Module 7 : Applications of Integration - I

Lecture 20 : Definition of the power function and logarithmic function with positive base [Section 20.1]

Objectives

In this section you will learn the following :

- How to define the power (exponential) function for arbitrary positive base.

20.1 The Power function

In view of the property as in theorem 19.2.2 (iii), the function $\exp(x)$ is also called the power function with the natural base, i.e., Using the functions $\exp(x)$ and $\ln(x)$, we define the power function for any base $a > 0$.

20.1.1 Definition:

Let $a > 0$. For $x \in \mathbb{R}$, define
$a^x := \exp(x \ln a)$.
The function $x \mapsto a^x$ is called the exponential function with base a.

20.1.2 Theorem:

For $a > 0$, the function
$f : \mathbb{R} \to (0, \infty), f(x) = a^x$
has the following properties:

(i) $f(0) = 1$.

(ii) $f'(x) = (\ln a) f(x)$.

(iii) $\left(a^{x_1}\right)^{x_2} = a^{(x_1 x_2)} = \left(a^{x_2}\right)^{x_1}$.

(iv) If $a = 1$, then
$f(x) = 1$ for all $x \in \mathbb{R}$.

(v) For $a > 1$, the function $f(x)$ is a strictly increasing and concave upward with

$$\lim_{x \to \infty} a^x = \infty \text{ and } \lim_{x \to -\infty} a^x = 0.$$

(vi) For $a < 1$, the function a^x is strictly decreasing, concave upward with

$$\lim_{x\to\infty} a^x = 0 \text{ and } \lim_{x\to-\infty} a^x = \infty.$$

(v) For $a \neq 1, a^x$ is one-one and its range is $(0,\infty)$. If $g:(0,\infty)\to\mathbb{R}$ is the inverse of f, then

$$g(x) = \left(\frac{\ln x}{\ln a}\right) \text{for all } x \in (0,\infty).$$

The function g is denoted by $\log_a$. Thus,

$$\log_a(x) = \left(\frac{\ln x}{\ln a}\right) \text{ for all } x \in (0,\infty).$$

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(iv) If $a=1$, then

$f(x)=1$ for all $x\in\mathbb{R}.$

(v) For $a>1$, the function $f(x)$ is a strictly increasing and concave upward with

$$\lim_{x\to\infty} a^x = \infty \text{ and } \lim_{x\to-\infty} a^x = 0.$$

(vi) For $a<1$, the function a^x is strictly decreasing, concave upward with

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(v) For $a\neq 1, a^x$ is one-one and its range is $(0,\infty)$. If $g:(0,\infty)\to\mathbb{R}$ is the inverse of f, then

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The function g is denoted by $\log_a$. Thus,

$$\log_a(x) = \left(\frac{\ln x}{\ln a}\right) \text{ for all } x \in (0,\infty).$$

Proof:

The statement (i) follows since,

$$f(0) := \exp(0) = 1.$$

20.1.3 Definition:

The function $\log_a(x), x \in (0,\infty)$ is called the logarithmic function with base a.

20.1.4 Note:

For $a = e$, the power function e^x, is given by

$e^x = \exp(x \ln e) = \exp(x)$ for every $x \in \mathbb{R}$.

20.1.5 Examples:

(i) Consider the function

$$f(x) = x^x, x > 0.$$

Let us analyse if for regions of increasing / decreasing and concavity. Since,

$$f(x) = \exp(x \ln x),$$

$$f'(x) = \exp(x \ln x)\left(x.\frac{1}{x} + \ln x\right)$$

$$= x^x(1 + \ln(x)) > 0 \text{ for all } x > 0.$$

Thus $f(x)$ is a strictly increasing function.

Further

$$f''(x) = f'(x)\left(1 + \ln(x)\right) + f(x)\left(1 + \frac{1}{x}\right)$$

$$= x^x\left(1 + \ln(x)\right)^2 + \frac{x^x\left(x+1\right)}{x} > 0,$$

it is a concave up function.

(ii) Let

$$f(x) = \frac{\ln(x)}{x}, x > 0.$$

Then

$$f'(x) = -\frac{\ln(x)}{x^2} + \frac{1}{x^2} = 1 - \frac{\ln(x)}{x^2}.$$

Since, $f'(x) > 0$ for $(0, e)$ and $f'(x) < 0$ for $(0, \infty)$, we have

$f(x)$ is strictly increasing on $(0, e)$,

$f(x)$ is strictly decreasing on (e, ∞),

and f has a global minimum on $x = e$.

Thus, if $0 < a < b$, then

$$\frac{\ln(b)}{b} < \frac{\ln(a)}{a},$$

$i.e., a\ln(b) < b\ln(a)$

$i.e., \ln(b^a) < \ln(a^b).$

Since $\ln(x)$ is a monotonically increasing function, above implies that

$$b^a < a^b$$

In particular, since $e < 3 < \pi$, we get the following inequalities:

$$3^e < e^3, \pi^e < e^\pi, \pi^3 < 3^e.$$

PRACTICE EXERCISES

1. Analyse the following functions for the region of increase/ decrease:

(i) $f(x) = x^{\frac{2}{x}}, x > 0.$

(ii) $f(x) = \log_2\left(\frac{x_2}{x-1}\right), x > 1.$

2. Sketch a graph of the following functions:

(i) $f(x) = 3^x, x > 0.$

(ii) $f(x) = 2^{-1/x}, x > 0.$

(iii) $f(x) = \left(\frac{1}{2}\right)^{x^2}, x > 0.$

3. Find equation of the tangent to the following at the indicated points

(i) $f(x) = \log_4 x + \log_4 x^2$ at $x = 1.$

(ii) $f(x) = (\log_2 x)(\log_4 x)$ at $x = 1.$

Recap

In this section you have learnt the following

- How to define the power (exponential) function for arbitrary positive base.

Section - 20.2

Objectives

In this section you will learn the following :

- How derivative function is useful in computing limits of the form $\frac{0}{0}, \frac{\infty}{\infty}$ and so on.

20.2 L'Hopital's Rule

In this section we shall see how the derivative function is useful in computing limits of certain types of functions.

20.2.1 Example:

Consider the function

$$f(x)=\frac{\sqrt{x^2+5}-3}{x^2-4},\ x\neq\pm 2.$$

We want to analyse $\lim_{x\to 2} f(x)$. We note that simply putting $x=2$ in the above formula does not help, as f is not defined at $x=2$, and

$$g(x):=\sqrt{x^2+5}-3=0=h(x)=x^2-4, \text{ for } x=2.$$

However, both $g(x)$ and $h(x)$ are differentiable at $x=2$. Thus, we may assume that near $x=2$,

$$f(x)\cong\frac{g(a)+(x-2)g'(a)}{h(a)+(x-2)h'(a)}=\frac{\dfrac{x}{\sqrt{x^2+5}}}{2x}.$$

Thus, we should expect that

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\left(\frac{x}{\sqrt{x^2+5}}\right)\left(\frac{1}{2x}\right)=\frac{1}{6}.$$

In fact, the above does hold and is made precise in the next theorem.

20.2.2 Theorem (L'Hopital Rule $\frac{0}{0}$ form)

Let $g,h:(a,b)\to\mathbb{R}$ be functions such that the following hold:

(i) $\lim_{x\to a^+} g(x)=0=\lim_{x\to a^+} h(x)$.

(ii) $g(x)$ and $h(x)$ are differentiable in (a,b) with $h'(x)\neq 0$.

(iii) $\lim_{x\to a^+}\left(\frac{g(x)}{h(x)}\right)$ exists, says L.

Then,

$$\lim_{x \to a^+}\left(\frac{g(x)}{h(x)}\right) = \lim_{x \to a^+}\left(\frac{g'(x)}{h'(x)}\right) = L.$$

20.2.4 Note:

(i) Theorem 20.2.2 is applicable for functions g, h defined in intervals of the type (c, a) with $x \to a^+$ replaced by

$x \to a^-$:

$$\lim_{x \to a^-}\frac{g(x)}{h(x)} = \lim_{x \to a^-}\frac{g'(x)}{h'(x)},$$

whenever the right hand side limit exists.

(ii) In theorem 20.2.2, since $g(x) = h(x) = 0$ for $x = a$, the limit

$$\frac{\lim_{x \to a^+} g(x)}{\lim_{x \to a^+} h(x)} = \frac{0}{0} \ .$$

This is called indeterminate form $\frac{0}{0}$.

(iii) The conclusion of theorem 20.2.2 is also true when $x \to +\infty$, or $x \to -\infty$, or $L = +\infty / -\infty$.

20.2.5 Examples:

(i) Let $g(x) = sin\ x$ and $h(x) = x$, then the $\lim_{x \to 0}\frac{sin\ x}{x}$ is of the form $\frac{0}{0}$, and by theorem 20.2.2,

$$\lim_{x \to 0}\left(\frac{sin\ x}{x}\right) = \lim_{x \to 0}\left(\frac{cos\ x}{1}\right) = 1.$$

(ii) We want to compute $\lim_{x \to +\infty}\left(\frac{1}{x^{\frac{4}{3}}\ Sin\left(\frac{1}{x}\right)}\right)$. Let

20.2.2 Theorem (L'Hopital Rule $\frac{0}{0}$ form)

Let $g, h : (a, b) \to \mathbb{R}$ be functions such that the following hold:

(i) $\lim_{x \to a^+} g(x) = 0 = \lim_{x \to a^+} h(x)$.

(ii) Both $g(x)$ and $h(x)$ are differentiable in (a, b) with $h'(x) \neq 0$.

(iii) $\lim\limits_{x\to a^+}\left(\dfrac{g(x)}{h(x)}\right)$ exists, says L.

Then,

$$\lim_{x\to a^+}\left(\frac{g(x)}{h(x)}\right)=\lim_{x\to a^+}\left(\frac{g'(x)}{h'(x)}\right)=L.$$

Proof :

The proof is an application of Cauchy's Mean Value theorem.

Let α, β be arbitrary real numbers with

$a<\alpha<\beta<b.$

We note that $h(\beta)\neq h(\alpha)$, for otherwise by Roll's theorem we will have $h'(\gamma)=0$ for some $\alpha<r<\beta$, which is not possible by (ii). Thus,by Cauchy's mean value theorem for $g,h:[\alpha,\beta]\to\mathbb{R}$, there exists a point $c\in(\alpha,\beta)$ such that

$$\frac{g(\beta)-g(\alpha)}{h(\beta)-h(\alpha)}=\frac{g'(c)}{h'(c)}\cdot\ (1)$$

By (iii), given $\epsilon>0$, there exists $\delta>0$ such that

$$L-\epsilon<\frac{g'(x)}{h'(x)}<L+\epsilon, \text{for all } x\in(a,a+\delta).$$

Now from (1), for $\beta<a+\delta$, we get

$$L-\epsilon<\frac{g(\beta)-g(\alpha)}{h(\beta)-h(\alpha)}<L+\epsilon, \text{for } a<\beta<a+\delta.$$

Since, this holds for all $a<\alpha<\beta$, letting $\alpha\to a$, we have $g(\alpha)\to 0, h(\alpha)\to 0,$ and hence

$$L-\epsilon\ \leq\frac{g(\beta)}{h(\beta)}\ \leq\ L+\epsilon, \text{for } a<\beta<a+\delta.$$

Hence,

$$\lim_{\beta\to a^+}\frac{g(\beta)}{h(\beta)}=L.$$

20.2.3 Examples:

(i) Let

$$g(x) = \sqrt{x^2+5} - 3 \text{ and } h(x) = x^2 - 4 \;.$$

Then, both $g(x)$ and $h(x)$ satisfy the conditions of the theorem in the interval say $(2,3)$. Thus,

$$\lim_{x\to 2^+}\left(\frac{\sqrt{x^2+5}-3}{x^2-4}\right) = \lim_{x\to 2^+}\left(\frac{x}{\sqrt{x^2+5}}\right)\left(\frac{1}{2x}\right) = \frac{1}{6}.$$

$$g(x) := x^{-\frac{4}{3}} \; and \; h(x) = \sin\left(\frac{1}{x}\right).$$

Then, $\displaystyle\lim_{x\to+\infty}\frac{g(x)}{h(x)}$ is of the form $\dfrac{0}{0}$, and hence

$$\lim_{x\to\infty}\left(\frac{x^{-\frac{4}{3}}}{\sin\frac{1}{x}}\right) = \lim_{x\to\infty}\left(\frac{-\frac{4}{3}x^{-\frac{4}{3}-1}}{\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}\right)$$

$$= \lim_{x\to\infty}\left(\frac{4}{3x\cos\left(\frac{1}{x}\right)}\right)$$

$$= 0.$$

(iii) Let us compute

$$\lim_{x\to 0}\frac{e^x-1}{x^3}$$

The required limit is of the form $\dfrac{0}{0}$. Thus

$$\lim_{x\to 0}\frac{e^x-1}{x^3} = \lim_{x\to 0}\frac{e^x}{3x^2} = +\infty.$$

We give below (without proof) some other terms of the L'Hopital's rule:

20.2.6 Theorem (L'Hopital's rule, $\dfrac{\infty}{\infty}$ form):

Let $g, h : (a,b) \to \mathbb{R}$ be differentiable functions such that $h'(x) \neq 0$ for every x and

(i) $\displaystyle\lim_{x\to a^+} g(x) = \pm\infty = \lim_{x\to a^+} h(x).$

(ii) $\lim_{x \to a^+} \frac{g'(x)}{h'(x)} = M$, exists,

where $M \in \mathbb{R}$ or $M = +\infty / -\infty$.

Then

$$\lim_{x \to a^+} \left(\frac{g(x)}{h(x)} \right) = M$$

20.2.7 Note:

(i) The conclusions of theorem 20.2.6 also holds when $x \to a^+$ is replaced by $x \to a^-$, provided g, h are defined

in some interval (c, a).

(ii) The conclusion of theorem 20.2.6 also holds when g, h are defined in some interval $(a, +\infty)$ and $x \to +\infty$ (or

in some interval $(-\infty, a)$ and $x \to -\infty$).

(iii) L'Hopital's rule also becomes applicable to indeterminate forms like $\infty - \infty$, $0(-\infty)$, 1^∞ and so on. In most of

these cases, we can bring the required limit to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

20.2.8Examples:

(i) $\lim_{x \to 0^+} \frac{\ln (sin\ x)}{\ln (x)}$ $\left(\frac{-\infty}{\infty} \text{form} \right)$

$$= \lim_{x \to 0^+} \frac{\frac{cos\ x}{sin\ x}}{\frac{1}{x}}$$

$$= \lim_{x \to 0^+} \left(\frac{x}{sin\ x} \right) (cos\ x)$$

$$= 1.$$

(ii) $\lim_{x \to \infty} \left(\frac{\ln (x)}{x} \right)$ $\left(\frac{\infty}{\infty} \text{form} \right)$

$$= \lim_{x \to \infty} \left(\frac{1}{x} \right)$$

$$= 0$$

(iii) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{sin\ x} \right)$ $(\infty - \infty)$

$$= \lim_{x\to 0^+}\left(\frac{\sin x - x}{x \sin x}\right) \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\cos x - 1}{\sin x + x\cos x}\right) \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{-\sin x}{2\cos x - x\sin x}\right)$$

$$= \frac{0}{2} = 0\,.$$

(iv) $\lim_{x\to 0^+}\left(x \ln x\right) \qquad \left(0-(-\infty)\right)$

$$= \lim_{x\to 0^+}\left(\frac{\ln x}{\frac{1}{x}}\right) \qquad \left(\frac{-\infty}{+\infty}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{1}{x}\right)\left(-x^2\right)$$

$$= 0\,\cdot$$

(v) To compute $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ which is of the form 1^∞, we write

$$f(x) := \left(1+\frac{1}{x}\right)^x\,\cdot$$

using the fact that ln(x) in conditions,

Applying $\ln$ function to both sides, and taking limit,

$$\lim_{x\to\infty}\left(\ln f(x)\right) = \lim_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)^x\right)$$

$$= \lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right) \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{1}{x}}\right)\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\left(\frac{1}{1+\frac{1}{x}}\right) = 1.$$

Hence,

$$\lim_{x\to\infty}\ln\left(f\left(x\right)\right) = 1.$$

Since $\exp\ (x)$ is a continuous function, we get

$$\lim_{x\to\infty} f\left(x\right) = \exp(1) = e\,.$$

Hence,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e\,\cdot$$

20.2.11 Remark

(i) L'Hopital's rule should not be taken as a golden rule, applicable always. For example

$\lim_{x\to\frac{\pi}{2}}\frac{\tan\ x}{\sec\ x}$ is of the form $\frac{\infty}{\infty}$, since

$$\frac{d^2}{dx^2}\left(\tan\ x\right) = \tan\ x, \frac{d^2(sec\ x)}{dx^2} = \sec\ x,$$

L'Hopital's rule will yield no result.

(ii) Consider

$$I = \lim_{x\to\infty}\frac{x^2 + sin\ x}{x^2}\,\cdot$$

It is of the form $\frac{\infty}{\infty}$,and if we apply L'Hopital's rule twice, we get

$$I = \lim_{x \to \infty} \frac{2x + \cos x}{2x}, \qquad \left(\frac{\infty}{\infty} \text{form} \right)$$
$$= \lim_{x \to \infty} \frac{2 - \sin x}{2},$$

and the last limit does not exist. However, from this we cannot conclude that I does not exist.

In fact,

$$\lim_{x \to \infty} \left(\frac{x^2 + \sin x}{x^2} \right) = \lim_{x \to \infty} \left(1 + \frac{\sin x}{x^2} \right)$$
$$= 1 + \lim_{x \to \infty} \left(\frac{\sin x}{x^2} \right)$$
$$= 1,$$

because

$$\left| \frac{\sin x}{x^2} \right| \le \frac{1}{x^2} \to 0 \text{ as } x \to \infty.$$

QUIZ

PRACTICE EXERCISES

1. Evaluate the following limits:

(i) $\lim_{x \to \frac{\pi}{2}} (\sin x - \tan x).$

(ii) $\lim_{x \to \infty} \left(x - \sqrt{x + x^2} \right).$

(iii) $\lim_{x \to 0} \frac{\cos x - 1 + \left(x^2 / 2 \right)}{x^4}.$

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x + 2, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) := \begin{cases} x + 1, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0, \end{cases}.$$

Show that $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = 1$, but $\lim_{x \to 0} \frac{f(x)}{g(x)} = 2$. Does this contradict L'Hopital's Rule?

3. Evaluate the following:

(i) $\lim_{x \to 0^+} x \ln x.$

(ii) $\lim_{x \to \infty} \frac{\ln(\ln x)}{\ln x}$.

(iii) $\lim_{x \to \infty} \frac{x^5}{e^x}$.

(iv) $\lim_{x \to 0} \frac{3^{\sin x} - 1}{x}$.

4. Using L'Hopital's Rule, analyze the convergence of the sequence whose nth term is given below. In case it is

convergent, find the limit.

(i) $\left(1 + \frac{a}{n}\right)^n, a \in \mathbb{R}$.

(ii) $\frac{n!}{10^n}$.

(iii) $\left(\frac{n}{n+1}\right)^n$.

(iv) $\frac{\ln n}{n^{\frac{1}{n}}}$.

5. Suppose f'' exists at $x = a$, compute

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h^2}.$$

Recap

In this section you have learnt the following

- How derivative function is useful in computing limits of indeterminate the form $\frac{0}{0}$, $\frac{\infty}{\infty}$ and so on.