Module 16 : Line Integrals, Conservative fields Green's Theorem and applications

Lecture 46 : Line integrals [Section 46.1]

Objectives

In this section you will learn the following :

- How to define the integrals of a scalar field over a curve.

46.1 Line integrals

In this section we describe a natural generalization of the notion of definite integral, called the line integral. This notion finds many applications.

46.1.1 Definition:

Let

$f : D \subset \mathbb{R}^3 \to \mathbb{R}$

and C be a curve in $\mathbb{R}^3$ with parameterization

$\mathbf{r} : [a,b] \to \mathbb{R}^3$, where $\mathbf{r}(t) \in D$ for $t \in [a,b]$.

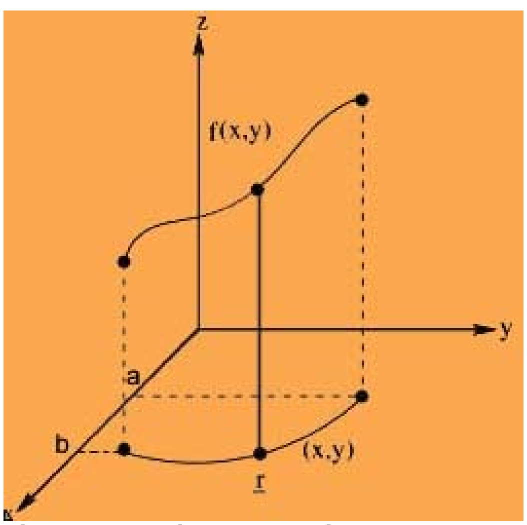

Figure 177. Line Integral

Let $\mathbf{r}$ have the arc length parametrization $\mathbf{r}(s), a \le s \le b$. Then the function

$$s \to f(\mathbf{r}(s)),\ a \le s \le b$$

is a scalar-valued continuous function on the interval $[a,b]$ for both f and $\mathbf{r}$ are continuous. Thus, the integral

$$\int_c f ds := \int_{s=a}^{s=b} (f o\, \mathbf{r})(s)\, ds$$

is well-defined. It and is called the **line integral** of f over C.

The line integral being a definite integral, has the following properties.

46.1.2 **Theorem :**

If

$$f : D \subset \mathbb{R}^3 \to \mathbb{R}$$

is continuous and C *is a simple, regular curve in* D *with a parameterization* $\mathbf{r}(t), t \in [c,d]$ *, then*

$$\int_C f\ ds = \int_c^d f(x(t), y(t), z(t))\, \| r'(t) \|\, dt.$$

Since

$$\int_c f\ ds = \int_a^b f(x(s), y(s), z(s))\, ds, \qquad \text{----------(1)}$$

and

$$\frac{ds}{dt} = \| \mathbf{r}'(t) \|, \qquad \text{----------(2)}$$

from (1) and (2) we have

$$\int_c f\ ds = \int_a^b f(x(t), y(t), z(t))\, \| \mathbf{r}'(t) \|\, dt.$$

46.1.3 **Note:**

In defining $\int_c f\ ds$, implicitly we have assumed that the arc length increases as the variable increases. This is normally, called the **positive orientation** on C. The opposite orientation will give a change of sign for $\int_c f ds$.

46.1.4 **Examples:**

1. Let us evaluate

   $$\int_C (1 + xy^2) ds,$$

   where C is the line segment from $(0,0)$ to $(1,2)$ in $\mathbb{R}^2$.

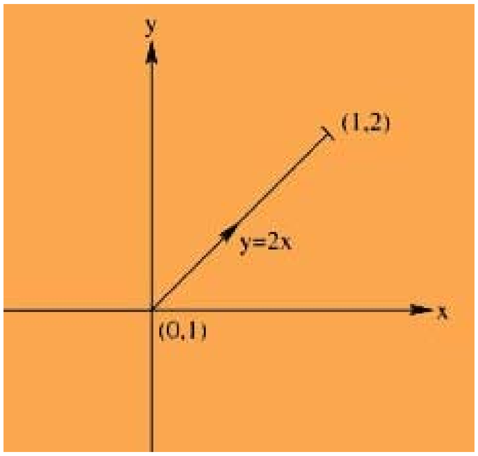

Figure: The line segment

To move from $(0,0)$ to $(1,2)$, let us choose the parameterization

$$\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j},\ 0 \le t \le 1.$$

Then,

$$\|\mathbf{r}'(t)\| = \sqrt{5}.$$

Hence,

$$\int_C (1 + xy^2)\,ds = \int_0^1 (1 + 4t^3)\sqrt{5}\,dt = 2\sqrt{5}.$$

2. Let us calculate

$$\int_C f\ ds\ \text{ for } f(x,y,z) = xy + z^3$$

where C is the circular helix

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k},\ \text{ from } (1,0,0) \text{ to } (-1,0,\pi).$$

Note that, to move from $(1,0,0)$ to $(-1,0,\pi)$ along $\mathbf{r}(t)$, t varies our $[0,\pi]$.

Since

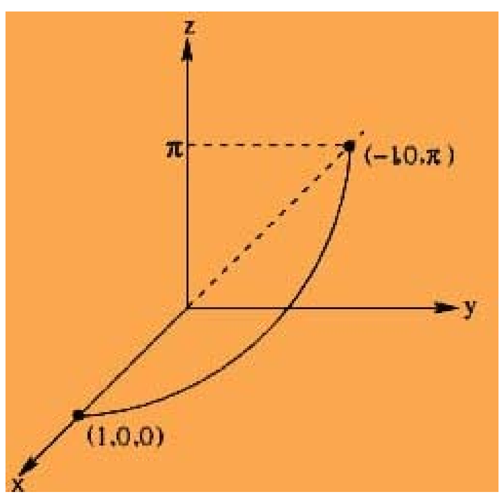

Figure: Circular helix

$$\|r'(t)\| = \sqrt{1+1} = \sqrt{2},\ 0 \le t \le \pi,$$
we have

$$\int_C f\,ds = \sqrt{2}\int_0^{\pi}(\cos t\sin t + t^3)\,dt$$
$$= \sqrt{2}\left[\frac{\sin^2 t}{2} + \frac{t^4}{4}\right]_0^{\pi}$$
$$= \frac{\sqrt{2}\pi^4}{4}.$$

46.1.5 **Theorem (Properties of the line integral):**

1. $\int_C (f+g)\,ds = \int_C f\,ds + \int_C g\,ds$
2. $\int_C (\alpha f)ds = \alpha\left(\int_C f ds\right)$
3. *If C consists of finite number of pieces $C_1, C_2, \ldots, C_n$, where each C_i is regular, then* $\int_c f ds := \sum_{i=1}^{n}\left(\int_{c_i} f ds\right).$

Proof
We assume these properties.

46.1.6 Example:

Let us compute

$\int_C f\,ds$ where $f(x,y,z) = x + \sqrt{y} - z^2,$
and the path C given by

$y = x^2$ from $O(0,0,0)$ to $A(1,1,0)$ and the line segment from $A(1,1,0)$ to $B(1,1,1).$

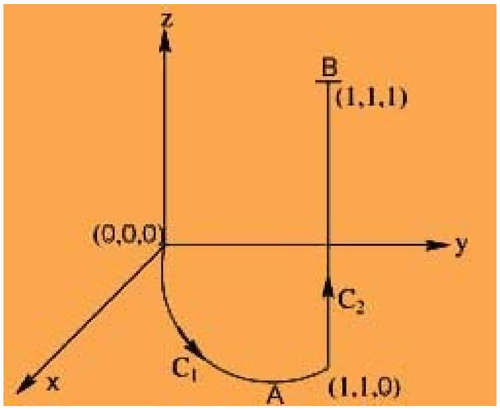

Figure: The curve $C = C_1 \cup C_2$

We can think of C as two pieces, C_1 from O *to* A along $y = x^2$ with parameterization given by

$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j},\ 0 \le t \le 1,$

and the piece C_2 the path from *A* to *B* along the line segment joining them, with parameterization given by

$\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\,\mathbf{k},\ 0 \le t \le 1.$

Thus, the curve C consists of two pieces C_1 and C_2, both of which arc regular, and hence

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$$
$$= \int_0^1 (t+t)\sqrt{1+(2t)^2}\,dt + \int_0^1 (2-t^2)\,dt$$
$$= \frac{1}{4}\int_0^1 (1+4t^2)^{\frac{1}{2}}\,(8t)\,dt + \left[2t - \frac{t^3}{3}\right]_0^1$$
$$= \frac{1}{4}\left[\frac{(1+4t^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 + \left[2 - \frac{1}{3}\right]$$
$$= \frac{1}{4}\times\frac{2}{3}\left[\sqrt{5}-1\right] + \frac{5}{3} = \frac{1}{6}\left[\sqrt{5}-1+10\right].$$

46.1.7 **Definition :**

Let C be a smooth parametric curve with a parameterization $\mathbf{r}(t), t \in [a,b]$. Consider the curve

$\tilde{\mathbf{r}}(t) := \mathbf{r}(b-(t-a)), t \in [a,b].$

Then, $\tilde{\mathbf{r}}$ is also a smooth curve. Geometrically,

$\{\tilde{\mathbf{r}}(t) \mid t \in [a,b]\} = \{\mathbf{r}(t) \mid t \in [a,b]\}.$

However, $\tilde{\mathbf{r}}$ traverses the path C backwards, i.e., the initial-point of $\mathbf{r}$ is the final-point of $\tilde{\mathbf{r}}$ and vice-verse. The curve $\tilde{\mathbf{r}}$ is called the **reverse** of C, and is denoted by $-C$.

46.1.8 **Theorem :**

Let $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ be a scalar field. If $\int_C f\,ds$ exists, then $\int_{-C} f\,ds$ also exists and

$\int_{-C} f\,ds = -\left(\int_C f\,ds\right).$

PROOF

Follows from the fact that

$$\frac{d\tilde{\mathbf{r}}(t)}{dt} = -\frac{d\mathbf{r}(t)}{dt}, \text{ for every } t \in [a,b].$$

46.1.9 Definition :

Let C be a smooth curve in $D \subset \mathbb{R}^3$ with parameterization

$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, t \in [a,b].$

1. For a continuous scalar field $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$, define

$$\int_C f\,dx := \int_{t=a}^{t=b} f(x(t), y(t), z(t))\left(\frac{dx}{dt}\right)dt,$$

$$\int_C f\,dy := \int_{t=a}^{t=b} f(x(t), y(t), z(t))\left(\frac{dy}{dt}\right)dt,$$

and

$$\int_C f\, dz := \int_{t=a}^{t=b} f(x(t), y(t), z(t)) \left(\frac{dz}{dt}\right) dt.$$

2. For a continuous vector field $\mathbf{F} : D \subset \mathbb{R}^3 \to \mathbb{R}$, with

$$\mathbf{F} := f_1\,\mathbf{i} + f_2\,\mathbf{j} + f_3\,\mathbf{k},$$

define

$$\int_C \mathbf{F} \cdot d\mathbf{r} := \int_C f_1\, dx + \int_C f_2\, dx + \int_C f_3\, dz,$$

Called the **line integral** of $\mathbf{F}$ over C .

46.1.10 **Note :**

The line integral $\int_c \mathbf{F}.d\mathbf{r}$ depends not both upon the orientation (positive or reverse) of C, also upon the initial and the final points of C .

46.1.11 Example :

Let C_1 and C_2 be smooth curves given by

$$\mathbf{r}_1(t) := t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}, 0 \le t \le 1$$

and

$$\mathbf{r}_2(t) := t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, 0 \le t \le 1.$$

Then, C_1 and C_2 both have initial point $(0,0,0)$ and final point $(1,1,1)$. Further, for the vector field

$$\mathbf{F} := yz\,\mathbf{i} + xz\,\mathbf{j} + x^2 y\,\mathbf{k},$$

we have

$$\int_{C_1} yz\, dx + xz\, dy + yx^2\, dz = \int_0^1 (t^2\, dt + t^2\, dt + t^3\, dt)$$

$$= \left[\frac{2t^3}{3}\right]_0^1 + \left[\frac{t^4}{4}\right]_0^1 = \frac{11}{12}.$$

And

$$\int_{C_2} yz\, dx + xz\, dy + yx^2\, dz = \int_0^1 (t^4\, dt + t^4 (2t\, dt) + t^4 (3t^2\, dt))$$

$$= \left[\frac{t^5}{5}\right]_0^1 + \left[\frac{2t^5}{5}\right]_0^1 + \left[\frac{3t^6}{6}\right]_0^1 = \frac{11}{10}.$$

For Quiz refer the WebSite

Practice Exercises

Evaluate the following line integrals :

1. $\int_C \left(x^2 - y + 3z\right) ds$, where C is the line segment going $(0,0,0)$ with $(1,2,1)$.

Answer: $5/\sqrt{6}$

2. $\int_C \frac{1}{1+x^2}\, ds$, where C is the curve $r(t) = t\,\mathbf{i} + \frac{2}{3} t^{3/2}\,\mathbf{j}, 0 \le t \le 3$.

   Answer: 2

3. For the given vector field F and the curve C, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$:

   $\mathbf{F}(x,y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$,

   C is the circle $x^2 + y^2 = 4$.

   $\mathbf{F}(x,y,z) = x^2 y\,\mathbf{i} + (x-z)\,\mathbf{j} + xyz\,\mathbf{k}$,

   C is the curve $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2\,\mathbf{k}, 0 \le t \le 1$

   Answer:
   - 0
   - $-\frac{17}{15}$.

4. Let $f(x,y) = x - y$ and C be the curve $r(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j}$. Compute the following : $\int_C f\,dx, \int_C f\,dy$

   Answer: $0, -\frac{1}{2}$

5. Compute the following :

   1. $\int_C y\,dx + x^2\,dy,$

      where C is the arc of the parabola $y = 4x - x^2$ from $(4,0)$ to $(1,3)$.

   2. $\int_C zy\,dx\; x3\,dy + xydz$

      where C is the curve $\mathbf{r}(t) = e^t\,\mathbf{i} + e^t\,\mathbf{j} + e^t\,\mathbf{k}$ from $0 \le t \le 1$

   Answer:
   - $\frac{9}{2}$
   - $1 - e^3$

Recap

In this section you have learnt the following

- How to define the integrals of a scalar field over a curve.

[Section 46.2]

Objectives

In this section you will learn the following :

- How to use line integral to compute areas of some surfaces.
- Physical applications of line integrals.

46.2 Applications of line integral

46.2.1 Surface area of a thin sheet :

Suppose we have a surfaces S whose base is a curve C in the xy-plane and its height at any point $(x,y) \in C$ in the value $z = f(x,y)$, where f is some function which domain includes C.

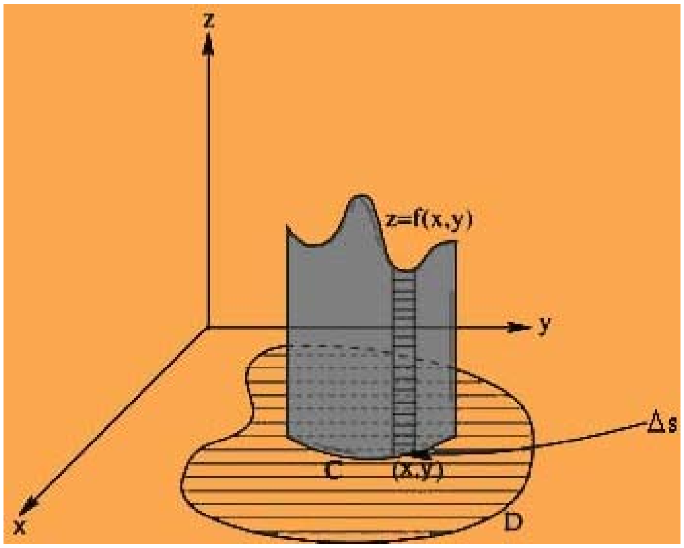

Figure 181. Surface with base C and height $z = f(x,y)$

We can think of this surface as made up of small vertical strips with base Δs and height $f(x,y)$. The area of this strip in approximately given by $f(x,y)\Delta s$. Thus, the total area of this surface can be defined to be

$$\lim_{\Delta s \to} \left(\sum f(x,y) \Delta s \right),$$

whenever it exists. This limit is nothing but $\int_C f ds$. Thus, the area of the surface s can be defined to be

$$\text{Area of } S := \int_C f\, ds.$$

46.2.2 Example:

Let us compute the area of the surface

S with base the circle $x^2 + y^2 = 1$ in the $xy-$plane

extending upward to

the parabolic cylinder $z = 1 - x^2$ at the top.

The required area is given by

$$A = \int_C (1 - x^2)\, ds,$$

where C is the circle with arc-length parameterization :

$$\mathbf{r}(s) = \cos s\, \mathbf{i} + \sin s\, \mathbf{j},\ 0 \le s \le 2\pi.$$

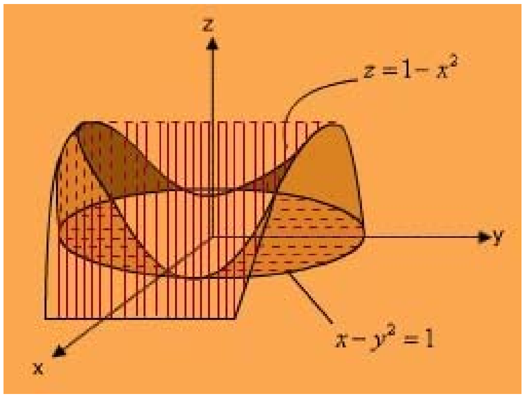

Figure: Surface with base the circle and height $z = 1 - x^2$

Thus,

$$A = \int_0^2 \pi(1 - \cos^2 s) ds = \int_0^2 \pi \sin s\, ds = \pi$$

46.2.3 Mass and Center of gravity of a thin wise :

Consider a thin wire in the shape of a curve C in space.

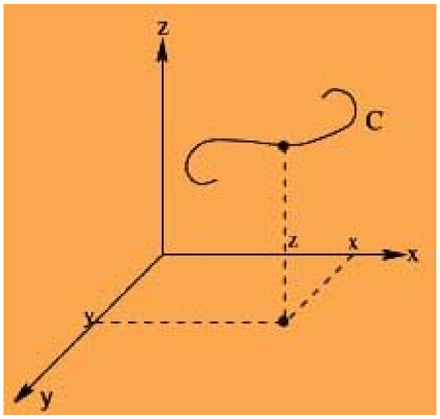

Figure: A piece of wire C

If $f(x, y, z)$ represents the mass per unit length of the wire, then the mass of a small portion Δs of the wire, is given by

$\Delta M := f(x,y,z)\Delta s.$

Thus, we can define the total mass of the wire to be

$$M := \int_C f\, ds.$$

Similarly, we can define the moments of the wire C about the coordinate planes as follows

$$M_{xy} := \int_c zf(x,y,z)\, ds,\; M_{yz} := \int_c xf(x,y,z)\, ds,\; M_{zx} := \int_c yf(x,y,z)\, ds.$$

Finally, the point $(\bar{x}, \bar{y}, \bar{z})$, called the center of mass of the wire, in defined by

$$\bar{x} := \frac{M_{yz}}{M},\; \bar{y} := \frac{M_{zx}}{M},\; \bar{z} := \frac{M_{xy}}{M}.$$

46.2.4 Work done along a curve:

Consider a force $\mathbf{F}$ being applied to a body to move it along a curve C from a point A to a point B.

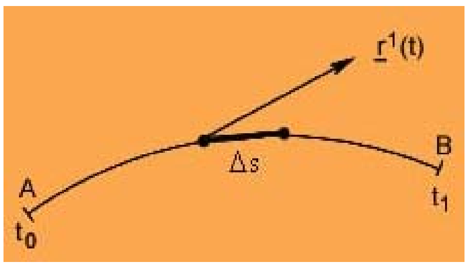

Figure: 177

If $\mathbf{r}(t), t_0 \le t \le t_1,$ is a parameterization of $C,$ then the amount of work done to move the body by a small distance Δs along the curve is given by

$$\left(\mathbf{F}\cdot\frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}\right)\Delta s,$$

since, $\mathbf{F}\cdot\mathbf{r}'(t)$ is the tangential component of force. Thus $W,$ the **total work** done in moving the body along $C,$ is given by

$$W = \int_C \left(\mathbf{F}\cdot\frac{r'}{\Delta\mathbf{r}'\Delta}\right) ds$$

$$= \int_C \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\right) ds$$

$$= \int_C \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{t=t_0}^{t_1} \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right) dt. \qquad \text{---------- (37)}$$

If $\mathbf{F}$ has components $F_1, F_2, \text{and } F_3,$ i.e.,

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}, \text{ and } \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},$$

then equation (37) can also be written as

$$W = \int_{t=t_0}^{t_1} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$
$$:= \int_{t=t_0}^{t_1} F_1\, dx + F_2\, dy + F_3\, dz.$$

46.2.5 Circulation of a fluid along a curve:

Let

$$\mathbf{v} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$$

be the velocity field of a fluid flowing through a region D in space. Let C be a curve inside the region D.

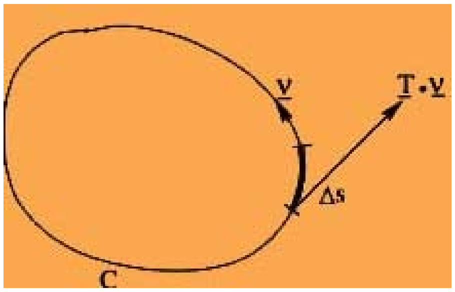

Figure: Flux along Δs

Then the tangential component of $\mathbf{v}$ at a point an the curve is given by

$$\mathbf{v} \cdot \mathbf{T}, \text{ where } \mathbf{T} \text{ is the unit} - \text{tangent vector to } C$$

at that point. For a small portion Δs of the curve, the quantity $(\mathbf{v} \cdot \mathbf{T})\Delta s$ represents the flow of the fluid flow across the small portion Δs. Thus, the **total flow** of the fluid along the curve C is given by

$$\text{Total flow along } C := \int_C (\mathbf{v} \cdot \mathbf{T})\, ds.$$

If the curve C is a closed curve, then the above integral is called the **circulation** of the flow along the curve.

46.2.6 Flux across a plane curve:

Consider a fluid flowing in a region D in the plane. Let $\mathbf{v}$ be the velocity vector of the fluid and $\rho(x,y)$ be its density at a point $(x,y) \in D$.

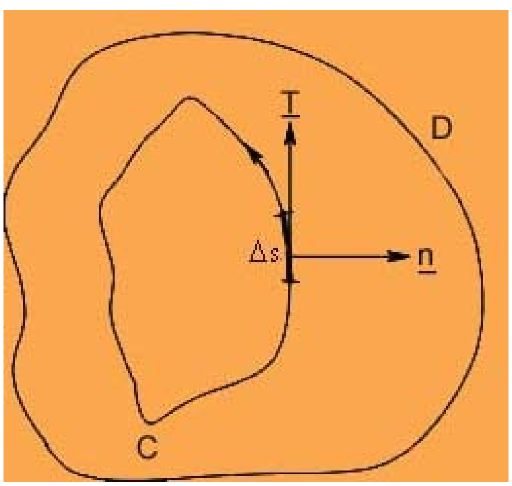

Figure: Flux across Δs

Then, the vector field

$$\mathbf{F}(x,y) := \rho(x,y)\, \mathbf{v}(x,y), \;\; (x,y) \in D$$

represents the rate of change of mass, per unit time across a unit length. Let be a curve in the domain . Then the rate of

C D

change of mass of the fluid across a small portion Δs of the curve is given by

$(\mathbf{F} \cdot \mathbf{n})\Delta s,$

where $\mathbf{n}$ is the unit normal vector to the curve. Thus, the total mass flow across whole of C is given by

$$\text{Total flow across } C := \int_C (\mathbf{F} \cdot \mathbf{n})\, ds,$$

called the **flux** of the fluid flow across C.

Practice Exercises

1. Compute the area of the surface with base on the curve C in the xy-plane and at the point (x,y) in C, the height being $z = f(x,y)$ for the following:

    1. $f(x,y) = xy$, C is the part of the unit circle $x^2 + y^2 = 1$ from $(1,0)$ to $(0,1)$.
    2. $f(x,y) = 3x$, C is the parabola $y = x^2, 0 \le x \le 2$.
    3. $f(x,y) = 2 + \frac{1}{2}(3y - 4y^3)$, C is the unit circle.

    Answer:

    (i) $\frac{1}{2}$

    (ii) $\dfrac{17\sqrt{17}-1}{4}$

    (iii) 4π

2. Find the work done by a force field $\mathbf{F}(x,y,z)$, moving along a curve C a given below:

    1. $F(x,y,z) = \frac{x}{2}\mathbf{i} - \frac{y}{2}\mathbf{j} + \frac{k}{4}$, C is $\mathbf{r}(t) = \cos \mathbf{i} + \sin \mathbf{j} + t\,\mathbf{k}$, $0 \le t \le 3\pi$.
    2. $F(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j}$, C is $\mathbf{r}(t) = 3t^2\,\mathbf{i} + t\,\mathbf{j} + \mathbf{k}, 0 \le t \le 1$. :

    Answer:

    (i) $3\pi/4$

    (ii) 5

3. Find the circulation of $\mathbf{v}(x^2+y^2)\,(\mathbf{i}+\mathbf{j})$ along the curve $\mathbf{r}\,(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, 0 \le t \le 2\pi$

    Answer: 0

4. Find the flux of the vector field $F\,(x,y) = y^3\,\mathbf{i} + x^5\,\mathbf{j}$ across the boundary of the unit square $[0,1]\times[0,1]$

    Answer: 0

Recap

In this section you have learnt the following

- How to use line integral to compute areas of some surfaces.
- Physical applications of line integrals.