Module 14 : Double Integrals, Applilcations to Areas and Volumes Change of variables

Lecture 42 : Change of variables [Section 42.1]

Objectives

In this section you will learn the following :

- The change of variables formulae.

42 .1 Change of Variables

Recall that we had proved the following result for Riemann integration, called the integration by substitution: Let $f:[a,b]\to\mathbb{R}$ be continuous, $g:[c,d]\to[a,b]$ differentiable and $g':[c,d]\to\mathbb{R}$ Riemann integrable.If $g(c)=a$ and $g(d)=b$, then

$$\int_a^b f(x)dx = \int_c^d f(g(t))\ g'(t)\ dt.$$

To obtain a similar result for double integrals, we make the following definition.

42.1.1Definition:

Let Ω in $\mathbb{R}^2$ be an open set and

$$g:\Omega\to\mathbb{R}^2,\ g(u,v):=(g_1(u,v),g_2(u,v))\ \text{ for }\ (u,v)\in\Omega,$$

where

$$g_1, g_2:\Omega\to\mathbb{R}$$

are such that both have partial derivatives in Ω. Then, the **Jacobian** of the function g is the function

$$J:\Omega\to\mathbb{R}$$

defined by

$$J(P) := \frac{\partial(g_1, g_2)(P)}{\partial(u,v)} := \det \begin{pmatrix} \frac{\partial g_1(P)}{\partial u} & \frac{\partial g_1(P)}{\partial v} \\ \frac{\partial g_2(P)}{\partial u} & \frac{\partial g_2(P)}{\partial v} \end{pmatrix}, P \in \Omega.$$

We state next the change of variables formula for double integrals without proof.

42.1.2 Theorem (Change of variable):

Let D be an elementary regions in $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be continuous. Let Ω be an open set in $\mathbb{R}^2$ and

$$g : \Omega \to \mathbb{R}^2,\ g = (g_1, g_2),$$

be a one-one function such that the following holds:

(i) Both g_1 and g_2 have continuous partial derivatives in Ω.

(ii) The Jacobian function J of g does not vanish at any point of Ω.

(iii) There exists $E \subset \Omega$ is such that E is an elementary region and $g(E) = D$. Then

$$\iint_D f(x,y)\, d(x,y) = \iint_E f(g_1(u,v), g_2(u,v))\, |J(u,v)|\, d(u,v).$$

See picture on the next page

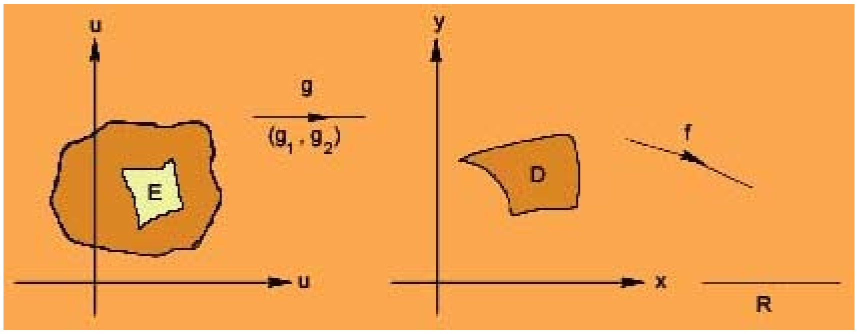

Figure: Change of variables

42.1.3 Note:

(i) The Jacobian may be thought of as a ‘magnification factor' for areas. While employing the change of variables

result for double integrals, one keeps in mind that after a change of variables, the integrand should be simpler and/or the domain over which integration is to take place should be simpler (for example, a rectangular domain) so that the computations are reduced

(ii) The change of variable formula extends to domains D which are unions of finite number of non-overlapping

elementary regions.

(iii) The Jacobian function can also be defined for suitable functions of three or more variables and a

corresponding and there exists a corresponding change of variable formula.

42.1.4 Examples:

(i) Let us find the area of the region D in the xy-plane bounded by the lines

$x+y=1$, $x-y=0$, $x-y=-4$ and $x+y=4$.

The region is the parallelogram

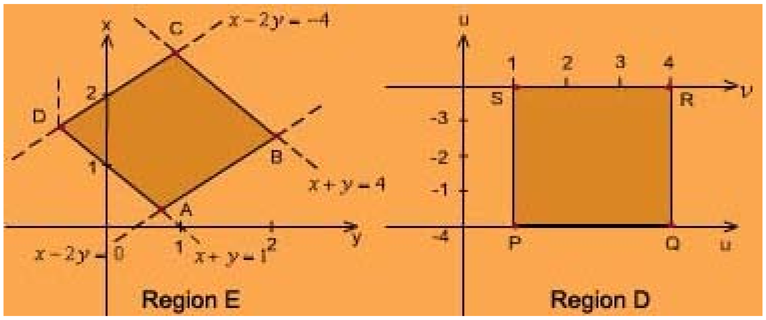

Figure: Change variable

Let us consider the transformation

$u = x + y$ and $v = x - 2y$.

Then, the transformation

$$g(u,v) = (x,y), \text{ where } x = \frac{2u+v}{3}, y = \frac{4-v}{3},$$

will take the region E to D, where to find E, we note that the g takes the line

$u = 1$ to $x + y = 1$,

$u = 4$ to $x + y = 4$,

$v = 0$ to $x - 2y = 0$,

$v = -4$ to $x - 2y = -4$,

Thus, E such that $g(E) = D$ is given by the rectangle bounded by the lines

$u = 1$, $u = 4$, $v = 0$ and $v = -4$.

Further, the jabcobian of g is

$$J(u,v) = \det\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} = -\frac{1}{3}.$$

Thus, by change of variables

$$Area(D) = \iint_D d(x,y)$$

$$= \iint_E |J(u,v)|\, d(u,v) = \frac{1}{3} \iint_E d(u,v) = \frac{1}{3} Area\,(E) = 4.$$

(ii) Let

$$D = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 2, \frac{y}{2} \le x \le \frac{y+4}{2}\right\}$$

$u = 1$ to $x + y = 1$,

$u = 4$ to $x + y = 4$,

$v = 0$ to $x - 2y = 0$,

$v = -4$ to $x - 2y = -4$,

Thus, E such that $g(E) = D$ is given by the rectangle bounded by the lines

$u = 1$, $u = 4$, $v = 0$ and $v = -4$.

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$$Area(D) = \iint_D d(x,y)$$

$$= \iint_E |J(u,v)|\, d(u,v) = \frac{1}{3} \iint_E d(u,v) = \frac{1}{3} Area\ (E) = 4.$$

(ii) Let

$$D = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 2, \frac{y}{2} \le x \le \frac{y+4}{2}\right\}$$

and

$$f(x,y) = y^3 (2x - y)\, e^{(2x-y)^2} \text{ for } (x,y) \in D.$$

Then

$$\iint_D f(x,y)\, d(x,y) = \int_0^2 \left[\int_{y/2}^{(y+4)/2} y^3 (2x-y) e^{(2x-y)^2}\, dx \right] dy.$$

The above integral as such is difficult to compute. The integrand suggest the following change of variables. Let

$$u := 2x - y \text{ and } v = y.$$

Then

$x := g_1(u,v) = \frac{(u+v)}{2},\ \ y := g_2(u,v) = v.$

note that $g(u,v) = (x,y)$ is one-one and the Jacobian function for g is given by

$$J(u,v) = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = \frac{1}{2}.$$

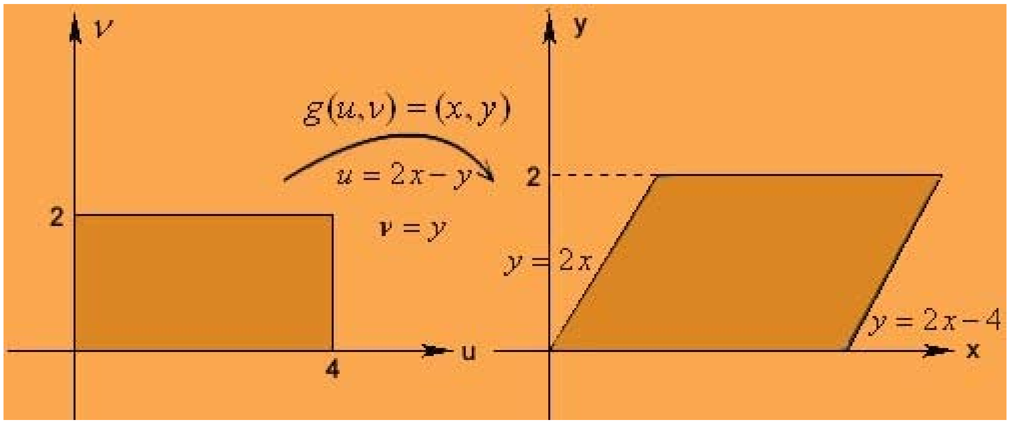

Figure : Change of variable

To compute E such that $g(E) = D$, we compute $g^{-1}(D)$ as follows: g^{-1} maps

the line $y = 0$ to $v = 0$,

the line $y = 2x$ to $v = u + y = u + v$, i.e., $u = 0$,

the line $y = 2x - 4$ to $v = (u+v) - 4$, i.e., $u = 4$,

and

the line $y = 2$ to $v = 2$.

Thus, if

$E := \{(u,v) \in \mathbb{R}^2 \mid 0 \le u \le 4,\ 0 \le v \le 2\},$

then $g(E) = D$. Thus, by the change of variables formula, we have

$$\begin{aligned}\iint_D f(x,y)\,d(x,y) &= \iint_E v^3 u\, e^{u^2}\left(\frac{1}{2}\right) d(u,v) \\ &= \frac{1}{2}\int_0^2 \left[\int_0^4 v^3 u\, e^{u^2}\, du\right] dv \\ &= \frac{1}{2}\int_0^2 v^3 \left(\frac{e^{16}-1}{2}\right) dv \\ &= e^{16} - 1.\end{aligned}$$

Practice Exercises :

1. Evaluate

$$\iint_D \left(x^2 - y^2\right) dxdy$$

by making the change variables

$$u = \frac{x+y}{2}, v = \frac{x-y}{2}$$

Answer

2. Evaluate the integral $\iint_D (x-y)^2 \sin^2(x+y) d(x,y)$, where D is the parallelogram with vertices at $(\pi,0)$, $(2\pi,\pi)$, $(\pi,2\pi)$ and $(0,\pi)$.

Answer

3. Determine the are of the region R in the xy-plane bounded by the hyperbolas $xy=1, xy=9$ and the lines $y=x, y=4x$.

[Hint : Use the transformation $x = u/v, y = uv$].

Answer

4 Let D be the region in $\mathbb{R}^2$ bounded by the lines

$x=0$, $x=4$, $2y-x=2$ and $2y-x=4$.

Using the transformation

$x=4u$, $y=2u+3v$,

Compute

$$\iint_D x\,y\,dx\,dy \text{ and } \iint_D (x-y)\,dxdy$$

Answer

Recap

In this section you have learnt the following

- The change of variables formulae.

[Section 42.2]

Objectives

In this section you will learn the following :

- The change of variable formula in $\mathbb{R}^2$ from cartesian to polar coordinates.
- The change of variable formula in $\mathbb{R}^3$ from Cartesian to cylindrical coordinates.

- The change of variable formula in $\mathbb{R}^2$ from Cartesian to spherical coordinates.

42.2 Change of variables to polar, cylinderical and spherical coordinates

In this section we give some important applications of the general change of variables formula proved in the previous section.

42.2.1 Definition:

Consider the transformation $g:(0,\infty)\times[0,2\pi)\to\mathbb{R}^2$ defined by

$g(r,\theta)=(x,y),$ where $x=r\cos\theta, y=r\sin\theta.$

The map g is a one-to-one map and is called the **polar coordinates transformation** . For every point P in the plane with Cartesian coordinates (x,y), the above map associates an ordered pair, (r,θ), called the **polar coordinates** of the point P.

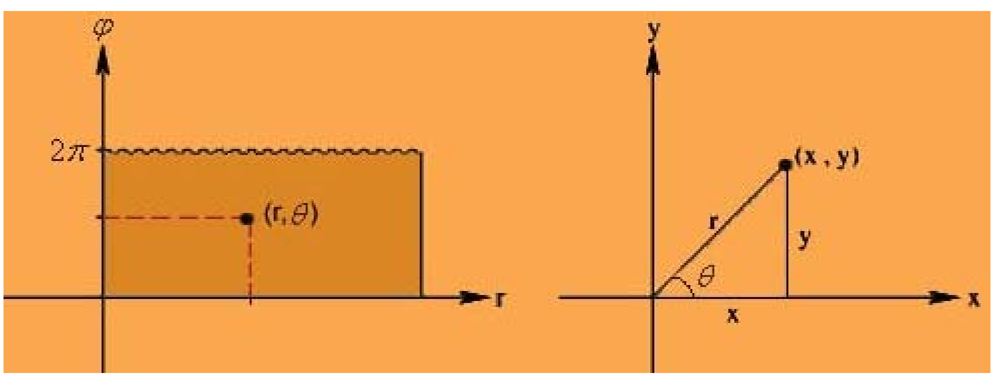

Figure : Polar to Cartesian coordinates

42.2.2 Theorem (Change of variables from rectangular to polar

Let $f:\mathbb{R}^2\to\mathbb{R}$. *Let*

$D\subset(0,\infty)\times[0,2\pi),$ and $E=g(D),$

where g is the polar coordinates transformation. If f is integrable over E, then fog is integrable over D and

$$\iint_E f(x,y)dxdy = \iint_D f(r\cos\theta, r\sin\theta)\, r\, d(r,\theta).$$

42.2.3Examples:

(i) Let us evaluate

$$\iint_D (x^2+y)\, d\,(x,y)$$

where D is the annular region lying between the two circles $x^2+y^2=1$ and $x^2+y^2=4$.

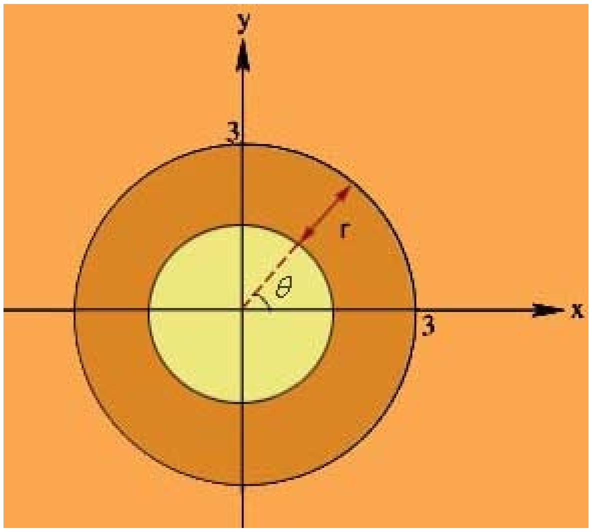

Figure 153. Region D

Form the figure, it is clear that if we write

$$E=\{(r,\theta)\,|\,1\le r\le 2\,,0\le\theta\le 2\pi\},$$

then $g(E)=D$, where g is the polar coordinate transformations. Thus

$$\begin{aligned}\iint_D (x+y)\, d(x,y) &= \int_0^{2\pi}\int_1^2 r^2(\cos^2\theta+\sin\theta)\, r\, dr\, d\theta\\ &= \int_0^{2\pi}\left(\int_1^2\left(r^3\cos^2\theta + r^2\sin\theta\right)dr\right)d\theta\\ &= \int_0^{2\pi}\left[\frac{15}{4}\cos^2\theta+\sin\theta\right]d\theta\\ &= \frac{15\pi}{8}.\end{aligned}$$

(ii) Let $a>0, b>0$, and

$$D=\left\{(x,y)\in \mathbb{R}^2 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}.$$

Let

$$f(x,y)=y^2 \text{ for } (x,y)\in D.$$

To evaluate

$$\iint_D f(x,y)\,d(x,y),$$

we make the change of variables to generalized polar coordinates

$$x:=g_1(r,\theta):=a\,r\cos\theta \text{ and } y:=g_2(r,\theta):=b\,r\sin\theta.$$

The Jacobian of this transformation $g=(g_1,g_2)$ is given by

$$J(r,\theta)=\det\begin{pmatrix} a\cos\theta & -a\,r\sin\theta \\ b\sin\theta & b\,r\cos\theta \end{pmatrix}=r\,a\,b\,(\cos^2\theta+\sin^2\theta)=rab.$$

If we set

$$E=\{(r,\theta)\mid 0\le r\le 1, 0\le\theta\le 2\pi\}=[0,1]\times[0,2\pi],$$

then $g(D)=E$, and by the change of variables formula,

$$\begin{aligned}\iint_D f(x,y)\,d(x,y) &= \iint_E (b\,r\sin\theta)^2 r\,ab\,d(r,\theta)\\ &=\int_0^1\left[\int_0^{2\pi} a\,b^3\,r^3\sin^2\theta\,d\theta\right]dr\\ &=\int_0^1 a\,b^3\,r^3\,\pi\,dr\\ &=\frac{a\,b^3\,\pi}{4}.\end{aligned}$$

42.2.4

(i) **Triple integral in cylindrical coordinates :**

A point $P(x,y,z)\in\mathbb{R}^3$ can also be described in terms of **cylindrical coordinates** (r,θ,z), where (r,θ) are the polar coordinates the point Q, the projection of P onto xy-plane. Thus

$$0\le r<\infty,\ 0\le\theta<2\pi \text{ and } z\in\mathbb{R}.$$

These coordinates are related to the cartesian coordinates by the relations :

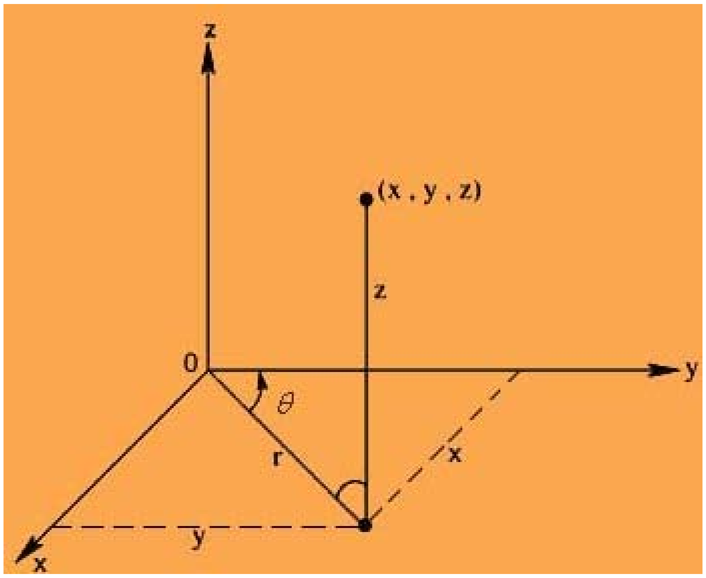

Figure: Cylindrical coordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

The transformation

$g(r,\theta,z) = (x,y,z)$

is called the **cylindrical coordinates transformation** . If $g(E) = D$, then by the change of variables formula, since $J(r,\theta,z) = r$, we have

$$\iiint_D f(x,y,z)\, d(x,y,z) = \iiint_E (f \circ g)(r,\theta,z)\,|J(r,\theta,z)|\, d(r,\theta,z)$$
$$= \iiint_E (f \circ g)(r,\theta,z)\, r\, dr\, d\theta\, dz$$

This change of variable is useful when the domain D is spherical or cylindrical in nature.

(ii) **Triple integral in spherical coordinates :**

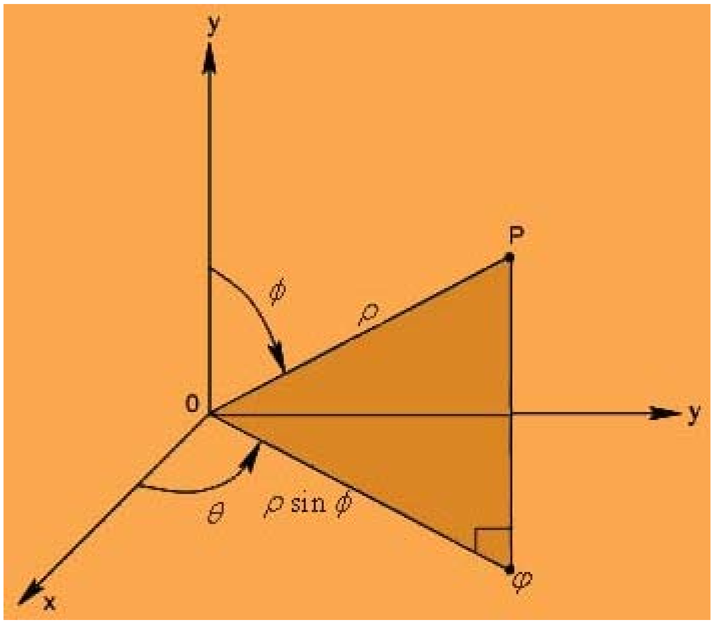

Figure: Spherical coordinates

Another way of representing points in space is by **spherical coordinates** , (ρ,θ,ϕ) , where for a point P in space, ρ is the magnitude OP , θ is the polar angle of the projection on of P onto the xy -plane, and ϕ is the angle between the line OP and the positive z -axis.

The spherical coordinates are related to the Cartesian coordinates by

$$\left.\begin{aligned} x &= \rho\sin\phi\cos\theta,\\ y &= \rho\sin\phi\sin\theta,\\ z &= \rho\cos\phi \end{aligned}\right\} \begin{aligned} \rho &> 0,\\ 0 \le \theta &< 2\pi,\\ 0 \le \phi &< \pi. \end{aligned}$$

If we denote the above transformation by $g(r,\theta,\phi)=(x,y,z)$, then

$$|J_g(r,\theta,\phi)| = \left|\det\begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\theta\cos\phi \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\sin\phi \end{bmatrix}\right| = \rho^2\sin\phi.$$

Thus, if $g(E)=D$, then for every integral function f over D ,

$$\iiint_D f(x,y,z)\,d(x,y,z) = \iiint_E (f\circ g)(r,\theta,\phi)\,\rho^2\sin\phi\,d\theta\,d\theta\,d\phi$$

42.2.5Examples :

(i) Find the volume of the solid region D cut from the sphere

$x^2+y^2+z^2=1$
from the cylinder
$x^2+(y-1/2)^2=1/4$ ·

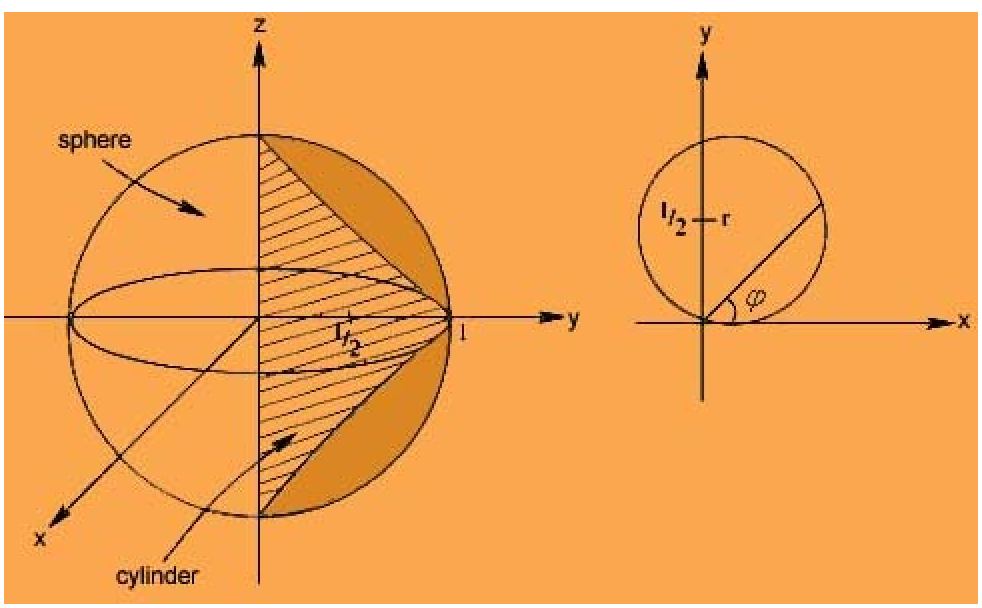

Figure: Region D in spherical coordinates

The required volume is

$$\iiint_D 1\,dv$$

In cylindrical coordinates, D can be described by

$$E=\left\{(r,\theta,z)\,|\,r=\sin\theta, 0\le\theta<\pi\ge-\sqrt{1-r^2}\le z\le\sqrt{1-r^2}\right\}$$

Thus,

$$\begin{aligned}
\iiint_D dv &= \int_0^{\pi}\int_0^{\sin\theta}\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\,d(r,\theta,z)\\
&=2\int_0^{\pi/2}\int_0^{\sin\theta}\left(\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} 1\,dz\right) r\,dr\,d\theta\\
&=2\int_0^{\pi/2}\left(\int_0^{\sin\theta} 2\,r\sqrt{1-r^2}\,dr\right)d\theta\\
&=2\int_0^{\pi/2}\left[-\frac{2}{3}(1-r^2)^{3/2}\right]_0^{\sin\theta} d\theta\\
&=\frac{4}{3}\int_0^{\pi/2}(1-\cos^3\theta)\,d\theta\\
&=\frac{4}{3}\int_0^{\pi/2}\left(1-\cos\theta+\cos\theta\sin^2\theta\right)d\theta\\
&=\frac{4}{3}\left[\theta-\sin\theta+\frac{\sin^3\theta}{3}\right]_0^{\pi/2}\\
&=\frac{4}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right].
\end{aligned}$$

(ii) Let us find the volume of the solid D cut from the sphere

$$x^2+y^2+z^2=9$$

by the cone

$$z=\sqrt{x^2+y^2}\,.$$

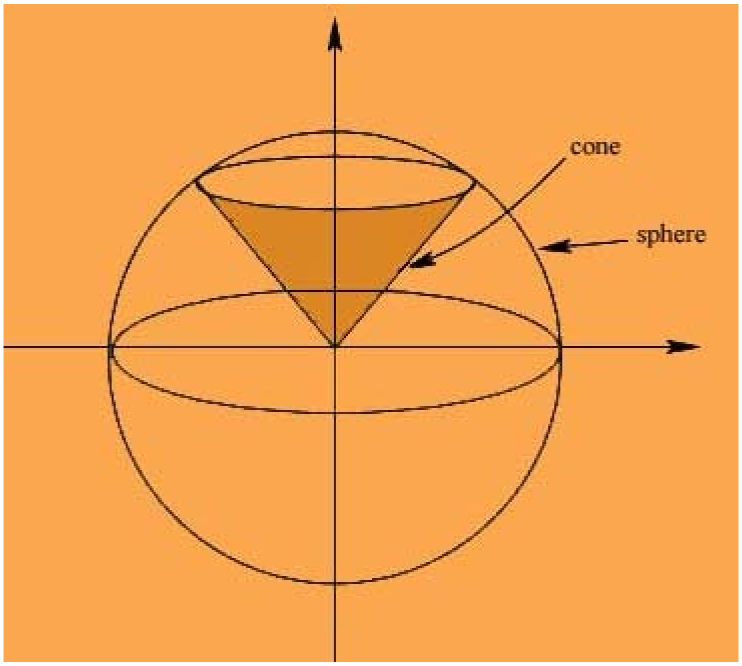

Figure: The region D

In spherical coordinates, the equation of the sphere is $\rho = 3$ and the equation of the cone is

$$\begin{aligned}\rho\cos\phi = z &= \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} \\ &= \rho\sin\phi,\end{aligned}$$

i.e., the cone is described by $\tan\phi = 1$. Thus D can be described as

$$D = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4\}.$$

Hence, the required volume is

$$\begin{aligned}\iiint_D 1\,dv &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^2\sin\phi \\ &= \int_0^{2\pi}\left(\int_0^{\pi/4} 9\sin\phi\,d\phi\right)d\theta \\ &= 9\int_0^{2\pi}\left[-\cos\phi\right]_0^{\pi/4}d\theta \\ &= 9\int_0^{2\pi}\left(1-\frac{\sqrt{2}}{2}\right)d\theta \\ &= 9\pi(2-\sqrt{2}).\end{aligned}$$

Practice Exercises

(1) Using polar coordinates find the volume of the solid region D bounded above by the hemisphere

$$z = \sqrt{16 - x^2 - y^2}$$

and below by the disc in the xy-plane

$$x^2 + y^2 \le 4.$$

Answer

(2)(i) Using the change of variables formula for polar coordinates, find for every $r > 0$.

$$I(r) := \iint_{D(r)} e^{-(x^2+y^2)}\, d(x,y),$$

where

$$D(r) = \{(x,y) \mid x^2 + y^2 \le r^2\}.$$

(ii) Using (i), show that

$$\lim_{r\to\infty} I(r) = \pi.$$

(iii) Using (i), deduce that

$$\iint_{D+(r)} e^{-(x^2+y^2)}\, d(x,y) = \frac{\pi(1-e^{-r^2})}{4},$$

where

$$D^+(r) = \{(x,y) \mid x^2 + y^2 \le r^2, x \le 0, y \ge 0\}.$$

Hence, deduce

$$\lim_{r\to\infty}\left(\iint_{D^+(r)} e^{-(x^2+y^2)}\, d(x,y)\right) = \frac{\pi}{4}.$$

(iv) For every $r > 0$, let

$$J(r) := \{(x,y) \mid |x| \le r, |y| \le r\}.$$

Show that

$$I(r) < J(r) < I(\sqrt{2r}), \text{for every } r > 0.$$

Hence deduce that

$$\lim_{r\to\infty} J_r = \pi.$$

(3) Using polar coordinates, evaluate

$$\iint_D \sin\theta\, dA,$$

where D is the region in the first Guardant that is out side the circle $r = 2$ and inside the cardioid $r = 2(1+\cos\theta)$

Answer

(4) Use cylindrical coordinate to evaluate

$$\int_{-3}^{+3}\int_{-\sqrt{a-x^2}}^{\sqrt{a-x^2}}\int_0^{a-x^2-y^2} x^2\, dz\, dy\, dx$$

Answer

(5) Use spherical coordinate to evaluate

$$\iiint_D f(x,y,z)\,dv$$

where

$$f\ (x,y,z) = z^2\ \sqrt{x^2+y^2+z^2}\ ,$$

and D is the solid bounded above by the hemisphere $z=\sqrt{1-x^2-y^2}$ and below by the disc in the xy-plane

$$x^2+y^2\leq 1$$

Answer

Recap

In this section you have learnt the following

- The change of variable formula in $\mathbb{R}^2$ from cartesian to polar coordinates.
- The change of variable formula in $\mathbb{R}^3$ from Cartesian to cylindrical coordinates.
- The change of variable formula in $\mathbb{R}^2$ from Cartesian to spherical coordinates.