Module 18 : Stokes's theorem and applications

Lecture 53 : Stokes' theorem for general domains [Section 53.1]

Objectives

In this section you will learn the following :

- How to verify the conclusion of Stokes' theorem for given vector fields and surfaces.

53.1 Verification of Stokes' theorem

To verify the conclusion of Stokes' theorem for a given vector field $\mathbf{F}$ and a surface S, one has to compute the surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,ds \qquad \text{----------(88)}$$

for a suitable choice of $\mathbf{n}$, and accordingly decide the positive orientation on the boundary curve $\partial(S) = C$. Finally, compute

$$\int_C (\nabla\times\mathbf{F})\cdot d\mathbf{r} \qquad \text{----------(89)}$$

and check that (88) and (89) are equal.

53.1.1 Example :

Let us verify Stokes' s theorem for

$$\mathbf{F} = x^2\mathbf{i} + 4\,x\,y^3\mathbf{j} + y^2\,x\,\mathbf{k}$$

for the surface S, the rectangular region in the plane $z = y$ with vertices $(0,0,0), (1,0,0), (1,3,3), (0,3,3)$. We have

$$\text{curl}\,(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & 4xy^3 & xy^2 \end{vmatrix} = 2xy\,\mathbf{i} - y^2\,\mathbf{j} + 4\,y^3\,\mathbf{k}.$$

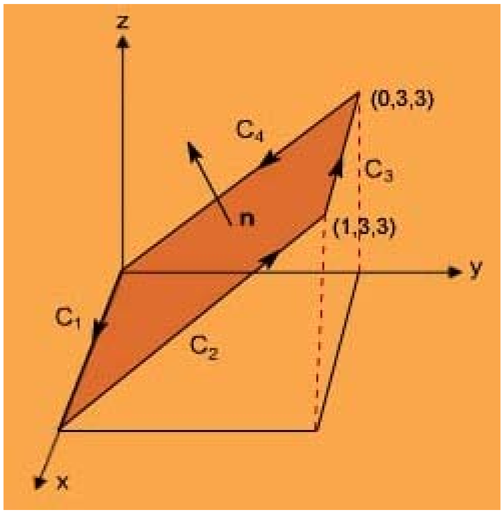

Figure: Surface S

The surface is $z = f(x,y) = y$, where $(x,y) \in R$,

$$R = \{(x,y) \in R^2 \mid 0 \le x \le 1 \text{ and } 0 \le y \le 3\}.$$

Thus,

$$\iint_S (\ \text{curl}\ \ \mathbf{F} \cdot \mathbf{n})\, dS = \ \iint_R (\ \text{curl}\ \ \mathbf{F}) \cdot \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k} \right) dxdy,$$

where

$$\mathbf{n} = -\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k},$$

is the upward unit normal as the z -component is positive on S . Thus,

$$\iint_S (\ \text{curl}\ \ \mathbf{F} \cdot \mathbf{n})\, ds = \int_{x=0}^{1} \ \int_{y=0}^{3} \left(2xy\,\mathbf{i} - y^2\mathbf{j} + 4y^3\,\mathbf{k}\right) \cdot (-\,\mathbf{j} + k)\, dxdy$$

$$= \int_0^1 \left(\int_0^3 \left(y^2 + 4y^3\right) dy \right) dx$$

$$= \int_0^1 \left[\frac{y^3}{3} + y^4 \right]_0^3 dx$$

$$= 90. \qquad \text{---------(90)}$$

The selected orientation on S induces the anti-clockwise orientation on $C = \partial(S)$, as shown in figure. The boundary C consists of four parts:

$C_1 : x\,\mathbf{i},\ 0 \le x \le 1,$

$C_2 : \mathbf{i} + y\,\mathbf{j} + y\,k,\ 0 \le y \le 3,$

$C_3 : (1-x)\,\mathbf{i} + 3\,\mathbf{j} + 3k,\ 0 \le x \le 1,$

$C_4 :\ (3-y)\,\mathbf{j} + (3-y)\,\mathbf{k},\ 0 \le y \le 3.$

Thus

$$\oint_C \mathbf{F}.d\boldsymbol{r} = \oint_C x^2dx + 4xy^3dy + y^2xdz$$
$$= \sum_{i=1}^{4} \oint_{C_i} x^2dx + 4xy^3dy + y^2xdz\,.$$

Since

$$\oint_{C_1} x^2dx + 4xy^3dy + y^2xdz = \int_0^1 x^2dx = \frac{1}{3},$$

$$\oint_{C_2} x^2dx + 4xy^3dy + y^2xdz = \int_{y=0}^{3} \left(4y^3 + y^2\right)dy = \left[y^4 + \frac{y^3}{3}\right]_0^3 = 90,$$

$$\oint_{C_3} x^2dx + 4xy^3dy + y^2xdz = \int_{x=1}^{0} x^2dx = -\frac{1}{3},$$

$$\oint_{C_4} x^2dx + 4xy^3dy + y^2xdz = \int_{y=3}^{0} 0dy = 0,$$

we have

$$\oint_C \mathbf{F}.d\boldsymbol{r} = 90. \qquad \text{----------(91)}$$

Equations (90) and (91) verify Stokes' theorem,

53.1.2 Example:

Let us verify Stokes' theorem for the following:

$$\mathbf{F} = -\,y\,\mathbf{i} + 2y\,z\,\mathbf{j} + y^2\,\mathbf{k},$$

S to be the surface of the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

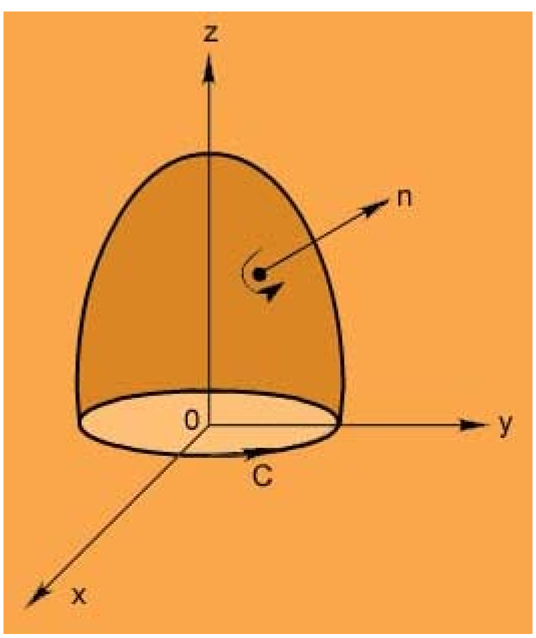

Figure: Surface S with boundary C

We note that

$$\nabla\times\mathbf{F}=\begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & 2yz & y^2 \end{vmatrix}=\mathbf{k}.$$

The upper half of the sphere is given by

$$z=\sqrt{1-x^2-y^2}:=h\,(x,y),\ (x,y)\in D$$

where $D=\left\{(x,y)\mid x^2+y^2<1\right\}$. Let us select the normal to S to be

$$\mathbf{n}=-h_x\mathbf{i}-h_y\mathbf{j}+\mathbf{k}.$$

Then,

$$\begin{aligned}\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS &= \iint_D(\nabla\times\mathbf{F})\cdot\left[-h_x\,\mathbf{i}-h_y\,\mathbf{j}+\mathbf{k}\right]dA\\ &= \iint_D dA\\ &=\pi.\end{aligned}$$

For the selected normal on S, the orientation on $\partial S=C$, the circle $x^2+y^2=1$ is the anticlockwise orientation. Since, as $z=0, x=\cos t, y=\sin t, 0\le t\le 2\pi,$ on C, we have

$$\begin{aligned}\oint_C\mathbf{F}.d\mathbf{r} &=\oint_C\left(-yi+2yz\mathbf{j}+y^2\mathbf{k}\right).\left(idx+\mathbf{j}dy+\mathbf{k}dz\right)\\ &=\oint_C -y\,dx\\ &=\int_0^{2\pi}\left(-\sin t\right)\left(-\sin t\right)dt\\ &=\int_0^{2\pi}\sin^2 t\ dt\\ &=\pi.\end{aligned}$$

Practice Exercises :

1. Verify Stokes' theorem for the vector field

   $$\mathbf{F}=2z\mathbf{i}+3x\mathbf{j}+5y\mathbf{k}$$
   where S is the surface of the paraboloid $z=4-x^2-y^2$ for which $z\ge 0$.

   Answer: 12π

2. Verify Stokes' theorem for

   $$\mathbf{F}=(x-y)\,\mathbf{i}+(y-z)\,\mathbf{j}+(z-x)\,\mathbf{k},$$
   S being the plane $x+y+z=1$ in the first octant.

   Answer: $\dfrac{3}{2}$

3. Verify Stokes' theorem for

   $\mathbf{F} = (x-y)\,\mathbf{i} + (x-z)\,\mathbf{j} + (z-z)\,\mathbf{k},$
   S being the surface of the cone $z^2 = x^2 + y^2$, intercepted by the sphere
   $x^2 + (y-a)^2 + z^2 = a^2, z \geq 0.$

   Answer: $\dfrac{\pi a^2}{2}$

Recap

In this section you have learnt the following

- How to verify the conclusion of Stokes' theorem for given vector fields and surfaces.

[Section 53.2]

Objectives

In this section you will learn the following :

- How to extend the conclusion of Stokes' theorem for general domains.

53.2 Extending Stokes' theorem

53.2.1 Surface with more than one boundary components:

In section 52.2 we proved Stokes' theorem for piecewise smooth orientable surfaces where boundary consisted of a simple closed piecewise smooth curve. These are the surface which have no holes. For example this does not include surface such as shown in the figure

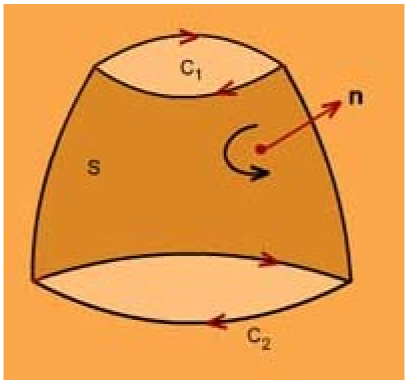

Figure: Orientable domain with multiple boundaries

The boundary of this surface consists of two disjoint carves C_1 and C_2. These surfaces are called surface with holes'. In a way analogue to the extension of the green's theorem, Stokes' theorem can be extended to such surfaces, giving suitable orientation to the various components of the boundary. We give some examples to illustrate this.

53.2.2 Example:

Let S be the surface consisting of the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 0$ and below the plane $z = 4$. The boundary of this surface consists of two components:

$$\partial S = C_1 \cup C_2,$$

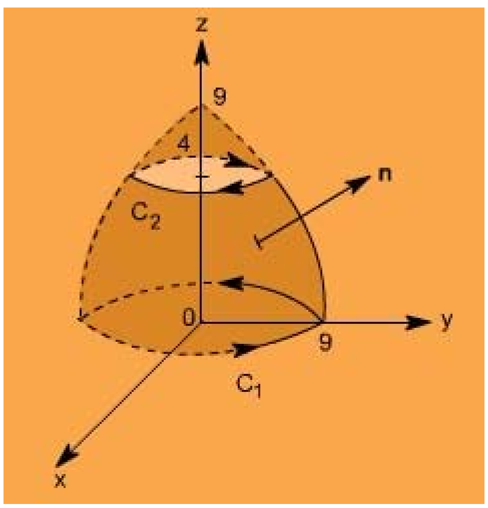

Figure: Part of paraboloid

where C_1 is the circle $x^2 + y^2 = 9$ in the xy-plane and C_2 is the circle $x^2 + y^2 = 4$ in the plane $z = 4$. Let

$$\mathbf{F} = (z - y)\,\mathbf{i} + (z + x)\,\mathbf{j} - (x + y)\,\mathbf{k}.$$

Then

$$\nabla \times \mathbf{F} = -2\,\mathbf{i} + 2\,\mathbf{j} + 2\,\mathbf{k}.$$

For S, a parameterizations is given by

$$r\,(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (4 - x^2 - y^2)\,\mathbf{k}, (x, y) \in D,$$

where

$$D = \left\{(x, y) \,|\, 4 \le x^2 + y^2 \le 9\right\}.$$

If we select the normal to S to be

$\mathbf{r}_x \times \mathbf{r}_y = 3\,x\,\mathbf{i} + 2\,y\,\mathbf{j} + \mathbf{k},$

then

$$\begin{aligned}\iint_S (\nabla F \cdot \mathbf{n})\,ds &= \iint_D \nabla F \cdot (\mathbf{r}_x \times \mathbf{r}_y)\,dxdy \\ &= \iint_D (-4x + 4y + 2)\,dxdy \\ &= \iint_D 2\,dxdy \\ &= (18-8)\,\pi \\ &= 10\pi.\end{aligned}$$

On the other hand,

$$\oint_{\partial S} \mathbf{F} = \oint_{C_1} \mathbf{F}.d\boldsymbol{r} + \oint_{C_2} \mathbf{F}.d\boldsymbol{r}$$

where C_1 gets the anticlockwise orientation and C_2 gets the clockwise orientation. Since

$$\begin{aligned}\oint_{C_1} \mathbf{F}.d\boldsymbol{r} &= \oint_C (x-y)\,dx + (z+x)\,dy \\ &= \oint (zdy - ydx) \\ &= 2 \times \text{ Area of the disk } x^2 + y^2 \le 9, z = 0 \\ &= 18\pi,\end{aligned}$$

and

$$\begin{aligned}\oint_{C_1} \mathbf{F}.d\boldsymbol{r} &= \oint_{C_2} (z-y)\,dx + (z+x)\,dy \\ &= \oint_{C_2} (xdy - ydx) + \oint_{C_2} (4dx + 4dy) \\ &= -8\pi.\end{aligned}$$

Thus

$$\oint_{\partial(S)} \mathbf{F}.d\boldsymbol{r} = 18\pi - 8\pi = 10\pi$$

verifying Stokes' theorem.

Practice Exercises :

1. Consider S, the portion of the cylinder $x^2 + y^2 = 3$ between the planes $x + y + z = 1$ and the xy-plane, the vector field being

   $$\mathbf{F} = -\,y^3\,\mathbf{i} + x^3\,\mathbf{j} - z^3\,\mathbf{k}.$$

   Using extended form of the Stokes' theorem, write

   $$\iint_S (\nabla \times F) \cdot \mathbf{n}\,dS$$

   as a sum of line integrals and compute the value.

   Answer:

$$\oint_{C_1} F.dr + \oint_{C_2} F.d\mathbf{r}$$

where

$$C_1 \text{ is } x^2 + y^2 = 3, z = 0$$

and

$$C_2 \text{ is } x^2 + y^2 = 3, z = 1 - x - y.$$

Recap

In this section you have learnt the following

- How to extend the conclusion of Stokes' theorem for general domains.