Module 1 : Real Numbers, Functions and Sequences

Lecture 1 : Real Numbers, Functions [Section 1.1 : Real Numbers]

Objectives

In this section you will learn the following

- Axiomatic definition of real numbers.
- Properties of real numbers.

1.1.1 The Real Numbers :

Real Numbers are the elements of a set, denoted by $\mathbb{R}$, with the following properties:

1) Algebraic properties of real numbers:

There are two binary operations defined on $\mathbb{R}$, one called addition, denoted by $(x,y)\mapsto x+y$, and the other called multiplication, denoted by $(x,y)\mapsto xy$, with the usual algebric properties: for all $x,y,z\in\mathbb{R}$

- $x+y=y+x$; $xy=yx$ $(commutative\ law)$.
- $x+(y+z)=(x+y)+z$; $x(yz)=(xy)z$ $(associative\ law)$.
- $x(y+z)=xy+xz$; $(y+z)x=yx+zx$ $(distributive\ law)$.

There exist two distinct elements in $\mathbb{R}$, denoted by 0 and 1, with following properties:

$0+x=x$ for all $x\in\mathbb{R}$; $1x=x$ for all $0\neq x\in\mathbb{R}$.
The elements 0, read as **zero**, is called the **additive identity** and 1, read as **one**, is called the **multiplicative identity**.

- For every $x\in\mathbb{R}$, there exists unique element $-x\in\mathbb{R}$ such that $x+(-x)=0$; for $x\neq 0$ in $\mathbb{R}$, there exists unique element $x^{-1}\in\mathbb{R}$ such that $x\,x^{-1}=1$.

2) Order properties of real numbers:

There exists an **order**, denoted by **<**, between the elements of $\mathbb{R}$ with the following properties:

- For $x,y\in\mathbb{R}$, one and only one of the following relations hold : $x<y$, $x=y$, $y<x$.
- $x,y>0\Rightarrow xy>0$ and $x+y>0$.

- $x > y$ and $y > z \Rightarrow x > z$.

There are two more properties that real numbers have which we shall describe later **:**

3) Archimedean property
4) Completeness property

Geometrically, set of all points on a line represent the set of all real numbers. There are some special subsets of $\mathbb{R}$ which are important. These are the familiar number systems.

1.1.2 $\mathbb{N}$ **, the set of Natural Numbers:**

Recall that, there exist unique elements 0, 1 $\in \mathbb{R}$ such that $0 + x = x$, for $x \in \mathbb{R}$ and $1x = x$, for all $0 \neq x \in \mathbb{R}$. One can show that $0 < 1$. The set $\mathbb{N}$ is the 'smallest' subset of $\mathbb{R}$ having the property **:** $1 \in \mathbb{N}$ and $n + 1 \in \mathbb{N}$, whenever $n \in \mathbb{N}$ **.** This is also called the **Principle of Mathematical Induction**. One can show that such a subset of $\mathbb{R}$ exists, and is unique. Elements of $\mathbb{N}$ are called **natural numbers**. We shall use the familiar notation, $\mathbb{N} = \{1,2,....,\}$. Geometrically, we can select any arbitrary point *O* on the real line and associate it with $0 \in \mathbb{R}$. Equidistant points on the right of *O* can be labeled as $1, 2, 3, \ldots$

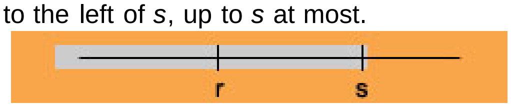
The set $\mathbb{N}$ has the following properties, which we shall assume:

- $n \geq 1$ for all $n \in \mathbb{N}$.
- For every $n \in \mathbb{N}, n > 1$ and $n - 1 \in \mathbb{N}$.
- For every $n \in \mathbb{N}$, there is no element $m \in \mathbb{N}$ such that $n < m < n + 1$.
- Archimedean property

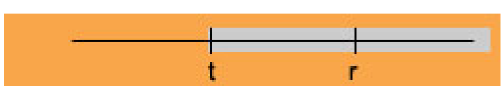
  For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.

1.1.3 Definition:

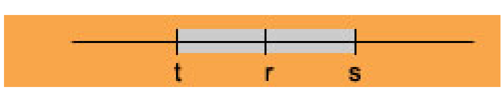
(i) A subset $E \subseteq \mathbb{R}$ is said to be **bounded above** if there exists $s \in \mathbb{R}$ such that $r \leq s$ for all $r \in E$. That is, all the elements of *E* lie to the left of *s*, up to *s* at most.



(ii) Similarly, we say $E \subseteq \mathbb{R}$ is **bounded below** if there exists $t \in \mathbb{R}$ such that $t \leq r$ for all $r \in E$.



(iii) A set E is said to be **bounded** if it is both bounded above and below, i.e., there exist *s, t* $\in \mathbb{R}$ such that $t \leq r \leq s$ for every $r \in E$.



1.1.4 **Example :**

$\mathbb{N}$ is bounded below by 1. In fact, every $E \subseteq \mathbb{N}$ is bounded below. Archimedian property says that $\mathbb{N}$ is not bounded above.

1.1.5 $\mathbb{Z}$ **, the set of Integers :**

For every $n \in \mathbb{N}$, let $-n$ be the unique element of $\mathbb{R}$ such that $n+(-n)=0$. Let

$$\mathbb{Z} := \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$$

Elements of $\mathbb{Z}$ are called integers. Clearly, $\mathbb{Z}$ is neither bounded above nor bounded below.

1.1.6 $\mathbb{Q}$, the set of rational numbers :

For every $n \in \mathbb{N}$, let $n^{-1} \in \mathbb{R}$ be such that $n(n^{-1})=1$. The element n^{-1} is also denoted by $\frac{1}{n}$. Let

$$\mathbb{Q} := \{ nm^{-1} \mid n \in \mathbb{Z}, m \in \mathbb{N} \}$$

The set $\mathbb{Q}$ is called the set of rational numbers and the elements of the set $\mathbb{R} \setminus \mathbb{Q}$ are called the irrational numbers.
Both, the rational and the irrational numbers have the following denseness property *:*

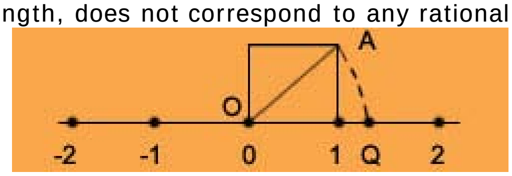
1.1.7 Denseness of rational and the irrational numbers:

For every real numbers x and y, with $x<y$,there exist a rational α and an irrational β such that $x<\alpha<y$ and
$x<\beta<y$.

1.1.8 Note: Why real numbers?

At this stage one can ask the following questions: What is the need to work with real numbers? Can one not work always with rational numbers? How real numbers are different from rational numbers? You will see answer to some of these questions in this course. Hopefully, you would have realized by now that arithmetic is necessary for day-to-day life.

Also, you would have seen (in your school courses) that there does not exist any rational *r* such that r^2 = 2. (This was discovered by the Greek mathematicians in 500 B.C.) This is one of the reasons why mathematicians were forced to invent a set of numbers which is 'bigger' than that of rationals, and which satisfy equations of the type $x^n = 2$ for all $n \in \mathbb{N}$. The property of the real numbers that distinguishes them from the rational numbers, is called the completeness property, which we shall discuss in section 1.6 of lecture 3. Geometrically, rational numbers when represented by points on the line, do not cover every point of the line. For example, the point *Q* on the line such that *OQ* is equal to *OA*, the length of the diagonal of a square with unit length, does not correspond to any rational number.



Thus, some gaps are left when rational numbers are represented as points on a horizontal line. Filling up these gaps is the "completeness property" of the real numbers. We will make it mathematically precise in the next section. These gaps are the irrational numbers.

1.1.9 Intervals :

We describe next another important class of subsets of $\mathbb{R}$, called **intervals**. For *a, b* $\in \mathbb{R}$ with $a<b$, we write

$(a, b) := \{ x \in \mathbb{R} \mid a < x < b \}$, $[a, b] := \{ x \in \mathbb{R} \mid a \leq x \leq b \}$,

$(a, b] := \{ x \in \mathbb{R} \mid a < x \leq b \}$, $[a, b) := \{ x \in \mathbb{R} \mid a \leq x < b \}$,

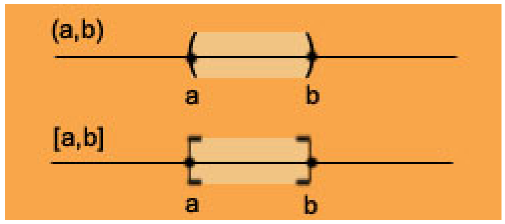
$(a, +\infty) := \{ x \in \mathbb{R} \mid a < x \}$, $[a, +\infty) := \{ x \in \mathbb{R} \mid a \leq x \}$,

$(-\infty, a] := \{ x \in \mathbb{R} \mid a \geq x \}$, $(-\infty, a) := \{ x \in \mathbb{R} \mid a > x \}$,

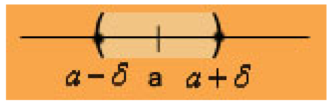
When $a = b$, interval $(a, a) := \emptyset$ and $[a, a] := \{ a \}$. Intervals of the type *(a ,b]* are called **left-open right-closed intervals**.

Similiarly, intervals of the type *[a, b)* are called **left-closed right-open Intervals**. And intervals of the type *(a, b)*, *(-∞ , a)*, *(a, +∞)* are called **open intervals** and of the type [*a,b]*, *(-∞,a]*, *(a, +∞* } are called closed intervals. Note that -∞, +∞ are just symbols, and not numbers.

As stated above, shall assume that the set of real numbers can be identified with points on the straight line. If point *O* represent the number *0*, then points on the left of *O* represent negative real numbers and points on the right of *O* represent positive real numbers. Intervals are part of the line as shown:



This identification is useful in visualizing various properties of real numbers. An open interval of the type $(a-\delta, a+\delta)$ is called an δ - neighborhood of $a \in \mathbb{R}$.



For Quiz refer the WebSite

Practice Exercises 1.1 : Real Numbers

(1) Using the Principle of Mathematical Induction, prove the following for all $n \in \mathbb{N}$:

(i) $n > 1$ implies $n - 1 \in \mathbb{N}$.

(ii) For $x \in \mathbb{R}$ with $x > 0$, if $x + n \in \mathbb{N}$, then $x \in \mathbb{N}$.

(iii) $m + n$, $m n \in \mathbb{N}$ for all $m, n \in \mathbb{N}$.

(2) Using the Principle of induction prove the following:

(i) If $a_1 = 1, a_n = \dfrac{3a_{n-1}+2}{6} \;\forall\; n \geq 2$, then $a_n \geq \dfrac{2}{3}$ and $a_n \geq a_{n+1} \;\forall\; n \geq 1$.

(ii) If $a_1 = 1, a_2 = 2$ and $a_{n+1} = a_n + a_{n-1} \forall\; n \geq 2$, then

$$a_n < \left(\frac{1+\sqrt{5}}{2}\right)^n \text{ for all } n \geq 1 .$$

(3) Show that the product of any two even (odd) integers is also an even (odd) integer.

(4) Find an irrational number between 3 and 4.

Recap

In this section you have learnt the following :

- Axiomatic definition of real numbers and the construction of other number systems.

- The algebraic and order properties of real numbers.

- The Archemedian property of real numbers.

[Section 1.2 : Functions]

Objectives

In this section you will learn the following

- Concept of a function.
- The absolute value function.
- Concept of a bijective function.

1.2 FUNCTIONS :

You already have some familiarity with the concept of a function. Function is a kind relation between various objects. For example, the volume V of a cube is a function of its side x ; in physics velocity v of a body at any time t is a function of its initial velocity and time, and acceleration; and so on. In mathematics, a function is defined as follows:

1.2.1 **Definition :**

(i) For sets A and B, a **function** from A to B, denoted by $f : A \rightarrow B$, is a correspondence which assigns to every element $x \in A$, a unique element $f(x) \in B$. The value of the function f at an element x in A is denoted by $f(x)$, which is an element in B. This is indicated by $x \mapsto f(x) \in B$.

(ii) For a function $f : A \rightarrow B$, the set A is called the **domain** of f and the subset $f(A) = \{f(x) : x \in A\}$ of B, (set of images of f) is called the **range** of f. If $B \subseteq \mathbb{R}$, then f is said to be **real-valued**. If also $A \subseteq \mathbb{R}$, then the **natural domain** of f is the set of all $x \in \mathbb{R}$ for which $f(x) \in \mathbb{R}$**.**

1.2.2 **Examples :**

(i) Let $f(x) = x^3$. Then, f has natural domain $\mathbb{R}$. Its range is also $\mathbb{R}$, because for any given y , if $x := y^{\frac{1}{3}}$ then we get $f(x) = y$.

(ii) Let $f(x) = 1/(x-1)$. Then, f has natural domain $\{x \in \mathbb{R} : x \neq 1\}$, and its range is given by $\{y \in \mathbb{R} : y \neq 0\}$.

(iii) Let $f(x)=\sqrt{x^2-1}$ .Then, f has natural domain $\{x\in\mathbb{R}:|x|\geq 1\}$, and its range is given by $\{y\in\mathbb{R}:y\geq 0\}$.

1.2.3 *Definition :*

(i) A real-valued function f is said to be **bounded** if its range is a bounded subset of $\mathbb{R}$, that is, there is some $\alpha\in\mathbb{R}$ such that $|f(x)|\leq\alpha$ for all $x\in A$.

(ii) Let $f: A\to B$. The graph of f is the set :

$$G(f)=\{(x,f(x)):x\in A\}\subseteq A\times B.$$

(iii) Let $f: A\mapsto B$ be a function. We say f is one-one (or injective) if

$$f(a_1)\neq f(a_2),\ whenever\ a_1,a_2\in A\ and\ a_1\neq a_2 .$$

(iv) *If f is one -one, the function $f^{-1}: f(A)\to A$, defined by $f^{-1}(b)=a$ for $b=f(a)\in f(A)$, is called the* inverse function.

(v) We say f is onto (or surjective) *if $f(A)=B$.*

*(vi) A function f is called **bijective** if it is injective as well as surjective.*

(vii) If $f:A\to B$ *and* $g:C\to D$ *with* $f(A)\subseteq C$, then the composite function is the function go $f:A\to D$ defined by $(gof)(a)=g(f(a))$ for $a\in A$.

1.2.4 Note:

For $A,B\subseteq\mathbb{R}$, saying that a function $f:A\to B$, is onto means that the horizontal line at every point $y\in B$ meets the graph of the function at least once. Similarly, saying that the function is one -one means that the horizontal line at every point $y\in B$ meets the graph of the function at most once.

Click here to see an interactive visualization: Applet 1.1 (avaliable on website)

Click here to see an interactive visualization: Applet 1.2 (avaliable on website)

1.2.5 Absolute Value function :

The function $|\cdot| : \mathbb{R}\to[0,+\infty)$ defined by

$$|x|=\begin{cases}x \text{ if } x\geq 0,\\ -x \text{ if } x\leq 0,\end{cases}$$

is called the **absolute value function** and $|x|$ is called the **absolute value** of $x\in\mathbb{R}$.

1.2.6 Theorem :

For every $x,y,z\in\mathbb{R}$ the following results are true:

(i) $|x| \geq 0$ and $|x| = 0$ *if and only if* $x = 0$, *(that is,* $|x| = 0$ if $x = 0$, *and conversely* $x = 0$ if $|x| = 0$ *).*

(ii) $|xy| = |x|\,|y|$.

(iii) $|x| \leq y$ if and only if $-y \leq x \leq y$.

(iv) $|x+y| \leq |x| + |y|$, **(triangle inequality)**.

(v) $\big||x| - |y|\big| \leq |x-y|$.

Proof:

The proofs of (i) follows obviously from the definition of absolute value.

To prove (ii) we consider different cases. The required property is obvious if both x, y 0 .
In case $x > 0$ and $y < 0$) , we have
$|xy| = -(xy) = x(-y) = |x||y|$
The other cases can be analyzed similarly.

To prove (iii) suppose $|x| \leq y$. If $x \geq 0$, then
$-y \leq 0 \leq x = |x| \leq y$
If $x \geq 0$, then
$-x = |x| \leq y$
Thus, $-y \leq x \leq 0 \leq y$
Conversely, let $-y \leq x \leq y$. If $x \geq 0$, then $|x| = x \leq y$. If $x \leq 0$, then $|x| = -x \leq y$. Thus (iii) holds.
To prove (iv), note that
$-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$.
Adding the two we get
$-(|x| + |y|) \leq x + y \leq (|x| + |y|)$
and hence by (iii), $|x+y| \leq |x| + |y|$.
Finally, to prove (v) , note that by (iv)
$|x| - |y| \leq |x-y|$ and $|y| - |x| \leq |x-y|$.
Hence (v) follows from (iii).

1.2.7 Note :

For real numbers x , y and α > 0 , the inequality | x - y | $\leq$ α means that

$$y - \alpha \leq x \leq y + \alpha,$$

i.e., $x \in [\, y - \alpha \,,\, y + \alpha \,]$.

1.2.8 Definitions (Algebra of real valued functions) :

Let $A \subseteq \mathbb{R}$ and $f, g : A \to \mathbb{R}$.

(i) The **sum** of f and g is the function $f + g : A \to \mathbb{R}$, defined by

$$(f+g)(x) := f(x) + g(x),\ x \in A.$$

(ii) The **product** of f and g is the function $fg : A \to \mathbb{R}$, defined by

$$(fg)(x) := f(x)\,g(x),\ x \in A.$$

(iii) If $g(x) \neq 0\ \forall\ x \in A$, the **quotient** of f and g is the function $f/g : A \to \mathbb{R}$, defined by

$$(f/g)(x) := f(x)/g(x),\ x \in A.$$

(iv) For any real number α, the function αf is the function $\alpha f : A \to \mathbb{R}$, defined by

$$(\alpha f)(x) := \alpha f(x),\ x \in A.$$

(v) For any integer $n \geq 0$ and real numbers $a_0, a_1, \ldots, a_n$, the function $p : A \to \mathbb{R}$ defined by

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n,\ x \in A,$$

is called a **polynomial function**. When every a_i is an integer, it is called an **integral polynomial function**.

1.2.9 Definition:

A function f is called an **algebraic function** if it can be obtained from integral polynomial functions in finite number of operations involving one or more of the following operations:

sums / products / quotients / taking inverse. Functions which are not algebraic are called **transcendental functions**.

1.2.10 Note :

The most important among the transcendental functions are the **natural logarithmic function** and its inverse, the **exponential function**, and the **trigonometric functions**. We shall define these functions rigorously in module 6. For the time being, to illustrate various concepts, we shall assume the knowledge about trigonometric functions.

In particular we shall assume the following inequalities: For $0 < \theta < \dfrac{\pi}{2}$.

$$-\theta < \sin\theta < \theta,\quad -\theta < 1 - \cos\theta < \theta,\quad \theta\cos\theta < \sin\theta < \theta$$

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For Quiz refer the website.

Practice Exercises : Functions

(1) Let X be any set. For a subset A *of* X, the **indicator function** of A, denoted by χ_A, is the function $\chi_A : X \to \{0,1\}$ defined by

$$\chi_A(x) = \begin{cases} 0, if\ x \notin A, \\ 1, if\ x \in A \end{cases}.$$

Prove the following statements for :

$$A, B \subseteq X$$

(i) $\chi_{A \cap B} = \chi_A \chi_B$.

(ii) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$.

(iii) $\chi_{A \Delta B} = \left|\chi_A - \chi_B\right|, \text{ where } A \Delta B = (A \cup B) \setminus (A \cap B) := (A \cap B^c) \cup (A^c \cap B)$.

(2) Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by

$$f(n, m) = 2^{n-1}(2m-1) \text{ for } n, m \in \mathbb{N}.$$

Show that f is a one-one function. Is it onto also?
(This shows that the set $\mathbb{N}$ x $\mathbb{N}$ has as many points as $\mathbb{N}$ has !)

(3) Let $f : A \to B$. For any set $E \subseteq B$, let $f^{-1}(E) : \{a \in A \mid f(a) \in E\}$.

The set $f^{-1}(E)$ is called the **preimage** of E .

Prove the following: For $E, F \subseteq B$,

(a) $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$.

(b) $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.

(c) $f^{-1}(E \setminus F) = f^{-1}(E) \setminus f^{-1}(F)$.

(d) f is one one if and only if for every $b \in B$, $f^{-1}(\{b\})$ is at most a singleton set.

(e) If $f^{-1}(B) = A$, then f need not be onto.

Historical Comments :

You will be surprised to know that the concept of function is not very old. Till 1837 it was believed that every function could be traced with "free motion of the hand". A mathematician **Gustav Dirichlet** in 1837 gave the definition of function which we use most often now. He also gave the example the function $f : [0,1] \to \mathbb{R}, f(x) := 0$ if $x \in [0,1]$ is irrational and $f(x) = 1$ if $x \in [0,1]$ is a rational. (This is the indicator function of the set of rationals in $[0,1]$). Do you think you can graph the function?

Recap

In this section you have learnt the following

- Concept of a function.
- Concept of a bijective function.

- The notion of absolute value functions, algebraic and transcendental functions.