Module 8 : Applications of Integration - II

Lecture 24 : Volume of solids of revolution by washer method [Section 24.1]

Objectives

In this section you will learn the following :

- How to find the volume of a solid of revolution by the washer method.

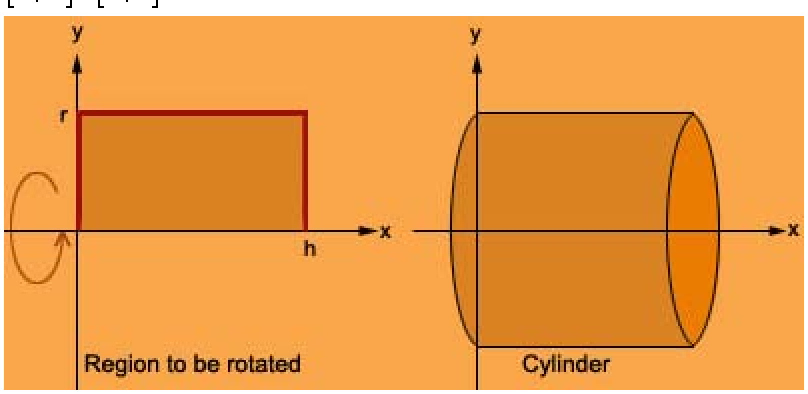
24.1 Volumes of solids revolution the washer Method

24.1.1 Definition:

A solid is called a solid of revolution if it is generated by revolving a plane region about an axis.

24.1.2 Examples:

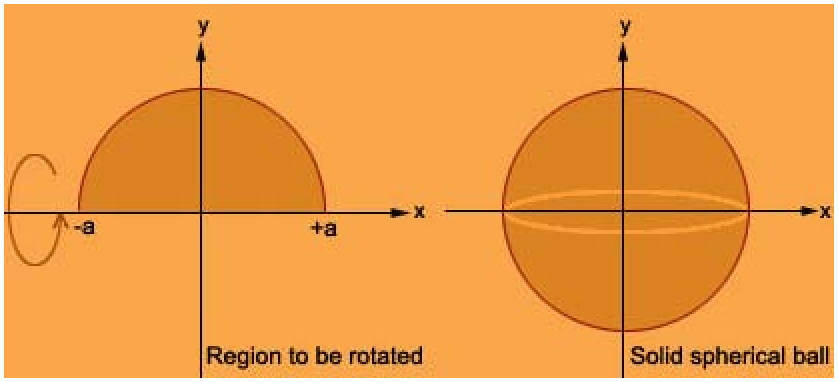
(i) A solid right circular cylinder of radius r and height h can be generated by revolving the rectangle $[0,h]\times[0,r]$ about the $x-$axis.



(ii) A solid spherical ball of radius a can be generated by revolving the semi-disc

$$\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le a^2, y\ge 0\}$$

about the $x-$axis



We describe next the slice method, as discussed in the previous section, as applied to solids of revolution.

24.1.3 Washer Method:

Let D be the plane region between the two curves given by the functions

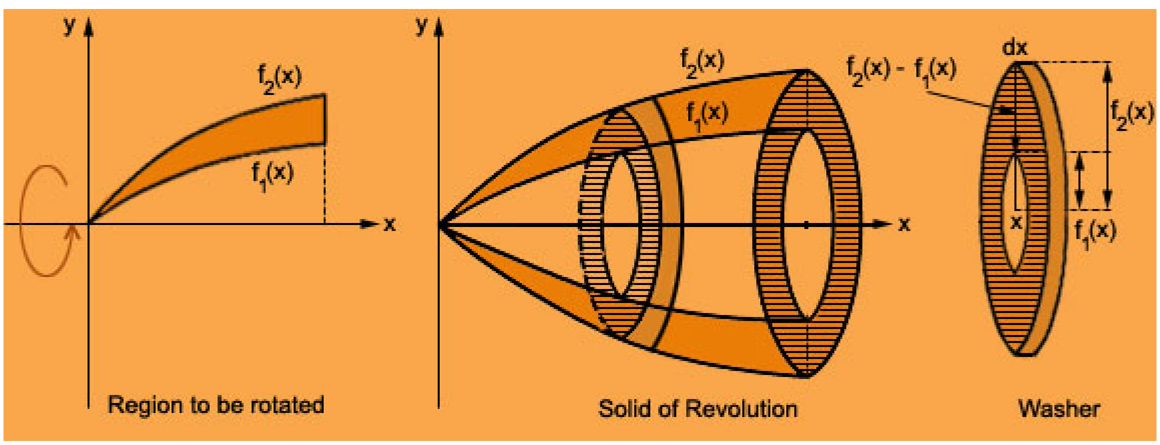
$$f_1, f_2 : [a,b] \rightarrow \mathbb{R}$$

which are Riemann integrable and

$$0 \le f_1(x) \le f_2(x) \text{ for all } x \in [a,b].$$

Then, for the solid obtained by revolving D about the $x-$axis, the slice of thickness dx at x is a circular washer of inner radius $f_1(x)$ and of outer radius $f_2(x)$. Thus, its cross sectional area is

$$A(x) = \pi[f_2(x)^2 - f_1(x)^2].$$



Therefore, the volume of the corresponding solid of revolution is given by

$$V = \pi\int_a^b [f_2(x)^2 - f_1(x)^2]\,dx.$$

24.1.4 Note:

For a solid of revolution the slice is taken in planes perpendicular to the axis of revolution. For example, if a plane region D between the two curves given by functions

$$x = f_1(y), x = f_2(y), 0 \le f_1(x) \le f_2(x) \text{ for } c \le y \le d,$$

is revolved around $y-$axis, then the slice of thickness dy by a plane perpendicular to $y-$axis at a point y is a circular washer of inner radius $f_1(y)$ and of outer radius $f_2(y)$. Thus, its cross sectional area is

$$A(y) = \pi[f_2(y)^2 - f_1(y)^2],$$

and the volume of the corresponding solid of revolution is given by

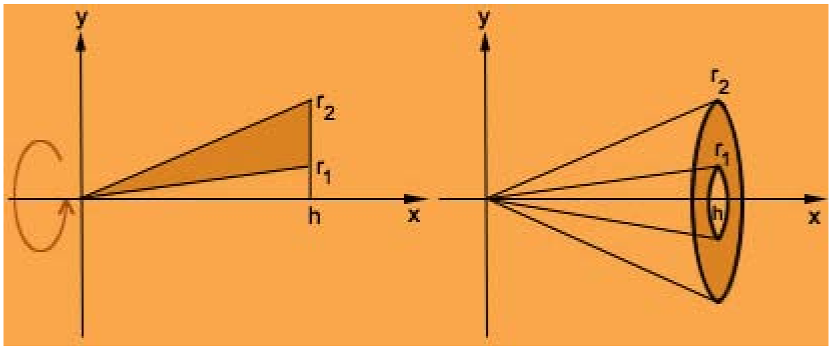
$$V = \pi\int_c^d [f_2(y)^2 - f_1(y)^2]\,dy.$$

24.1.5 Examples:

(i) Consider a solid obtained by revolving about the $x-$axis the triangular region in the first quadrant bounded by

the lines.

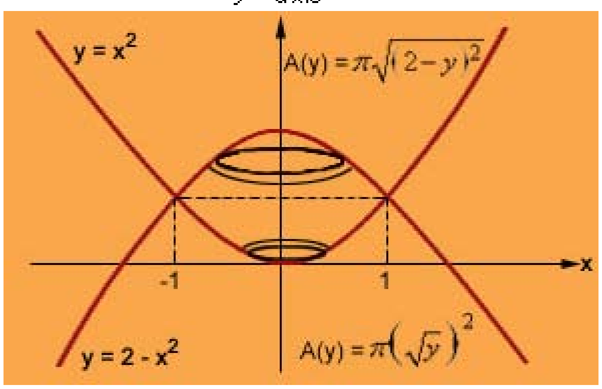
$$x = h, y = \frac{r_1 x}{h}, \text{ and } y = \frac{r_2 x}{h}, r_2 > r_1.$$



Thus, by the Washer Method, its volume equals

$$V = \pi\int_0^h \frac{(r_2^2 - r_1^2)x^2}{h^2}dx = \frac{\pi(r_2^2 - r_1^2)}{h^2}\left[\frac{h^3}{3}\right] = \frac{\pi(r_2^2 - r_1^2)h}{3}.$$

(ii) Let D be the region in the first quadrant above the curve $y = x^2$ and below the curve $y = 2 - x^2$ recovered around $y-$axis



Note that to apply the washer method, we have to split the solid in two pieces: for $0 \le y \le 1$ and $1 \le y \le 2$. Thus, the required volume is given by

$$V = \pi\int_0^1 \left(\sqrt{y}\right)^2 dy + \pi\int_1^2 \left(\sqrt{2-y}\right)^2 dy = \pi\left(\frac{1}{2} + \frac{1}{2}\right) = \pi.$$

[Visualizations of volume by washer method]

PRACTICE EXERCISES

1. Find the volume of the solid when the region enclosed by given curves is revolved about $x-$axis .

(i) $y = x^2 + \frac{1}{2}, y = x, x = 2$

(ii) $y = \sqrt{\cos x}, x = \frac{\pi}{4}, x = \frac{\pi}{2}, y = 0.$

(iii)

$y=\sqrt{x}$, $y=1$, $x=4$ and $y=1$.

2. Find the volume of the solid when the region enclosed by the given curves is revolved about $y-$axis

(i) $y=x^2$; $y=2x$ in the first quadrant.

(ii) $x=\sqrt{2\sin 2y}, 0\le y\le \frac{\pi}{2}; x=0.$

(iii) $y=x^2+1, y=0, x=0, x=1$

3. A solid spherical ball of radius R is cut into two pieces by a plane at a distance h from its center. Find the volume of the two pieces.

4. The disk
$x^2+(y-b)^2\le a^2$, where $a<b$,
is revolved about the $x-$axis to generate a solid torus. Find the volume of this solid torus by the Washer Method.

5. A solid is generated by rotating the region under the graph of a continuous function,
$y=f(x), 0\le x\le a$, where $f(x)\ge 0$
about the $x-$axis. If its volume, for any given a, is equal to a^2+a, find $f(x), 0\le x\le a$

6. Find the volume of the solid generated by revolving the region bounded by the curves

$y=3-x^2$ and $y=-1$
about the line $y=-1$ by the washer method.

7. A round hole of radius $\sqrt{3}$ centimeters is bored through the center of a solid ball of radius 2 centimeters. Find
the volume cut out.

Recap

In this section you have learnt the following

How to find the volume of a solid of revolution by the washer method.

Applications of Integration - II

Lecture 24 : Volume of solids of revolution by Shell method [Section 24.2]

Objectives

In this section you will learn the following :

How to find the volume of a solid of revolution by the shell method.

24.2 Volume of solid of revolution by Shell Method :

In this section we describe another method of finding the volume of solids of revolution.

24.2.1 Definition:

Let D be the plane region between the two curves given by functions

$$f_1, f_2 : [a,b] \to \mathbb{R}$$

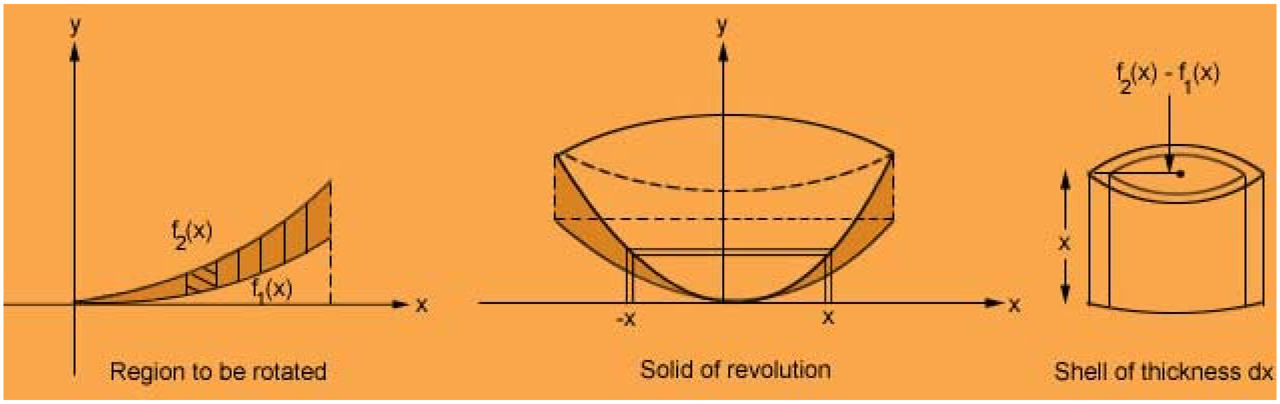
which are Riemann integrable and

$$f_1(x) \le f_2(x) \quad \text{for all } x \in [a,b]$$

Suppose that $a \ge 0$, so that D lies on or to the right of the y-axis. Then the volume of the solid generated by revolving D about the y-axis is defined to be

$$V = 2\pi \int_a^b x\left[f_2(x) - f_1(x)\right] dx \cdot$$ ----------

(15)



24.2.2 Note:

(i) This formula is motivated by the fact that if a thin vertical slice of thickness dx of the region D at a point x is

revolved about the y-axis, then we get a cylindrical shell of radius x and height $\left[f_2(x) - f_1(x)\right]$.

Thus, the volume of this thin shell is given by $2\pi x\left[f_2(x) - f_1(x) dx\right]$

Hence, the required volume is given by equation (15).

(ii) A notable difference between the Washer Method and the Shell Method is that in the former we take slices

perpendicular to the axis of revolution while in the latter we take slices parallel to the axis of revolution.

(iii) Let D be the region between the two curves given by Riemann integrable functions

$$x = g_1(y),\ x = g_2(y),\ y \in [c, d],\ \text{ where } g_1(y) \le g_2(y) \text{ for all } y \in [c, d]$$

Suppose $c \ge 0$ and the region D is revolved about the x-axis, then analogous to equation (15), the volume of this solid of revolution is given by

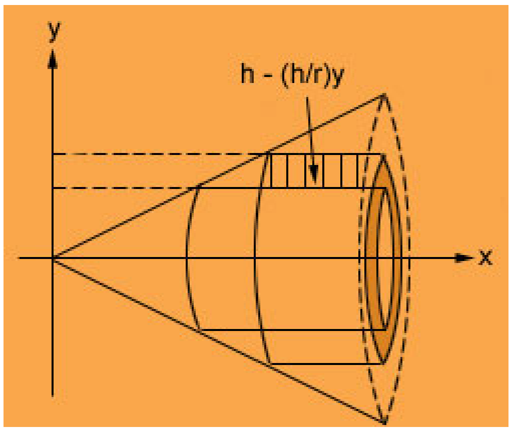
$$V = 2\pi \int_c^d y\left[g_2(y) - g_1(y)\right] dy \cdot$$

24.2.3Examples

(i) A solid cone of height h and radius r is obtained by revolving about the x-axis the triangular region bounded

by the lines

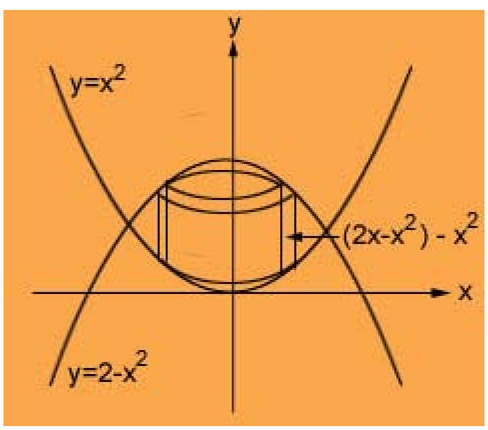
$$y = 0,\ x = h \text{ and } y = rx/h$$



Thus, the Shell Method its volume is given by

$$V = 2\pi \int_0^r y\left[h - \frac{h}{r}y\right] dy = 2\pi\left[\frac{hr^2}{2} - \frac{h}{r} \times \frac{r^3}{3}\right] = \frac{1}{2}\pi r^2 h$$

(ii) Let D be the region in the first quadrant above the curve $y = x^2$ and below the curve $y = 2 - x^2$.

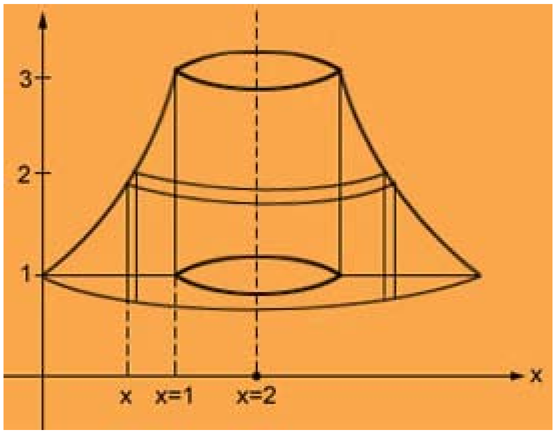


Then, the volume of the solid obtained by revolving D about the y-axis, by the Shell Method, is given by

$$V = 2\pi \int_0^1 x\left[\left(2 - x^2\right) - x^2\right] dx = 4\pi \int_0^1 \left(x - x^3\right) dx = \pi$$

(iii) Consider the solid obtained revolving the region bounded by the functions

$$y = x^2 + x + 1,\ y = 1 \text{ and } x = 1,$$



about the line $x=2$. To find its volume by shell method we consider a thin strip of thickness dx at a point x, $0 \le x \le 1$. This strip has height $x^2+x+1-1$ and is at a distance $2-x$ from the axis. Thus, the volume of the thin shell obtained by revolving this strip about $x=2$ is

$$2\pi(2-x)\left(x^2+x\right)dx$$

Hence, the required volume is

$$V = 2\pi \int_0^1 (2-x)\left(x^2+x\right)dx$$

$$= 2\pi \int_0^1 \left(-x^3+x^2+2x\right)dx$$

$$= \frac{26\pi}{12}.$$

Visualization of Shell method

PRACTICE EXERCISES

1. Find the volume of the solid of revolution by shell method, when the region enclosed by the given curve is revolved about the y-axis:

(i) $y=x$ and $y=x^2$.

(ii) $y=x-x^3$ and $x-$axis, $a \le x \le 1$.

2. Find the volume of the solid of revolution, by shell method, of the region enclosed by the given curves being revolved about x-axis:

(i) $y^2=x, y=1, x=0$.

(ii) $y=4x-x^2, x=0, y=0$.

3. Using shell method, find the volume of the solid obtained by revolving the area bounded by the curves

$y=x^2, y=4x-x^2$
about the line $x=4$.

4. Find the volume of the solid generated when the region bounded by the curves

$y=3-x^2$ and $y=-1$

is revolved about the line $y=-1$, by the shell Method.

5.The disk

$x^2+(y-b)^2 \leq a^2$, where $a<b$,

is revolved about the x-axis to generate a solid torus. Find the volume of this solid torus by the shell Method.

Recap

In this section you have learnt the following

- How to find the volume of a solid of revolution by the shell method.