Module 15 : Vector fields, Gradient, Divergence and Curl

Lecture 44 : Gradient Divergence and Curl [Section 44.1]

Objectives

In this section you will learn the following :

- The divergence of a vector field.
- The curl of a vector field.
- Their physical significance.

Divergence of a vector field.

Curl of a vector field.

44.1 Divergence of a vector field

44.1.1 Definition

Let $\mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector-field with components F_1, F_2, F_3. Then, the scalar field

$$\operatorname{div}(\mathbf{F}) : D \subseteq \mathbb{R}^3 \to \mathbb{R},$$

defined by

$$\operatorname{div}(\mathbf{F})(P) := \frac{\partial F_1}{\partial x}(P) + \frac{\partial F_2}{\partial y}(P) + \frac{\partial F_3}{\partial z}(P),\ P \in D,$$

is called the **divergence** of the vector-field F.

44.1.2 Example

1. **Divergence of an inverse square vector-field:**

   Let

   $$\mathbf{F}(x,y,z) = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2+y^2+z^2)^{3/2}},\ (x,y,z) \neq (0,0,0).$$

   Then,

   $$\mathbf{F}(x,y,z) = \frac{x}{r^3}\mathbf{i} + \frac{y}{r^3}\mathbf{j} + \frac{3}{r^3}\mathbf{k},$$

   where

   $$r^2 = (x^2+y^2+z^2).$$

   It is easy to see that

   $$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial r}{\partial z} = \frac{z}{r}.$$

   Thus

   $$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right).$$

   As

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)=\frac{r^3-x(3r^2)(x/r)}{(r^3)^2}=\frac{1}{r^3}-\frac{3x^2}{r^5},$$
$$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)=\frac{1}{r^3}-\frac{3y^2}{r^5},\ \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)=\frac{1}{r^3}-\frac{3z^2}{r^5},$$

we get

$$\begin{aligned}\operatorname{div}(\mathbf{F})&=\frac{3}{r^3}-\frac{3x^2+3y^2+3x^2}{r^5}\\&=\frac{3}{r^3}-\frac{3r^2}{r^5}=0.\end{aligned}$$

2. Let
$$\mathbf{F}(x,y,z)=e^{xy}\ \mathbf{i}-\cos y\,\mathbf{j}+\sin^2 z\,\mathbf{k}.$$

Then

$$\begin{aligned}\operatorname{div}(\mathbf{F})&=\frac{\partial}{\partial x}(e^{xy})+\frac{\partial}{\partial y}(\cos y)+\frac{\partial}{\partial z}(\sin^2 z)\\&=ye^{xy}-\sin y+2\sin z\cos z.\end{aligned}$$

44.1.3 Example (Continuity equation of fluid flow):

Consider the motion of a fluid in a region in which there are no sources or sinks, i.e., neither the fluid is being produced nor is destroyed. Let $\rho(x,y,z,t)$ denote the density of the fluid at a point (x,y,z) in the region at time t. In other words, we are assuming that the fluid is **compressible** . Let

$$\mathbf{v}=v_1\ \mathbf{i}+v_2\ \mathbf{j}+v_3\ \mathbf{k}$$

be the velocity vector field of the fluid . Then, the quantity

$$\mathbf{u}(x,y,z,t):=\rho(x,y,z,t)\ \mathbf{v}(x,y,z)$$

is called the **flux** of the fluid at the point (x,y,z) at time t. Note that, $\mathbf{u}(x,y,z,t)$ is a vector having same direction as that of $\mathbf{v}(x,y,z)$ and the magnitude of $\mathbf{u}$ represents the flow of unit mass of the liquid per unit area, per unit time. This comes from the dimension considerations of $\mathbf{u}$, which are

$$\frac{\text{mass}}{\text{unit volume}}\times\frac{\text{distance}}{\text{time}}=\frac{\text{mass}}{(\text{unit volume})\ (\text{unit time})}.$$

One would like to write the equation of the fluid flow. For this, consider a small portion, a rectangular parallelepiped W of dimensions Δx, Δy, Δz with sides parallel to axes, in the fluid. We calculate the change in mass in the region w by computing the outward flow.

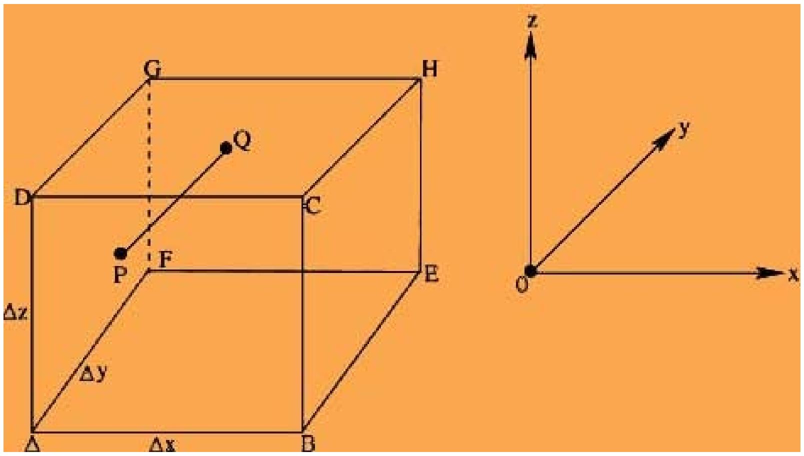

Figure 167. Fluid flow across a small parallelepiped

The mass of the fluid entering through the face $ABCD$ during a short time interval Δt at a point P is given by

$$\rho u_2(P)\,\Delta x\,\Delta z\,\Delta t,$$

as $\rho u_2(P)$ is the mass crossing over per unit area, in the positive direction of the $y-$axis, in unit time. The mass of the fluid leaving in the direction of the y -axis, across the face $EFGH$, is given at a point Q, by

$$\rho u_2(Q)\,\Delta x\,\Delta z\,t.$$

Thus, the net change in mass in the direction of the y -axis is given by

$$\rho\,(u_2(Q)-u_2(P))\,\Delta x\,\Delta z\,\Delta t = \left(\frac{u_2(Q)-u_2(P)}{\Delta y}\right)\Delta x\,\Delta y\,\Delta z\,\Delta t.$$

Hence, the net change in all directions is given by

$$\rho\left(\frac{\Delta u_1}{\Delta x}+\frac{\Delta u_2}{\Delta y}+\frac{\Delta u_3}{\Delta z}\right)\Delta x\,\Delta y\,\Delta z\,\Delta t,$$

where $\Delta u_1 = u_1(P)-u_1(Q)$, and so on. On the other hand, the rate of change of density is $\partial\rho/\partial t$, and hence the loss of mass in time Δt across parallelepiped is

$$-\frac{\partial\rho}{\partial t}\,\Delta x\,\Delta y\Delta z\,\Delta t.$$

Since there are no sinks or sources, we have, as Q approaches P,

$$\lim_{Q\to P}\left(\left(\frac{\Delta u_1}{\Delta x}+\frac{\Delta u_2}{\Delta y}+\frac{\Delta u_3}{\Delta z}\right)\right)=-\frac{\partial\rho}{\partial t},$$

i.e.,

$$\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z}\right)=-\frac{\partial\rho}{\partial t},$$

i.e.,

$$\mathrm{div}(\boldsymbol{u})=\mathrm{div}(\rho\mathbf{v})=-\frac{\partial\rho}{\partial t}$$

This is called the **continuity equation** of a compressible fluid flow without sinks or sources. The fluid flow is said to be **steady** , if ρ is independent of time. In that case $\partial\rho/\partial t = 0$ and hence the equation of flow is

$$\operatorname{div}(\rho\mathbf{v}) = 0.$$

If ρ is also a constant, i.e., the fluid has uniform density (incompressible), we have the equation to be $\operatorname{div}(\mathbf{v}) = 0$. This is also the necessary condition for the incompressibility of the fluid flow.

44.1.4 Visualizing Divergence:

We saw in the previous example that if we treat a vector-field $\mathbf{v}$ as the velocity-field of a steady flow of an incompressible fluid flow, then $\operatorname{div}(\mathbf{v}) = 0$ at a point means that the flow has no source or sink. We say fluid flow has **source** at a point P if $\operatorname{div}(P) > 0$ at that point and has a **sink** at a point P if $\operatorname{div}(P) < 0$ at that point.

Thus, if we represent $\mathbf{v}(P)$ as a vector (arrow), then at a point P where there is a sink, there are more arrows going in that point than the number of arrows that going out of it. At a source point the opposite happens, i.e., there are more arrows going out than coming in. Or, we can say that the flow is 'diverging' at that point. One can also treat $\mathbf{v}$ as a force field. Then $\mathbf{v}(P)$ as an arrow indicates the acceleration of a point P. See an interactive visualization at the end of the section.

44.1.5 Note:

Note that in examples 44.1.2 and 44.1.3, we represented physical quantities in terms of vectors, which of course depend upon coordinate systems. For example, our definition of divergence depended upon the vector representation

$$\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}.$$

of the vector field $\mathbf{F}$. Does that mean that physical phenomenon depend upon the choice of coordinates? One can show that this is not so. In fact, all that quantities like dot-product, cross product, divergence are independent of the choice of coordinates.

44.1.6 Symbolic representation of divergence:

Recall that, the divergence of a scalar field f was represented using the operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} := \sum \mathbf{i}\frac{\partial}{\partial x}.$$

We can use this operator to represent divergence of a vector field. For a vector field

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k},$$

$$\begin{aligned}\operatorname{div}(\mathbf{F}) = \nabla\cdot\mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},\end{aligned}$$

where the last equality is as if we have taken the dot product of ∇ with $\mathbf{F}$. One also writes above as

$$\nabla\cdot\mathbf{F} := \sum\left(\mathbf{i}\cdot\frac{\partial\mathbf{F}}{\partial x}\right) := \sum\left(\mathbf{i}\frac{\partial}{\partial x}\right)\cdot\mathbf{F}.$$

We describe next the properties of divergence with respect to various operations.

44.1.7 Theorem:

Let f, g be differentiable scalar fields and $\mathbf{v},\mathbf{w}$ be a differentiable vector field. Then the following hold:

1. $\operatorname{div}(k\mathbf{v}) = k\operatorname{div}(\mathbf{v})$, for all $k \in \mathbb{R}$.

2. $\operatorname{div}(\mathbf{v}+\mathbf{w}) = \operatorname{div}(\mathbf{v}) + \operatorname{div}(\mathbf{w})$.

3. $\operatorname{div}(f\mathbf{v}) = f\operatorname{div}(\mathbf{v}) + \mathbf{v}\cdot\nabla(f)$.

4. $\operatorname{div}(f\nabla(g)) = f(\nabla^2(g)) + \nabla(f)\cdot\nabla(g)$,

   where

   $$\nabla^2(g) = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2},$$

   *is called the **Laplacian** of* g.

5. $\operatorname{div}(\mathbf{V}\times\mathbf{W}) = \sum(\mathbf{i}\times\frac{\partial \mathbf{F}}{\partial x})\cdot\mathbf{V} - \sum(\mathbf{i}\times\frac{\partial \mathbf{W}}{\partial x})\cdot\mathbf{F}$.

PROOF

Proof of (i), (ii) are easy and are left as exercises. To prove (iii), note that

$$\begin{aligned}
\nabla\cdot(f\mathbf{v}) &= \sum \mathbf{i}\left[\frac{\partial}{\partial x}(f\mathbf{v})\right] \\
&= \sum \mathbf{i}\cdot\left[f_x\mathbf{v} + f\,\frac{\partial \mathbf{v}}{\partial x}\right] \\
&= (\sum i f_x)\cdot\mathbf{v} + f(\nabla\cdot\mathbf{v}) \\
&= \nabla f\cdot\mathbf{v} + f(\Delta\cdot\mathbf{v}).
\end{aligned}$$

The identity (iv) follows from (iii) with $V = \nabla g$. Finally, to prove (v) note that, using scalar triple product, we get

$$\begin{aligned}
\nabla\cdot(\mathbf{v}\times\mathbf{w}) &= \sum i\cdot\frac{\partial}{\partial x}(\mathbf{v}\times\mathbf{w}) \\
&= \sum i\cdot\left[\frac{\partial \mathbf{v}}{\partial x}\times\mathbf{w} + \mathbf{v}\times\frac{\partial \mathbf{w}}{\partial x}\right] \\
&= \sum i\cdot\left[\frac{\partial \mathbf{v}}{\partial x}\times\mathbf{w}\right] + \sum i\cdot\left[\mathbf{v}\times\frac{\partial \mathbf{w}}{\partial x}\right] \\
&= \sum i\cdot\left[\frac{\partial \mathbf{v}}{\partial x}\times\mathbf{w}\right] + \sum i\cdot\left[\mathbf{v}\times\frac{\partial w}{\partial x}\right] \\
&= \left(\sum i\cdot\frac{\partial \mathbf{v}}{\partial x}\right)\cdot\mathbf{w} - \left(\sum i\times\frac{\partial \mathbf{v}}{\partial x}\right)\cdot\mathbf{w}.
\end{aligned}$$

In part (iv) of the above theorem, for a vector field $\mathbf{v}$, the vector

$$\sum \mathbf{i}\times\frac{\partial \mathbf{v}}{\partial x}$$

plays an important role in various representations. We shall analyze it next. This also gives a method of generating new vector-fields out of given ones.

44.1.8 Definition

Let

$\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$

be a vector-field in a domain $D \subset \mathrm{IR}^3$. Define a vector field $\mathrm{curl}(\boldsymbol{v}) : D \rightarrow \mathrm{IR}^3$, by

$$\mathrm{curl}(\mathbf{v}) := \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_2}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}.$$

This is called the **curl** of the vector field $\mathbf{v}$. Another convenient representation of $\mathrm{curl}(\mathbf{v})$ is the following.

$$\mathrm{curl}(\mathbf{v}) := \begin{vmatrix} \mathbf{i} & \mathbf{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} =: \nabla \times \mathbf{v}$$

Here, ∇ is treated as a vector with components $\partial/\partial x, \partial/\partial y, \partial/\partial z$ and $\nabla \times \mathbf{v}$ is treated as the cross product. We also write

$$\nabla \times \mathbf{v} = \sum \left(\mathbf{i} \times \frac{\partial \mathbf{v}}{\partial x}\right).$$

44.1.9 Note:

Once again, through the definition of $\mathrm{curl}(\mathbf{v})$ is in terms of components of $\mathbf{v}$ which depend upon the choice of a coordinate system, one can show that the definition of $\mathrm{curl}(\mathbf{v})$ does not depend upon the choice of the coordinate system.

We give an example to illustrate the importance for curl operator.

44.1.10 Example:

We saw in example 43.12 (ii), that for the rotation of a rigid body about an axis in space,its velocity vector at a point P is given by

$\mathbf{v} = \mathbf{w} \times \mathbf{r}$,

where $\mathbf{w}$ is a vector along the axis of rotation and $\mathbf{r}$ is the position vector of P. In case we choose the coordinate system to be right handed cartesian coordinates with $z-$axis along the axis of rotation with $\mathbf{w} = w\mathbf{k}$, where w is the angular speed, then

$$\begin{aligned} \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & w \\ x & y & z \end{vmatrix} \\ &= w(-y\,\mathbf{i} + x\,\mathbf{j}). \end{aligned}$$

$$\operatorname{curl}(\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix}$$

$$= 2w\,\mathbf{k}$$

$$= 2\,\mathbf{w}.$$

Thus,

$\operatorname{curl}(\mathbf{v}) = 0$ if and only if $\mathbf{w} = \mathbf{0}$, i.e., $w = 0$,

i.e., there is no rotation of the body.

The above example motivates our next definition.

44.1.11 Definition:

A vector field $\mathbf{v}$ is said to be **irrotational** if $\operatorname{curl}(\mathbf{v}) = 0$.

44.1.12 Example:

1. Let $\mathbf{F}(x,y,z) = 2xy\,\mathbf{i} + (x^2+z^2)\,\mathbf{j} + 2zy\,\mathbf{k}$.

   Then

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+z^2 & 2zy \end{vmatrix}$$

$$= (2z-2z)\,\mathbf{i} - (0-0)\,\mathbf{j} + (2x-2x)\,\mathbf{k}$$

$$= 0.$$

Hence, $\mathbf{F}$ is irrotational.

(ii) Let $\mathbf{F}$ be any vector-field which has a potential f, i.e., for every $(x,y,z) \in D$

$\mathbf{F}(x,y,z) = (\nabla f)(x,y,z)$,

for some twice continuously differentiable scalar field f.

Then

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\mathbf{j}$$

$$+\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}$$

$$= 0.$$

Thus, $\mathbf{F}$ is irrotational. Hence, we have shown that every vector-field which has a potential is irrotational.

We state next some properties of the curl operator which show that it behaves like a differential operator.

44.1.13 Theorem:

Let $\mathbf{u}, \mathbf{v}$ *be continuously differentiable vector-fields and* f *a continuously differentiable Scalar-fields. Then the following hold:*

1. $\operatorname{curl}(\mathbf{u}+\mathbf{v}) = \operatorname{curl}(\mathbf{u}) + \operatorname{curl}(\mathbf{v}).$
2. $\operatorname{curl}(f\mathbf{u}) = f \ \operatorname{curl}(\mathbf{u}) + (\nabla f) \times \mathbf{u}.$
3. $\nabla \times (\mathbf{u} \times \mathbf{v}) = \operatorname{curl}(\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot u)\mathbf{v},$ *where*

$$\mathbf{v} \cdot \nabla = \sum \mathbf{i} \cdot \frac{\partial}{\partial x}$$

and

$$\mathbf{u} \cdot \nabla = \sum \mathbf{i} \cdot \frac{\partial}{\partial x}.$$

PROOF

1. $\operatorname{curl}(\mathbf{u}+\mathbf{v}) = \sum \mathbf{i} \times (\mathbf{u}+\mathbf{v}) = \sum (\mathbf{i} \times \mathbf{u}) + \sum (\mathbf{i} \times \mathbf{v}) = \operatorname{curl}(\mathbf{u}) + \operatorname{curl}(\mathbf{v}).$
2.

$$\begin{aligned}\operatorname{curl}(f\mathbf{u}) &= \sum \mathbf{i}\times\frac{\partial(f\mathbf{u})}{\partial x} \\ &= \sum \mathbf{i}\times\left[f\,\frac{\partial \mathbf{u}}{\partial x}+\frac{\partial f}{\partial x}\mathbf{u}\right] \\ &= \sum \mathbf{i}\times\left(f\,\frac{\partial y}{\partial x}\right)+\sum \mathbf{i}\times\left(\frac{\partial f}{\partial x}\mathbf{u}\right) \\ &= f\left(\sum \mathbf{i}\times\frac{\partial \mathbf{u}}{\partial x}\right)+\sum\left(\frac{\partial f}{\partial x}\mathbf{i}\right)\times\mathbf{u} \\ &= f\ \operatorname{curl}(\mathbf{u})+(\nabla f)\times\mathbf{u}\end{aligned}$$

3. $$\begin{aligned}\operatorname{curl}(\mathbf{u}\times\mathbf{v}) &= \sum \mathbf{i}\times\frac{\partial(\mathbf{u}\times\mathbf{v})}{\partial x} \\ &= \sum \mathbf{i}\times\left[\frac{\partial \mathbf{u}}{\partial x}\times\mathbf{v}+\mathbf{u}\times\frac{\partial \mathbf{v}}{\partial x}\right] \\ &= \sum \mathbf{i}\times\left(\frac{\partial \mathbf{u}}{\partial x}\times\mathbf{v}\right)+\sum i\times\left(\mathbf{u}\times\frac{\partial v}{\partial x}\right)\end{aligned}$$

Now using the identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(a\cdot b)\mathbf{c},$$

we get

$$\begin{aligned}\operatorname{curl}(\mathbf{u}\times\mathbf{v}) &= \sum\left[(\mathbf{i}\cdot\mathbf{v})\frac{\partial \mathbf{u}}{\partial x}-(\mathbf{i}\cdot\frac{\partial \mathbf{u}}{\partial x})\mathbf{v}\right] \\ &\quad+\sum\left[(\mathbf{i}\cdot\frac{\partial \mathbf{v}}{\partial x})\mathbf{u}-(i\cdot\mathbf{u})\frac{\partial \mathbf{v}}{\partial x}\right] \\ &= \left(\sum(\mathbf{i}\cdot\mathbf{v})\frac{\partial}{\partial x}\right)\mathbf{u}-\left(\sum(\mathbf{i}\cdot\mathbf{u})\frac{\partial}{\partial x}\right)\mathbf{v} \\ &\quad+\left[\left(\sum\mathbf{i}\cdot\frac{\partial}{\partial x}\right)\mathbf{v}\right]\mathbf{u}-\left[\left(\sum\mathbf{i}\cdot\frac{\partial}{\partial x}\right)\mathbf{u}\right]\mathbf{v} \\ &= (\mathbf{v}\cdot\nabla)\,\mathbf{v}-(\mathbf{u}\cdot\nabla)\mathbf{v}+(\nabla\cdot\mathbf{v})\mathbf{u}-(\nabla\cdot\mathbf{u})\mathbf{v}.\end{aligned}$$

Visualization of Divergence

Visualization of rotational vector fields

For Quiz refer the WebSite.

Practice Exercises

1. **Calculate the divergence of the following:**

   1. $\boldsymbol{F}(x,y) = xe^{y}\ \boldsymbol{i} - \left(\dfrac{y}{x+y}\right)\boldsymbol{j}.$
   2. $\boldsymbol{F}(x,y,z) = x^3y^2z\ \boldsymbol{i} + x^2z\ \boldsymbol{j} + x^2y\ \boldsymbol{k}$
   3. $\boldsymbol{F}\times\boldsymbol{G}$, where $\boldsymbol{F}(x,y,z) = \mathbf{i} + 2x\ \boldsymbol{j} + 3y\ \boldsymbol{k},\ \boldsymbol{G}(x,y,z) = x\mathbf{i} - y\ \boldsymbol{j} + z\ \boldsymbol{k}.$

   Answer:

   (i) $e^{y} - \dfrac{x}{(x+y)^2}$.

(ii) $3x^2y^2z$

(iii) $2z+3x$

2. Calculate the curl of the following:

   1. $\boldsymbol{F}(x,y,z) = yz\,\boldsymbol{i} + xz\,\boldsymbol{j} + xy\,\boldsymbol{k}$.

   2. $\boldsymbol{F} \times \boldsymbol{G}$, where $\boldsymbol{F}(x,y,z) = x\,\boldsymbol{i} - z\,\boldsymbol{k}$, $\boldsymbol{G}(x,y,z) = x^2\,\boldsymbol{i} + y\,\boldsymbol{j} + z^2\,\boldsymbol{k}$.

   3. Where $\boldsymbol{F}(x,y,z) = xyz\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$.

Answer:

(i) 0

(ii) $\left(x - 2xz + x^2\right)\mathbf{i} - 2y\,\mathbf{j} + \left(z + z^2 - 2xz\right)\mathbf{k}$

(iii) $-xy\,\mathbf{j} + xz\,\mathbf{k}$

3. Let

$$\boldsymbol{r}(x,y,z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k} \text{ for } (x,y,z) \neq 0,$$

and let

$$r(x,y,z) := \|\boldsymbol{r}(x,y,z)\|.$$

Prove the following:

   1. $\nabla(\ln(r)) = \dfrac{\boldsymbol{r}}{r^2}$.

   2. $\nabla(\dfrac{1}{r}) = -\dfrac{\boldsymbol{r}}{r^3}$.

   3. $\nabla r^n = nr^{n-2}\boldsymbol{r}$.

4. For $\boldsymbol{r}$ and r as in exercise (3) above, show that

   1. $\nabla \cdot \left(\dfrac{\boldsymbol{r}}{r^3}\right) = 0$.

   2. $\nabla \cdot (r^n \boldsymbol{r}) = (n+3)r^n$.

   3. $\nabla \times (r^n \boldsymbol{r}) = 0$.

5. Show that the vector field

$$\boldsymbol{F}(x,y,z) = x^2y\,\boldsymbol{i} + 3\,\boldsymbol{j} + xyz\,\boldsymbol{k}$$

is not incompressible.

6. Show that the following vector fields are not conservative:

   1. $\boldsymbol{F}(x,y,z) = x^3 y^2 z\boldsymbol{i} + x^2 z\boldsymbol{j} + x^2 y\boldsymbol{k}$.
   2. $\boldsymbol{F}(x,y,z) = y(\cos x)\boldsymbol{i} + x(\sin y)\boldsymbol{j}$.

7. Show that the following vector fields are not irrotational:

   1. $\boldsymbol{F}(x,y,z) = 2xz^2\boldsymbol{i} + \boldsymbol{j} + xy^3 z\boldsymbol{k}$.
   2. $\boldsymbol{F}(x,y,z) = (x+y)\boldsymbol{i} + (x,z)\boldsymbol{j} + (zx)\boldsymbol{k}$.

Recap

In this section you have learnt the following

- The divergence of a vector field.
- The curl of a vector field.
- Their physical significance.