

Module 4 : Local / Global Maximum / Minimum and Curve Sketching

Lecture 11 : Absolute Maximum / Minimum [Section 11.1]

Objectives

In this section you will learn the following :

- How to find points of absolute maximum / minimum for a function.

11.1 Absolute Maximum/Minimum:

11.1.1 Definition:

Let $f : [a, b] \rightarrow \mathbb{R}$.

(i) A point $c_1 \in [a, b]$ is called a point of absolute maximum for f if

$$f(c_1) \geq f(x) \text{ for all } x \in [a, b].$$

The value $f(c_1)$ is called the absolute maxima of f .

(ii) A point $c_2 \in [a, b]$ is called a point of absolute minimum for f if

$$f(c_2) \leq f(x) \text{ for all } x \in [a, b].$$

The value $f(c_2)$ is called the absolute minima of f .

Recall that for every continuous function $f : [a, b] \rightarrow \mathbb{R}$ there exist points in $c_1, c_2 \in [a, b]$ such that $f(c_1)$ is absolute maximum and $f(c_2)$ is absolute minimum. We shall analyze method of finding such points.

11.1.2 Definition:

For a function $f : [a, b] \rightarrow \mathbb{R}$, a point $x \in [a, b]$ is called a critical point of f if $x \in [a, b]$ and either f is not differentiable at x , or f is differentiable at x with $f'(x) = 0$.

11.1.3 Theorem:

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then, f assumes its absolute maximum as well as its absolute minimum at some points which are either critical point of f or are the end points a and b .

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Proof:

Since f is continuous, there exists $c_1, c_2 \in (a, b)$ such that $f(c_1)$ is absolute maximum and $f(c_2)$ is absolute minimum. Suppose that neither of them is any one of the end points a or b . Then, $c_1, c_2 \in (a, b)$. Note that, c_1 is a point of local maximum and c_2 is a point of local minimum also. Now either f is not differentiable at c_1 or if f is differentiable at c_1 , then $f'(c_1) = 0$ by lemma 9.1.4. Similarly, either f is not differentiable at c_2 or if f is differentiable at c_2 , then, $f'(c_2) = 0$, again by lemma 9.1.4.

**11.1.4 Method for locating absolute maximum/ minimum:**

Above theorem tells us that to explore the absolute maximum/ minimum f on $[a, b]$, we need only to find the critical points of f in $[a, b]$, and compare the values of f at these points.

11.1.5 Examples:

(i) Let

$$f(x) = \begin{cases} -x, & \text{if } -1 \leq x \leq 0 \\ 4x - 2x^2, & \text{if } 0 < x \leq 2. \end{cases}$$

We note that f is not differentiable at $x = 0$, since

$$f'_-(0) = -1 \text{ and } f'_+(0) = 4.$$

Further,

$$f'(x) = \begin{cases} -1, & \text{if } -1 \leq x < 0 \\ 4(1-x), & \text{if } 0 < x < 2 \end{cases}$$

So $f'(x) = 0$ for $x = 1$. Thus, the critical points for f are $x = -1, 0, +1, 2$ and

$$f(-1) = 1, f(0) = 0, f(1) = 2, f(2) = 0.$$

Hence, f assumes its maximum at $x = 1$, and its minimum at $x = 0$ and $x = 2$.

(ii) Let us find local maximum/ local minimum, absolute maximum and absolute minimum of f given by

$$f(x) = |x - x^2|, \quad x \in [-2, 2].$$

Since f is continuous, it is bounded and hence, attains its absolute maximum and absolute minimum.

Note that $x(1-x) \geq 0$ if and only if $x \geq 0$ and $1-x \geq 0$ or $x \leq 0$ and $1-x \leq 0$. Thus,

$$f(x) = \begin{cases} x(1-x) & \text{if } x \in [0, 1], \\ -x(1-x) & \text{if } x \in [-2, 0] \cup [1, 2] \end{cases}$$

Since f is not differentiable at $x=0, 1$, and $f'(x) = 0$, for $x = 1/2$, the critical points of f , are $-2, 0, 1/2, 1$ and 2 . Further,

$$f(-2) = 6, f(0) = 0, f(1) = 0, f(1/2) = 1/4, f(2) = 2$$

So, f has absolute maximum at $x = -2$ and absolute minimum at $x = 0$ and $x = 1$. Since

$$f''(1/2) = -2 \times \frac{1}{2} = -1 < 0,$$

f has local maximum at $x = 1/2$. The absolute maximum for f is 6 and absolute minimum is 0.

11.1.6 Remarks:

- (i) Even if a function f is continuous at $x=c$ and has a global (local) maximum at c , it is not necessary that f will be increasing on $(c-\delta, c)$ or decreasing on $(c, c+\delta)$ for some $\delta > 0$. For example, consider the function

$$f(x) = -\left|x \sin \frac{1}{x}\right| \text{ for } x \neq 0 \text{ and } f(0) = 0.$$

The function is continuous at $x=0$, and has global maximum at $x=0$. But f takes all values between -1 and 0 at points arbitrarily close to $x=0$. Hence it is neither increasing nor decreasing in any neighborhood of the point $x=0$.

(See figure in the next slide)

- (ii) Even if a function f has a local maximum at a point c and $f''(c)$ exists, it need not imply that $f''(c) < 0$. For example consider the function $f(x) = -x^4$ for $x \in \mathbb{R}$. It has local (in fact global) minimum at the point $x=0$, and $f''(0) = 0$.

(iii) A function f can have a local (global) maximum / minimum at a point $c \in (a, b)$ without being differentiable or

even continuous at c . For example, let

$$f(x) = \begin{cases} 1, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

Then f has global minimum at $x = 0$, but f is not continuous at $x = 0$.

11.1.7 Examples:

(i) Let us analyze absolute maximum/ minimum of the function

$$f(x) = x + \frac{4}{x}, \quad x > 0.$$

Though the function f is continuous, its domain is not a closed and bounded interval. Thus, we are not sure that the function has maximum or minimum. However, f is differentiable and

$$f'(x) = 1 - \frac{4}{x^2} = 0$$

gives $x = 2$ in the domain of f . Clearly,

$$f'(x) < 0 \text{ for } 0 < x < 2 \text{ and } f'(x) > 0 \text{ for } x > 2.$$

Thus, f is strictly increasing in every interval of the form $(-a, 2)$, and is strictly decreasing in every interval of the form $(2, +a)$, for every $a > 0$. Since f is continuous at $x = 2$, this implies that

$$f(x) \geq f(2) \text{ for every } x > 0.$$

Hence, f has global minimum at $x = 2$.

(ii) Let

$$f(x) = x^{\frac{5}{2}}(x-1)^{\frac{4}{3}}, \quad x \in [0, 1].$$

We want to find absolute maxima and absolute minima of f . Since

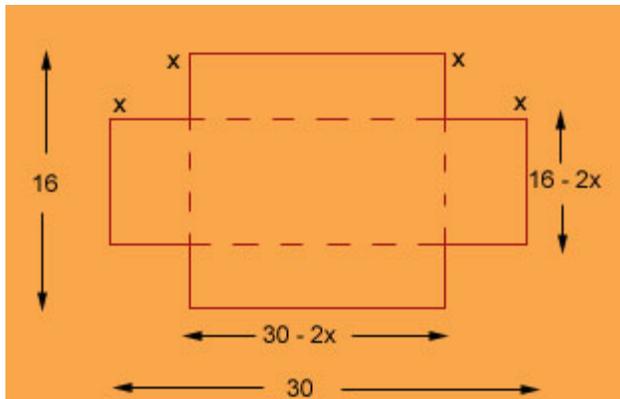
$$\begin{aligned}
 f'(x) &= \frac{5}{2}x^{\frac{3}{2}}(x-1)^{\frac{4}{3}} + x^{\frac{5}{2}}\frac{4}{3}(x-1)^{\frac{1}{3}} \\
 &= x^{\frac{3}{2}}(x-1)^{\frac{1}{3}} \left[\frac{5(x-1)}{2} + \frac{4x}{3} \right] \\
 &= x^{\frac{3}{2}}(x-1)^{\frac{1}{3}} \left(\frac{23x}{6} - \frac{5}{2} \right)
 \end{aligned}$$

The critical points of f in the given domain are $x = 0$, $x = 1$ and

$$\frac{23x}{6} - \frac{5}{2} = 0, \text{ i.e. } x = \frac{15}{23}.$$

Comparing the values of f at 0 , $\frac{15}{23}$ and 1 , we get absolute minima to be $f(0) = f(1) = 0$ and absolute maxima to be $f\left(\frac{15}{23}\right)$.

- (iii) A rectangular box, open at the top, is made from a metal sheet of 16cm x 30cm by cutting out four squares of equal size from the four corners and bending the sides. We want to find the size of the square so that the box has largest volume.



Let x be the side of the each square to be removed. Then, $0 \leq x \leq 8$. The volume of the box is

$$V(x) = (16 - 2x)(30 - 2x)x = 4x^3 - 92x^2 + 480x,$$

where $0 \leq x \leq 8$. To maximize $V(x)$, note that

$$\begin{aligned}
 V'(x) &= 12x^2 - 184x + 480 \\
 &= 4(3x - 10)(x - 12).
 \end{aligned}$$

Thus, the critical points of $V(x)$ are $x = \frac{10}{3}$ and $x = 12$. Since, only $0 \leq \frac{10}{3} \leq 8$,

$$V(0) = 0, V(8) = 0, V\left(\frac{10}{3}\right) = \frac{19600}{27} > 0,$$

the maximum value for $V(x)$ is $V\left(\frac{10}{3}\right)$.



PRACTICE EXERCISES

1. Find the global maximum and the global minimum of f on $[-2, 5]$, where

$$f(x) = 1 + 12|x| - 3x^2.$$

2. Analyze the function f for points of absolute maximum/minimum:

$$f(x) = \begin{cases} 1 - |x|, & \text{for } |x| \leq 1, \\ 0, & \text{for } 1 < |x| \leq 2. \end{cases}$$

3. Find points of absolute maximum and absolute minimum of the function

$$f(x) = 4 \sin x + 3 \cos x,$$

in the intervals $[0, \pi/2]$.

4. A window is to be made in the form of a rectangle surmounted by a semicircular portion with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass and the semicircular portion is to be of colored glass admitting only half as much light per square foot as the clear glass. If the total perimeter of the window frame is to be p feet, find the dimensions of the window which will admit the maximum light.

5. The cost of per hour of fuel to run a truck is $\frac{v^2}{25}$ Rupees, where v is the speed in km/hour. There is an additional cost of Rs.1000/- per hour to run the truck. What should be the speed to minimize the cost per km.

6. Find the ratio of the height to the diameter of the base of cylinder of maximum volume that can be inscribed in a sphere of radius R .

7. Find the maximum and the minimum value of the function

$$f(x) = x^{10}(x-1)^{12}, x \in [0, 1].$$

8. Find p, q such that $f(x) = x^2 + px + q$ has absolute maxima / minima at $x = 1$, with $f(1) = 3$, in the interval $[0, 2]$.

9. Locate absolute maxima and minima of $f(x) = 2 \sin 2x + \sin 4x$, $x \in [0, 2\pi]$, and hence for $x \in \mathbb{R}$.

10. Let $y = f(x)$ be given by

$$x(t) = t - \sin t$$

$$y(t) = 2 - 2 \cos t.$$

Find absolute maximum/ minimum for f for $t \in [0, 2\pi]$.

Recap

In this section you have learnt the following

- How to find points of absolute maximum / minimum for a function.

