Module 4 : Local / Global Maximum / Minimum and Curve Sketching

Lecture 12 : Asymptoes [Section 12.1]

Objectives

In this section you will learn the following :

- The notion of asymtotes for a function.
- How to sketch the graph a function.

12.1 Asymtotes

In this section we analyze the behavior of a function $f(x)$ when $x \to +\infty / -\infty$ and the situations when $f(x) \to +\infty / -\infty$. We first look at an example.

12.1.1 Examples:

Consider the function

$$f(x) := \frac{1}{x}, x \in (0,1)$$

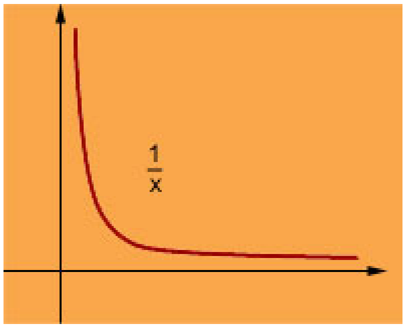
Since, $f(x) := 1/x > 0$ for every $x > 0$, the graph of the function lies above the x-axis.
Further,

$$f'(x) := \frac{-1}{x^2} < 0, \text{ and } f''(x) = \frac{2}{x^3} > 0, \text{ for all } x > 0.$$

Thus, $f(x)$ is a strictly decreasing, concave up function. Thus, it is natural to ask, what does the graph look like for all large values of x, and for all small values of x? We note that

$$\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0 \text{ and } \lim_{x\to 0}\left(\frac{1}{x}\right) = \infty .$$

We can interpret these as follows: as x increases to $+\infty$, $f(x)$ comes closer and closer to x-axis, the line $y = 0$, and as x decreases to 0, $f(x)$ comes closer and closer to y-axis, the line $x = 0$. Thus the graph of $f(x)$ is as follows:



This motivates our next definition.

12.1.2 Definitions:

Let $A \subset \mathbb{R}$ and $y = f(x)$ be a real valued function defined on A.

(i) A line $y = b$ is called a horizontal asymptote on the left to $y = f(x)$ if

$$\lim_{x \to -\infty} f(x) = b$$

In other words, the graph of $f(x)$ approaches the line $y = b$ as we keep moving to the left on the x-axis.

(ii) A line $y = b$ is called a horizontal asymptote on the right to $y = f(x)$ if

$$\lim_{x \to \infty} f(x) = b.$$

In other words, the graph of $f(x)$ approaches the line $y = b$ as we keep moving to the right on the x-axis.

(iii) A line $x = a$ is called a vertical asymptote from left at $x = a$ to $f(x)$ if

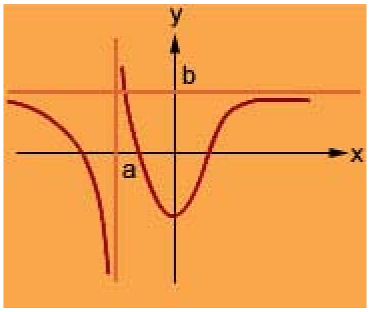
$$\lim_{x \to a^-} f(x) = \pm\infty$$

i.e., the function $f(x)$ approaches the vertical line $x = a$ as x approaches the point a from the left on the
x-axis.

(iv) A line $x = a$ is called a vertical asymptote from right at $x = a$ to $f(x)$ if

$$\lim_{x \to a^+} f(x) = \pm\infty,$$

i.e., the function $f(x)$ approaches the vertical line $x = a$ as x approaches the point a the right on the x-axis.



12.1.3 Examples:

Consider the function

$$f(x) = \frac{x}{x-1} \text{ for } x \neq 1.$$

Since,

$$\lim_{x \to 1^+} f(x) = \infty \text{ and } \lim_{x \to 1^-} f(x) = -\infty,$$

$x = 1$ is a vertical asymptote to $f(x)$ at $x = 1$, both from the left as well as from the right.

Also, since

$$\lim_{x \to \infty} \left(f(x) - 1\right) = \frac{1}{x-1} = 0 \text{ and } \lim_{x \to -\infty} \left(f(x) - 1\right) = \frac{1}{x-1} = 0,$$

the line $y = 1$ is a horizontal asymptote both from the left as well as from the right. Further

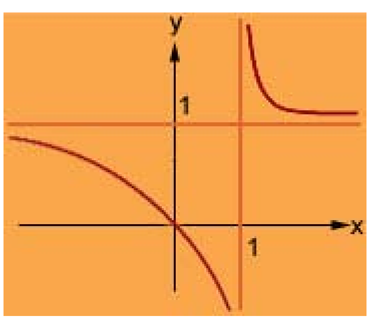
$$f'(x) = \frac{-1}{(x-1)^2} < 0 \text{ for all } x \neq 1.$$

Thus, $f(x)$ is strictly decreasing in $(-\infty, 1)$ and in $(1, \infty)$. Note that, it is not defined at $x = 1$.

Next,

$$f''(x) = \frac{2(x-1)}{(x-1)^4}, \text{ for all } x \neq 1.$$

Since, $f''(x) < 0$ for all $x < 1$, and $f''(x) > 0$ for all $x > 1$, $f(x)$ is strictly concave downward in $(-\infty, 1)$ and is strictly concave upward in $(1, \infty)$. Hence, $x = 1$ is a point of inflection and the graph of f is as follows:

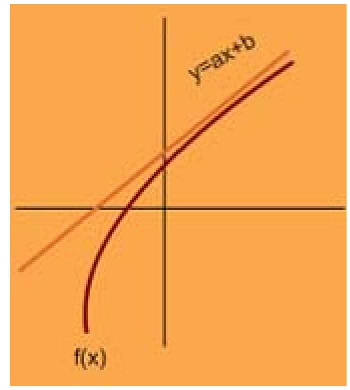


We state next an extension of the idea of a horizontal asymptote.

12.1.4 Definition (Oblique Asymptote) :

(i) A line is called an oblique asymptote from left to if

$y = ax+b$ $y = f(x)$



$$\lim_{x\to+\infty}\left[f(x)-(ax+b)\right]=0,$$

i.e., the graph $f(x)$ approaches the line $y=(ax+b)$ as x approaches $+\infty$.

(ii) A line $y = ax+b$ is called an oblique asymptote from right to $y = f(x)$ if

$$\lim_{x\to-\infty}\left[f(x)-(ax+b)\right]=0,$$

i.e., the graph $f(x)$ approaches the line $y=(ax+b)$ as x approaches $-\infty$.

Note that if $a=0$, then the oblique asymptote is in fact the horizontal asymptote.

12.1.5 **Example:**

Consider the function

$$f(x)=\frac{x^2-1}{2x+4},\quad x\neq-2.$$

We can write

$$y := f(x)=\left(\frac{x}{2}-1\right)+\frac{3}{2x+4},$$

and we have,

$$\lim_{x\to\pm\infty}\left[y-\left(\frac{x}{2}-1\right)\right]=0.$$

Hence, the line

$y=\frac{x}{2}-1$ is an oblique asymptote to $f(x)$

both from the left as well as from the right. Moreover,

$$\lim_{x\to-2}\left(\frac{x^2-1}{2x+4}\right)=0,$$

and therefore, $x=-2$ is a vertical asymptote, both from left as well as from right. Since,

$$f'(x)=\frac{1}{2}-\frac{6}{(2x+4)^2} \text{ for all } x\neq-2,$$

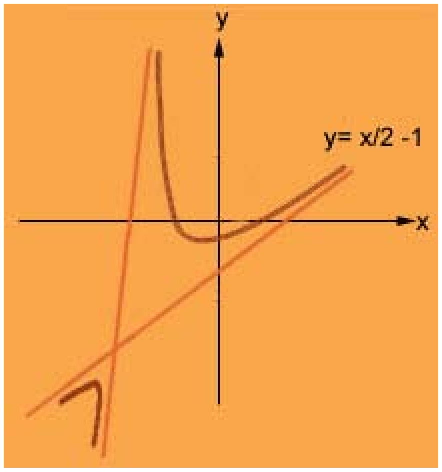
the critical points of f are given by

$$f'(x)=0 \text{ i.e., } 12=(2x+4)^2 \text{ i.e., } x=-2\pm\sqrt{3}.$$

Next,

$$f''(x)=\frac{24(2x+4)}{(2x+4)^4} \text{ for all } x\neq-2.$$

Thus, for $x<-2,\ f''(x)>0$, and hence $f(x)$ is concave up for $x>-2$. For $x<-2,\ f''(x)<0$, and hence $y=f(x)$ is concave down for $x<-2$. Thus, $x=2$ is a point of inflection. Its is easy to check that f has a local maximum at $x=-2-\sqrt{3}$ and local minimum at $x=-2+\sqrt{3}$.



12.1.6 Remark:

Consider a rational function

$$f(x)=\frac{p(x)}{q(x)},\ x \text{ such that } q(x)\neq 0,$$

where p and q are polynomial functions with $q \neq 0$ and

$$\deg\left(p(x)\right)=\deg\left(q(x)\right)+1.$$

Then, dividing $p(x)$ by $q(x)$, we can write

$$f(x)=(ax+b)+\frac{r(x)}{q(x)},$$

where $r(x)$ is a polynomial with $\deg(r)<\deg(q)$. Since,

$$\frac{r(x)}{q(x)}\to 0 \text{ as } x\to\infty \text{ or as } x\to-\infty,$$

we get

$$\lim_{x\to\pm\infty}\left[f(x)-(ax+b)\right]=0,$$

i.e., the line $y=ax+b$ will an asymptote, both from left and right, to the curve $y=f(x)$. Note that, if $\deg p \leq \deg q$, then $a=0$, and we obtain a horizontal asymptote.

Click here to see interactive visualization : Applet 12.1

Interactive Review Quiz on Derivative Tests

PRACTICE EXCERCES

(1) Determine the horizontal/ vertical/ oblique asymptotes to the following:

(i) $f(x)=\dfrac{2x^2}{1-x}$.

(ii) $f(x)=\dfrac{x^3-3x^2}{x^2-1}$.

(iii) $f(x)=\dfrac{(x+2)(2-3x)}{(2x+3)^2}$.

(iv) $f(x)=\dfrac{x^3-1}{x^2-x-2}$.

(v) $f(x)=\dfrac{x^2-2x}{x^3+1}$.

ANSWERS

(2) A function $f(x)$ is said to approach asymptotically to a polynomial $q(x)$ if

$$\lim_{x\to+\infty}\left[f(x)-q(x)\right]=0 \text{ or } \lim_{x\to-\infty}\left[f(x)-q(x)\right]=0$$

Prove the following:

(i) $f(x)=\dfrac{x^3+1}{x}$ asymptotically approaches the polynomial $y=x^2$.

(ii) $f(x)=\dfrac{2+3x-x^3}{x}$ asymptotically approaches the polynomial $3-x^2$.

(3) Let $f(x)=\dfrac{p(x)}{q(x)}$ is a rational function such that

$$p(x)=-q(x)r(x)+d(x)$$

where $dy(d(x))<\deg(r(x))$. Show that $f(x)$ will approach $q(x)$ asymptotically.

Recap

In this section you have learnt the following

- The notion of asymtotes for a function.
- How to sketch the graph a function.

[Section 12.2]

Objectives

In this section you will learn the following :

How to sketch a curve using the various results proved so far.

12.2 Curve sketching

In this section we shall see how the various tools of calculus developed so far help us to draw a graph of the function $y = f(x)$. The aim is to get a visualization of the function from is formula $y = f(x)$. In addition to the various properties that we have analyzed so far some more properties of interest to sketch the graph of a function are the following:

12.2.1 Symmetries:

(i) A function $y = f(x)$ is symmetric with respect to y-axis if $y = f(x) = f(-x)$ for every x in the domain of f.

Such a function is also called an even function. For such a function, one need to draw the graph of $y = f(x)$ only $x \geq 0$, and reflect the graph about y-axis to get it for $x \leq 0$.

(ii) A function $y = f(x)$ is symmetric with respect to origion if $f(-x) = -f(x)$ for every x. Such a function is also

called an odd function . For such a function also, one need to draw the graph for $x \geq 0$ only. For $x \leq 0$, its graph is obtained by reflecting against both x-axis and then y-axis.

12.2.2 Examples:

(i) The function $f(x) = x^2$ is symmetric about y-axis as $f(-x) = (-x)^2 = x^2 \; \forall \, x$.

(ii) The function $f(x) = 2x^3 - x$ is symmetric about origion as

$$\begin{aligned} f(-x) &= 2(-x)^3 - (-x) \\ &= -2x^3 + x \\ &= -(2x^3 - x) \\ &= -f(x) \end{aligned}$$

12.2.3 Intercepts:

(i) For a function $y = f(x)$, the values x such that $f(x) = 0$ are called the zeros , or the x- intercepts of

.

f

This means that the points $(x, 0)$ lie on the graph of f.

(ii) Similarly, for a function $y = f(x)$, the point y such that $y = f(0)$ is called the y-intercept for the function $y = f(x)$. This means that the point $(0, f(0))$ is on the graph of f.

Note that while a function $y = f(x)$ can have more than one x-intercepts, it can have only one y-intercept.

12.2.4 Example:

For the function $y = 2x^3 - x$, the x-intercepts are given by

$$2x^3 - x = 0,$$

i.e., $x(2x^2 - 1) = 0$

i.e., $x = 0,\ x = \pm\frac{1}{\sqrt{2}}$

The y-intercept is $y = 0$.

12.2.5 Periodicity:

A function $y = f(x)$ is said to be periodic with period p if

$$f(x+p) = f(x), \text{for every } x \text{ in the domain of } f.$$

Periodicity means that, one need to analyze the graph of $f(x)$ for x atmost in an interval of length p. Everywhere else it will be a replica of the same graph in intervals of length p.

Now we can give an algorithm that helps us to sketch graphs of functions.

12.2.6 Algorithm for curve sketching:

- We list the steps one can follow to draw a graph of a function $y = f(x)$.

  Locate x-intercepts and y-intercept.
  Look for symmetry.

- Look for periodicity.
- Analyze continuity.
- Analyze differentiability : existence of f', f''.

- Locate intervals of increase, decrease.
- Find critical points.
- Analyse behaviour of f, f', f'' at critical points to locate points of local extrema.
- Locate regions of convexity, concavity.
- Locate points of inflection.
- Analyse $\lim\limits_{x \to \pm\infty} f(x)$.
- Locate horizontal asymptotes: $\lim\limits_{x \to \pm\infty} f(x) = b$.
- Locate vertical asymptotes: $\lim\limits_{x \to \pm a} f(x) = \pm\infty$.
- Locate oblique asymptotes.

12.2.7 Example:

Let $y = f(x)$ be a function with the following properties:

(i) $y = f(x)$ is defined for all x.

(ii) $f(x) = 0$ for $x = -2$ and $x = -1$.

(iii) $y-$intercept is $+2$, *i.e.* $(0, +2)$ is on the graph.

(iv) $f(x)$ is twice differentiable with

(a) $f(x) > 0$ for $x < -1$ and $x > +1$.

(b) $f(x) < 0$ for $x \in (-1, 1)$.

(c) $f''(-1) < 0$, $f''(+1) > 0$, $f(-1) = 4, f(+1) = 0$.

(d) $f''(x) < 0$ for $x < 0$; $f''(x) > 0$ for $x > 0$.

(v) $\lim\limits_{x \to +\infty} f(x) = +\infty$; $\lim\limits_{x \to -\infty} f(x) = -\infty$.

We want to sketch the graph of $f(x)$ with the given data. The graph is smooth; continuous; passing through points $(-1,4), (0,2), (1,0)$; with local maximum at $x = -1$ and local minimum at $x = 0$. Further, it is strictly concave down in $(-\infty, 0)$ strictly concave up in $(0, \infty)$ with a point of inflection at $x = 0$. It has no asymptotes, $f(x)$ decrease to $-\infty$ as $x \to \infty$ and $f(x)$ increases to $+\infty$ as $x \to +\infty$. Thus a graph of f is given by:

12.2.8 Let us sketch the curve . We make the following observations:

$$f(x) = \frac{x^3}{x^2+1}, x \in \mathbb{R}$$

(i) $f(x)$ is defined for every $x \in \mathbb{R}$.

(ii) $f(x)$ is differentiable for all $x \in \mathbb{R}$.

(iii) For $x = 0, y = f(x) = 0$. Thus the curve passes through $(0,0)$.

(iv) $f(-x) = \frac{(-x)^3}{(-x)^2+1} = -\frac{x^3}{x^2+1} = -f(x), x \in \mathbb{R}$. Thus, f is an odd function.

(v)
$$f'(x) = \frac{\left(x^2+1\right)\left(3x^2\right) - x^3(2x)}{\left(x^2+1\right)^2}$$

$$= \frac{3x^4+3x^2-2x^4}{\left(x^2+1\right)^2}$$

$$= \frac{x^4+3x^2}{\left(x^2+1\right)^2}$$

$$= \frac{x^2(x^2+3)}{\left(x^2+1\right)^2}$$

Since $f'(x) > 0 \ \forall \ x \in \mathbb{R}$, $f(x)$ is a strictly increasing function. Further $f'(x) = 0 \ iff \ x = 0$. Thus $x = 0$ is the only critical point. However, $x = 0$ is neither a local maxima, nor a local minima.

(vi)
$$f''(x) = \frac{\left(x^2+1\right)^2\left(4x^3+6x\right) - \left(x^4+3x^2\right)\left(2\left(x^2+1\right)2x\right)}{\left(x^2+1\right)^4}$$

$$= \frac{\left(x^2+1\right)\left[\left(x^2+1\right)\left(4x^3+6x\right) - \left(x^4+3x^2\right)4x\right]}{\left(x^2+1\right)^4}$$

$$= \frac{2x\left(x^2+1\right)\left[\left(x^2+1\right)\left(2x^2+3\right) - 2\left(x^4+3x^2\right)\right]}{\left(x^2+1\right)^4}$$

$$= \frac{2x\left(x^2+1\right)\left[2x^4+2x^2+3x^2+3-2x^4-6x^2\right]}{\left(x^2+1\right)^4}$$

$$= \frac{2x\left(x^2+1\right)\left(-x^2+3\right)}{\left(x^2+1\right)^4}$$

Thus,

$$f''(x) > 0 \text{ if } -\sqrt{3} < x < 0 \text{ or } x > +\sqrt{3}$$

and

$$f''(x) < 0 \text{ if } x < -\sqrt{3} \text{ or } 0 < x < \sqrt{3}.$$

Thus, we conclude that

$f(x)$ is strictly concave up in intervals $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$,

and

$f(x)$ is strictly concave down in $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$.

The points $x = -\sqrt{3}$, $x = 0$ and $x = \sqrt{3}$ and are points of inflection.

(vii) $$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{x^3}{x^2+1}\right)$$

$$= \lim_{x\to-\infty}\left(x - \frac{x}{x^2+1}\right)$$

$$= \lim_{x\to-\infty}\left(x - \frac{1}{x+\frac{1}{x}}\right)$$

$$= -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} -f(x) = -\lim_{x\to-\infty} f(x) = +\infty.$$

Thus,

$f(x)$ keeps strictly decreasing to $-\infty$ as $x \to -\infty$

and

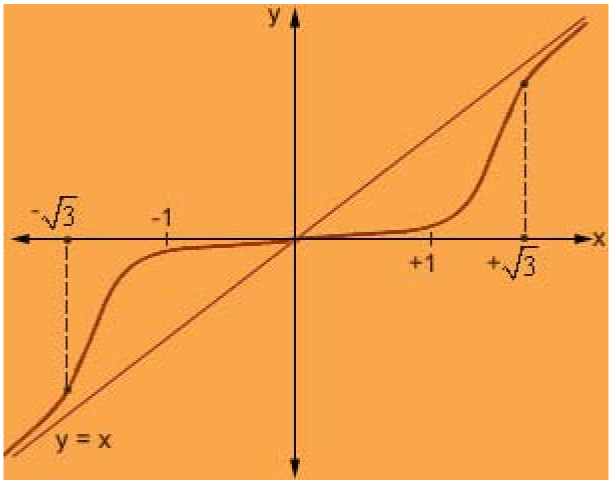
$f(x)$ keeps strictly increasing to $+\infty$ as $x \to +\infty$.

In fact

$$\lim_{x\to\pm\infty}(f(x) - x) = 0.$$

Thus, the line $y = x$ is an asymptote to $f(x)$.

Thus a sketch of $y = f(x)$ is given by



[Derivative review puzzle]

PRACTICE EXERCISES

1. Sketch a continuous curve $y = f(x)$ having the following the given properties.

(i) $f(x)$ is defined for all real x and

$$f(-2) = 8, f(0) = 4, f(2) = 0.$$

Further, $f'(2) = f'(-2) = 0$;

$$f'(x) > 0 \text{ for } |x| > 2 \text{ and } f'(x) < 0 \text{ for } |x| < 2;$$

and

$$f''(x) < 0 \text{ for } x < 0 \text{ and } f''(x) > 0 \text{ for } x > 0.$$

(ii) $f(x)$ is defined for all real x, with $f(0) = 0$, $f(4) = 0$; $f'(x)$ exists for all x except at $x = 0$ and $f'(x) = 0$ for $x = 1$ with $f(1) = -3$. Further $f'(x) > 0$ for $x > 1$, $f'(x) < 0$ for $x < 1$. The second derivative $f''(x)$ is such that $f''(x) < 0$ for $x < 0$, $f''(x) > 0$ for $x > 0$.

2. Sketch the following curves after locating intervals of increase/ decrease, intervals of concavity upward/ downward,

points of local maxima/ minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

(i) $y = 1 + 12|x| - 3x^2,\ x \in [-2, 5]$.

(ii) $y = 2x^3 + 2x^2 - 2x - 1$.

(iii) $y = \dfrac{x^2}{x^2+1}$.

(iv) $y = 3x^{\frac{2}{3}} - 2x$.

(v) $y = |x^2 - 6x + 5|$.

(vi) $y = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$.

3. Find constants a, b, c, d such that the function

$$f(x) = ax^3 + bx^2 + cx + d,\ x \in \mathbb{R},$$

has a local maximum at $x = -1$, a point of inflection at $x = 1$, and satisfy $f(-1) = 10,\ f(1) = -6$.

4. Given

$$f'(x) = 6(x-1)(x-2)^2(x-3)^3(x-4)^4 \text{ for all } x \in \mathbb{R}$$

find all $c \in \mathbb{R}$ at which f has a local maximum, a local minimum or a point of inflection.

5. Give an example of function $f : (0,1) \to \mathbb{R}$ having the following properties

(i) f is strictly increasing and concave upward.

(ii) f is strictly increasing and concave downward.

(iii) f is strictly decreasing and concave upward.

(iv) f is strictly decreasing and concave downward.

(v) f is strictly increasing and has a point of inflection at $x = \frac{1}{2}$.

6. For

$$f(x) = ax^2 + bx + c,\ a \neq 0,$$

analyse the graph by describing the region of increase, decrease, local extrema, convexity and behaviour as $x \to \pm\infty$.

Recap

In this section you have learnt the following

- How to sketch a curve using the various results proved so far.