}_v)(u, v) \neq 0,$
   everywhere on R. From now onwards, we shall assume this condition. Such surfaces are called **regular surfaces** .

49.2.3 Example:

Find the area of the parabolic cylinder

$y = x^2$ in the first octant bounded by the planes $z = 2$ and $y = \dfrac{1}{4}$.

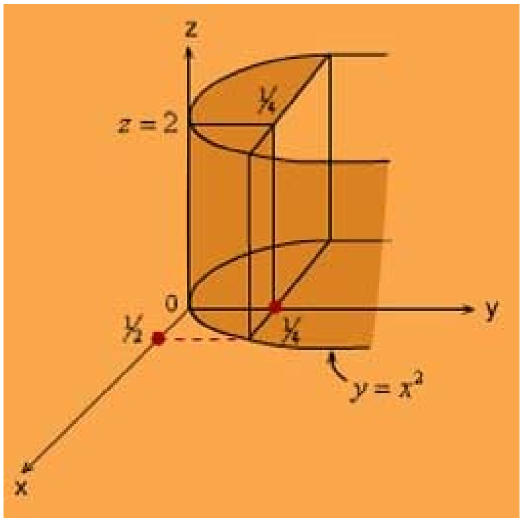

Figure: The surface y = x

We can parameterize the surface as follows:

$$\mathbf{r}\,(x,z) = x\,\mathbf{i} + x^2\,\mathbf{j} + z\,\mathbf{k},\ (x,z) \in R\,,\ 0 \le z \le 2,$$

where

$$R = \left\{(x,z) \in \mathbb{R}^2 \mid 0 \le x \le 1/2, 0 \le z \le 2\right\}.$$

Thus,

$$\mathbf{r}_x = \mathbf{i} + 2x\,\mathbf{j} \text{ and } \mathbf{r}_z = \mathbf{k}.$$

Thus

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= 2x\,\mathbf{i} - \mathbf{j}.$$

Hence,

$$\left|\mathbf{r}_x \times \mathbf{r}_z\right| = \sqrt{1 + 4x^2}.$$

Thus, the area of the surface is

$$\int_{x=0}^{1/2} \int_{z=0}^{2} \left(1 + 4x^2\right) dx\,dz$$

49.2.4 Surface area for explicit representation:

For a surface $S,$ explicitly representable in the form

$z = h(x,y), (x,y) \in R,$

we saw in example 49.1.8 that a parametric representation of S is

$\mathbf{r}\,(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + h(x,y)\,\mathbf{k}, (x,y) \in R,$

and we have

$\left|\mathbf{r}_x \times \mathbf{r}_y\right|^2 = 1 + h_x^2 + h_y^2.$

Hence,

$$\text{surface area }(S) = \iint_R \sqrt{1+h_x^2+h_y^2}\,dxdy$$

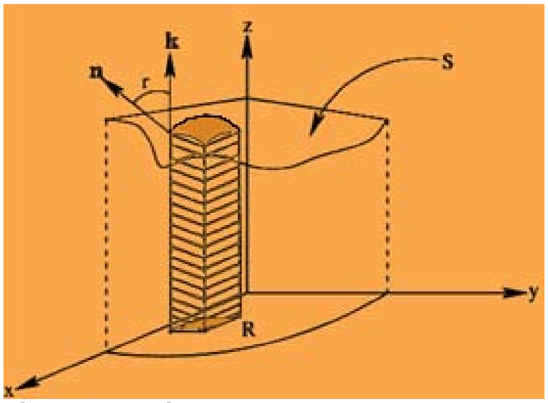

Figure: Caption text

If γ denotes the angle between the positive z-axis and the vector $\mathbf{r}_x \times \mathbf{r}_y$ Then,

$$(\mathbf{r}_x \times \mathbf{r}_y)\cdot\mathbf{k} = (-h_x\,\mathbf{i} - h_y\,\mathbf{j} + \mathbf{k})\cdot\mathbf{k} = 1. \qquad \text{----------(62)}$$

On the other hand

$$(\mathbf{r}_x \times \mathbf{r}_y)\cdot\mathbf{k} = |\mathbf{r}_x \times \mathbf{r}_y|\cos\gamma.$$

Thus,

$$|\mathbf{r}_x \times \mathbf{r}_y|\cos\gamma = 1 \qquad \text{----------(63)}$$

i.e.,

$$\sec\gamma = |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1+4x^2+4y^2}.$$

From (62) and (63) it follows that $\cos\gamma > 0$, and hence $0 \le \gamma < \pi/2$. Thus, if γ denote the acute angle between the z-axis and the undirected normal, then we also have

$$\text{surface area of } S = \iint_R \sqrt{1+h_x^2+h_y^2}\,dx\,dy = \iint_R \sec\gamma\,dx\,dy,$$

where R is the projection of S on to the xy plane. Also,

$$(\mathbf{r}_x \times \mathbf{r}_y)\cdot k = 1 = \cos\gamma\,|\mathbf{r}_x \times \mathbf{r}_y|$$

implying

$$\mathbf{n}\cdot\mathbf{k} = \frac{(\mathbf{r}_x \times \mathbf{r}_y)\cdot k}{|\mathbf{r}_x \times \mathbf{r}_y|} = \cos\gamma.$$

Thus,

$$\sec\gamma = \frac{1}{|\mathbf{n}\cdot\mathbf{k}|}, \text{ where } \mathbf{n} = \frac{\nabla F}{|\nabla F|},$$

and $F(x,y,z) = z - h(x,y)$.

49.2.5 Example:

Let us find the surface area of the surface

$$z^2 = x^2 + y^2, 0 \le z \le 1.$$

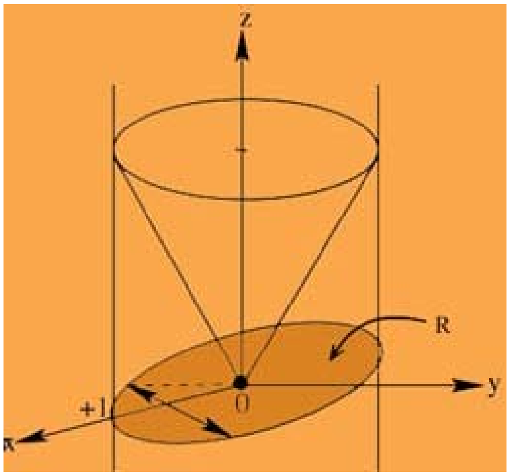

Figure: Surface $z^2 = x^2 + y^2, 0 \le z \le 1$

The surface is given by $F(x,y,z) = x^2 + y^2 - z^2 = 0$. Thus,

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{x\,\mathbf{i} + y\,\mathbf{j} - z\,\mathbf{k}}{z\sqrt{2}} \text{ and } \mathbf{n}\cdot\mathbf{k} = -\frac{1}{\sqrt{2}}.$$

Hence,

$$\sec\gamma = \frac{1}{|\mathbf{n}\cdot\mathbf{k}|} = \sqrt{2},$$

and the projection of the surface on to the xy-plane is

$$R = \left\{(x,y) \mid x^2 + y^2 = 1\right\}$$
$$= \{(x,y) \mid 0 \le x \le 1,\ 0 - \sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$$

Thus,

$$\text{Surface area} = \iint_R \sec\gamma\, dx\, dy$$
$$= \sqrt{2}\iint_R 1\, dx\, dy$$
$$= \sqrt{2}\int_{-1}^{+1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\right) dx = \sqrt{2}\pi$$

49.2.6 Example

Find the area of the portion of cylinder $(x-a)^2 + y^2 = a^2$ that lies inside the upper part of sphere the $x^2 + y^2 + z^2 = 4a^2$.

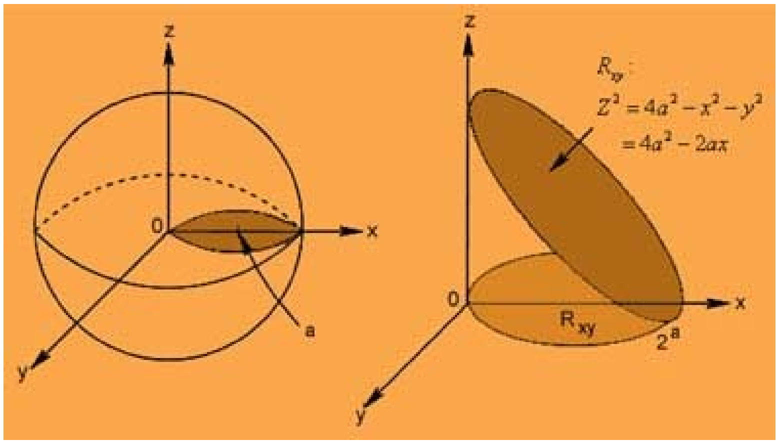

Figure: Surface of sphere inside cylinder

Projection of the cylinder onto xy plane is

$$R_{xy} = \{(x,y) \mid (x-a)^2 + y^2 \le a^2\}.$$

The surface, cylinder , is given by $F(x,y,z) = x^2 + y^2 - 2ax = 0.$ Thus,

$$\nabla F = (2x - 2a)\,\mathbf{i} + 2y\,\mathbf{j}.$$

Since $\nabla F \cdot \mathbf{k} = 0,$ we should not project the surface onto the xy plane. Let us project S onto the xz plane. This projection is given by

$$R_{xz} = \left\{(x,z) \mid 0 \le x \le 2a,\ z = \sqrt{4a^2 - 2a\,x}\right\}.$$

Since,

$$\begin{aligned} |\nabla F| &= \sqrt{(2x-2a)^2 + (2y)^2} \\ &= 2\sqrt{(x-a)^2 + y^2} \\ &= 2a \end{aligned}$$

and

$$\nabla F \cdot \mathbf{j} = 2y = 2\sqrt{a^2 - (a-x)^2},$$

we have

$$\mathbf{n} \cdot \mathbf{j} = \frac{\nabla F \cdot \mathbf{j}}{|\nabla F|} = \frac{\sqrt{a^2 - (a-x)^2}}{a}.$$

Thus, the required

$$\begin{aligned} \text{surface area} &= \iint_{R_{xz}} \frac{a}{\sqrt{a^2 - (a^2 - x)^2}}\, dx\, dz \\ &= a \int_{x=0}^{x=2a} \int_{z=0}^{z=\sqrt{4a^2 - 2a\,x}} \left(\frac{1}{\sqrt{a^2 - (a-x)^2}} \right) dz\, dx. \end{aligned}$$

49.2.7 Remark:

If the surface S is represented explicitly by

$$y = g\,(x,z),\ (x,z) \in R,$$

then the corresponding formulae for the surface area are

$$\text{Surface Area} = \iint_R \sec\beta \, dx\, dz$$

where β is the acute angle between the y - axis and the unit normal. Similarly, if the surface is representable explicitly as

$$x = f(y,z),\ (y,z) \in R,$$

then

$$\text{Surface Area} = \iint_R \sec\alpha \, dy\, dz,$$

where α is the acute angle between the x - axis and the unit normal.

49.2.8 Example:

Consider the sphere given by
$\mathbf{r}(u,v) = a\cos v\cos u\,\mathbf{i} + a\sin u\cos v\,\mathbf{j} + a\sin v\,\mathbf{k}$, $0 \le u \le 2\pi$, $0 \le v \le \pi$.

Then,
$\mathbf{r}_u = -a\sin u\cos v\,\mathbf{i} + a\cos u\cos v\,\mathbf{j}$

and
$\mathbf{r}_v = -a\sin v\cos u\,\mathbf{i} - a\sin u\sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$.

Thus,

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u\cos v & a\cos u\cos v & 0 \\ -a\sin u\sin v & -a\sin u\sin v & a\cos v \end{vmatrix}$$

$$\begin{aligned} &= (a^2\cos^2 v\cos u)\,\mathbf{i} + (a^2\sin u\cos^2 v)\,\mathbf{j} \\ &+ (a^2\sin u\sin v\cos v + a^2\cos^2 u\sin v\cos v)\,\mathbf{k} \\ &= (a^2\cos^2 v\cos u)\,\mathbf{i} + (a^2\sin u\cos^2 v)\,\mathbf{j} + (a^2\sin u\cos v)\,\mathbf{k}. \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathbf{r}_u \times \mathbf{r}_v\| &= a^4\cos^4 v\cos^2 u + a^4\sin^2 u\sin^4 v + a^4\sin^2 u\cos^2 v \\ &= a^4\left[\cos^4 v + \sin^2 v\cos^2 v\right] \\ &= a^4\cos^2 v[\cos^2 v + \sin^2 v] \\ &= a^4\cos^2 v. \end{aligned}$$

Thus, the surface area of the sphere is given by

$$\begin{aligned} \text{Surface area} &= a^2\pi\int_0^{2\pi} |\cos v|\, dv = \int_0^{2\pi}\int_0^{\pi} a^2\cos v\, du\, dv \\ &= 4\pi a^2\int_0^{\pi/2} \cos v\, dv \\ &= 4\pi a^2. \end{aligned}$$

49.2.9 Surface area for implicit equations:

Let S be a surface defined implicitly by $F(x,y,z) = 0$. Let S_{xy} be the projection of S onto the xy-plane. Suppose, the surface can be explicitly given by

$z = h(x, y), (x, y) \in S_{xy}$.

Then, we have

$$h_x = -\frac{\partial F/\partial x}{\partial F/\partial z} \text{ and } h_y = -\frac{\partial F/\partial y}{\partial F/\partial z},$$

provided $\frac{\partial F}{\partial Z} \neq 0$. Thus,

$$\begin{aligned}\text{Surface area} &= \iint_{S_{xy}} \sqrt{1 + h_x^2 + h_y^2}\, dx\, dy \\ &= \iint_{S_{xy}} \left(\frac{\sqrt{\frac{\partial F^2}{\partial z} + \frac{\partial F^2}{\partial x} + \frac{\partial F^2}{\partial y}}}{\frac{\partial F}{\partial z}} \right) dx\, dy.\end{aligned}$$

49.2.10 Surface area for surface of revolution:

A method of generating a surface in $\mathbb{R}^3$ is to rotate a curve about an axis. Suppose C is the curve in the xz plane given by

$$z = f(x), a \leq x \leq b,$$

being rotated about the z -axis to get a surface S. We want to find the surface are of this surface. Consider a small portion Δs of the curve. When rotated, this will give a cylinder of radius x and of height Δs. Thus, its surface area is given by

$$2\pi\, x\, \Delta s.$$

Thus, the total area of the surface is defined to be

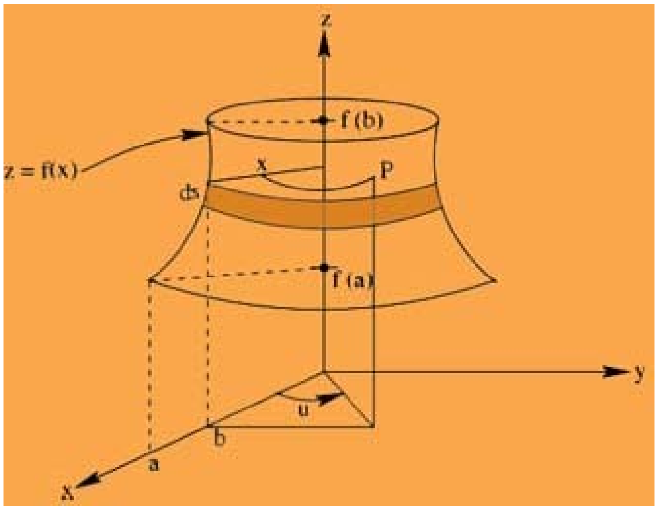

Figure: Surface area of surface of revolution

$$\text{Surface area} := \int_{x=a}^{x=b} 2\pi\, x\, ds = 2\pi \int_a^b x \sqrt{1 + (f'(x))^2}\, dx.$$

The surface S can be represented in parametric from, using cylindrical coordinates, as

$$\mathbf{r}\,(u, v) = u \cos v\, \mathbf{i} + u \sin v\, \mathbf{j} + f\,(u)\, \mathbf{k}, a \leq u \leq b, 0 \leq v \leq 2\pi$$

Thus,

$$\mathbf{r}_u = \cos v\, \mathbf{i} + \sin v\, \mathbf{j} + f'(u)\, \mathbf{k},$$

$\mathbf{r}_v = -u \sin v \, \mathbf{i} + u \cos v \, \mathbf{j}.$

Hence,

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u & \sin v & f'(u) \\ -u \sin v & u \cos v & 0 \end{vmatrix} = (u \cos v \, f'(u)) \, \mathbf{i} + (u \sin v \, f'(u)) \, \mathbf{j} + \mathbf{k} \, (u).$$

This gives

$$\begin{aligned} \| r_u \times \mathbf{r}_v \|^2 &= u^2 \cos^2 v \, f'(u)^2 + u^2 \sin^2 u \, f'(v) + v^2 \\ &= u^2 \, (1 + f'(u)^2). \end{aligned}$$

Hence, we have

$$\begin{aligned} \text{Surface area} &= \int_0^{2\pi} \int_a^b u \sqrt{1 + f'(v)^2} \, du \, dv \\ &= 2\pi \int_a^b u \, \sqrt{1 + f'(v)^2} \, du \, dv. \end{aligned}$$

Similar formulae can be obtained when if we consider the surface obtained by revolving the same curve $z = f(x)$ about x-axis or y-axis.

49.2.11 1^{st} fundamental form for curves on surfaces :

Suppose $\mathbf{r} = \mathbf{r}(t), t \in I$ is a curve on a surface S with a parameterization $(u, v) \mapsto r(u, v)$. Then, the curve is given by

$$t \mapsto \mathbf{r} \, (t) = \mathbf{r} \, (u(t), v(t)), t \in I.$$

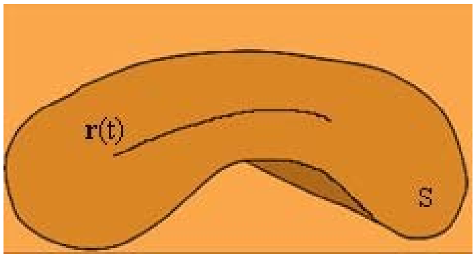

Figure: Curve on a surface

Suppose, we want to find its length. Since, the curve is given by

$$\mathbf{r} = \mathbf{r} \, (t) = \mathbf{r}(u(t), v(t)), \, t \in I,$$

we have

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}_u \frac{du}{dt} + \mathbf{r}_v \frac{dv}{dt}.$$

We write this as

$$d\mathbf{r} = \mathbf{r}_u \, du + \mathbf{r}_v \, dv.$$

Since,

$$\left\| \frac{d\mathbf{r}}{dt} \right\|^2 = \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt},$$

we write

$$\begin{aligned}\| d\mathbf{r} \|^2 &= (\mathbf{r}_u\, du + \mathbf{r}_v\, dv)\cdot(\mathbf{r}_u du + \mathbf{r}_v dv)\\ &= \left(\mathbf{r}_u \cdot \mathbf{r}_v\right) du^2 + 2(\mathbf{r}_u \cdot \mathbf{r}_v)\, du\, dv + (\mathbf{r}_u \cdot \mathbf{r}_v)\, dv^2\\ &:= E\, du^2 + 2F\, du\, dv + G\, du^2,\end{aligned}$$

where

$E = \mathbf{r}_u \cdot \mathbf{r}_v$, $F = r_u \cdot \mathbf{r}_v$, $G = \mathbf{r}_u \cdot \mathbf{r}_v$.

Thus, the length of $\mathbf{r}\,(u(t)),(v(t)), t \in I = [a,b]$ is given by

$$\ell = \int_a^b \sqrt{\| d\mathbf{r} \|^2}\, dt = \int_a^b \left(\sqrt{E\, du^2 + 2F\, du\, dv + G\, dv^2}\right) dt$$

Practice Exercises

1. Find the area of the plane $2x+6y+3z=6$ cut by the cylinder $x^2+y^2=1$.

   Answer: $\frac{7\pi}{3}$

2. Find the area of the portion of the cone $x^2+y^2=z^2$, above the xy-plane cut off by the cylinder $x^2+y^2=2ax$.

   Answer: $\sqrt{2}\pi a^2$

3. Find the area of the surface common to the cylinders $x^2+y^2=a^2$ and $y^2+z^2=a^2$.

   Answer: $16a^2$

4. Find the area of the portion of the sphere $x^2+y^2+z^2=4a^2$ that lies inside the cylinder $x^2+y^2=2ax$.

   Answer: $16a^2\left(\frac{\pi}{2}-1\right)$

Recap

In this section you have learnt the following

- How to define the integrals of a scalar field over a curve.