Module 3 : Electromagnetism
Lecture 13 : Magnetic Field

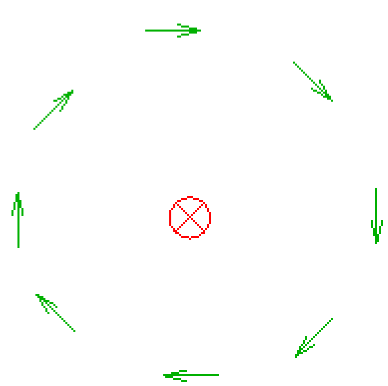
Objectives

In this lecture you will learn the following

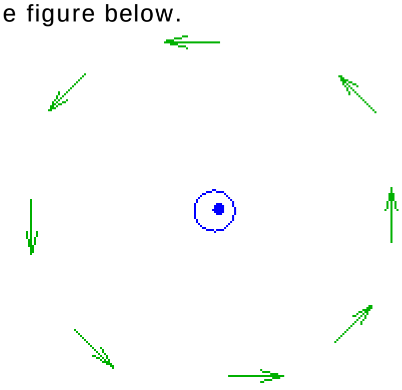
- Electric current is the source of magnetic field.
- When a charged particle is placed in an electromagnetic field, it experiences a force, called Lorentz force.
- Motion of a charged particle in a magnetic field and concept of cyclotron frequency.
- Find the trajectory of a charged particle in crossed electric and magnetic field.

MAGNETIC FIELD

Electric charges are source of electric fields. An electric field exerts force on an electric charge, whether the charge happens to be moving or at rest.
One could similarly think of a magnetic charge as being the source of a magnetic field. However, isolated magnetic charge (or magnetic monopoles) have never been found to exist. Magnetic poles always occur in pairs (dipoles) - a north pole and a south pole. Thus, the region around a bar magnet is a magnetic field.
What characterizes a magnetic field is the qualitative nature of the force that it exerts on an electric charge. The field does not exert any force on a static charge. However, if the charge happens to be moving (excepting in a direction parallel to the direction of the field) it experiences a force in the magnetic field.
It is not necessary to invoke the presence of magnetic poles to discuss the source of magnetic field. Experiments by Oersted showed that a magnetic needle gets deflected in the region around a current carrying conductor. The direction of deflection is shown in the figure below.



Current into the page



Current out of the page

Thus a current carrying conductor is the source of a magnetic field. In fact, a magnetic dipole can be considered as a closed current loop.

Electric Current and Current Density

Electric current is the rate of flow of charges in electrical conductor. In a conductor the charges may have random motion. However, the net drift velocity of the charges is zero, giving a zero net current. In the presence of an external force field, the charges move with a net non-zero drift velocity, which gives a current.

The direction of current has been defined, conventionally, as the direction in which the positive charges move. In case of metallic conductors, the current is caused by flow of negatively charged electrons, whose direction of motion is opposite to the direction of current. In electrolytes, however, the current is due to flow of both positive and negative ions.

The current density at a position $\vec{r}$ is defined as the amount of charge crossing a unit cross-

$$\vec{J}(\vec{r})$$

sectional area per unit time. In terms of the net drift velocity of the charges (taken opposite to the net drift velocity of electrons), $\vec{v}$,

$$\vec{J} = \rho\vec{v}$$

where ρ is the volume density of the mobile charges.

The integral of the current density over a surface defines electric current, which is a scalar.

$$I = \int_S \vec{J} \cdot d\vec{S}$$

The unit of electric current is Ampere (= coulomb/sec) and that of the current density is A/m ^{2}. For a thin wire with a small cross sectional area, the current density $\vec{J}$ may be taken as uniform. In this case,

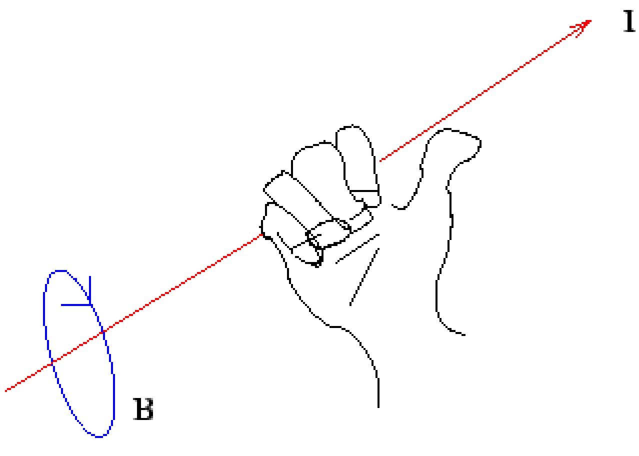
$$I = | J | S$$

where S is the component of the area vector parallel to the direction of current.

The Right Hand Rule

The direction of magnetic field due to a current carrying conductor is given by the right hand rule.

If one clasps the conductor with one's right hand in such a way that the thumb points in the direction of the current (i.e. in the direction opposite to the direction of electron flow) then, the direction in which the fingers curl gives the direction of the magnetic field due to such a conductor.



The Lorentz Force

We know that an electric field $\vec{E}$ exerts a force $q\vec{E}$ on a charge q. In the presence of a magnetic field $\vec{B}$, a charge q experiences an additional force $\vec{F}_m = q\vec{v} \times \vec{B}$

where $\vec{v}$ is the velocity of the charge. Note that

- There is no force on a charge at rest.

- A force is exerted on the charge only if there is a component of the magnetic field perpendicular to the direction of the velocity, i.e. *the component of the magnetic field parallel to* $\vec{v}$ *does not contribute to* $\vec{F}_m$.

- $\vec{v} \cdot \vec{F}_m = 0$, which shows that the magnetic force does not do any work.

In the case where both $\vec{E}$ and $\vec{B}$ are present, the force on the charge q is given by

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

This is called **Lorentz force** after H.E. Lorentz who postulated the relationship. It may be noted that the force expression is valid even when $\vec{E}$ and $\vec{B}$ are time dependent

Unit of Magnetic Field

From the Lorentz equation, it may be seen that the unit of magnetic field is Newton-second/coulomb-meter, which is known as a Tesla (T). (The unit is occasionally written as Weber/m ^{2} as the unit of magnetic flux is known as Weber). However, Tesla is a very large unit and it is common to measure $\vec{B}$ in terms of a smaller unit called Gauss,

$$1\mathrm{T} = 10^4\ \mathrm{G}$$

It may be noted that $\vec{B}$ is also referred to as magnetic field of induction or simply as the induction field. However, we will use the term ``magnetic field".

Motion of a Charged Particle in a Uniform Magnetic Field

Let the direction of the magnetic field be taken to be z- direction,

$$\vec{B} = B\hat{k}$$

we can write the force on the particle to be

$$\vec{F}_m = m\frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B}$$

The problem can be looked at qualitatively as follows. We can resolve the motion of the charged particle into two components, one parallel to the magnetic field and the other perpendicular to it. Since the motion parallel to the magnetic field is not affected, the velocity component in the z-direction remains constant.

$$v_z(t) = v_z(t = 0) = u_z$$

where $\vec{u}$ is the initial velocity of the particle. Let us denote the velocity component perpendicular to the direction of the magnetic field by $v_\perp$. Since the force (and hence the acceleration) is perpendicular to the direction of velocity, the motion in a plane perpendicular to $\vec{B}$ is a circle. The centripetal force necessary to sustain the circular motion is provided by the Lorentz force

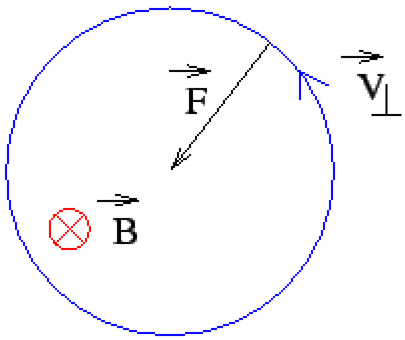
$$\frac{mv_\perp^2}{R} = \mid q \mid v_\perp B$$

where the radius of the circle R is called the Larmor radius, and is given by

$$R = \frac{mv_\perp}{|q|B}$$

The time taken by the particle to complete one revolution is $T = \dfrac{2\pi R}{v_\perp}$

The cyclotron frequency ω_c is given by $\omega_c = \dfrac{2\pi}{T} = \dfrac{|q|B}{m}$



$\otimes$ $\vec{B}$ Magnetic field into the page.

Figure shows directions of force and velocity for a positive charge.

Motion in a Magnetic Field - Quantitative

Let the initial velocity of the particle be $\vec{u}$. we may take the direction of the component of $\vec{u}$ perpendicular to $\vec{B}$ as the $x-$direction, so that

$$\vec{u} = (u_x, 0, u_z)$$

Let the velocity at time t be denoted by $\vec{v}$

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

we can express the force equation in terms of its cartesian component

$$\begin{aligned} m\frac{dv_x}{dt} = q(v_yB_z - v_zB_y) &= qv_yB \\ m\frac{dv_y}{dt} = q(v_zB_x - v_xB_z) &= -qv_xB \\ m\frac{dv_z}{dt} = q(v_xB_y - v_yB_x) &= 0 \end{aligned}$$

where we have used $B_x = B_y = 0$ and $B_z = B$.

The last equation tells us that no force acts on the particle in the direction in which $\vec{B}$ acts, so that

$$v_z = \text{constant} = u_z$$

The first two equations may be solved by converting them into second order differential equatons. This is done by differentiating one of the equations with respect to time and substituting the other equation in the resulting second order equation. For instance, the equation for v_x is given by

$$\frac{d^2v_x}{dt^2} = \frac{qB}{m}\frac{dv_y}{dt} = -\frac{q^2B^2}{m^2}v_x$$

The equation is familiar in the study of simple harmonic motion. The solutions are combination of sine and cosine functions.

$$v_x = A\sin\omega_c t + B\cos\omega_c t$$

where

$$\omega_c = \frac{qB}{m}$$

is called the *cyclotron frequency* and A and B are constants. These constants have to be determined from initial conditions. By our choice of x and y axes, we have

$$v_x(t=0) = u_x$$

so that $B = u_x$. Differentiating the above equation for u_x,

$$\begin{aligned}\frac{dv_x}{dt} &= A\omega_c\cos\omega_c t - B\omega_c\sin\omega_c t\\ &\equiv \frac{qB}{m}v_y = \omega_c v_y\end{aligned}$$

Since $v_y = 0$ at $t = 0$, we have $A = 0$. Thus, the velocity components at time t are given by

$$\begin{aligned}v_x &= u_x\cos\omega_c t\\ v_y &= -u_x\sin\omega_c t = u_x\sin(\omega_c t + \frac{\pi}{2})\end{aligned}$$

which shows that v_x and v_y vary harmonically with time with the same amplitude but with a phase difference of $\pi/2$. Equation of the trajectory may be obtained by integrating the equations for velocity components

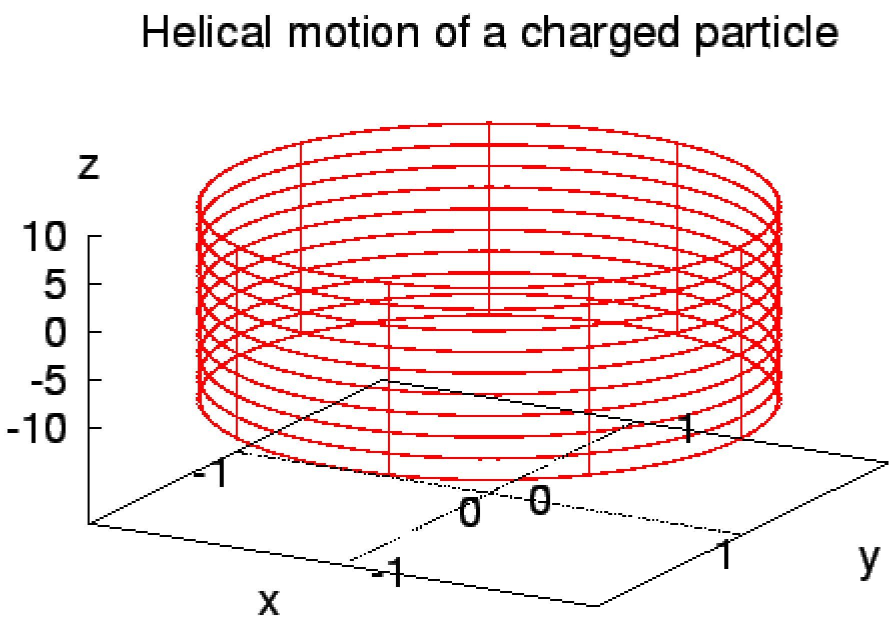
$$\begin{aligned}x(t) &= \frac{u_x}{\omega_c}\sin\omega_c t + x_0\\ y(t) &= \frac{u_x}{\omega_c}\cos\omega_c t + y_0\end{aligned}$$

$$z(t) = u_z t + z_0$$

where x_0, y_0 and z_0 are constants of integration representing the initial position of the particle. The equation to the projection of the trajectory in the x-y plane is given by

$$(x - x_0)^2 + (y - y_0)^2 = \frac{u_x^2}{\omega_c^2}$$

which represents a circle of radius u_x/ω_c, centered about (x_o, y_0). As the z- component of the velocity is constant, the trajectory is a helix.



A plot of the motion of a charged particle in a constant magnetic field.

Motion in a crossed electric and magnetic fields

The force on the charged particle in the presence of both electric and magnetic fields is given by

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Let the electric and magnetic fields be at right angle to each other, so that,

$$\vec{E} \cdot \vec{B} = 0$$

If the particle is initially at rest no magnetic force acts on the particle. As the electric field exerts a force on the particle, it acquires a velocity in the direction of $\vec{E}$. The magnetic force now acts sidewise on the particle.

For a quantitative analysis of the motion, let $\vec{E}$ be taken along the x-direction and $\vec{B}$ along z-direction. As there is no component of the force along the z-direction, the velocity of the particle remains zero in this direction. The motion, therefore, takes place in x-y plane. The equations of motion are

$$m\frac{dv_x}{dt} = q(\vec{E} + \vec{v} \times \vec{B})_x = qE + qBv_y \quad (1)$$

$$m\frac{dv_y}{dt} = q(\vec{E} + \vec{v} \times \vec{B})_y = -qBv_x \qquad (2)$$

As in the earlier case, we can solve the equations by differentiating one of the equations and substituting the other,

$$m\frac{d^2v_x}{dt^2} = qB\frac{dv_y}{dt} = -\frac{q^2B^2}{m}v_x$$

which, as before, has the solution

$$v_x = A\sin\omega_c t$$

with $\omega_c = qB/m$. Substituting this solution into the equation for v_y, we get

$$v_y = a\cos\omega_c t + C$$

Since $v_y = 0$ at $t = 0$, the constant $C = -A$, so that

$$v_y = A(\cos\omega_c t - 1)$$

The constant A may be determined by substituting the solutions in eqn. (1) which gives

$$mA\omega_c\cos\omega_c t = qE + qBA(\cos\omega_c t - 1)$$

Since the equation above is valid for all times, the constant terms on the right must cancel, which gives $A = E/B$. Thus we have

$$\begin{aligned} v_x &= \frac{E}{B}\sin\omega_c t \\ v_y &= \frac{E}{B}(\cos\omega_c t - 1) \end{aligned}$$

The equation to the trajectory is obtained by integrating the above equation and determining the constant of integration from the initial position (taken to be at the origin),

$$\begin{aligned} x &= \frac{E}{B\omega_c}(1 - \cos\omega_c t) \\ y &= \frac{E}{B\omega_c}(\sin\omega_c t - \omega_c t) \end{aligned}$$

The equation to the trajectory is

$$\left(x - \frac{E}{B\omega_c}\right)^2 + \left(y - \frac{Et}{B}\right)^2 = \frac{E^2}{B^2\omega_c^2}$$

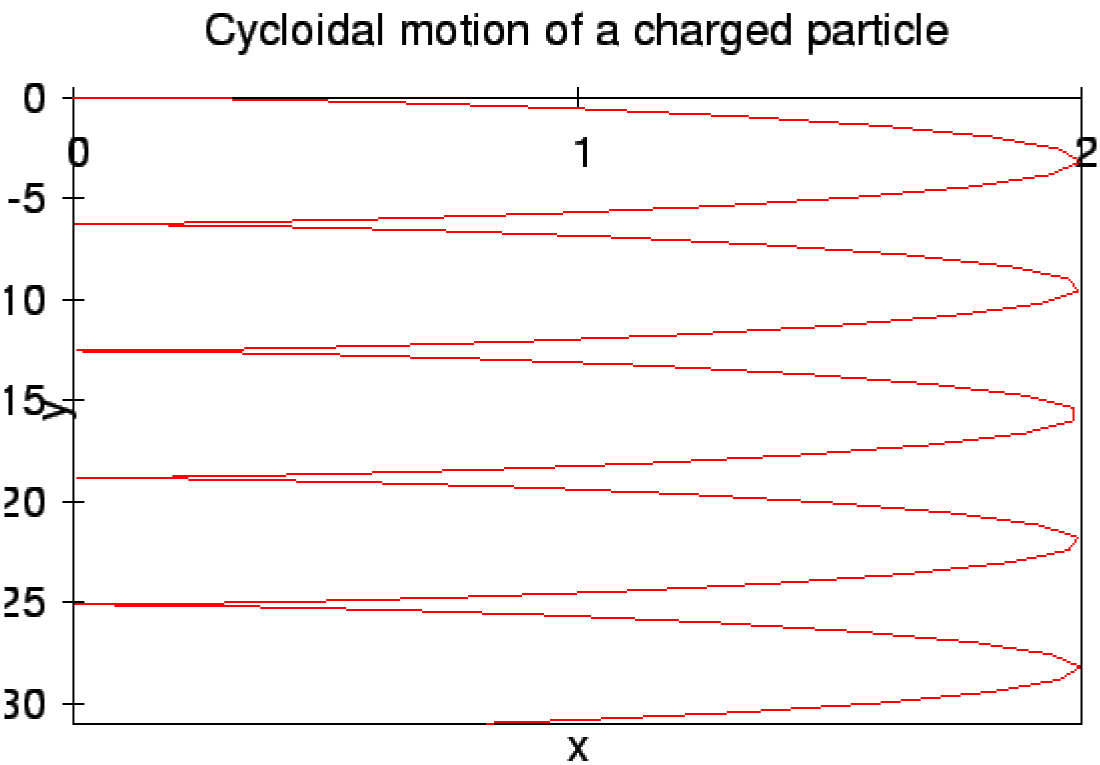
which represents a circle of radius

$$R = \frac{E}{B\omega_c}$$

whose centre travels along the negative y direction with a constant speed

$$v_0 = \frac{E}{B}$$

The trajectory resembles that of a point on the circumference of a wheel of radius R, rolling down the y-axis without slipping with a speed v_0. The trajectory is known as a cycloid.



Exercise 1

Find the maximum value of x attained by the particle during the cycloidal motion and determine the speed of the particle at such points.

(Ans. $x_{max} = 2E/B\omega_c$ speed $2E/B$.)

Recap

In this lecture you have learnt the following

- Just as a static electric charge is source of an electric field, the source of magnetic field is a moving charge, i.e., electric current.
- When a charged particle is is placed in a magnetic field, it experiences a velocity dependent force. The force acts perpendicular to the direction of motion of the charge. it experiences a force, called Lorentz force.
- The trajectory of a charged particle moving in a umiform magnetic field is circular.
- The time period of revolution of a charged particle in a magnetic field depends on the strength of the magnetic field and the charge to mass ratio of the particle.

- In a crossed electric and magnetic field a charged particle moves in a helical path.