Module 9 : Infinite Series, Tests of Convergence, Absolute and Conditional Convergence, Taylor and Maclaurin Series

Lecture 26 : Absolute convergence [Section 26.1]

Objectives

In this section you will learn the following :

- More tests that help in analyzing convergence of series of numbers.

26.1 More tests of convergence:

We describe next a generalization of the comparison test. For that, we need the following result which allows one to compare terms of two sequences.

26.1.1Lemma (Limit Comparison):

Let $\{a_n\}_{n\geq 1}$ *and* $\{b_n\}_{n\geq 1}$ *be sequences of positive real numbers such that*

$$l := \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) \quad \textit{exists}\,.$$

(i) *If* $l \neq 0$, *then there exists positive scalars,* α, β *and* $n_0 \in \mathbb{N}$ *such that*

$$\alpha\, b_n \leq a_n \leq \beta b_n \quad \textit{for all } n \geq n_0.$$

(ii) *If* $l = 0$, *then*

$$0 < a_n < b_n \quad \textit{for all } n \geq n_0.$$

PROOF

26.1.1Lemma:

Let $\{a_n\}_{n\geq 1}$ *and* $\{b_n\}_{n\geq 1}$ *be sequences of positive real numbers such that*

$$l := \lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) \quad \textit{exists}\,.$$

(i) If $l \neq 0$, then there exists positive scalars, α, β and $n_0 \in \mathbb{N}$ such that

$$\alpha b_n \leq a_n \leq \beta b_n \text{ for all } n \geq n_0.$$

(ii) If $l = 0$, then

$$0 < a_n < b_n \text{ for all } n \geq n_0.$$

Proof:

If $l \neq 0$, we can find $n_0 \in \mathbb{N}$ such that
$\frac{l}{2} \leq \frac{a_n}{b_n} \leq \frac{3l}{2}$ for $n \geq n_0$.
The required claim follows with
$\alpha := \frac{l}{2}$ and $\beta = \frac{3l}{2}$.
This proves (i).

In case $l = 0$, given $1 > \in > 0$, there exists n_0 such that
$\frac{a_n}{b_n} < \in$ for $n \geq n_0$.
Thus,
$0 < a_n < \in b_n < b_n$ for $n \geq N$.

26.1.2Examples:

(i) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \geq 1$. Then

$0 < \frac{1}{n} \leq \frac{1}{n^p}$ for every $n \in \mathbb{N}$.

Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (example 25.1.4 (iv)), the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is also divergent for $p \geq 1$.

(ii) Consider the series

$$\sum_{n=1}^{\infty} \frac{n+5}{n^2 - 2n + 5}$$

Apparently, the n th term of the series behaves like $\frac{1}{n}$. Let us consider

$a_n = \frac{n+5}{n^2 - 2n + 5}$ and $b_n = \frac{1}{n}$,
Then

$$\frac{a_n}{b_n} = \frac{n^2 + 5}{n^2 - 2n + 5} = \frac{1 + \frac{5}{n}}{1 - \frac{2}{n} + \frac{5}{n^2}}.$$

Thus

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1$$

Since the series $\sum_{n=1}^{\infty} a_n$ is also divergent.

(iii) Consider the series

$$\sum_{n=1}^{\infty} \frac{3n^2-2n+4}{n^4-n^3+2}.$$

The n^{th} - term of the series will behave like $\frac{3n^2}{n^4} = \frac{3}{n^2}$. In fact, if we take

$a_n = \frac{3n^2-2n+4}{n^4-n^3+2}, b_n = \frac{1}{n^2}$, then

$$\frac{a_n}{b_n} = \frac{3n^2-2n^2+4}{n^4-n^3+2} \text{ x } \frac{n^2}{1}$$

$$= \frac{3n^4-2n^3+4n^2}{n^4-n^3+2}$$

$$= \frac{3 - \frac{2}{n} - \frac{4}{n^2}}{1 - \frac{1}{n} - \frac{2}{n^2}}.$$

Hence

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{3}{1} \neq 0.$$

Since, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, the given series $\sum_{n=1}^{\infty} a_n$ is also convergent.

In comparison test, or the limit comparison test, one needs to guess the convergent / divergence and then select an appropriate series to compare. Some convergence test which are more intrinsic are given next.

26.1.3Theorem (The ratio Test):

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that

$$l = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right).$$

Then the following hold:

(i) If $l < 1$, then the series is convergent.

(ii) If $l > 1$ or $l = +\infty$, then the series is divergent.

(iii) If $l = 1$, the series may converge or diverge.

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Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms such that

$l = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$.

Then the following hold:

(i) If $l < 1$, then the series is convergent.

(ii) If $l > 1$ or $l = +\infty$, then the series is divergent.

(iii) If $l = 1$, the series may converge or diverge.

Proof

(i) For $l < 1$, select $\in > 0$ such that $0 < l + \in < 1$, and choose $N \in \mathbb{N}$ such that

$\left|\frac{a_{n+1}}{a_n} - l\right| < \in$ for $n \geq N$.

Then

$l - \in < \frac{a_{n+1}}{a_n} < l + \in$ for $n \geq N$.

Thus

$a_{n+1} < (\,l + \in)\ a_n$ for $n \geq N$.

Hence, for $k \geq N+1$

$a_k < (\,l + \in)\, a_{k-1} \cdots < (l + \in)^{k-N}\, a_N$

Since $\sum_{k=N+1}^{\infty} (l + \in)^{k-N} a_N$ is a convergent geometric series, as $0 < (\,l + \in) < 1$, by comparison test, $\sum_{k=1}^{\infty} a_k$ is convergent.

26.1.4 Theorem (Root test):

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and suppose that

$$l := \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$$

Then the following hold:

(i) If $l < 1$, then the series is convergent.

(ii) If $l > 1$ or, $l = +\infty$ the series is divergent.

(iii) If $l = 1$, the series may converge or diverge.

26.1.4 Theorem (Root test):

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and suppose that

$l := \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$,

Then the following hold:

(i) If $l<1$, then the series is convergent.

(ii) If $l>1$ or, $l=+\infty$, the series is divergent.

(iii) If $l=1$, the series may converge or diverge.

Proof:

By definition, for $\varepsilon>0$ given, we can choose $N\in\mathbb{N}$ such that

$$l-\varepsilon<(a_n)^{\frac{1}{n}}<l+\varepsilon \text{ for all } n\geq N$$

In case $l<1$, we start with $\varepsilon>0$ such that $0<\alpha:=l+\varepsilon<1$. Then

$$(a_n)^{\frac{1}{n}}<\alpha \text{ for all } n\geq N,$$

i.e.,

$$a_n<\alpha^n \text{ for all } n\geq N.$$

Since, $0<\alpha<1$, the series $\sum_{n\geq N}\alpha^n$ is a convergent series. Thus by comparison test, $\sum_{n=1}^{\infty}a_n$ is also convergent. In case, $\infty>l>1$, we can start with $\varepsilon>0$ such that $1<(l-\varepsilon)$. Then

$$(a_n)^{\frac{1}{n}}>(l-\varepsilon) \text{ for all } n\geq N.$$

26.1.5 Examples:

(i) Consider the series

$$\sum_{n=1}^{\infty}\frac{1}{n!} \text{ and } \sum_{n=1}^{\infty}\frac{n^n}{n!}.$$

Since

$$\lim_{n\to\infty}\left(\frac{\frac{1}{n+1!}}{\frac{1}{n!}}\right)=\lim_{n\to\infty}\left(\frac{1}{n+1}\right)=0<1,$$

the series $\sum_{n=1}^{\infty}\frac{1}{n!}$ is convergent by ratio test. Also

$$\lim_{n\to\infty}\left(\frac{(n+1)^{n+1}}{n+1!}\times\frac{n!}{n^n}\right)=\lim_{n\to\infty}\left(\frac{(n+1)^n}{n}\right)=e>1$$

the series $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ is divergent by ration test.

(ii) Consider the series

$$\sum_{n=1}^{\infty}\left(\frac{4n+5}{2n-1}\right)^n.$$

For this series, the convergence/ divergence is difficult to analyze using, ratio test. However,

$$\lim_{n\to\infty}\left[\left(\frac{4n+5}{2n-1}\right)^n\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(\frac{4n+5}{2n-1}\right)$$

$$= 2 > 1.$$

Thus, the series is divergent by the root test.

We close this section by another test.

26.1.6 Theorem (Integral Test):

Let $f:[1,\infty)\to\mathbb{R}$ be a positive continuous decreasing function with

$$f(x) := a_n,\ n \geq 1$$

Then either both

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)dx$$

converge or diverge.

PROOF

26.1.6 Theorem (Integral Test):

Let $f:[1,\infty)\to\mathbb{R}$ be a positive continuous decreasing function with
$f(x) := a_n,\ n \geq 1$
Then either both
$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$
converge or diverge.

Proof

For $n \in \mathbb{N}$, consider the interval $[1, n]$ with the partition
$P_n := 1, 2, \ldots, n$.
Then, since is decreasing,

$$L(P_n, f) = \sum_{i=2}^{n} f(i) \text{ and } U(P_n, f) = \sum_{i=1}^{n-1} f(i)$$

Thus, if

$$S_n := \sum_{i=1}^{n} a_i,\ n \geq 1,$$

then

$$S_n - a_1 = \sum_{i=2}^{n} f(i) = L(P_{n_1} f) \leq \int_1^n f(x)dx \leq U(P_{n_1} f) := \sum_{i=1}^{n-1} f(i) = S_n \qquad \text{---------(*)}$$

In case $\int_1^{\infty} f(x)\, dx$ is convergent, we have for $n \geq 1$

$$S_n \leq \int_1^{\infty} f(x)\, dx + a_1$$

Since f is positive, is monotonically increasing and hence it is convergent. Conversely, if

$\{S_n\}_{n\geq 1}$

$\lim_{n\to\infty} S_n$ exists, then by the Sandwich theorem, (*) implies that $\int_0^\infty f(x)\,dx$ is convergent.

26.1.7 **Examples:**

(i) p-Series:

Consider the series

$$\sum_{p=1}^{\infty} \frac{1}{n^p}, p \geq 0.$$

Obviously, the series is divergent for $p = 0$, as $a_n = 1$ for even n. If we consider the function

$$f : [1, \infty) \to \mathbb{R},$$

$$f(x) = \frac{1}{x^p},\ x \geq 1,$$

then f is a continuous, positive, decreasing, function. Further, see example . . . ,

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent for $p > 1$ and divergent for $0 < p \leq 1$.

Thus,

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent for $0 \leq p \leq 1$.

(ii) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}.$$

To analyze the convergence/ divergence of this series, we can proceed as follows: Since

$\frac{1}{2n-1} > \frac{1}{2n}$ for every $n \geq 1$,

and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (p=1 for the p-series), by comparison test, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ is also divergent. We could directly apply the integral test with $f(x) = \frac{1}{2x-1}$, $x \geq 1$. As

$$\int_1^\infty \frac{dx}{2x-1} = \lim_{k\to+\infty} \int_1^k = \lim_{k\to\infty}\left[\frac{1}{2} l_n(2k-1)\right] = +\infty,$$

we can conclude that the series

$\sum_{n=1}^{\infty} \frac{1}{2n-1} = \sum_{n=1}^{\infty} f(n)$ is divergent.

26.1.8 Note (Basic strategy for testing convergence):

(i) As a general rule, check $\lim_{n\to\infty} a_n$. If

$\lim_{n\to\infty} a_n \neq 0$, the series is divergent.

If $\lim_{n\to\infty} a_n = 0$ try convergent tests as suggested next.

(ii) If $\{a_n\}_{n\geq 1}$ is a decreasing sequence of positive terms, such that $f(n) = a_n$ for some function $f:[1,0) \to \mathbb{R}$

try Integral test.

(iii) If a_n is a rational function, or is some root of n, try limit comparison test.

(iv) Some of the standard series for comparison test are: Geometric series, p-series.

(v) Ratio test is useful if a_n has factorial/ powers of n

(vi) Root test is useful, if it is series to find n^{th} root of a_n

Practice Exercises

1. Using limit comparison test, determine the convergence/ divergence of the following series:

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+4}$.

(ii) $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2(n+3)^2}$.

(iii) $\sum_{n=1}^{\infty} \frac{ln(n)}{n^{\frac{5}{2}}}$ (Hint $\ln(n)$ glows more slowly than n^{α} for every $\alpha > 0$)

Answers

2. Analyze the convergence of the following series using the ratio test:

(i) $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

(ii) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$.

(iii) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$.

Answers

3. Analyze the convergence of the following using the root test:

(i) $\sum_{n=1}^{\infty} \left(\frac{2n+1}{n-1}\right)$.

(ii) $\sum_{n=1}^{\infty} \frac{n}{4^n}$.

(iii) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

Answers

4. Prove the following:

(i) For the p-series, both the ratio test and the root test tail, however the series is convergent.

(ii) For the series, for $p > 0$,

for $p > 0$,

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(x))^p},$$

both the ratio test and the root test tail, but the series is divergent (by comparison test).

5. Canchy's Condensation Test

Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence of positive terms. Let

$$S_n = a_1 + a_2 + \dots + a_n \text{ and } t_n = a_1 + 2a_2 + \dots + 2^n a_{2^n}.$$

Prove the following:

(i) For every $n \geq 1$,

$$S_{2^n} \leq t_n \leq 2S_{2^n}.$$

(ii) Deduce that the series $\sum_{n=1}^{\infty} a_n$ is convergent if and if the series $\sum_{n=1}^{\infty} 2^n a_{2^n}$ is convergent.

6. Using exercise (5), deduce that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n \ln(n)^p}$$

are convergent for $p > 1$ and divergent for $0 \geq p \geq 1$.

Recap

- In this section you have learnt the following

More tests that help in analyzing convergence of series of numbers.

Series, Tests of Convergence, Absolute and Conditional Convergence, Taylor and Maclaurin Series

Lecture 26 : Conditional convergence [Section 26.2]

Objectives

In this section you will learn the following :

- Absolute convergence of series.
- Conditional Convergence of series.

26.2 Absolute and Conditional Convergence

In the previous section we saw that most of the convergence tests were applicable for series with positive terms. When, this is not the case, series can behave differently. In example 25.1.4(V) we saw that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is convergent, while the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is not convergent. To analyze such occurrences in detail , we make the following definition.

26.2.1 Definition:

Let $\sum_{n=1}^{\infty} a_n$ be a series of real members.

(i) We say $\sum_{n=1}^{\infty} a_n$ is ***absolutely convergent*** if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

(ii) We say the series $\sum_{n=1}^{\infty} a_n$ is ***absolutely divergent*** if $\sum_{n=1}^{\infty} |a_n|$ is divergent.

(iii) We say the series $\sum_{n=1}^{\infty} a_n$ is ***conditionally convergent*** if $\sum_{n=1}^{\infty} a_n$ is convergent, but $\sum_{n=1}^{\infty} |a_n|$ is not convergent.

(iv) We say the series $\sum_{n=1}^{\infty} a_n$ is an ***alternating series*** if either

$$\begin{cases} a_n > 0 \text{ for } n \text{ even} \\ a_n < 0 \text{ for } n \text{ odd} \end{cases}$$

or

$$\begin{cases} a_n < 0 \text{ for } n \text{ even} \\ a_n > 0 \text{ for } n \text{ odd.} \end{cases}$$

26.2.2 Note (Tests for absolute convergence):

The tests of section 26.2.1 namely, the comparison test, limit comparison test, ratio test, and root test, all are tests for absolute convergence.

26.2.3 Examples:

(i) The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

(ii) The series

$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$$

is absolutely convergent, since

$$\left| \frac{\cos(n)}{n^2} \right| \le \frac{1}{n^2}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

(iii) Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^2\, 2^n}{n!}.$$

Let

$$a_n := (-1)^n \frac{2^n}{n!}.$$

Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{2}{n+1} \right) = 0 < 1.$$

Then by ratio test, the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

The relation between convergence and absolute convergence of a series is described in the next theorem.

26.2.4 Theorem:

If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is also convergent.

26.2.4 Theorem:

If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is also convergent.

Proof

Let

$b_n := a_n + |a_n|,\ n \ge 1$

Then

$$b_n = \begin{cases} 0 & \text{if } a_n < 0 \\ 2|a_n| & \text{if } a_n > 0. \end{cases}$$

Thus,

$b_n \le 2|a_n|$ for every n

Since $\sum_{n=1}^{\infty}|a_n|$ is convergent, by comparison test, $\sum_{n=1}^{\infty}b_n$ is also convergent. Hence, by theorem 25.2.4, since

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - \sum |a_n|,$$

26.2.5 Examples:

Let

$$a_n = \begin{cases} 1 & \text{if } \quad n = 1 \\ \dfrac{(-1)^n}{2^n} & \text{if } \quad n \text{ is a prime} \\ \dfrac{1}{2^n} & \text{otherwise.} \end{cases}$$

Note that, $\sum_{n=1}^{\infty} a_n$ is not a geometric series. However, $\sum_{n=1}^{\infty}|a_n|$ is a geometric series with common-ratio $\frac{1}{2}$. Hence, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and thus is itself convergent.

Finally, we give a test which helps us to analyze convergence of an alternating series.

26.2.6 Theorem (Alternating series test):

Let $\sum_{n=1}^{\infty} a_n$ be an alternating series such that

(i) $|a_1| \ge |a_2| \ge |a_3| \ge \ldots \ge |a_n| \ge \ldots$

(ii) $\lim\limits_{n \to \infty} |a_n| = 0$. Then $\sum_{n=1}^{\infty} a_n$ is convergent.

26.2.7 Examples:

(i) Consider the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Clearly, for

$$a_n = \frac{(-1)^{n+1}}{n},$$

the sequence $\{|a_n|\}_{n \ge 1}$ is decreasing and

$$\left|a_n\right| \to 0 \text{ as } n \to \infty$$

Hence, the above series is convergent.

(ii) Consider the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n^2}$$

This is an alternating series with

$$a_n = (-1)^{n+1} \frac{2^n}{n^2}$$

Since

$$\lim_{n \to \infty} \left|a_n\right| = \lim_{n \to \infty} \left(\frac{2^n}{n^2} \right)$$

$$= \lim_{n \to \infty} \left(\frac{n 2^{n-1}.2}{2n} \right)$$

$$= +\infty,$$

the series is divergent.

\

(iii) Consider alternating series

$$\sum_{n=1}^{\infty} (-1)^n \ln\left(1 + \frac{1}{n}\right).$$

Let

$$f(x) = \ln\left(1 + \frac{1}{x}\right) \text{ for } x > 0, \text{ and } a_n = (-1)^n \ln\left(1 + \frac{1}{n}\right)$$

Then

$$f(n) = \left|a_n\right|, n \geq 1$$

Since

$$f'(x) = \frac{1}{x(1+x)} < 0 \text{ for } x > 0,$$

f is a monotonically decreasing function. Thus

$$\left|a_{n+1}\right| = f(n+1) < f(n) = \left|a_n\right| \text{ for all } n.$$

Further

$$\lim_{n \to \infty} \left|a_n\right| = \lim_{n \to \infty} \ln\left(1 + \frac{1}{n}\right)$$

$$= \ln(1)$$

$$= 0.$$

Hence, by alternating series test, the above series is convergent.

26.2.8 Note:

(i) The alternating series test not only gives the convergence of the series, in fact, if

$$S = \sum_{n=1}^{\infty} a_n,$$

then

$$\left|S - S_n\right| \leq a_{n+1} \text{ for all } n.$$

(ii) If $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series, and any rearrangement of the series does not affect its convergence or its sum. However, this is not the case with an alternating series. In fact, if a alternating series is convergent, then by a suitable rearrangement, it can be made to converge to any given real numbers. For more elaboration reader may consult any book on Real Analysis.

PRACTICE EXCERCISES

1. Show that the following alternating series are convergent.

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{ln(n)}$.

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n-1}}$.

(iv) $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt{n}}$.

2. Show that the following alternating series are absolutely convergent

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

(ii) $\sum_{n=1}^{\infty} \frac{\sin \frac{(2n-1)^{\pi}}{2}}{n^{\frac{3}{2}}}$.

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!}$.

(iv) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^3+1}$.

3. Show that the following series are conditionally convergent:

(i) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1+ln(n)}$.

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{2+5n}$.

(iii) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt{n}}$.

(iv) $\sum_{n=1}^{\infty} (-1)^{n} \left(\sqrt{n}-\sqrt{n+1}\right)$.

4. Prove the following statements:

(i) If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then

$$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|.$$

(ii) If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both absolutely convergent, then so are the series $\sum_{n=1}^{\infty} (a_n + b_n)$ and $\sum_{n=1}^{\infty} (a_n - b_n)$.

5. Let $\sum_{n=1}^{\infty} a_n$ series. Define for all $n \ge 1$,

$$a_n^+ := \begin{cases} a_n & \text{if } a_n \ge 0 \\ 0 & \text{if } a_n < 0. \end{cases}$$

$$a_n^- := \begin{cases} 0 & \text{if } a_n \ge 0 \\ -a_n & \text{if } a_n < 0. \end{cases}$$

The series $\sum_{n=1}^{\infty} a^+{}_n$ is called the positive part of the series and the series $\sum_{n=1}^{\infty} a_n^-$ is called the negative part of the series. Prove the following:

(i) $a_n = a_n^+ - a_n^-$,
$|a_n| = a_n^+ - a_n^-$

(ii) If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then both $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent series (of non negative terms).

Recap

In this section you have learnt the following

- Absolute convergence of series.
- Conditional Convergence of series.