Module 3 : MAGNETIC FIELD
Lecture 19 : Time Varying Field

Objectives

In this lecture you will learn the following

- Relate time varying magnetic field with emf generated.
- Define mutual inductance and calculate it in simple cases.
- Define self inductance.
- Calculate energy stored in a magnetic field.

Time Varying Field

Even where there is no relative motion between an observer and a conductor, an emf (and consequently an induced current for a closed conducting loop) may be induced if the magnetic field itself is varying with time as flux change may be effected by change in magnetic field with time. In effect it implies that a changing magnetic field is equivalent to an electric field in which an electric charge at rest experiences a force.

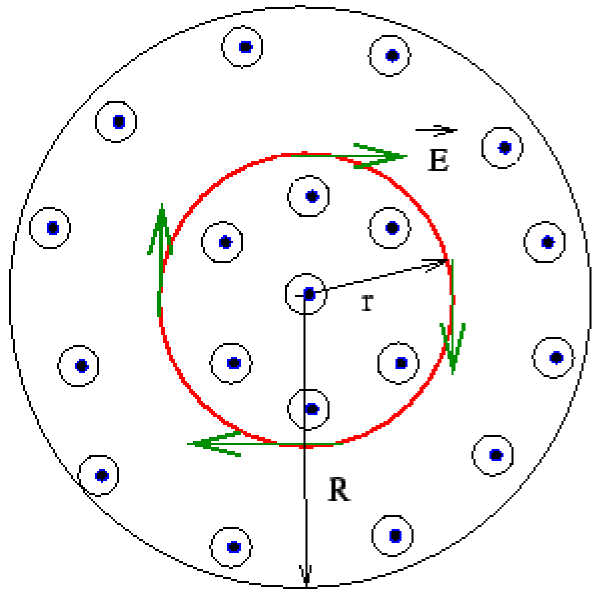
Consider, for example, a magnetic field $\vec{B}$ whose direction is out of the page but whose magnitude varies with time. The magnetic field fills a cylindrical region of space of radius R. Let the magnetic field be time varying and be given by

$$B = \begin{cases} B_0 \sin \omega t \ \text{ for } r \le R \\ 0 \qquad\quad \text{ for } \ r > R \end{cases}$$

Since $\vec{B}$ does not depend on the axial coordinate z as well as the azimuthal angle ϕ, the electric field is also independent of these quantities. Consider a coaxial circular path of radius $r \le R$ which encloses a time varying flux. By symmetry of the problem, the electric field at every point of the cicular path must have the same magnitude E and must be tangential to the circle.

Thus the emf is given by $\mathcal{E} = \int \vec{E} \cdot \vec{dl} = 2\pi r E$

By Faraday's law
$$\begin{aligned} \mathcal{E} &= -\frac{d\Phi}{dt} = -\frac{d}{dt}[\pi r^2 B(t)] \\ &= -\pi r^2 \frac{dB(t)}{dt} \end{aligned}$$



Equating these, we get for $r \leq R$,$E = -\frac{r}{2}\frac{dB}{dt} = -\frac{B_0\omega}{2} r\cos\omega t$

For $r > R$, the flux is $\Phi = \pi R^2 B(t)$, so that $\mathcal{E} = -\frac{d\Phi}{dt} = -\pi R^2 \frac{dB}{dt}$

and the electric field foir $r > R$ is $E = -\frac{R^2}{2r}\frac{dB}{dt} = -\frac{B\omega R^2}{2r}\cos\omega t$

Exercise 1

A conducting circle having a radius R_0 at time $t = 0$ is in a constant magnetic field B perpendicular to its plane. The circle expands with time with its radius becoming $R = R_0(1+\alpha t^2)$ at time t. Calculate the emf developed in the circle.

(Ans. $-4\pi R_0^2 \alpha t(1+\alpha t^2)B$)

Mutual Inductance

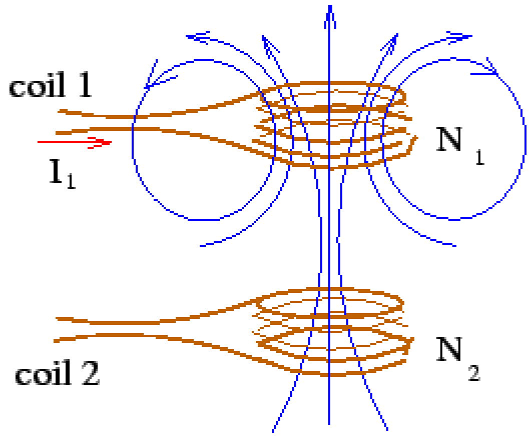
According to Faraday's law, a changing magnetic flux in a loop causes an emf to be generated in that loop. Consider two stationary coils carrying current. The first coil has N_1 turn and carries a current I_1. The second coil contains N_2 turns. The current in the first coil is the source of a magnetic field $\vec{B}_1$ in the region around the coil. The second loop encloses a flux $N_2\Phi_2 = N_2\int_S \vec{B}_1 \cdot d\vec{S}_2$, where S is the surface of one turn of the loop. If the current I_1 in the first coil is varied, $\vec{B}_1$,and consequently Φ_2 will vary with time.

The variation of Φ_2 causes an emf $\mathcal{E}_2$ to be developed in the second coil. Since B_1 is proportional to I_1, so is . The emf, which is the rate of change of flux is, therefore, proportional to ,

Φ_2 dI_1/dt

$$\mathcal{E}_2 = -M_{21}\frac{dI_1}{dt}$$

where M_{21} is a constant, called the *mutual inductance* of the two coils, which depends on geometrical factors of the two loops, their relative orientation and the number of turns in each coil.



Analogously, we can argue that if the second loop carries a current I_2 which is varied with time, it generates an induced emf in the first coil given by $\mathcal{E}_1 = -M_{12}\frac{dI_2}{dt}$

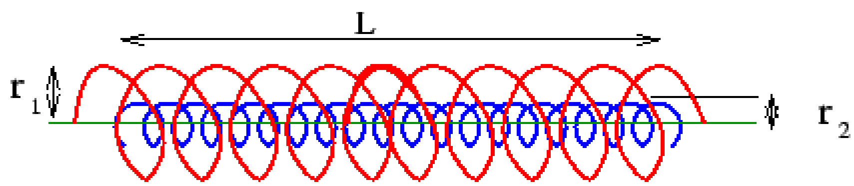
For instance, consider two concentric solenoids, the outer one having n_1 turns per unit length and inner one with n_2 turns per unit length. The solenoids are wound over coaxial cylinders of length L each. If the current in the outer solenoid is I_1, the field due to it is $B_1 = \mu_0 n_1 I_1$, which is confined within the solenoid. The flux enclosed by the inner cylinder is

$$\begin{aligned} N_2\Phi_2 &= N_2\pi r_2^2 B_1 \\ &= n_2 L\pi r_2^2\cdot(\mu_0 n_1 I_1) \\ &= \mu_0 n_1 n_2 L\pi r_2^2 I_1 \end{aligned}$$

If the current in the outer solenoid varies with time, the emf in the inner solenoid is

$$\mathcal{E}_2 = -N_2\frac{d\Phi_2}{dt} = -\mu_0 n_1 n_2 L\pi r_2^2\frac{dI_1}{dt}$$

so that $M_{21} = \mu_0 n_1 n_2 L\pi r_2^2$



If, on the other hand, the current I_2 in the inner solenoid is varied, the field due to it $B_2 = \mu_0 n_2 I_2$ which is non-zero only within the inner solenoid. The flux enclosed by the outer solenoid is, therefore,

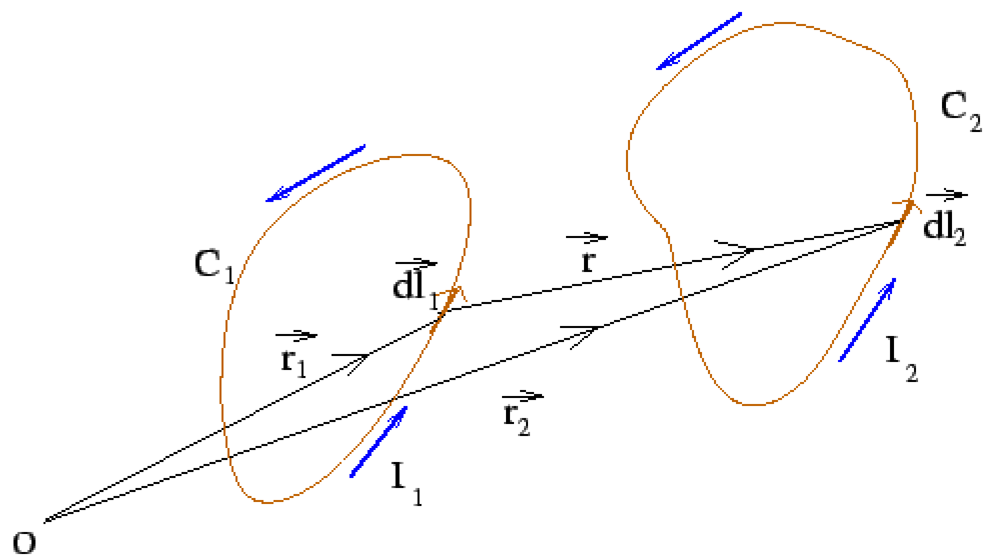
$$N_1\Phi_1 = (n_1 L)\pi r_2^2 \mu_0 n_2 I_2$$

If I_2 is varied, the emf in the outer solenoid is $\mathcal{E}_1 = -\mu_0 n_1 n_2 L\pi r_2^2 \frac{dI_2}{dt}$ giving

$M_{12} = \mu_0 n_1 n_2 L\pi r_2^2$

One can see that $M_{12} = M_{21}$.

This equality can be proved quite generally from Biot-Savart's law. Consider two circuits shown in the figure.



The field at $\vec{r_2}$, due to current in the loop C_1 (called the *primary*) is $\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l}_1 \times \hat{r}}{|\, r\, |^2}$

where $\vec{r} = \vec{r_2} - \vec{r_1}$. We have seen that $\vec{B}_1$ can be expressed in terms of a vector potential $\vec{A}_1$, where

, by Biot-Savart's law $\vec{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{l}_1}{|\, r\, |}$

The flux enclosed by the second loop, (called the *secondary*) is

$$
\begin{aligned}
\Phi_2 &= \int_{S_2} \vec{B}_1 \cdot d\vec{S}_2 \\
&= \int_{S_2} (\nabla \times \vec{A}_1) \cdot d\vec{S}_2 \\
&= \oint_{C_2} \vec{A}_1 \cdot d\vec{l}_2 \qquad \textbf{Stoke's law} \\
&= \frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mid r \mid}
\end{aligned}
$$

Clearly, $M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l}_1 \cdot d\vec{l}_2}{\mid r \mid}$

It can be seen that the expression is symmetric between two loops. Hence we would get an identical expression for M_{12}. This expression is, however, of no significant use in obtaining the mutual inductance because of rather difficult double integral.

Thus a knowledge of mutual inductance enables us to determine, how large should be the change in the current (or voltage) in a primary circuit to obtain a desired value of current (or voltage) in the secondary circuit. Since $M_{21} = M_{12}$, we represent mutual inductance by the symbol M. The emf $\mathcal{E}_s$ in the secondary circuit is given by $\mathcal{E}_s = -M\frac{dI_p}{dt}$, where I_p is the variable current in the primary circuit.

Units of M is that of Volt-sec/Ampere which is known as Henry (h)

Example 22

Consider two parallel rings C_1 and C_2 with radii R_1 and R_2 respectively with a separation d between their centres. Radius of C_2 is much smaller than that of C_1, so that the field experienced by C_2 due to a current in C_1 may be taken to be uniform over its area. Find the mutual inductance of the rings.

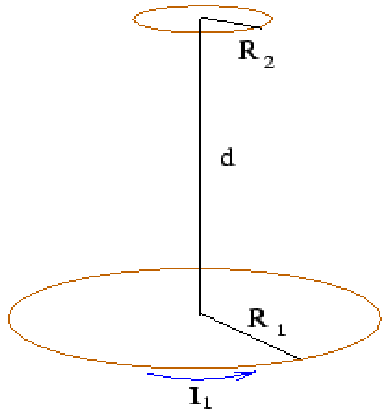
Solution :

The field experienced by the smaller ring may be taken to be given by the expression for the magnetic field of a ring along its axis. We had earlier shown that at a distance d from the centre of the ring, the field along the axis is given by

$$B_1 = \frac{\mu_0}{2} \frac{R_1^2}{(R_1^2 + d^2)^{3/2}} I_1$$

The flux enclosed by C_2 is $\Phi_2 = \frac{\mu_0}{2} \frac{R_1^2}{(R_1^2 + d^2)^{3/2}} I_1 \cdot \pi R_2^2$

By Faraday's law the emf in C_2 is $\mathcal{E}_2 = -\frac{\mu_0}{2} \frac{R_1^2}{(R_1^2 + d^2)^{3/2}} \pi R_2^2 \frac{dI_1}{dt}$



which gives $M_{21} = \frac{\mu_0\pi}{2}\frac{R_1^2 R_2^2}{(R_1^2+d^2)^{3/2}}$

The expression above is obviously not symmetrical between the loops. This is because of our assumption of uniform field over C_2. The approximation will be legitimate if the dimensions of C_2 is negligibly small, i.e. if C_2 is taken to be a dipole, $d >> R_1$, so that from C_1, the other loop looks like a point. In such a case,

$$M_{21} = \frac{\mu_0\pi}{2d^3}R_1^2 R_2^2$$

In the next example we assume that the current is changing in the dipolar loop and determine the emf generated in the larger loop.

Example 23

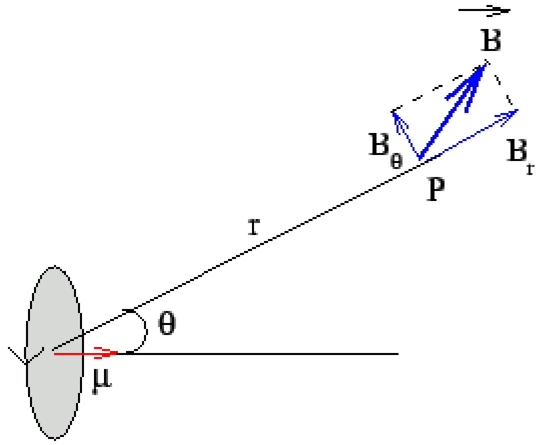
By considering the current in C_2 to be time varying determine the change in flux of the larger coil and hence determine the mutual inductance.

Solution :

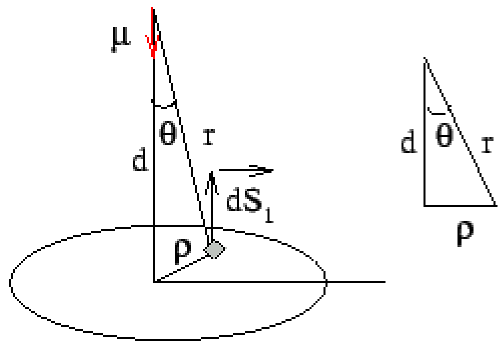
Since the field over the larger loop cannot be considered uniform, we need to use expressions for the magnetic field due to a magnetic dipole. The field is conveniently expressed in terms of its radial and tangential components. For a point dipole, the field components are given by expressions similar to the ones we derived for electric dipole. By replacing the electric dipole moment p by the magnetic dipole moment μ and permittivity factor $1/4\pi\epsilon_0$ by the permeability of vacuum $\mu_0/4\pi$, we have

$$B_r = \frac{\mu_0}{4\pi}\frac{2\mu\cos\theta}{r^3}$$
$$B_\theta = \frac{\mu_0}{4\pi}\frac{\mu\sin\theta}{r^3}$$

In the figure, the plane of the loop is normal to the page and the current direction is anticlockwise as seen from the right, so that the magnetic moment vector is as shown.



Consider an element of area $\rho d\rho d\phi$ of the larger ring at the position (ρ,ϕ) where the angle is measured with respect to some reference line in the plane of the ring. From the figure it can be seen that



$$\begin{aligned} r^2 &= d^2+\rho^2 \\ \tan\theta &= \frac{\rho}{d} \end{aligned}$$

so that,

$$\begin{aligned} \cos\theta &= \frac{d}{\sqrt{\rho^2+d^2}} \\ \sin\theta &= \frac{\rho}{\sqrt{\rho^2+d^2}} \end{aligned}$$

The angle between $d\vec{S}_1$ and $\underline{\vec{r}}$ is θ so that

$$\begin{aligned} \vec{B}\cdot d\vec{S}_1 &= B_r\cos\theta\rho d\rho d\phi + B_\theta\sin\theta\rho d\rho d\phi \\ &= \frac{\mu_0}{2}I_2R_2^2\frac{\cos^2\theta}{r^3}\rho d\rho d\phi + +\frac{\mu_0}{4}I_2R_2^2\frac{\sin^2\theta}{r^3}\rho d\rho d\phi \end{aligned}$$

where we have substituted $\mu = I_2R_2^2$ for the magnetic moment of the current loop. The flux enclosed by C_1 is obtaied by integrating over ϕ from 0 to 2π and over ρ from 0 to R_1 .

The integration over ϕ gives 2π. Using the expressions for $\sin\theta, \cos\theta$ and r in terms of the variable ρ, we get, for the flux $\Phi = \int \vec{B}\cdot d\vec{S}_1$

$$\begin{aligned} \Phi &= \mu_0 I_2 R_2^2 \pi \int_0^{R_1} \left[\frac{d^2 \rho}{(\rho^2+d^2)^{3/2}} + \frac{\rho^3}{2(\rho^2+d^2)^{5/2}} \right] d\rho \\ &= \frac{\mu_0 I_2 R_2^2 \pi}{2} \int_0^{R_1} \frac{2d^2+\rho^2}{(\rho^2+d^2)^{5/2}} \rho d\rho \end{aligned}$$

The integration can be easily performed by substituting $x = d^2+\rho^2$. After a bit of an algebra one gets

$$\Phi = -\frac{\mu_0 I_2 R_2^2 \pi}{2} \left[(\rho^2+d^2)^{-1/2} + \frac{d^2}{3}(\rho^2+d^2)^{-3/2} \right]_0^{R_1}$$

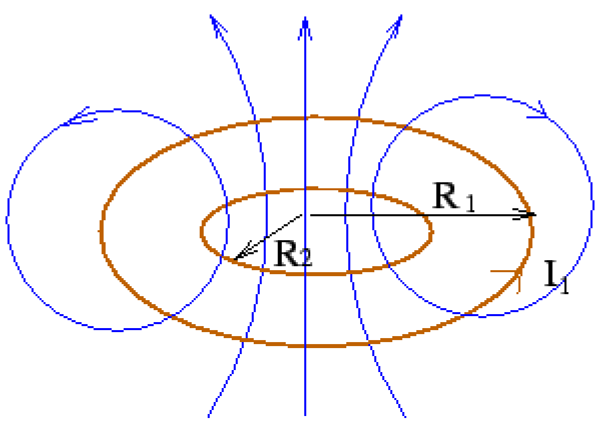
On substituting the limits and using a binomial expansion to retain leading order term when $d >> R_1$, we get

$$\Phi = \frac{\mu_0 R_1^2 R_2^2 \pi}{2d^3} I_2$$

which gives the same expression for mutual inductance as in the previous example.

Exercise 1

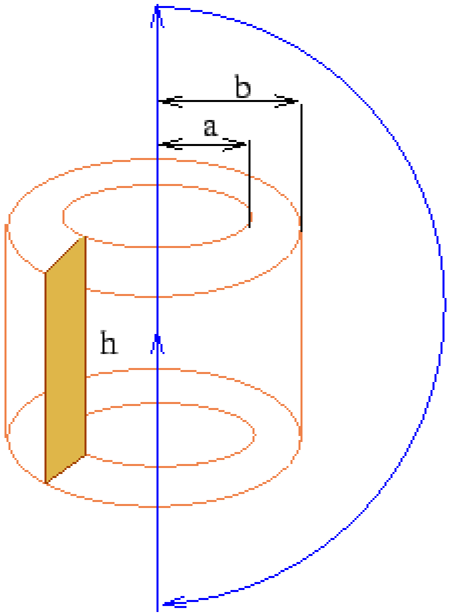
The figure shows two coplanar and concentric rings of radii R_1 and R_2 where $R_1 \gg R_2$. Determine the mutual inductance of the coils. Solve the problem by considering the current to be changing in either of the coils.

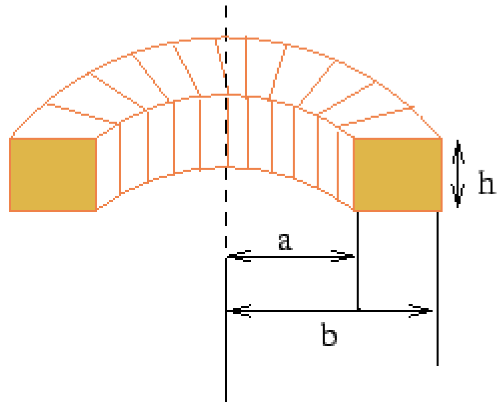


(Ans. $\mu_0 \pi R_2^2 / 2R_1$).

Exercise 2

A toroidal coil of rectangular cross section, with height h has N tightly wound turns. The inner radius of the torus is a and the outer radius b. A long wire passes along the axis.





The ends of the wire are connected by a semi-circular arc. Find the mutual inductance. Show explicitly that $M_{21} = M_{12}$.

(Hint : When the current flows in the turns of toroid, the field at a distance r from the toroid axis is $\mu_0 NI/2\pi r$. The semi-circular area traps flux only in one rectangular turn of height h and width $b - a$.

Answer : ($\mu_0 Nh/2\pi \ln(b/a)$.)

Self Inductance :

Even when there is a single circuit carrying a current, the magnetic field of the circuit links with the circuit itself. If the current happens to be time varying, an emf will be generated in the circuit to oppose the change in the flux linked with the circuit. The opposing voltage acts like a second voltage source connected to the circuit. This implies that the primary source in the circuit has to do additional work to overcome this **back emf** to establish the same current. The induced current has a direction determined by Lenz's law.

If no ferromagnetic materials are present, the flux is proportional to the current. If the circuit contains N turns, Faraday's law gives

$$N\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

where L is known as *Self Inductance* of the circuit. By definition, L is a positive quantity. From the above it follows, on integrating,

$$N\Phi = LI + \text{ constant}$$

Since $\Phi = 0$ when $I = 0$, the constant is zero and we get

$$L = \frac{N\Phi}{I}$$

The self inductance can, therefore, interpreted as the amount of flux linked with the circuit for unit current. The emf is given by

$$\mathcal{E} = -N\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

Example-24

Obtain an expression for the self inductance in a toroid of inner radius a, outer radius b and height h.

Solution :

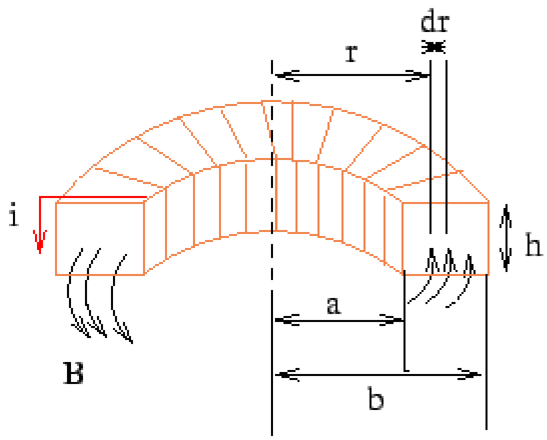
We have seen that the field inside the toroid at a distance r from the axis of the toroid is given by $B(r) = \mu_0 NI/2\pi r$.

The flux through one turn of the coil is the integral of this field over the cross section of the coil

$$\Phi(\text{one turn}) = h\int_a^b \frac{\mu_0 NI}{2\pi r}dr = \frac{\mu_0 INh}{2\pi}\ln(\frac{b}{a})$$

The flux threading N turns is

$$\Phi = \frac{\mu_0 IN^2 h}{2\pi}\ln(\frac{b}{a})$$



The self inductance is thus given by

$$L = \frac{\Phi}{I} = \frac{\mu_0 N^2 h}{2\pi}\ln\frac{b}{a}$$

Energy Stored in Magnetic Field

Just as capacitor stores electric energy, an inductor can store magnetic energy. To see this consider an L-R circuit in which a current I_0 is established. If the switch is thrown to the position such that the battery gets disconnected from the circuit at $t = 0$, the current in the circuit would decay. As the inductor provides back emf, the circuit is described by

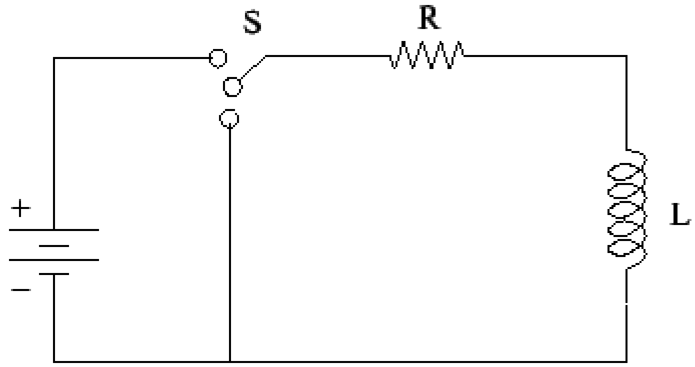
$$L\frac{dI}{dt} + IR = 0$$

With the initial condition $I = I_0$, the solution of the above is

$$I = I_0 \exp(-Rt/L)$$

As the energy dissipated in the circuit in time dt is $RI^2 dt$, the total energy dissipated from the time the battery is disconnected is

$$U = \int_0^\infty RI^2 dt = \int_0^\infty RI_0^2 \exp(-2Rt/L)dt = \frac{1}{2}LI_0^2$$



Thus the energy initially stored must have been $(1/2)LI_0^2$. If an inductor carries a current I, it stores an energy $(1/2)LI^2$. Thus the toroidal inductor discussed earlier stores an energy

$$\frac{\mu_0 N^2 h I^2}{4\pi}\ln\frac{b}{a}$$

when it carries a current I. We eill now show that this is also equal to the volume integral of $B^2/2\mu_0$.

Consider the magnetic field in the toroid at a distance r from the axis. We have seen that the magnetic field B is given by $\mu_0 NI/2\pi r$. Thus the value of $B^2/2\mu_0$ at this distance is $\mu_0 N^2 I^2/8\pi^2 r^2$.

Considering the toroid to consist of shells of surface area $2\pi rh$ and thickness dr, the volume of the shell is $2\pi rhdr$. The volume integral of $B^2/2\mu_0$ is therefore,

$$\begin{aligned}\int \frac{B^2}{2\mu_0} d^3r &= \int_a^b \frac{\mu_0 N^2 I^2}{8\pi^2 r^2} 2\pi rhdr \\ &= \int_a^b \frac{\mu_0 N^2 h I^2}{4\pi}\frac{1}{r}dr \\ &= \frac{\mu_0 N^2 h I^2}{4\pi}\ln\frac{b}{a}\end{aligned}$$

which is exactly the expression for the stored energy derived earlier.

Recap

In this lecture you have learnt the following

- The emf generated in one circuit due to a changing current in a second circuit is due to mutual inductance between the circuits.
- Mutual inductance is symmetric.
- In a few simple cases mutual inductance was calculated.
- A circuit gives rise to a back emf because of change in the current in the same circuit. A circuit can have a self inductance.
- Magnetic field can store energy and the energy density of the field can be calculated from a knowledge of the self inductance.