# Module 2 : Electrostatics
## Lecture 12 : Conductors and Dielectric

**Objectives**

In this lecture you will learn the following

- Properties of capacitors fielled with dielectric

- Force on a dielectric

- Calculation of effective capacitance for capacitor filled with different dielectrics
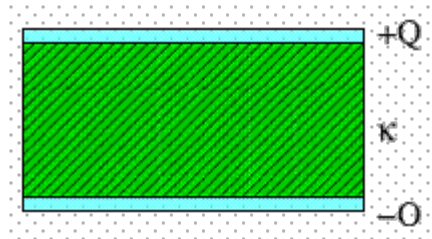
**Capacitance filled with Dielectric**

If a material of dielectric constant $\kappa$ is inserted between the plates of a capacitor, the field $\vec{E}$ is reduced by a factor $\kappa$. The potential between the plates also reduces by the same factor $\kappa$.

$$\phi \longrightarrow \frac{\phi}{\kappa}$$

Thus the capacitance

$$C = \frac{Q}{\phi}$$

**increases** by a factor $\kappa$.



**Example 20**

A parallel plate capacitor with plate separation 3.54mm and area 2m$^2$ is initially charged to a potential difference of 1000 volts. The charging batteries are then disconnected. A dielectric sheet with the same thickness as that of the separation between the plates and having a dielectric constant of 2 is then inserted between the capacitor plates. Determine (a) the capacitance , (b) potential difference across the capacitor plates, (c) surface charge density (d) the electric field and (e) displacement vector , before and after the insertion of the dielectric .

**Solution :**

(a) The capacitance before insertion of the dielectric is

$$C = \epsilon_0 \frac{A}{d} = 8.85 \times 10^{-12} \frac{2}{3.54 \times 10^{-3}} = 5 \times 10^{-9} \ F$$

After the insertion the capacitance doubles and becomes $10^{-8}$ F.

(b) Potential difference between the plates before insertion is given to be 1000 V. On introducing the dielectric it becomes half, i.e. 500 V.

(c) The charge on each capacitor plate was $Q = CV = 5 \times 10^{-6}$ coulomb, giving a surface charge density

of $2.5 \times 10^{-6}$ C/m$^2$. The free charge density remains the same on introduction of the dielectric.

(d) The electric field strength $E$ is given by

$$E = \frac{\sigma}{\epsilon_0} = 2.8 \times 10^5 \text{ volts/meter}$$

The electric field strength is reduced to $1.4 \times 10^5$ volt/meter on insertion.

(e) The displacement vector remains the same in both cases as the free charge density is not altered. It is given by $D = \sigma = 2.5 \times 10^{-6}$ C/m$^2$.

**Example 21**

The parallel plates of a capacitor of plate dimensions $a \times b$ and separation $d$ are charged to a potential

difference $\phi$ and battery is disconnected. A dielectric slab of relative permittivity $\kappa$ is inserted between the

plates of the parallel plate capacitor such that the left hand edge of the slab is at a distance $x$ from the left most edge of the capacitor. Calculate (a) the capacitance and (b) the force on the dielectric.
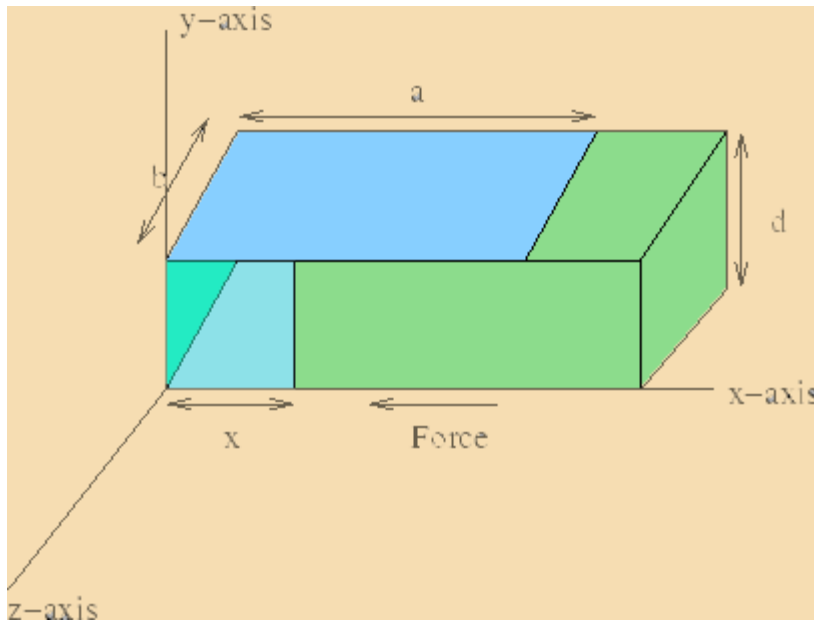


**Solution :**

Since the battery is disconnected, the potential difference between the plates will change while the charge remains the same. Since the capacitance of the part of the capacitor occupied by the dielectric is increased by a factor $\kappa$, the effective capacitance is due to two capacitances in parallel,

$$C = \epsilon_0 \frac{b}{d}[x + (a - x)\kappa]$$

The energy stored in the capacitor is

$$U = \frac{Q^2}{2C} = \frac{Q^2}{2} \frac{d}{b\epsilon_0} \frac{1}{x + (a - x)\kappa}$$

Let $F$ be the force *we* need to apply in the x-direction to keep the dielectric in place. For an infinitisimal increment $dx$ of $x$, we have to do an amount of work $Fdx$ of work, which will increase the energy strored in the field by $dU$, so that

$$F = \frac{dU}{dx}$$

the differentiation is to be done, keeping the charge $Q$ constant. Thus

$$F = \frac{dQ^2}{2b\epsilon_0} \frac{\kappa - 1}{[x + (a - x)\kappa]^2}$$

Since $\kappa > 1$, $F$ is positive. This means the electric field pulls the dielectric inward so that an external agency has to apply an outward force to keep the dielectric in position. Since the initial potential difference $\phi$ is given by $Q/C$, one can express the force in terms of this potential

$$F = \frac{\epsilon_0 b}{2d} \phi^2 (\kappa - 1)$$

This is the force that the external agency has to apply to keep the left edge of the dielectric at $x$. The force with which the capacitor pulls the dielectric in has the same magnitude.

**Example 22**

In the above example, what would be the force if the battery remained connected ?

**Solution :**

If the battery remained connected $Q$ does not remain the same, the potential $\phi$ does. The battery must do work to keep the potential constant. It may be realised that the force exerted on the dielectric in a particular position depends on the charge distribution (of both free and bound charges) existing in that position and the force is independent of whether the battery stays connected or is disconnected. However, in order to calculate the force with battery remaining connected, one must, explicitly take into account the work done by the battery in computing the total energy of the system. The total energy $U$ now has two parts, one the work done by the external agency $Fdx$ and the other the work done by the battery, viz., $\phi dQ$ where $dQ$ is the extra charge supplied by the battery to keep the potential constant. Thus

$$U = Fdx + \phi dQ$$

which gives

$$F = \frac{dU}{dx} - \phi \frac{dQ}{dx}$$

Since $\phi$ is constant, we have

$$U = \frac{1}{2}C\phi^2$$
$$Q = C\phi$$

Using these

$$F = \frac{1}{2}\phi^2\frac{dC}{dx} - \phi^2\frac{dC}{dx} = -\frac{1}{2}\phi^2\frac{dC}{dx}$$

(Note that if the work done by the battery were negnected, the direction of $F$ will be wrong, though, because we have used linear dielectrics, the magnitude, accidentally, turns out to be correct !)
In the previous example, we have seen that
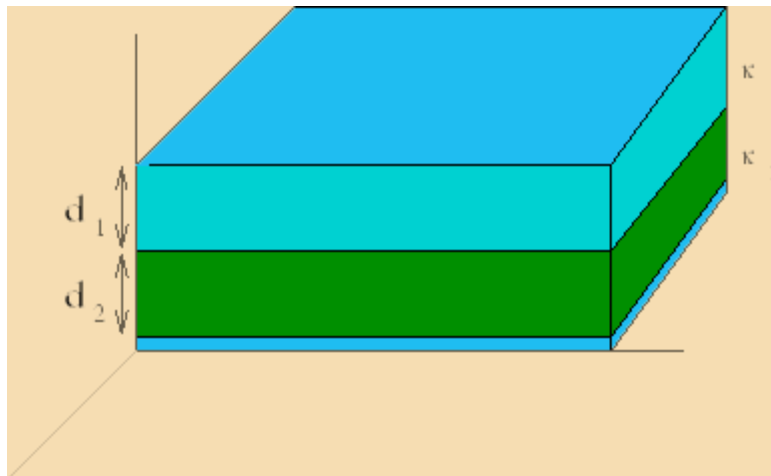
$$C = \frac{b\epsilon_0}{d}[x + (a - x)\kappa]$$

giving

$$\frac{dC}{dx} = \frac{b\epsilon_0}{d}(1 - \kappa)$$

which is negative. Thus $F$ is positive, as before,

$$F = \frac{b\epsilon_0\phi^2}{d}(\kappa - 1)$$

**Example 23**

The space between the plates of a parallel plate capacitor is filled with two different dielectrics, as shown. Find the effective capacitance.



**Solution :**

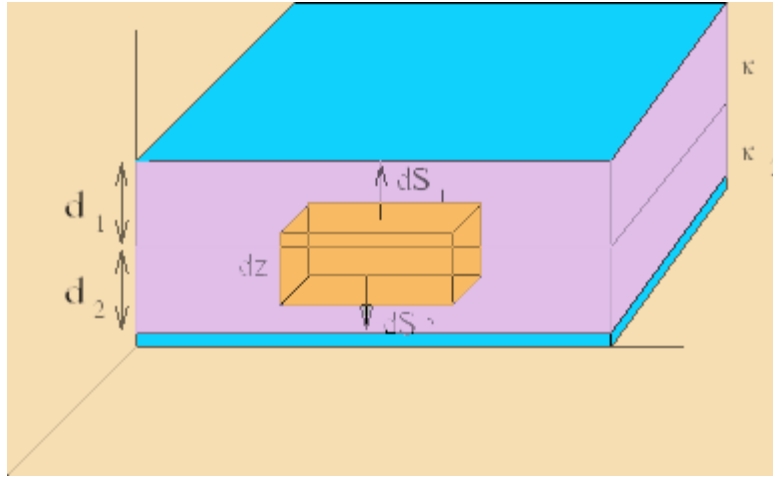Take a Gaussian pill-box as shown. We have

$$\int \vec{D} \cdot d\vec{S} = \rho^{free} = 0$$

as there are no free charges inside the dielectric. Contribution to the integral comes only from the faces of the

pill-box parallel to the plates and            . Hence,

$$d\vec{S}_1 = d\vec{S}_2$$

$$D_1 = D_2 = \sigma$$

where $\sigma$ is the surface density of free charges.



Let $\phi_1$ be the potential difference between the upper plate and the interface between the dielectric and $\phi_2$ that

between the interface and the lower plate. We have

$$
\begin{aligned}
\phi &= \phi_1 + \phi_2 \\
&= E_1 d_1 + E_2 d_2 \\
&= \frac{D_1}{\kappa_1 \epsilon_0} d_1 + \frac{D_2}{\kappa_2 \epsilon_0} d_2 \\
&= \frac{\sigma d_1}{\kappa_1 \epsilon_0} + \frac{\sigma d_2}{\kappa_2 \epsilon_0}
\end{aligned}
$$

Thus the effective capacitance is given by

$$
\begin{aligned}
C = \frac{Q}{\phi} &= \frac{\sigma A}{\frac{\sigma}{\epsilon_0}\left(\frac{d_1}{\kappa_1} + \frac{d_2}{\kappa_2}\right)} \\
&= \frac{A\epsilon_0}{\frac{d_1}{\kappa_1} + \frac{d_2}{\kappa_2}} \\
&= \frac{C_1 C_2}{C_1 + C_2}
\end{aligned}
$$

where $C_1$ and $C_2$ are the capacitances for parallel plate capacitors with one type of dielectric with separations

$d_1$ and $d_2$ between the plates respectively.


**Example 24**

A capacitor consists of an inner conducting sphere of radius $R$ and an outer conducting shell of radius $2R$.
The space between the spheres is filled with two different linear dielectrics, one with a dielectric constant

from $r = R$ to $r = 1.5R$ and the other with dielectric constant $2\kappa$ from $r = 1.5R$ to $r = 2R$. The outer shell has a charge $-Q$ while the inner conductor has a charge $+Q$. Determine the electric field for $r > 0$ and find the effective capacitance.

**Solution :**

The electric field is radially symmetric and may be obtained by applying Gauss's law for the displacement vector

$$\int \vec{D} \cdot d\vec{S} = 4\pi r^2 D = Q^{free}$$

where $Q^{free}$ is the free charge enclosed within a sphere of radius $r$. For $r < R$, the field is zero as the free charges are only on the surface of the inner cylinder.
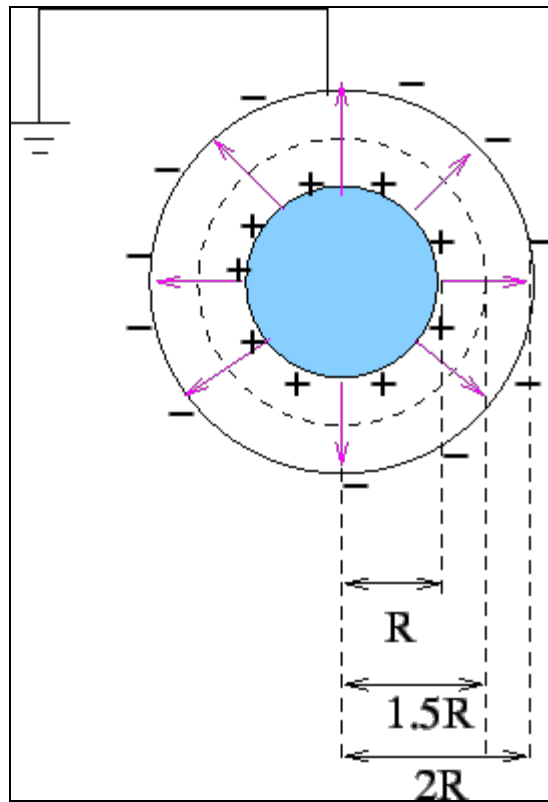


$r < 1.5R$, the electric field is

$$E = \frac{D}{\kappa\epsilon_0} = \frac{Q}{4\pi\,\epsilon_0\,\kappa r^2}$$

and for $1.5R < r < 2R$,

$$E = \frac{D}{2\kappa\epsilon_0} = \frac{Q}{8\pi\epsilon_0\kappa r^2}$$

For          , the field is zero. The fields are radial with the ineer sphere at a higher potential. The potential

$$r > 2R$$

difference is calculated by taking the taking the line integral of the electric field along any radial line.

$$
\begin{aligned}
\Delta\phi &= \int \vec{E} \cdot \vec{dl} = \int_{R}^{2R} E\, dr \\
&= \int_{R}^{1.5R} \frac{Q}{4\pi\kappa\epsilon_0 r^2}\, dr + \int_{1.5R}^{2R} \frac{Q}{8\pi\kappa\epsilon_0 r^2}\, dr \\
&= \frac{5Q}{48\pi\kappa\epsilon_0}
\end{aligned}
$$

The effective capacitance is

$$C = \frac{Q}{\Delta\phi} = \frac{48\pi\kappa\epsilon_0}{5}$$

## Example 25

A parallel plate capacitor has charge densities $\pm\sigma$ on its plates which are separated by a disance $d$. The space between the capacitor plates is filled with a linear but inhomogeneous dielectric. The dielectric constant varies with distance from the positive plate linearly from a value 1 to a value 2 at the negative plate. Determine the effective capacitance.

**Solution :**

As the dielectric is linear,

$$D = \epsilon_0 \kappa E = \epsilon_0 \left(1 + \frac{x}{d}\right) E$$

As the insertion of dielectric does not affect free charges, the displacement vector $\vec{D}$ remains the same as it would in the absence of the dielectric. Thus $\vec{D} = \hat{\imath}\sigma$.
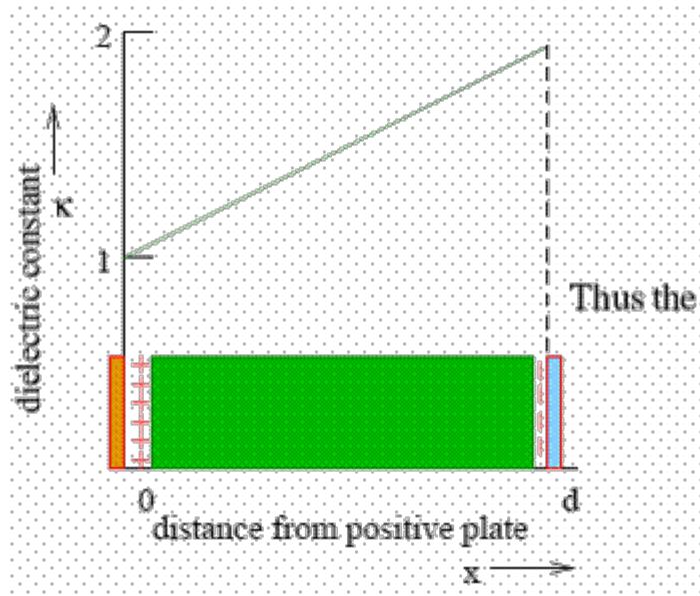


Thus the electric field $\vec{E}$ is given by

$$\vec{E} = \hat{\imath}\frac{\sigma a}{\epsilon_0(x+d)}$$

The field close to $x = d$ is given by $E = \sigma/2\epsilon_0$, which shows that adjacent to the negative plate there is a positive charge density $\sigma/2$. To find the effective capacitance, we find the potential difference between the plates by integrating the electric field

$$\phi = \int_0^d E \, dx = \frac{\sigma d}{\epsilon_0} \int_0^d \frac{dx}{d+x} = \frac{\sigma d}{\epsilon_0} \ln 2$$

so that

$$C = \frac{Q}{\phi} = \frac{A\sigma}{\phi} = \frac{A\epsilon_0}{d \ln 2}$$

The polarization $P$ is given by

$$P = D - \epsilon_0 E = \frac{\sigma x}{x+d}$$

The volume density of bound charges, given by $\nabla \cdot \vec{P} = -\rho_b$ is found as follows :

$$\rho_b = -\sigma \frac{d}{dx}\left[\frac{x}{x+d}\right] = -\frac{\sigma d}{(x+d)^2}$$

The bound charge density on the surface, given by $\hat{n} \cdot \vec{P} = P$, has a value $\sigma/2$ on the dielectric adjacent to the negative plate ( $x = d$). As the dielectric is charge neutral, this requires a net volume charge of $-\sigma/2$ in the dielectric. This can be verified by integrating over the volume charge density $\rho_b$ given above.

**Exercise**
A parallel plate capacitor of plate area $S$ and separation $d$ , contains a dielectric of thickness $d/2$ and of dielectric constant 2, resting on the negatve plate.
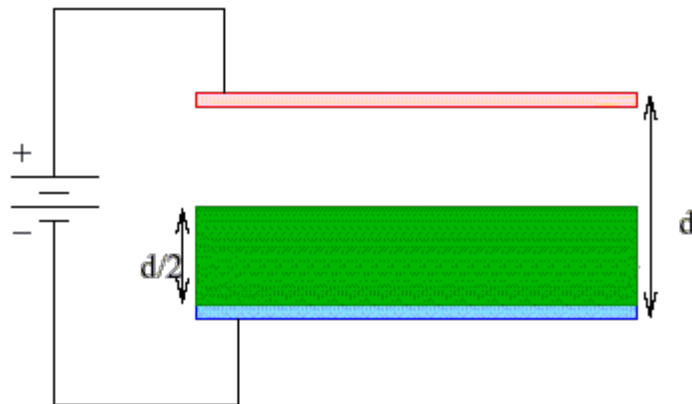


A potential difference of $\phi$ is maintained between the plates. Calculate the electric field in the region between the plates and the density of bound charges on the surface of the dielectric.

[Ans. field in empty region $= 4\phi/3d$, within dielectric $= 2\phi/3d$, bound charge density $= 2\epsilon_0\phi/3d$]

The permittivity of a medium filling the space between the plates of a spherical capacitor with raddi $a$ and $b$ ( $b > a$) is given by

$$\epsilon = \begin{cases} 2\epsilon_0 & a \leq r \leq (a+b)/2 \\ 4\epsilon_0 & (a+b)/2 \leq r \leq b \end{cases}$$

Find the capacitance of the capacitor, distribution of surface bound charges and the total bound charges in the dielectric.

[Ans. $C = 8\pi\epsilon_0 \left[\dfrac{1}{a} - \dfrac{1}{a+b} - \dfrac{1}{2b}\right]^{-1}$, bound charges on dielectric surface with radii $a, (a+b)/2$

and $b$ are respectively $-\sigma/2, 3\sigma a^2/(a+b)^2$ and $3\sigma a^2/4b^2$]

**Recap**
In this lecture you have learnt the following

- When a dielectric is inserted inside a dielectric, the electric fieldin the capacitor is reduced by a factor known as the dielectric constant.

- The capacitance of a dielectric is increased by insertion of a dielectric.

- The dielectric experiences force in the capacitor due to the electric field.

- Effective capacitance of various combinations of capacitors filled with dielectric were calculated.