Module 5 : MODERN PHYSICS

Lecture 23 : Particle and Waves

Objectives

In this lecture you will learn the following

- Radiation (light) exhibits both wave and particle nature.
- Laws governing black body radiation, like Stefan's law and Wien's law.
- Inadequacy of wave theory in explaining blackbody radiation spectrum.
- Planck's hypothesis on atoms absorbing radiation in quanta of energy.

Particle and Waves

In classical physics have come to regard matter and waves as two disctinct entities. A **particle** is an idealized point object which is characterized by

- a mass
- position (a particle cannot be in more than one position at the same time)
- momentum

In practice entities like electrons, protons, atoms, molecules etc. are approximated as particles. In classical dynamics it is also common to consider macroscopic objects like a billiard ball as a particle. Particle transmit energy from one point in space to another by collisions with other particles during which transfer of momentum also takes place.
Wave is an extended disturbance in space which can transmit energy from one point to another without imparting a net motion to the medium through which it propagates. Examples of waves are mechanical waves like sound waves, water waves etc. which require a material medium to propagate and electromagnetic waves (light waves, radio waves, x-rays etc.) which can propagate in space without requiring a medium. A wave is characterized by

- wavelength
- frequency

Traditionally, the wave and particle properties have been considered distinct. For instance, the following phenomena can be only understood in terms of wave properties :

- Interference
- Diffraction
- Polarization

Similarly, the process of collision or the concept of temperature as energy of vibrating molecules are understood in terms of particle properties. However, some experimental observations made in the late 19th century and early 20th century seemed to indicate that the strict behavioural pattern stated above is not always valid.

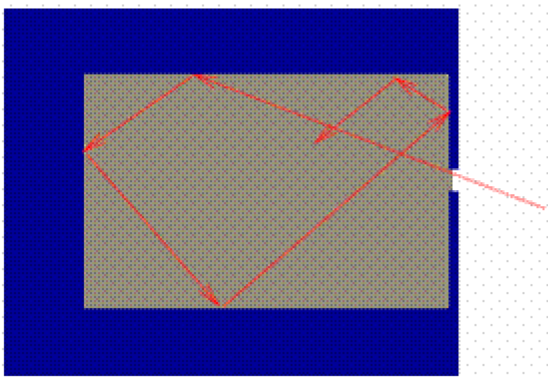
Particle Nature of Waves

Light was accepted to have wave nature in view of well established experiments on diffraction. However, **Photoelectric Effect** could be understood only by assuming that light consisted of streams of particles possessing energy and momentum. The first phenomenon which was observed to be in disagreement with the wave nature of light is the **black body radiation** problem.

Black Body Radiation :

A black body, by definition, is an object which absorbs all radiation that fall on it. Since it does not reflect any light, it appears black.

In a laboratory, one could approximate a blackbody by a cavity with highly polished walls. If the walls of the cavity has a small hole, any radiation that enters through the hole gets trapped in the cavity. Stars may also be approximated as black bodies as any radiation directed at them gets absorbed.



A black body is also a perfect emitter of radiation. It can emit at all wavelengths. However, the radiation from a black body is observed to obey the following two laws :

- **Stefan's Law :** The intensity of emitted radiation for a given wavelength is proportional to the fourth power of the temperature of the black body.

- **Wien's law :** For a given temperature, the spectrum of emitted radiation has maximum intensity for a wavelength λ_{max}, which is inversely proportional to the temperature of the black body. Thus relatively colder bodies appear red as their maximum intensity is in the red end of the spectrum while hotter bodies appear bluish. Because of this, when we heat a metal wire it firs becomes red hot and then as the temperature increases it become "**white hot**".

In classical physics, radiation is considered as waves and the calculation of radiant energy emitted by a black body is carried out in the following steps.

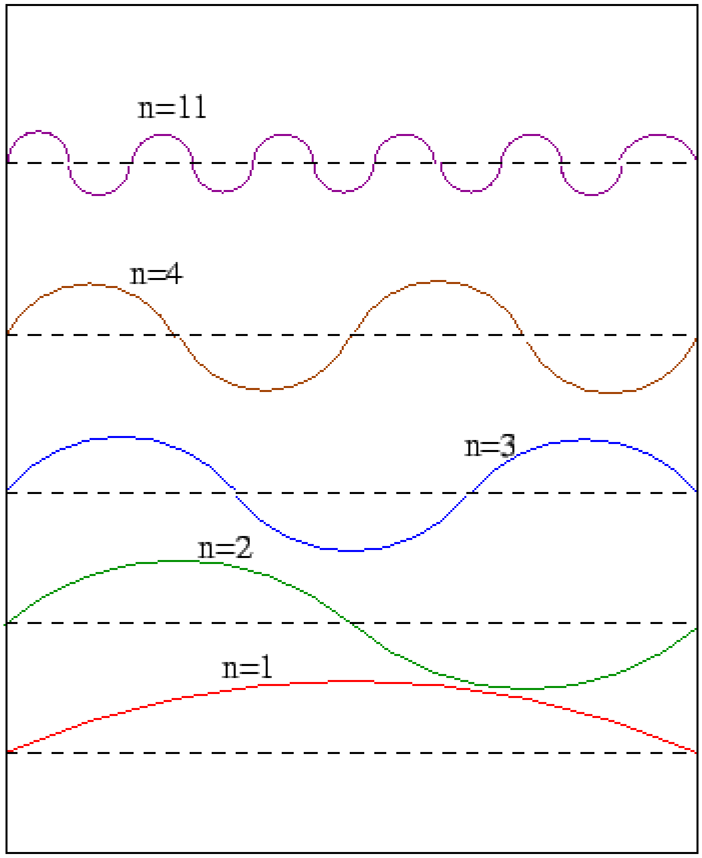
(1) We consider the black body to be in the shape of a cubical metal cavity of side L with a small hole in it.

Any radiation which falls on the hole is lost inside the cavity. The radiation which emerges from the hole has the characteristics of the radiation that is trapped inside the cavity.

(2) The waves inside the cavity form standing wave pattern with nodes at the walls of the cavity since the electric field must vanish inside a metal.

If we consider standing waves in one dimension, the electric field having nodes at $x = 0$ and $x = L$ is given by $E = E_0 \sin\left(n\frac{\pi}{L}x\right)\sin(\omega t)$ where n is a positive integer. The pattern of the standing wave is shown. The frequency $\nu = \omega/2\pi$ is given by

$$\nu = \frac{nc}{2L}$$



Standing Wave Pattern in a Cavity

Extending to three dimensions, the electric field is given by

$$E = E_0 \sin\left(n_x \frac{\pi}{L} x\right) \sin\left(n_y \frac{\pi}{L} y\right) \sin\left(n_z \frac{\pi}{L} z\right) \sin(\omega t)$$

where $\{n_x, n_y, n_z\}$ is a set of positive integers. (If any of these inegers is zero, it gives zero field. Taking negative values of the integers do not give different fields as it amounts to simply multiplying E_0 by a sign factor.) Substituting Eqn. (1) in the electromagnetic wave equation

$$\nabla^2 E = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$$

we get

$$\left(\frac{n_x^2 \pi^2}{L^2} + \frac{n_y^2 \pi^2}{L^2} + \frac{n_z^2 \pi^2}{L^2}\right) E = \frac{\omega^2}{c^2} E$$

The frequency $\nu = \omega/2\pi$ is given by

$$\nu = \frac{c}{2L} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

For a given frequency, the equation above represents a sphere of radius in the three

$$R = 2L\nu/c$$

dimensional space of n_x, n_y and n_z and each value of ν represents a distinct point in this space. Since n_x, n_y, n_z can only take integral values, the number of points per unit volume is one. If we treat ν as a continuous variable, the number of modes for frequency less than some given ν is given by

$$N(\nu) = 2 \times \frac{1}{8}\frac{4\pi R^3}{3}$$

$$= \frac{8\pi V}{3c^3}\nu^3$$

where V is the volume of the cavity. In the above, the factor of 1/8 comes because we are restricted to the positive octant as n_x, n_y, n_z can only be positive. The factor of 2 takes into account the fact that there are two transverse modes. The number of modes in the frequency interval ν and $\nu + d\nu$ is

$$N(\nu)d\nu = N(\nu + d\nu) - N(\nu)$$

$$= \frac{8\pi V}{3c^3}[(\nu + d\nu)^3 - \nu^3]$$

$$= \frac{8\pi V}{c^3}\nu^2 d\nu$$

(3) As the average energy of a mode is kT, the radiant energy density, which is defined as the average energy per unit volume is given by

$$u(\nu)d\nu = \frac{8\pi kT}{c^3}\nu^2 d\nu \quad (2)$$

This is known as **Rayleigh - Jeans' Law**

Exercise 1

Show that, in terms of the wavelength interval, the Rayleigh Jeans' law can be expressed as

$$u(\lambda)d\lambda = \frac{8\pi kT}{\lambda^4}d\lambda$$

(4) The radiant intensity can be obtained from the expression for the energy density by multiplying the above expression by $c/4$. The curious factor of 1/4 arises because

- At any instant, on an average, half of the waves are directed towards the wall of the cavity and another half is directed away from it. This gives a factor of 1/2.

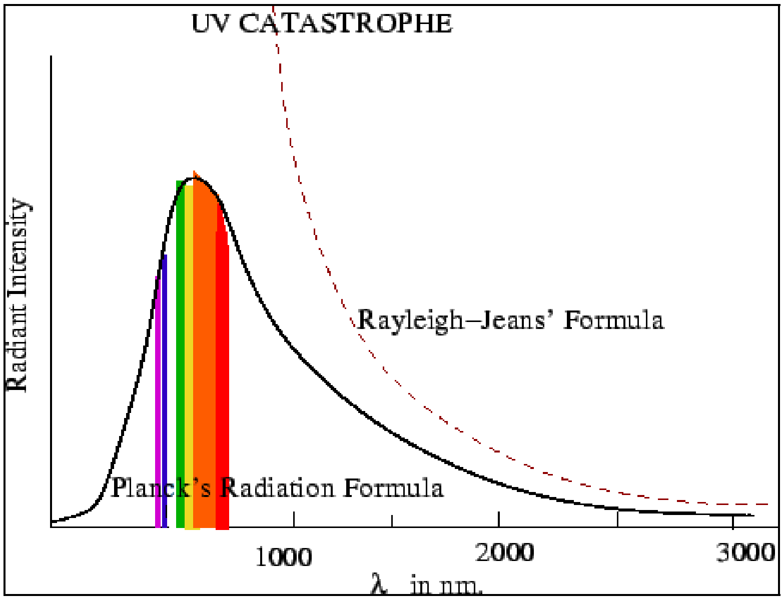
We need to average over all angles. In computing the radiant power, we get a factor of $\cos^2\theta$, which averages to 1/2. The radiant intensity is given by

$$I(\lambda)d\lambda = \frac{2\pi c}{\lambda^4} kT d\lambda$$

Black Body Radiation :

Rayleigh- Jeans' law is roughly in agreement with the thermal radiation curves at long wavelengths. However, at short wavelengths, it gives infinite energy density as $u(\lambda \longrightarrow \infty$ as $\lambda \longrightarrow 0$. This is clearly unphysical.

The failure of the classical wave theory to explain the observed radiation curve in the ultraviolet end of the electromagnetic spectrum is known as **ultraviolet catastrophe** .



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Planck's Theory :

In 1900, Max Planck suggested that *oscillating atoms could emit or absorb energy in tiny bursts of energy called* ***quanta*** . The energy of the quanta is proportional to the frequency of radiation. Planck's suggestion imparts a discrete or particle nature to radiation. If the frequency of radiation is ν, the energy of the quantum associated with it is

$$\varepsilon = h\nu$$

where the constant of proportionality h is called **Planck's constant** . Its value in SI units is 6.626×10^{-34} J-s. Thus the possible energy of a mode with frequency ν is $nh\nu$ where $n = 0, 1, 2 \ldots$. According to Boltzmann distribution, the probability of a mode having an energy E at a temperature T is given by $\exp(-\beta E)$, where $\beta = 1/kT$. Here, k is the Boltzmann constant and T is the absolute temperature. Thus the average energy of a mode is

$$\bar{\varepsilon} = \frac{\sum_{n=0}^{\infty} nh\nu \exp(-nh\beta\nu)}{\sum_{n=0}^{\infty} \exp(-nh\beta\nu)}$$

$$= \frac{h\nu}{\exp(h\nu\beta) - 1} \qquad (3)$$

Exercise 2

Prove Eqn. (3).

(Hint : Treat β as a continuous variable and show that the right hand side is $-\frac{\partial}{\partial\beta}\ln\sum\exp(-nh\nu\beta)$).

Using $\bar{\mathcal{E}}$ to be the average energy of the mode instead of kT, the energy density is given, instead of Eqn. (2), by

$$u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu\beta}-1}d\nu \quad (4)$$

Exercise 3

Show that Eqn. (4) reduces to Rayleigh - Jeans' expression for long wavelengths i.e. as $\nu \longrightarrow 0$. [Hint : use $e^x \approx 1+x$ for $x \ll 1$]

Exercise 4

Show that, in terms of wavelength, the expression for radiant intensity is given by

$$I(\lambda)d\lambda = \frac{2\pi hc^2}{\lambda^5}\frac{1}{\exp(hc\beta/\lambda)-1} \quad (5)$$

Example-1

Find the temperature for which the radiant energy density at a wavelength of 200 nm is four times that of the density at 400 nm.

Solution

$$\frac{u(\lambda_1}{u(\lambda_2)} = \left(\frac{\lambda_2}{\lambda_1}\right)^5 \frac{\exp(hc/\lambda_2 kT)-1}{\exp(hc/\lambda_1 kT)-1} = 4$$

Substituting values of h, c, k, λ_1 and λ_2, we get

$$32\frac{\exp(3.6\times 10^4/T)-1}{\exp(7.2\times 10^4/T)-1} = 4$$

which gives, on simplification $\exp(3.6\times 10^4/T) = 7$. On solving, the temperature works out approximately to be 18,500 K.

Stefan's Law

The power radiated by the black body per unit area is

$$U = \int_0^\infty I(\lambda)d\lambda$$

$$= 2\pi hc^2\int_0^\infty \frac{1}{\lambda^5}\frac{1}{\exp(hc\beta/\lambda)-1}d\lambda$$

To evaluate the integral, substitute $x = hc\beta/\lambda$, so that $dx = -hc\beta/\lambda^2 d\lambda$. We get

$$U = 2\pi hc^2 \frac{1}{(hc\beta)^4} \int_0^\infty \frac{x^3}{e^x - 1} dx$$

The value of the integral $\int_0^\infty x^3/(e^x - 1)$ is known to be $\pi^4/15$, so that

$$U = \frac{2\pi (kT)^4}{h^3 c^2} \frac{\pi^4}{15} = \sigma T^4$$

where

$$\sigma = \frac{2\pi^5 k^4}{15 h^3 c^2} = 5.67 \times 10^{-8} \text{ J.K}^4/\text{m}^2.\text{s}$$

is known as Stefan's constant.

For a body with emissivity (the ratio of radiation emitted by a body to that predicted by PLanck's law for an ideal black body) e, the power radiated from a unit area of the surface is

$$\sigma e T^4$$

In addition to emitting radiation, a body at temperature T also absorbs radiation. If the surrounding temperature is T_0, the power absorbed per unit surface area is

$$\sigma e(T^4 - T_0^4)$$

Example-2

Estimate the radiant energy emitted by a blackbody at 6000 K.

Solution

According to Stefan's law the radiant power emitted per unit area is

$$\sigma T^4 = 5.67 \times 10^{-8} \times (6000)^4 = 7.35 \times 10^7 \;\; \text{W/m}^2$$

Example-3

Estimate the fraction of radiant power of Example 1 which is emitted in the visible region of the spectrum.

Solution

According to Planck's radiation formula, the power per unit area is given by

$$P = \frac{2\pi h}{c^2} \int_{\nu_1}^{\nu_2} \nu^3 \frac{1}{\exp(h\nu/kT) - 1} d\nu$$

Substituting $x = h\nu/kT$, the expression reduces to

$$P = \frac{2\pi h}{c^2} \left(\frac{kT}{h}\right)^4 \int_{x_1}^{x_2} dx x^3 \frac{1}{e^x - 1}$$

where x_1 and x_2 are respectively the upper and the lower limits of x corresponding to visible spectrum.

Taking $\nu_1 = 4.3 \times 10^{14}$ Hz and $\nu_2 = 7.5 \times 10^{14}$ Hz, we get $x_1 = 3.46$ and $x_2 = 6$. Thus

$$P = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \int_{3.46}^{6} dx x^3 \frac{1}{e^x - 1}$$

$$= 1.126 \times 10^7 \int_{3.46}^{6} dx x^3 \frac{1}{e^x - 1}$$

The integral above has to be done numerically, for instance, by Simpson's method. A crude estimate gives the value of the integral to be approximately 2.41. Thus $P = 2.7 \times 10^7$ watts, which is about 36% of the total emitted radiation.

Exercise 5

A spherical black body of radius 2m is at 27 $^\circ$ C. Find the power radiated.

[Ans. 22077 watts]

Exercise 6

Total energy radiated from a blackbody source is collected for one minute and is used to heat a quantity of water. The temperature of water is found to increase from 20°C to 20.5°C. If the absolute temperature of the blackbody were doubled and the experiment repeated with the same quantity of water at 20°C, find the temperature of water.(Ans. 28 $^\circ$ C)

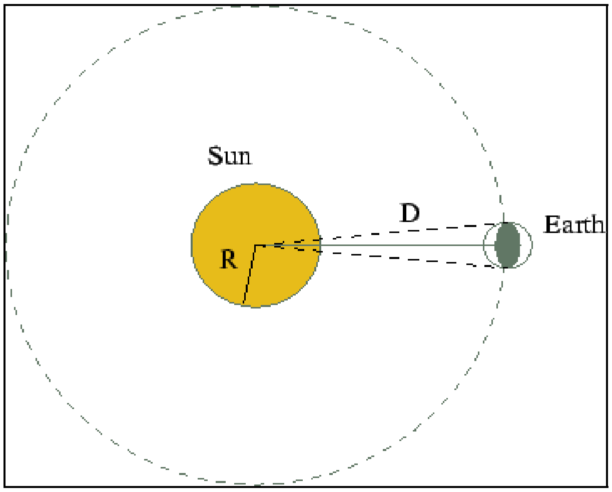
Example-4

The earth receives 1.4 kW of power from the sun. Assume that both earth and the sun to be black bodies. If the radius of the sun is 6.95×10^8 m and the earth-sun distance is 1.5×10^{11} m, calculate the temperature of the sun.

Solution

According to Stefan's law, the power radiated by the Sun per unit area is σT^4. If R is the radius of the sun, the total power radiated is $4\pi R^2 \sigma T^4$.

This power radiates outward from the sun. If at a distance D, the power received per unit area is P_e, the total power is equal to the surface area of a sphere of radius D times this amount. Thus,



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$$4\pi R^2 \sigma T^4 = P_e \times 4\pi D^2$$

Thus

$$T^4 = P_e \frac{D^2}{\sigma R^2}$$

Substituting $D = 1.5 \times 10^{11}$ and $R = 6.95 \times 10^8$ m, we get $T = 5823^\circ$ K.

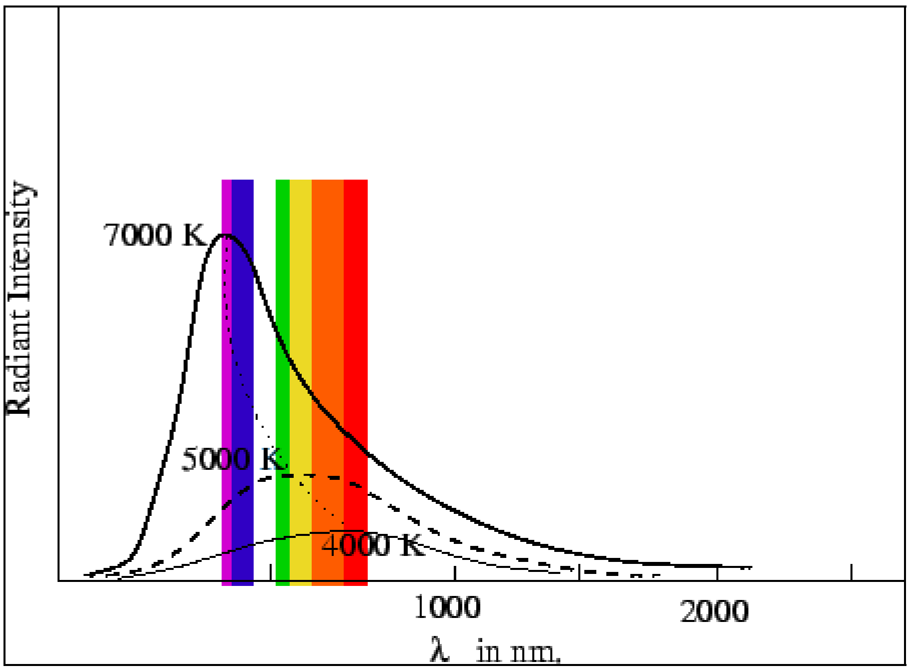
Exercise 7

Using the above distances and the calculated temperature of the sun, estimate the equilibrium temperature of the earth.

(Hint : First determine the total amount of power collected by the earth by observing that πR_E^2 section of the earth collects all the power falling on the earth. In equilibrium, this amount is equal to the power radiated from the earth..Ans. 278.7 K.)

Wien's Displacement Law :

The wavelength at which the radiant intensity is maximum is inversely proportional to the temperature of the black body. According to this law, hotter objects emit most of their radiation at shorter wavelength, which would make them appear more bluish. Similarly, cooler objects radiate in the red end of the spectrum, making them appear red.



See the animation

The radiant intensity at a given temperature has a maximum when

$$\frac{dI(\lambda)}{d\lambda} = (2\pi hc^2)\frac{\partial}{\partial\lambda}\left[\frac{1}{\lambda^5}\frac{1}{\exp(hc/\lambda kT) - 1}\right]$$

$$= (2\pi hc^2)\left[\frac{-5}{\lambda^6}\frac{1}{\exp(hc/\lambda kT) - 1} - \frac{1}{\lambda^5}\frac{exp(hc/\lambda kT)\left(\frac{-hc}{kT\lambda^2}\right)}{(\exp(hc/\lambda kT) - 1)^2}\right]$$

$$= 0$$

which gives

$$\frac{hc}{kT\lambda}\frac{1}{1-\exp(-hc/\lambda kT)} = 5$$

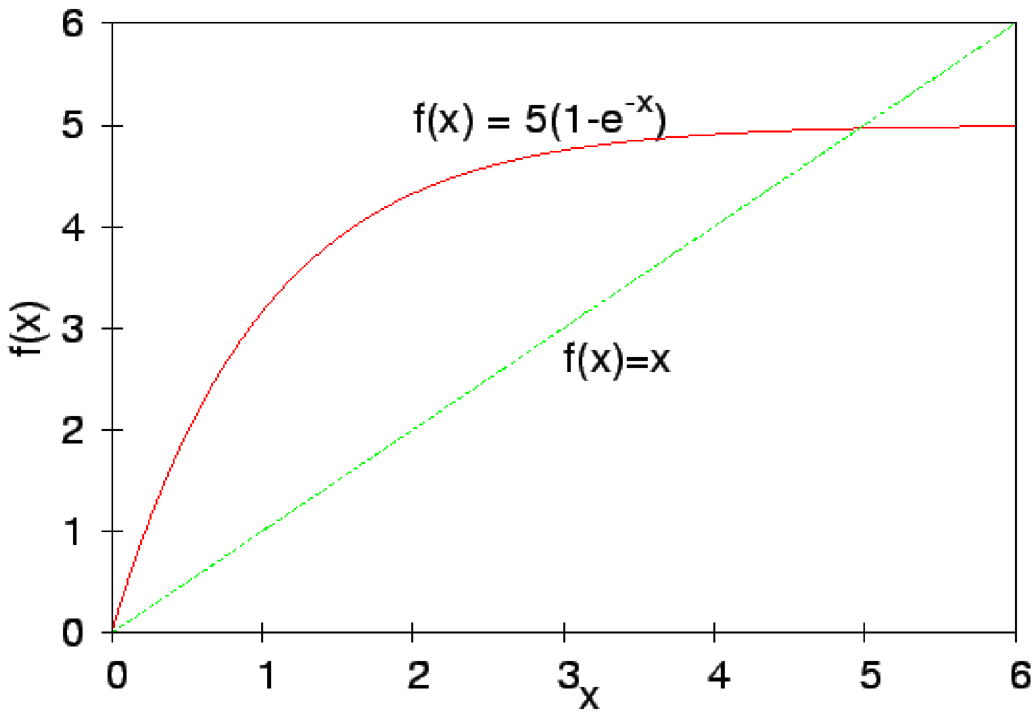
This equation is to be solved numerically. Substituting $x = hc/kT\lambda$, the equation becomes

$$5(1-e^{-x}) = x$$

Black Body Radiation :

Numerical solution of this equation gives

$$x \approx 4.9652$$



[**See the animation**]()

Substituting the values of h, c and k we get Wien's law

$$\lambda_{max} T = 2.9\times 10^{-3}\ \mathrm{mK}$$

where λ_{max} is the wavelength at which the radiation intensity is maximum at a temperature T. If λ_{max} is expresses in cm, the relationship is given by

$$\lambda_{max} = \frac{0.29}{T}$$

Exercise 8

The surface temperature of the sun is about 6000 K. What is the wavelength at which the sun emits its peak radiation intensity ?

(Ans. 483 nm)

Exercise 9

Taking the mean temperature of the surface of the earth to be 10 $^{\circ}$ C, calculate the wavelength at which the earth emits maximum radiation.

(Ans. 10 μ, i.e. the earth emits mostly in infrared.)

Example 5

The exercise above shows that the sun emits mostly in the visible region. Compare the total intensity of radiation emitted by a star of similar size as the sun whose surface temperature is 7200 K.

Solution

The total intensity is given by Stefan's law,

$$\frac{T_{star}^4}{T_{sun}^4} = \left(\frac{7200}{6000}\right)^4 = 2.07$$

However, the star emits its peak intensity in the blue end of the spectrum as the wavelength at which the radiation intensity being inversely proportional to the temperature is given by

$$\lambda_{max} = \frac{0.29}{7200} = 4.02 \times 10^{-5}\ \text{ cm} \approx 402\ \text{ nm}$$

Exercise 11

The black body spectrum of an object A has its peak intensity at 200 nm while that of another object of same shape and size has its peak at 600 nm. Compare radiant intensities of the two bodies.

(Ans. A radiates 81 times more than B)

Cosmic Microwave Background

According to the **big bang theory** , the universe, at the time of creation was a very hot and dense object. Subsequently it expanded, bringing down the temperature, the present temperature of the universe isapproximately 2.7 K. As a result, the peak intensity of radiation is given by

$$\lambda_{max} = \frac{0.29}{2.7} \approx 0.1 \text{ cm}$$

The wavelength lies in the microwave region. This is known as the **cosmic microwave background.** The energy density at this temperature is obtained by multiplying σT^4 by $4/c$, which gives 4.01×10^{-14} J/m ^{3}.

Recap

In this lecture you have learnt the following

- Traditional picture of light being a wave cannot explain several phenomena such as blackbody radiation and photoelectric effect.
- A blackbody is a an object whic absorbs all the radiation that falls on it.
- A blackbody is also a perfect emitter, i.e. it emits all the radiation that it absorbs.
- Intensity of radiation emitted by a blackbody is proportional to the fourth power of its temperature (Stefan's law).
- The wavelength at which the emitted radiation has the maximum intensity is inversely proportional to its temperature (Wien's law).

- Classical Rayleigh-Jeans' formula leads to the ultraviolet catastrophe at short wavelength.

- Planck proposed that oscillating atoms emit or absorb radiation in quanta. Using Boltzmann distribution, he derived a formula for radiation which satisfactorily explains the blackbody radiation spectrum.