Module 13 : Maxima, Minima and Saddle Points, Constrained maxima and minima

Lecture 38 : Second derivative test for local maxima / minima & saddle points [Section 38.1]

Objectives

In this section you will learn the following :

- The second derivative test for locating points of local maximum/minimum and saddle points.

38 .1 Second derivative test for local maxima/minima and saddle points

In order to obtain sufficient conditions for local maxima, local minima and saddle points, we need the following notion.

38.1.1 Definition

Let $f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ and $(x_0, y_0) \in D$ be an interior point. Let all the second order partial derivatives of f at (x_0, y_0) exist. Then

$$\Delta f(x_0, y_0) := \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}$$

is called the **discriminant (or hessian)** of f at (x_0, y_0).

38.1.2 Note:

If

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

then

$$\Delta f(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2.$$

For example, this would be the case if f_{xy} and f_{yx} are continuous at (x_0, y_0). Further, in such a case, if $\Delta f(x_0, y_0) > 0$, then both $f_{xx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$ are nonzero and have the same sign.

38.1.3Theorem (Discriminant Test):

Assume that the first and second order partial derivatives of f exist and are continuous at every point of $B_r(x_0, y_0)$. Suppose

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

Then, we have the following:

(i) The function f has a

local maximum at (x_0, y_0) if $\Delta f(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$ (or $f_{yy} > 0$).

(ii) The function f has a

local minimum at (x_0, y_0) if $\Delta f(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$ (or $f_{yy} > 0$).

(iii) The function f has a

saddle point at (x_0, y_0) if $\Delta f(x_0, y_0) < 0$.

We assume his result, the interested reader may refer a book on advanced calculus book.

38.1.4Example:

Let

$$f(x, y) = 4xy - x^4 - y^4 \text{ for } (x, y) \in \mathbb{R}.$$

Then,

$$f_x(x_0, y_0) = 4(y_0 - x_0^3) \text{ and } f_y(x_0, y_0) = 4(x_0 - y_0^3).$$

Thus,

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

gives us

$$(x_0, y_0) = (0, 0), (1, 1) \text{or} (-1, -1).$$

Further,

$$f_{xx}(x_0, y_0) = -12x_0^2,\ f_{xy}(x_0, y_0) = 4,\ f_{yy}(x_0, y_0) = -12y_0^2$$

and

$\Delta f(x_0, y_0) = 16(9{x_0}^2{y_0}^2 - 1).$

Hence,

$\Delta f(0,0) = -16 < 0,$ implying that $(0,0)$ is a saddle point.

Also,

$\Delta f(1,1) = \Delta f(-1,-1) = 128 > 0,$

and

$f_{xx}(1,1) = f_{xx}(-1,-1) = -12 < 0.$

Thus,

f has a local maximum at $(1,1)$ as well as at $(-1,-1).$

Further,

along the line $x = y, f(x,x) = 2x^2(z - x^2) > 0,$ for every x

and

along the line $x = -y, f(x,-x) = -2x^2(2 - x^2) < 0$ for all $0 < x^2 < 1.$

Thus,

the point $(0,0)$ is a saddle point for f.

38.1.5 Remark:

The Discriminant test is inconclusive at (x_0, y_0) when

$f_x(x_0, y_0) = f_y(x_0, y_0) = \Delta f(x_0, y_0) = 0.$

For details see next examples.

38.1.6 Examples:

(i) The function

$f(x,y) = -(x^4 + y^4)$
has a local maximum at $(0,0)$ even though $f_x(0,0) = f_y(0,0) = \Delta f(0,0) = 0.$

(ii) On the other hand the function

$f(x,y) = x^4 + y^4$

has a local minimum at $(0,0)$ and $f_x(0,0) = f_y(0,0) = \Delta f(0,0) = 0.$

(iii) The function

$$f(x,y) = 4x^3y - 4xy^3,$$

satisfies the property that $f_x(0,0) = f_y(0,0) = \Delta f(0,0) = 0$. Further, if we consider the curve

$$(x_1(t), y_1(t)) = (t, -t/2), t \in \mathbb{R}.$$

Along this curve,

$$f(x_1(t), y_1(t)) = -\frac{7t^4}{4} \text{ for every } t \in \mathbb{R}.$$

Thus, we can find points (x, y) close to $(0,0)$ where $f(x,y) < f(0,0) = 0$.
Similarly, along the curve

$$(x_1(t), y_1(t)) = (t, t/2), t \in \mathbb{R}$$

the function is given by

$$f(x_1(t), y_1(t)) = \frac{7t^4}{4} \text{ for every } t \in \mathbb{R}.$$

Thus, we can find points (x, y) close to $(0,0)$ where $f(x,y) > f(0,0) = 0$. Hence, f has a saddle point at $(0,0)$.

Practice Exercises

(1) Show that the following functions have local minima at the indicated points.

(i) $f(x,y) = x^4 + y^4 + 4x - 32y - 7, (x_0, y_0) = (-1, 2)$.

(ii) $f(x,y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3, (x_0, y_0) = (0,0)$.

(2) Analyze the following functions for local maxima, local minima and saddle points:

(i) $f(x,y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$.

(ii) $f(x,y) = x^3 - 3xy^2$.

(iii) $f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.

(iv) $f(x,y) = x^3 + y^3 - 3xy + 15$.

(3) Let

$$f(x,y) = y^2 - 4x^2y - 3x^4.$$

(i) Show that $(0,0)$ is a critical point for f, but the second derivative test fails.

(ii) Show that $(0,0)$ is neither a point of local maximum nor local minimum

(iii) Show that along every line through the origin, f has a local minimum at $(0,0)$

(iv) Show that along the curve $y = 2x^2$, f has a local maximum at $(0,0)$.

(4) Let

$$f(x,y) = x^2 y^2, (x,y) \in \mathbb{R}^2.$$

Find all the critical points of f and analyze them for being points of local maximum/minimum/saddle point. Every point along x-axis and y-axis is a point of local minimum.

(5) Let

$$f(x,y) = 4x^2 e^y - 2x^4 - e^4.$$

Show that f has only two critical points, both being points of local maximum. (Note that for a function of a single variable, between any two points of local maximum, there must be a point of local minimum).

Recap

In this section you have learnt the following

- The second derivative test for locating points of local maximum/minimum and saddle points.