Module 2 : Limits and Continuity of Functions

Lecture 5 : Continuity [Section 5.1 : Extending the Limit concept]

Objectives

In this section you will learn the following

- The meaning of $f(x) \to L$ as $x \to +\infty / -\infty$.
- The meaning of $f(x) \to +\infty / -\infty$ as $x \to c$.

5.1 Extending the limit concept

5.1 .1 Definition :

(i) We write $\lim\limits_{x \to \infty} f(x) = l$ if for every $\varepsilon > 0$ there is some $x_0 > 0$ such that for every

$$x \in \mathbb{R},\ x > x_0 \Rightarrow |f(x) - l| < \varepsilon.$$

(ii) We write $\lim\limits_{x \to -\infty} f(x) = l$ if for every $\varepsilon > 0$ there is some $x_0 > 0$ such that $x \in \mathbb{R}$,

$$x < -x_0 \Rightarrow |f(x) - l| < \varepsilon.$$

5.1 .2 Examples :

(i) $\lim\limits_{x \to \infty} \frac{1}{x} = 0 = \lim\limits_{x \to -\infty} \frac{1}{x}$, since for a given ε, we can choose $x_0 = \frac{1}{\varepsilon}$ as per requirements.

(ii) $\lim\limits_{x \to \infty} \frac{x^2 + 2x + 3}{4x^2 + 5x + 6} = \lim\limits_{x \to \infty} \frac{1 + \frac{2}{x} + \frac{3}{x^2}}{4 + \frac{5}{x} + \frac{6}{x^2}} = \frac{1}{4}$.

5.1 .3 Definitions :

(i) $\lim\limits_{x \to c} f(x) = \infty$ if for every $\alpha > 0$, there exists some $\delta > 0$ such that

implies .

$$0<|x-c|<\delta \qquad f(x)>\alpha$$

(ii) $\lim_{x\to c} f(x) = -\infty$ if for every $\alpha>0$, there exists some $\delta>0$ such that

$$0<|x-c|<\delta \text{ implies } f(x)<-\alpha .$$

Similarly, $f(x)\to\infty$ or $f(x)\to-\infty$ as $x\to+\infty$ or $x\to-\infty$ can be defined.

5.1 .4 Examples :

(i) $\lim_{x\to 0,\, x>0} \frac{1}{x} = +\infty$ and $\lim_{x\to 0,\, x<0} \frac{1}{x} = -\infty.$

(ii) $\lim_{x\to\infty} x^2 = +\infty$, $\lim_{x\to\infty,\, x>0} x^3 = +\infty$ and $\lim_{x\to\infty,\, x<0} x^3 = -\infty$

QUIZ please reffer web site for quize

Practice Exercises : Extension of the limit concept

1) Let $\lim_{x\to c} f(x) = +\infty$ *and* $\lim_{x\to c} g(x) = l$.

Show that $\lim_{x\to c}\left[f(x)+g(x)\right] = +\infty$ *and* $\lim_{x\to c}\left[f(x)-g(x)\right] = +\infty$.

2) Let $0<|g(x)|\le M$ *for* $x\in(c-\delta,\ c+\delta)$, *for some* $\delta>0$.

If $\lim_{x\to c} f(x) = -\infty$, what can you say about $\lim_{x\to c}\left[f(x)g(x)\right]$?

3) Interpret geometrically the following :

(i) $\lim_{x\to c} f(x) = +\infty$.

(ii) $\lim_{x\to\infty} f(x) = l$.

(iii) $\lim_{x\to\infty}\left[f(x)-(ax+b)\right] = 0$.

(4) Let $f(x) = \dfrac{(x-1)(x-3)}{(x-2)(x-4)}, \quad x\ne 2,4.$

Compute $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, $\lim_{x\to 2} f(x)$ *and* $\lim_{x\to 4} f(x)$.

Recap

In this section you have learnt the following

- The meaning of $\lim_{x\to+\infty} f(x) = L$, $\lim_{x\to-\infty} f(x) = L$.
- The meaning of $\lim_{x\to c} f(x) = +\infty$, $\lim_{x\to c} f(x) = -\infty$.

Objectives

In this section you will learn the following

- The concept of contiinuity of a function at a point.

Elementary properties of continuous functions.

5.2 Continuity of functions

Recall that for a function f , which may or may not be defined at a point c , if $\lim_{x\to c} f(x)$ exists, then it is the value that the function f is expected to take in consideration of its values at points in a neighborhood of c. Suppose, the function is actually defined at c . In case the value expected of the function at x = c exists and is equal to the actual value $f(c)$, it is natural to say that there is a continuity in the behaviour of f . This motivates our next definition.

5.2.1 Definition:

Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}$ and c be an interior point of A. We say f is continuous at c if $\lim_{x\to c} f(x)$ exists and is equal to $f(c)$. Equivalently f is continuous at c if for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$x \in A \ \ and \ \ |x-c| < \delta \ \ \Rightarrow \ \ |f(x) - f(c)| < \epsilon .$$

We say f is continuous on A if f is continuous at each $c \in A$. If $A = [a,b]$ we say f that is continuous at a if $\lim_{x\to a^+} f(x)$ exists and is equal to $f(a)$, and f is continuous at b if $\lim_{x\to b^-} f(x)$ exists and is equal to $f(b)$.

5.2.2 Examples:

(i) The function $f(x) = x$ is continuous at every point. For example we can take $\delta = \varepsilon$, for any given ε to claim that

$$|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon .$$

(ii) The function $f(x) = |x|$ is continuous everywhere as $\big| |x| - |c| \big| \le |x - c|$, and hence once again we can choose $\delta = \varepsilon$ to claim that $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.

(iii) The trigonometric functions $f(x) = \sin x \ \ and \ \ f(x) = \cos x$ are continuous on $\mathbb{R}$. To see this note that

$$\sin x - \sin c = 2\sin\frac{x-c}{2}\cos\frac{x+c}{2}, \ \cos x - \cos c = -2\sin\frac{x-c}{2}\sin\frac{x+c}{2},$$

$$|\sin\theta| \le |\theta|, \ |\sin\theta| \le 1, \ \ and \ \ |\cos\theta| \le 1 \ \ for\ all \ \ \theta \in \mathbb{R}$$

Thus, we can choose $\delta \le \varepsilon$ for any given ε to satisfy the given requirement of continuity.

(iv) Let $f(x) = \sqrt[n]{x}, \ x \in [0, \infty), \ where \ \ n \ge 1.$ For any given $\varepsilon > 0$, let $\delta > 0$ be chosen such that

$\delta < \in^{n}$. Then, for $c = 0, |x - c| < \delta$ implies that $\left|\sqrt[n]{x}\right| < \delta^{\frac{1}{n}} < \varepsilon$.

Hence, f is continuous at $c = 0$. For $c = 1$, given $0 < \varepsilon < 1$, $(1-\varepsilon) < x < (1+\varepsilon)$ implies that

$$(1-\varepsilon) < (1-\varepsilon)^{\frac{1}{n}} < \sqrt[n]{x} < (1+\varepsilon)^{\frac{1}{n}} < 1+\varepsilon,$$

$$i.e., \quad |x-1| < \delta = \varepsilon \;\Rightarrow\; |\sqrt[n]{x} - 1| < \varepsilon.$$

Hence, $\lim\limits_{x \to 1}\left(\sqrt[n]{x}\right) = 1$. Finally, for any $c \in (0, \infty)$ using limit theorems, we have

$$\lim_{x \to c}\left(\sqrt[n]{x}\right) = \sqrt[n]{c}\lim_{y \to 1}\left(\sqrt[n]{y}\right) = \left(\sqrt[n]{c}\right).$$

Hence, $f(x) = \sqrt[n]{x}$, $x \in [0, \infty)$ is continuous everywhere on $[0, \infty)$.

(v) The function $f(x) = [x]$, is not continuous at c, if c is an integer since the left hand limit is not equal to the right hand limit.

(vi) Consider the function $f(x) = 1$ if x is rational, and $f(x) = 0$ if x is irrational. It is discontinuous at every $c \in \mathbb{R}$. To see this note that given $c \in \mathbb{R}$, we can choose $\{x_n\}_{n \ge 1}$ and $\{y_n\}_{n \ge 1}$ both conveying to c, where each x_n is a rational an each y_n is an irrational. Then,

$f(x_n) \to 0 \;\; while \;\; f(y_n) \to 1$. Thus,

$\lim\limits_{x \to c} f(x)$ does not exist.

(vii) The function

$$f(x) = \sin\frac{1}{x} \;\; for \;\; x \neq 0 \quad and \quad f(0) = 0,$$

is discontinuous at $x = 0$. To see this, let

$$x_n := \frac{1}{n\pi} \;\; and \;\; y_n := \frac{1}{(2n + 1/2)\pi}, n \ge 1.$$

Then, $x_n \to 0$, $y_n \to 0$. However, $f(x_n) \to 0 \;\; while \;\; f(x_n) \to 1$. Hence, $\lim\limits_{x \to 0} f(x)$ does not exist.

5.2 .3 Theorem:

Let $A \subseteq \mathbb{R}, c \in A$ and $r \in \mathbb{R}$. Let $f, g : A \to \mathbb{R}$ be functions,both continuous at $x = c$. Then the following holds:

(i) $f \pm g$ and fg are continuous at $x = c$.

(ii)If $g(c) \neq 0$, $then \;\; f/g$ is defined in a neighbourhood of c and is continuous at c.

5.2 .3 Theorem:

Let $A \subseteq \mathbb{R}, c \in A$ and $r \in \mathbb{R}$. Let $f, g : A \to \mathbb{R}$ be functions,both continuous at .

Then the following holds:

(i) $f \pm g$ and fg are continuous at $x = c$.

(ii)If $g(c) \neq 0$, $then \;\; f/g$ is defined in a neighbourhood of and is continuous at .

Proof:

Follows from theorem 3.1.3

5.2 .4 **Examples:**

(i) It follows from the above theorem that every polynomial function $p:\mathbb{R}\to\mathbb{R}$ with

$$p(x)=a_n x^n + a_{n-1}x^{n-1}+\ldots+a_1 x + a_0,$$

where $n\in\mathbb{N}$ *and* $a_0, a_1, \ldots, a_n \in \mathbb{R}$ is continuous. Also, any rational function , that is, a function of the form $f(x)=p(x)/q(x)$ where p and q are polynomial functions, is continuous at every point $c\in\mathbb{R}$ for which $q(c)\neq 0$.

(ii) All trigonometric functions (and also, rational functions in them) are continuous wherever they are defined.

5.2 .5 **Theorem:**

Let $A,B\subseteq\mathbb{R}, f:A\to\mathbb{R}, g:B\to\mathbb{R}$ with range $f(A)\subseteq B$. For $c\in A$, if f is continuous at c and g is

continuous at $f(c)$, then gof is continuous at c .

5.2 .5 Theorem:

Let $A,B\subseteq\mathbb{R}, f:A\to\mathbb{R}, g:B\to\mathbb{R}$ with $f(A)\subseteq B$. For $c\in A$, if f is continuous at c and g is continuous at $f(c)$, then gof is continuous at .

Proof:

Let $c\in A$ and $x_n\to c, x_n\in A$.

Then, by the continuity of f, $f(x_n)\to f(c)$.

Now, by the continuity of g, $g(f(x_n))\to g(f(c))$.

Thus, gof is continuous at $x=c$.

5.2 .6 **Example:**

As an application of theorems 5.2.3 and 5.2.5, it follows that the function $F:\mathbb{R}\to\mathbb{R}$ defined by

$F(x)=x^3\cos^4\left(\sqrt{|x|}\right)+\left|\sin^2(\cos 2x)\right|$ is continuous.

5.2 .7 Note:

Geometrically, saying that a function $f:[a,b]\to\mathbb{R}$ is continuous on the interval, means that there is no break in the graph of the function, we can draw its graph on paper starting with $(a,f(a))$ to $(b,f(b))$ without lifting the pen.

please refer website for quize

Practice Exercises 5.2 : Continuity of a function

(1) Discuss the continuity of the following functions :

(i) $f(x) = \sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$.

(ii) $f(x) = x\sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$.

(iii) $f(x) = (x^3 - 3x - 2)/(x - 2)$, if $x \neq 2$ and $f(2) = 1$.

(2) Discuss continuity of f at x = 2, where $f:[1,3] \to \mathbb{R}$ is such that $\frac{x}{[x]} \leq f(x) \leq \sqrt{6-x}$ for all

$x \in [1,3]$, $f(2) = 1$ and is continuous on $[1,2) \cup (2,3]$.

(3) Let $f:(-2,2) \to \mathbb{R}$ be continuous function such that in every neighbourhood of 0, there exists a point where
f takes the value 0. Show that $f(0) = 0$.

(4) Let $f:\mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every $c \in \mathbb{R}$.

(5) Let $f:\mathbb{R} \to \mathbb{R}$, and $\alpha \in \mathbb{R}$ be such that for every $x_1, x_2 \in \mathbb{R}$
$|f(x_1) - f(x_2)| \leq \alpha|x_1 - x_2|$. Show that f is continuous.

(6) Construct a function $f:[0,1] \to [0,1]$ such that f is one-one, onto but not continuous.

(7) Let $f:[a,b] \to \mathbb{R}$ satisfy the following:

(i) $|f(x)| \leq M$ for some M, for all $x \in [a,b]$

(ii) $f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}\{f(x_1) + f(x_2)\}$ for every $x_1, x_2 \in [a,b]$ with $a \leq x_1 \leq x_2 \leq b$.

Show that f is continuous.

Optional Exercises:

Show that the function f in Problem 4 satisfies the following relation:
$f(kx) = kf(x)$,

(a) for all $k \in \mathbb{N}, x \in \mathbb{R}$.

(b) for all $k \in \mathbb{Z}, x \in \mathbb{R}$.

(c) for all $k \in \mathbb{Q}, x \in \mathbb{R}$.

(d) for all $k \in \mathbb{R}, x \in \mathbb{R}$.

Deduce that $f(x) = \alpha x \; \forall \; x \in \mathbb{R}$ and for some $\alpha \in \mathbb{R}$.

Historical Comments

The idea of limit existed implicitly in the 'method of exhaustion' – a method used by the Greek mathematicians (500-200 B.C.) such as Eudoxes and Archimedes. It was Isaac Newton who first talked explicitly about limits in his work on Calculus. The concept of limit was finally defined by Cauchy (1789-1857). Cauchy worked as a civil engineer in Napoleon's army. Despite his busy schedule, he helped local authorities in conducting school examinations and continued doing his research. He produced about 500 research papers in diverse branches of mathematics.

Recap

In this section you have learnt the following

- The concept of continuity of a function at a point.

- Various theorems analyzing algera of continuous functions.

Congratulations ! You have finished Lecture - 5. To view the next Lecture select it from the left hand side of the page.