Module 7 : Applications of Integration - I

Lecture 19 : Definition of the natural logarithmic function [Section 9.1]

Objectives

In this section you will learn the following :

- How integration helps us to define the natural logarithm function and derive its properties.

19.1 Definition of the natural logarithmic function :

Recall, that in elementary calculus, one uses the formula

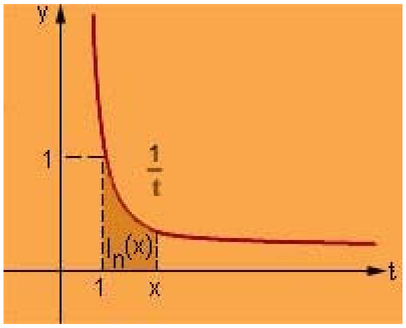
$$\frac{d}{dx}(\ln x) = \frac{1}{x}, x > 0$$,

without really knowing the function $\ln x$. We can use this formula, and the fundamental theorem of calculus, to define the $\ln(x)$. Since the function $f(t) = 1/t, t \in (0, \infty)$ is continuous on $(0, \infty)$, it is integrable, i.e., $\int_1^x 1/t\, dt$ exists for every $x \in (0, \infty)$. This motivate our definition:

19.1.1 Definition:

For $x \in (0, \infty)$, define the natural logarithm of x by

$$\ln x := \int_1^x 1/t\, dt \cdot$$



We show that the above function is the familiar logarithmic function.

19.1.2 Theorem:

The function $\ln : (0, +\infty) \to \mathbb{R}$ has the following properties:

(i) $\ln 1 = 0$.

(ii) $\ln(x) > 0$ for $x > 1$ and $\ln x < 0$ for $0 < x < 1$.

(ii) $\ln x$ is differentiable and

$$\ln'(x) = \frac{1}{x} \text{ for all } x \in (0, \infty).$$

(iii) $\ln(xy) = \ln x + \ln y$ for all $x, y \in (0, \infty)$.

(iv) $\ln(x^r) = r(\ln x)$ for all $r \in \mathbb{Q}$ and $x \in (0, \infty)$.

(v) $\ln(x)$ is strictly increasing and concave downward on $(0, \infty)$. In particular, $\ln(x)$ is a one-one function.

(vi) $\ln x \to \infty$ as $x \to \infty$ and $\ln x \to -\infty$ as $x \to 0$. In particular, it is an onto function and takes every real value exactly once.

PROOF

19.1.2 Theorem:

Proof:

It is obvious from the definition that $\ln 1 = 0$. Further, $\ln(x)$ being the integral of a positive function,
$\ln(x) > 0$ for $x > 1$.
For $0 < x < 1$,
$$\ln x = -\int_1^x \frac{1}{t}\,dt < 0.$$
This proves (i).

Since $1/x, x > 0$ is a continuous function, by FCT-II, $\ln x$ is differentiable and
$$\ln'(x) = \frac{1}{x} \text{ for all } x \in (0, \infty),$$

proving (ii). To prove (iii), fix $y > 0$ arbitrarily. Let
$f(x) := \ln(yx)$ and $g(x) := \ln y + \ln x$ for every $x > 0$.
Then
$$f'(x) = \frac{1}{x} = g'(x) \text{ for every } x > 0.$$
Thus,
$f(x) = g(x) + c$ for every x and a constant c.
Since, $f(1) = \ln y = g(1)$, we have $c = 0$. Hence,
$\ln(xy) = \ln x + \ln y$ for all $x, y \in (0, \infty)$.
This proves (iii).

19.1.3 Definition:

By Theorem 19.1.2 (iv) above, there is a unique positive real number, denoted by e, whose natural logarithm is equal to 1, i.e., $\ln(e) = 1$. This positive real number e is one of the Euler's number.

19.1.4 **Note:**

(i) From Theorem 19.1.2 (iv), we have

$\ln 2 < 1 = \ln(e) = 1 < 2 < 2\ln(2) = \ln(4)$

hence, by the intermediate property and the fact, that $\ln(x)$ is a strictly increasing, one-one function, we get $2 < e < 4$. In fact one can show that $2 < e < 3$.

(ii) Recall that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists. Let it be denoted by l. Since $\ln(x)$ is continuous, we have

$$\begin{aligned}\ln(l) &= \lim_{n\to\infty}\left(\ln\left(1+\frac{1}{n}\right)^n\right)\\ &= \lim_{n\to\infty}\left[n\left(\ln\left(1+\frac{1}{n}\right)\right)\right]\\ &= \lim_{n\to\infty}\left[\frac{\ln(1+\frac{1}{n})-\ln(1)}{1/n}\right]\\ &= \frac{d}{dx}(\ln(x))\Big|_{x=1} = 1 = \ln(e).\end{aligned}$$

Hence $l = e$.

(iii) It can be shown that the number e is an irrational number and is also given by

$$e = \lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right) \cong 2.71828\ldots.$$

19.1.5 Examples

(i) Let

$$f(x) = \frac{(x-2)}{\sqrt{x^2+1}}, x \neq 2.$$

We want to find $f'(x)$ for $x \neq 0$. We can use the $\ln$ function to do this. Since

$$\begin{aligned}\ln(f(x)) &= \ln\left(\frac{x-2}{\sqrt{x^2+1}}\right)\\ &= \ln(x-2) - \ln\left((x^2+1)^{\frac{1}{2}}\right)\\ &= \ln(x-2) - \frac{1}{2}\ln\left(x^2+1\right),\end{aligned}$$

we have, by Chain rule,

$$\frac{f'(x)}{f(x)} = \frac{1}{x-2} - \frac{2x}{2\left(x^2+1\right)}$$

$$= \frac{2x^2+2-2x^2+4x}{2(x-2)\left(x^2+1\right)}$$

$$= \frac{2(2x+1)}{2(x-2)\left(x^2+1\right)}.$$

Hence,

$$f'(x) = \frac{(x-2)\left(x^2+1\right)^{-\frac{1}{2}}(2x+1)}{(x-2)\left(x^2+1\right)}$$

$$= \frac{2x+1}{\left(x^2+1\right)^{\frac{3}{2}}}.$$

PRACTICE EXERCISES

1. Let $f(x)$ be a differentiable function of x, $f(x) \neq 0$ for any x. Show that the function $ln\left(\left|f(x)\right|\right)$ is

differentiable with

$$\frac{d}{dx}\left(ln\left|f(x)\right|\right) = \frac{f'(x)}{f(x)}.$$

2. Compute the following:

(i) $\lim\limits_{x \to 2^+} ln(x-2).$

(ii) $\lim\limits_{x \to 3^-} ln\left(x^3(x-2)\right).$

(iii) $\lim\limits_{x \to 0}\left[\frac{d}{dx}\left(ln\left|Cos\, x\right|\right)\right]$

3. Analyse the following functions for relative maxima/minima and inflection points:

(i) $f(x) = ln(x) - x^2.$

(ii) $f(x) = ln\left|x\right|.$

(iii) $f(x) = 3\, ln(x-1)^{\frac{1}{3}}.$

Recap

In this section you have learnt the following

- How integration helps us to define the natural logarithm function and derive its properties.

Section - 19.2

Objectives

In this section you will learn the following :

How to define the natural exponential function and drive its properties.

19.2 Exponential function

We saw in the previous section that the function $\ln : (0,\infty) \longrightarrow \mathbb{R}$ is one-one and onto. Thus it has inverse.

19.2.1Definition:

The exponential function, denoted by exp is the one-one, onto function $\exp : \mathbb{R} \longrightarrow (0,\infty)$ such that

$\exp(y) = x$ if and only if $\ln x = y, y \in \mathbb{R}, x \in (0,\infty)$.

The exponential function, as defined above is also called the exponential function to the natural base. The properties of the exponential function are given in the next theorem.

19.2.2 Theorem:

The function $\exp : \mathbb{R} \longrightarrow (0,\infty)$ has the following properties:

(i) $\exp\,(0) = 1\, and\, \exp(1) = e$

(ii) The function $\exp(y)$ is differentiable and

$$\frac{d}{dy}(\exp(y)) = \exp(y).$$

(iii) For $y_1, y_2 \in \mathbb{R}, \exp\left(y_1 + y_2\right) = \left(\exp y_1\right)\left(\exp y_2\right)$.

(iv) The function $\exp(y)$ is strictly increasing and concave upward on $\mathbb{R}$

(v) $\lim\limits_{y\to\infty} \exp y = \infty$ *and* $\lim\limits_{y\to-\infty} \exp y = 0.$

19.2.2 Theorem:

The function $\exp : \mathbb{R} \to (0,\infty)$ has the following properties:

(i) $\exp\,(0) = 1$ *and* $\exp(1) = e$

(ii) The function $\exp(y)$ is differentiable and

$$\frac{d}{dy}(\exp(y)) = \exp(y).$$

(iii) For $y_1, y_2 \in \mathbb{R}, \exp(y_1 + y_2) = (\exp y_1)(\exp y_2).$

(iv) The function $\exp(y)$ is strictly increasing and concave upward on $\mathbb{R}$

(v) $\lim\limits_{y\to\infty} \exp y = \infty$ *and* $\lim\limits_{y\to-\infty} \exp y = 0.$

Proof:

That $\exp\,(0) = 1$ and $\exp(1) = e$,follow from the fact that $\ln(1) = 0$ and $\ln(e) = 1$.

The differentiability of $\exp\,(x)$ follows from the theorem on the derivative of the inverse function. Further, its derivative at a point $y = \ln(x)$ is given by

$$\frac{d}{dy}(\exp(y)) = \frac{1}{\frac{d}{dx}(\ln(x))} = \frac{1}{1/x} = x = \exp(y).$$

This proves (ii). To prove (iii), let $x_1 = \exp(y_1), x_2 = \exp(y_2).$

Then,
$y_1 + y_2 = \ln(x_1) + \ln(x_2) = \ln(x_1 x_2).$
Hence,
$\exp(y_1 + y_2) = x_1 x_2 = (\exp y_1)(\exp y_2).$

PRACTICE EXERCISES

1. Show that the following functions on $\mathbb{R}$ are one-one, onto. Find a formula for the inverse function.

(i) $f(x) = \exp\,(3x).$

(ii) $f(x) = \exp\,(3x).$

(iii) $f(x) = \exp\,(x) - 2.$

(iv) $f(x) = \exp\,(x-2).$

2. Sketch the following functions:

(i) $f(x) = \exp\,(-x),\ x \in \mathbb{R}.$

(ii) $f(x) = \exp(-2x)$, $x \in \mathbb{R}$.

(iii) $f(x) = \dfrac{\exp(x)}{1+\exp(x)}$, $x \in \mathbb{R}$.

3. Find linear and quadratic approximations to the following functions near $x = 0$.

(i) $f(x) = \exp(3x)$.

(ii) $f(x) = \exp\left(-\dfrac{x^2}{2}\right)$.

4. Show that if $f : \mathbb{R} \to (0, \infty)$ is any differentiable function with f' continuous such that

$f'(x) = f(x)$ for all $x \in \mathbb{R}$ and $f(0) = 1$.
then $f(x) = \exp(x)$ for all $x \in \mathbb{R}$.

5. Let f be a twice differentiable functions with f'' continuous and

$$f''(x) = \frac{1}{2}(\exp(x) + \exp(-x)), f(0) = 1, f(0) = 1, f'(0) = 0,$$

Find $f(x)$.

6. Show that for any $a, b \in \mathbb{R}$,

$$\frac{\exp(a)}{\exp(b)} = \exp(a-b).$$

7. Let $a > 1$ be any real number.

Show that

$$\int_1^a ln(x)dx + \int_0^{\ln(a)} \exp(x)dx = a\, l_n(a).$$

Interpret this geometrically as the areas.

8. Let a, b be real numbers with $a < b$. Prove the following:

(i) $\exp\left(\dfrac{ln(a) + l_n(b)}{2}\right) < \exp(x) < \dfrac{a+b}{2}$

for every $x \in [ln(a), ln(b)]$.

(ii) Using (i) and the fact that

$$\int_{\ln(a)}^{\ln(b)} \exp(x)dx = b - a,$$

deduce that

$$\sqrt{ab} < \frac{b-a}{ln(b) - ln(a)} < \frac{a+b}{2}.$$

9. Show that the functions $f(x) = \exp(x)$ and $g(x) = -x$ intersect only at one point and at that point the tangents

to f and g are perpendicular to each other.

10. Analyse if $f(x) = 2 + e^{x}$ and $g(x) = ln(x-2)$ are inverse of each other or not.

Recap

In this section you have learnt the following

- How to define the natural exponential function and drive its properties.