Module 1 : A Crash Course in Vectors
Lecture 1 : Scalar And Vector Fields

Objectives

In this lecture you will learn the following

- Learn about the concept of field
- Know the difference between a scalar field and a vector field.
- Review your knowledge of vector algebra
- Learn how an area can be looked upon as a vector
- Define position vector and study its transformation properties under rotation.

SCALAR AND VECTOR FIELDS

This introductory chapter is a review of mathematical concepts required for the course. It is assumed that the reader is already familiar with elementary vector analysis.

Physical quantities that we deal with in electromagnetism can be scalars or vectors.
A scalar is an entity which only has a magnitude. Examples of scalars are mass, time, distance, electric charge, electric potential, energy, temperature etc.
A vector is characterized by both magnitude and direction. Examples of vectors in physics are displacement, velocity, acceleration, force, electric field, magnetic field etc.

A field is a quantity which can be specified everywhere in space as a function of position. The quantity that is specified may be a scalar or a vector. For instance, we can specify the temperature at every point in a room. The room may, therefore, be said to be a region of ``temperature field" which is a scalar field because the temperature $T(x, y, z)$ is a scalar function of the position. An example of a scalar field in electromagnetism is the electric potential.

In a similar manner, a vector quantity which can be specified at every point in a region of space is a vector field. For instance, every point on the earth may be considered to be in the gravitational force field of the earth. we may specify the field by the magnitude and the direction of acceleration due to gravity (i.e. force per unit mass) $g(x, y, z)$ at every point in space. As another example consider flow of water in a pipe. At each point in the pipe, the water molecule has a velocity $\vec{v}(x, y, z)$. The water in the pipe may be said to be in a velocity field. There are several examples of vector field in electromagnetism, e.g., the electric field $\vec{E}$, the magnetic flux density $\vec{B}$ etc.

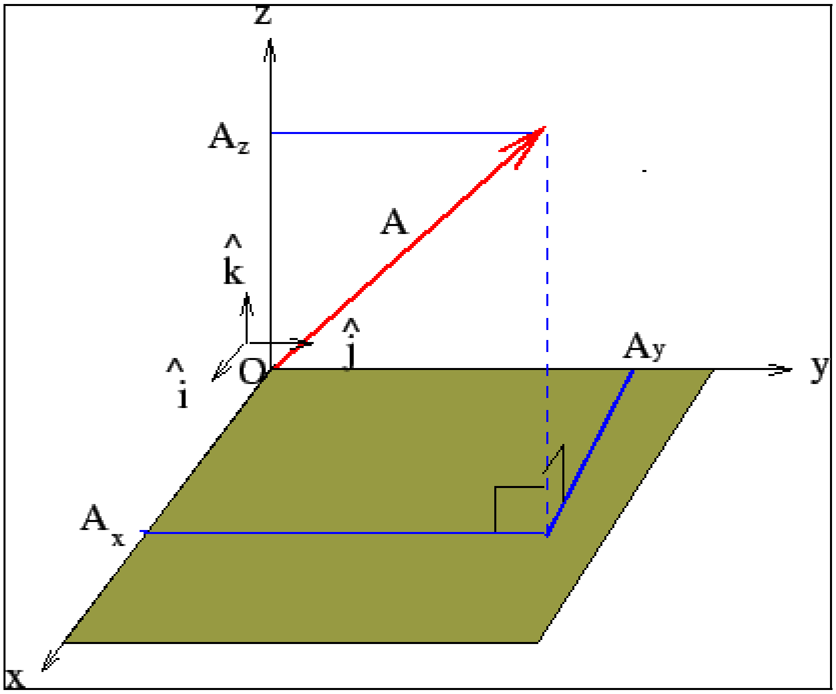
Elementary Vector Algebra :

Geometrically a vector is represented by a directed line segment. Since a vector remains unchanged if it is shifted parallel to itself, it does not have any position information.

A three dimensional vector can be specified by an ordered set of three numbers, called its *components* . The magnitude of the components depend on the coordinate system used. In electromagnetism we usually use cartesian, spherical or cylindrical coordinate systems. (Specifying a vector by its components has the advantage that one can extend easily to n dimensions. For our purpose, however, 3 dimension would suffice.)

A vector $\vec{A}$ is represented by (A_x, A_y, A_z) in cartesian (rectangular) coordinates. The magnitude of the vector is given by

$$\mid A \mid = \sqrt{A_x^2 + A_y^2 + A_z^2}$$



[Click here for Animation]

A unit vector in any direction has a magnitude (length) 1. The unit vectors parallel to the cartesian x, y and z coordinates are usually designated by $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ respectively. In terms of these unit vectors, the vector $\vec{A}$ is written

$$\vec{A} = \hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z$$

Any vector in 3 dimension may be written in this fashion. The vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ are said to form a *basis* . In fact, any three non-colinear vectors may be used as a basis. The basis vectors used here are perpendicular to one another. A unit vector along the direction of $\vec{A}$ is

$$\hat{A} = \frac{\vec{A}}{|A|}$$

Vector Addition

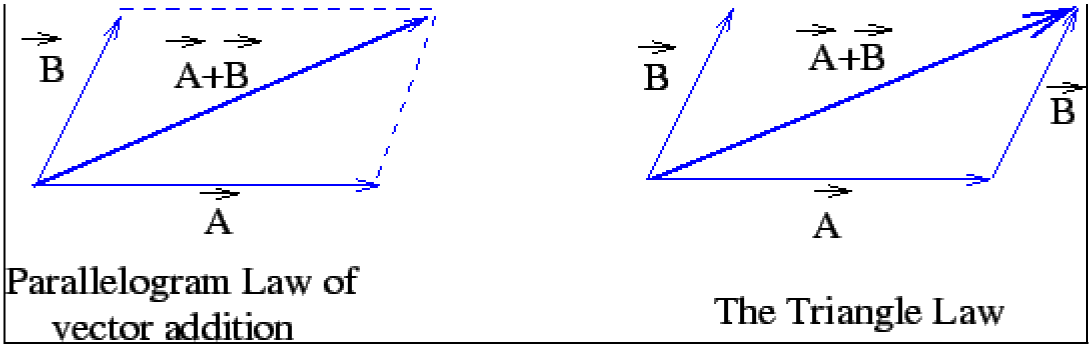
Sum of two vectors $\vec{A}$ and $\vec{B}$ is a third vector. If

$$\begin{aligned} \vec{A} &= (A_x, A_y, A_z) \\ \vec{B} &= (B_x, B_y, B_z) \end{aligned}$$

then

$$\vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z)$$

Geometrically, the vector addition is represented by parallelogram law or the triangle law, illustrated below.



Parallelogram Law of vector addition

The Triangle Law

Scalar Multiplication

The effect of multiplyimg a vector by a real number c is to multiply its magnitude by c without a change in direction (except where c is negative, in which case the vector gets inverted). In the component representation, each component gets multiplied by the scalar

$$c\vec{A} = (cA_x, cA_y, cA_z)$$

Scalar multiplication is distributive in addition, i.e.

$$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$$

Two vectors may be multiplied to give either a scalar or a vector.

Scalar Product (The Dot products)

The dot product of two vectors $\vec{A}$ and $\vec{B}$ is a scalar given by the product of the magnitudes of the vectors times the cosine of the angle ($\leq 0 \leq \theta$) between the two

$$\vec{A} \cdot \vec{B} = \mid A \mid\mid B \mid \cos\theta$$

In terms of the components of the vectors

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Note that

Dot product is commutative and distributive

$$\begin{aligned} \vec{A} \cdot \vec{B} &= \vec{B} \cdot \vec{A} \\ \vec{A} \cdot (\vec{B} + \vec{C}) &= \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \end{aligned}$$

Two vectors are orthogonal if

$$\vec{A} \cdot \vec{B} = 0$$

Dot products of the cartesian basis vectors are as follows

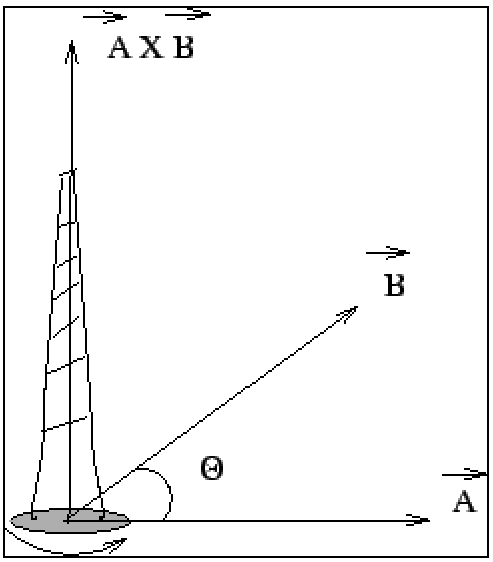
$$\begin{aligned} \hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} &= 1 \\ \hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} &= 0 \end{aligned}$$

Exercise 1

Show that the vectors $\vec{A} = 3\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$ and $\vec{B} = 2\hat{\imath} + 4\hat{\jmath} + 7\hat{k}$ are orthogonal.

Vector Product (The Cross Product)

The cross product of two vectors $\vec{A}\times\vec{B}$ is a vector whose magnitude is $|A||B|\sin\theta$, where θ is the angle between the two vectors. The direction of the product vector is perpendicular to both $\vec{A}$ and $\vec{B}$. This, however, does not uniquely determine $\vec{A}\times\vec{B}$ as there are two opposite directions which are so perpendicular. The direction of $\vec{A}\times\vec{B}$ is fixed by a convention, called the Right Hand Rule.



Right Hand Rule :

Stretch out the fingers of the right hand so that the thumb becomes perpendicular to both the index (fore finger) and the middle finger. If the index points in the direction of $\vec{A}$ and the middle finger in the direction of $\vec{B}$ then, $\vec{A}\times\vec{B}$ points in the direction of the thumb. The rule is also occasionally called the *Right handed cork screw rule* which may be stated as follows. If a right handed screw is turned in the direction from $\vec{A}$ to $\vec{B}$, the direction in which the head of the screw proceeds gives the direction of the cross product.

In cartesian basis the cross product may be written in terms of the components of $\vec{A}$ and $\vec{B}$ as follows.

$$\begin{aligned}\vec{A}\times\vec{B} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_k \\ B_x & B_y & B_k \end{vmatrix} \\ &= \hat{\imath}(A_yB_z - A_zB_y) + \hat{\jmath}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)\end{aligned}$$

The following points may be noted :

Vector product is anti-commutative, i.e.,

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

Two vectors are parallel if their cross product is zero.

Vector product is distributive

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

The cross product of cartesian basis vectors are as follows

$$\begin{aligned}\hat{\imath} \times \hat{\jmath} &= \hat{k} \\ \hat{\jmath} \times \hat{k} &= \hat{\imath} \\ \hat{k} \times \hat{\imath} &= \hat{\jmath}\end{aligned}$$

and

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$$

Exercise 2

Find a unit vector which is perpendicular to the plane containing the vectors $2\hat{\imath} - \hat{\jmath} - \hat{k}$ and $\hat{\imath} + 2\hat{\jmath} + \hat{k}$

[Ans. $(1/\sqrt{35}(\hat{\imath} - 3\hat{\jmath} + 5\hat{k})$]

Exercise 3

Vector $\vec{A} = 3\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$ and $\vec{B} = 6\hat{\imath} + \alpha\hat{\jmath} + \beta\hat{k}$. Find the values of α and β such that the vectors are parallel

[Ans. $\alpha = -10, \beta = 4$.]

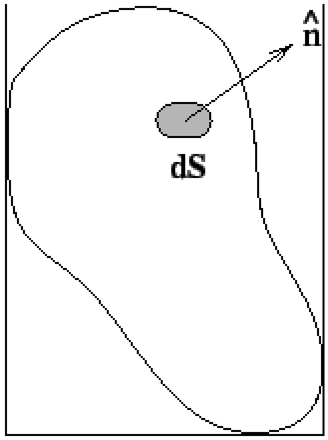
Area as a Vector Quantity

The magnitude of the vector also happens to be the area of the parallelogram formed by the vectors $\vec{A}$ and $\vec{B}$. The fact that a direction could be uniquely associated with a cross product whose magnitude is equal to an area enables us to associate a vector with an area element. The direction of the area element is taken to be the outward normal to the area. (This assumes that we are dealing with one sided surfaces and not two sided ones like a Möbius strip.

For an arbitrary area one has to split the area into small area elements and sum (integrate) over such elemental area vectors

$$\vec{S} = \int d\vec{S}$$

A closed surface has zero surface area because corresponding to an area element $d\vec{S}$, there is an area element $-d\vec{S}$ which is oppositely directed.



[Click here for Animation]

Scalar and Vector Triple Products

One can form scalars and vectors from multiple vectors. Scalar and vector triple products are often useful.

The scalar triple product of vectors $\vec{A}, \vec{B}$ and $\vec{C}$ is defined by

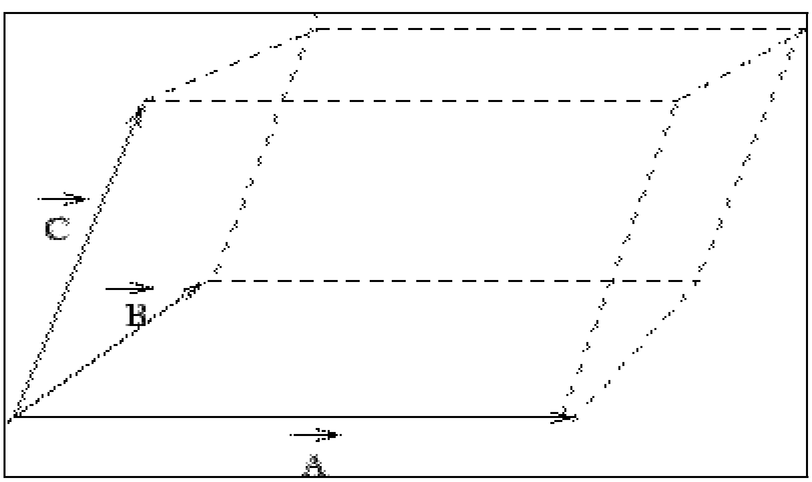
$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

Note that the scalar triple product is the same for any cyclic permutation of the three vectors $\vec{A}, \vec{B}$ and $\vec{C}$. In terms of the cartesian components, the product can bew written as the determinant

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_k \\ B_x & B_y & B_k \\ C_x & C_y & C_k \end{vmatrix}$$

Since $\vec{B}\times\vec{C}$ gives the area of a parallelogram of sides $\vec{B}$ and $\vec{C}$, the triple product $\vec{A}\cdot(\vec{B}\times\vec{C})$ gives the volume of a parallelopiped of sides $\vec{A}$, $\vec{B}$ and $\vec{C}$.

The vector triple product of $\vec{A}, \vec{B}$ and $\vec{C}$ is defined by $\vec{A}\times(\vec{B}\times\vec{C})$. Since cross product of two vectors is not commutative, it is important to identify which product in the combination comes first. Thus $\vec{A}\times(\vec{B}\times\vec{C})$ is not the same as $(\vec{A}\times\vec{B})\times\vec{C}$.



[Click here for Animation]

Example 1

Express an arbitray vector $\vec{A}$ as a linear combination of three non-coplanar vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

Solution :

Let $\vec{A} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$. Since the cross product $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$, its dot product with both vectors is zero. Taking the dot product of $\vec{A}$ with $\vec{b} \times \vec{c}$, we have

$$\vec{A} \cdot (\vec{b} \times \vec{c}) = \alpha\vec{a} \cdot (\vec{b} \times \vec{c})$$

which gives

$$\alpha = \frac{\vec{A} \cdot (\vec{b} \times \vec{c})}{\vec{a} \cdot (\vec{b} \times \vec{c})}$$

The coefficients β and γ may be found in a similar fashion.

Exercise 4

Prove the following vector identity which is very useful and often used

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{C})$$

For ease of remembering this formula is often known as bac-cab formula.

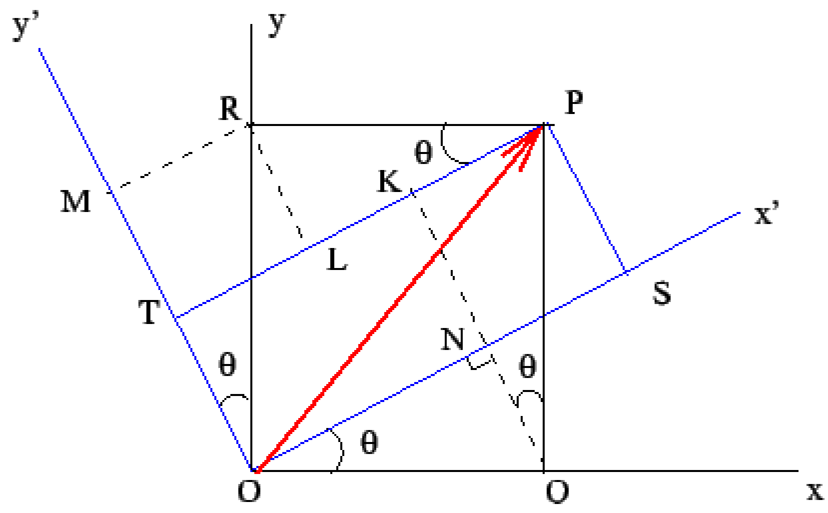
Position Vector and its Transformation under Rotation

Though a general vector is independent of the choice of origin from which the vector is drawn, one defines a vector representing the position of a particle by drawing a vector from the chosen origin O to the position of the particle. Such a vector is called the *position vector* . As the particle moves, the position vector also changes in magnitude or direction or both in magnitude and direction. Note, however, though the position vector itself depends on the choice of origin, the displacement of the particle is a vector which does not depend on the choice of origin.

In terms of cartesian coordinates of the point , the position vector is

$$\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$$

We will now derive the relationship between the x, y, z and the corresponding values x', y', z' in a coordinate system which is rotated with respect to the earlier coordinate system about an axis passing through the origin. For simplicity consider the axis of rotation to be the z-axis so that the z coordinate does not change.



[Click here for Animation]

In the figure P is the foot of the perpendicular drawn from the tip of the position vector on to the x-y plane. Since the axis of rotation coincides with the z-axis, the z coordinate does not change and we have $z' = z$. The figure shows various angles to be equal to the angle of rotation θ by use of simple geometry. One can easily see

$$
\begin{aligned}
x' = OS &= ON + NS \\
&= ON + KP \\
&= OQ\cos\theta + PQ\sin\theta \\
&= x\cos\theta + y\sin\theta \\
y' = OT &= OM - TM \\
&= OM - RL \\
&= OR\cos\theta - RP\sin\theta \\
&= y\cos\theta - x\sin\theta
\end{aligned}
$$

Since any vector can be parallely shifted to the origin, its tranformation properties are identical to the transformation properties of the position vector. Thus under rotation of coordinate system by an angle θ about the z-axis the components of a vector $\vec{A}$ transform as follows :

$$
\begin{aligned}
A_{x'} &= A_x\cos\theta + A_y\sin\theta \\
A_{y'} &= -A_x\sin\theta + A_y\cos\theta \\
A_{z'} &= A_z
\end{aligned}
$$

Exercise 5

Show that the cross product of vectors satisfy the transformation property stated above.

Recap

In this lecture you have learnt the following

- A field is a quantity that can be specified at every point in a certain region of space. A field ay be a scalar or a vector
- An area element can be regarded as a vector because in addition to having a magnitude a direction can be associated with it. The direction is conventionally chosen as the outward normal to the area element.

- Expressions for scalar and vector triple product were obtained.

- While a vector like displacement does not depend on the origin of coordinate system, position vector depends on the origin. The transformation properties of a position vector under rotation of coordinate system was studied.