Module 2 : Limits and Continuity of Functions

Lecture 4 : Limit at a point

Objectives

In this section you will learn the following

- The sequential concept of limit of a function.
- The $\varepsilon - \delta$ definition of the limit of a function.

4 Limit and Continuity of Functions

Recall that, our aim is to understand a function $f: A \subseteq \mathbb{R} \to \mathbb{R}$ by analyzing various properties of f. For example, one would like to analyze:

Does the 'graph' of f have any 'breaks' ?

In this lecture we shall analyze the most important and fundamental concept: limit of a function, and shall see how it helps us to answer the above question.

4.1 Limit of a function concept :

Let us start with the following problem:

How to predict a suitable value of a function at a point, which may or may not be in its domain, by analyzing its values at points in the domain which are near the given point?

Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$. Let $c \in \mathbb{R}$, c may or may not be an element of A. The question we want to answer is the following : Can we predict some 'suitable' value l for f at c by looking at the values of f at points close to c in A? To answer this, let us assume that f is defined at all points sufficiently near c (may be not at c), for otherwise we have no data on the basis of which we can predict.

For example, this is true when A is an open interval or $c \in I \subseteq A$ where I is an open interval.

Next, we should clarify as to what do we mean by saying that a real number $l \in \mathbb{R}$ is a 'suitable value' for f at c ?

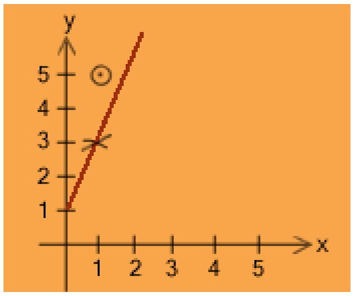
One way of interpreting this is to demand that the values $f(x)$ comes closer to the number l as the point x comes 'closer' to c. This immediately raises the following question: How do we interpret this mathematically ? A natural way of doing this is to say that this closeness is achieved iteratively, i.e., we can come close to any point $\mathbb{R}$ via sequences.

So if we approach c by any sequence of points in A, say $\{c_n\}_{n\geq 1}$ with $c_n \to c$, then we would like sequences of values of f at $x = c_n$ to converge to the same value, namely l, i.e., $f(c_n) \to l$. In that case we can predict the value l for f at the point c.
Let us look at some examples.

4.1 .1Example :

i) Consider a function $f:[0,3] \to \mathbb{R}$ defined as :

$$f(x) = \begin{cases} 2x+1 & \text{if } 0 \leq x \leq 3,\, x \neq 1 \\ 5 & \text{if } x = 1 \end{cases}$$



Clearly, f is defined at all points near $x=1$. Though f is defined at $x=1$ also, our aim is to predict a suitable value for f at $x=1$ by analyzing its values at points near $x=1$. For example, let us approach the point $x=1$ by a sequence, i.e., consider any sequence $\{c_n\}_{n\geq 1}$ of points in the domain of f such that $c_n \neq 1$ for all $n \geq 1$ and $c_n \to 1$. Then, $f(c_n) = 2c_n + 1$. Since $c_n \to 1$, it follows, from the limit theorems of sequences (see section 3.2.1), that $f(c_n) = (2c_n + 1) \to 3$. Hence, we can say that the natural value that f should take at $x=1$ is 3 .

Click here to see an interactive visualization: Applet 2.1

(ii) Let $f(x) = [x], x \in \mathbb{R}$, the greatest integer function. Clearly, $f(x) = 0$ for $1/2 < x < 1$ and $f(x) = 1$ for
$1 < x < 3/2.$

Thus, if we take a sequence $\{c_n = 1 - 1/3n\}_{n\geq 1}$, then clearly, $c_n \to 1$ and $f(c_n) \to 0$, as $f(c_n) = 0 \;\forall\; n \geq 1$. On other hand, if we take sequence $\{c_n = 1 + 1/3n\}_{n\geq 1}$, then again $c_n \to 1$, but $f(c_n) \to 1$, as $f(c_n) = 1 \;\forall\; n \geq 1$. Thus, we cannot predict a single value for f at $x=1$.

Click here to see an interactive visualization: [Applet 2.2]()

This motivates the following definition.

4.1 .2 Definition :

Let I be an open interval of $\mathbb{R}$ and $c \in I$. Let $A = I \setminus \{c\}$. Let $f : A \to \mathbb{R}$. We say that f has limit at c if there is a real number l with the property that $f(c_n) \to l$, for every sequence $\{c_n\}_{n \geq 1}$ with $c_n \to c$.

Such l is unique (see exercise 3), whenever it exists and is denoted by $\lim\limits_{x \to c} f(x)$.

In view of the algebra of limits for sequences (see section 3.2), we have the following theorems.

4.1 .3 Theorem (Algebra of limits):

Suppose $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ exist. Then the following hold:

(i) $\lim\limits_{x \to c} (f+g)(x) = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$.

(ii) $\lim\limits_{x \to c} (fg)(x) = \lim\limits_{x \to c} f(x) \lim\limits_{x \to c} g(x)$.

(iii) For any real number α, $\lim\limits_{x \to c} (\alpha f)(x) = \alpha \lim\limits_{x \to c} f(x)$.

(iv) If $\lim\limits_{x \to c} g(x) \neq 0$, then $\lim\limits_{x \to c} (f/g)(x) = \lim\limits_{x \to c} f(x) / \lim\limits_{x \to c} g(x)$.

PROOF

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Proof:

Follows from the Limit Theorems for sequences. We leave the details as an exercise.

4.1 .4 Sandwich Theorem :

Suppose $f, g, h : (c-r, c+r) \to \mathbb{R}$ are functions such that $f(x) \leq g(x) \leq h(x)$ for all $x \in (c-r, c) \cup (c, c+r)$. for some r > 0.

If $\lim\limits_{x \to c} f(x) = l = \lim\limits_{x \to c} h(x)$, then $\lim\limits_{x \to c} g(x) = l$.

Proof

4.1 .4 Sandwich Theorem :

Suppose $f(x) \le g(x) \le h(x)$ for all $x \in (c-r, c) \cup (c, c+r)$. for some r > 0.
If $\lim_{x\to c} f(x) = l = \lim_{x\to c} h(x)$, then $\lim_{x\to c} g(x) = l$.

Proof:

Follows from the Sandwich Theorem for sequences.

Next we look at another way of describing the statement that a function has a limit at point. To predict the value of
a function f at a point c we have to analyze the values $f(x)$ of the function as x approaches c. In our
definition above, we used the concept of sequences $c_n \to c$. One can directly use the notion of distance for
this. Suppose we want to analyse whether a number l is the natural value expected of f at $x = c$ or not?
At a point x near c, $x \ne c, |f(x) - l|$ is the error one will be making for being not equal to value expected. If l
is the value expected, then one would like to make this error small, smaller than any given value. Let us say that
this error is less than a given value $\in > 0$ for all points sufficiently close to c . Let us look at an example.

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If l is the value expected, then one would like to make this error small, smaller than any given value. Let us say that this error is less than a given value $\varepsilon > 0$ for all points sufficiently close to c. Let us look at an example.

4 Limit and Continuity of Functions

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How to predict a suitable value of a function at a point, which may or may not be in its domain, by analyzing its values at points in the domain which are near the given point?

Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$. Let $c \in \mathbb{R}$, c may or may not be an element of A. The question we want to answer is the following : Can we predict some 'suitable' value l for f at c by looking at the values of f at points close to c in A? To answer this, let us assume that f is defined at all points sufficiently near c (may be not at c), for otherwise we have no data on the basis of which we can predict.

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4.1.5 Example:

Consider the function $f : [0,3] \longrightarrow \mathbb{R}$ with

$$f(x) = \begin{cases} 2x+1 & if\ 0 \le x \le 3, x \ne 1 \\ 5 & if\ x = 1 \end{cases}$$

Natural value expected of f at 1, by looking at values near 1, is 3 and not 5 .

For example, the error

$$\left|f(x) - 3\right| = \left|2x - 2\right| < \frac{1}{10}$$

whenever the point x is close to 1 by distance $\frac{1}{20}$. In other words, $\forall\ x \in [0,3]$,

$$0 < |x - 1| < \frac{1}{20} \Rightarrow \left|f(x) - 3\right| < \frac{1}{10}.$$

In fact, if we want $f(x)$ close to $l = 3$ by a distance (error) at most ε (any positive real number), then $\forall\ x \in [0,3]$

$$0 < |x-1| < \varepsilon / 2 \Rightarrow \left|f(x) - 3\right| < \varepsilon ,$$

i.e., given any $\varepsilon > 0$ we can choose $\delta = \varepsilon / 2 > 0$ such that $f(x)$ is close to 3 by distance ε whenever x is close to 1 by distance δ .

This motivates our next definition.

4.1.6 Definition :

Let I be an open interval of $\mathbb{R}$ and $c \in I$. Let $A = I \setminus \{c\}$. Let $f : A \longrightarrow \mathbb{R}$. A real number l is called an $\varepsilon - \delta$ limit of f as x tends to c if the following hold: given any real number $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$x \in A,\ 0 < |x - c| < \delta \Rightarrow \left|f(x) - l\right| < \varepsilon.$$

Such a l, whenever it exists, is unique (see excercise 3) and is denoted by $\lim\limits_{x \to c} f(x)$.

Click here to see an interactive visualization: Applet 2.3

Let us look at some examples.

4.1 .7 Examples :

(i) Let $f(x) = x^3$ if $x \neq 2$ and $f(2) = 1$. Then, $\lim_{x\to 2} f(x) = 8$. Indeed,

$$|x^3 - 8| = |x-2||x^2 + 2x + 4|.$$

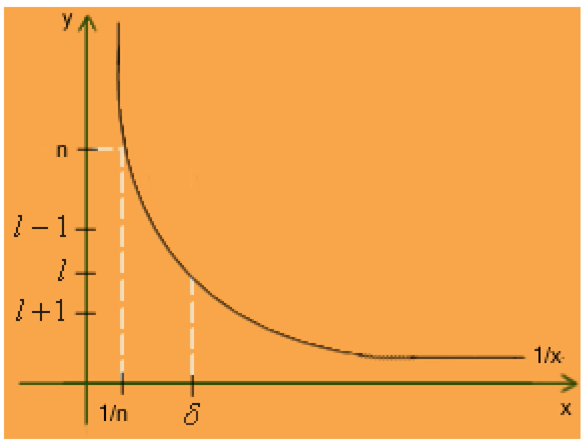
We find an upper bound for $|x^2 + 2x + 4|$ when x is close to 2 , say $|x-2| < 1$, that is $1 < x < 3$. Then,

$$|x^2 + 2x + 4| < 9 + 6 + 4 = 19 .$$

Thus, given any $\varepsilon > 0$, we may take $\delta = \min\{1, \varepsilon/19\}$ and then,

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 8| < 19|x - 2| < \varepsilon .$$

(ii) Let $f(x) = \frac{1}{x}$ if $x \neq 0$. . We claim that $\lim_{x\to 0} f(x)$ does not exist.



Suppose, $\lim_{x\to 0} f(x)$ exists and the limit is l . Then, for $\varepsilon = 1, \exists \delta > 0$ such that

$$0 < |x| < \delta \Rightarrow \left|\frac{1}{x} - l\right| < \varepsilon = 1 .$$

In particular, for $0 < x < \delta$,

$$l - 1 < \frac{1}{x} < l + 1 .$$

That is,

$$\frac{1}{x} < l + 1 \quad \text{for every } 0 < x < \delta .$$

This is not possible, for example, we can choose positive integer n such that $\frac{1}{n} < \delta$, but

$$f\left(\frac{1}{n}\right) = n > l + 1 .$$

Hence, $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Click here to see an interactive visualization : Applet 2.4

Before proceeding further, we show that the existence of limit is equivalent to the existence of the $\varepsilon - \delta$ limit.

4.1 .8 Theorem :

For a function $f: A \to \mathbb{R}$, the $\varepsilon - \delta$ limit exists at a point c if and only if $\lim_{x \to c} f(x) = l$, i.e., for every sequence $\{x_n\}_{n \geq 0}$ with $\lim_{n \to \infty} x_n = c, x_n \neq c$ and $x_n \in A$ for all $n \in \mathbb{N}$, we have $\lim_{n \to \infty} f(x_n) = l$.

Proof:

Assume that $\lim_{x \to c} f(x) = l$ and $\lim_{n \to \infty} x_n = c$, $x_n \neq c$ and $x_n \in A$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $0 < |x - c| < \delta \Rightarrow |f(x) - l| < \in$. Next, for this δ choose $n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow |x_n - c| < \delta$. Then, for $n > n_0$, $0 < |x_n - c| < \delta$ implies $|f(x_n) - l| < \varepsilon$. Hence, $\lim_{x \to c} f(x) = l$.

Conversely, suppose that the $\varepsilon - \delta$ limit of f at c does not exist. Then, there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there is some $x \in A$ with

$0 < |x - c| < \delta$, but $|f(x_n) - l| \geq \in$.

In particular, for each $n \in \mathbb{N}$ there is some $x_n \in A$ with

$0 < |x_n - c| < \frac{1}{n}$, but $|f(x_n) - l| \geq \in$.

Then $x_n \neq c$ and $x_n \in A$ for all $n \in \mathbb{N}$, $\lim_{n \to \infty} x_n = c$, but $\lim_{n \to \infty} f(x_n) \neq l$. This is a contradiction. Hence the $\varepsilon - \delta$ limit of f at c exists and is equal to l.

4.1.9 Note :

(i) $\lim_{x \to c} f(x)$ depends on the values of f at points near c . The function f may or may not be defined at c.

Even if f is defined at c , $\lim_{x \to c} f(x)$ may or may not exist. Even if $\lim_{x \to c} f(x)$ exist, it need not be equal to $f(c)$.

(ii) To find $\lim_{x \to c} f(x)$, one has to make a guess and then prove it.

Let us note that, $\lim_{x \to c} f(x) = l$ means that for a given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$,

$$0 < |x - c| < \delta \quad \text{implies} \quad |f(x_n) - l| < \varepsilon.$$

Equivalently, $\forall \varepsilon > 0$ there exists $\delta > 0$ such that

$$x \in (c - \delta, c) \text{ implies } |f(x) - l| < \varepsilon \quad \text{and} \quad x \in (c, c + \delta) \text{ implies } |f(x) - l| < \varepsilon.$$

This motivates our next definition.

4.1.10 Definitions :

Let I be an open interval, $c \in I$ and $A = I \setminus \{c\}$. Let $f: A \subseteq \mathbb{R} \to \mathbb{R}$.

(i) We say f has left-hand limit at a point $x = c \in A$, if there is a real number l with the property that for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$x \in A, c - \delta < x < c \Rightarrow |f(x) - l| < \varepsilon.$$

We write this as $\lim_{x \to c^-} f(x) = l$, and call l to be the left-hand limit of f at $x = c$.

(ii) We say a function f has right-hand limit l at a point x =c if there is a real number l with the property that

for every $\varepsilon > 0$ there is some $\delta > 0$ such that

$$x \in A, c < x < c + \delta \Rightarrow |f(x) - l| < \varepsilon .$$

We write this as $\lim_{x \to c^+} f(x) = l$, and call l to be the right-hand limit of f at $x = c$.

The above remarks tell us the following :

4.1.11 Theorem :

Let $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that $(c - r, c) \cup (c, c + r)$ is contained in A for some $r > 0$. Then $\lim_{x \to c} f(x)$ exists and is equal to l if and only if $\lim_{x \to c^-} f(x) = l$ as well as $\lim_{x \to c^+} f(x) = l$. That is the limit of a function at a point exists and is equal to l if and only if both, the left- hand and the right hand limits exist and are equal to l.

4.1.12 Examples :

(i) If $f(x) = [x], x \in \mathbb{R}$, the greatest integer function, then

$$\lim_{x \to 1^-} f(x) = 0 \text{ and } \lim_{x \to 1^+} f(x) = 1.$$

Thus, $\lim_{x \to 1} f(x)$ does not exist.

(ii) Let $f(x) = \begin{cases} +1 \text{ if } & x \geq 0 \\ -1 \text{ if } & x < 0. \end{cases}$

Then,

$$\lim_{x \to 0+} f(x) = +1 \text{ and } \lim_{x \to 0-} f(x) = -1.$$

Thus, $\lim_{x \to 1} f(x)$ does not exist.

4.1 .13 Example :

Let $f(x) = \sin \frac{1}{x}, x \neq 0$. To analyze $\lim_{x \to 0} f(x)$, consider

$$x_n = \frac{1}{n\pi}, n \geq 1 .$$

Then, $x_n \to 0$ and

$$f(x_n) = \sin(n\pi) = 0 \to 0 .$$

However, if we consider

$$y_n = \frac{2}{(4n+1)\pi}, \text{ then } y_n \to 0 \text{ and for every } n \geq 1$$

$$f(y_n) = \sin \frac{(4n+1)\pi}{2} = 1 .$$

Hence, $f(y_n) \to 1$. Thus, though both $\{x_n\}_{n \geq 1}$ and $\{y_n\}_{n \geq 1}$ converge to 0, but

$\{f(x_n)\}_{n\geq 1}$, and $\{f(y_n)\}_{n\geq 1}$ and converge to different limits.

Hence, limit does not exist, by the previous theorem.

4.1 .14 Note :

Theorems similar to that of theorem 2.1.3 hold for left-hand and right-hand limits.

4.1.15Examples :

Let $0 < \theta < \frac{\pi}{2}$.

(i) Since $-\theta < \sin\theta < \theta$, we get $\lim_{\theta\to 0^+} \sin\theta = 0$.

(ii) Since $-\theta < 1 < -\cos\theta$, $\lim_{\theta\to 0^+} \cos\theta = 1$.

(iii) Since $\theta\cos\theta < \sin\theta < \theta$, we get $\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$.

Using the above properties and changing θ to $-\theta$, it is easy to show that

$$\lim_{\theta\to 0} \sin\theta = 0, \lim_{\theta\to 0} \cos\theta = 1, \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

please refer website for quiz

Practice Exercises : Limits of Functions

1. For the following functions $f(x)$, given $\varepsilon > 0$, find some $\delta > 0$ such that $|f(x) - l| < \varepsilon$, whenever $0 < |x - c| < \delta$, where

(i) $f(x) = x^2 + 1, c = 1, l = 2$.

(ii) $f(x) = x\sin\frac{1}{x}, c = 0, l = 0$.

(iii) $f(x) = \frac{2x^2 + 6x + 5}{x + 5}, c = -5, \ l = -4$.

(iv) $f(x) = \begin{cases} 4 - 2x, x < 1, \\ 6x - 4, x > 1, \end{cases} \quad c = 1, l = 1.$

2. Do the following limits exist? If so, find them.

(i) $\lim_{x\to 0} \frac{|x|}{x+1}$ (ii) $\lim_{x\to 1} (|x| - x)$ (iii) $\lim_{x\to\infty} \frac{x^4}{x^4 - 7x^3 + 2}$ (iv) $\lim_{x\to 0} x[x]$ (v) $\lim_{x\to 0} \frac{\sin x}{|x|}$.

3. Show that limit of a function is unique whenever it exists.

4. Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to c} f(x) = 0$. Prove or disprove the following statements:

(i) .

$$\lim_{x \to c}\left[f(x)g(x)\right] = 0$$

(ii) $\lim_{x \to c}\left[f(x)g(x)\right] = 0$, if g is bounded on $\left\{x \in \mathbb{R} : 0 < \left|x - c\right| < \delta\right\}$ for some $\delta > 0$.

(iii) $\lim_{x \to c}\left[f(x)g(x)\right] = 0$, if $\lim_{x \to c} g(x)$ exists.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for some $\lim_{h \to 0}\left[f(\alpha + h) - f(\alpha - h) \right] = 0$. Does this imply that $\lim_{x \to \alpha} f(x)$ exists? Analyze the converse.

6. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0$,

where $a_n, \ldots, a_0, b_m, \ldots, b_0$ are real numbers with $a_n \neq 0$ and $b_m \neq 0$. Show that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \begin{cases} 0, \text{ if } m > n \\ \dfrac{a_m}{b_m} \text{ if } m = n, \end{cases}$$

and that $\dfrac{f(x)}{g(x)} \to \infty$ if $m > n$ and $\dfrac{a_n}{b_m} > 0$, while $\dfrac{f(x)}{g(x)} \to -\infty$ if $m < n$ and $\dfrac{a_n}{b_m} > 0$.

7. Let $f(x) \geq \alpha$ for all $x \in (a, a + \delta)$, where $\delta > 0$. If $\lim_{x \to a^+} f(x) = l$, show that $l \geq \alpha$.

8. Let $f : (a,b) \to \mathbb{R}$ and $c \in (a,b)$. Prove that if $\lim_{x \to c} f(x) > \alpha$, then there is some $\delta > 0$ such that $f(c + h) > \alpha$ for all $0 < |h| < \delta$.

Recap

In this section you have learnt the following

- The sequential definition of limit of a function at a point.
- The $\varepsilon - \delta$ definition of limit of a function at a point.
- The equivalence of the two definitions.