Module 17 : Surfaces, Surface Area, Surface integrals, Divergence Theorem and applications

Lecture 50 : Surface Integrals [Section 50.1]

Objectives

In this section you will learn the following :

- How to define the integrals of a scalar field over a surface.

50.1 Surface Integrals :

Similar to the integral of a scalar field over a curve, which we called the line integral, we can define the integral of a vector-field over a surface.

Let S be a surface in space with finite surface area. Let f be a continuous scalar-field defined on the surface S .We can subdivide S into smaller portions, say $S_1, S_2, \ldots, S_n$ having areas $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$, and form the sum

$$\sigma_k := \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta S_k,$$

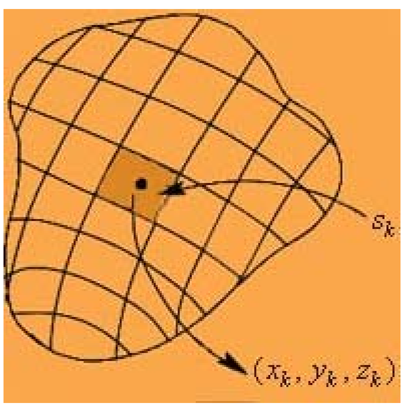

Figure: Subdivision of the surface

where $(x_k, y_k, z_k) \in S_k$, is selected arbitrarily. By refining the patches into more smaller patches such that $\max(\Delta S_k) \to 0$, if σ_k approaches a limit, we call it the surface integral of f over S , and denote it by

$$\iint_S f\ (x,y,z)dS.$$

50.1.1 Definition :

Let S be a surface with parameterization

$$\mathbf{r} : R \subseteq \mathbb{R}^2 \to \mathbb{R}^3, (u,v) \mapsto \mathbf{r}(u,v), (u,v) \in R.$$

If $\mathbf{r}(u,v)$ is continuous and R is closed and bounded, then for a continuous function $f : R \to \mathbb{R}^3$, we can define

$$\iint_S f\ (x,y,z)\ dS := \iint_R f(x(u,v), y(u,v), z(u,v)) \| \mathbf{r}_u \times \mathbf{r}_v \| \, du\, dv,$$

called the **surface integral** of f over the surface S.

50.1.2 Example :

Let us evaluate the surface integral

$$\iint_S y^2\, dS,$$

where S is the sphere

$$\left\{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \right\}.$$

We give S the spherical coordinate parameterization

$$\mathbf{r}\ (\theta,\phi) = \cos\theta \sin\phi\, \mathbf{i} + \sin\theta\, \sin\phi + \cos\phi\, \mathbf{k},\ (\theta,\phi) \in [0,2\pi] \times [0,\pi].$$

Then

$$\mathbf{r}_\theta = -\sin\theta \sin\phi\, \mathbf{i} + \cos\theta\, \sin\phi\, \mathbf{j},$$

and

$$\mathbf{r}_\phi = \cos\theta\, \cos\phi\, \mathbf{i} + \sin\theta\, \cos\phi\, \mathbf{j} - \sin\phi\, \mathbf{k}.$$

Thus

$$\begin{aligned} \mathbf{r}_\theta \times \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta \sin\phi & \cos\theta \sin\phi & 0 \\ \cos\theta \cos\phi & \sin\theta \cos\phi & -\sin\phi \end{vmatrix} \\ &= \left(-\sin^2\phi\, \cos\theta\right) \mathbf{i} - \left(+\sin^2\phi\, \sin\theta\right) \mathbf{j} \\ &\quad + \left(-\sin^2\theta\, \sin\phi \cos\phi - \cos^2\theta\, \sin\phi \cos\theta\right) \mathbf{k}. \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathbf{r}_\theta \times \mathbf{r}_\phi\|^2 &= \sin^4\phi\, (\cos^2\theta + \sin^2\theta) + \sin^2\phi \cos^3\phi \\ &= \sin^2\phi\, (\sin^2\phi + \cos^2\phi) = \sin^2\phi. \end{aligned}$$

Thus,

$$\|\mathbf{r}_\theta \times \mathbf{r}_\phi\| = \sin\phi.$$

This gives, for $R = [0,2\pi] \times [0,\pi]$,

$$\iint_S y^2 ds = \iint_R \sin^2\phi \sin^2\theta \sin\phi \, d\theta \, d\phi$$
$$= \int_0^{2\pi} \int_0^{\pi} \sin^3\phi \sin^2\theta \, d\theta \, d\phi$$
$$= \int_0^{2\pi} \left[\int_0^{\pi} \sin^3 d\phi\right] \sin^2\theta \, d\theta$$
$$= \int_0^{2\pi} \left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi} \sin^2\theta \, d\theta$$
$$= \frac{4}{3}\int_0^{2\pi} \sin^2\theta \, d\theta$$
$$= \frac{4}{3}\left[\frac{\theta}{2} + \sin 2\theta\right]_0^{2\pi}$$
$$= \frac{4\pi}{3}.$$

50.1.3 Surface Integral for surfaces in explicit form :

For a smooth surface given explicitly as

$$S = \{(x,y,z) \mid z = h(x,y) \text{ for } (x,y) \in D\},$$

a parameterization is given by

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + h(x,y)\,\mathbf{k}, \ (x,y) \in D.$$

Since,

$$\|\mathbf{r}_x \times \mathbf{r}_y\| = \sqrt{1 + h_x^2 + h_y^2}$$

we have

$$\iint_S f(x,y,z)\, dS = \iint_R f(x,y,h(x,y)) \sqrt{1 + h_x^2 + h_y^2}\, dx\,dy.$$

Similarly, if S is given by

$$S = \{(x,y,z) \mid x = g(y,z), (y,z) \in R\},$$

then

$$\iint_S f(x,y,z)\, dS = \iint_R f(g(y,z),y,z) \sqrt{1 + g_y^2 + g_z^2}\, dy\,dz.$$

Finally, if S is given by

$$S = \{(x,y,z) \mid y = k(x,y), (x,y) \in R\},$$

then

$$\iint_S f(x,y,z)\, dS = \iint_R f(x,k(x,z),z) \sqrt{1 + k_x^2 + k_z^2}\, dx\,dz.$$

50.1.4 Example :

Let us evaluate the integral

$$\iint_R f dS,$$

where $f(x,y,z)=z^2$ and S is the surface of the cone $z^2=x^2+y^2$ between the planes $z=1$ and $z=2$.
We can give the surface the following parameterization:

$$\mathbf{r}(x,y)=x\,\mathbf{i}+y\,\mathbf{j}+\left(\sqrt{x^2+y^2}\right)\mathbf{k},\ (x,y)\in R,$$

where R is the projection of the surface on the xy-plane,

$$R:=\left\{(x,y)\mid 1\le x^2+y^2\le 2\right\}.$$

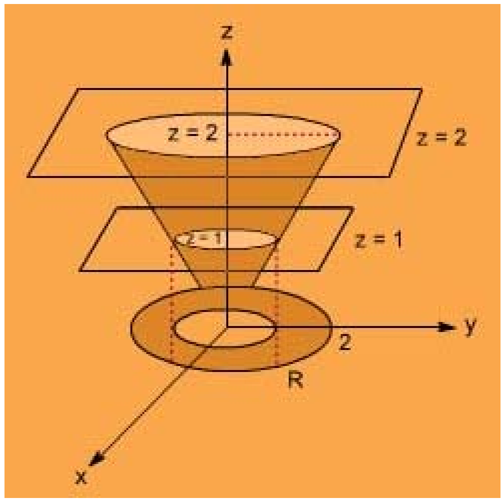

Figure: Surface S and its projection R

Since $\|\mathbf{r}_x\times\mathbf{r}_y\|=\sqrt{2}$, we have

$$\begin{aligned}\iint_S f\,dS &= \iint_R (x^2+y^2)\,\sqrt{2}\,dx\,dy\\ &=\int_{r=1}^{2}\int_0^{2\pi}\sqrt{2}\,r^2 r\,dr\,d\theta \text{(Using polar coordinates)}\\ &=2\sqrt{2}\,\pi\int_1^2 r^3 dr\\ &=2\sqrt{2}\,\pi\left[\frac{16}{4}-\frac{1}{4}\right]\\ &=\frac{15\sqrt{2}\,\pi}{2}.\end{aligned}$$

50.1.5 Note :

Recall that, for a surface S given explicitly by $z=h(x,y),(x,y)\in D$, the surface integral of a scalar field f over S is given by

$$\iint_S f\,dS = \iint_D f(x,y,h(x,y))\,\sqrt{1+h_x^2+h_y^2}\,dx\,dy,$$

where D is the projection of S onto the xy-plane. Thus,

$$\iint_S f(x,y,z)\left(\frac{1}{\sqrt{1+h_x^2+h_y^2}}\right)dS = \iint_D f\,(x,y,h(x,y))\,dx\,dy. \qquad ----------(64)$$

Since S has parameterization

$$\mathbf{r}(x,y)=x\,\mathbf{i}+y\,\mathbf{j}+h(x,y)\,\mathbf{k},\ (x,y)\in D,$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = -h_x\,\mathbf{i} - h_y\,\mathbf{j} + \mathbf{k}, \qquad \text{----------(65)}$$

this gives,

$$\sqrt{1+h_x^2+h_y^2} = \|\mathbf{r}_x \times \mathbf{r}_y\|. \qquad \text{----------(66)}$$

Using (65), we have

$$1 = (\mathbf{r}_x \times \mathbf{r}_y)\cdot\mathbf{k} = \|\mathbf{r}_x \times \mathbf{r}_y\| \cos\gamma, \qquad \text{----------(67)}$$

where γ is the acute angle between $\mathbf{r}_x \times \mathbf{r}_y$, the normal to S, and $\mathbf{k}$. From (66) and (67), we have

$$\cos\gamma = \frac{1}{\sqrt{1+h_x^2+h_y^2}}.$$

Hence, (64) gives us the relation

$$\iint_S f\,\cos\gamma\,dS = \iint_D f(x,y,h(x,y))\,dx\,dy.$$

Practice Exercises

1. Evaluate the surface integral

   $$\iint_S (y^2+2yz)\,dS,$$

   where S is portion of the plane $2x+y+2z=6$ in the first octant

   Answer: $\frac{24z}{2}$

2. Evaluate

   $$\iint_S (x+z)\,dS$$

   where S is the portion of the cylinder $y^2+z^2=9$ in the first octant between the planes $x=0$ and $x=4$.

   Answer: $12\pi+36$

3. Evaluate

   $$\iint_s x\sqrt{y^2+4}\,dS,$$

   where S is the portion of the cylinder $y^2+4z=16$ cut by the planes $x=0$, $x=1$ and $z=0$.

   Answer: $\frac{56}{3}$

Recap:

In this section you have learnt the following

- How to define the integrals of a scalar field over a surface.

[Section 50.2]

Objectives

In this section you will learn the following :

- Some application of the surface integrals.

50.2 Applications of surface integrals :

50.2.1 Mass and center of mass of a surface.

Consider a surface S of density (mass per unit area) $\rho(x,y,z),(x,y,z)\in S$. Then the mass of S can be defined to be

$$M := \iint_S \rho(x,y,z)\, dS,$$

The moments of S about the three axes planes is defined by

$$\iint_S x\,\rho(x,y,z)\, dS\,,\ \iint_S y\,\rho(x,y,z)\, dS\,,\ \iint_S z\,\rho(x,y,z)\, dS\,.$$

Further the point $(\bar{x},\bar{y},\bar{z})$ is called the **center of mass** of S, where

$$\bar{x} = \frac{\iint_S x\,\rho(x,y,z)\, dS}{\iint_S \rho(x,y,z)\, dS}\,,$$

$$\bar{y} = \frac{\iint_S y\,\rho(x,y,z)\, dS}{\iint_S \rho(x,y,z)\, dS}\,,$$

$$\bar{z} = \frac{\iint_S z\,\rho(x,y,z)\, dS}{\iint_S \rho(x,y,z)\, dS}\,.$$

50.2.2 Flux of a fluid across a surface

Let $\mathbf{V}(x,y,z)$ represent the velocity field of a fluid flow in space at a point (x,y,z). Let $\rho(x,y,z)$ be its density at (x,y,z). Then $\mathbf{F}(x,y,z)=\rho(x,y,z)\,\mathbf{V}(x,y,z)$,

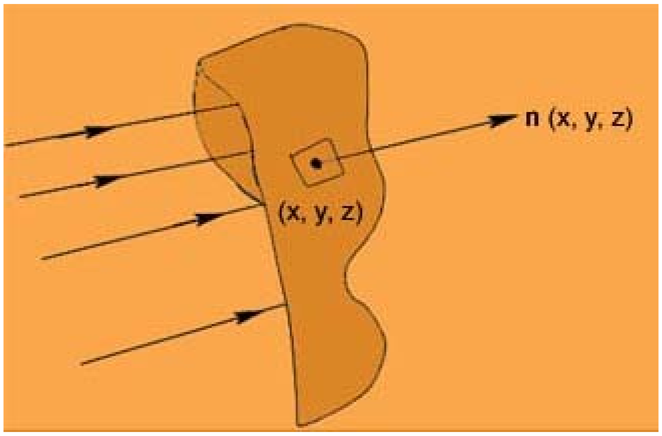

Figure: Flow across a surface

represents the **flux-density** (mass per unit area per unit time) of the flow. Consider a surface S in the flow. If S is smooth, then the flux-density across a small patch ΔS of the surface at a point $(x,y,z)\in S$ is given by the normal component of $\mathbf{F}$, i.e., $\mathbf{F}\cdot\mathbf{n}$. Thus, the mass of the fluid flow across ΔS can be taken to be $(\mathbf{F}\cdot\mathbf{n})\Delta S$, where $\mathbf{n}$ is the unit normal at (x,y,z). Thus, the total mass of the fluid crossing across the surface S can be defined to be

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS. \qquad \text{----------(68)}$$

In order to be able to do so, it becomes necessary to ensure that the function

$$(\mathbf{F}\cdot\mathbf{n})(x,y,z),\,(x,y,z)\in S$$

is integrable over S. For example, this will be so if $(x,y,z)\mapsto(\mathbf{F}\cdot\mathbf{n})(x,y,z)$ is continuous. For this, we can assume that $\mathbf{F}$ is continuous. Thus, to be able to define (68), we should be able to say that our surface S is such that at every point $(x,y,z)\in S$, three exist unit normal $\mathbf{n}(x,y,z)$ which varies continuously as (x,y,z) very over S. This motivates our next definition:

50.2.3 Definition :

A surface S is said to be **orientable** if there exists a continuous vector-field

$$(x,y,z)\mapsto\mathbf{n}(x,y,z),\,(x,y,z)\in S$$

such that $\mathbf{n}(x,y,z)$ is the unit normal vector to S at $(x,y,z)\in S$.

Orientability of a surface essentially means that there are two sides of the surface.

50.2.4 Examples :

1. Every simple closed surface is orientable, we can have a continuous inward or an outward normal to the surface.
   For example, surfaces like sphere, ellipsoid, etc, are all orientable, with a continuous normal pointing in the region enclosed or pointing away from the region enclosed.

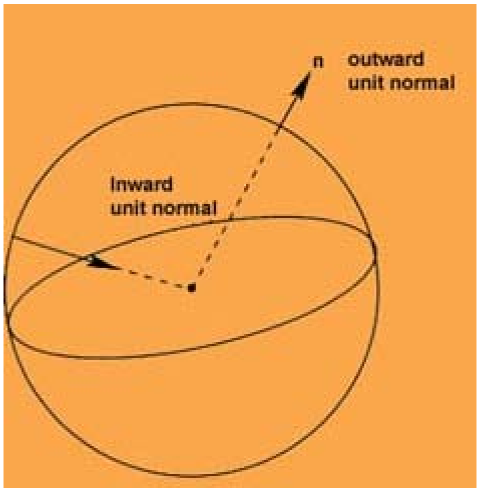

Figure: Sphere with inward and outward normal

2. If S is the boundary of an annulus region in space, it is orientable. For example, the surface enclosing two concentric spheres is orientable (note, it is not connected).

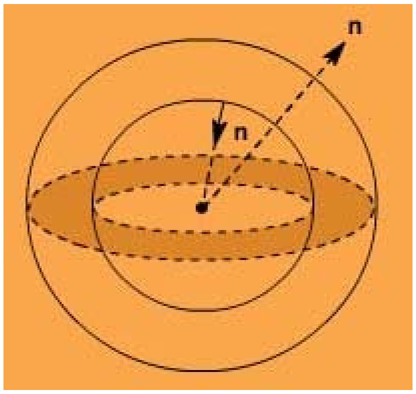

Figure: Boundary of annulus region

3. **Möbius strip:** The surface as shown below is not orientable. It is not possible to define a continuous normal along, say, the curve C

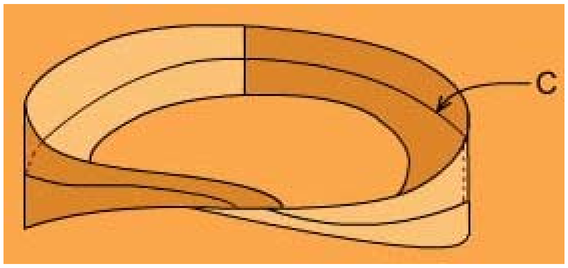

Figure: Möbius strip

50.2.5 Definition :

Let S be an oriented surface with the continuous unit normal $\mathbf{n}(x,y,z), (x,y,z) \in S$. Let $\mathbf{F}$ be a continuous vector field on S. Then the integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$$

is called the **flux-integral** of $\mathbf{F}$ over the surface S.

Physically, $\iint_S (\mathbf{F}.\mathbf{n}) dS$ represents the flux of the fluid with flux density $\mathbf{F}$ across the surface S in the direction of the chosen normal.

50.2.6 Example:

Let

$$\mathbf{F} = xz\,\mathbf{i} + yz\,\mathbf{j} + x^2\mathbf{k}$$

and

$$S = \{(x,y,z) \mid x^2 + y^2 + z^2 = a^2\},$$

oriented with outward unit normal. We want to compute

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS.$$

We can write $S = S_1 \cup S_2$, where S_1 is the upper hemisphere and S_2 is the lower hemisphere. The upper part S_1 parameterized as

$$\mathbf{r}\,(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (\sqrt{a^2 - x^2 - y^2})\,\mathbf{k},\ (x,y) \in R = \left\{x^2 + y^2 \le a^2\right\}.$$

Thus, for S_1,

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial}{\partial x}\left(\sqrt{a^2 - x^2 - y^2}\right)\mathbf{i} - \frac{\partial}{\partial y}\left(\sqrt{a^2 - x^2 - y^2}\right)\mathbf{j} + \mathbf{k}$$

$$= \frac{x}{\sqrt{a^2 - x^2 - y^2}}\,\mathbf{i} + \frac{y}{\sqrt{a^2 - x^2 - y^2}} + \mathbf{k},$$

and this is the outward normal for the upper hemisphere as the $\mathbf{k}$ component is positive. Similarly, the surface S_2, has parameterization

$$\mathbf{r}\,(x,y) = x\,\mathbf{i} + y\,\mathbf{j} - (\sqrt{a^2 - x^2 - y^2})\,\mathbf{k}\,,\ (x,y) \in R\left\{x^2 + y^2 \le a^2\right\}.$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial}{\partial x}\left(-\sqrt{a^2 - x^2 - y^2}\right)\mathbf{i} - \frac{\partial}{\partial y}\left(-\sqrt{a^2 - x^2 - y^2}\right)\mathbf{j} + \mathbf{k}$$

But, this is not the outward normal, as the $\mathbf{k}$ component is positive. In fact, the outward normal for S_2 is given by

$$-\mathbf{r}_x \times \mathbf{r}_y = -\left(\frac{x}{\sqrt{a^2 - x^2 - y^2}}\,\mathbf{i} - \frac{y}{\sqrt{a^2 - x^2 - y^2}} + \mathbf{k}\right).$$

Thus,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$$

$$= \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS. \qquad \text{----------(69)}$$

The integrand of first integral in the right hard side of (69) is

$$\frac{x\left(\sqrt{a^2-x^2-y^2}\right)x}{\sqrt{a^2-x^2-y^2}}+\frac{y\left(\sqrt{a^2-x^2-y^2}\right)y}{\sqrt{a^2-x^2-y^2}}+x^2=2\,x^2+y^2$$

Similarly, the integrand of the second integral in (69) is

$$\frac{x\left(-\sqrt{a^2-x^2-y^2}\right)x}{\sqrt{a^2-x^2-y^2}}+\frac{y\left(-\sqrt{a^2-x^2-y^2}\right)y}{\sqrt{a^2-x^2-y^2}}-x^2=-2\,x^2-y^2$$

Thus, from (69)

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=0.$$

An alternate way of analyzing the above problem is the following. First of all, the surface S can also be described by the implicit equation

$$f\,(x,y,z)=x^2+y^2+z^2-a^2=0.$$

Since

$$\nabla f=2\,x\,\mathbf{i}+2\,y\,\mathbf{j}+2\,z\,\mathbf{k},$$

a unit normal to S is given by

$$\begin{aligned}\mathbf{n}&=\pm\frac{\nabla f}{\|\nabla f\|}\\&=\pm\frac{2\,x\,\mathbf{i}+2\,y\,\mathbf{j}+2\,z\,\mathbf{k}}{2\sqrt{x^2+y^2+z^2}}\\&=\pm\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{x^2+y^2+z^2}\\&=\pm(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}).\end{aligned}$$

Clearly, $\mathbf{n}$ with the positive sign is the unit outward normal to S. Thus

$$\begin{aligned}\iint \mathbf{F}\cdot\mathbf{n}\,dS&=\iint_S\frac{\left(x\,z\,\mathbf{i}+y\,z\,\mathbf{j}+x^2\mathbf{k}\right)\cdot\left(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}\right)}{a}\,dS\\&=\frac{1}{a}\iint_S\left(x^2\,z+y^2\,z+x^2\,z\right)dS\\&=\frac{1}{a}\iint_S\left(x^2+y^2+z^2\right)z\,dS\\&=\frac{a^2}{a}\iint_S\,dS\\&=\left(\iint_{S_1}z\,dS-\iint_{S_2}z\,dS\right)\\&=0.\end{aligned}$$

50.2.7 Example:

Consider the surface S to be the boundary of the region

$$\left\{(x,y,z)\,|\,1\le x^2+y^2+z^2\le 4\right\}.$$

Let us evaluate

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS,$$

for $\mathbf{n}$ to be the outward normal on S and

$$\mathbf{F}(x,y,z) = -\frac{x}{r^3}\,\mathbf{i} - \frac{y}{r^3}\,\mathbf{j} - \frac{z}{r^3}\,\mathbf{k},$$

where

$$r = \sqrt{x^2+y^2+z^2}$$

The surface S_1 is the outer sphere of radius 1 and the inner sphere S_2 of radius 2.

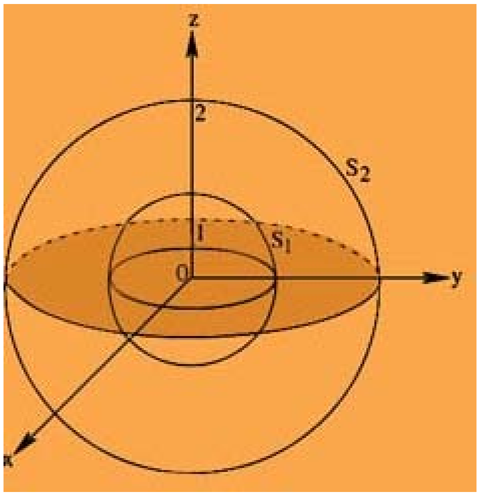

Figure: The surface S

As in previous example, the outward unit normal for S_2 given by

$$\mathbf{n} = \frac{\nabla\,\mathbf{f}}{\|\nabla\mathbf{f}\|} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{r}.$$

Thus, for S_2 we have

$$\begin{aligned}
\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_{S_2}\left(-\frac{x}{r^3}\,\mathbf{i} - \frac{y}{r^3}\,\mathbf{j} - \frac{z}{r^3}\,\mathbf{k}\right)\cdot\left(\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{r}\right)dS \\
&= \iint_{S_2} -\left(x^2+y^2+z^2\right)dS \\
&= -\frac{1}{r^2}\int_{S_1} dS \\
&= -4\pi.
\end{aligned}$$

Similarly for S_1, the outward unit normal is

$$\mathbf{n} = \frac{\nabla\,F}{\|\nabla\,F\|} = \frac{-x\,\mathbf{i}-y\,\mathbf{j}-z\,\mathbf{k}}{r}.$$

Thus,

$$\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_1}\left(-\frac{x}{r^3}\mathbf{i}-\frac{y}{r^3}\mathbf{j}-\frac{z}{r^3}\mathbf{k}\right)\left(\frac{-x\,\mathbf{i}+-y\,\mathbf{j}+-z\,\mathbf{k}}{r}\right) dS$$

$$=+\iint_{S_1} dS$$

$$=4\pi.$$

Hence

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = 0.$$

50.2.8 Note:

1. Note that for an orientable surface, if $\mathbf{n}(x,y,z)$ is one choice of continuous unit normal vector to S, then $-\mathbf{n}(x,y,z)$ is also another choice of continuous unit normal to S. The flux integral changes sign if we change

   one selection to other. When we select positive sign, we call $\mathbf{n}$ as the **positive unit normal**, and $-\mathbf{n}$ will be called the negative-unit normal. Thus, for an orientable surface S with parametrization $\mathbf{r}(u,v),(u,v)\in D$, we have

   $$\iint_S (\mathbf{F}\cdot\mathbf{n})\, ds = \pm\iint_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\, dudv,$$

   depending upon one choice of the unit normal.

2. **Special forms of flux-integral**

   Let us look at the special cases of S. Suppose S is given explictely by $z=g(x,y),(x,y)\in D$. Then, a parametrization of S is given by

   $$\mathbf{r}\,(x,y) = x\,\mathbf{i}+y\,\mathbf{j}+g\,(x,y)\,\mathbf{k}.$$

   Thus,

   $$\mathbf{r}_x\times\mathbf{r}_y = -g_x\,\mathbf{i}-g_y\,\mathbf{j}+\mathbf{k},$$

   and hence for the positive orientation of S

   $$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \iint_D \mathbf{F}\cdot\left(-g_x\,\mathbf{i}-g_y\,\mathbf{j}+\mathbf{k}\right) dxdy$$

   Thus, if

   $$\mathbf{F} = P\,\mathbf{i}+Q\,\mathbf{j}+R\,\mathbf{k},$$

   then

   $$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \iint_D \left(-P\,g_x - Q\,g_y + R\right) dxdy,$$

   where $\mathbf{n}$ is the positive unit normal. If we write $G(x,y,z)=z-g(x,y)$, then

   $$\mathbf{r}_x\times\mathbf{r}_y = -g_x\,\mathbf{i}-g_y\,\mathbf{j}+\mathbf{k} = \nabla G.$$

   Hence, for the choice of positive oriented normal on S, given by $z=g(x,y)$ and $G(x,y,z)=z-g(x,y)$,

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = \iint_s (\mathbf{F}\cdot\nabla G)\,dS$$

Similar formula holds if S is represented as $y = h(x,z)$ or $x = k(y,z)$.

3. There is another representation possible for the flux-integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$.

Let the continuous normal $\mathbf{n}$ have direction cosines $\cos\alpha, \cos\beta, \cos\gamma$, i.e.,

$$\mathbf{n} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

Then, for $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = \iint_S P\cos\alpha\,ds + \iint_S Q\cos\beta\,ds + \iint_S R\cos\gamma\,dS\,.$$

While evaluating, care must be taken the integrals on right hand side since S is oriented. Suppose, we select the positive orientation for the normal. Then for

$$S: z = g(x,y), (x,y)\in D\,,$$

$$\iint_S R\cos\gamma\,dS = \begin{cases} \iint_D R(x,y,g(x,y))\,dxdy & \text{if} \quad \cos\gamma > 0 \\ -\iint_D R(x,y,g(x,y))\,dxdy & \text{if} \quad \cos\gamma < 0. \end{cases}$$

(iv) If $\mathbf{F}$ and $\mathbf{r}$ are expressed in terms of their components:

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k},$$

$$\mathbf{r}\,(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}\,,\ (u,v)\in D,$$

then

$$\mathbf{r_u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\,,$$

$$\mathbf{r_v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\,,$$

and hence

$$\begin{aligned}\mathbf{r}_u\times\mathbf{r}_v = &\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)\mathbf{i}\\ &+\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\mathbf{j}\\ &+\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\mathbf{k}.\end{aligned}$$

Thus, for positive orientation of the surface,

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = \iint_D P\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)dudv$$
$$+ \iint_D Q\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)dudv$$
$$+ \iint_D R\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)dudv.$$

The three integrals on the right hand side are represented as follows

$$\iint_S P(x,y,z)\,dy\wedge dz := \iint_D P(\mathbf{r}(u,v))\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)dudv,$$

$$\iint_S Q(x,y,z)\,dz\wedge dx := \iint_D Q(\mathbf{r}(u,v))\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)dudv,$$

and

$$\iint_S R(x,y,z)\,dx\wedge dy := \iint_D R(\mathbf{r}(u,v))\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\right)dudv,$$

Note that the order of the notation $dx\wedge dy$, etc, is important. Thus, in the above notations, the flux integral of
$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k},$
is written as

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\,dS = \iint_S \left(P\,dy\wedge dz + Q\,dz\wedge dx + P\,dx\wedge dy\right).$$

50.2.9 Examples :

1. Let us find the flux of $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$

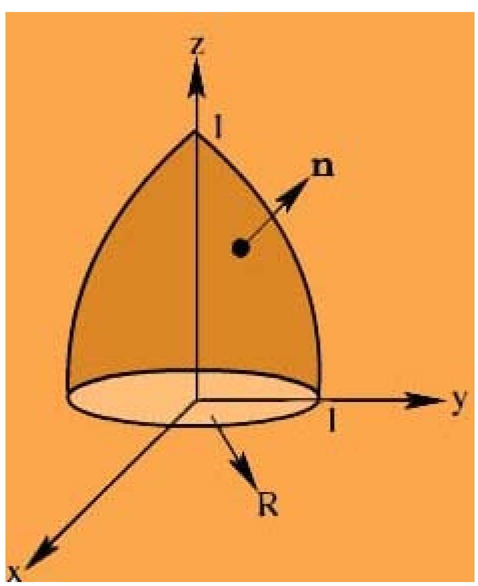

Figure: Cone above the xy-plane

outward across S, the portion of the cone $z = 1 - x^2 - y^2$, that lies above the xy-plane. The surface S is given by $G(x,y,z) = z + x^2 + y^2 - 1 = 0$. Thus, the normal vector is

$$\pm \nabla G = (2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}).$$

Note that for the outward normal, the z component is always positive. So, we choose

$$\nabla G = 2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}$$

for S. Hence,

$$\begin{aligned}\text{flux across } S \text{ is} &= \iint_R (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot (2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k})\,dxdy \\ &= \iint_R \left(x^2 + y^2 + 1\right) dxdy \\ &= \int_0^1 \int_0^{2\pi} (1 + r^2)\, r\, dr\, d\theta \\ &= \frac{3\pi}{2}\end{aligned}$$

2. Let us compute the flux of the vector field

$$\mathbf{F}(x,y,z) = 3z^2\mathbf{i} + 6\mathbf{j} + 6xz\mathbf{k}$$

across parabolic cylinder S given by

$$y = x^2,\ 0 \le x \le 2,\ 0 \le z \le 3.$$

We parameterize the surface as

$$r(x,z) = x\,\mathbf{i} + x^2\,\mathbf{j} + z\,\mathbf{k},\ (x,z) \in D = [0,2] \times [0,3].$$

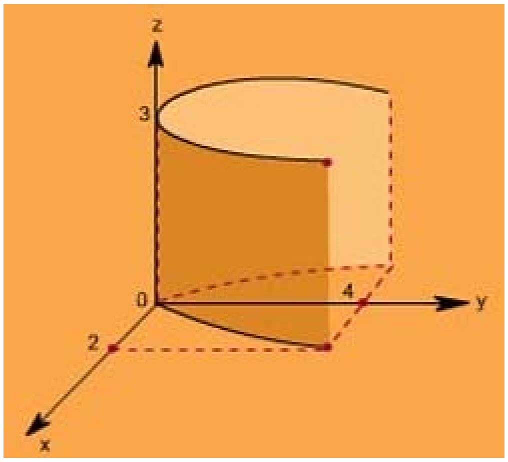

Figure: Parabolic cylinder

Then,

$$\mathbf{r_x} \times \mathbf{r_y} = (\mathbf{i} + 2x\,\mathbf{j}) \times (\mathbf{k}) = 2x\,\mathbf{i} - \mathbf{j}.$$

Thus the positive oriented normal is

$$\mathbf{n} = \frac{2x\,\mathbf{i} - \mathbf{j}}{\sqrt{5}}.$$

The flux integral along this orientation is

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \iint_D \left(3z^2\,\mathbf{i} + 6\,\mathbf{j} + 6\,xz\right)\cdot(2\,x\,\mathbf{i} - \mathbf{j})\, dxdz$$
$$= \int_D \left(6\,xz^2 - 6\right) dxdz$$
$$= \int_0^3 \left(\left[6\,\frac{x^2}{2}\,z^2 - 6x\right]_0^2\right) dz$$
$$= \int_0^3 \left(12\,z^2 - 12\right) dz = 72.$$

Let us evaluate the same flux integral using the formula

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \iint_S \left(P\cos\alpha\right) dS + \left(Q\cos\beta\right) dS + \left(R\cos\gamma\right) dS$$

In this case,

$$\mathbf{n} = \frac{1}{\sqrt{5}}\,(2\,x\,\mathbf{i} - \mathbf{j}) = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}.$$

Thus, $\cos\alpha > 0$, while $\cos\beta < 0$. Hence, the required flux integral along the positive orientation is

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \iint_{D'} P\, dydz - \iint_{D'} P\, dzdx$$
$$= \iint_{D'} 3\,z^2\, dydz - \iint_{D'} 6\, dzdx$$

Since, the surface S is $\sqrt{y}\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},\ (y,z)\in D'$, where

$$D' = \left\{(y,z)\,|\,0\le y\le 4,\ 0\le z\le 3\right\},$$

we have

$$\iint_{D'} 3\,z^2\, dydz = 3\int_0^4 \left(\int_0^3 z^2\, dz\right) dy = 3\int_0^4 \frac{27}{3}\, dy = 108,$$

and

$$\iint_{D'} 6\, dzdx = \int_0^3\int_0^2 6\, dzdx = 6\times 3\times 2 = 36.$$

Hence, the required flux is given by

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = 108 - 36 = 72.$$

Practice Exercises

1. Evaluate the surface integral

   $$\iint_S \mathbf{F}\cdot d\mathbf{S},$$

   where S is the surface given by

   $$r(\varphi,\phi) = \cos\varphi\sin\phi\,\mathbf{i} + \sin\varphi, \sin\phi\,\mathbf{j} + \cos\varphi\,\mathbf{k}, 0\le\varphi\le 2\pi, 0\le\phi\le\pi,$$

   and

$\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$

Answer: -4π

2. Compute the flux of the vector field

$\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

across the surface S that is the portion of the paraboloid

$z = 4 - x^2 - y^2,$

lying above the xy-plane, oriented by the upward unit normal.

Answer: 24π

3. Show that the flux of the universe square vector field

$\mathbf{F}(x,y,z) = \dfrac{\mathbf{r}}{\|\mathbf{r}\|^3}, \mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$

across the sphere R

$x^2 + y^2 + z^2 = 4$

towards the outward unit normal is given by 4π

4. Find the coordinates of the center of mass of the surface out from the cylinder

$y^2 + z^2 = 9, z \geq 0,$ by the planes $x = 0$ and $x = 3.$

Answer: $\overline{x} = \dfrac{3}{2}, 5 = 0, \overline{z} = \dfrac{6}{\pi}$

5. Let $\mathbf{F}$ be a vector field such that $\mathbf{F} \cdot \mathbf{r} = 1$ for all (x,y,z) on the unit sphere $S.$

Show that

$\iint_S \mathbf{F} \cdot d\mathbf{S} = 2\pi^2$

Recap

In this section you have learnt the following

- Some application of the surface integrals.