Module 5 : Linear and Quadratic Approximations, Error Estimates, Taylor's Theorem, Newton and Picard Methods

Lecture 13 : Linear Approximations [Section 13.1]

Objectives

In this section you will learn the following :

- Approximating a given function by linear functions.

13.1 Linear Approximations

The aim of this module is to see how the concept of differentiation leads to the idea of approximating a given complicated function by simpler one.

13.1.1 Definitions:

Let $f : A \longrightarrow \mathbb{R}$ be a function such that $f'(a)$, its derivative at a , exists. Then $f'(a)$ is the slope of the tangent to the graph of f at $x = a$. The equation of the tangent line is given by

$$L(x;a) = f(a) + f'(a)(x-a).$$

For x near a , the linear function $L(x;a)$ is a reasonable approximation of $f(x)$. We call $L(x;a)$ the linear (or tangent line) approximation of f for x near a . The error committed in this process of approximation is given by:

$$e_1(x;a) := f(x) - L(x;a) = f(x) - (f(a) + f'(a)(x-a))$$

Thus,

$$f(x) - f(a) = (x-a)f'(a) + e_1(x;a).$$

Note that, the increment $f(x) - f(a)$ in f is equal to the tangent line increment $f'(a)(x-a)$ plus the error $e_1(x;a)$.

13.1.2 Note:

(i) This process of approximation of f near a is also known as the method of linearization, for it allows us to approximate a given function $f(x)$ in a small interval around a point $x=a$ by a linear function: the tangent line.

(ii) The error $e_1(x;a) \mapsto 0$ as $x \to a$. In fact, it vanishes at a much faster rate, since

$$\frac{e_1(x;a)}{x-a} := \frac{\left(f(x)-f(a)\right)}{x-a} - f'(a)$$

this implies that

$$\lim_{x \mapsto a}\left(\frac{e_1(x;a)}{x-a}\right) = 0.$$

Thus, not only $e_1(x;a) \to 0$ as $x \to a$, it does so at a rate faster than at which $(x-a) \to 0$.

13.1.3 Examples:

(i) For a linear function $f(x) = mx + c$, its linear approximation at $x=a$ is given by

$$L(x;a) = f(a) + f'(a)(x-a) = m(x) + c = f(x)$$

the function itself. For example, the perimeter of a circle is a function of its radius r, given by $p(r) = 2\pi r$ is a linear function of r.

(ii) The surface area of a sphere is a function of its radius r i.e., $S(r) = 4\pi r^2$. The linear approximation is

$$L(x;a) = S(a) + S'(a)(x-a) = 4\pi a^2 + 8\pi a(x-a),$$

and the error is

$$e_1(h;a) := S(a+h) - S(a) - S'(a)h = 4\pi h^2.$$

(iii) Let c be any real number and k be any rational number. Let

$$f(x) = (1+cx)^k \text{ for } x \in \mathbb{R},\ x \neq \frac{-1}{c}.$$

Then, the linear approximation of f for x near $a = 0$ is

$$L(x;0) = 1 + kcx.$$

Thus, for example, if

$$f(x) = \frac{1}{1-x}, \text{ then } L(x;0) = 1 + x,$$

and if

$$f(x) = \sqrt[3]{1+2x}, \text{ then } L(x;0) = 1 + \frac{2x}{3},$$

This is useful in finding approximate values of roots. For example, for $k=5$

$$\sqrt[5]{2}=\sqrt[5]{1+1}\approx 1+\frac{1}{5} = \frac{6}{5} = 1.2,$$

and for $k=55$,

$$\sqrt[55]{1.0002}=\sqrt[55]{1+1.0002}\approx 1+\frac{.0002}{5} = 1+0.00004 = 1.00004.$$

13.1.4 Examples:

Consider the function

$$f(x)=\frac{x}{1+x}, x\neq -1.$$

Suppose, we want to find linear approximation of $f(x)$ at $x=1.3$. Since, linear approximation works over an interval near $x=a$, we consider the linear approximations of $f(x)$ for x near $a=1$. We do this to make our computations simpler. Since,

$$f'(x)=\frac{1}{1+x^2}, \; x\neq -1$$

We have

$$L(x,1)=f(1)+f'(1)(x-1)=\frac{1}{2}+\frac{1}{4}(x-1)=\frac{x+1}{4}.$$

Hence, a linear approximation to $f(x)$ at $x=1.3$ is $L(1.3,1)=0.575$, whereas the actual value is $f(1.03)=0.513$.

13.1.5 Note:

For a function $y=f(x)$, through $L(x,a)$ gives an approximate value of $f(x)$ at points x near a, it cannot be used directly in many situations. The reason being, the calculation of $L(x,a)$ requires the values of $(x-a)$ which may not be known. However, in many cases, we have a bound for $(x-a)$ and one would like to calculate a bound for the error $f(x)-f(a)$ which is approximately $f'(a)(x-a)$. For example, if $\alpha\leq(x-a)\leq\beta$, then, we can take $\alpha f'(a)\leq f'(a)(x-a)\leq \beta f'(a)$.

13.1.6 Example:

Consider the experiment of finding the volume of a sphere of radius r. Suppose its radius is measured with an error of at most $\pm\frac{1}{100}cm$. We want to estimate the error in computing the volume. The volume as a function of the radius is $f(r)=\frac{4}{3}\pi r^3$.

Thus, $f'(r)=4\pi r^2$. Now if r_a is the actual radius and r_0 is the measured radius, then we know

$$-\frac{1}{100}\leq (r_0-r_a) \leq\frac{1}{100}.$$

Since

$$f(r_0)-f(r_a)\approx f'(r_0)(r_a-r_0),$$

the error made in measuring the volume will be

.

$$-\frac{1}{100}f'(r_0) \le f(r_0) - f(r_a) \le \frac{1}{100}f'(r_0)$$

Thus, if the measured value of the radius is $r_0 = 10\,cm$, then the error bounds for the measurement of the volume will be,

$$-4\pi \le f(r_0) - f(r_a) \le +4\pi,$$

as $f'(r_0) = 4\pi(10)^2$. Similarly, if we say that 0.05% error was made in measuring the radius, then

$$\frac{r_0 - r_a}{r_0} \text{ x } 100 = 0.05$$

The corresponding estimated percentage error in measuring the volume is given by

$$\frac{f(r_0) - f(r_a)}{f(r_0)} \text{ x } 100 \simeq \frac{f'(r_0)(r - r_0)}{f(r)} \text{ x } 100$$

$$= \frac{4\pi r_0^2 (r - r_0)}{\frac{4}{3}\pi r_0^3} \text{ x } 100$$

$$= \frac{3(r - r_0)}{r_0} \text{ x } 100$$

$$= 3 \text{ x } 0.05$$

$$= 0.15$$

PRACTICE EXCERCISES

1. Find linearization's of the following:

   $(i)\ f(x) = x^2 - 2x + 5,$ near $a = 0$ and $a = 2.$

   $(ii)\ f(x) = \sqrt{x}$ near $a = 2.$

   $(iii)\ f(x) = \sqrt{x^2 + 4}$ near $a = -2.$

   $(iv)\ f(x) = \sin 2x + \cos x^2$ near $a = 0.$

   $(v)\ f(x) = \tan(x^3/3)$ near $a = 0.$

2. Use the tangent line increment for suitable functions near suitable points to find an approximate value of

$$(8.01)^{\frac{4}{3}} + (8.01)^2 - (8.01)^{-\frac{1}{3}}$$

3. Let $f(x) = \sqrt{1+x}$ and let $L(x)$ be its linearization near $x = 0$. Show that

$$\lim_{x \to 0}\left(\frac{f(x)}{L(x)}\right) = 1$$

   What do you conclude from this?

4. For $a \in \mathbb{R}$, let

$$F(x) = c_0 + c_1(x - a) \text{ for } x \in \mathbb{R}.$$

   If a function f is differentiable at a, show that F is the linearization of f near a if and only if

$f(a) = F(a) = c_0$, and $f'(a) = F'(a) = c_1$.

This shows that among all the linear functions F, linearization is the only one with the property that

$$\lim_{x \to 0}\left(\frac{f(x) - F(x)}{x - a}\right) = 0.$$

5. The side of a cube is measured to be $20\,cm$ with a possible error of $\pm(.5)\,cm$.

(i) Use the tangent line approximation to estimate the error made in calculating the volume.

(ii) Estimate the percentage of errors in measuring the side and the volume.

Recap

In this section you have learnt the following

- Approximating a given function by linear functions.

[Section 13.2]

Objectives

In this section you will learn the following :

- How to estimate the error in linear approximations.

13.2 Error estimate for linear approximations

We want to analyze the questions:
How well $L(x;a)$ approximates $f(x)$? Can we get a bound for the error?

To investigate this, we must estimate the error $e_1(x;a)$. To do this, we need the following extension of the Lagrange's Mean Value Theorem, which we shall assume without proof.

13.2.1Theorem (Extended Mean Value:)

Let $f:[a,b]\to\mathbb{R}$, *be a differentiable function such that* f' *is contiuous on* $[a,b]$ *and* f'' *exists on* (a,b). *Then there exists* $c_2\in(a,b)$ *such that*

$$f(b)-f(a)=f'(a)(b-a)+f''(c_2)\frac{(b-a)^2}{2}$$

As a consequence of this theorem, we have the following:

13.2.2Corollary (Error estimate for linear approximation):

Let $f:[c,d]\to\mathbb{R}$. *Let* $a,x\in[c,d], a\neq x$ *and* I *be the closed interval joining* a *and* x. *Suppose the following are satisfied:*

(i) *The functions* $f,f':I\to\mathbb{R}$ *are continuous.*

(ii) *For every* c *between* a *and* x,

$$f''(c) \text{ exists and } |f''(c)|\leq M_2(x)$$

Then

$$|e_1(x;a)|=|f(x)-L(x;a)|\leq M_2(x)\left(\frac{|x-a|^2}{2}\right).$$

13.2.2 **Corollary (Error estimate for linear approximation):**

Proof:

If $a<x$, put $b=x$ in the theorem 13.2.1. and obtain

$$f(x)=f(a)+f'(a)(x-a)+f''(c_2)\frac{(x-a)^2}{2}.$$

When $x<a$, define a function $g:[x,a]\to\mathbb{R}$ by

$g(t):=f(a+x-t),\ t\in[x,a]$.

Then, g satisfies the conditions of the theorem 13.2.1 on the interval $[x,a]$. Further,

$g'(t)=-f'(a+x-t)$ and $g''(t)=f''(a+x-t)$

Thus, by theorem 13.2.1, there is some $c_2\in(x,a)$ such that

$$g(a)=g(x)+g'(a)(x-a)+g''(\tilde{c}_2)\frac{(x-a)^2}{2},$$

i.e.,

$$f(x)=f(a)+f'(a)(x-a)+f''(c_2)\frac{(x-a)^2}{2},$$

where $c_2=a+x-\tilde{c}_2$. Thus, in either case, we have a point c_2 between a and x such that

$$f(x)=f(a)+f'(a)(x-a)+f''(c_2)\frac{(x-a)^2}{2}.$$

Hence,

$$|f(x)-(f(a)-f'(a)(x-a))|=|e_1(x;a)|\leq M_2(x)\frac{|x-a|^2}{2}.$$

13.2.3Example:

Consider the function

(i) $$f(x) = \frac{1}{1-x} \text{ for } x \neq 1.$$

Then,

$$f'(x) = -\frac{1}{(1-x)^2}, \; x \neq 1$$

and

$$f''(x) = \frac{2}{(1-x)^3}, \text{ for } x \neq 1.$$

The linear approximation for $f(x)$ near the point $x = 0$ is given by $L(x, 0) = 1 + x$, as in example 13.1.3 (ii). Let us estimate the error for a point $x < 1$. We consider two cases:

Case (i): $0 < x$

In this case, for $0 < c < x$, we have

$$|f''(c)| = \left|\frac{2}{(1-c)^3}\right| \leq \frac{2}{(1-x)^3} := M_2(x)$$

Thus,

$$|\epsilon_1(x; 0)| \leq \frac{x^2}{1-x^3}$$

For example,

$$|e_1(x; 0)| \leq (0.1)^2 / (0.9)^3 < 0.014, \text{ for } 0 < x < 0.1.$$

Case (ii): $x < 0$

In this case, for $x < c < 0$, we have

$$|f''(c)| < 2 := M_2(x),$$

and hence

$$|e_1(x; 0)| \leq |x|^2$$

For example,

$$|e_1(x; 0)| \leq 0.01, \text{ for } -0.1 < x < 0$$

Click here to see an interactive visualization : Applet 13.1

PRACTICE EXERCISES

1. For the following functions find the linear approximation near $a = 0$. Also find an upper bounds for the error

$e_1(x,0)$ which are valid for all $x \in (0, \delta)$, and the one which is valid for all $x \in (-\delta, 0)$:

(i) $f(x) = \sqrt{1+x},\ x \geq -1,\ \text{and}\ \delta = 0.01.$

(ii) $f(x) = \dfrac{x}{x^2+1},\ \ \delta = 0.5.$

(iii) $f(x) = \tan x,\ \ \delta = \pi/4.$

2. Let

$$f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} \text{ for } x > 0.$$

Find the linear approximation $L(x)$ of $f(x)$ near $a = 4$ and estimate the error $f(4.41) - L(4.41)$.

3. Let

$$f(x) = \sqrt{x} + \sqrt{x+1} - 4.$$

Show that there is a unique $x_0 \in (3,4)$ such that $f(x_0) = 0$. Using the linear approximation of f near 3, find an approximation x_1 of x_0. Find x_0 exactly and determine the error $|x_1 - x_0|$.

Recap

In this section you have learnt the following

How to estimate the error in linear approximations.