Module 6 : Definition of Integral

Lecture 18 : Approximating Integral : Trapezoidal Rule [Section 18.1]

Objectives

In this section you will learn the following :

Mid point and the Trapezoidal methods for approximately the integral.

18.1 Approximating integral : Trapezoidal rule

In the previous section we saw how the existence of an anti-derivative for a function f helps us to evaluate its integral. However, it is not always possible to identify an anti-derivative of a given function. For example, the function

$$f(x) = \frac{\sin x}{x}\, x \in [1, 2]$$

or the function

$$g(x) = \frac{x^2+2}{\sin x},\ x \in [\pi/4,\ \pi/2]$$

are both integrable functions since each is continues. However, it is not possible to identify explicitly a function F such that $F' = f$ or a function G such that $G' = g$. In such cases, when we can not compute the integral of an integral function exactly, the next best possible thing to do is to find an approximate value of the integral. We describe some methods to do so.

18.1.1 Definition Approximating f by step function:

Let $f : [a, b] \to \mathbb{R}$ be integrable and $P_n = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ such that

$$x_i - x_{i-1} = h_n := \frac{b-a}{n}, i = 1, \ldots, n.$$

Let

$$y_i = f(x_i), i = 0, 1, \ldots, n.$$

To find an approximate value of the integral $\int_a^b f(x)\,dx$, the idea is to replace f, on each subinterval $[\,x_{i-1}, x_i\,]$ by a simpler function whose integral can be found easily, and use that integral as an approximation. Mainly, we use polynomials functions to approximate f on each subinterval.

Let

$$c_i \in \left[\, x_{i-1}\,,\, x_i \,\right], i = 1, \ldots, n.$$

On each subinterval $\left[\, x_{i-1}\,,\, x_i \,\right]$ let us replace f by the constant function $f\left(c_i\right)$. Then, the integral of this new function on $\left[a,b\right]$ is the sum of the areas of the rectangles and is given by

$$h_n \sum_{i=1}^{n} f\left(c_i\right)$$

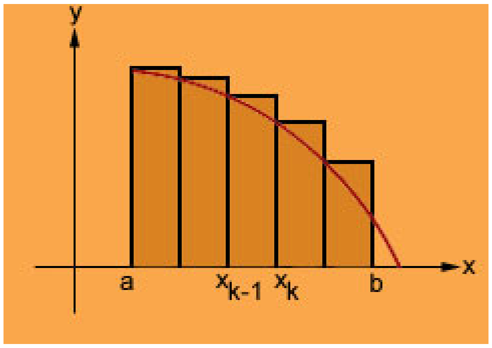
Note that this new function is well-defined on each subinterval $\left[\, x_{k-1}\,,\, x_k \,\right]$, and is called a step function. Some particular cases are the following:

(i) Left-end point rule:

If $c_i := x_{i-1}$ for every $i = 1, 2, \ldots, n,$ then

$$L_n \;:=\; h_n \sum_{i=1}^{n} f\left(x_{c-1} - 1\right)$$

is called the left-endpoint approximation for the integral $\int_a^b f\left(x\right) dx$.

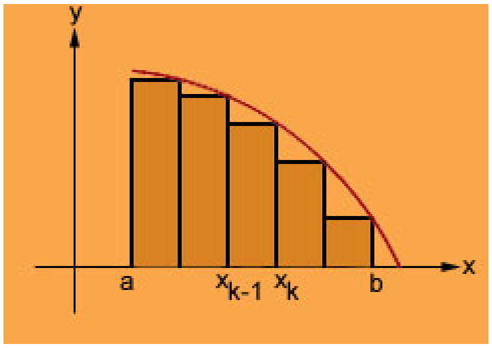


(ii) Right-endpoint rule:

If $c_i := x_i$ for every $i = 1, 2, \ldots, n,$ then

$$R_n \;:=\; h_n \sum_{i=1}^{n} f\left(x_i - 1\right)$$

is called the right-endpoint approximation for the integral $\int_a^b f\left(x\right) dx$.



18.1.2 Mid point rule:

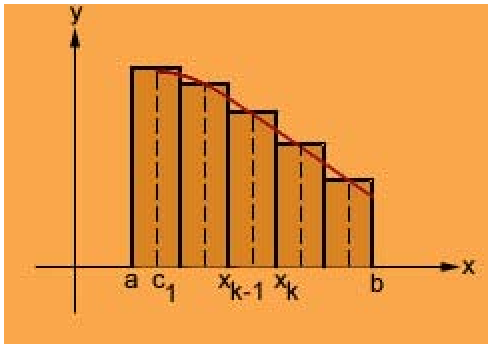
In case,

$$c_i = \frac{\left(x_{i-1}+x_i\right)}{2} \text{ for every } i = 1, 2, \ldots, n,$$

then,

$$M_n := h_n \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right),$$

is called the mid-point approximation for the integral $\int_a^b f(x)\,dx$.



18.1.3 Note:

Note that each of

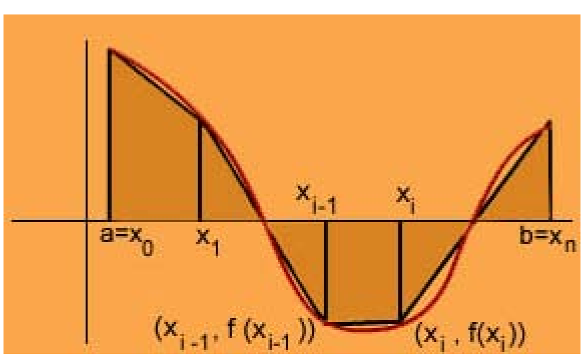
$$L_n, R_n, M_n \to \int_a^b f(x)\,dx \text{ as } n \to \infty.$$

Thus, each of them is an approximation for $\int_a^b f(x)\,dx$.

18.1.4 **Trapezoidal Rule:**

For every n, let us approximate f in each subinterval $\left[x_{i-1}, x_i\right]$ by the linear function g_n obtained by joining the points $\left(x_{i-1}, f\left(x_{i-1}\right)\right)$ and $\left(x_i, f\left(x_i\right)\right)$ by straight lines. Let

$$T_n := h_n\left(\sum_{i=1}^{n} \frac{f\left(x_i\right)+f\left(x_{i-1}\right)}{2}\right).$$



Geometrically, T_n is the sum of areas of the n trapezoids as shown in the figure. This is called

Trapezoidal approximation to $\int_a^b f(x)\,dx$.

18.1.5 Note:

(i) Note that for every n,

$$L(P_n, f) \le T_n \le U(P_n, f).$$

Since f is integrable, we have by Sandwich theorem,

$$T_n \to \int_a^b f(x)\,dx, \text{ as } n \to \infty.$$

Thus, we can use T_n as an approximation to $\int_a^b f(x)\,dx$.

(ii) It can be shown that if $f:[a,b] \to \mathbb{R}$ is a concave down function, then for every n,

$$T_n \le \int_a^b f(x)\,dx \le M_n,$$

and if f is concave up, then

$$M_n \le \int_a^b f(x)\,dx \le T_n.$$

18.1.6 Error estimates for the mid-point and the trapezoidal rules:

If $f:[a,b] \to \mathbb{R}$ is such that f' is continues on $[a,b]$ and $\left|f''(x)\right| \le \alpha$ for all $x \in (a,b)$, then it can be shown that for every $n \ge 1$, the following hold:

(i) $\left|M_n - \int_a^b f(x)\,dx\right| \le \dfrac{(b-a)^3 \alpha}{24n^2}$.

(ii) $\left|T_n - \int_a^b f(x)\,dx\right| \le \dfrac{(b-a)^3 \alpha}{12n^2}$.

18.1.7 Example

Let

$$f(x) = \frac{1}{x},\ x \in [1, 2].$$

For $n \ge 1$ fixed, consider the partition $P_n = \{x_0, x_1, \dots, x_n\}[1, 2]$, where

$$x_i = 1 + \frac{i}{n},\ i = 0, 1 \dots, n,\ h_n := \frac{1}{n}.$$

Then

$$M_n = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{(x_{i-1}+x_i)/2}$$
$$= \frac{1}{n}\sum_{i=1}^{n} \frac{2}{2+\left(\frac{2i-1}{n}\right)}$$
$$= 2\sum_{i=1}^{n} \frac{1}{2(n+i)-1}$$

and

$$T_n = \frac{1}{2n}\left[\frac{1}{1}+2\left(\frac{1}{\left(1+\frac{1}{n}\right)}+\frac{1}{\left(1+\frac{2}{n}\right)}+\ldots+\frac{1}{\left(2-\frac{1}{n}\right)}\right)+\frac{1}{2}\right]$$
$$= \frac{3}{4n}+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n-1}.$$

Since,

$$|f''(x)| = \left|\frac{2}{x^3}\right| \le 2, \text{ for all } x\in[1,2],$$

we have

$$\left|M_n - \int_1^2 \frac{1}{x}\,dx\right| \le \frac{1}{12n^2},$$

and

$$\left|T_n - \int_1^2 \frac{1}{x}\,dx\right| \le \frac{1}{6n^2}.$$

Thus, to approximate $\int_1^2 \frac{1}{x}\,dx$ with an error less than 10^{-3}, we need to choose n such that

$\frac{1}{12n^2} < 10^{-3}$, i.e., $n \ge 10$ for the mid-point method

and

$\frac{1}{6n^2} < 10^{-3}$, i.e., $n \ge 13$ for the trapezoidal method.

Click here to see a visualization: Applet 18.1

PRACTICE EXERCISES

1. Use $n = 10$ to approximate the following integrals by the mid-point rule and the trapezoidal rule. Find the actual

value and the error. Compare then with the error-estimates as given by 18.1.6:

(i) $\int_1^4 \frac{1}{\sqrt{x}}\,dx$.

(ii) $\int_{-1}^{+1} \frac{1}{2x+3}\,dx$.

(iii) (the actual value of the integral is).

$\int_0^1 \frac{4}{1+x^2}\,dx$ π

(iv) $\int_0^2 \sqrt{4-x^2}\,dx$.

2. Verify, for $n=4$, the claim in 18.1.5 (ii) for the following:

(i) $\int_0^1 x^2\,dx$

(ii) $\int_{-1}^0 x^3\,dx$

3. For the integrals in exercise 1, find a suitable number n of sub intervals for (ii) the mid point approximation and

the trapezoidal approximation, to ensure that the absolute error will be less than the given value:

(i) 5×10^{-5} for (i)

(ii) 10^{-5} for (ii)

(iii) 10^{-6} for (iii)

(iv) 10^{-4} for (iv)

4. Find the trapezoidal approximation with $n=4$ for $\int_0^1 x^{\frac{3}{2}}\,dx$. What can you say about the error-estimate?

Recap

In this section you have learnt the following

- Mid point and the Trapezoidal methods for approximately the integral.

[Section 18.2]

Objectives

In this section you will learn the following :

- Simpson's method for approximating the integral.

SIMPSON'S RULE

18.2 Approximating integrals: Simpson's Rule

In the trapezoidal rule, the integrand f is approximated on each subinterval of the partition by a linear function, i.e., a polynomial of degree 1. We can also use a quadratic polynomial for this as follows.

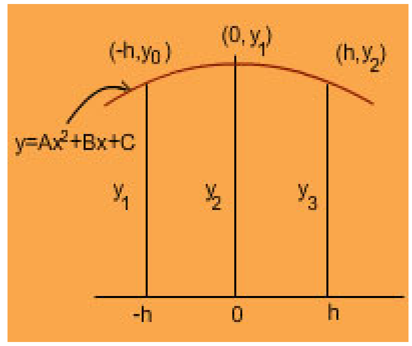
18.2.1 Proposition:

For a given function f, let

$$P(-h, 0), Q(0, y_1), R(h, y_2)$$

be three points on the the graph f.

If $y = Ax^2 + Bx + C$ is a quadratic function passing through the points P, Q and R,



then

$$C = y_1 \text{ and } A = \frac{y_0 - 2y_1 + y_2}{2h^2}.$$

Further, the area A_p bounded by the graph of this quadratic (a parabola), the x-axis and the lines $x = -h$ and $x = h$ is given by

$$A_p = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$

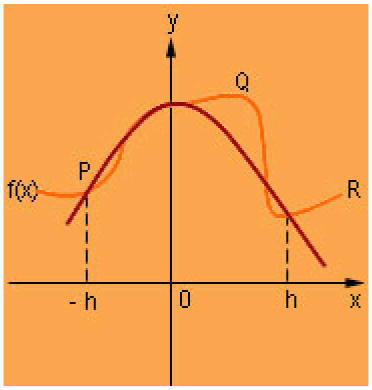
PROOF

18.2.1 Proposition:

For a given function f, let
$P(-h, 0), Q(0, y_1), R(h, y_2)$
be three points on the the graph f.

If $y = Ax^2 + Bx + C$ is a quadratic function passing through the points P, Q and R, then
$C = y_1$ and $A = \frac{y_0 - 2y_1 + y_2}{2h^2}$.
Further, the area A_p bounded by the graph of this quadratic (a parabola), the -axis and the lines $x = -h$ and $x = h$ is given by
$A_p = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$.

Proof:



Since P, Q and R lie on the graph of $y = f(x)$ and also on the quadractic curve, we have $y_1 = C, y_0 = Ah^2 - Bh + C$ and $y_2 = Ah^2 + Bh + C$.
On solving these equations, we obtain

$$A = \frac{y_0 - 2y_1 + y_2}{2h^2}$$

To calculate let us assume, without loss of generality that $x_1 = 0$. Then $x_0 = -h, x_2 = h$.
Now the required area is given by

$$\begin{aligned} A_p &:= \int_{-h}^{h} \left(Ax^2 - Bx + C\right) \\ &= \frac{2Ah^3}{3} + 2Ch \\ &= h\left(\frac{y_0 - 2y_1 + y_2}{3} + 2y_1\right) \\ &= \frac{h}{3}\left(y_0 - 4y_1 + y_2\right). \end{aligned}$$

18.2.2 Simpson's rule:

Given a function $f : [a, b] \to \mathbb{R}$, to approximate $\int_a^b f(x)\, dx$, for $n \geq 1$, fixed, we consider the partition $P_n = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ having odd number of subintervals, i.e, n is even, with

$$x_i - x_{i-1} = h_n := \frac{b-a}{n}, i = 1, \ldots, n,$$

Let

$$y_i = f(x_i), i = 0, 1, \ldots, n,$$

We replace f on each subinterval $[x_{i-1}, x_{i+1}]$ by quadratic curve passing through points (x_{i-1}, y_{i-1}), (x_i, y_i) and $(x_{i+1} - y_{i+1})$, $i = 1, 3, \ldots, n-1$. Using propsition 18.2.1, the integral of this new function is the sum of the areas under the quadratic curves, given by:

$$\begin{aligned}S_n &:= \frac{h_n}{3}\sum_{i=1, i\, odd}^{n-1}\left(y_{i-1}+4y_i+y_{i+1}\right)\\ &= \frac{h_n}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+\ldots+2y_{n-2}+4y_{n-1}+y_n\right)\\ &= \frac{h_n}{3}\left[y_0+4\left(y_1+y_3+\ldots+y_{n-1}\right)+2\left(y_2+y_4+\ldots+y_{n-2}\right)++y_n\right].\end{aligned}$$

The number S_n is called Simpson's approximation to $\int_a^b f(x)\,dx$ and the method is called the Simpson's rule.

18.2.3 Note:

Note that

$$S_n = \frac{1}{6}\left[\left(y_0+y_2+\ldots+y_{n-2}\right)2h_2+4\left(y_1+y_3+\ldots+y_{n-1}\right)2h_n+\left(y_2+y_4+\ldots+y_n\right)2h_n\right].$$

Hence

$$\begin{aligned}\lim_{n\to\infty} S_n &= \frac{1}{6}\left[\int_a^b f(x)\,dx+4\int_a^b f(x)\,dx+\int_a^b f(x)\,dx\right]\\ &= \int_a^b f(x)\,dx.\end{aligned}$$

18.2.4Error Estimates for Simpson's Rule:

Let $f:[a,b]\to\mathbb{R}$ be note that the following holds:

(i) $f^{(4)}(x)$ exists on $[a,b]$ and $\left|f^{(4)}(x)\right|\le\beta$ for all $x\in(a,b)$.

(ii) $f^{(3)}(x)$is continious on $[a,b]$.

Then the following holds:

If f is such that $f^{(3)}(x)$ is continues on $[a,b]$ and $\left|f^{(4)}(x)\right|\le\beta$ for all $x\in(a,b)$, then it can be shown that the following holds:

$$S_n-\int_a^b f(x)\,dx \le \frac{(b-a)^5\beta}{180n^4}.$$

18.2.5Example:

Let

$$f(x)=\frac{1}{x},\ x\in[1,2].$$

Then,

$$\left|f^{(4)}(x)\right| = \left|\frac{24}{x^5}\right| \le 24 \text{ for all } x\in[1,2].$$

For $n\ge 1$ fixed, n even, consider the partition

$P_n = \{x_0, x_1,\ldots,x_n\}$ of $[1,2]$, where $x_i = 1+\frac{i}{n}, i=0,1,\ldots,n, h_n := \frac{1}{n}$.

Then

$$S_n := \frac{1}{3n}\left[1+4\left(\frac{1}{\left(1+\frac{1}{n}\right)}+\frac{1}{\left(1+\frac{3}{n}\right)}+\ldots+\frac{1}{\left(2-\frac{1}{n}\right)}\right)+2\left(\frac{1}{\left(1+\frac{2}{n}\right)}+\frac{1}{\left(1+\frac{4}{n}\right)}+\ldots+\frac{1}{\left(2-\frac{2}{n}\right)}\right)\right]$$

$$=\frac{1}{3n}+\frac{4}{3}\left(\frac{1}{n+1}+\frac{1}{n+3}+\ldots+\frac{1}{2n-1}\right)+\frac{2}{3}\left(\frac{1}{n+2}+\frac{1}{n+4}+\ldots+\frac{1}{2n-2}\right).$$

Hence

$$\left|S_n - \int_1^2 \frac{1}{x}\,dx\right| \le \frac{2}{15n^4}.$$

To approximate $\int_1^2 \frac{1}{x}\,dx$ by Simpson's rule with an error less than 10^{-3}, we need to choose n such that

$$\frac{2}{15n^4} < 10^{-3}, \text{ i.e., } n \ge 4.$$

18.2.6 Note:

To compute the approximations M_n, T_n and S_n of $\int_a^b f(x)\,dx$, we need not know the function on the whole interval $[a, b]$, the values of f only at the partition points $x_0, x_1, \ldots, x_n$ are needed.

Click here to see a visualization: Applet 18.2

PRACTICE EXERCISES

1. Use $n = 10$ to approximate the following integrals by the Simpson's rule. Find the actual value and the error.

   Compare then with the error-estimates as given by 18.2.4:

(i) $\int_1^4 \frac{1}{\sqrt{x}}\,dx$.

(ii) $\int_{-1}^{+1} \frac{1}{2x+3}\,dx$.

(iii) $\int_0^1 \frac{4}{1+x^2}\,dx$ (the actual value of the integral is π).

(iv) $\int_0^2 \sqrt{4-x^2}\,dx$.

2. Verify, for $n = 4$, the claim in 18.2.3 for the following:

(i) $\int_0^1 x^2\,dx$

(ii) $\int_{-1}^0 x^3\,dx$

3. Use Simpson's rule to compute an approximate value for

$$\int_0^2 x^3 dx$$

for n = and compare it with the actual value. Are they same? Can you justify your observations.

4. Use $n=4$ to estimate $\int_0^1 x^{\frac{2}{3}} dx$ by Simpson's rule and compare it with that of trapezoidal rule. Find the actual error. Can you find a bound for the error estimate? Justify.

Recap

In this section you have learnt the following

- Simpson's method for approximating the integral.