Module 3 : Differentiation and Mean Value Theorems

Lecture 9 : Roll's theorem and Mean Value Theorem

Objectives

In this section you will learn the following :

- Roll's theorem
- Mean Value Theorem
- Applications of Roll's Theorem

9.1 Roll's Theorem

We saw in the previous lectures that continuity and differentiability help to understand some aspects of a function:

- Continuity of f tells us that its graph does not have any breaks.
- Differentiability of f tells us that its graph has no sharp edges.

In this section, we shall see how the knowledge about the derivative function f' help to understand the function f better.

9.1.1 Definitions:

Let $A \subset \mathbb{R}$ and $f : A \to \mathbb{R}$.

(i) We say f is increasing in A if

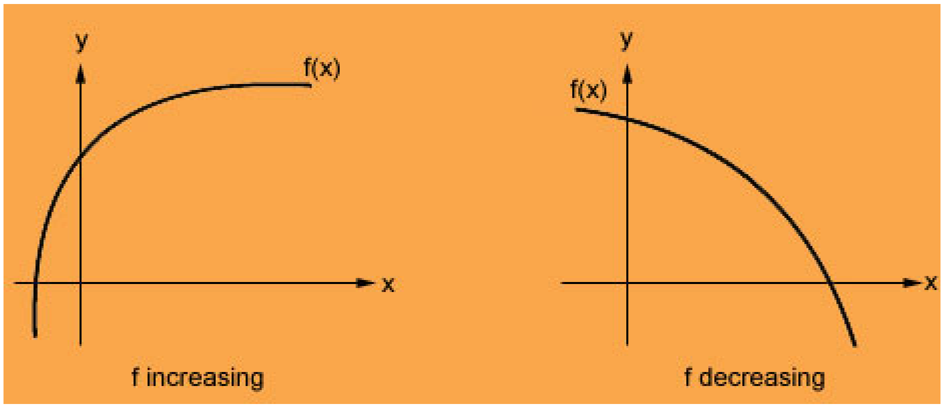
$x_1, x_2 \in A$ and $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$.

(ii) We say f is decreasing in A if

$x_1, x_2 \in A$ and $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$.

(iii) We say that f is strictly increasing/ decreasing if inequalities in (i) / (ii) are strict.

Geometrically,



f increasing f decreasing

9.1.2 Definitions:

Let $f:[a,b]\to\mathbb{R}$.

(i) We say f has a local maximum at $c\in(a,b)$, if there exists $\delta>0$ such that for

$$x\in A,\, c-\delta<x<c+\delta \text{ implies } f(x)\le f(c).$$

(ii) We say f has a local minimum at $c\in(a,b)$ if there exists $\delta>0$ such that for

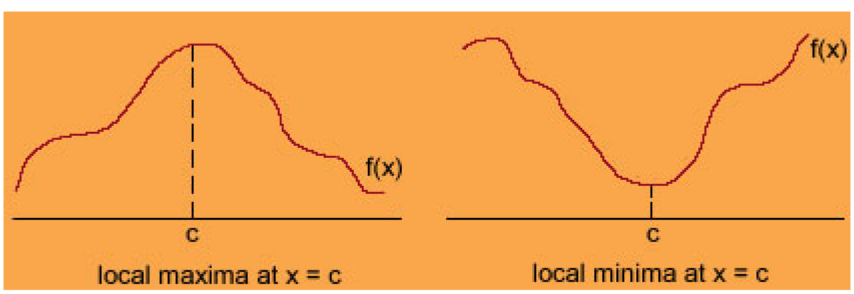
$$x\in A,\, c-\delta<x<c+\delta \text{ implies } f(x)\ge f(c).$$

(iii) We say f has a local maximum at a, if there exists $\delta>0$ such that for

$$x\in(a,b),\, a<x<a+\delta \text{ implies } f(x)\le f(a).$$

(iv) We say f has a local minimum at a, if there exists $\delta>0$ such that for

$$x\in(a,b),\, b-\delta<x<b \text{ implies } f(x)\ge f(b).$$



local maxima at x = c local minima at x = c

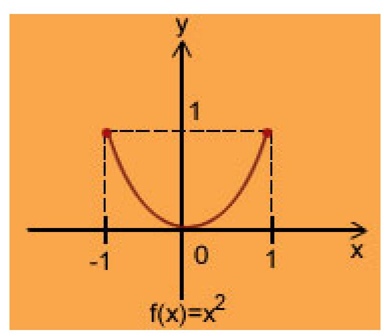
9.1.3 Examples:

(i) Let $f(x)=x^2, -1\le x\le 1$. Then f has a local minimum at $x=0$, since

$$f(0)=0\le x^2 \ \forall\ x\in[-1,1].$$

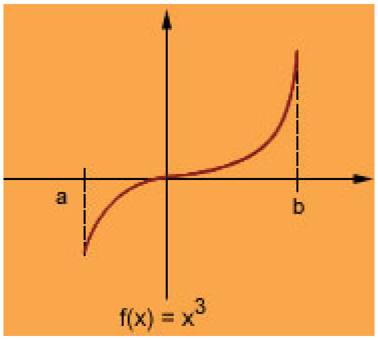
Also, f has a local maximum at $x=1$ and $x=-1$, because

$$f(-1)=f(1)=1\ge f(x)\ \forall x[-1,1].$$

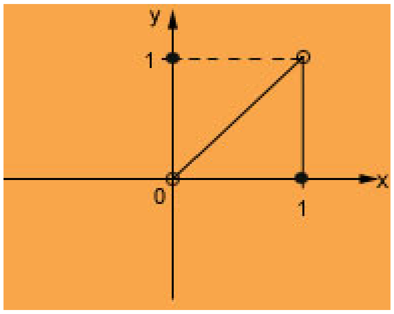


(ii) The function $f(x)=x^3$, $a \le x \le b$ is always increasing, since $\forall\, x,y \in [a,b]$,

$x^3 < y^3$ if $x<y$



(iii) Let $f(x)=\begin{cases} 1 & \text{if } x=0, \\ x & \text{if } 0<x<1, \\ 0 & \text{if } x=1. \end{cases}$



Then f is increasing in $(0,1)$, f has a local maximum at $x=0$ and a local minimum at $x=1$.

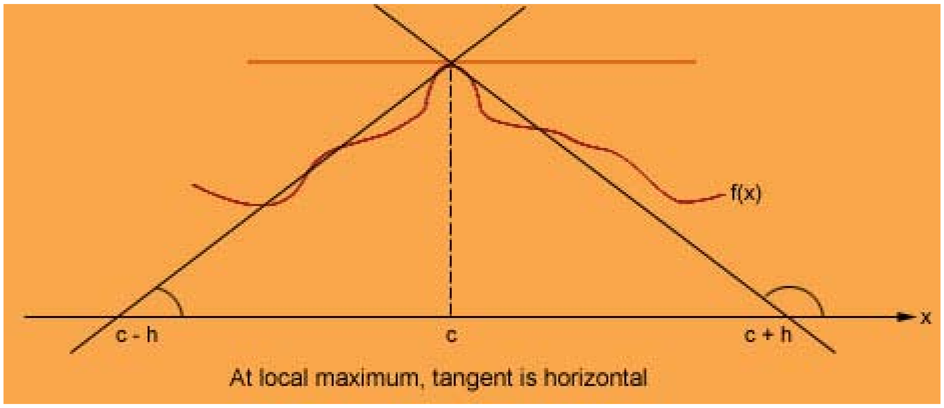
9.1.4 Lemma (Necessary condition for local extremum):

If $f:[a,b]\to\mathbb{R}$ is differentiable at $c\in(a,b)$ and has a local maximum or a local minimum at c, then $f'(c)=0$.

9.1.4 Lemma (Necessary condition for local extremum):

If $f:[a,b]\to\mathbb{R}$ is differentiable at $c\in(a,b)$ and has a local maximum or a local minimum at ,
then $f'(c)=0$.

Proof:



Suppose f has a local maximum at $c\in(a,b)$. Using definition, there is a $\delta>0$ such that $f(x)\le f(c)$ for every $x\in(c-\delta,c+\delta)\subset(a,b)$.
Thus,

$$f'(c)=\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}\ge 0 \quad \text{and} \quad f'(c)=\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}\le 0.$$

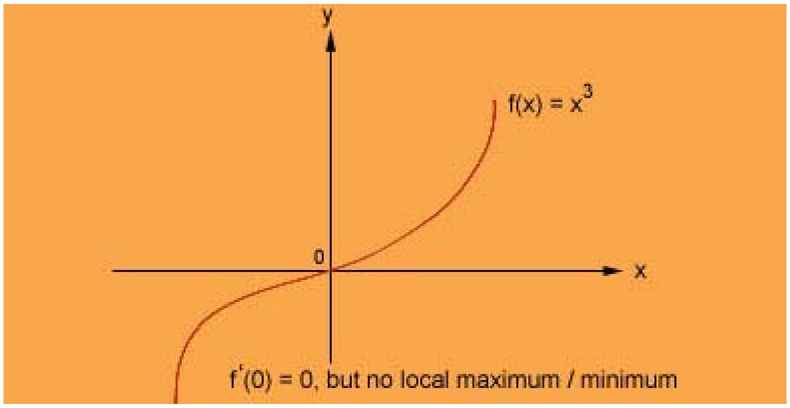
Hence, $f'(c)=0$. The case of a local minimum at is similar.

Click here to see a visualization : Applet 9.1

9.1.5 Remark:

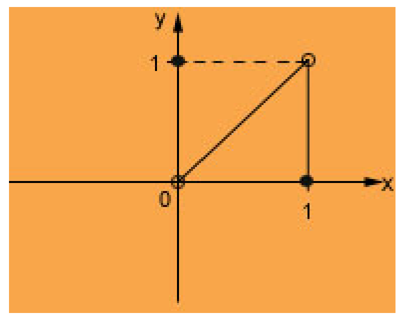
(i) Above Lemma gives only a necessary condition for a function to have local maximum or minimum at a point. The conditions are not sufficient, i.e., the converse need not hold. For example, let $f(x) = x^3$.

Then $f'(0) = 0$ but f has no maximum/minimum at 0.



(ii) Lemma holds only for c bening an interior point. If c is an end point, then f can have a local max/min at $x = c$ without derivative being zero. For example, the function

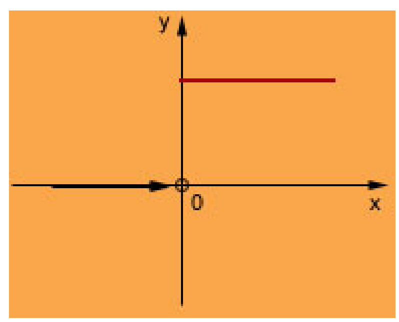
$f(x) = x, x \in [0, 1]$ has local maxima at $x = 1$ and local minima at $x = 0$ with $f'(0^+) = f'(1^-) = 1$.



(iii) f can have a local maximum/minimum at a point without being differentiable or even being continuous. For example, let

$$f(x) = \begin{cases} +1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Then f has local maximum at $x = 0$, but f is not even continuous at $x = 0$.



An important consequence of lemma 9.1.4 is the following:

9.1.6 Rolle's Theorem

If $f : [a,b] \to \mathbb{R}$ is continuous, f' exists on (a,b) and $f(a) = f(b)$, then there exists at least one point $c \in (a,b)$ such that $f'(c) = 0$.

9.1.6 Rolle's Theorem:

If $f : [a,b] \to \mathbb{R}$ is continuous, f' exists on (a,b) and $f(a) = f(b)$, then there exists at least one number $c \in (a,b)$ such that $f'(c) = 0$.

Proof:

Let $c_1, c_2 \in [a,b]$ be such that

$f(c_1) = \max\{f(x) : x \in [a,b]\}$ and $f(c_2) = \min\{f(x) : x \in [a,b]\}$

Note that such points c_1, c_2 exist as f is continuous on $[a,b]$. Either, c_1 or c_2 is an interior point of $[a,b]$ in which case

$f'(c_1) = 0$ or $f'(c_2) = 0$,

by the preceding lemma. If not, then both c_1 and c_2 are end points of $[a,b]$. Now,

$f(a) = f(b)$ implies $f(c_1) = f(c_2)$,

and hence, f is constant on $[a,b]$. Thus, $f'(c) = 0$ for every $c \in (a,b)$.

9.1.7 Examples:

(i) Let $f(x) = x^2 - 2x, [0,2]$.

Then f is differentiable on $[0,2]$ and $f(0) = f(2) = 0$. Thus, by Roll's theorem, there exists $c \in (0,2)$ such that $f'(c) = 0$.

In our case,

$$f'(x) = 2x - 2 = 0 \text{ implies } x = 1.$$

Thus for $c = 1 \in (0,2)$, $f'(c) = 0$.

(ii) Let $f(x) = x^4 - 2x^2$, $x \in [-1,1]$. Since f is differentiable on $[-1,1]$ and $f(1) = -1 = f(-1)$, by Roll's theorem, there exists $c \in (-1,1)$ such that $f'(c) = 0$. In our case

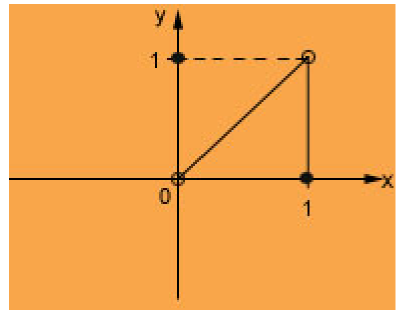
$$\begin{aligned} f'(x) &= 4x^3 - 4x \\ &= 4x(x^2 - 1). \end{aligned}$$

Thus, $f'(x) = 0$ will hold for $x = 0, \pm 1$. However, $c = 0 \in (-1,+1)$ only satisfies the required conclusion of Roll's theorem.

9.1.8 Remark:

In Rolle's Theorem, the continuity condition for the function f on the closed

interval $[a,b]$ is essential, it cannot be weekend. For example, let



$$f:[0,1]\to\mathbb{R}, f(x)=x \text{ if } 0\le x<1 \text{ and } f(1)=0.$$

Then, f is continuous on $(0,1)$,

$$f(0)=f(1) \text{ but, } f'(x)=1\neq 0 \text{ for every } x\in(0,1).$$

Click here to see a visualization: Applet 9.2

9.1.9 Examples:

(i) Let us see how Roll's Theorem is helpful in locating zeros of polynomials. Let

$$f(x)=x^4+2x^3-2, x\in[0,1].$$

Note that, f is continuous with

$$f(0)=-2<0 \; and \; f(1)=1>0.$$

Thus, by the intermediate value property, f has at least one root in $(0,1)$. Suppose that f has two roots c_1, c_2 in $[0,1]$. Then by Roll's Theorem, $f'(c)=0$ for some $c\in(c_1,c_2)$.

But

$$f'(x)=4x^3+6x^2>0 \text{ for all } x\in(0,1).$$

Hence, f can have at most one root in $[0,1]$ implying, f has a unique root in $(0,1)$.

(ii) Let

$$f(x)=|x|, x\in[-1,1].$$

Then, f is continuous on $[-1,1]$. Even though, $f(-1)=f(1)$,

$$f'(x)\neq 0 \text{ for any } x\in[-1,1].$$

In fact,

$$f'(x)=1 \text{ or } -1 \text{ for } x\neq 0.$$

This does not contradict Rolle's theorem, since $f'(0)$ does not exist.

(iii) Let $f(x)=x, x\in[0,1]$. Then f is continuous on $[0,1]$, f' exists but is nonzero on (0,1). This does not contradict Rolle's theorem since $f(0)\neq f(1)$.

We prove next an extension of the Rolle's theorem.

9.1.10 Lagrange's Mean Value Theorem (MVT):

Let $f:[a,b]\to\mathbb{R}$ be a continuous function such that f' exists on (a,b). Then there is at least one point
$c\in(a,b)$ such that

$$f(b)-f(a)=f'(c)(b-a).$$

9.1.10 Lagrange's Mean Value Theorem (MVT):

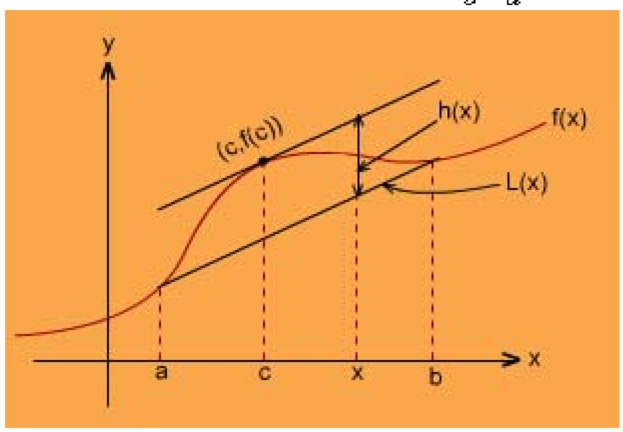
Let $f:[a,b]\to\mathbb{R}$ be *a continuous function such that* f'*exists on* (a,b)*. Then there is at least one point* $c\in(a,b)$ *such that*

$$f(b)-f(a)=f'(c)(b-a).$$

Proof :

The idea is to apply Rolle's theorem to a suitable function $h:[a,b]\to\mathbb{R}$ such that

$h(a)=h(b)$ and $h'(x)=f'(x)-\dfrac{f(b)-f(a)}{b-a}\ \forall x.$



From the figure, it is clear that such a $h(x)$ should be the difference between $f(x)$ and $L(x)$, the line joining $(a,f(a)$ and $(b,f(b))$. Thus, we consider for $x\in[a,b]$

$$h(x)=f(x)-\left[f(a)+\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)\right].$$

Observe that is continuous on $[a,b]$, differentiable on (a,b), and $h(b)=0$, $h(a)=0$, i.e., $h(a)=h(b)$.

Hence, by Rolle's theorem, $h'(c)=0$ for some $c\in(a,b)$ i.e.,

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Click here to see an applet : Applet 9.1

9.1.11 Physical Interpretation of MVT:

Let $f:[a,b]\to\mathbb{R}$ denote the distance traveled by a body from time $t=a$ to $t=b$. Then, the average speed of a moving body between two points A, at $t=a$, and B, at $t=b$, is

$$\text{Average speed}=\alpha:=\frac{f(b)-f(a)}{b-a}.$$

The mean value theorem says that there exists a time point $t=c$ in between $t=a$ and $t=b$ when the speed of the body is is actually α km/sec .

9.1.12 Theorem (Some Consequences of MVT) :

(i) Let f be differentiable on (a,b). If $f'(x)=0$ for all $x\in(a,b)$, then f is constant on (a,b)

(ii) Let f and g be differentiable on (a,b). If
$g'(x)=f'(x)$ *for all* $x\in(a,b)$,
then there exists a real constant C such that
$g(x)=f(x)+C\ \forall\ x\in(a,b)$.

(iii) Let f be continuous on $[a,b]$ and differentiable on (a,b).
If $m, M\in\mathbb{R}$ are such that $m\le f'(x)\le M$ for all $x\in(a,b)$, then

$m(b-a)\le f(b)-f(a)\le M(b-a)$

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(iii) Let f be continuous on $[a,b]$ and differentiable on (a,b).
If $m, M\in\mathbb{R}$ are such that $m\le f'(x)\le M$ for all $x\in(a,b)$, then

$$m(b-a)\le f(b)-f(a)\le M(b-a)$$

Proof :

To see (i) let $a_1, b_1\in(a,b), a_1<b_1$. Then, by the mean value theorem, f on $[a_1,b_1]$,

$$f(b_1)-f(a_1)=(b_1-a_1)f'(c)$$

for some $c\in(a_1,b_1)$, and hence $f(b_1)=f(a_1)$. This proves (i).
Statement (ii) follows from (i) and statement (iii) follows obviously from the mean value theorem.

9.1.13Example (Approximating square roots):

Mean value theorem finds use in proving inequalities. For example, for $n\in\mathbb{N}$, consider the function

$$f(x)=\sqrt{x},\ x\in[n,n+1].$$

We have, by the mean value theorem,

$$\sqrt{n+1}-\sqrt{n}=f(n+1)-f(n)=f'(c)=1/\left(2\sqrt{c}\right),$$

for some $c\in\mathbb{R}$ such that $n<c<n+1$. Hence,

$$1/\left(2\sqrt{n+1}\right) < \sqrt{n+1} - \sqrt{n} < 1/\left(2\sqrt{n}\right).$$

For $n = 1$, this gives $\sqrt{2} < 1.5$. Similarly, for $n = 3$ and $n = 4$, we get

$$\sqrt{3} < 1.75 \text{ and } \sqrt{5} < 2.25.$$

We give yet another extension of Rolle's Theorem.

9.1.14 **Theorem (Cauchy's Mean Value Theorem):**

Let f, g defined on $[a, b]$ be continuous functions such that both are differentiable on (a, b). Then, there exists $c \in (a, b)$ such that

$$[f(b) - f(a)]\, g'(c) = f'(c)[g(b) - g(a)].$$

PROOF

9.1.14Theorem (Cauchy's Mean Value Theorem):

Let f, g defined on $[a, b]$ be continuous functions such that both are differentiable on (a, b). Then, there exists $c \in (a, b)$ such that

$$[f(b) - f(a)]\, g'(c) = f'(c)[g(b) - g(a)].$$

Proof :

If $g(b) = g(a)$, we apply Rolle's Theorem to g to get a point $c \in (a, b)$ such that $g'(c) = 0$. Then

$$[f(b) - f(a)]\, g'(c) = 0 = f'(c)[g(b) - g(a)].$$

In the case $g(b) \neq g(a)$, define $h : [a, b] \to \mathbb{R}$ by

$$h(x) := f(x) - \alpha\, g(x),$$

where α is so chosen that $h(b) = h(a)$, i.e.,

$$\alpha = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Now an application of Rolle's Theorem to h gives $h'(c) = 0$, for some $c \in (a, b)$. Thus,

$$0 = h'(c) = f'(c) - \alpha g'(c) = \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right) g'(c)$$

which gives the required equality.

Practice Exercise : Rolle's theorem and mean value theorems

(1) Show that the following functions satisfy conditions of the Rolle's theorem. Find a point c, as given by the Rolle's theorem for which $f'(c) = 0$:

(i) $f(x) = (x-1)(x-2)(x-3),\ x \in [1, 3]$.

(ii) $f(x) = \dfrac{x^2 - 1}{x - 2},\ x \in [-1, 1]$.

(iii) .

$$f(x)=\frac{x}{2}-\sqrt{x},\ x\in[0,4]$$

(2) Verify that the hypothesis of the Mean Value theorem are satisfied for the given function on the given interval. Also find all points c given by the theorem:

(i) $f(x)=x^3-x-4$, $x\in[-1,2]$.

(ii) $f(x)=2x+\frac{1}{x}$, $x\in[3,4]$.

(iii) $f(x)=x\left(x^2-x-2\right)$, $x\in[-1,1]$.

(3) Let $f(t)=At^2+Bt+C$ be the distance traveled by a body for $t\in[a,b]$. Show that the average speed of the body is always attained at the mid point : $t=\frac{a+b}{2}$.

(4) Let p and q be two real numbers with $p>0$. Show that the cubic x^3+px+q has exactly one real root.

(5) Show that the cubic x^3-6x+3 has all roots real.

(6) Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)$ and $f(b)$ are of different signs and $f'(x)\neq 0$ for all $x\in(a,b)$, then there is a unique $x_0\in(a,b)$ such that $f(x_0)=0$.

(7) Consider the cubic $f(x)=x^3+px+q$, where p and q are real numbers. If $f(x)$ has three distinct real roots, then show that $4p^3+27q^2<0$ by proving the following:

(i) $p<0$.

(ii) f has maxima at $-\sqrt{\frac{-p}{3}}$ and minima at $\sqrt{\frac{-p}{3}}$.

(iii) $f\left(-\sqrt{\frac{-p}{3}}\right)f\left(\sqrt{\frac{-p}{3}}\right)<0$.

(8) Let $n\in\mathbb{N}$ and $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists in (a,b). If f vanishes at $n+1$ distinct points in $[a,b]$, then show that $f^{(n)}$ vanishes at least once in (a,b).

(9) Let f,g,h be continuous on $[a,b]$ and differentiable on (a,b). Show that there is some $c\in(a,b)$ such that

$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ g(a) & g(b) & g'(c) \\ h(a) & h(b) & h'(c) \end{vmatrix}=0.$$

Deduce that if $h(x)=1$ for all $x\in[a,b]$, we obtain the conclusion of Cauchy's Mean Value Theorem, i.e., $\left[f(b)-f(a)\right]g'(c)=f'(c)\left[g(b)-g(a)\right]$. What does the result say if $g(x)=x$ and $h(x)=1$ for all $x\in[a,b]$?

(10) Use the Mean Value Theorem to prove $|\sin a - \sin b| \le |a-b|$ for all $a, b \in \mathbb{R}$.

(11) Let $f:[0,\pi/2] \to \mathbb{R}$ be continuous and satisfy $f'(x) = 1/(1+\cos x)$ for all $x \in (0,\pi/2)$. If $f(0)=3$, estimate $f(\pi/2)$.

(12) Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=a$ and $f(b)=b$, show that there exist distinct c_1, c_2 in (a,b) such that $f'(c_1),\ f'(c_2)=2$. Formulate and prove a similar result for n points $c_1,......,c_n$ in (a,b).

(13) Let $a>0$ and f be continuous on $[-a,a]$. Suppose that $f'(x)$ exists and $f'(x) \le 1$ for all $x \in (-a,a)$. If $f(a)=a$ and $f(-a)=-a$, show that $f(0)=0$.

(14) In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.

(i) $f''(x)>0$ for all $x \in \mathbb{R}, f'(0)=1, f'(1)=1$

(ii) $f''(x)>0$ for all $x \in \mathbb{R}, f'(0)=1, f'(1)=2$

(iii) $f''(x)>0$ for all $x \in \mathbb{R}, f'(0)=1, f(x) \le 100$ for all $x>0$

(iv) $f''(x)>0$ for all $x \in \mathbb{R}, f'(0)=1, f(x) \le 1$ for all $x<0$.

(15) (Intermediate value Property for f'): Let f be differentiable on $[a,b]$. Show that the function f' has the Intermediate Value Property on $[a,b]$. (Hint : If $f'(a)<r<f'(b)$, then the function g defined by $g(x)=f(x)-rx, x \in [a,b]$, does not assume its minimum at or a at b.)

Recap

In this section you have learnt the following :

- Roll's theorem
- Mean Value Theorem
- Applications of Roll's Theorem