Module 6 : Definition of Integral

Lecture 16 : Integral from upper and lower sums [Section 16.1]

Objectives

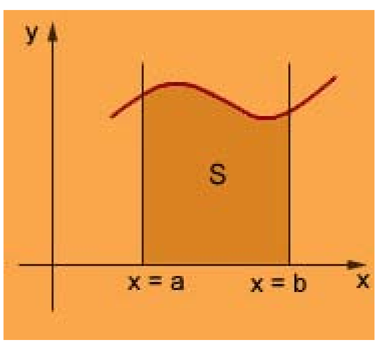
In this section you will learn the following :

How to define the integral of a function.

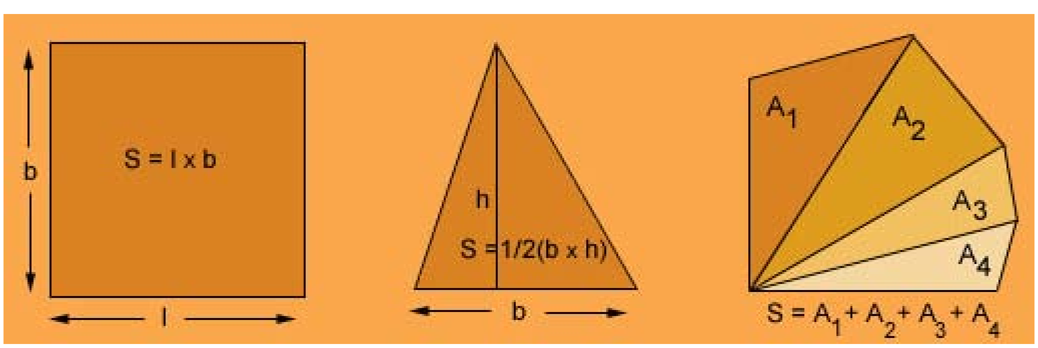
16.1 Integral from upper and lower sums

We start by analyzing the following:

The Area Problem: The problem is to find the area of a region S in the plane that is bounded by the curve
$y = f(x)$ from $x = a$ to $x = b$ and the $x-$axis.



This raises a natural question: What is the meaning of the word 'area'? It is easy to answer this question for region with straight sides, e.g., a rectangle, triangle, and a polygon.



However, it is not as easy for general regions as is for the regions with straight side. But it is possible to find an approximation to area using concept of area of the rectangles. The idea is to fill up (cover up) the required area by rectangles with sides parallel to the axes. To do this, let us make some definitions.

16.1.1Definition:

Let $[a,b]$ be a closed bounded interval.

(i) Let

$$a = x_0 < x_1 < \ldots < x_n = b$$

be points in the interval $[a,b]$. Then,

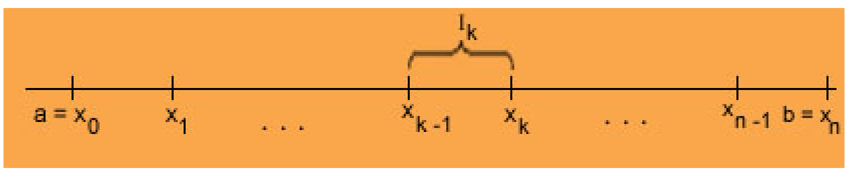
$$P = \{ x_0 = a, x_1, \ldots, x_n = b \}$$

is called a partition of the interval $[a,b]$.

(ii) A partition P with $n+1$ points divides the interval $[a,b]$ into n closed subintervals:

$$[a = x_0, x_1], [x_0, x_1] \ldots, [x_{n-1}, x_n].$$

A typical subintervals $I_k = [x_{k-1}, x_k]$ is called the k th-subinterval of the partition.



(iii) For each $1 \le k \le n$, the number

$$\Delta_k := |x_k - x_{k-1}|$$

is called the length of the subinterval $[x_{k-1}, x_k]$.

(iv) For a partition P, the number

$$\|P\| := \max\{ \Delta_k \mid 1 \le k \le n \},$$

is called the norm of the partition P.

Assumptions:

In order to define the area bounded by the graph of a function $f:[a,b] \to \mathbb{R}$, the $x-axis$ and the lines $x = a$ and $x = b$, we assume that f is a continuous function. As a consequence of this property not only f is bounded on every closed subinterval of $[a,b]$ in fact it attains its bounds in that interval.

16.1.2Definition:

Let $f:[a,b] \to \mathbb{R}$ be a continuous function and let

$$P = \{ x_0 = a, x_1, \ldots, x_n = b \}$$

be a partition of $[a,b]$.

(i) Let

$$m := \min\{f(x) \mid x \in [a,b]\},$$
$$M := \max\{f(x) \mid x \in [a,b]\},$$

$$M_k := \max\{f(x) : x \in [x_{k-1}, x_k]\}$$

and

$$m_k := \min\{f(x) : x \in [x_{k-1}, x_k]\}$$

for $1 \le k \le n$

(ii) Let

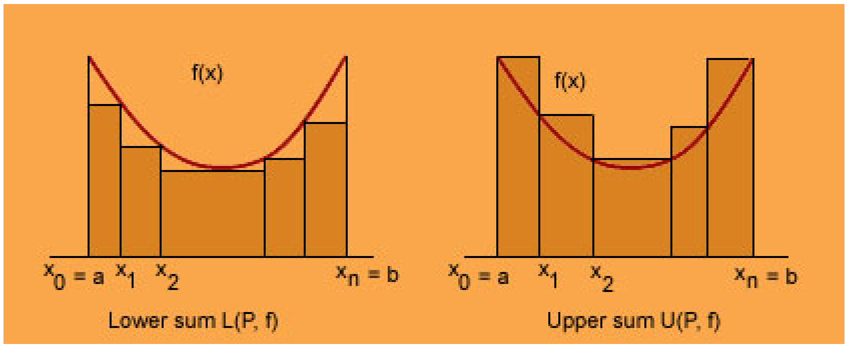
$$U(P,f) := \sum_{k=1}^{n} M_k \Delta_k \text{ and } L(P,f) := \sum_{k=1}^{n} m_k \Delta_k.$$

The sum $U(P,f)$ is called upper sum of f and the sum $L(P,f)$ is called lower sum of f for the partition P.

16.1.3 Note:

For $f(x) \ge 0, x \in [a,b]$, each $M_k \ge 0$ and $M_k \Delta_k$ is the area of the rectangle with base Δ_k and height M_k. The number $U(P,f)$ is the sum of the areas of all such rectangles. These rectangles cover of the region $S = \{(x,y) \mid a \le x \le b, y = f(x)\}$. Similarly, each $m_k \ge 0$ and $m_k \Delta_k$ is the area of the rectangle with base Δ_k and height m_k. The number $L(P,f)$ is the sum of all these rectangles which try to fill up the region $S = \{(x,y) \mid a \le x \le b, y = f(x)\}$. The sum $L(P,f)$ under estimates the 'area' of S and $U(P,f)$ over estimates the area of S, i.e.,

$$M(b-a) \ge U(P,f) \ge L(P,f) \ge m(b-a), \text{ for every partition } P \text{ of } [a,b].$$



Geometrically, the required area 'Area(S)' of the region S is captured between $U(P,f)$ and $L(P,f)$ i.e., for every partition P of $[a,b]$.

$$U(P,f) \ge \text{area }(S) \ge L(P,f). \qquad (1)$$

The natural question that arises is the following:

- Can we improve the approximations in (1) so that upper and lower sums come closer to the actual 'area(S)'?

To answer this, let us observe the following:

16.1.4 Lemma:

Let P and Q be two partitions of $[a,b]$ such that

$$P=\left\{a=x_0,\,x_1,\ldots,\,x_n=b\right\}$$

and

$$Q=\left\{a=x_0,\ldots,x_{i-1},x^*,x_i,\ldots,\,x_n=b\right\},$$

i.e., Q has all the points of P and an extra point x^* in the $i\,th$ subinterval. Then

$$U(P,f)\ \geq\ U(Q,f),\ \text{and}\ \ L(P,f)\leq\ L(Q,f).$$

PROOF

16.1.4 Lemma:

Let P and Q be two partitions of $[a,b]$ such that

$$P=\left\{a=x_0,\,x_1,\ldots,\,x_n=b\right\}$$

and

$$Q=\left\{a=x_0,\ldots,x_{i-1},x^*,x_i,\ldots,\,x_n=b\right\},$$

i.e., Q has all the points of P and an extra point x^* in the $i\,th$ subinterval. Then

$U(P,f)\geq U(Q,f),$ and $L(P,f)\leq L(P,f).$

Proof:

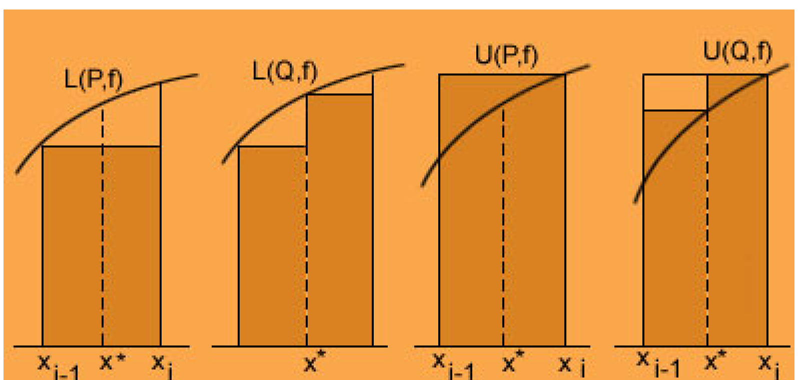


Figure 37: Caption Text

Since $\max\left\{f(x):x\in\left[x_{i-1},x_i\right]\right\}$ is greater than or equal to both $\max\left\{\left\{f(x):x\in\left[x_{i-1},x^*\right]\right\}\right\}$ and $\max\left\{f(x):x\in\left[x^*,x_i\right]\right\}$ clearly,

$$U(P,f)\ \geq\ U(Q,f),\ \text{and}\ \ L(P,f)\leq\ L(Q,f).$$

Thus, $U(P,f)$ becomes smaller and $L(P,f)$ becomes larger if we add more points to the partition P.

16.1.5 Definition:

(i) Let P and Q be two partition of $[a,b]$ such that every point of P is also a point of Q. Then we say Q is a
refinement of P.

(ii) A sequence $\{P_n\}_{n\geq 1}$ of partitions of $[a,b]$ is called a sequence of refinement partitions if for all $n \geq 1, P_{n+1}$ is a
refinement of P_n.

16.1.6Examples:

(i) For an interval $[a,b]$, a natural sequence $\{P_n\}_{n\geq 1}$ of refinement partitions is given by

$$P_n := \{a = x_0, x_1, \dots, x_{2^n} = b\}, \text{ where } x_i - x_{i-1} = \frac{b-a}{2^n} \text{ for every } i.$$

(ii) For an interval $[a,b]$, let,

$$P_n := \{a = x_0, x_1, \dots, x_{2^n} = b\}, \text{ where}$$
$$x_i - x_{i-1} = \frac{b-a}{n} \text{ for every } i.$$

Then $\{P_n\}_{n\geq 1}$ is not a sequence of refinement partition of $[a,b]$.

16.1.7Theorem:

Let $m, M \in \mathbb{R}$ be such that $m \leq f(x) \leq M$ for all $x \in [a,b]$ and let $\{P_n\}_{n\geq 1}$ be a sequence of refinement partitions of $[a,b]$. Then the following hold:

(i) The sequence $\{U(P_n, f)\}_{n\geq 1}$ is monotonically decreasing and is bounded below by $m(b-a)$.

(ii) The sequence $\{L(P_n, f)\}_{n\geq 1}$ is monotonically increasing and is bounded above by $M(b-a)$.

(iii) Both $\{U(P_n, f)\}_{n\geq 1}$ and $\{L(P_n, f)\}_{n\geq 1}$ are convergent sequences. In fact, for f is continuous,

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) =: A,$$

and the limit is independent of the sequence $\{P_n\}_{n\geq 1}$ of refinement partition of $[a,b]$.

PROOF

16.1.7 Theorem:

Let $m, M \in \mathbb{R}$ be such that $m \leq f(x) \leq M$ for all $x \in [a,b]$ and let $\{P_n\}_{n\geq 1}$ be a sequence of refinement
partitions of $[a,b]$. Then the following hold:

(i) The sequence $\{U(P_n, f)\}_{n\geq 1}$ is monotonically decreasing and is bounded below by $m(b-a)$.

(ii) The sequence $\{L(P_n, f)\}_{n\geq 1}$ is monotonically increasing and is bounded above by $M(b-a)$.

(iii) Both $\{U(P_n, f)\}_{n\geq 1}$ and $\{L(P_n, f)\}_{n\geq 1}$ are convergent sequences. In fact, for f is continuous,

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) =: A,$$

and the limit is independent of the sequence $\{P_n\}_{n\geq 1}$ of refinement partition of $[a,b]$.

Proof:

Proofs of (i) and (ii) follow directly from the lemma 16.1.6. Thus, by the completeness property of $\mathbb{R}$, both $\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f)$ exist. That for a continuous function, both these sequences converge to a common limit, and that limit is independent of the sequence $\{P_n\}_{n\geq 1}$ of refinement partitions of $[a,b]$ is technical, and we assume this.

This motivates for the following definition:

16.1.8 Definition:

For a continuous function $f:[a,b]\to\mathbb{R}$, the real number A as given by theorem 16.1.7 (ii) is called the definite integral or just the integral of f over $[a,b]$ and is denoted by

$$\int_a^b f(x)dx$$

16.1.9 Remarks:

(i) The notions of upper sums and lower sums, which we analyzed for continuous functions, can in fact be defined for any bounded function $f:[a,b]\to\mathbb{R}$. However, for such functions part (iii) of the above theorem need not hold. One says a bounded $f:[a,b]\to\mathbb{R}$ is Riemann-integral or simply integral, if there exists a sequence $\{P_n\}_{n\geq 1}$ of refinement partitions of $[a,b]$ and a real number A such that

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f).$$

In fact, if for a function f, there exists some sequence $\{P_n\}_{n\geq 1}$ of refinement partitions such that

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) = A,$$

then for every sequences of refinement partitions, the upper and the lower sum sequences converge to the same limit, namely, A. Thus, theorem 16.1.7 (iii) says the following:

Every continuous function $f:[a,b]\to\mathbb{R}$ is integrable.

In fact, if $f:[a,b]\to\mathbb{R}$ is a bounded function such that f has only finite number of discontinuities, say at $c_i, 1\leq i\leq k$, with $c_1 < c_2 < \ldots < c_k$, then it can be shown that f is integrable on $[a,b]$ and

$$\int_a^b f(x)dx = \sum_{i=1}^{k}\left(\int_{c_i}^{c_i+1} f(x)dx\right).$$

We shall assume this fact also.

(ii) Suppose $f:[a,b]\to\mathbb{R}$ is integrable. Then, by definition, there exists a sequence $\{P_n\}_{n\geq 1}$ of refinement partitions and a number A such that

$$\lim_{n\to\infty} U(P_n, f) = A = \lim_{n\to\infty} L(P_n, f).$$

In particular,

$$\lim_{n\to\infty}\left[U(P_n, f) - L(P_n, f)\right] = 0.$$

The converse of this statement also holds and we shall assume it.

16.1.10 Theorem:

A bounded function $f:[a,b]\to\mathbb{R}$ is integrable if and only if there exists a sequence $\{P_n\}_{n\geq 1}$ of refinement partitions of $[a,b]$ such that

$$\lim_{n\to\infty}\left[U(P_n, f) - L(P_n, f)\right] = 0.$$

16.1.11 Note:

In view of the above theorem, to check that a function $f:[a,b]\to\mathbb{R}$ is integrable, it is enough to produce a sequence $\{P_n\}_{n\geq 1}$ of refinement partitions of $[a,b]$ such that

$$\lim_{n\to\infty}\left[U(P_n, f) - L(P_n, f)\right] = 0.$$

16.1.12 Example:

(i) Let

$$f(x) = x^2, a \leq x \leq b.$$

We show that f is an integrable function. For $n \geq 1$, consider

$$P_n = \left\{a = x_0, x_1, \ldots, x_{2^n} = b\right\}$$

be a partition of $[a,b]$ obtained by dividing $[a,b]$ into 2^n equal parts, i.e.,

$$x_k - x_{k-1} = \frac{b-a}{2^n} \text{ for } k = 1, \ldots, 2^n.$$

Then, $\{P_n\}_{n\geq 1}$ is a sequence of refinement partitions and for all k, we have

$$m_k = Min\left\{f(x) = x^2 \mid x_{k-1} \leq x \leq x_k\right\} = x^2{}_{k-1},$$

$$M_k = Max\left\{f(x) = x^2 \mid x_{k-1} \leq x \leq x_k\right\} = x^2{}_k.$$

Thus,

$$
\begin{aligned}
U(P_n,f)-L(P_n,f) &= \sum_{k=1}^{2^n}\left(M_k-m_k\right)\\
&= \sum_{k=1}^{2^n}\left(x_k^2-x_{k-1}^2\right)\left(\frac{b-a}{2^n}\right)\\
&= \frac{b-a}{2^n}\left(\sum_{k=1}^{2^n}\left(x_k^2-x_{k-1}^2\right)\right)\\
&= \frac{b-a}{2^n}\left(b^2-a^2\right).
\end{aligned}
$$

Thus, $\lim\limits_{n\to\infty}\left[U(P_n,f)-L(P_n,f)\right]=0.$

Hence, $f(x)=x^2$ is integrable in $[a,b]$.

16.1.13 Note:

Proceeding on the same lines as in the above example, one can show that every monotonically increasing / decreasing (not necessarily continuous) function $f:[a,b]\to\mathbb{R}$ is integrable.

16.1.14 Example:

Let $f:[0,1]\to\{0,1\}$ be defined by

$$
f(x)=\begin{cases}1, & \text{if } x\in[0,1] \text{ is a rational,}\\ 0, & \text{if } x\in[0,1] \text{ is a irrational.}\end{cases}
$$

Let $P=\{0=x_0,x_1,\dots,x_n=1\}$ be any partition of $[0,1]$. Since there is a rational and an irrational number between any two real numbers, we have

$$
U(P,f)=1, \text{ and } L(P,f)=0.
$$

Thus, for every sequence $\{P_n\}_{n\geq 1}$ of refinement partitions of $[a,b]$

$$
\lim_{n\to\infty}U(P_n,f)=1\neq 0=\lim_{n\to\infty}L(P_n,f).
$$

Hence, f is a bounded function which is not integrable.

PRACTICE EXERCISES

(1) For the following partitions of $[0,1]$, compute $\|P\|$:

(i) $P_1=\left\{0,\frac{1}{7},\frac{1}{3},1\right\}.$

(ii) $P_2=\left\{0,\frac{1}{7},\frac{1}{4},\frac{3}{4},1\right\}.$

(iii) $P_3=P_1\cup P_2.$

Let be any two partitions of . Let . Show that is also a partition of and

(2) P_1, P_2 $[a,b]$ $P = P_1 \cup P_2$ P $[a,b]$ it is a

refinement of both P_1 and P_2 . (It is called the common refinement of P_1 and P_2). Further show that

$$\|P\| \le \min\{ \|P_1\|, \|P_2\| \}.$$

(3) Let $\{P_n\}_{n\ge1}$ be a sequence of partitions of $[a,b]$. Define

$$Q_1 := P_1,\ Q_n := P_n \cup Q_{n-1}, n \ge 2.$$

Show that $\{Q_n\}_{n\ge1}$ is a sequence of refinement partitions of $[a,b]$.

(4) Verify the claims of lemma 16.1.4 for the following:

$f(x) = x^2, 0 \le x \le 1,$

$P = \left\{0, \frac{1}{7}, \frac{1}{5}, \frac{2}{3}, 1\right\}, Q = \left\{0, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{3}, 1\right\}.$

(5) Using 16.1.5, show that every constant function $f:[a,b] \to \mathbb{R}$ is integrable. Using this and remark 16.1.9(i),

show that function

$f(x) = 1$ if $x \in [0,1]$ and $f(x) = 2$ if $x \in [1,2]$,

is Riemann integrable on $[0,2]$. Compute $\int_0^2 f(x)dx$ also.

(6) Show that every monotone function (not necessarily continuous) is integrable on every interval $[a,b]$.

Recap

In this section you have learnt the following

How to define the integral of a function.

[Section 16.2]

Objectives

In this section you will learn the following :

How to define the integral of a function as a limit of Riemann sums.

16.2 Integral as a Limit of Riemann sums

Through theorem 16.1.7 allows us to check whether a function is integrable or not, it is not very convenient to find $\int_a^b f(x)\,dx$. For this, we consider another way of approximating the required area.

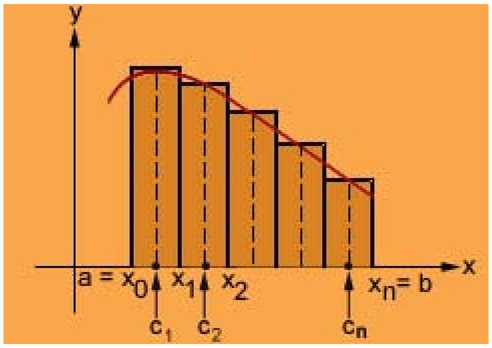
16.2.1 Definition:

Consider a function $f:[a,b]\to\mathbb{R}$. Given any partition

$$P=\{a=x_0<x_1,\ldots<x_n=b\}$$

of $[a,b]$, for $1\le k\le n$, choose $c_k\in\left[x_{k-1},x_k\right]$ arbitrarily and define the sum

$$S(P,f):=\sum_{k=1}^{n}f(c_k)(x_k-x_{k-1}).$$



The product $f(c_k)(x_k-x_{k-1})$ is the area of the rectangle over the interval $\left[x_{k-1},x_k\right]$ with height $f\left(c_k\right)$. The sum $S\left(P,f\right)$ is called a Reimann sum of f with respect to the partition P and the choice of the points $c_1,c_2,\ldots,c_n$.

Note that for every partition P, the sum $S\left(P,f\right)$ depends not only on the partition P, but also on the choice of points $c_k\in\left[x_{k-1},x_k\right]$. However, for every partition P, the following holds:

$$U\left(P,f\right)\ge S\left(P,f\right)\ge L\left(P,f\right).$$

We hope that as we make $\|P\|$ smaller and smaller, the Riemann sum $S\left(P,f\right)$ will approximate the required area better and better. This actually does happen for integrable functions. In fact, we have the following:

16.2.2 Theorem (Riemann):

A function $f:[a,b]\to\mathbb{R}$ is integrable if and only if there exists $A\in\mathbb{R}$ such that for every sequence $\{P_n\}_{n\geq 1}$ of partitions of $[a,b]$ with $\|P_n\|\to 0$, and every sequence $\{S(P_n,f)\}_{n\geq 1}$ of Riemann sums

$$\lim_{n\to\infty} S(P_n,f)=A.$$

Further, in that case,

$$A=\int_a^b f(x)dx$$

16.2.3 Note:

(i) In view of the above theorem, for a function f, which we know is integrable, to compute $\int_a^b f(x)dx$ we can use

any convenient sequence $\{P_n\}_{n\geq 1}$ of partition of $[a,b]$ with $\|P_n\|\to 0$ and compute

$$\int_a^b f(x)dx=\lim_{n\to\infty} S(P_n,f)$$

(ii) To define Riemann sums for a function $f:[a,b]\to\mathbb{R}$, we do not require f to be bounded. However, it can be
proved that, if f is integrable, then f is also bounded.

16.2.4 Example:

Let

$$f(x)=x^2,\ a\leq x\leq b$$

We know that f is a continuous function, and hence it is integrable. To compute its integral, for every $n\geq 1$, consider the partition $P_n=\{a=x_0, x_1,\ldots,x_n=b\}$ obtained by dividing $[a,b]$ into n equal parts, i.e.,

$$x_k-x_{k-1}=\left(\frac{b-a}{n}\right), 1\leq k\leq n.$$

Note that,

$$\|P_n\|=\left(\frac{b-a}{n}\right)\to 0 \text{ as } n\to\infty.$$

Let us choose $c_k := x_k\in\left[x_{k-1},x_k\right]$ for all $k=1,2,\ldots$

Let $\Delta=(b-a)/n$. Then,

$$S(P_n,f)=U(P_n,f)=\sum_{k=1}^{n} x_k^2\left(x_k-x_{k-1}\right)$$
$$=\Delta\left[\sum_{k=1}^{n}(a+k\Delta)^2\right]$$
$$=\Delta\left[\sum_{k=1}^{n}a^2+\sum_{k=1}^{n}2ak\Delta+\sum_{k=1}^{n}k^2\Delta^2\right]$$
$$=\Delta\left[a^2n+2a\Delta\left(\frac{n(n+1)}{2}\right)+\Delta^2\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$
$$=a^2(b-a)+a(b-a)^2\left(\frac{n+1}{n}\right)+\frac{(b-a)^3}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right).$$

Thus,

$$\lim_{n\to\infty}S(P_n,f)=a^2(b-a)+a(b-a)^2+\left(\frac{(b-a)^3}{3}\right)$$
$$=\frac{b-a}{3}\left(3a^2+3a(b-a)+(b-a)^2\right)$$
$$=\left(\frac{b^3-a^3}{3}\right).$$

Hence,

$$\int_a^b f(x)dx=\frac{b^3-a^3}{3}.$$

Next, we describe some of the important properties of the integral.

16.2.5Theorem (Properties of Integral):

Let $f,g:[a,b]\to\mathbb{R}$ be bounded functions.

(i) If f is integrable and $f(x)\geq 0\ \forall\ x$, then

$$\int_a^b f(x)dx\geq 0.$$

(ii) If f is integrable, then $|f|$ is integrable and

$$\left|\int_a^b f(x)dx\right|\leq\int_a^b|f(x)|dx.$$

(iii) If f is integrable and $k\in\mathbb{R}$, then kf is integrable and

$$\int_a^b (kf)(x)dx=k\left(\int_a^b f(x)dx\right).$$

(iv) If f and g are integrable, then $f\pm g$ are integrable and

$$\int_a^b (f\pm g)(x)\,dx=\int_a^b f(x)\,dx\pm\int_a^b g(x)\,dx.$$

(v) If f and g are integrable and $f(x)\geq g(x)$ for all $x\in[a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

(vi) If f is integrable on $[a,b]$, then f is integrable over every interval $[c,d] \subseteq [a,b]$ and

$$\int_a^b f(t)\,dt = \int_a^c f(t) \; + \int_c^b f(t)dt \;, a \leq c \leq b.$$

This is called the additive property of the integral.

(we define $\int_a^a f(t)\,dt = 0$ for every f).

(vii) If f and g are integrable, then fg is also integrable.

16.2.5 Theorem (Properties of Integral):

Let $f, g : [a,b] \to \mathbb{R}$ be bounded functions.

(i) If f is integrable and $f(x) \geq 0 \;\forall\; x$, then $\int_a^b f(x)dx \geq 0.$

(ii) If f is integrable, then $|f|$ is integrable and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$

(iii) If f is integrable and $k \in \mathbb{R}$, then kf is integrable and $\int_a^b (kf)(x)dx = k\left(\int_a^b f(x)dx\right)$.

(iv) If f and g are integrable, then $f \pm g$ are integrable and $\int_a^b (f \pm g)(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

(v) If f and g are integrable and $f(x) \geq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$

(vi) If f is integrable on $[a,b]$, then f is integrable over every interval $[c,d] \subseteq [a,b]$ and

$$\int_a^b f(t)\,dt = \int_a^c f(t) \; + \int_c^b f(t)dt \;, a \leq c \leq b.$$

This is called the additive property of the integral.

(we define $\int_a^a f(t)\,dt = 0$ for every f).

(vii) If f and g are integrable, then fg is also integrable.

Proof:

Proofs of all these properties follow from the properties of limits. Though not difficult, the proofs are technical. We shall assume them. Interested reader can refer a book on Real Analysis.

16.2.6 Theorem (Mean Value Property for Definite Integrals):

If $f : [a,b] \to \mathbb{R}$ is continuous, then there is at least one point $c \in [a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

16.2.6Theorem (Mean Value Property for Definite Integrals):

If $f:[a,b]\to\mathbb{R}$ is continuous, then there is at least one point $c\in[a,b]$ such that
$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Proof:

Since f is continuous on $[a,b]$, it is integrable. Let
$m:=\min\{f(x)\mid x\in[a,b]\}$ and $M:=\max\{f(x)\mid x\in[a,b]\}$.
Then,
$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a),$$
i.e.,
$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

Thus,
$$\frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\in R(f),$$
the range of f. Thus, by the intermediate value property for continuous functions, there exists a point $c\in[a,b]$ such that
$$f(c)=\frac{1}{b-a}\left(\int_a^b f(t)dt\right).$$

16.2.7 Note:

(i) Average Value of a function:

For a integrable function $f:[a,b]\to\mathbb{R}$, the number
$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$
can be thought of as the average value of f in the interval $[a,b]$. Theorem 16.2.6 says that the average value of a function is attained at some point if the function is continuous.

(ii) Consider $f:[-1,1]\to\mathbb{R}$ given by $f(x)=0$ if $-1\le x\le 0$ and $f(x)=1$ if $0<x\le 1$.

The average value of f equals $1/2$. But $f(c)\ne 1/2$ for any $c\in[-1,1]$. Thus, the continuity hypothesis in theorem 16.2.6 can not be dropped.

PRACTICE EXERCISES

1. Find an interval $[a,b]$ a function $f:[a,b]\to\mathbb{R}$, and a partition $P_n, n\ge 1$, for which S_n given below is

the
Riemann sum $S(P_n, f)$:

(i) $S_n = \frac{1}{n^{5/2}}\left(\sum_{i=1}^{n} i^{3/2}\right)$.

(ii) $S_n = \sum_{i=1}^{n} \frac{n}{i^2+n^2}$.

(iii) $S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{in+n^2}}$.

(iv) $S_n = \frac{1}{n}\sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right)$.

(v) $S_n = \frac{1}{n}\left\{\sum_{i=1}^{n}\left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^{2}\right\}$.

2. Assuming that f is integrable on a suitable interval, express

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right]$$

as an integral for a suitable function $f:[a,b]\to\mathbb{R}$.

3. For the function

$$f(x) = x^3, 0 \le x \le b$$

obtain a Riemann sum $S(P_n, f)$ for a suitable sequence $\{P_n\}_{n\ge 1}$ of partitions and compute

$$\int_0^b x^3 dx \cdot$$

4. Let $f:[-a,+a]\to\mathbb{R}$ be a nonzero integrable function such that

$$f(-x) = -f(x) \text{ for every } x\in[-a,+a].$$

Show that

$$\int_0^b f(x)\,dx = 0$$

even though $f(x)\neq 0$ for any $x\in[-a,+a]$.

5. Let $f:[a,b]\to\mathbb{R}$ be an integrable function such that

$$f(x) = 0 \text{ for all } x\in[a,c)\cup(d,b],$$

where $a \le c < d \le b$. Show that

$$\int_a^b f(x)\,dx = \int_c^d f(x)\,dx \cdot$$

6. Prove that for all $h \in \mathbb{R}$ and $m = 1, 2, \ldots,$

$$2\sin\left(\frac{h}{2}\right)\left(\sin h + \sin 2h + \ldots + \sin(mh)\right) = \cos\left(\frac{h}{2}\right) - \cos\left(m + \frac{1}{2}\right)h.$$

Hence, find $\int_0^{\frac{\pi}{2}} \sin x \, dx$ by computing the Riemann sums $S(P_n, f)$ for a suitable sequence $\{P_n\}_{n \geq 1}$ of partitions and taking limits.

7. Let $f : [a, b] \to \mathbb{R}$ be a non-negative continuous function such that $f(c) > 0$, for some $c \in [a, b]$. Show that there exist

$\alpha, \delta > 0$, such that

$$f(x) \geq \alpha, \text{ for all } x \in [c - \delta, c + \delta].$$

Hence, deuce that

$$\int_a^b f(x) \, dx > 0$$

Hence,

if f is non-negative continuous and $\int_a^b f(x) \, dx = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

8. Give an example of a nonzero integrable function $f : [a, b] \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in [a, b]$, but

$$\int_a^b f(x) \, dx = 0 \cdot$$

Recap

In this section you have learnt the following

How to define the integral of a function as a limit of Riemann sums.