Objectives

In this section you will learn the following :

- How to find the area of the surfaces generated by revolving a plane curve.

23.1 Area of a Surface of Revolution

23.1.1 Definition:

Consider a curve C given by the graph of a function

$$f:[a,b]\to\mathbb{R},\ a\leq x\leq b\cdot$$

Let S be the surface generated by revolving about this curve x-axis. The area of this surface is defined to be

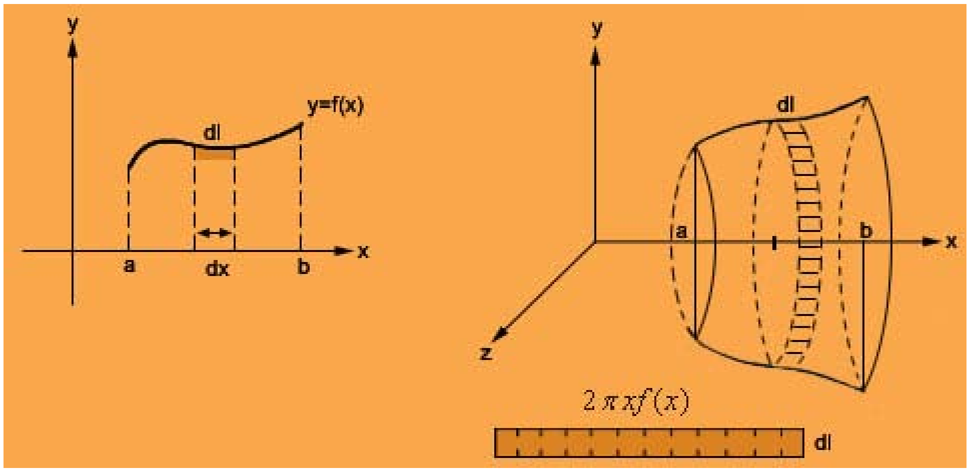
$$\text{Surface area of } S = 2\pi\int_a^b |f(x)|\ \sqrt{1+f'(x)^2}\ dx \qquad \text{----------}$$

(14)

23.1.2 Note:

The reason for defining the surface area as above is the following: consider the surface area generated by a small piece of the curve, say dl, near the point $(x, f(x))$. This surface can be regarded as a cylinder of radius $f(x)$ and of hight dl. Thus, the area of the surface generated by revolving this small piece is given by $2\pi|f(x)|dl$. If the curve C is smooth, i.e., f is differentiable and f' is continuous, then using the arc length formula, we have

$$dl = \sqrt{1+f'(x)^2}\ dx\cdot$$



Thus, it is reasonable to define the total surface area by equation (14).

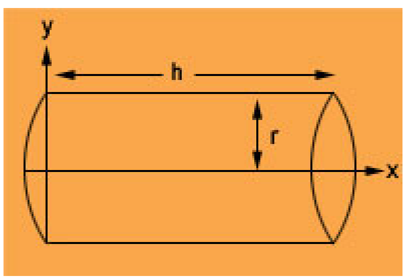
23.1.3 Examples:

(i) A cylinder of radius r and height h can be obtained by revolving the line

$$y = r, \ 0 \le x \le h,$$

around the x-axis. Hence, the surface area of this cylinder is given by

$$2\pi \int_0^h |r| \sqrt{1+0^2}\, dx = 2\pi r h \cdot$$

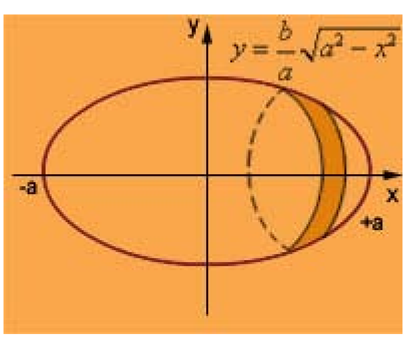


(ii) Consider the surface obtained formed by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the x-axis $(a > b)$. This surface is called *ellipsoid*. To find its surface area, it suffices to consider the portion of the ellipse where $y \ge 0$. Thus, the the curve being revolved is given by

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \ -a \le x \le a$$



Hence, the surface area of the ellipsoid is equal to

$$2\pi \int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2}\left[\sqrt{1+\frac{b^2x^2}{a^2\left(a^2-x^2\right)}}\right] dx$$

$$= \frac{2\pi b}{a}\int_{-a}^{a}\sqrt{a^2-c^2x^2}\;dx$$

$$= 2\pi ab\left(\sqrt{1-c^2}+\frac{\sin^{-1}c}{c}\right),$$

where

$$c=\frac{\sqrt{a^2-b^2}}{a}$$

denotes the eccentricity of the ellipse.

23.1.4 Definition :

Consider a smooth curve C given in the parametric form by

$$x = x(t),\; y = y(t), \alpha \le t \le \beta$$

Suppose the curve C lies on one side of x-axis, and is revolved about x-axis. Then, the surface area S of the surface of revolution is defined to be

$$S = 2\pi\int_{\alpha}^{\beta} y(t)\sqrt{x'(t)^2+y'(t)^2}\;dt$$

Similarly, if C lies on one side of y-axis, and is revolved about y-axis, then the surface area S of the surface of revolution is defined to be

$$S = 2\pi\int_{\alpha}^{\beta} x(t)\sqrt{x'(t)^2+y'(t)^2}\;dt\,.$$

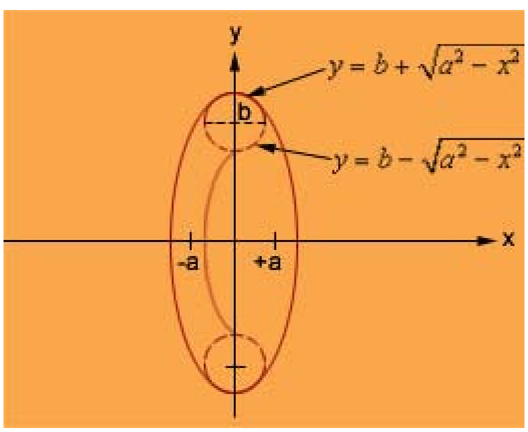
23.1.5 Example

Consider the the surface generated by revolving the circle

$$x^2+(y-b)^2=a^2, (0<a<b),$$

about x-axis. To find its surface area, we can use the parametric equations for the circle:

$$x=a\cos t, y=b+a\sin t,\; 0\le t\le 2\pi.$$



Thus, the surface area of the torus is given by

$$\text{Surface area of torus} = 2\pi \int_0^{2\pi} (b + a\sin t)\sqrt{(-a\sin t)^2 + (a\cos t)^2}\ dt$$
$$= 2\pi a \int_0^{2\pi} (b + a\sin t)\,dt$$
$$= 4\pi^2 ab.$$

Visualization of some surfaces of revolution

PRACTICE EXERCISES

1. For the following curves, find the area of the surface generated by revolving the curve about the specified axis:

(i) $y = \frac{1}{2}\left(x^2 + 1\right),\ 0 \le x \le 1$

about y-axis.

(ii) $y = \frac{x^3}{3} + \frac{1}{4x},\ 1 \le x \le 3$

about the line $y = -1$.

(iii) $x = \frac{y^4}{4} + \frac{1}{8y^2},\ 1 \le y \le 2$

about y-axis.

2. Show that for a hollow spherical object of radius R, the surface area of a slice of thickness h depends only

upon R and h, and not on the position from where the slice is cut.

3. Find the area of the surface generated by revolving the parametric curve

$$x(t) = t^2, y(t) = 2t,\ 0 \le y \le 4.$$

4. The curve

$$x(\theta) = a(1 - \sin\theta)\ \ y = a(1 - \cos\theta),\ 0 \le \theta \le 2\pi,$$

is revolved about x-axis. Fine the area of the surface so generated.

5. Consider a smooth curve C given by the function

$$y = f(x), a \le c \le b \ \text{ with } \ f(x) \ge 0 \text{ for every } x.$$

Let k and K denote the absolute minimum and the absolute maximum, respectively, of f. Let L denote the length of the curve, A denote the area under the curve, and S denote the area of the surface generated by revolving C about x-axis.

(i) Show that

$$2\pi k \le S \le 2\pi K.$$

(ii) Show that

$$2\pi A \le S.$$

(iii) For what f, is $2\pi A \le S$?

Recap

In this section you have learnt the following

How to find the area of the surfaces generated by revolving a plane curve.

Applications of Integration - II

Lecture 23 : Volume of solids by slicing [Section 23.2]

Objectives

In this section you will learn the following :

How to find the volume of a solid by the method of slicing.

23.2 Volume of a Solid by slicing

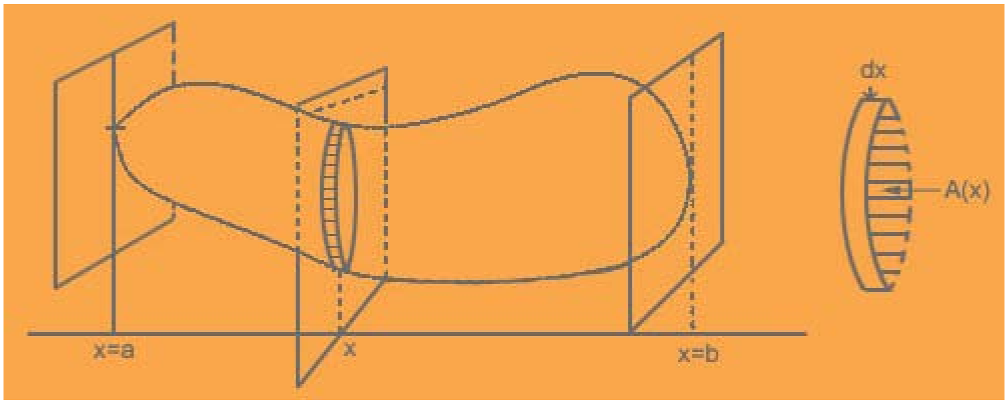
A general definition of the volume of a solid in 3-space is usually given via triple integrals. In this section we give definition of volume in the following cases.

23.2.1 Slice Method:

Consider a solid in space, bounded by the planes $x = a,\ x = b$. Consider a portion of this solid cut by two planes, both perpendicular to x-axis at distances x and $x + \Delta x$ for $a \leq x \leq x + \Delta x$. This will give us a slice of the solid of thickness Δx. Let $A(x)$ be the area of cross-section of this slice by the plane at distance x. Then the volume of the this slice is $A(x)\Delta(x)$. Thus, we can define the volume of the solid to be

$$V = \int_a^b A(x)\ dx,$$

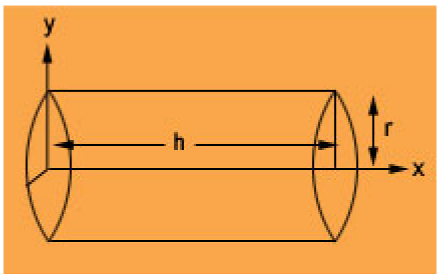
provided $x \mapsto A(x), x \in [a, b]$ is Riemann integrable.



23.2.2 Examples:

(i) Consider a solid right circular cylinder of radius r and height h. This can be written as the set

$$\left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, 0 \le x \le h,\ y^2 + z^2 \le r^2\right\}.$$



Thus, the slice of this solid at every point x is a disk of radius r, with its area $A(x) = \pi r^2$. Hence, the volume of the cylinder is

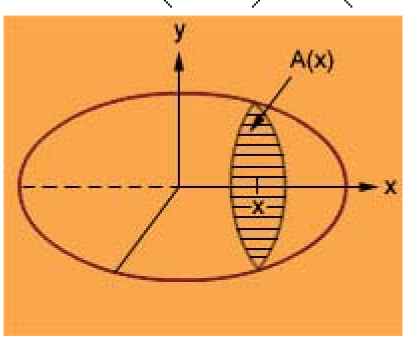
$$\int_0^h \pi r^2\, dx = \pi r^2 h.$$

(ii) Consider the solid enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \cdot$$

For any $-a < x < a$, the area of cross section by a plane perpendicular to x-axis is the area in the yz-plane of the region enclosed by the ellipse

$$\frac{y^2}{b^2\left(1 - \frac{x^2}{a^2}\right)} + \frac{z^2}{c^2\left(1 - \frac{x^2}{a^2}\right)} = 1 \quad .$$



Hence, for $-a \le x \le a$, the area $A(x)$ is given by

$$A(x) = \pi\left(b\sqrt{1-\frac{x^2}{a^2}}\right)\left(c\sqrt{1-\frac{x^2}{a^2}}\right) = \pi bc\left(1-\frac{x^2}{a^2}\right),$$

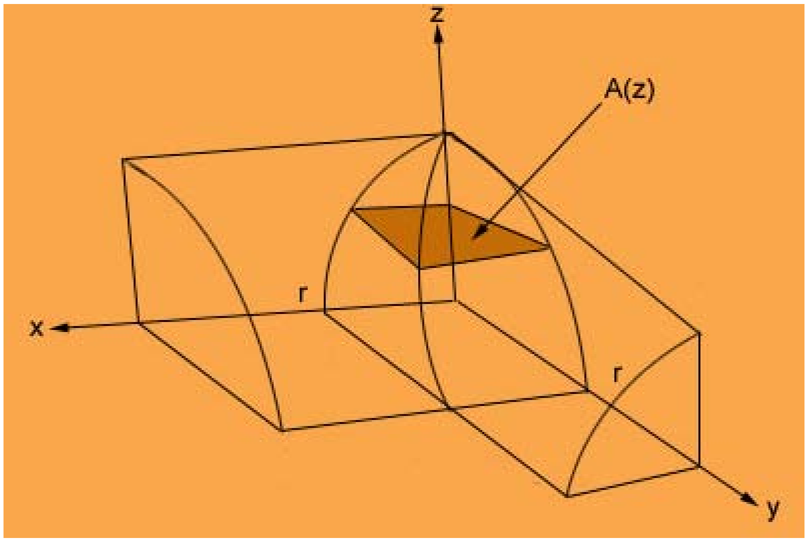
And $A(a) = 0 = A(-a)$. Therefore, the volume of the solid enclosed by the ellipsoid equals

$$V = \pi bc\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)dx = \pi bc\left(2a-\frac{2a^3}{3a^2}\right) = \frac{4}{3}\pi abc.$$

Note that, in particular, if $a = b = c$, the ellipsoid is a sphere and the volume of a spherical ball of radius a is $\frac{4}{3}\pi a^3$.

(iii) Consider two right circular cylinders $x^2 + z^2 = r^2, y^2 + z^2 = r^2$.

We want to find the volume of the solid common to both these cylinders. One eight of the solid is as shown below:



The section of this portion of the solid by planes perpendicular to $z-$axis at a point z is a square of area of cross-section $A(z)$ given by

$$A(z) = (r^2 - z^2).$$

Thus, the required volume is

$$v = 8\int_0^r (r^2 - z^2)\,dz = \frac{16r^3}{3}.$$

PRACTICE EXERCISES

1. Find the following:

(i) Volume of the right pyramid of height h and square base of side a.

(ii) The volume of a curved wedge cut from a cylinder of radius 3 by two planes. One perpendicular to the axis of

the cylinder and second at an angle of $\frac{\pi}{4}$ to the center of the cylinder.

2. The base of a certain solid is the disk $x^2 + y^2 \leq 1$. Find the volume of the solid if each section of the solid cut

out by a plane perpendicular to the $x-$axis is

:

(i) an isosceles right angled triangle with one leg in the base of the solid.
(ii) an equilateral triangle.
(iii) a semi-circle.

3. The cross sections of a certain solid by planes perpendicular to the $x-$axis are circles with diameters extending from the curve $y=x^2$ to the curve $y=8-x^2$. The solid lies between the points of intersection of these two curves. Find its volume.

4. A twisted solid is generated as follows. A fixed line l in $3-$space and a square of side s in a perpendicular to l are given. One vertex of the square is on l. As this vertex moves a distance h along l, the square turns through a full revolution with l as the axis. Find the volume of the solid generated by this motion. What would the volume be if the square had turned through two full revolutions in moving the same distance along the line l?

Recap

In this section you have learnt the following

How to find the volume of a solid by the method of slicing.

Applications of Integration - II

Lecture 23 : Volume of solids by slicing [Section 23.2]

Objectives

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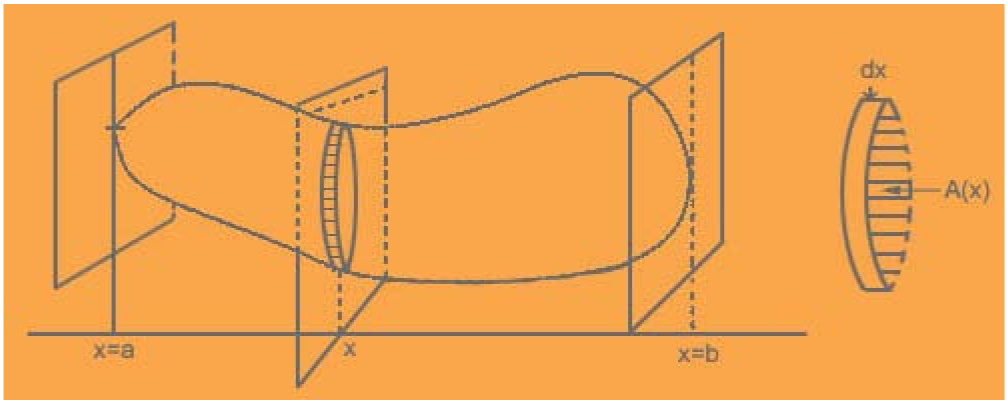

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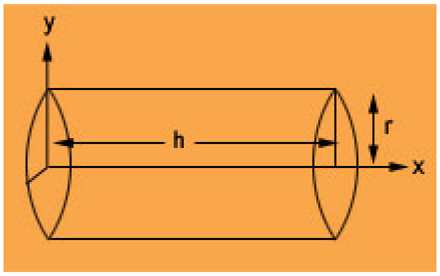

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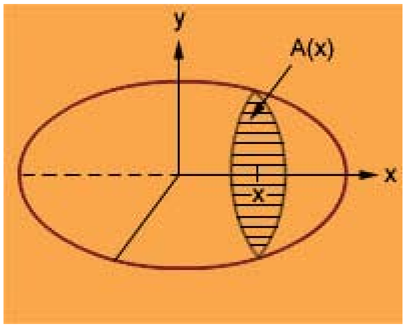

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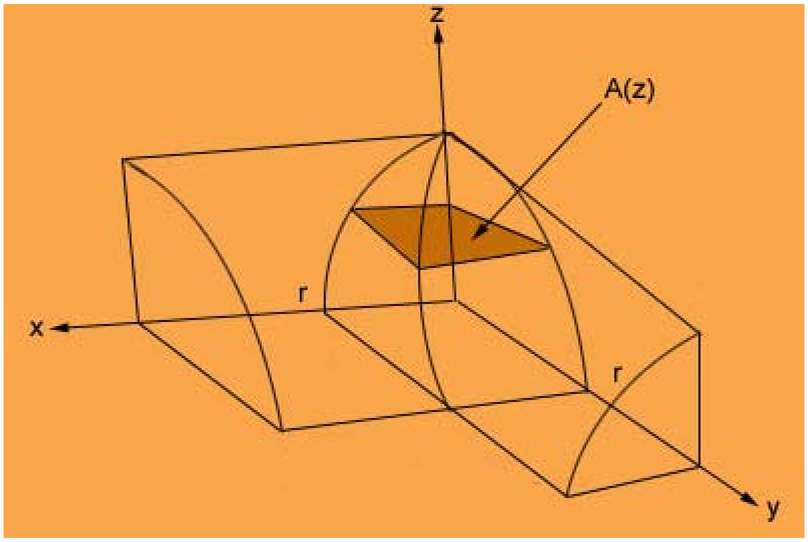

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