Module 14 : Double Integrals, Applilcations to Areas and Volumes Change of variables

Lecture 40 : Double integrals over rectangular domains [Section 40.1]

Objectives

In this section you will learn the following :

- The concept of double integral over rectangular domains.

40 .1Double integrals

In section 16.1 the concept of Riemann integral of a function was developed to define and compute the 'area' of a region bounded by the graph of a bounded function on an interval $[a,b]$, the x-axis, the ordinates $x=a$, and $x=b$. Analogously, we attempt to find the 'volume' of a region which lies above the xy-plane, and is bounded by the planes $x=a, x=b, y=c, y=d$ and is under the surface defined by a nonnegative bounded function $z=f(x,y)$ defined on the rectangle

$$D:=[a,b]\times[c,d]=\{(x,y)\in \mathbb{R}^2 \mid x\in[a,b], y\in[c,d]\}.$$

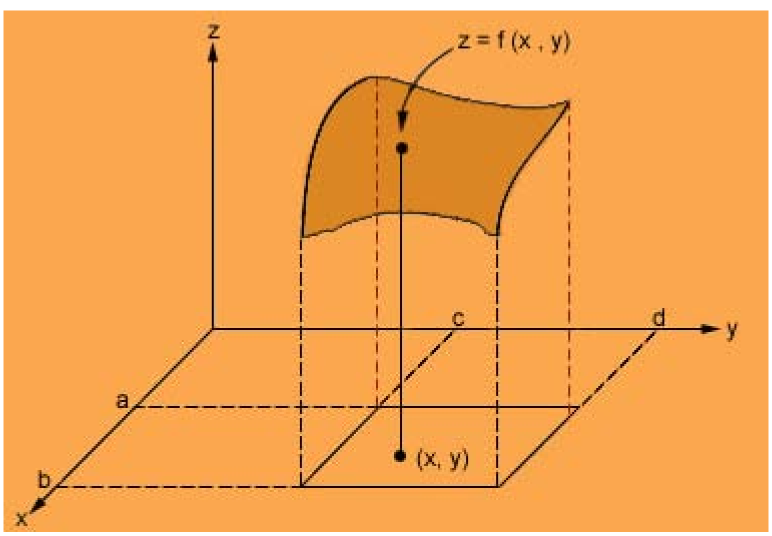


Figure: Volume below a surface

Let

$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$
be a bounded function. To find the volume below the surface S,
$z = f(x, y), (x, y) \in D$

and above the xy-plane, we can treat it as the volume of a cylinder with base D, and height $z = f(x, y)$ at any point $(x, y) \in D$. To approximate this , we proceed as in the case of a single variable. We partition the domain D into smaller rectangles by lines parallel to the axes at distances Δx and Δy apart. Let us number these smaller rectangles as $A_1, \ldots, A_n$. Let ΔA_i denote the area of A_i. Choose a point $(x_i, y_i) \in A_i$. Then
$f(x_i, y_i)\,\Delta A_i$
represents an approximation to the volume below the surface $z = f(x, y), (x, y) \in A_i$.

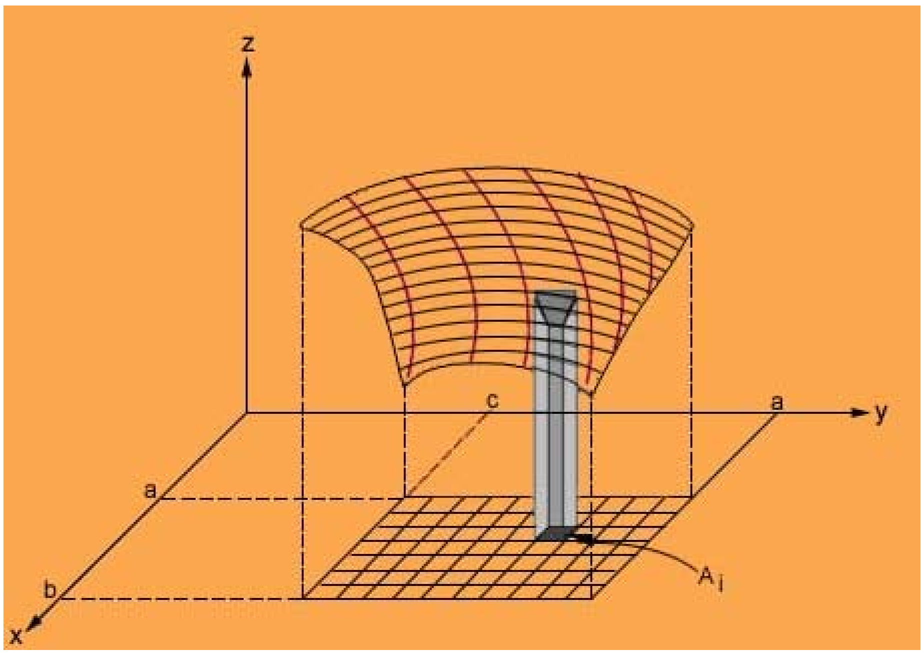


Figure: Volume above a small rectangle

Thus, the sum

$$S(P, f) := \sum_{i=1}^{n} f(x_i, y_i)\,\Delta A_i$$

will represent an approximation to the required volume. If we make our partition of D into smaller rectangles, we can expect that $S(P, f)$ will give better and better approximations, and eventually give the required volume in the limiting case. This motivates our next definition.

40.1.1Definition:

Let
$D := [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid x \in [a, b], y \in [c, d]\}$.

(i) For a partition P of D into rectangles obtained by lines parallel to the axes, at distances Δx and Δy apart,

let
$\| P \| := \max\{\Delta x, \Delta y\}$.

The number $\| P \|$ is called the **norm** of the partition.

(ii) Let $f : D \to \mathbb{R}$ be a bounded function. We say f is **double integrable** over ?

if
$$\lim_{\|P\|\to 0} S(P,f) \text{ exists.}$$

In this case, the above limit is called the **double integral** of f on D and is denoted by

$$\iint_{[a,\, b]\times[c,\, d]} f(x,y)\, d(x,y)$$

40.1.2Examples:

(i) Let $D := [a,b]\times[c,d]$ and

$$f : D \to \mathbb{R},\ f(x,y) := r, \text{for every } (x,y)\in D,$$

where r is a constant. Then, for any partition P of $[a,b]$ into smaller rectangles,

$$S(P,f) = r(b-a)(d-c).$$

Hence, f is double integrable and

$$\iint_{[a,b]\times[c,d]} f(x,y)d(x,y) = r(b-a)(d-c).$$

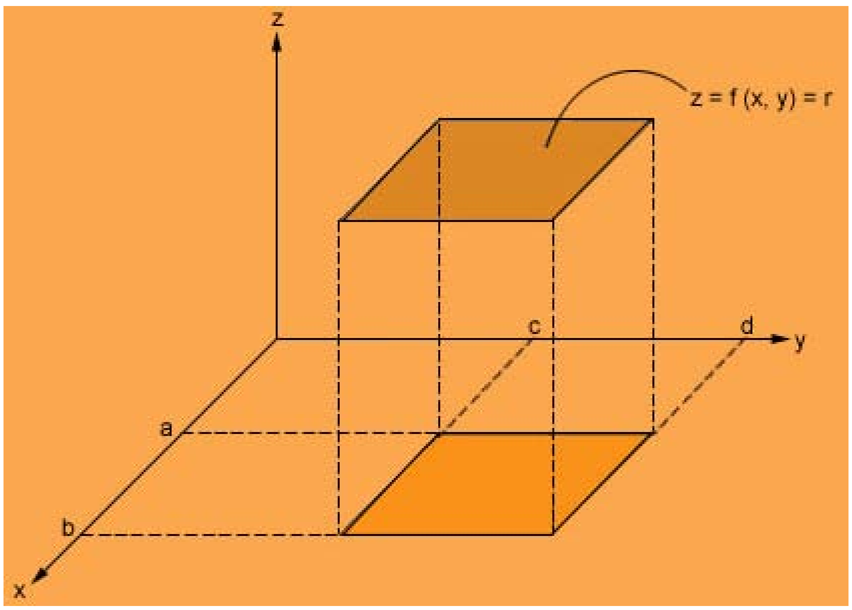


Figure: Volume below a plane

(ii) Let $D = [a,b]\times[c,d]$ and $f : D \to \mathbb{R}$ be defined by: for $(x,y)\in D$,

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ or } y \text{ is irrational,} \\ 1, & \text{if both } x \text{ and } y \text{ are rational.} \end{cases}$$

Consider any partition P of $[a,b]\times[c,d]$ into smaller rectangles. For every sub-rectangle ΔA in this partition,

$$S(P,f) = 1, \text{if we choose}(x,y)\in\Delta A \text{ with both } x, y \text{ as rationales.}$$

And

$$S(P,f) = 0, \text{ if we choose } (x,y)\in\Delta A \text{ with both } x, y \text{ as irrationals.}$$

Thus,

$\lim_{\|P\|\to 0} S(P,f)$ cannot exist.

Hence, the function f, though bounded, is not double integrable over D.

The double integral has properties similar to that of the Riemann integral, which we sate without proof.

40.1.3Theorem (Properties of double integrable):

Let $f, g : D=[a,b]\times[c,d] \to \mathbb{R}$ be bounded functions. Then, the following hold:

(i) *If f is continuous, then it is double integrable.*

(ii) *If f is double integrable over D and $D = D_1 \cup D_2$, where D_1, D_2 are non- overlapping rectangles, then*

$$\iint_D f(x,y)\, d(x,y) = \iint_{D_1} f(x,y)\, d(x,y) + \iint_{D_2} f(x,y)\, d(x,y).$$

(iii) *If f is double integrable, then so are the functions αf and $|f|$, $\alpha \in \mathbb{R}$, with*

$$\int_D (\alpha f)(x,y)\, d(x,y) = \alpha\left(\iint_D f(x,y)\, d(x,y)\right).$$

$$\left| \iint_D f(x,y)\, d(x,y)\right| \leq \iint_D \left|f(x,y)\right| d(x,y).$$

(iv) *If f, g are double integrable, then so are $f+g$, fg with*

$$\iint_D (f+g)(x,y)\, d(x,y) = \iint_D f(x,y)\, d(x,y) + \iint_D g(x,y)\, d(x,y).$$

40.1.4Definition:

If $f:[a,b]\times[c,d]\to\mathbb{R}$ is double integrable then

$$\iint f(x,y)\,d(x,y) \geq 0, \text{ if } f(x,y)\geq 0 \text{ for all } (x,y)\in[a,b]\times[c,d].$$

In this case, we define this double integral to be the **volume** of the region under the surface defined by $z=f(x,y)$ and bounded by the planes $x=a, x=b, y=c, y=d$ as well as the xy-plane.

The evaluation of a double integral can often be reduced to a repeated evaluation of Riemann integrals by the following theorem:

40.1.5Fubini's Theorem:

Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be double integrable.

(i) *If for each fixed $x\in[a,b]$, the function given by $y\mapsto f(x,y)$ is Riemann integrable on $[c,d]$, then the function $A:[a,b]\to\mathbb{R}$ defined by*

$$A(x) = \int_c^d f(x,y)dy$$

is Riemann integrable on $[a,b]$ and

$$\int_a^b \left[\int_c^d f(x,y)dy\right]dx = \int_a^b A(x)\,dx = \iint_{[a,\,b]\times[c,\,d]} f(x,y)\,d(x,y).$$

(ii) *If for each fixed $y\in[c,d]$, the function given by $x\mapsto f(x,y)$ is Riemann integrable on $[a,b]$, then the function $B:[c,d]\to\mathbb{R}$ defined by*

$$B(y)=\int_a^b f(x,y)dx$$

is Riemann integrable on $[c,d]$ *and*

$$\int_c^d\left[\int_a^b f(x,y)dx\right]dy=\int_c^d B(y)\,dy=\iint_{[a,b]\times[c,d]} f(x,y)\,d(x,y).$$

40.1.6 Note:

The integrals

$$\int_a^b\left[\int_c^d f(x,y)dy\right]dx \text{ and } \int_c^d\left[\int_a^b f(x,y)dx\right]dy$$

are known as the **iterated integrals** of f. Fubini's theorem says that if f is double integrable on $[a,b]\times[c,d]$, then the double integral of f equals either of the iterated integral of f. Geometrically, this means that when the volume of a region under the surface over a rectangle is well-defined, it can be found either by calculating the areas

$$A(x)=\int_c^d f(x,y)dy,\ \ a\le x\le b,$$

of cross sections of the region perpendicular to the x-axis, or by calculating the areas

$$B(y)=\int_a^b f(x,y)dx,\ \ c\le y\le d,$$

of cross sections of the region perpendicular to the y-axis, and adding them.

See Figures on next page

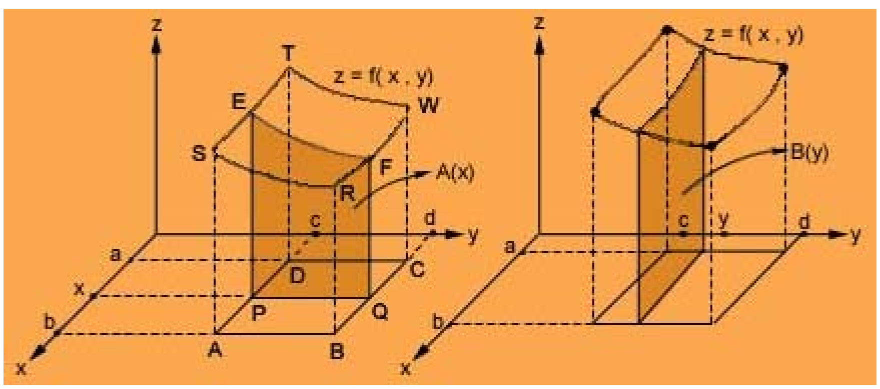


Figure: Iterated integrals

40 .1.7 Example:

Let $\phi:[a,b]\to\mathbb{R}$ and $\psi:[c,d]\to\mathbb{R}$ be integrable functions. Let $D=[a,b]\times[c,d]$ and

$$f(x,y)=\phi(x)\psi(y),(x,y)\in D.$$

Then, by Fubini's theorem, f is integrable over D and

$$\iint_G f\ (x,y)\ d\ (x,y)=\left(\int_a^b\ \phi(x)\ dx\right)\left(\int_c^d \psi(y)\ dy\right).$$

For example, for $s \geq 0, t \geq 0$, let

$$\phi(x) = x^s, x \in [a,b] \text{ and } \psi(y) = y^t, y \in [c,d].$$

Then

$$f(x,y) = x^s y^t \text{ for } (x,y) \in [a,b] \times [c,d],$$

and

$$\iint_{[a,b] \times [c,d]} x^s y^t \, d(x,y) = \left(\int_a^b x^s \, dx\right)\left(\int_c^d y^t \, dt\right) = \left(\frac{b^{s+1} - a^{s+1}}{s+1}\right)\left(\frac{d^{r+1} - c^{r+1}}{r+1}\right).$$

40.1.8Example :

Consider

$$\iint_D (x^2 + 2y) \, d\,(x,y),$$

where $D = [0,1] \times [0,1]$. Then, for every fixed x, $0 \leq x \leq 1$,

$$A(x) = \int_0^1 (x^2 + 2y) \, dy = \left[x^2 \, y + \frac{2y^2}{2}\right]_0^1 = x^2 + 1,$$

and hence

$$\iint_D (x^2 + 2y) \, d\,(x,y)$$

$$= \int_0^1 A(x) \, dx$$

$$= \int_0^1 (x^2 + 1) \, dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^1$$

$$= \frac{4}{3}.$$

Similarly, for every fixed $y, 0 \leq y \leq 1$,

$$B(y) = \int_0^1 (x^2 + 2y) \, dx = \left[\frac{x^3}{3} + 2xy\right]_0^1 = \frac{1}{3} + 2y.$$

Thus

$$\iint_D (x^2 + 2y) \, d(x,y) = \int_0^1 \left(\frac{1}{3} + 2y\right) dy$$

$$= \left[\frac{y}{3} + y^2\right]_0^1$$

$$= \frac{4}{3}.$$

Practice Exercises

(1) Evaluate the following integrals where $D = [0,1] \times [0,1]$:

(i) $\iint_D x^2 y^2 \cos(x^3) \, d(x,y).$

(ii) $\iint_D \frac{x}{(xy+1)^2} d(x,y).$

Answers

(2) Compute the following:

(i) $\iint_D xy^3 \exp(x^2 b^2)\, d(x,y)$, where $D=[1,3]\times[1,2]$.

(ii) $\iint_D x\sqrt{1-x^2}\, d(x,y)$, where $D=[0,1]\times[2,3]$.

Answers

(3) Using suitable order of integration, evaluate

$$\iint_D x \cos^2 \pi x \, d\,(x,y),$$

where

$$D=\left[0,\frac{1}{2}\right]\times[0,\pi]$$

Answers

Recap

In this section you have learnt the following

- The concept of double integral over rectangular domains.

[Section 40.2]

Objectives

In this section you will learn the following :

The concept of double integral over general domains.

40.2.1Definition

Let D be a bounded subset of $\mathbb{R}^2$ and $f: D \to \mathbb{R}$ be a bounded function. Define $f^*: \mathbb{R}^2 \to \mathbb{R}$ by

$$f^*(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D \\ 0, & \text{if } (x,y) \notin D. \end{cases}$$

Let

$$R = [a,b] \times [c,d]$$

be any rectangle in $\mathbb{R}^2$ which includes D. We say f is **double integrable** if f^* is double integrable on R, on D and

$$\iint_D f(x,y)\, d(x,y) := \iint_R f^*(x,y)\, d(x,y).$$

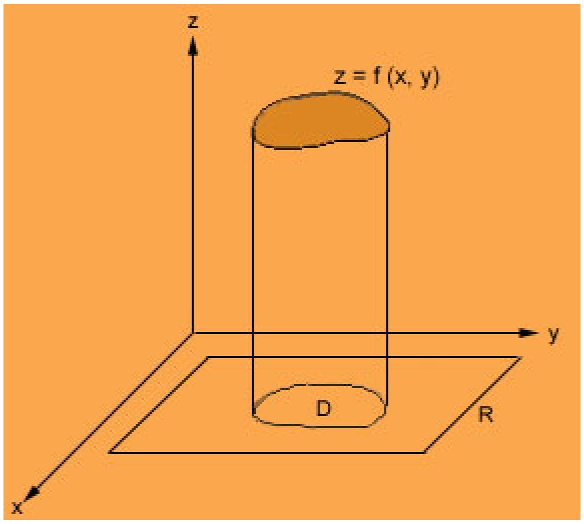


Figure: Double integral of f

40.2.2Note:

(i) By the domain additivity property, it follows that this definition does not depend on the choice of the rectangle
$R = [a,b] \times [c,d]$ containing D.

(ii) The extended notion of integral has properties similar to that of theorem $40.1.3$.

40.2.3Example:

Let

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}$$

and

$$f: D \to \mathbb{R} \text{ be given by } f(x,y) = x^2 + y^2 \text{ for } (x,y) \in D.$$

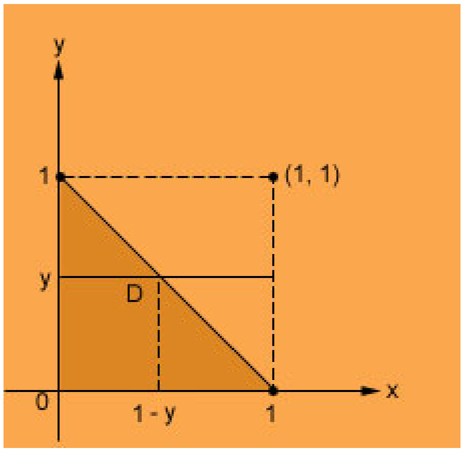
Then

$$D \subset R = [0,1] \times [0,1]$$

and f being continuous on D, is double integrable over R. Thus,

$$\iint_D f(x,y)\,d(x,y) = \iint_{[0,1]\times[0,1]} f^*(x,y)\,d(x,y)$$
$$= \int_0^1 \left[\int_0^1 f^*(x,y)dx \right] dy.$$

Fix $y \in [0,1]$. Since



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$$f^*(x,y) = \begin{cases} x^2 + y^2, & \text{for all } x \in [0, 1-y], \\ 0, & \text{for } x \in (1-y, 1], \end{cases}$$

we have

$$\int_0^1 f^*(x,y)\,dx = \int_0^{1-y} f^*(x,y)\,dx + \int_{1-y}^1 f^*(x,y)\,dx$$
$$= \int_0^{1-y} (x^2+y^2)\,dx$$
$$= \frac{(1-y)^3}{3} + y^2(1-y).$$

Hence

$$\iint_D f(x,y)d(x,y) = \int_0^1 \left[\frac{(1-y)^3}{3} + y^2(1-y) \right] dy = \frac{1}{6}.$$

We describe next a method of computing the double integrals for special regions.

40.2.4Definition:

Let $D \subseteq \mathbb{R}^2$.

(i) Let

$$D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\},$$

where

$$\phi_1, \phi_2 : [a,b] \to \mathbb{R}$$

are continuous functions. Then D is called a **type I elementary region** in $\mathbb{R}^2$.

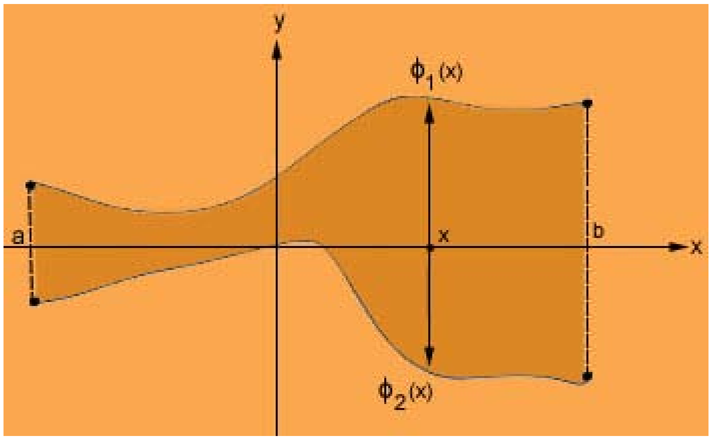

Figure: **Type I elementary region**

(ii) Let

$$D=\left\{(x,y)\in \mathbb{R}^2 \mid c\le y\le d, \psi_2(y)\right\},$$

where

$$\psi_1,\psi_2:[c,d]\to \mathbb{R}$$

are continuous functions. Then D is called a **type-II elementary region** in $\mathbb{R}^2$.

40.2.5 Example

The disk $D=\{(x,y)\in \mathbb{R}^2 \mid x^2+y^2\le 1\}$ is a type-I elementary region in $\mathbb{R}^2$ since

$$D=\left\{(x,y)\in \mathbb{R}^2 : -1\le x\le 1, -\sqrt{1-x^2}\le y\le \sqrt{1-x^2}\right\}$$

We can also visualize the disc as a type-II elementary region as follows

$$D=\left\{(x,y)\in \mathbb{R}^2 : -1\le y\le 1, -\sqrt{1-y^2}\le x\le \sqrt{1-y^2}\right\}.$$

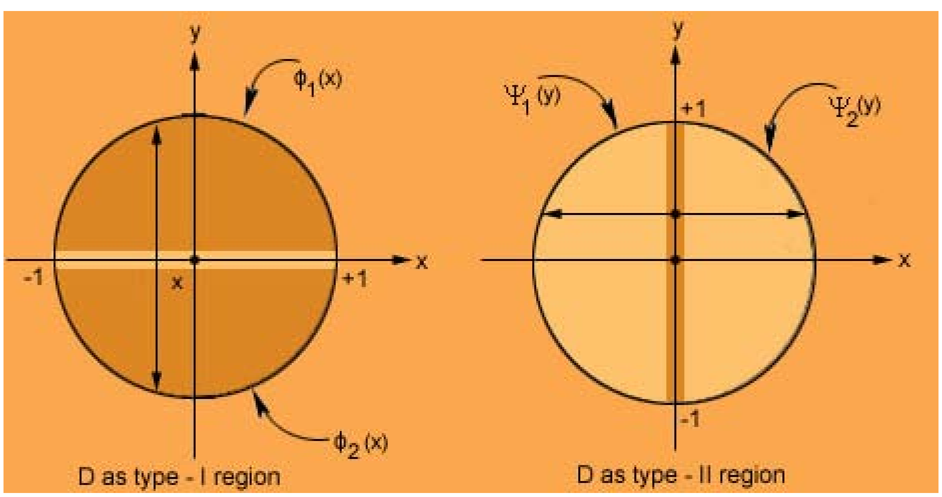

Figure: D **as type-I and type-II region.**

The double integral over elementary region is given by:

40.2.6 Fubini's Theorem:

Let $D\subset \mathbb{R}^2$ be bounded closed and $f:D\to \mathbb{R}$ is bounded continuous function.

(i) If D is a type-1 elementary region

$$D=\{(x,y)\in \mathbb{R}^2 \mid a\le x\le b, \phi_1(x)\le y\le \phi_2(x)\},$$

where

$$\phi_1,\phi_2:[a,b]\to\mathbb{R}$$

are continuous functions, then f is double integrable on D and

$$\iint_D f(x,y)\,d(x,y)=\int_a^b A(x)dx=\int_a^b\left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy\right]dx.$$

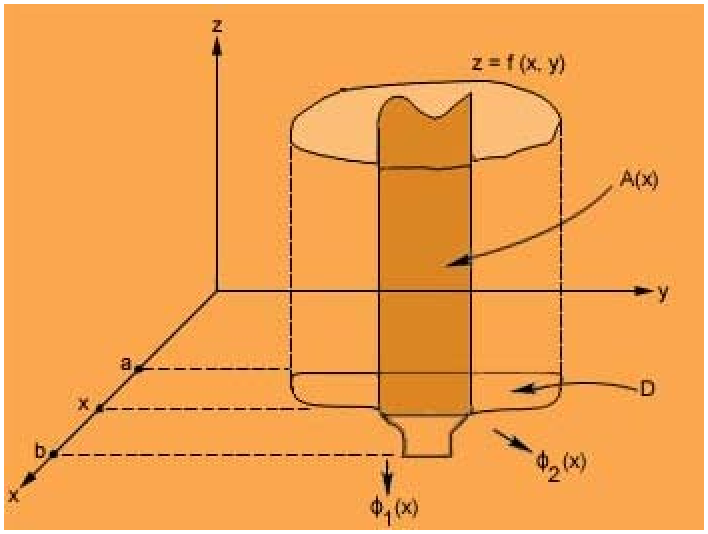

Figure: Type - I region

(ii) *If D is a type-II elementary region*

$$\{(x,y)\in \mathbb{R}^2 \mid c\le y\le d, \psi_1(y)\le x\le \psi_2(y)\},$$

then

$$\iint_D f(x,y)d(x,y)=\int_c^d B(y)dy=\int_c^b B(y)dy=\int_c^d\left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right]dy.$$

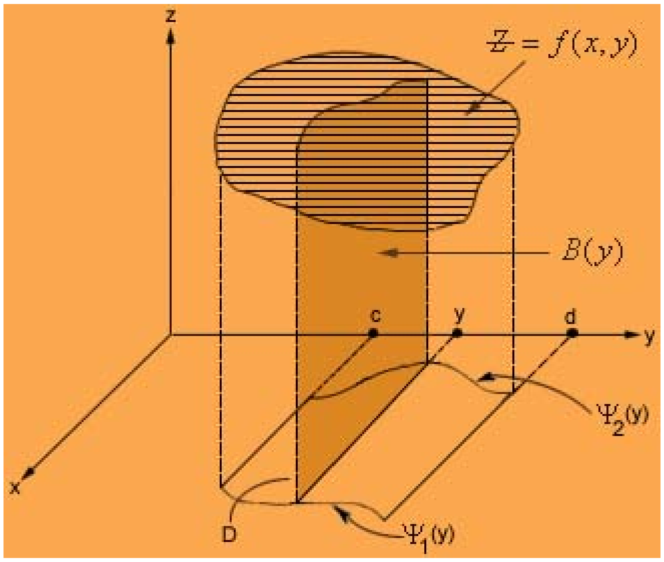

Figure: Type-II region

40.2.7 Examples:

(i) Let

$$D = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0,\ x^2 + 2y^2 \leq 4\}$$

and

$$f(x, y) = y \ \text{ for } \ (x, y) \in D.$$

If we express D as a type-I domain

$$D = \{(x, y) \mid -2 \leq x \leq 2,\ 0 \leq y \leq [(4 - x^2)/2]^{1/2}\},$$

then

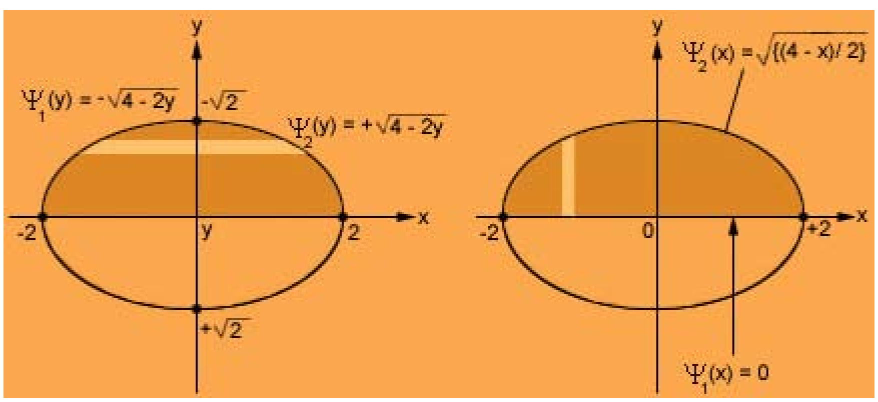

Figure: D as a type-I and type-II domain

$$\iint_D f(x, y)\, d(x, y) = \int_{-2}^{2} \left[\int_0^{[(4-x^2)/2]^{1/2}} y\, dy \right] dx$$

$$= \int_{-2}^{2} \frac{1}{2} \left(\frac{4 - x^2}{2} \right) dx$$

$$= \frac{8}{3}.$$

$$D=\{(x,y)\mid 0\le y\le\sqrt{2},-(4-2y)^{1/2}\le x\le(4-2y)^{1/2}\},$$

then the required integral is given by,

$$\iint_D f(x,y)\,d(x,y)=\int_0^{\sqrt{2}}\left[\int_{-(4-2y^2)^{1/2}}^{(4-2y^2)^{1/2}} y\,dx\right]dy$$
$$=\int_0^{\sqrt{2}} y\left[2\sqrt{4-2y^2}\right]dy$$
$$=\frac{8}{3},$$

where to compute the last integral one has to make suitable substitutions making it a bit more difficult than the one in the previous case.

40.2.8 Note:

Computation of

$$\iint_D f(x,y)d(x,y)$$

may be easier with suitable choice of order of integration.

40.2.9 Example

Find

$$\iint_D (x+2y)d(x,y),$$

where D is the region bounded by the parabolas

$$Y=2x^2 \text{ and } y=1+x^2.$$

To express the region analytically, we proceed as follows:

Step (i):

Sketch the region D:

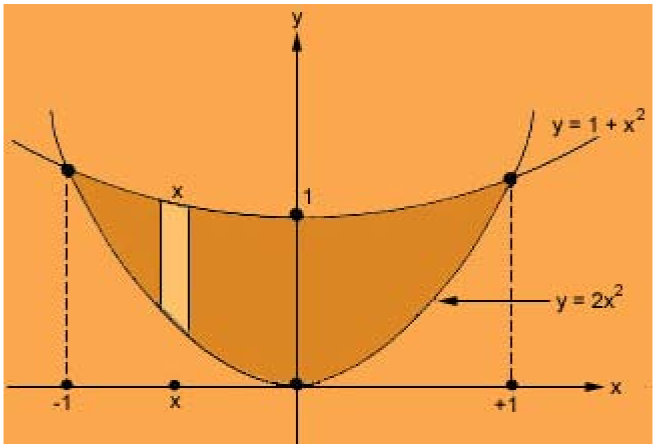

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Figure: D as a type-I region

Step (ii):

To find the limits of integration, consider vertical and horizontal lines through the origin. Since vertical lines throughout the region go from $y=2x^2$ to $y=1+x^2$, we should integrate first with respect to the variable y, i.e., express D as a type-I region as

$$D=\{(x,y)\mid -1\le x\le 1, 2x^2\le y\le 1+x^2,\}$$

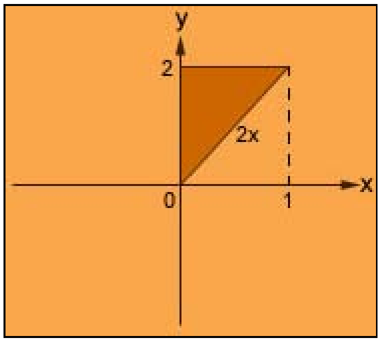
and hence

$$\iint_D (x+2y)\,d(x,y)=\int_{x=-1}^{x=+1}\left(\int_{y=2x^2}^{y=1+x^2}(x+y)dy\right)dx.$$

Note that if one tries to express D as a type-II region, one lands up in problem as the horizontal lines do not remain inside the region throughout.

(ii) Let

$$D=\{(x,y)\mid 0\le x\le 1, 0\le y\le 2x\}$$



and

$$f(x,y)=e^{x^2} \text{ for } (x,y)\in D.$$

Then, D is a type-I region, and hence

$$\iint_D f(x,y)\,d(x,y)=\int_0^1\left[\int_0^{2x}e^{x^2}dy\right]dx=\int_0^1 2xe^{x^2}dx=e-1.$$

We could also describe D as a type-II region

$$D=\{(x,y)\mid 0\le y\le 2, y/2\le x\le 1\}.$$

Then

$$\iint_D f(x,y)\,d(x,y)=\int_0^2\left[\int_{y/2}^1 e^{x^2}dx\right]dy,$$

but the integral

$$\int_{y/2}^1 e^{x^2}dx$$

cannot be evaluated.

40.2.10 Note:

A complicated region D can often be divided into elementary ones in order to evaluate

$$\iint_D f(x,y)\,d(x,y).$$

Here are some examples:

(i) Let D be the triangle in $\mathbb{R}^2$ with vertices $(-2,3),(2,3)$ and $(0,1)$. As a type-II region

$$D=\{(x,y)\mid 1\le y\le 3, 1-y\le x\le y-1\},$$

however, as it is not a type-I region, but a union of two non-overlapping type-II regions: $D=D_1\cup D_2$, where

$$D_1=\{(x,y)\mid -2\le x\le 0, -x+1\le y\le 3\}$$

$D_2 = \{(x,y) \mid 0 \le x \le -2, x+1 \le y \le 3\}.$

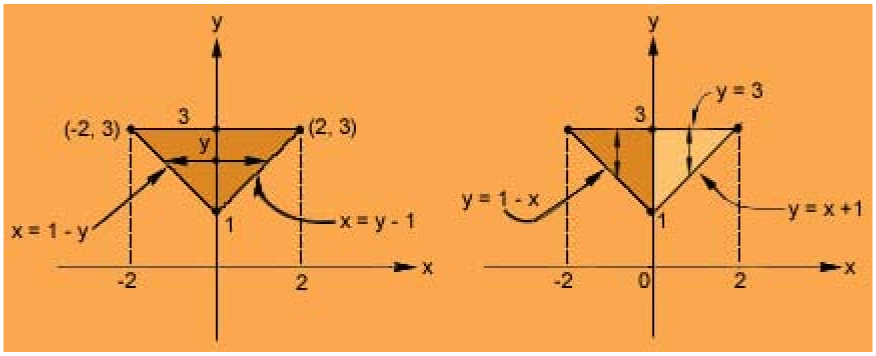


Figure: D as type-I and as union of type-II regions

In general a domain D can be split into smaller regions of type-I/II by drawing lines parallel to axes:

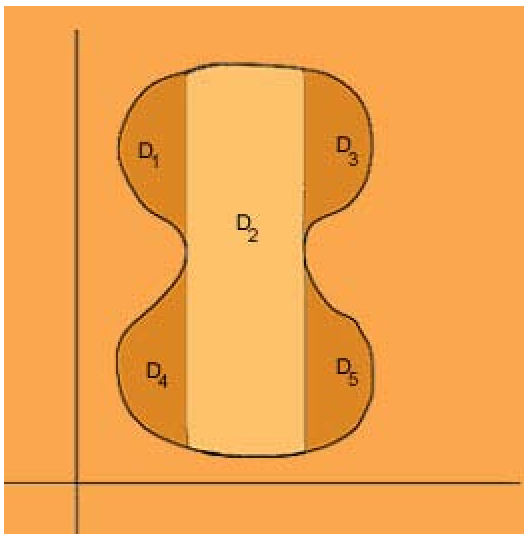


Figure: Caption text

the domain D can be expressed as

$$D = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5,$$

where the domains

$$D_1, D_3, D_4, D_5 \quad \text{are type} - \text{II domains}$$

and domain

$$D_2 \ \text{ is a type} - \text{I domain}.$$

Recap

In this section you have learnt the following

- The concept of double integral over general domains.