Module 14 : Double Integrals, Applilcations to Areas and Volumes Change of variables

Lecture 41 : Triple integrals [Section 41.1]

Objectives

In this section you will learn the following :

- The concept of triple integral.

41 .1 Triple integrals

41.1.1Definition:

For a continuous function $f : D \to \mathbb{R}$ where D is a cubical region

$D = [a,b] \times [c,d] \times [e,f] \subset \mathbb{R}^3$,

the triple integral of F over D, denoted by

$\iiint_D f(x,y,z)\,dV$

can be defined in a manner similar to that of the double integral. We divide the region D into n rectangular cells $C_i, i = 1, 2, \ldots, n,$ where the volume of the i th cell is $\Delta V_i \cdot$ We select any arbitrary point $(x_i, y_i, z_i) \in C_i$ and consider the limit of the (Riemann) sums

$$J_n := \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i,$$

the limit being taken as $n \to \infty$ with $\max\{\Delta V_i \,|\, 1 \le i \le n\} \to 0.$ The triple integral is this limit, whenever it exists. As in the two variables case, this can be extended to closed bounded regions $D \subset \mathbb{R}^3.$ We omit the details. The triple integral is also written as

$\iiint_D f(x,y,z)\,dx\,dy\,dz$.

Triple integral satisfies properties similar to the ones satisfied by double integrals, see theorem 40.1.3.

The evaluation of triple integrals becomes possible because of the three dimensional version of Fubini's theorem.

41.1.2Theorem (Fubini's):

Suppose $D \subset \mathbb{R}^3$ *is closed and bounded, given by*

$$D = (x, y, z) \mid (x, y) \in R \subseteq \mathbb{R}^2,\ g(x, y) \le z \le h(x, y),$$

where $g, h : R \to \mathbb{R}$ *are continuous functions. Let* $f : D \to \mathbb{R}$ *be continuous. Then,*

$$\iiint_D f(x,y,z)\,dV = \iint_R \left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz \right) dx\,dy$$

41 .1.3 **Note:**

Note in theorem 41.1.2, the region is bounded below by the surface $z = g(x, y)$, above by the surface $z = h(x, y)$ and on the side by a cylinder parallel to z-axis.

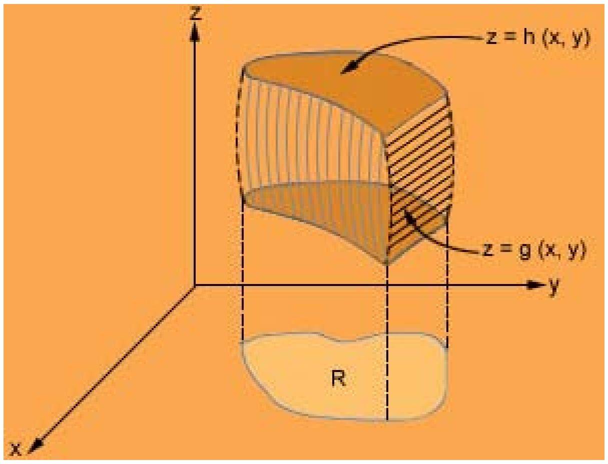

Figure: Region with projection in xy-plane

The region D has the property that it can be projected onto the xy-plane, in this case the set $R \subset \mathbb{R}^2$ is the projection. This theorem reduces the computation of the triple integral to that of an ordinary integral and a double integral. For example, if R itself can be further expressed as a type-I elementary domain as

$$R = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\},$$

where

$$\phi_1, \phi_2 : [a, b] \to \mathbb{R}$$

are continuous functions, then

$$\iiint_D f(x,y,z)\,dV = \int_{x=a}^{x=b}\left(\int_{y=\phi_1(x)}^{y=\phi_2(x)}\left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz\right)dy\right)dx$$

Similarly if $R \subset \mathbb{R}^2$ can be further expressed as a type-II domain as

$$R = \{(x,y)\in\mathbb{R}^2 \mid c \le y \le d, \psi_1(y) \le x \le \psi_2(y)\},$$

where

$$\psi_1, \psi_2 : [c,d] \to \mathbb{R}$$

are continuous functions, then

$$\iiint_D f(x,y,z)\,dV = \int_{y=c}^{y=d}\left(\int_{x=\psi_1(y)}^{x=\psi_2(y)}\left(\int_{g(x,y)}^{h(x,y)} f(x,y,z)\,dz\right)dx\right)dy.$$

These are called the **iterated integrals** with respect to the projection of D onto the xy-plane. In all it may be possible to express a triple integral in six iterated integral, corresponding to projections of D onto the three coordinate planes. Regions $D \subseteq \mathbb{R}^3$ which can be projected onto a coordinate plane are called **elementary regions**

41.1.4Examples:

(i) Let us compute

$$\iiint_D dv,$$

where D is the solid in $\mathbb{R}^3$ bounded by the ellipsoid

$$4x^2 + 4y^2 + z^2 = 16.$$

We can interpret D as

$$D = \left\{(x,y,z) \mid (x,y)\in R, -2\sqrt{4-x^2-y^2} \le z \le 2\sqrt{4-x^2-y^2}\right\},$$

where R is the projection of D onto xy-plane.

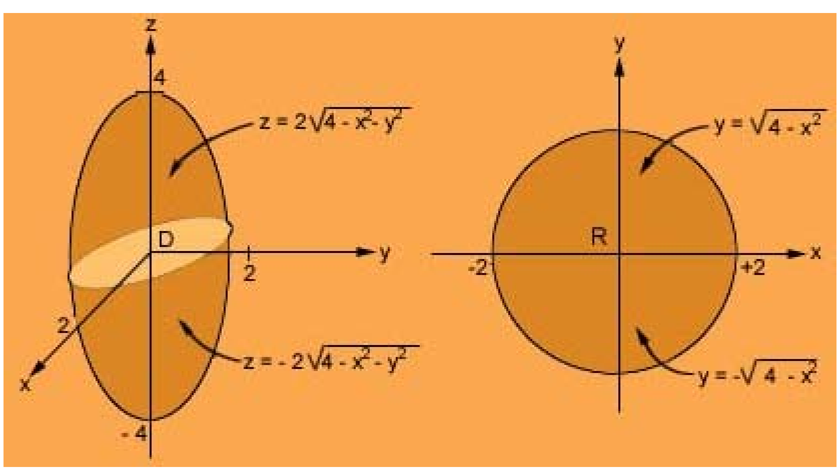

Figure: D and its projection in xy-plane

Since is the region in the xy-plane bounded by the circle $4x^2 + 4y^2 = 16$, we can write

$$D = \left\{(x,y) \in \mathbb{R}^2 \mid -2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}\right\}$$

Thus

$$\begin{aligned}
\iiint_D dv &= \int_{-2}^{+2} \left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(\int_{-2\sqrt{4-x^2-y^2}}^{2\sqrt{4-x^2-y^2}} 1dz \right) dy \right) dx \\
&= 8 \int_0^2 \left(\int_0^{\sqrt{4-x^2}} \left(\int_0^{2\sqrt{4-x^2-y^2}} 1dz \right) dy \right) dx \\
&= 16 \int_0^2 \left(\int_0^{\sqrt{4-x^2}} \sqrt{(4-x^2)-y^2}\, dy \right) dx \\
&= 8 \int_0^2 \left[y\sqrt{4-x^2-y^2} + (4-x^2) \sin^{-1}\left(\frac{y}{\sqrt{4-x^2}} \right) \right]_0^{4-x^2} dx \\
&= 8 \int_0^2 \left[0 + (4-x^2) \sin^{-1}(1) - 0 - 0 \right] dx \\
&= 8 \int_0^2 (4-x^2)\left(\frac{\pi}{2}\right) dx \\
&= 4\pi \left[4x - \frac{x^3}{3} \right]_0^2 \\
&= \frac{64\pi}{3}
\end{aligned}$$

Let us evaluate

$$\iiint_D z dv,$$

where D is the solid wedge in the first actant cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$.

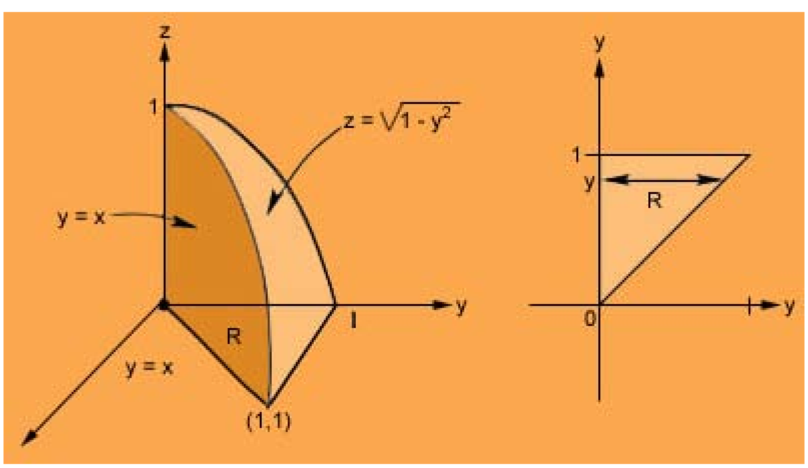

Figure: D and its projection R onto xy-plane

The region D can be written as

$$D=\left\{(x,y,z)\mid(x,y)\in \mathbb{R},0\le z\le\sqrt{1-y^2}\right\},$$

where R is the projection of the wedge on the xy-plane and is given by

$$R=\{(x,y,z)\mid 0\le x\le,x\le y\le 1\}=\{(x,y)\mid 0\le y\le 1,0,\le x\le y\}$$

Hence

$$\begin{aligned}\iiint_D zdv &= \iint_R\left(\int_0^{\sqrt{1-y^2}} zdz\right)d\,(x,y)\\ &= \iint_R \frac{1-y^2}{2}\,d\,(x,y).\end{aligned}$$

For further, computation, if treat R as a type-II region, then

$$\begin{aligned}\iiint_D zdv &= \int_0^1\left(\int_0^y \frac{1-y^2}{2}\,dx\right)dy\\ &= \frac{1}{2}\int_0^1 (y-y^3)\,dy\\ &= \frac{1}{8}.\end{aligned}$$

Practice Exercises

(1) For the following integrals, sketch the region of integration and express the integral in six iterated integrals:

(i) $\iiint_D dv$, where D is the solid in $\mathbb{R}^3$ in the first octant bounded by the coordinate planes, the plane
$y+z=1$ and the vertical plane $x=2$.

(ii) $\iiint_D 1dv$
where D is the solid in $\mathbb{R}^3$ is the paraboloid cylinder bounded on the side by the paraboloid $y=x^2$, top by the plane $x+y=1$ and the bottom by the xy- plane.

Answers

(2) Consider the integral

$$\int_0^{\sqrt{\pi/2}}\left(\int_x^{\sqrt{\pi/2}}\left(\int_1^3 \sin\,(y^2)\,dz\right)dy\right)dx.$$

Sketch the region D over which the function $f(x,y,z)=\sin(y^2)$ is being integrated.
Interchange the order of integration for the variables x by and evaluate the above

integral.

Answer

(3) Evaluate $\iiint_D 1\, dv = \frac{3\pi}{\sqrt{2}}$, where D is the solid bounded by the paraboloid

$z = 5x^2 + 5y^2$ and $z = 6 - 7x^2 - y^2$.

Answer

Recap

In this section you have learnt the following

- The concept of triple integral.

Module 14 : Double Integrals, Applilcations to Areas and Volumes Change of variables

Lecture 41 : Applications of multiple integrals [Section 41.2]

Objectives

In this section you will learn the following :

- How to compute the area of a region in $\mathbb{R}^2$.
- How to compute the volume of a region in $\mathbb{R}^3$.

41 .2 Applications of Multiple integrals

41.2.1Definition :

Let $D \subseteq \mathbb{R}^2$ be a closed bounded region.

(i) If the double integral

$A := \iint_D 1 \, d(x,y)$ exists,

Then A is called the **area of the region** D.

(ii) Let D be a solid in $\mathbb{R}^3$ which has projection R in xy-plane and is bounded above by the surface

and below by -plane. If the double integral

$z = f(x,y)$ xy

$V := \iint_D f(x,y)\, d(x,y)$ exists,

then V is called the volume of the solid D.

41.2.2Examples :

(i) Let us find the area bounded by the graphs of the functions

$f(x) = \sin x$, $g(x) = \cos x$, $\pi/4 \le x \le 5\pi/4$.

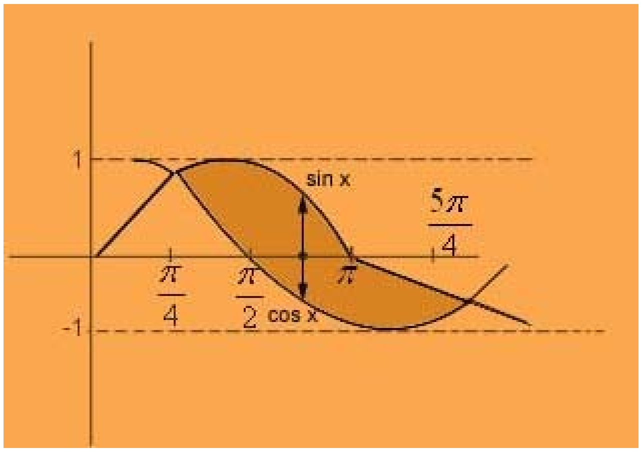

Figure: Area between $\sin x$ **and** $\cos x$

The region D is given by

$$D = \{(x,y) \mid \pi/4 \le x \le 5\pi/4,\ \cos x \le y \le \sin x\}$$

The required area is

$$\iint_D 1\, d(x,y) = \int_{\pi/4}^{5\pi/4}\left(\int_{\cos x}^{\sin x} dy\right)dx$$

$$= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$$

$$= 2\sqrt{2}.$$

(ii) Let us find the volume of the cylinder $x^2 + y^2 = 4$ bounded by the planes $z = 0$ and $z = 4 - y$.

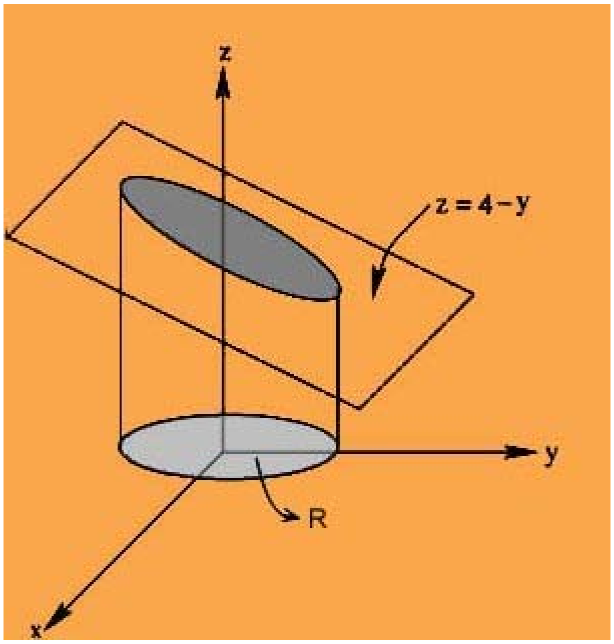

Figure: D and its projection in xy-plane

Since, the solid has projection onto xy-plane to be

$$R=\left\{(x,y)\in \mathbb{R}^2 \mid x^2+y^2=4\right\},$$

the required volume is

$$\begin{aligned}\iint_D (4-y)\, d\,(x,y) &= \int_{-2}^{+2}\left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-y)\,dy\right)dx\\ &= \int_{-2}^{+2}\left[4y-\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx\\ &= \int_{-2}^{+2} 8\sqrt{4-x^2}\,dx\\ &= 16\pi.\end{aligned}$$

41.2.3 **Example:**

Find the volume of the region D enclosed between the two surfaces

$z=x^2+3y^2$ and $z=8-x^2-y^2$.

The two surfaces intersect on the cylinder
$x^2+3y^2=8-x^2-y^2$ or $x^2+2y^2=4$.

The region D projects onto the region R in the xy-plane enclosed by the ellipse having the same equation. Thus,
$D=\{(x,y,z) \mid -2\le x\le 2, -\sqrt{(4-x^2)^2}\le y\le \sqrt{(4-x^2)^2}, x^2+3y^2\le z\le 8-x^2-y^2\}.$

Hence, by Fubini's theorem, the required volume is

$$V = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \left[2(8-2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right] dx$$

$$= \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4-x^2)^{3/2}\, dx$$

$$= 8\pi\sqrt{2}.$$

41.2.4Definition :

For a closed solid $D \subseteq \mathbb{R}^3$, the triple integral

$$\iiint_D 1\, dv$$

defines the volume of the solid.

41.2.5Note :

The notion of volume given by 42.2.1 when D is the region

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 \mid (x,y) \in R,\ g(x,y) \le z \le h(x,y) \right\},$$

for in this case

$$\iiint_D 1\, dv = \iint_R \left(\int_{g(x,y)}^{h(x,y)} 1\, dz \right) d(x,y)$$

$$= \iint_R \left[h(x,y) - g(x,y) \right] d(x,y).$$

41.2.6Example :

(i) Let us find the volume of the solid enclosed by the paraboloid

$z = 5x^2 + 5y^2$ and $z = 6 - 7x^2 - y^2$

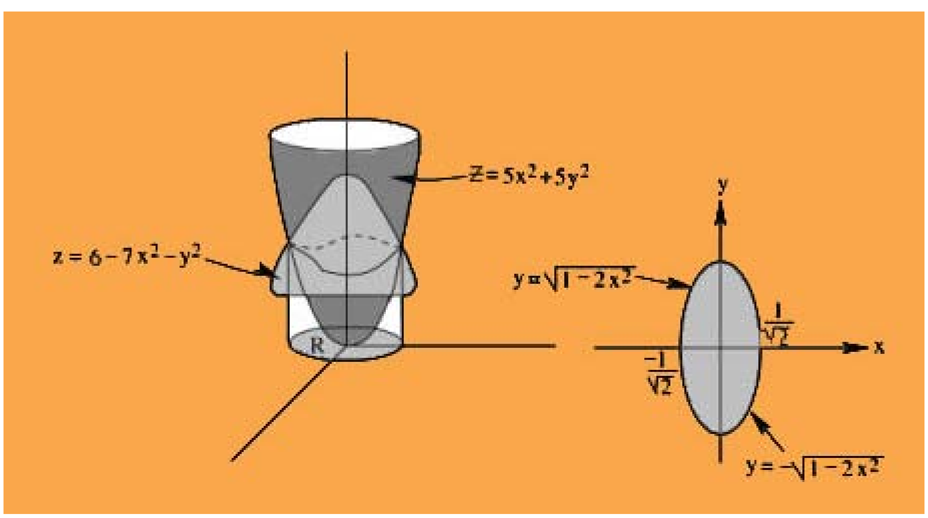

Figure: D and its projection in xy-plane

To find the projection of the enclosed solid D, we first find the curve of intersection of the two surfaces. This is given by

$$5x^2 + 5y^2 = 6 - 7x^2 - y^2,$$

i.e.,

$$2x^2 + y^2 = 1.$$

Thus, the projection of D onto the xy-plane is given by

$$R = \{(x, y) \in \mathbb{R}^2 \mid 2x^2 + y^2 = 1\}$$
$$= \left\{(x, y) \in \mathbb{R}^2 \mid -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}, -\sqrt{1-2x^2} \le y \le \sqrt{1-2x^2}\right\}.$$

Hence, the required volume is

$$V = \iiint_D 1\, dv = \int_{-1/\sqrt{2}}^{+1/\sqrt{2}} \left(\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \left(\int_{5x^2+5y^2}^{6-7x^2-y^2} dz \right) dy \right) dx.$$

This, as computed earlier, the volume is $3\pi/\sqrt{2}$ units.

41.2.7 Note (Some physical applications):

(i) **Average of a function over a region** :

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be integrable over D. Then

$$Avg\ (f) := \frac{\iint_D f(x, y)\, d(x, y)}{\iint_D 1\, d\ (x, y)}.$$

is called the **average** of f over D. Similarly if $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ is integrable, then

$$Avg\ (f) := \frac{\iiint_D f(x, y, z) dv}{\iiint_v 1\, dv},$$

is called the average of f over D.

(ii) **Mass, center of mass of a planer body** :

Let a planer body occupy a region $D \subseteq \mathbb{R}^2$ and have density (mass per unit area) to be $f(x, y)$ at a point (x, y) in the body. Then the mass of the body is given by

$$Mass\ (D) = \iint_D f(x, y)\, d(x, y).$$

Further the point $(\bar{x},\bar{y})$, where

$$\bar{x} = \frac{\iint_D x\, f(x,y)\, d(x,y)}{\iint_D f(x,y)\, d(x,y)}$$

and

$$\bar{y} = \frac{\iint_D y\, f(x,y)\, d(x,y)}{\iint_D f(x,y)\, d(x,y)}$$

is called the **center of mass** of the body. This is the point where the whole mass of the body can be assumed to be concentrated. For $D \subseteq \mathbb{R}^3$, if $f(x,y,z)$ is the density (mass per unit volume) at a point $(x,y,z) \in D$, then the mass of D is defined by

$$M(D) = \iiint_D f(x,y,z)\, dv$$

and the center of mass $(\bar{x},\bar{y},\bar{z})$ is given by

$$\bar{x} = \frac{\iiint_D x\, f(x,y,z)\, dv}{\iiint_D f(x,y,z)\, dv},$$

$$\bar{y} = \frac{\iiint_D y\, f(x,y,z)\, dv}{\iiint_D f(x,y,z)\, dv},$$

$$\bar{z} = \frac{\iiint_D z\, f(x,y,z)\, dv}{\iiint_D f(x,y,z)\, dv}.$$

(iii) **Moments of Inertia :**

Let D be a planer body with density $P(x,y)$ for $(x,y) \in D$. Then

$$I_x := \iint_D y\, f(x,y)\, d(x,y),$$

$$I_y := \iint_D x\, f(x,y)\, d(x,y),$$

are called the moment of inertia of D about x-axis and about the y-axis, respectively. Similarly, for a solid $D \subseteq \mathbb{R}^3$, the scalars

$$I_x := \iiint_D (y^2 + z^2)\, f(x,y,z)\, dv,$$
$$I_y := \iiint_D (x^2 + z^2)\, f(x,y,z)\, dv,$$
and
$$I_z := \iiint_D (x^2 + y^2)\, f(x,y,z)\, dv,$$

are called the **moments of inertia** of D about the x-axis, y-axis, and the z-axis, respectively. The quantities

$$I_{xy} := \iiint_D z^2\, f(x,y,z)\, dv,$$
$$I_{yz} := \iiint_D x^2\, f(x,y,z)\, dv,$$
and
$$I_{zx} := \iiint_D y^2\, f(x,y,z)\, dv,$$

\

are sometimes called the **moments of inertia** of the body about the xy-plane, plans, and zx-plane respectively.

Practice Exercises

(1) Compute the area bounded by the lines $2y + x = 0$, $y + 2x = 0$ and $x + y = 1$.

Answer

(2) Find the volume of the solid whose base is the region in the xy-plane bounded by the parabola $y = 4 - x^2$
and the line $y = 3x$, and whose top is bounded by the plane $z = x + 4$.

Answer

(3) Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ using double integral over a region
in the plane. (Hint: Consider the part in the first octant.)

Answer

(4) Find the total mass of a solid D for which the density $P(x,y,z)$ is given:

(i) $P(x,y,z) = 2(x^2 + y^2)$, D is the cylinder $x^2 + y^2 \le 4, 0 \le z \le 6$.

(ii) $P(x,y,z) = xy$, D is the solid tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$.

Answer

(5) Find the moment of inertia I_x of a solid D with uniform density $P(x,y,z) = 1$ where D is the solid cylinder
$y^2 + z^2 \le a^2$, $0 \le x \le h$.

Answer

(6) Find the centroid of the region hounded above by the sphere

$$x^2 + y^2 + z^2 = 2a^2$$

and below by the paraboloid

$$az = x^2 + y^2 , a > 0$$

Answer

Recap

In this section you have learnt the following

- How to compute the area of a region in $\mathbb{R}^2$.
- How to compute the volume of a region in $\mathbb{R}^3$.