Module 10 : Scaler fields, Limit and Continuity

Lecture 30 : Continuity of scaler fields [Section 30.1]

Objectives

In this section you will learn the following :

- The notion of continuity for scalar fields.

30 .1 Continuity of scalar fields:

Following the case of function of a single variable, we define the notion of continuity for functions of two variables: the value expected of f at a point is actual value of f at the point.

30.1.1 Definition:

Let $D \subseteq \mathbb{R}^2, (x_0, y_0) \in D$ and $f : D \to \mathbb{R}$. We say f is continuous at (x_0, y_0) if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$(x, y) \in D \cap B\big((x_0, y_0), \delta\big) \Rightarrow |f(x, y) - f(x_0, y_0)| < \varepsilon.$$

Thus, for a limit point (x_0, y_0) of D,

$$f \text{ is continuous at } (x_0, y_0) \text{ if and only if } \lim_{(x,y)\to(x_0,y_0)} f(x, y) \text{ exists and is equal to } f(x_0, y_0).$$

As a consequence of the corresponding theorem on limits, we have the following:

30.1.2 Theorem:

Let $D \subseteq \mathbb{R}^2, (x_0, y_0) \in D$ *and* $f, g : D \to \mathbb{R}.$

(i) *The function* f *is continuous at* (x_0, y_0) *if for every* $\varepsilon > 0$ *, there exists* $\delta > 0$ *such that*

$$(x, y) \in D, \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \text{ implies that } | f(x, y) - f(x_0, y_0) | < \varepsilon.$$

(ii) *The function* f *is continuous at* (x_0, y_0) *if and only if for every sequence* $\{(x_n, y_n)\}$ *of points of* D *such*

that
$(x_n, y_n) \to (x_0, y_0)$, *we have* $f(x_n, y_n) \to f(x_0, y_0)$.

(iii) *If both* f *and* g *are continuous at* (x_0, y_0) *and* $r \in \mathbb{R}$ *, then the functions*

$$f+g,\ f-g,\ rf \text{ and } fg$$

are also continuous at (x_0, y_0). *Further, in case* $g(x_0, y_0) \neq 0$, *the function* f/g *is defined in a neighborhood of* (x_0, y_0) *and is continuous there.*

Another useful result is the following:

30.1.3 Theorem:

Let $f : D \to \mathbb{R}$ *be continuous at* (x_0, y_0) *and* E *be a subset of* $\mathbb{R}$ *containing the range of* f. *If* $g : E \to \mathbb{R}$ *is continuous at* $f(x_0, y_0)$, *then the composite* $gof : D \to \mathbb{R}$ *is also continuous at* (x_0, y_0).

30.1.3 Theorem:

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Proof

Let $\varepsilon > 0$ be given. Since g is continuous at $f(x_0, y_0)$, we can find $\delta > 0$ such that

$\| (x,y) - f(x_0, y_0) \| < \delta$ implies $| g(x,y) - g(f(x_0, y_0)) | < \varepsilon$ ----------(1)

Also, f being continuous at (x_0, y_0), we can find $\delta_1 > 0$ such that

$\| (s,t) - (x_0, y_0) \| < \delta_1$ implies $| f(s,t) - f(x_0, y_0) | < \delta$ ----------(2)

From (1) and (2), taking $(x,y) = f(s,t)$, we have

$\| (s,t) - f(x_0, y_0) \| < \delta$ implies $| g(s,t) - g(f(x_0, y_0)) | < \varepsilon$.

Hence, gof is continuous at (x_0, y_0).

30.1.4 Examples:

(i) Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R} \text{ defined by } f(x,y) := \sin(xy).$$

Since, the functions $(x,y) \mapsto xy$ and $\theta \mapsto \sin\theta$, are both continuous at every $(x_0, y_0) \in \mathbb{R}^2$, by the above theorems, $f(x,y)$ is also continuous at every point.

(ii) Consider the function $f:\mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(0,0)=0 \text{ and } f(x,y)=\frac{x^2y}{x^4+y^2} \text{ for } (x,y)\neq(0,0).$$

It is continuous at every $(x_0,y_0)\neq(0,0)$, by the above theorems. It is discontinuous at the point $(0,0)$. For example, if $(x,y)\to(0,0)$ along the parabola $y=ax^2 (a\neq 0)$, then

$$f(x,y)\to\frac{a}{1+a^2}\neq 0,$$

which is different for different parabolas.

(iii) Consider the function $f:\mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(0,0)=0 \text{ and } f(x,y)=\frac{xy}{\sqrt{x^2+y^2}} \text{ if } (x,y)\neq(0,0).$$

It is continuous at every $(x_0,y_0)\neq(0,0)$ by the above theorems. It is also continuous at $(0,0)$. To see this, note that

$$|xy|=|x||y|\leq\left(\sqrt{x^2+y^2}\right)|y| \text{ for every } (x,y)\in\mathbb{R}^2.$$

Thus, given any $\varepsilon>0$, if we take $\delta=\varepsilon$, then

$$(x,y)\in B\big((0,0),\delta\big) \text{ implies } |f(x,y)|<\varepsilon.$$

Practice Exercises

1. Find a suitable value for the function $f(x,y)$ at the indicated point so that it is continuous at that point:

(i) $f(x,y)=\sec x\tan y$, at $(0,\pi/4)$.

(ii) $f(x,y)=\dfrac{x\sin y}{x^2+1}$, at $(1,0)$.

(iii) $f(x,y)=\dfrac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$, at $(0,0)$.

Answers

(2) Show that the following functions cannot be defined at the stated points to make them continuous:

(i) $f(x,y)=\dfrac{x}{\sqrt{x^2+y^2}}$, at $(0,0)$.

(ii) , at $(0,0)$.

$$f(x,y)=\frac{x-y}{x+y}$$

(iii) $f(x,y)=\dfrac{x^3-xy^2}{x^2+y^2}$, at $(0,0)$.

(3) Using limit theorems, show that the following functions are continuous everywhere in their domain of definition

(i) $f(x,y)=\tan^2(xy)+\tan^2(yz)+\tan^2(zx)$.

(ii) $$f(x,y)=\begin{cases}\dfrac{\sin^2(x^2+y^2)}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0)\\ 1 & \text{if} \quad (x,y)=(0,0).\end{cases}$$

(iii) $$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0),\\ 0 & \text{if} \quad (x,y)=(0,0).\end{cases}$$

[Answers]

(4) Let $g,h:\mathbb{R}\to\mathbb{R}$ be continuous functions. Prove the following functions are continuous everywhere on $\mathbb{R}^2$.

(i) $f(x,y)=g(x)+h(y)$.

(ii) $f(x,y)=\sin(g(x)h(y))$.

(iii) $f(x,y)=g(g(x)+g(h(y)))$.

(iv) $f(x,y)=\min\{g(x),h(y)\}$.

(v) $f(x,y)=\max\{g(x),h(y)\}$.

(vi) $f(x,y)=|g(x)+h(y)|$.

(5) Using definition, examine the following functions for continuity:

(i) $$f(x,y)=\begin{cases}\dfrac{y\sqrt{x^2+y^2}}{|y|}, & \text{if } y\neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

(ii)

$$f(x,y)=\begin{cases} x\sin\dfrac{1}{x}+\dfrac{1}{y}\sin y, & \text{if } x\neq 0, y\neq 0, \\ x\sin\dfrac{1}{x}, & \text{if } x\neq 0, y=0, \\ y\sin\dfrac{1}{y}, & \text{if } x=0, y=0, \\ 0, & \text{if } x=0, y=0. \end{cases}$$

[Answers]

(6) Let

$$f(x,y)=\exp\left(\frac{-x^2}{y}\right), y>0.$$

Find, if possible, the values that f can be given at points $(x,0), x\in\mathbb{R}$ so that f becomes continuous at all points $\{(x,y)\,|\,x\in\mathbb{R}, y\geq 0\}$.

[Answers]

(7)

Let $D\subseteq\mathbb{R}^2$ and $(x_0,y_0)\in D$ be such that $B\big((x_0,y_0),r\big)\subseteq D$ for some $r>0$, i.e., (x_0,y_0) is an interior point of D. Let $\vec{\boldsymbol{u}}=(u_1,u_2)$ be a unit vector in $\mathbb{R}^2$. Then the equation of the line through (x_0,y_0) along the direction of u is given by

$$L=L\big((x_0,y_0),\vec{\boldsymbol{u}}\big)=\{(x_0+tu_1,y_0+tu_2)\,|\,t\in\mathbb{R}\}.$$

Let $\delta>0$ be such that

$$\{(x_0+tu_1,y_0+tu_2)\,|\,-\delta<t<\delta\}\subset L\big((x_0,y_0),\vec{\boldsymbol{u}}\big).$$

Let $f:D\subseteq\mathbb{R}^2\to\mathbb{R}$ and

$$h(t):=f(x_0+tu_1,y_0+tu_2), -\delta<t<\delta.$$

If f is is continuous at (x_0,y_0), show that the function $h(t)$ is continuous at $t=0$. Is the converse true?

(8) Let $f:D\subseteq\mathbb{R}^2\to\mathbb{R}$. Prove the following:

(i) f is continuous at $(x_0,y_0)\in D$ if and only if difference

$$f(x,y)-f(x_0,y_0)\to 0 \text{ as } (x,y)\to(x_0,y_0).$$

Let

$$h:=x-x_0, k:=y-y_0,$$

and

$$\varepsilon(h,k) := f(x_0+h, y_0+k) - f(x_0, y_0),$$

for $h,k \in \mathbb{R}$ such that $(x_0+h, y_0+k) \in D$. If f is continuous at (x_0, y_0), show

that

$$\varepsilon(h,k) \to 0 \text{ as } (h,k) \to (0,0).$$

(9) Let

$$d(h,k) = \sqrt{h^2+k^2}, (h,k) \in \mathbb{R}^2,$$

the distance of (h,k) from $(0,0)$. Clearly,

$$d(h,k) \to (0,0) \text{ as } (h,k) \to (0,0).$$

We say a function $f(x,y)$ converges to zero as $(x,y) \to (0,0)$, at least **at the same rate** as $d^{\alpha}(h,k)$, if there exists $\delta > 0$ and $C > 0$ such that

$$\left|\frac{f(h,k)}{d^{\alpha}(h,k)}\right| \le C \text{ for all } (h,k) \in B\big((0,0),\delta\big).$$

Prove the following:

(i) The function $f(x,y) = x$ and $g(x,y) = y$, converge to 0 at least at the same rate as $d(h,k)$.

(ii) If f and g converge to zero at least at the same rate as d, what can you say about the function $\alpha f + \beta g$ for $\alpha, \beta \in \mathbb{R}$.

(iii) The function $f(x,y) = ax^2 + bxy + cy^2$ converges to 0 at least at the same rate as d^2.

Recap

In this section you have learnt the following

- The notion of continuity for scalar fields.

[Section 30.2]

Objectives

In this section you will learn the following :

Theorem that tell us some properties of continuous scalar fields.

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30.2 Properties of continuous functions:

To state some more properties of continuous functions of two variables, we need some additional concepts about sets.

30.2.1 Definitions:

Let D be a subset of $\mathbb{R}^2$. A point $(x_0, y_0) \in \mathbb{R}^2$ is an **interior point** of D if

$B((x_0, y_0), r) \subset D$ for some $r > 0$.

We say $(x_0, y_0) \in \mathbb{R}^2$ is a **boundary point** of D if for every $r > 0, B((x_0, y_0), r)$ contains a point in D as well as a point not in D.

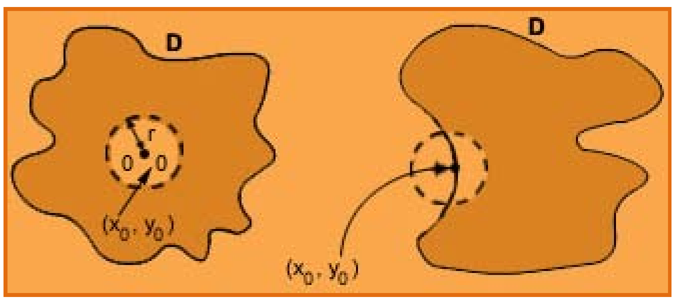

Figure 1. Interior point and boundary point

30.2.2 Definition:

Let D be a subset of $\mathbb{R}^2$.

(i) We say D is a **open** set if every point of D is an interior point of D.

(ii) We say D is a closed set if every boundary point of D is in D.

(iii) We say D is a **bounded** set if there exists some $r > 0$ such that
$D \subseteq B((x_0, y_0), r)$.

30.2.3 Example:

(i) In $\mathbb{R}^3$, each of the coordinate planes

$$\{(x,0,0) \mid x \in \mathbb{R}\}, \{(0,y,0) \mid y \in \mathbb{R}\}, \{(0,0,z) \mid y \in \mathbb{R}\}$$

are closed set. None of them is open.

(ii) The annulus ring

$$\left\{(x,y)\in \mathbb{R}^2 \,|\, a < x^2+y^2 < b\right\}$$

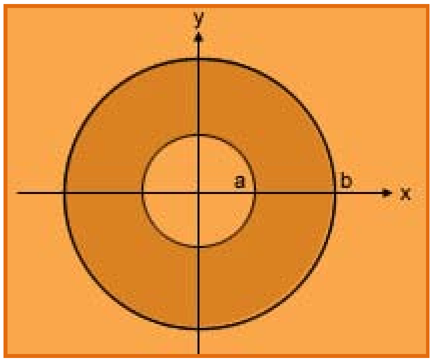

Figure 2. Bounded and open set

for $0 < a < b$, is open and bounded, while,

$$\left\{(x,y)\in \mathbb{R}^2 \,|\, a \le x^2+y^2 \le b\right\}$$

is both closed and bounded.

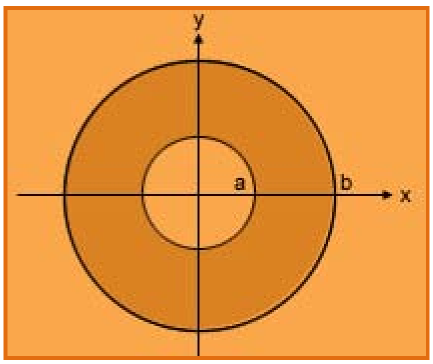

Figure 3. Closed and bounded set

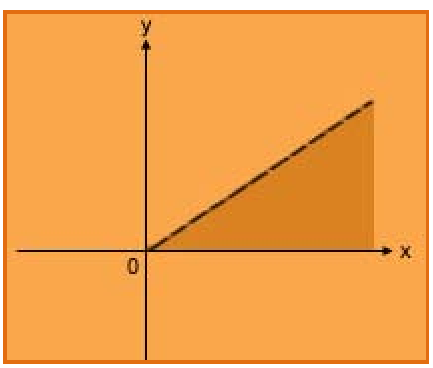

Figure 4. Neither open nor bounded set

(iii) The region

$$\left\{(x,y)\in \mathbb{R}^2 \,|\, 0 \le y < x\right\}$$

is neither open, nor bounded

Recall that, a continuous function of one variable f defined as a closed bounded interval had special properties, for example, the range of such a function f is also bounded and it attains its maximum and minimum values. Similar properties hold for special domains $D \subseteq \mathbb{R}^2$. We shall assume the following theorem:

30.2.4Theorem:

Let D be a closed and bounded subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a continuous function. Then the following holds:

(i) f *is a bounded function, that is, there is some $\alpha \in \mathbb{R}$ such that*
$| f(x, y) | \leq \alpha$ for all $(x, y) \in D.$

(ii) f *attains its bounds, that is, there are $(c_1, d_1), (c_2, d_2) \in D$ such that*

$$f(c_1, d_1) \geq f(x, y) \text{ for all } (x, y) \in D,$$
$$f(c_2, d_2) \leq f(x, y) \text{ for all } (x, y) \in D.$$

30.2.5 Note:

All the above notions and results have natural analogues for subsets of $\mathbb{R}^n, n \geq 2$ and continuous functions on them.

30.2.6 Example:

(i) Consider the function

$$f : D \to \mathbb{R},$$

where

$$D = \{(x, y) \mid y \neq 0\}$$

and

$$f\ (x, y) = \frac{x}{y}\ , (x, y) \in D.$$

The set D is an open subset of $\mathbb{R}^2$ which is not bounded. The function f is continuous on D, but is not bounded, for example $f(x, 1) = x$, can be made as large as we want, by choosing x large.

(ii) Consider the function

$$f : D \to \mathbb{R},$$

where

$$D = \{(x, y) \mid (x, y) \mid y \neq 0\}$$

and

$$f\ (x, y) = \cos y \sin x.$$

Even though D is not closed and bounded, the function f is bounded, in fact

$$| f\ (x, y) | \leq 1 \text{ for all } (x, y) \in D\ .$$

Practice Exercises

(1) Show that the following functions are continuous on the specified domains. Are these functions bounded?

(i) $f(x,y) = y\ln(1+x), \{(x,y) \mid x > -1\}$.

(ii) $f(x,y) = \sqrt{9-x^2-y^2}, \{x^2+y^2 \le 9\}$ ·

(iii) $f(x,y) = \sin(x^2+y^2+z^2), (x,y,z) \in \mathbb{R}^3$ ·

Answers

(2) Let $f, g : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be bounded functions. Show that $f+g, fg$ and αf are also bounded for all $\alpha \in \mathbb{R}$.

What can you say about f/g ?

Recap

In this section you have learnt the following

- Theorem that tell us some properties of continuous scalar fields.