Module 18 : Stokes's theorem and applications

Lecture 52 : Orienting the boundary of an orientable surface [Section 52.1]

Objectives

In this section you will learn the following :

- The motion of orienting the boundary of an orientable surface.

52 .1 Orienting the boundary of an orientable surface

In the previous section we extended green's theorem to Divergence theorem, which related surface integrals, over surfaces enclosing closed bounded regions, with triple integrals. One can also extend Green's theorem to surface integrals over surfaces which have boundaries as closed curves. Recall, we showed Green's theorem can be written as

$$\oint_C \mathbf{F}.\mathbf{dr} = \iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dxdy$$

In this section we shall extend the above relation to $\mathbb{R}^3$. To do that, we need to give orientations to curves which are boundaries of surfaces in a compatible way.

52.1.1 Definition:

Let S be a piecewise smooth orientable surface bounded by a simple closed curve C, i.e., $\partial S = C$. Let us select and fix an continuous unit normal on S and choose an orientation on C as follows :

1. Let Γ be any curve on the surface and let P be a point on the surface, inside Γ. Let $\mathbf{n}(P)$ denote the unit normal on S at P as per the orientation of S. Consider the orientation on Γ which looks anti-clockwise when viewed from the top of $\mathbf{n}(P)$. This is called the **positive-orientation** on Γ relative to the orientation on S

   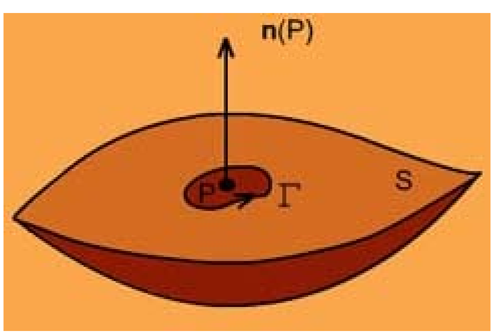

   Figure: Positive orientation on

2. We say that the selected orientation on $C = \partial(S)$ **positive orientation** on C if it has the following property:

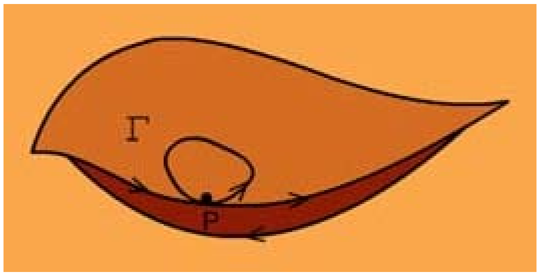

Figure: Positive orientation on $\partial(S)$

For any point $P \in C = \partial(S)$, if we consider a simple closed curve Γ on S which meets C only at P, then the orientations on Γ and C coincide. Thus, positive orientation on $C = \partial S$ can be described as follows: imagine a person walking along C with his head in the direction of the orientation of the surface (i.e., that of the unit normal to the surface). The person will be walking along the positive-direction if the surface lies on his left. The orientation opposite to the positive orientation is called the **negative orientation** on $C = \partial S$,

52.1.2 Note:

Another way of saying that the curve Γ on S has positive orientation with respect to the orientation on S is the following:

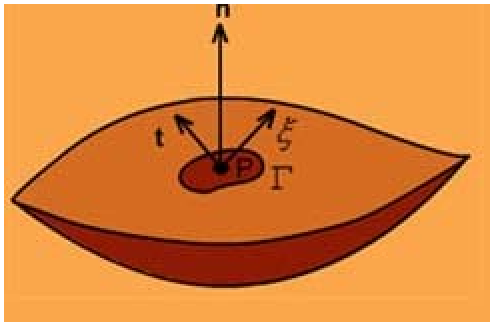

Figure: Right hand system on Γ

Γ is said to have positive orientation relative to orientation on S if at any point P of Γ, if we consider the vectors $\mathbf{t}$, the tangent vector to Γ at P, ξ, the outward unit normal to Γ at P, and $\mathbf{n}(P)$, the unit normal to S at P, then $\xi, \mathbf{t}$ and $\mathbf{n}(P)$ form a right-hand system, as shown in the figure below.

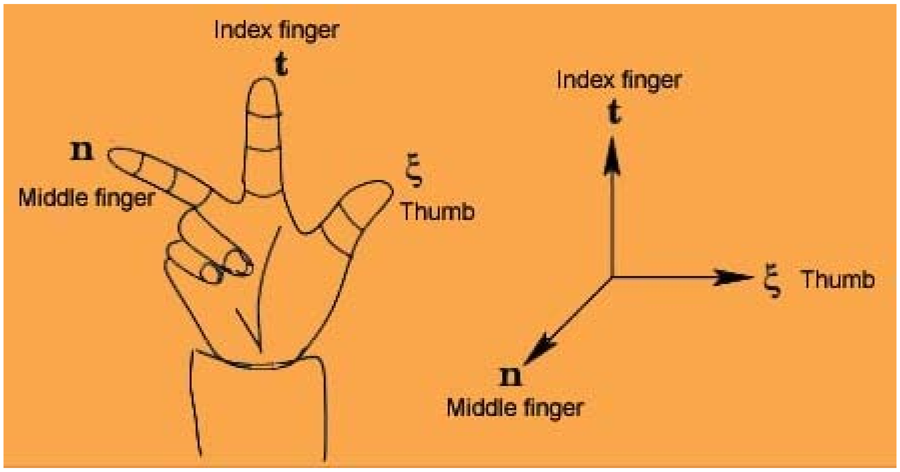

Figure: Right hand system

52.1.3 Examples:

1. Consider the surface

$S = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1, z = 0\}$.

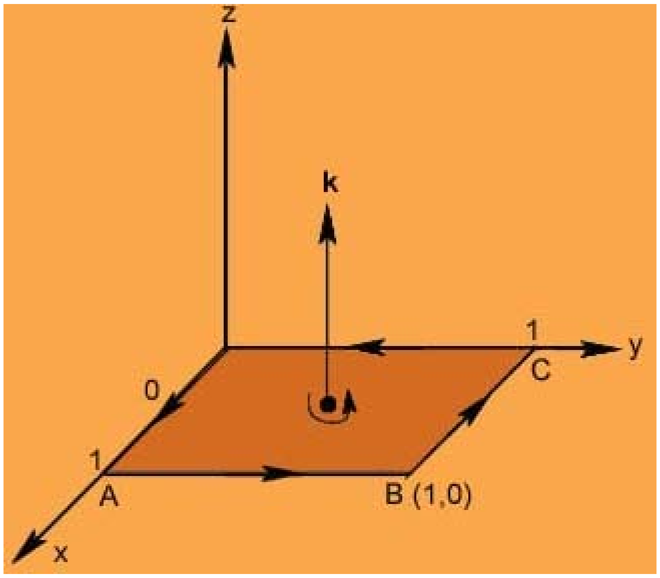

Figure: Positive orientation on $\partial(S)$

Let us select the orientation of S to be such that the direction of the unit normal to S at every point is that of $\mathbf{k}$. Then, the positive orientation to $C = \partial S$ relative to this orientation of S is as shown in the figure, going from 0 to A to B to C to O.

2. Consider the plane S in $\mathbb{R}^3$ given by the equation

$$x + y + z = 1, 0 \le x \le 1,\ 0 \le y \le 1, 0 \le z \le 1,$$

i.e.,

$$S = \{(x,y,z) \mid x + y + z = 1, 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1.\}$$

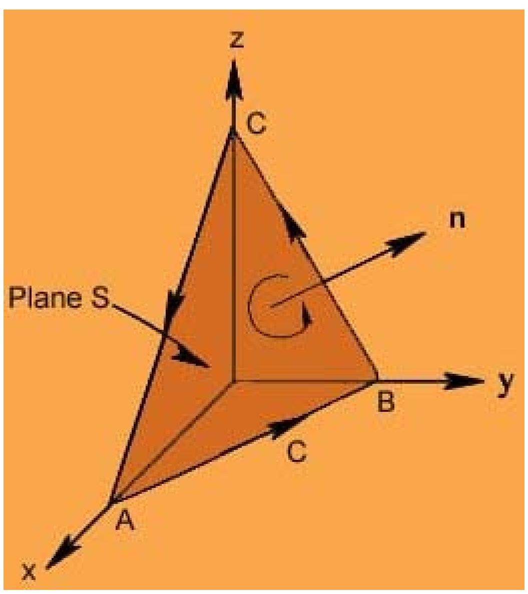

Figure: Positive orientation on $\partial(S)$

Let S be given the positive-orientation, i.e., if we write S as

$$r(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (1 - x - y)\,z\,\mathbf{k},\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1,$$

then

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{\|\mathbf{r}_x \times \mathbf{r}_y\|} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}.$$

Thus, the orientation on $C = \partial(S)$ is as shown is the figure, from A to B to C to A.

3. Consider S, the paraboloid,

$$z = 4 - x^2 - y^2, z \ge 0,$$

with upward normal orientation, i.e., geometrically the unit normal points upward relative to xy-plane. Then, the orientation to $C = \partial S,$ the circle $x^2 + y^2 = 4, z = 0,$ is the counter-clockwise orientation, as viewed from the top of positive z-axis.

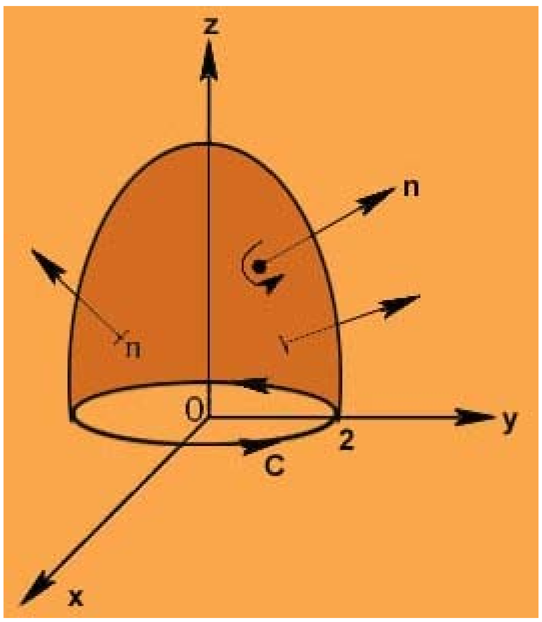

Figure: Positive orientation on the circle, $\partial(S)$

Practice Exercises

For the following surface S with the given choice of unit normal, find positive orientation on the boundary of S:

1. S is the cone $z = \sqrt{x^2 + y^2}, 0 \le z \le 2,$ with the orientation given by

$$\mathbf{n} = \frac{\cos\varphi}{\sqrt{2}}\mathbf{i} - \frac{\sin\varphi}{\sqrt{2}}\mathbf{j} + \frac{k}{\sqrt{2}},\ 0 \le \varphi \le 2\pi.$$

2. S is the hemisphere $x^2 + y^2 + z^2 = 16z, \ge 0,,$ with the orientation given by

$$\mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{4}.$$

3. S is the cylinder $x^2+y^2=4,\ 0\le z\le 1,$ closed at $z=1,$ with $\mathbf{n}$ to be the outward normal.

4. S is the surface of the cylinder $x^2+y^2=4$ bounded at the bottom by the xy- plane and at the top by the plane $x+z=6,$ with $\mathbf{n}$ to be the outward unit normal.

5. S is the annulus disc in the xy-plane bounded by the circles

   $x^2+y^2=16$ and $x^2+y^2=4,$

   with $\mathbf{n}$ parallel to $\mathbf{k}$.

Recap

In this section you have learnt the following

- The motion of orienting the boundary of an orientable surface.

[Section 52.2]

Objectives

In this section you will learn the following :

- The relation between surface integral and the line integral.

52.2 Stokes' theorem

We state now the theorem that relates surface-integral with line integral.

52.2.1 Theorem (Stokes'):

Let S be a piecewise smooth oriented surface and let its boundary be a piecewise-smooth simple closed curve C. Let $\mathbf{F}$ be continuously differentiable vector field in a region D including S. Then

$$\iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS = \oint_C (\mathbf{F}.dr) \qquad \text{----------(80)}$$

where $\mathbf{n}$ is a unit normal on S and C is given the positive orientation relative to the chosen orientation of S.

We shall prove the theorem only for simple surfaces, i.e., surface that can be explicitly written as

$z = f\ (x, y), (x, y) \in D'$

or

$y = g\ (x, y), (x, z) \in D^{\cdot}$

or

$x = h\ (y, z), (y, z) \in D^{\cdot}$.

First of all, suppose that S has a parameterizations

$(u, v) \mapsto \mathbf{r}(u, v), (u, v) \in G.$

Then, the normal to the surface is given by

$\mathbf{r}_u \times \mathbf{r}_v := N_1\ \mathbf{i} + N_2\ \mathbf{j} + N_3\ \mathbf{k}.$

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then

$$\iint_R (\mathbf{curl}\,\mathbf{F}) \cdot \mathbf{N}\, dS = \iint_G (\mathbf{curl}\,\mathbf{F}) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dudv$$

$$= \iint_G (\mathbf{curl}\,\mathbf{F}) \cdot (N_1\ \mathbf{i} + N_2\ \mathbf{j} + N_3\ \mathbf{k})\, dudv$$

$$= \iint_G \left(\frac{\partial R}{\partial y} - \frac{\partial \varphi}{\partial z} \right) N_1\, dudv + \iint_G \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) N_2\, dudv$$

$$+ \iint_G \left(\frac{\partial \varphi}{\partial x} - \frac{\partial P}{\partial y} \right) N_3\, dudv$$

$$= \iint_G \left(\frac{\partial P}{\partial z} N_2 - \frac{\partial P}{\partial y} N_3 \right) dudv + \iint_G \left(\frac{\partial \varphi}{\partial x} N_3 - \frac{\partial \varphi}{\partial z} N_1 \right) dudv$$

$$+ \iint \left(\frac{\partial R}{\partial y} N_1 - \frac{\partial R}{\partial x} N_2 \right) dudv. \qquad \text{----------(81)}$$

On the other hand

$$\oint_C F.dr = \oint_C Pdx + \oint_C Q.dy + \oint_C R.dz \qquad \text{----------(82)}$$

In view of (81) and (82), to prove (80) it sufficient to prove the following:

$$\iint_G \left(\frac{\partial P}{\partial z} N_2 - \frac{\partial P}{\partial y} N_3 \right) dudv = \oint_C Pdx, \qquad \text{----------(83)}$$

$$\iint_G \left(\frac{\partial Q}{\partial x} N_3 - \frac{\partial Q}{\partial z} N_1 \right) dudv = \oint_C Qdy, \qquad \text{----------(84)}$$

and

$$\iint_G \left(\frac{\partial R}{\partial y} N_1 - \frac{\partial R}{\partial x} N_2 \right) dudv = \oint_C Rdz, \qquad \text{----------(85)}$$

We shall prove (83) only, as the proofs of (84) and (85) are similar. Since S can be expressed explicitly as $z = f(x, y), (x, y) \in D'$, we can write a parameterizations of S as

$$\mathbf{r}\,(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + f\,(x, y)\,\mathbf{k},\ (x, y) \in D'.$$

Thus,

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = -f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k},$$

giving us,

$$N_1 = f_x, N_2 = f_y \text{ and } N_3 = 1.$$

Thus,

$$\iint_G \left(\frac{\partial P}{\partial z} N_2 - \frac{\partial P}{\partial y} N_3 \right) du dv = \iint_{D'} \left(-\frac{\partial P}{\partial z} f_y + \frac{\partial P}{\partial y} \right) dx dy. \qquad \text{----------(86)}$$

Note that for our choice of orientation of S, the normal $\mathbf{N}$ to S has positive $\mathbf{k}$-component. Thus, the curve $C = \partial(S)$ has counter-clockwise orientation. Hence, if we choose a parameterizations $\mathbf{r}'(t)$ for $\partial(D')$, traversed in the counter-clockwise direction, with

$$\mathbf{r}'\,(t) = x\,(t)\,\mathbf{i} + y\,(t)\,\mathbf{j},\ t \in I,$$

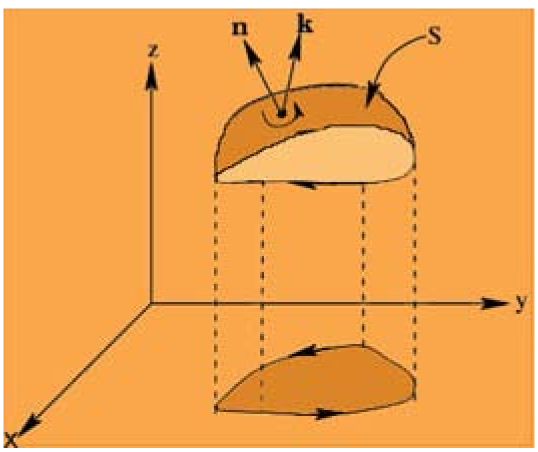

Figure 243. Orientations on $\partial(D')$ and $\partial(S)$

we have

$$\mathbf{r}\,(t) = x\,(t)\,\mathbf{i} + y\,(t)\,\mathbf{j} + f\,(x\,(t), y\,(t))\mathbf{k},\ t \in I.$$

This gives us parameterizations of C in the counter-clockwise direction. Further, if we write

$$\phi(x, y) := P(x, y, f(x, y)), (x, y) \in D',$$

then

$$\oint_C P dx = \int_{t\in I} P\big(x(t), y(t)\big)\frac{dx}{dt}dt$$
$$= \int_{t\in I} \phi\big(x(t), y(t)\big)\frac{dx}{dt}dt$$
$$= \oint_{\partial(D')} \phi(x,y)dx$$
$$= \iint_{D'}\left(-\frac{\partial\phi}{\partial y}\right)dxdy,$$

by Green's theorem. Since, by chain rule,

$$\frac{\partial \phi}{\partial y} = \frac{\partial P}{\partial y}\big(x, y, f(x,y)\big) = \frac{\partial P}{\partial y}(x,y,z) + \frac{\partial P}{\partial z}(x,y,z)\frac{\partial z}{\partial y}(x,y)$$
$$= \frac{\partial P}{\partial y}(x,y,z) + \frac{\partial P}{\partial z}(x,y,z)\frac{\partial f}{\partial y}(x,y).$$

Hence

$$\oint_C P dx = \iint_{D'}\left[\left(-\frac{\partial P}{\partial y}\right) + \frac{\partial P}{\partial z}\left(-f_y\right)\right]dxdy \qquad \text{----------(87)}$$

(86) and (87) prove (83).

Practice Exercises:

1. Let S_1 and S_2 be two orientable surfaces with a common boundary, $C = \partial S_1 = \partial S_2$. Let $\mathbf{n}_i$ be the unit normal on S_i be so chosen such that both $\mathbf{n}_1$ and $\mathbf{n}_2$ induce the same positive orientation on C. Let $\mathbf{F}$ be a smooth vector field with domain including S_1, S_2 and C. Show that $\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 \, dS = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, dS.$

2. Let f, g be scalar fields with continuous partial derivatives. Let S be a smooth surface such that S and $C = \partial(S)$ satisfy conditions of Stokes' s theorem. Prove the following:

   1. $\iint_S (\nabla f)\cdot(\nabla g)\cdot \mathbf{n}\, ds = \int_C (f\nabla g)\cdot d\mathbf{r}.$

   2. $\iint_S (\nabla f)\cdot(\nabla g)\cdot \mathbf{n}\, ds = \int_C (f\nabla g)\cdot d\mathbf{r}.$

   3. $\int_C (f\nabla g + g\nabla f)\cdot d\mathbf{r} = 0.$

3. Let $\mathbf{F}$ be a constant vector field, S a smooth oriented surface with unit normal $\mathbf{n}$, and $\partial S = C$. Show that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \frac{1}{2}\int_C (\mathbf{F}\times\mathbf{r})\cdot d\mathbf{r}$$

Recap

In this section you have learnt the following

- The relation between surface integral and the line integral.