Module 10 : Scaler fields, Limit and Continuity

Lecture 29 : Limit of scaler fields [Section 29.1]

Objectives

In this section you will learn the following :

The notion of limits for scalar fields.

29.1 Limits of scalar fields

Through level curves and contour lines give some information about the function $f(x,y)$, they are not sufficient to draw any definite conclusion about f. One would like to define concepts, as for functions of a single variable, which will help us analyze $f(x,y)$ analytically. Recall that, for a function f of one variable, the fundamental concept, which is used again and again to define various other concepts, is that of limit of f. Intuitively, as for a function of one variable, we say $f(x,y)$ approaches a limit $L \in \mathbb{R}$, as (x,y) approaches (a,b), if $f(x,y)$ lies 'arbitrarily close' to L for all points 'sufficiently close' to (a,b). To describe this concept of 'closeness' in $\mathbb{R}^2$ we have the notion of distance between points. So, it is natural to say that a function f of two variables has a limit L as $(x,y) \in \mathbb{R}^2$ 'approaches' $P(a,b)$ if the distance between $f(x,y)$ and L is 'arbitrarily' small for all points P sufficiently close to P. To make this definition meaningful, let us assume for the time being that f is defined at all points 'sufficiently close' to P, possibly not at P. That is, we assume that there exists some $r > 0$ such that $B_0((a,b),r) \subseteq D$. And we have the following definition.

29.1.1 Definition:

Let

$$f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

and $P(a,b) \in \mathbb{R}^2$ be such that $B_0(P,r) \subseteq D$ for some $r > 0$. We say $f(x,y)$ approaches a **limit** L as (x,y) approaches P, and write

$$\lim_{(x,y) \mapsto (a,b)} f(x,y) = L,$$

if for every $\varepsilon > 0$, there is some $0 < \delta < r$, such that

$$0 < \| (x,y) - (a,b) \| < \delta \text{ implies } | f(x,y) - L | < \varepsilon,$$

that is,

$$(x,y) \in B_0((a,b),\delta) \text{ implies } |f(x,y)-L| < \varepsilon.$$

29.1.2Examples:

(i) Let

$f(x,y) = x$ and $g(x,y) = y$ for all $x,y \in \mathbb{R}$.

Let $a,b \in \mathbb{R}$. To analyse $\lim_{(x,y)\to(a,b)} f(x,y)$, we first have to guess whether this limit can exists or not and if it exits, what is the possible value. Suppose it exists and is L. Then to prove that $\lim_{(x,y)\mapsto(a,b)} f(x,y) = L$, we have to find $\delta > 0$, for given $\varepsilon > 0$, such that

$$0 < \|(x,y)-(a,b)\| < \delta \text{ implies } |f(x,y)-L| = |x-L| < \varepsilon.$$

Since

$|x-L| \leq \|(x,y)-(L,M)\|$ for all $M \in R$,

it is obvious that if we take $L = a$ and chose $\delta = \varepsilon$, then for a given $\varepsilon > 0$,

$$0 < \|(x,y)-(a,b)\| < \delta \text{ implies } |x-a| < \|(x,y)-(a,b)\| < \delta = \varepsilon,$$

i.e.,

$|f(x,y)-a| < \varepsilon$ for $\|(x,y)-(a,b)\| < \delta$.

Hence,

$$\lim_{(x,y)\mapsto(a,b)} f(x,y) = a.$$

Similarly

$$\lim_{(x,y)\mapsto(a,b)} g(x,y) = b.$$

The functions f and g are **projection** maps on the x-axis and y-axis, respectively.

(ii)

Let

$$f(x,y) = \frac{xy}{x^2+1}, \quad (x,y) \in \mathbb{R}^2.$$

We claim that

$$\lim_{(x,y)\mapsto(0,0)} f(x,y) = 0.$$

Note that

$$|f(x,y)-0| = \left|\frac{xy}{x^2+1}\right| \leq |x||y| \leq \|(x,y)\|\|(x,y)\|.$$

Thus, given $\varepsilon > 0$, if we chose δ such that $\delta^2 < \varepsilon$, then

$$0 < \|(x,y)\| < \delta \text{ implies } \left|\frac{xy}{x^2+1}\right| \leq \delta^2 < \varepsilon.$$

Hence,

$$\lim_{(x,y)\mapsto(0,0)} f(x,y) = 0.$$

(iii)

Let

$$f(x,y)=\begin{cases} y\sin\dfrac{1}{x}, & \text{if } x\neq 0\\ \alpha, & \text{if } x=0,\end{cases}$$

where α is some fixed real number. Then

$$\lim_{(x,y)\mapsto(0,0)}\left(y\sin\frac{1}{x}\right)=0,$$

because given any $\varepsilon>0$, we can take $\delta=\varepsilon$ and note that

$$0<|x|<\delta,\ |y|<\delta \text{ implies } \left|y\sin\frac{1}{x}\right|\leq|y|<\delta=\varepsilon.$$

In the case of function of a real variable, the existence of limit at a point implied that both the left hand and right hand limit exist and are equal, That is, $f(x)$ approached the same value when x approached the point from the left or from the right. In $\mathbb{R}^2$, a point (a,b) can be approached to along many different paths, and if the $\lim_{(x,y)\mapsto(a,b)} f(x,y)=L$, then $f(x,y)$ should approach L irrespective of the path along which one approaches the point (a,b). Hence, we have the next theorem.

29.1.3Theorem (Path independence of the limit) :

Let $P(a,b)\in D\subseteq\mathbb{R}^2$ *be such that* $B_0(P,r)\subseteq D$ *for some* $r>0$. *Let*

$$f: D\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$$

be such that

$$\lim_{(x,y)\mapsto(a,b)} f(x,y)=L \text{ exists.}$$

If $g:\mathbb{R}\rightarrow D\subseteq\mathbb{R}^2$ *is any path such that*

$$\lim_{x\to a} g(x)=b,$$

and

$$h(x):=f(x,g(x)),\ x\in I,$$

then,

$$\lim_{x\to a} h(x)=L.$$

PROOF

29.1.3Theorem (Path independence of the limit) :

Let $P(a,b)\in D\subseteq\mathbb{R}^2$ *be such that* $B_0(P,r)\subseteq D$ *for some* $r>0$. *Let*

$$f: D\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$$

be such that

$$\lim_{(x,y)\mapsto(a,b)} f(x,y)=L \text{ exists.}$$

If $g:\mathbb{R}\rightarrow D\subseteq\mathbb{R}^2$ *is any path such that*

$$\lim_{x\to a} g(x)=b,$$

and

$$h(x) := f(x, g(x)),\ x \in I,$$

then,

$$\lim_{x \to a} h(x) = L.$$

Proof

Given $\varepsilon > 0$, choose $\delta_1 > 0$ such that

$0 < \| (x, y) - (a, b) \| < \delta_1$ implies $| f(x, y) - L | < \varepsilon$. ----------(19)

Also, choose $\delta_2 > 0,\ \delta_2 < \dfrac{\delta_1}{2}$, such that

$0 < | x - a | < \delta_2$ implies $| g(x) - b | < \dfrac{\delta_1}{2}$.

Then, for $0 < | x - a | < \delta_2$,

$$\| (x, g(x)) - (a, b) \| = \sqrt{(x-a)^2 + (g(x)-b)^2}$$
$$\leq \sqrt{\delta_2^2 + \frac{\delta_1^2}{4}} \leq \sqrt{\frac{\delta_1^2}{2}} < \delta_1.$$

Hence by (19), we have

$| f(x, g(x)) - L | < \varepsilon$, that is, $| h(x) - L | < \varepsilon$.

29.1.3 Theorem (Path independence of the limit) :

Let $P(a,b) \in D \subseteq \mathbb{R}^2$ *be such that* $B_0(P, r) \subseteq D$ *for some* $r > 0$. *Let*

$$f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

be such that

$$\lim_{(x,y) \mapsto (a,b)} f(x, y) = L \text{ exists.}$$

If $g : \mathbb{R} \to D \subseteq \mathbb{R}^2$ *is any path such that*

$$\lim_{x \to a} g(x) = b,$$

and

$$h(x) := f(x, g(x)),\ x \in I,$$

then,

$$\lim_{x \to a} h(x) = L.$$

Proof

Given $\varepsilon > 0$, choose $\delta_1 > 0$ such that

$0 < \| (x, y) - (a, b) \| < \delta_1$ implies $| f(x, y) - L | < \varepsilon$. ----------(19)

Also, choose $\delta_2 > 0,\ \delta_2 < \dfrac{\delta_1}{2}$, such that

$0 < | x - a | < \delta_2$ implies $| g(x) - b | < \dfrac{\delta_1}{2}$.

Then, for $0 < | x - a | < \delta_2$,

$$\| (x, g(x)) - (a, b) \| = \sqrt{(x-a)^2 + (g(x)-b)^2}$$
$$\leq \sqrt{\delta_2^2 + \frac{\delta_1^2}{4}} \leq \sqrt{\frac{\delta_1^2}{2}} < \delta_1.$$

Hence by (19), we have

$| f(x, g(x)) - L | < \varepsilon$, that is, $| h(x) - L | < \varepsilon$.

29.1.4 Note:

From the above theorem,

if $\lim_{(x,y)\mapsto(a,b)} f(x,y) = L$ exists,

then $f(x, g(x)) \to L$ as $x \to a$ along every path $y = g(x)$ through (a,b).

That is, a limit of $f(x,y)$ exists and is same along every path to (a,b). This gives us a test for the nonexistence of a limit.

29.1.5 Corollary (Test for non-existence of limit):

If for a function $f(x,y)$

either the limit of $f(x,y)$ at (a,b) does not exits along at least one path through (a,b)

or

$f(x,y)$ has different limits along at least two different paths through (a,b)

then

$\lim_{(x,y)\mapsto(a,b)} f(x,y)$ does not exits.

29.1.6 Example:

Consider the function

$$f(x,y) = \frac{x+y}{x-y}, \ x \neq y \text{ and } f(x,y) = 0, \ x = y.$$

To analyse $\lim_{(x,y)\to(0,0)} f(x,y)$, let us find this limit along some simple paths through $(0,0)$, for example, along $y = g(x) = mx$, the line through the origin with slop m. Note that

$$y \to 0 \text{ as } x \to 0.$$

Since

$$f(x, mx) = \frac{x+mx}{x-mx} = \frac{1+m}{1-m},$$

we have

$$\lim_{x\to 0} f(x, mx) = \frac{1+m}{1-m},$$

which is different along different lines. Hence,

$\lim_{(x,y)\mapsto(0,0)} f(x,y)$ does not exist.

To define $\lim_{(x,y)\to(a,b)} f(x,y)$ we required that f be defined at all points sufficiently close to (a,b), except possibly at (a,b). This condition can be relaxed. All we need is that f should be defined at least at points close to (a,b), as close as we want. More precisely, we have the following:

29.1.7 Definition:

Suppose $(a,b) \in D$ is such that

for all $r > 0$, $D \cap (B_0(a,b), r) \neq \varnothing$.

Such a point is called a **limit point** of D.

29.1.8 Example:

The point $(0,0)$ is a limit point each of the domains shown in the figure.

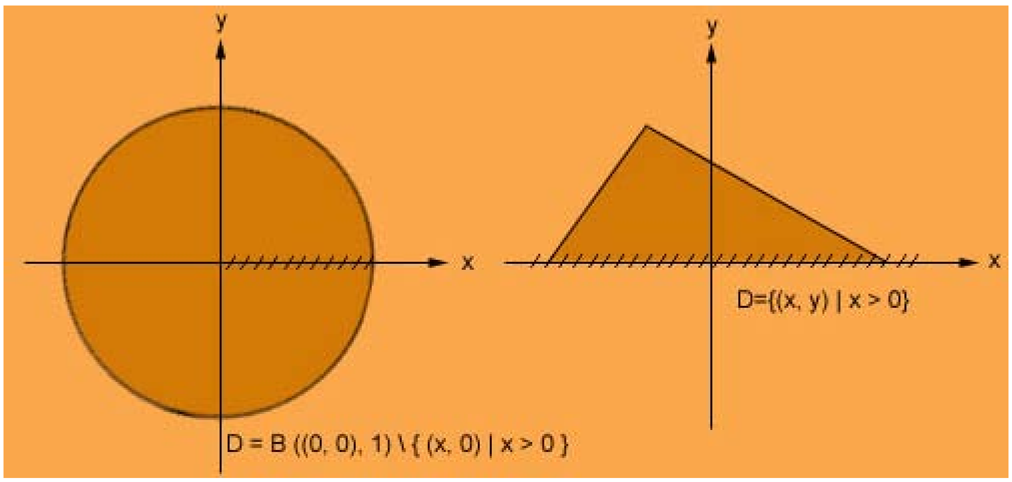

Figure 1. $(0,0)$ is a limit point

29.1.9 Note:

A limit point of a set D need not belong to D. However, if $(a,b) \in \mathbb{R}^2$ is a limit point of D, then there exists a sequence $\{P_n\}_{n \geq 1}$ of points of D converging to (a,b).

29.1.10 Definition:

Let $(x_0, y_0) \in \mathbb{R}^2$ is a limit point of $D \subseteq \mathbb{R}^2$ and let $f : D \subseteq \mathbb{R}^2$. We say

$$\lim_{(x,y) \mapsto (x_0,y_0)} f(x,y) = L$$

if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$(x,y) \in D \cap B_0((x_0,y_0),\delta) \text{ implies } |f(x,y) - L| < \varepsilon.$$

29.1.11 Example:

Let

$$f(x) = y \sin\left(\frac{1}{x}\right), \text{ for all } (x,y) \text{ with } x \neq 0.$$

The domain of f is the set

$$D = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0\}.$$

The point $(0,0) \notin D$, but is a limit point of D. For any given $\varepsilon > 0$, if we choose $\delta = \varepsilon$, then

$$0 < |x| < \delta \ \text{ implies } \ |f(x,y) - 0| = |x \sin \frac{1}{y}| < |x| < \delta = \varepsilon.$$

Hence,

$$\lim_{(x,y) \to (0,0)} f(x,y) = 0.$$

QUIZ

PRACTICE EXERCISES

1. Using $\varepsilon - \delta$ definition, analyze $\lim_{(x,y)\to(0,0)} f(x,y)$ for the following functions:

(i) $f(x,y) = \dfrac{xy}{x^2 - y^2}, x \neq \pm y.$

(ii) $f(x,y) = \dfrac{x^2 y}{x^2 + y^2}, (x,y) \neq (0,0).$

(iii) $f(x,y) = \dfrac{x^3 y}{x^6 + y^2}, (x,y) \neq (0,0).$

(iv) $f(x,y) = \dfrac{xy^3 - x^3 y}{x^2 + y^2}, (x,y) \neq (0,0).$

(v) $f(x,y) = \dfrac{\sin^2(x+y)}{|x| + |y|}, (x,y) \neq (0,0).$

(vi) $f(x,y) = ||x| - |y|| - |x| - |y|.$

Answers

2. Let

$$f(x,y) = \begin{cases} x \sin \dfrac{1}{y}, & \text{if } y \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists but $\lim_{y \to 0} f(x,y)$ does not exist for every x fixed.

3. Let $f : B_0\big((0,0), r\big) \to \mathbb{R}$ for some $r > 0$ be such that

$$\lim_{(x,y)\to(0,0)} f(x,y) = l.$$

If both $\lim_{x\to 0} f(x,y)$ and $\lim_{y \to 0} f(x,y)$ exists for all x, y is a neighborhood of (0,0), show that

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = l.$$

4. Let

$$f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2} \quad \text{for} \quad (x,y) \neq (0,0).$$

Show that both $\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$ and $\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = 0$ exist and are equal but $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Thus, the converse of exercise 3 above need not hold.

5. For

$$f(x,y) = \frac{x-y}{x+y}, x+y \neq 0,$$

analyse the existence of the following limits:

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right), \; \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) \; \text{ and } \; \lim_{(x,y)\mapsto(0,0)} f(x,y).$$

6. Another way of analysing $\lim_{(x,y)\to(0,0)} f(x,y)$ is to write

$$x = r\cos\theta, y = r\sin\theta,$$

and noting that limit of f as $(x,y) \longrightarrow (0,0)$ is equivalent to analysing

$$\lim_{(r\,\theta)\mapsto(0,0)} f(r\cos\theta, r\sin\theta).$$

Use this to analyse the following limits:

(i) $\displaystyle\lim_{(x,y)\mapsto(0,0)}\left(\frac{x^3-xy^2}{x^2+y^2}\right).$

(ii) $\displaystyle\lim_{(x,y)\mapsto(0,0)}\tan^{-1}\left(\frac{|x|+|y|}{x^2+y^2}\right).$

(iii) $\displaystyle\lim_{(x,y)\mapsto(0,0)}\left(\frac{y^2}{x^2+y^2}\right).$

Answers

Recap

In this section you have learnt the following

- The notion of limits for scalar fields.

[Section 29.2]

Objectives

In this section you will learn the following :

Theorem that help us to compute the limits for scalar fields.

29.2 Limit theorems

As in the case of function of a single variable, the following theorem is useful for computing limits.

29.2.1 Theorem:

Let $f, g : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $(x_0, y_0) \in D$ be a limit point of D. Then the following hold:

(i) $\lim_{(x,y)\mapsto(x_0,y_0)} f(x,y) = l$ if and only if for every sequence $\{(x_n, y_n)\}_{n\geq 0}$ in D such that $(x_n, y_n) \to (x_0, y_0)$ implies that $\lim\limits_{n\to\infty} f(x_n, y_n) = l.$

(ii) If

$$\lim_{(x,y)\mapsto(x_0,y_0)} f(x,y) = l, \quad \lim_{(x,y)\mapsto(x_0,y_0)} g(x,y) = m$$

and $r \in \mathbb{R}$ *,then*

$$\lim_{(x,y)\mapsto(x_0,y_0)} (f \pm g)(x,y) = l \pm m,$$

$$\lim_{(x,y)\mapsto(x_0,y_0)} (rf)(x,y) = rm,$$

$$\lim_{(x,y)\mapsto(x_0,y_0)} (fg)(x,y) = lm,$$

and if $m \neq 0$,

$$\lim_{(x,y)\mapsto(x_0,y_0)} \left(\frac{f}{g}\right)(x,y) = \frac{l}{m}.$$

Proof

Proofs of all the statements are similar to that of functions of a single variable. To exhibit this, we prove (i). Let $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = l$ *and choose* $\delta > 0$ *such that*

$$0 < \| (x,y) - (x_0, y_0) \| < \delta \text{ implies } | f(x,y) - l | < \varepsilon. \qquad \text{----------(20)}$$

Now, since $(x_n, y_n) \to (x_0, y_0)$, *choose* N *such that*

$$| (x_n, y_n) - (x_0, y_0) | < \delta \quad \forall n \geq N. \qquad \text{----------(21)}$$

From (20) and (21), we have

$$|f(x_n, y_n) - l| < \varepsilon \quad \forall\, n \geq N.$$

Hence

$$\lim_{n\to\infty} f(x_n, y_n) = l.$$

Conversely, let

$$(x_n, y_n) \to (x_0, y_0) \text{ imply } \lim_{n\to\infty} f(x_n, y_n) = l.$$

Suppose

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \neq l.$$

Then, there exists some $\varepsilon > 0$ *such that for all* $n \geq 1$, *there exists* $(x_n, y_n) \in D$ *with the property that*

$$\|(x_n, y_n) - (x_0, y_0)\| < \frac{1}{n} \text{ but } |f(x_n, y_n) - l| > \varepsilon.$$

Then, clearly for this sequence

$$(x_n, y_n) \to (x_0, y_0) \text{ but } f(x_n, y_n) \not\to l,$$

a contradiction. Hence

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = l.$$

We leave the proof of other parts as exercise for the reader.

Back

29.2.2Example:

Let us compute

$$\lim_{(x,y)\to(1,2)} \left(\frac{x+y}{x-y}\right).$$

Since, for $f(x,y) = x$ and $g(x,y) = y$,

$$\lim_{(x,y)\to(a,b)} f(x,y) = a \text{ and } \lim_{(x,y)\to(a,b)} g(x,y) = b,$$

using theorem 29.2.1,

$$\lim_{(x,y)\to(1,2)} \left(\frac{x+y}{x-y}\right) = \left(\frac{1+2}{1-2}\right) = -3.$$

29.2.3Theorem (Sandwich) :

Let $f, g, h : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ *and* $(x_0, y_0) \in D$ *be a limit point. Let there exist some* $\delta > 0$ *such that*

$$g(x,y) \leq f(x,y) \leq h(x,y) \text{ for all } (x,y) \in B_0((x_0,y_0), \delta) \cap D.$$

If

$$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L = \lim_{(x,y)\to(x_0,y_0)} h(x,y),$$

then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L.$$

Proof

Easy to prove and left as exercises.

29.2.4Example:

We want to compute

$$\lim_{(x,y)\to(0,0)} \left(\frac{\tan^{-1}(xy)}{xy} \right).$$

From the Maclaurin series expansion of $\tan^{-1}(\varphi)$, it follows that for φ in a neighborhood of $\varphi = 0$, except at 0,

$$1 - \frac{\varphi^3}{3} < \frac{\tan^{-1}\varphi}{\varphi} < 1.$$

Hence, in a neighborhood of $(0,0)$, except the axes we have

$$1 - \frac{x^2 y^2}{3} < \frac{\tan^{-1}(xy)}{xy} < 1.$$

Thus, using sandwich theorem,

$$\lim_{(x,y)\to(0,0)} \frac{\tan^{-1}(xy)}{xy} = 1.$$

PRACTICE EXERCISES

1. Analyze the following:

(i) $\lim_{(x,y,z)\to(0,0,0)} \left(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z \right)$

(ii) $\lim_{(x,y,z)\to(1,0,0)} \left(\frac{1}{|xz| + |y|} \right)$

Answers

2. Using Maclaurin series for $\cos\varphi$, deduce that

$$2|xy| - \frac{x^2 y^2}{6} < 4\left(1 - \cos\sqrt{|xy|} < 2|xy|\right)$$

and hence deduce that

$$\lim_{(x,y)\to(0,0)} \left(\frac{4 - 4\cos\sqrt{|xy|}}{|xy|} \right) = 2.$$

3. Using limit theorems, justify the following :

(i)

$$\lim_{(x,y)\to(0,0)} xy\left(\frac{x^2-y^2}{x^2+y^2}\right)=0$$

(ii) $\displaystyle\lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}=0.$

(iii) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} e^{x+y}\cos z=1.$

(iv) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{2+\cos x}=0.$

Recap

In this section you have learnt the following

- Theorem that help us to compute the limits for scalar fields.