Module 8 : Applications of Integration - II

Lecture 22 : Arc Length of a Plane Curve [Section 22.1]

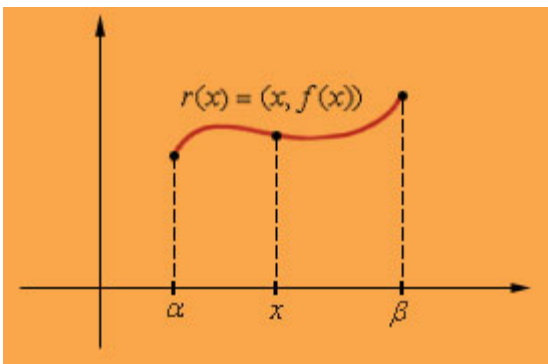### Objectives

In this section you will learn the following :

- How to find the length of a plane curve.

22.1 Arc Length of a Plane Curve

A plane curve is the function that describes the motion of a particle in plane. Since the position of a point is given by its coordinates, we have the following definition. We first consider the special case when the y-coordinate is a function of the x-coordinate.

22.1.1 Definition:

Let $f : [\alpha, \beta] \rightarrow \mathbb{R}$ be a function. We say $\gamma(x) = (x, f(x))$, $x \in [\alpha, \beta]$, is a plane curve given by the function f. Note that the image of the function γ if precisely the graph of f.

Our aim is to define the notion of length for a given curve.

22.1.2 Definition

Let a curve γ be given by function $f : [\alpha, \beta] \rightarrow \mathbb{R}$ where f is differentiable with f' continuous. Then the arc length of γ is defined to be

$$L(\gamma) = \int_a^b \sqrt{1 + f'(x)^2}\, dx \, .$$

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22.1.3 Note:

(i) The above definition is consistent with the notion of length for line segments. Indeed, for the line segment

joining points the (x_1, y_1) and (x_2, y_2), the curve given by the function

$$f(x) = \frac{y_2 - y_1}{x_2 - x_1} x + (y_1 - x_1), \ x \in [x_1, x_2].$$

Thus, the length of this line segment is given by

$$\int_{x_1}^{x_2} \sqrt{1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2} \ dt = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

(ii) The motivation for defining the arc length by equation (10) is because of the following. To find the the length of a
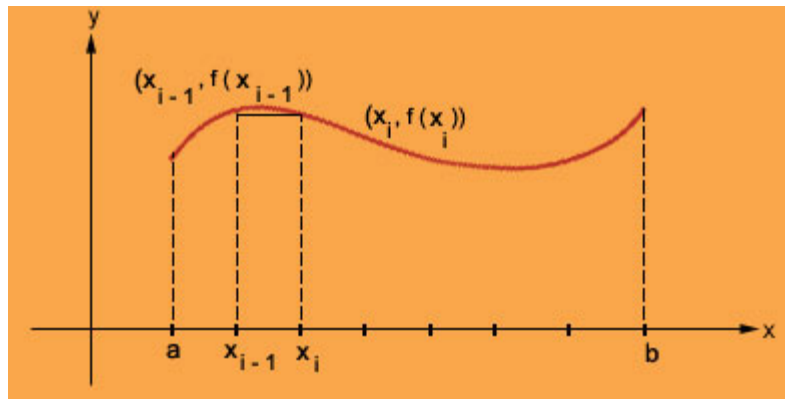
curve, we can approximate the curve by finitely many straight line segments. Let a curve γ is given by the graph of a function $f : [\alpha, \beta] \to \mathbb{R}$, i.e.,

$$\gamma(x) = (x, f(x)), \ x \in [a, b].$$

Let us approximate the curve by finitely many line segments. For this we consider a partition

$$P = \{x_0, x_1, \ldots, x_n\}$$

of the domain $[a, b]$ of the curve and consider the line segments joining the point $(x_{i-1}, f(x_{i-1}))$ with $(x_i, f(x_i))$ for $i = 1, 2, \ldots, n$. The sums of the lengths of these line segments give us an

approximation, $L(P, \gamma)$, to the length of the curve given by

$$L(P, \gamma) = \sum_{i=1}^{n} \sqrt{[x_i - x_{i-1}]^2 + [f(x_i) - f(x_{i-1})]^2}$$

In case f is differentiable and $\sqrt{1 + (f')^2}$ is Riemann integrable, using the Mean Value Theorem for f, it is easy to show that have

$$\lim_{\|(P)\| \to 0} L(P, f) = \int_a^b \sqrt{1 + f'(x)^2} \, dx.$$

22.1.4 Examples:

(i) The arc length of the curve given by the function

$$y = \frac{2x^6 + 1}{8x^2}, 1 \le x \le 2,$$

is given by

$$L = \int_1^2 \sqrt{1 + \left(x^3 - \frac{1}{4x^3}\right)^2} \, dx = \int_1^2 \left(x^3 + \frac{1}{4x^3}\right) dx = \frac{123}{32}$$

(ii) Let

$$g(y) = \sqrt{r^2 - y^2}, \ 0 \le y \le r \sin \theta,$$

where $r > 0$ and $0 \le \theta < \pi/2$ is fixed. Consider the curve given by this function. Geometrically, it is the arc of a circle of radius r marked by the points $(r, 0)$ and $(r \cos \theta, r \sin \theta)$.

Since $g'(y)^2 = y^2 / (r^2 - y^2)$,

the length of this portion of the circle is given by

$$L = \int_0^{r\sin\theta} \sqrt{1 + g'(y)^2} \, dy = \int_0^{r\sin\theta} \frac{r}{\sqrt{r^2 - y^2}} \, dy = \int_0^\theta r \, du = r\theta$$

Note that by symmetry, this formula for L holds also when $(\pi/2) \le \theta \le 2\pi$.

22.1.5 Note:

(i) Sometimes, it is convenient to represent a curve Y as the graph of a function $x = g(y), y \in [c,d]$. If g is differentiable with g' continuous, the arc length of Y is defined to be
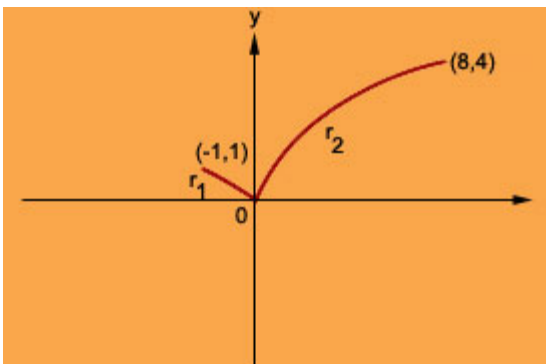
$$L(\gamma) := \int_c^d \sqrt{1 + f'(y)^2} \, dy$$

(ii) Sometimes, the length of a curve can be computed by splitting it into pieces, length of each of which can be computed, and adding the lengths of these pieces (see next example).

22.1.6 Example:

Consider the curve Y given by the function

$$f(x) = x^{\frac{2}{3}}, \ x \in [-1,8].$$

We cannot use the formula (10) directly to find the length of the curve in the specified interval as f is not differentiable at $x = 0$.

We can think Y as a sum of two curves Y_1 and Y_2 when Y_1 is given by the function

$$f_1(y) = -y^{\frac{3}{2}}, 0 \le y \le 1,$$

and y_2 is given by the function

$$f_2(y) = y^{\frac{3}{2}}, 0 \leq y \leq 4.$$

Thus, the required length is

$$L = L_1 + L_2$$

$$= \int_0^1 \sqrt{1+\left(f_1'(y)\right)^2}\,dy + \int_0^4 \sqrt{1+\left(f_2'(y)\right)^2}\,dy$$

$$= \int_0^1 \left(\sqrt{1+\frac{9}{4}y}\,dy\right) + \int_0^4 \left(\sqrt{1+\frac{9}{4}y}\,dy\right).$$

Making substitutions $4 = 1 + \dfrac{9y}{4}$ we have

$$L = \frac{4}{9}\int_1^{\frac{13}{4}} u^{\frac{1}{2}}\,du + \frac{4}{9}\int_1^{10} u^{\frac{1}{2}}\,du,$$

$$= \frac{8}{27}\left[u^{\frac{3}{2}}\right]_1^{\frac{13}{4}} + \frac{8}{27}\left[u^{\frac{3}{2}}\right]_1^{10}$$

$$= \frac{1}{27}\left[13\sqrt{3} + 80\sqrt{10} - 16\right].$$

22.1.7 Remark:

Intuitively, it may look that one should be able to find length of any curve. However, that is not the case. In our definition, we have given a sufficient condition for the same. Curves, whose lengths can be computed are called rectifiable curves. For more details the reader may refer a book on advanced calculus.

PRACTICE EXERCISES

1. Find the length of the curve given by the following functions:

(i) $f(x) = x^{\frac{3}{2}}, x \in [1,1]$.

(ii) $f(x) = \ln(\cos x), x \in \left[0, \frac{\pi}{4}\right]$.

(iii) $g(y) = (y-1)^{\frac{3}{2}}, x \in [0,8]$.

2. An electric cable is hanging between two poles that are 200 meters apart and cable is in the shape of the graph of

the function $f(x) = 75\left(e^{\frac{x}{150}} + e^{-\frac{x}{150}}\right)$.

Find the length of the cable.

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3. (i) Using mid point approximation with $n = 10$, find an approximation for the length of the curve given by the: function

$$f(x) = x^2, x \in [0,2]$$

(ii) Using Simpson's Rule with $n = 10$, find approximation to the length of the curve given by the function

$$f(x) = x^3, x \in [0,4].$$

4. Let the curve γ be given by a function $f(x), x \in [a,b]$ such that

$$m \leq f(x) \leq M \text{ for all } x \in [a,b].$$

Show that the arc length L of γ has the property

$$(b-a)\sqrt{1+m^2} \leq L \leq (b-a)\sqrt{1+m^2} \,.$$

Recap

In this section you have learnt the following

- How to find the length of a plane curve.

Applications of Integration - II

Lecture 22 : Arc length of a plane curve in parametric and polar forms [Section 22.2]

Objectives

In this section you will learn the following :

- How to find the length of a curve given in the parametric form

22.2 Arc Length of a plane Curve in parametric form

In general, the position of a particle moving in a plane is located by its coordinates which depend upon time. To be precise, we have the following:

22.2.1 Definition

(i) A curve in a plane is a function

$$\gamma: [\alpha, \beta] \to \mathbb{R}^2, \gamma = \big(x(t), y(t)\big), t \in [\alpha, \beta].$$

The functions $t \mapsto x(t)$ and $t \mapsto y(t)$ are called a ***parameterization*** of the curve.

(ii) A curve

$$\gamma: [\alpha, \beta] \to \mathbb{R}^2, \; \gamma(t) = (x(t), y(t)), t \in [\alpha, \beta]$$

is said to be **smooth** if both the functions $t \mapsto x(t)$ and $t \mapsto y(t)$ are differentiable with the derivative functions $x'(t)$ and $y'(t)$ continuous.

(iii) For a smooth curve $\gamma(t) = (x(t), y(t)), \; t \in [\alpha, \beta]$ such that

$$\gamma(t_1) \neq \gamma(t_2), \text{ for } t_1 \neq t_2$$

its arc length is defined by

$$L = \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2} \; dt .$$

22.2.2 Note:

In definition above, the condition

$$\gamma(t_1) \neq \gamma(t_2), \text{ for } t_1 \neq t_2.$$

(12)

22.2.3 Examples

Consider the curve γ given by

$$r(t) = (x(t), y(t)), \text{ where } x(t) = a\cos^3 t, y(t) = a\sin^3 t, t \in [0, 2\pi].$$

This curve is called the asteroid, and can also alternatively described as the curve given by the function

$$x^{2/3} + y^{2/3} = a^{2/3},$$

where $a > 0$. The length of this curve is given by

$$L = \int_0^{2\pi} \sqrt{\left(-3a\cos^2 t \sin t\right)^2 + \left(3a\sin^2 t \cos t\right)^2} \; dt$$

$$= \int_0^{2\pi} 3a \left|\cos t \sin t\right| dt$$

$$= 4\frac{3a}{2} \int_0^{\frac{\pi}{2}} \sin 2t \; dt$$

$$= 6a.$$

Another way the position of a particle in the plane can be located is by knowing its polar coordinates. This motivates our next definition.

22.2.4 Definition

(i) We say a function

$$r: [\alpha, \beta] \to \mathbb{R}, \; \theta \mapsto r(\theta).$$

defines a curve in polar form.

(ii) We say a curve,

$$r: [\alpha, \beta] \to \mathbb{R}$$

in polar form is smooth if r' exists and is continuous.

(iii) For a smooth curve

$$r:[\alpha, \beta] \to \mathbb{R}$$

in polar form with

$$\gamma(t_1) \neq \gamma(t_2), \text{ for } t_1 \neq t_2.$$

we define its arc length is by

$$L = \int_\alpha^\beta \sqrt{r(\theta)^2 + r'(\theta)^2} \; d\theta \cdot \qquad\qquad \text{---------}$$

(13)

22.2.5 Note:

Given a curve

$$r:[\alpha, \beta] \to \mathbb{R}$$

in polar form, we can represent it in parametric form by taking

$$x(\theta) = r(\theta)\cos\theta \text{ and } y(\theta) = r(\theta)\sin\theta, \theta \in [\alpha, \beta].$$

Note that,

$$\sqrt{x'(\theta)^2 + y'(\theta)^2} \; d\theta = \sqrt{[r'(\theta)\cos\theta - r(\theta)\sin\theta]^2 + [r'(\theta)\sin\theta + r(\theta)\cos\theta]^2}$$
$$= \sqrt{r(\theta)^2 + r'(\theta)^2}.$$

Thus, by equation (12) in definition 22.1.2, the arc length is same as in equation (13).

22.2.6 Note:

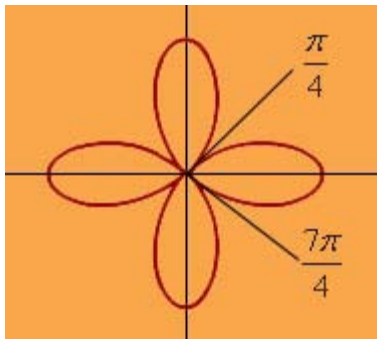The arc length of the cardioid $r = a(1 + \cos\theta)$, where $a > 0, 0 \le \theta \le \pi$ is

$$L = \int_0^{2\pi} \sqrt{a^2(1 + \cos\theta)^2 + a^2 \sin^2\theta} \; d\theta$$
$$= \int_0^{2\pi} \sqrt{2a^2(1 + \cos\theta)} \; d\theta$$
$$= 2a \int_0^{2\pi} |\cos(\theta/2)| \, d\theta$$
$$= 4a \int_0^\pi \cos(\theta/2) \, d\theta$$
$$= 8a.$$

22.2.7 Note:

For a curve in parametric form or polar form, if the corresponding functions are differentiable at all except finitely many points, then the arc length can be determined by cutting the curve in appropriate smooth "pieces" and summing their lengths. For example, consider the curve

$$r = \cos 2\theta, 0 \le \theta \le 2\pi$$

This is called the four-petal rose, with graph as follows:

The total length of the curve can be taken to be the sum of the four pieces of the curve: when θ-varies in intervals

$$\left[\frac{7\pi}{4}, \frac{\pi}{4}\right], \left[\frac{\pi}{4}, \frac{3\pi}{4}\right], \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right], \left[\frac{5\pi}{4}, \frac{7\pi}{4}\right]$$

PRACTICAL EXERCISES:

1. Find the arc length of the following curves given in the parametric form:

(i) $x(t) = \frac{t^3}{3}, y(t) = \frac{t^2}{2}, 0 \le t \le 1$.

(ii) $x(t) = \cos 2t, y(t) = \sin 2t, 0 \le t \le \frac{\pi}{2}$.

(iii) $x(t) = e^{-t} \cos t, y(t) = e^{-t} \sin t, 0 \le t \le \frac{\pi}{2}$.

(iv) $x(\theta) = a \cos^3 \theta, y(e) = a \sin^3 \theta, 0 \le \theta \le 2\pi$.

(v) $x(\theta) = a(\theta - \sin\theta), y(\theta) = a(1 - \cos\theta), 0 \le \theta \le 2\pi$.

2. Find the arc length of the following curve given in polar form:

(i) $r(\theta) = 2(1 - \cos\theta), 0 \le \theta \le 2\pi$.

(ii) $r = 1 + \sin\theta, 0 \le \theta \le 2\pi$.

3. The position of a moving particle at time t is given by

$$x(t) = \frac{1}{2}(2t+3)^{\frac{3}{2}}, y(t) = \frac{t^2}{2} + t.$$

Find the distance travelled by the particle between $t = 0$ and $t = 3$.

4. Two particles A and B are moving in the xy-plane. A starts at $(0,0)$ and moves along y-axis with some

    uniform speed. The particles B starts at $(1,0)$ and is moving towards y-axis, trying to catch A, along the path

$$y = \frac{1}{3}\left(x^{\frac{3}{2}} - 3x^{\frac{1}{2}} - 2\right).$$

    If B is moving with twice the speed of A, show that the distance covered by B is twice that of A when they meet.

Recap

In this section you have learnt the following

- How to find the length of a curve given in the parametric form