

Module 3 : MAGNETIC FIELD

Lecture 17 : Vector Potential

Objectives

In this lecture you will learn the following

- Define vector potential for a magnetic field.
- Understand why vector potential is defined in a gauge.
- Calculate vector potential for simple geometries.
- Define electromotive force and state Faraday's law of induction

Vector Potential

For the electric field case, we had seen that it is possible to define a scalar function ϕ called the "potential" whose negative gradient is equal to the electric field : $\nabla\phi = \vec{E}$. The existence of such a scalar function is a consequence of the conservative nature of the electric force. It also followed that the electric field is irrotational, i.e. $\text{curl } \vec{E} = 0$.

For the magnetic field, Ampere's law gives a non-zero curl

$$\text{curl } \vec{B} = \mu_0 \vec{J}$$

Since the curl of a gradient is always zero, we cannot express \vec{B} as a gradient of a scalar function as it would then violate Ampere's law.

However, we may introduce a vector function $\vec{A}(\vec{r})$ such that

$$\vec{B} = \nabla \times \vec{A}$$

This would automatically satisfy $\nabla \cdot \vec{B} = 0$ since divergence of a curl is zero. \vec{A} is known as vector potential. Recall that a vector field is uniquely determined by specifying its divergence and curl. As \vec{B} is a physical quantity, curl of \vec{A} is also so. However, the divergence of the vector potential has no physical meaning and consequently we are at liberty to specify its divergence as per our wish. This freedom to choose a vector potential whose curl is \vec{B} and whose divergence can be conveniently chosen is called by mathematicians as a choice of a *gauge*. If ψ is a scalar function any transformation of the type

$$\vec{A} \longrightarrow \vec{A} + \nabla\psi$$

gives the same magnetic field as curl of a gradient is identically zero. The transformation above is known as *gauge invariance*. (we have a similar freedom for the scalar potential ϕ of the electric field in the sense that

it is determined up to an additive constant. Our most common choice of ϕ is one for which $\phi \rightarrow 0$ at infinite distances.)

A popular gauge choice for $\text{vec}A$ is one in which

$$\nabla \cdot \vec{A} = 0$$

which is known as the "Coulomb gauge". It can be shown that such a choice can always be made.

Exercise 1

Show that a possible choice of the vector potential for a constant magnetic field \vec{B} is given by $\vec{A} = (1/2)\vec{B} \times \vec{r}$. Can you construct any other $\text{vec}A$?

(Hint : Take \vec{B} in z-direction, express $\text{vec}A$ in component form and take its curl.)

Biot-Savart's Law for Vector Potential

Biot-Savart's law for magnetic field due to a current element $d\vec{l}$

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{l} \times \hat{r}}{r^2} = -\frac{\mu_0 I}{4\pi} d\vec{l} \times \nabla\left(\frac{1}{r}\right)$$

may be used to obtain an expression for the vector potential. Since the element $d\vec{l}$ does not depend on the position vector of the point at which the magnetic field is calculated, we can write $d\vec{B} = \frac{\mu_0 I}{4\pi} \nabla \times \left(\frac{d\vec{l}}{r}\right)$

the change in sign is because $\nabla\left(\frac{d\vec{l}}{r}\right) = \nabla(1/r) \times d\vec{l}$.

Thus the contribution to the vector potential from the element $d\vec{l}$ is $d\vec{A} = \frac{\mu_0 I}{4\pi r} d\vec{l}$

The expression is to be integrated over the path of the current to get the vector potential for the system

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}}{r}$$

Example 16

Obtain an expression for the vector potential at a point due to a long current carrying wire.

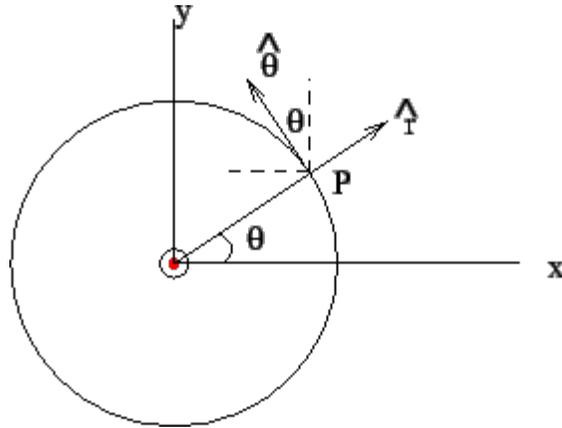
Solution :

Take the wire to be along the z-direction, perpendicular to the plane of the page with current flowing in a direction out of the page. The magnitude of the field at a point P is $\mu_0 I / 2\pi r$ with its direction being along

the tangential unit vector $\hat{\theta}$ at P, $\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\theta}$

The direction of \vec{B} makes an angle $(\pi/2) + \theta$ with the x direction, where $\tan \theta = y/x$. Thus

$$\begin{aligned}
\hat{\theta} &= \hat{i} \cos\left(\frac{\pi}{2} + \theta\right) + \hat{j} \sin\left(\frac{\pi}{2} + \theta\right) \\
&= -\hat{i} \sin \theta + \hat{j} \cos \theta \\
&= -\hat{i} \frac{y}{r} + \hat{j} \frac{x}{r}
\end{aligned}$$



Hence we have $\vec{B} = \frac{\mu_0 I}{2\pi} \frac{(-y\hat{i} + x\hat{j})}{x^2 + y^2}$

We wish to find a vector function $\text{vec } A$ whose curl is given by the above. One can see that the following function fits the requirement

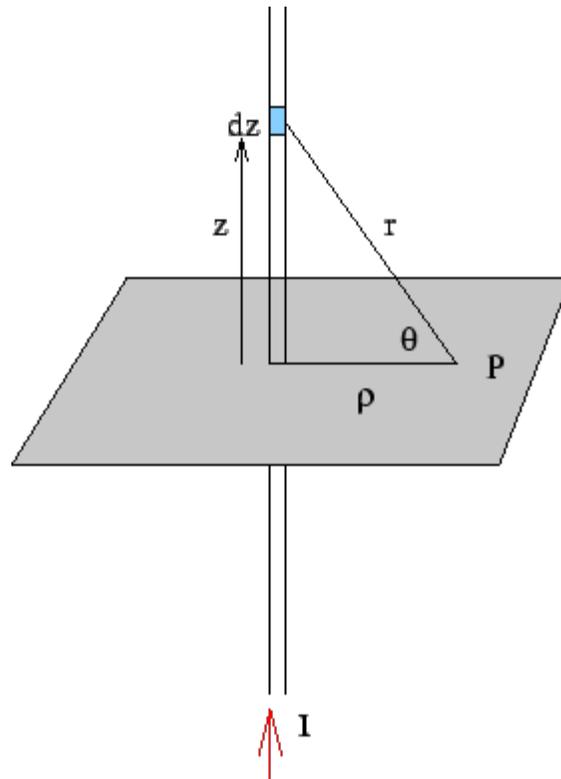
$$\vec{A} = -\frac{\mu_0 I}{4\pi} \ln(x^2 + y^2) \hat{k} \quad (1)$$

In the following, we will derive this directly from the expression for Biot-Savart's law. If ρ is the distance of P from an element of length dz at z of the wire, we have, $r^2 = x^2 + y^2 + z^2 = \rho^2 + z^2$

Thus $\vec{A} = \frac{\mu_0 I}{4\pi} \int \frac{dz}{(\rho^2 + z^2)^{1/2}}$

If the above integral is evaluated from $z = -\infty$ to $z = +\infty$, it diverges. However, we can eliminate the infinity in the following manner. Let us take the wire to be of length $2L$ so that

$$\vec{A} = \frac{\mu_0 I}{4\pi} \hat{k} \int_{-L}^L \frac{dz}{(\rho^2 + z^2)^{1/2}}$$



The integral is evaluated by substituting $z = \rho \tan \theta$, so that $dz = \rho \sec^2 \theta d\theta$. We get

$$\begin{aligned} \vec{A} &= \frac{\mu_0 I}{4\pi} \hat{k} \int_{-\alpha}^{\alpha} \sec \theta d\theta \\ &= \frac{\mu_0 I}{2\pi} \hat{k} \ln(\sec \alpha + \tan \alpha) \end{aligned}$$

where $\tan \alpha = L/\rho$.

In terms of L and ρ , we have $\sec \alpha = \frac{(L^2 + \rho^2)^{1/2}}{\rho} = \frac{L}{\rho} \left(1 + \frac{\rho^2}{L^2}\right)^{1/2} \approx \frac{L}{\rho} \left(1 + \frac{\rho^2}{2L^2}\right)$

Thus to leading order in L , $A = \frac{\mu_0 I}{2\pi} \hat{k} \ln(2L/\rho) = \frac{\mu_0 I}{2\pi} (\ln 2L - \ln \rho) \hat{k}$

As expected, for $L \rightarrow \infty$, the expression diverges. However, since $\text{vec} A$ itself is not physical while curl of $\text{vec} A$ is, the constant term (which diverges in the limit of $L \rightarrow \infty$) is of no consequence and $\text{vec} A$ is given by

$$A = -\frac{\mu_0 I}{2\pi} \hat{k} \ln \rho = -\frac{\mu_0 I}{4\pi} \ln(x^2 + y^2)$$

which is the same as Eqn. (1)

Example 17

Obtain an expression for the vector potential of a solenoid.

Solution :

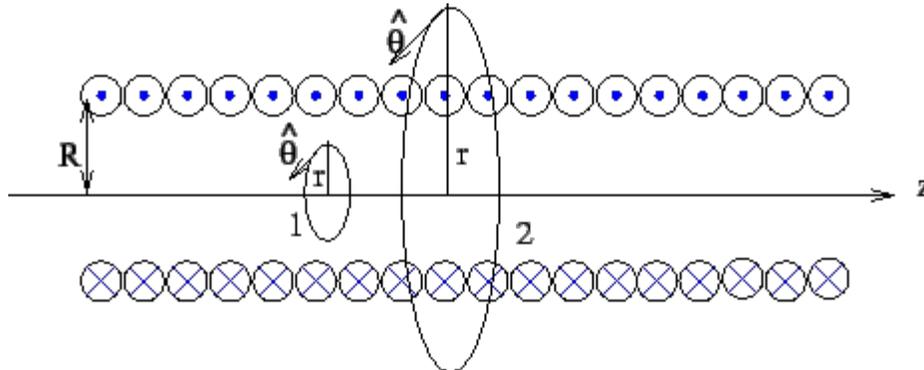
We had seen that for a solenoid, the field is parallel to the axis for points inside the solenoid and is zero

outside.

$$B = \mu_0 n I \hat{k} \text{ inside solenoid}$$

$$= 0 \text{ outside}$$

Take a circle of radius r perpendicular to the axis of the solenoid. The flux of the magnetic field is



$$\int \vec{B} \cdot d\vec{S} = \mu_0 n I \cdot \pi r^2 \text{ for } r \leq R$$

$$= \mu_0 n I \cdot \pi R^2 \text{ for } r \geq R$$

Since \vec{B} is axial, $\text{vec } A$ is directed tangentially to the circle. Further, from symmetry, the magnitude of $\text{vec } A$ is constant on the circumference of the circle.

Use of Stoke's theorem gives

$$\int \vec{B} \cdot d\vec{S} = \int \text{curl } \vec{A} \cdot d\vec{S}$$

$$= \oint \vec{A} \cdot d\vec{l}$$

$$= |A| 2\pi r$$

Thus

$$\vec{A} = \frac{\mu_0 n I \pi r^2}{2\pi r} \hat{\theta} = \frac{\mu_0 n I r}{2} \hat{\theta} \text{ for } r \leq R$$

$$= \frac{\mu_0 n I \pi R^2}{2\pi r} \hat{\theta} = \frac{\mu_0 n I R^2}{2r} \hat{\theta} \text{ for } r \geq R$$

where $\hat{\theta}$ is the unit vector along the tangential direction.

Exercise 2

Obtain an expression for the vector potential inside a cylindrical wire of radius R carrying a current I .

(Ans. $-\mu_0 I r^2 / 4\pi R^2$)

The existence of a vector potential whose curl gives the magnetic field directly gives

$$\text{div } B = 0$$

as the divergence of a curl is zero. The vector identity

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

can be used to express Ampere's law in terms of vector potential. Using a Coulomb gauge in which $\nabla \cdot \vec{A} = 0$, the Ampere's law $\nabla \times \vec{B} = -\mu_0 \vec{J}$ is equivalent to

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

which is actually a set of three equations for the components of $\text{vec}A$, viz.,

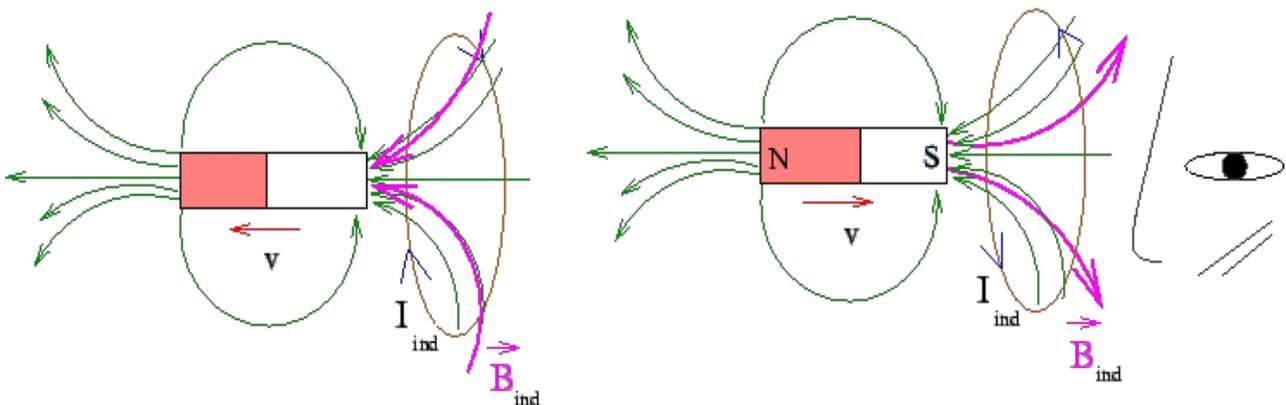
$$\begin{aligned} \nabla^2 A_x &= -\mu_0 J_x \\ \nabla^2 A_y &= -\mu_0 J_y \\ \nabla^2 A_z &= -\mu_0 J_z \end{aligned}$$

which are Poisson's equations.

Electromagnetic Induction :

We have seen that studies made by Oersted, Biot-Savart and Ampere showed that an electric current produces a magnetic field. Michael Faraday wanted to explore if this phenomenon is reversible in the sense whether a magnetic field could be source for a current in a conductor. However, no current was found when a conductor was placed in a magnetic field. Faraday and (Joseph) Henry, however, found that if a current loop was placed in a time varying magnetic field or if there was a relative motion between a magnet and the loop a transient current was established in the conducting loop. They concluded that the source of the electromotive force driving the current in the conductor is not the magnetic field but the changing magnetic flux associated with the loop. The change in flux could be effected by (i) a time varying magnetic field or by (ii) motion of the conductor in a magnetic field or (iii) by a combined action of both of these. The discovery is a spectacular milestone in the sense that it led to important developments in Electrical engineering like invention of transformer, alternator and generator.

Shortly after Faraday's discovery, Heinrich Lenz found that the direction of the induced current is such that it opposes the very cause that produced the induced current (i.e. the magnetic field associated with the induced current opposes the change in the magnetic flux which caused the induced current in the first place). Lenz's law is illustrated in the following.



In the figures the loops are perpendicular to the plane of the page. The direction of induced current is as seen towards the loop from the right. Note that the magnetic field set up by the induced current tends to increase the flux in the case where the magnet is moving away from the loop and tends to decrease it in the case where it is moving towards the loop,

Mathematically, Faraday's law is stated thus : the electromotive force is proportional to the rate of change of magnetic flux. In SI units, the constant of proportionality is unity.

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

where Φ is the flux associated with the circuit and the minus sign is a reminder of the direction of the current as given by Lenz's law. If the loop contains N turns, the equation becomes

$$\mathcal{E} = -N\frac{d\Phi}{dt}$$

Though the flux is a scalar, one can fix its sign by considering the sign of the area vector which is fixed by the usual right hand rule. The dot product of \vec{B} and $d\vec{S}$ then has a sign.

Recap

In this lecture you have learnt the following

- Vector potential for a magnetic field was defined.
- Vector potential can be determined up to an additive term which is gradient of a scalar function.
- Equivalent vector potentials which give the same magnetic field are connected by a gauge transformation.
- Vector potentials were calculated in a few simple cases.