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Module 12 : Total differential, Tangent planes and normals

Lecture 34 : Gradient of a scaler field [Section 34.1]

Objectives

In this section you will learn the following :

- The notions gradient vector
- The relation of gradient with the directional derivative

34 .1 Gradient of a scalar field

We have seen that for a function $f(x,y,z)$ the partial derivatives f_x, f_y, f_z, whenever they exist, play an important role. This motivates the following definition.

34.1.1Definition:

Let $(x_0, y_0, z_0) \in D \subseteq \mathbb{R}^3$ and $f : D \to \mathbb{R}$. If each of f_x, f_y and f_z exist at a point (x_0, y_0, z_0), then the vector $(f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0))$ is called the **gradient vector** of f at (x_0, y_0, z_0), and is denoted by

$$(\nabla f)(x_0, y_0, z_0) = (f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), \; f_z(x_0, y_0, z_0).$$

For a function of 2-variables, it is given by

$$(\nabla f)(x_0, y_0) := (f_x(x_0, y_0), f_y(x_0, y_0, z_0)).$$

34.1.2Theorem:

Let $(x_0, y_0, z_0) \in D \subseteq \mathbb{R}^3$ and $f : D \to \mathbb{R}$ be differentiable at (x_0, y_0, z_0).

(i) For every unit vector $u \in \mathbb{R}^3$, $(D_u)(x_0, y_0, z_0)$ exist and

$$(D_u f)(x_0, y_0, z_0) = (\nabla f)(x_0, y_0, z_0) \cdot \boldsymbol{u}.$$

(ii) *Suppose* D *is such that any two points in it can be joined by line segments parallel to axes and*

$(\nabla f)(x,y,z) = 0 \text{ for all } (x,y,z) \in D$, *then* f *is constant in* D.

PROOF

34.1.2Theorem:

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Proof

The proof of (i) follows from theorem 33.2.4. To prove (ii) first note that the given condition

$(\nabla f)(x,y,z) = 0 \text{ for all } (x,y,z) \in D$,

implies that

$\text{each of } f_x, f_y, f_z = 0 \text{ in } D.$

Let $A, B \in D$ be such that A and B can be joined by a path as shown in figure below, where AC, BC are parallel to axes.

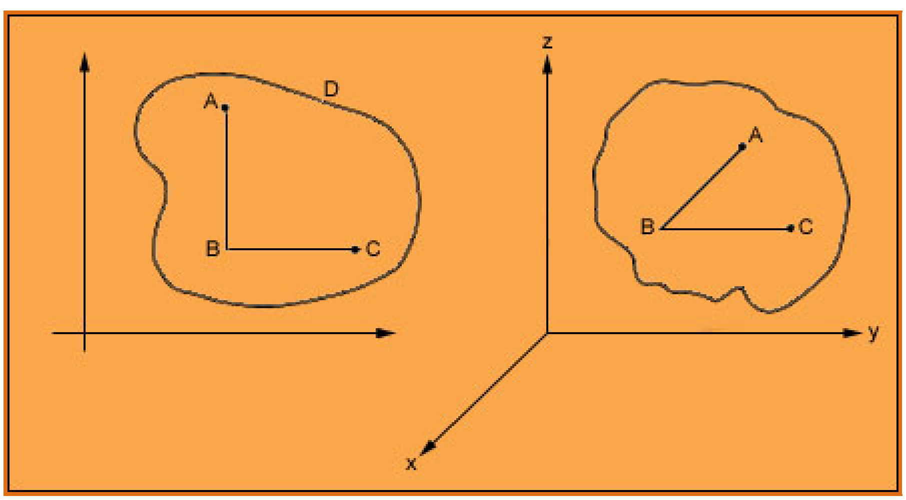

Figure 1

Then, by one variable case,
$f(A) = f(C) = f(B)$.
Thus, if any two points in D can be joined by a piecewise linear path, moving parallel to axes only, then
$(\nabla f)(x,y) = 0 \text{ for all } x, y \in D$ implies that f is constant in D.

34.1.3 Example:

Let

$f(x,y,z) = x^2 y - y z^3 + z.$

Then

$f_x\ (x,y,z) = 2xy$,

$f_y\ (x,y,z) = x^2 - z^3$,

$f_z\ (x,y,z) = -3yz^2 + 1.$

Obviously, each of f_x, f_y, f_z is a continuous function everywhere. Then, f is differentiable and for every unit vector $\mathbf{u}$

$(D_{\mathbf{u}}\ f)(x_0, y_0, z_0) = (\nabla f)(x_0, y_0, z_0) \cdot \mathbf{u}.$

For $2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$ example, if we want to find the directional derivative of f at the point $(1,-2,0)$, in the direction of the vector , than we take

$$\mathbf{u} = \frac{2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}}{\sqrt{a}} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

and

$f_x\ (1,-2,0) = -4$, $f_y\ (1,-2,0) = 1$, $f_z\ (1,-2,0) = 1.$

Thus

$$(D_u\ f)(1,-1,0) = (-4,1,1) \cdot \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = -3.$$

34.1.4 Remark:

The formula

$(D_{\boldsymbol{u}}\, f)(x_0, y_0, z_0) = (\nabla f(x_0, y_0, z_0)) \cdot \boldsymbol{u}$ may not hold if f_x, f_y either of f_y is discontinuous at (x_0, y_0, z_0).

For example, consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(0,0) = 0 \text{ and } f(x,y) = \frac{x^3}{x^2 + y^2} \text{ for } (x,y) \neq (0,0).$$

We have

$$\nabla f(0,0) = (1,0),$$

and for any unit vector $\boldsymbol{u} = (u_1, u_2)$,

$$(D_{\boldsymbol{u}} f)\ (0,0) = u_1^3.$$

Thus,

$$(D_{\underline{u}}\ f)(0,0) \neq \left(\nabla f(0,0)\right) \cdot \boldsymbol{u}, \text{ whenever } u_1 \neq 0, 1, -1.$$

Note that for $(x_0, y_0) \neq (0,0)$, we have

$$f_x(x_0, y_0) = \frac{x_0^4 + 3x_0^2 y_0^2}{(x_0^2 + y_0^2)^2} \quad \text{and} \quad f_y(x_0, y_0) = \frac{-2x_0^3 y_0}{(x_0^2 + y_0^2)^2}.$$

It is easy to see that both f_x and f_y are discontinuous at $(0,0)$.

We describe next some geometric properties of the gradient.

34.1.5Theorem:

Let $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ be differentiable at $(x_0, y_0, z_0) \in D$ so that that

$\nabla f(x_0, y_0, z_0) \neq (0,0,0).$

Let $\boldsymbol{u} = (u_1, u_2, u_3)$ be a unit vector. Then the following holds:

(i) *Near the point (x_0, y_0, z_0), the direction in which f increases most rapidly is that of $\nabla f(x_0, y_0, z_0)$.*

(ii) *Near the point (x_0, y_0, z_0), the direction in which f decreases most rapidly is the one opposite to that of $\nabla f(x_0, y_0, z_0)$.*

(iii) *Near the point (x_0, y_0, z_0), the directions perpendicular to that of $\nabla f(x_0, y_0)$ are the directions of no change*

in f.

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Proof

By definition, we have
$$(D_{\boldsymbol{u}} f)(x_0, y_0, z_0) = (\nabla f(x_0, y_0, z_0)) \cdot \boldsymbol{u} = |\nabla f(x_0, y_0, z_0)| \cos\theta,$$
where $\theta \in [0, \pi]$ is the angle between $\nabla f(x_0, y_0, z_0)$ and $\boldsymbol{u}$. Since $-1 \leq \cos\theta \leq 1$, we have
$(D_{\boldsymbol{u}} f)(x_0, y_0, z_0)$ is maximum when $\cos\theta = 1$, $\theta = 0$.
Thus, near (x_0, y_0, z_0),
$$\boldsymbol{u} = \frac{\nabla f(x_0, y_0, z_0)}{|\nabla f(x_0, y_0, z_0)|} \text{ is the direction in which } f \text{ increases most rapidly.}$$
The value of $(D_{\boldsymbol{u}} f)(x_0, y_0, z_0)$ is minimum when $\cos\theta = -1$, that is, when $\theta = \pi$. Thus, near

(x_0, y_0, z_0),

$$\boldsymbol{u} = -\frac{\nabla f(x_0, y_0, z_0)}{|\nabla f(x_0, y_0, z_0)|} \text{ is the direction in which } f \text{ decreases most rapidly.}$$

Finally, $(D_{\boldsymbol{u}} f)(x_0, y_0, z_0) = 0$ when $\cos\theta = 0$, that is, when $\theta = \pi/2$. Thus, near (x_0, y_0, z_0),

$$\boldsymbol{u} = \pm\frac{f_y(x_0, y_0, z_0)\mathbf{i} - f_x(x_0, y_0, z_0)\mathbf{j}}{|\nabla f(x_0, y_0, z_0)|} \text{ are the directions of no change in } f.$$

34.1.6 Note:

In case $\nabla f(x_0, y_0, z_0) = (0,0,0)$, we have $(D_{\boldsymbol{u}} f)(x_0, y_0, z_0) = 0$ for every $\boldsymbol{u}$, and hence near (x_0, y_0, z_0), , the f has no rate of change in all directions.

34.1.7 Example:

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by Suppose

$$f(x, y) = 4 - x^2 - y^2 \text{ for } (x, y) \in \mathbb{R}^2.$$

We have

$$f_x = -2x, f_y = -2y.$$

At $(x_0, y_0) = (1,1)$

$$\nabla f(1,1) = (-2,-2).$$

Thus, on the surface $z = f(x, y)$ near $(1,1)$,

$$\frac{\nabla f(1,1)}{|\nabla f(1,1)|} = \frac{(-2,-2)}{2\sqrt{2}} = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) \text{ is the direction of steepest ascent}$$

while in the reverse direction, namely,

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ is direction of steepest descent.}$$

The directions of no change are

$$\pm\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right).$$

Since $\nabla f(0,0) = (0,0)$, rate of change of f is zero in every direction at $(0,0)$.

34.1.8 Example:

Let

$$f(x, y) = 20 - 4x^2 - y^2$$

represent the temperature of a metallic sheet. Starting at the point $(2,1)$ let us find the continuous path

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},$$

that will give the direction of maximum increase in temperature. Since, the direction to this path at any time point t is

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j},$$

and that has to be of maximum increase of f , we should have

$\alpha\,\mathbf{r}'(t) = \nabla f$, for some scalar α.

That is,

$$\alpha\, x'(t)\,\mathbf{i} + \alpha\, y'(t)\,\mathbf{j} = -8\,x\,\mathbf{i} - -2\,y\,\mathbf{j},$$

i.e.,

$$\alpha\, x'(t) = -8\,x,\;\; \alpha\, y'(t) = -2\,y.$$

This gives us the differential equation

$$\frac{dy}{dx} = \frac{2y}{8x} = \frac{y}{4x}.$$

A solution to which is

$$x = ky^4\text{ , } k \text{ some scalar.}$$

Since, this passes through $(1,2)$, we have

$$2 = k.$$

Thus, the required path is $x = 2y^4$.

Practice Exercises

(1) Find the gradient for the following functions at the indicated point P and its directional derivative at P in the

direction of the indicated point Q:

(i) $f(x,y) = \sqrt{xy}\,e^y$, $P = (1,1),\ Q = (0,-1)$.

(ii) $f(x,y,z) = x^3y^2z^5 - 2xz + yz + 3x,\ P = (-1,-2,1),\ Q = (0,0,-1)$.

Answers

(2) For the following functions, find the direction of maximum increase at the indicated point:

(i) $f(x,y,z) = \sin xy + \cos yz,\ P = (-3,0,7).$

(ii) $f(x,y,z) = 2xyz + y^2 + z^2,\ P = (2,1,1).$

Answers

(3) The temperature at a point (x,y,z) on the surface of a body is given by

$$T(x,y,z) = 2\,x^2 - y^2 + 4\,z^2.$$

Find the rate of change of temperature at the point $P = (1,-2,1)$ in the direction of the vector $4\mathbf{i} - \mathbf{j} + 2k$.
In what direction at P, the temperature is decreasing most rapidly?

Answers

(4) If $z = f(x,y)$ is a differentiable function, where $x = x(t)$ and $y = y(t)$ are also differentiable with respect to

t, compute $\dfrac{dz}{dt}$ in terms of ∇z.

Answers

(5) Let $f(x,y)$ be a differentiable function such that

$(D_{\mathbf{u}}f)(x,y) = 0 = (D_{\mathbf{v}}f)(x,y)$, for all (x,y)

for any two fixed vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ such that $\mathbf{u} \neq \alpha \mathbf{v}$ for any constant α. Show that $(D_{\mathbf{w}}f)(x,y) \equiv 0$ for all $\mathbf{w} \in \mathbb{R}^2$.

(6) Let $f(x,y)$ be such that

(i) $f_x(x,y)$ and $f_y(x,y)$ exist for all $(x,y) \in B_r(1,2)$ for some $r > 0$ and are continuous at $(1,2)$.

(ii) The directional derivative of f at $(1,2)$ in the direction toward $(2,3)$ is $2\sqrt{2}$.

(iii) The directional derivative of f at $(1,2)$ in the direction toward $(1,0)$ is -3. Find $f_x(1,2), f_y(1,2)$ and the directional derivative of f at $(1,2)$ in the direction toward $(4,6)$.

Answers

(7) Let $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ be such that all f_x, f_y, f_z, g_x, g_y and g_z exist in $B_r((x_0, y_0))$, for some $r > 0$.

Prove the following:

(i) $(\nabla f)(f \pm g) = (\nabla f) \pm (\nabla g)$.

(ii) $\nabla (fg) = f(\nabla g) + (g(\nabla f)$.

(iii) $\nabla(\alpha f) = \alpha(\nabla f)$, for every $\alpha \in \mathbb{R}$.

Recap

In this section you have learnt the following

- The notions gradient vector
- The relation of gradient with the directional derivative