Module 1 : A Crash Course in Vectors
Lecture 3 : Line and Surface Integrals of a Vector Field

Objectives

In this lecture you will learn the following

- Line, surface and volume integrals and evaluate these for different geometries.
- Evaluate flux

Line and Surface Integrals of a Vector Field

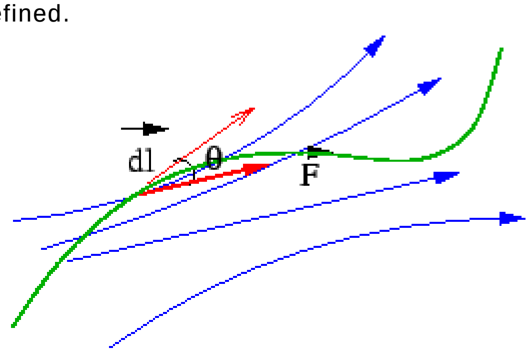
Since a vector field is defined at every position in a region of space, like a scalar function it can be integrated and differentiated. However, as a vector field has both magnitude and direction it is necessary to define operations of calculus to take care of both these aspects.

Line Integral :

If a vector field $\vec{F}$ is known in a certain region of space, one can define a line integral of the vector function may be defined as

$$\int_C \vec{F}\cdot\vec{dl}$$

where C is the curve along which the integral is calculated. Like the integral of a scalar function the integral above is also interpreted as a limit of a sum. We first divide the curve C into a large number of infinitisimally small line segments such that the vector function is constant (in magnitude and direction) over each such line segment. The integrand is then equal to the product of the length of the line segment and the component of $\vec{F}$ along the segment. The integration is defined as the limit of the sum of contributions from all such segments in the same manner as ordinary integration is defined.



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The concept of line integral is very useful in many branches of physics. In mechanics, we define work done by a force $\vec{F}$ in moving an object from an initial position A to a final position B as the line integral of the force along the curve joining the end points. Except in the case of conservative forces, the line integral depends on the actual path along which the particle moves under the force.

Example 8

A force $\vec{F} = zy\hat{\imath} + x\hat{\jmath} + z^2x\hat{k}$ acts on a particle that travels from the origin to the point $(1,2,3)$. Calculate the work done if the particle travels (i) along the path $(0,0,0)\to(1,0,0)\to(1,2,0)\to(1,2,3)$ along straightline segments joining each pair of end points (ii) along the straightline joining the initial and final points.

Solution :

Along the path $(0,0,0)\to(1,0,0)$, $\vec{dl} = \hat{\imath}dx$ and $y=0, z=0$. Since $F_x = 0$ along this segment, the integral along C_1 is zero. Along the path C_2 joining $(1,0,0)$ and $(1,2,0)$, and $x=1$.

$$\vec{dl} = \hat{j}dy$$

Along this path $F_y = x = 1$. The integral

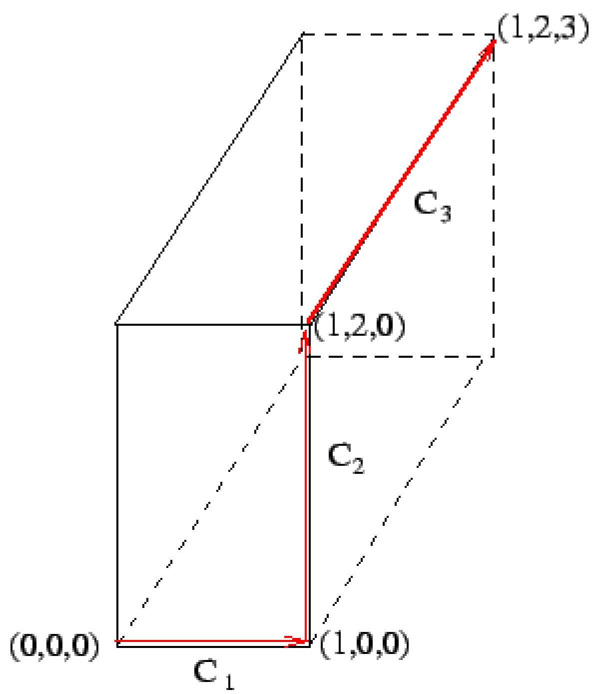
$$\int_0^2 F_y dy = \int_0^2 dy = 2$$

Along the third path connecting $(1,2,0)$ to $(1,2,3)$, $x = 1, y = 2$. $\vec{dl} = \hat{k}dz$ and $F_z = z^2 x = z^2$.

The line integral is

$$\int_0^3 z^2 dz = \frac{z^3}{3} \Big|_0^3 = 9$$

The work done is, therefore, $0 + 2 + 9 = 11$.



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In order to calculate the work done when the particle moves along the straightline connecting the initial and final points, we need to write down the equation to the line in a parametric form. If (x_1, y_1, z_1) and (x_2, y_2, z_2) are the end points, the equation is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = m$$

Substituting for the coordinates, we get the equation to the line as

$$\begin{aligned} x &= m \\ y &= 2m \\ z &= 3m \end{aligned}$$

Thus the differential elements are $dx = dm, dy = 2dm$ and $dz = 3dm$. The line integral is given by

$$
\begin{aligned}
\int \vec{F}\cdot \vec{dl} &= \int (F_x dx + F_y dy + F_z dz) \\
&= \int_0^1 (6m^2.dm + m.2dm + 9m^3.3dm) \\
&= 2m^3 + m^2 + 27\frac{m^4}{4} \mid_0^1 = 9.75
\end{aligned}
$$

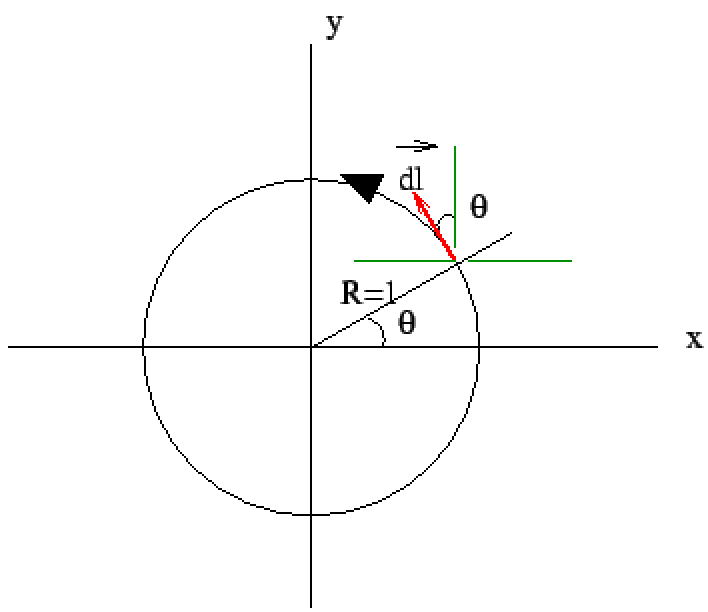
Example 9

Find the line integral of $\vec{F} = y\hat{\imath} + \hat{\jmath}$ over an anticlockwise circular loop of radius 1 with the origin as the centre of the circle.

Solution :

The length element dl has a magnitude $1.d\theta = d\theta$. Since the unit vector along $\vec{dl}$ makes an angle of $(\pi/2) + \theta$ with the positive $x-$ axis,

$$
\begin{aligned}
\vec{dl} &= \mid dl \mid \cos(\frac{\pi}{2} + \theta)\hat{\imath} + \mid dl \mid \sin(\frac{\pi}{2} + \theta)\hat{\jmath} \\
&= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}
\end{aligned}
$$



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In polar coordinates, $x = \cos\theta$ and $y = \sin\theta$ (since the radius is 1). Thus

$$
\begin{aligned}
\int \vec{F}\cdot \vec{dl} &= \int F_x dx + F_y dy \\
&= -\int_0^{2\pi} \sin^2\theta d\theta + \int_0^{2\pi} \cos\theta d\theta \\
&= -\pi + 0 = -\pi
\end{aligned}
$$

Exercise 1

A force $\vec{F} = xy\hat{\imath} + (x^2 - z^3)\hat{\jmath} - xz^2\hat{k}$ acts on a particle. Calculate the work done if the particle is taken from the point to the point along straight line segment connecting

$(0,0,0)$ $(2,1,3)$

$(0,0,0) \rightarrow (0,1,0) \rightarrow (2,1,0) \rightarrow (2,1,3)$. What would be the work done if the particle directly moved to the final point along the straightline connecting to origin.

(Ans. -16, -13.8.)

Exercise 2

A vector field is given by $\vec{F} = (2x+3y)\hat{\imath} + (3x+2y)\hat{\jmath}$

Evaluate the line integral of the field around a circle of unit radius traversed in clockwise fashion.

(Ans. 6π)

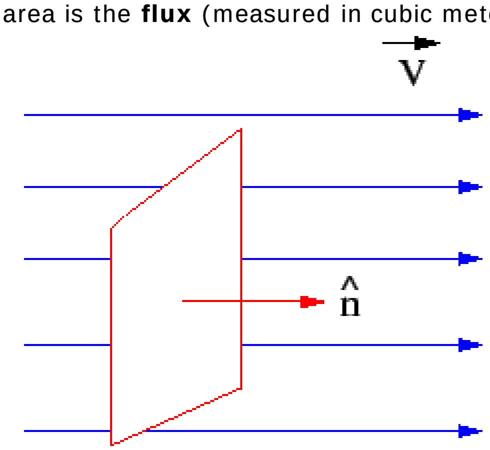
Exercise 3

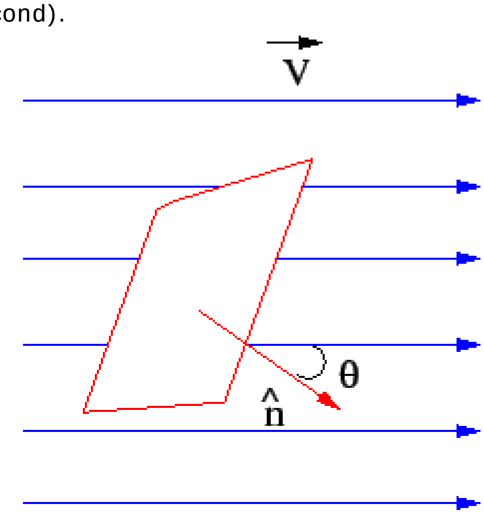
Evaluate the line integral of a *scalar function* xy along a parabolic path $y = x^2$ connecting the origin to the point $(1,1)$.

[Hint : Remember that the arc length along a curve is given by $\sqrt{(dx)^2+(dy)^2}$. The curve can be parametrized by $x = t$ and $y = t^2$.] [Ans. $(25\sqrt{5}+1)/120$]

Surface Integral :

We have seen that an area element can be regarded as a vector with its direction being defined as the outward normal to the surface. The concept of a surface integral is related to flow. Suppose the vector field represents the rate at which water flows at a point in the region of flow. The flow may be measured in cubic meter of water flowing per square meter of area per second. If an area is oriented perpendicular to the direction of flow, as shown in the figure to the left, maximum amount of water would flow through the surface. The amount of water passing through the area is the **flux** (measured in cubic meter per second).





If, on the other hand, the surface is tilted relative to the flow, as shown to the right, the amount of flux through the area decreases. Clearly, only the part of the area that is perpendicular to the direction of flow will contribute to the flux.

We define **flux** through an area element $\vec{dS}$ as the dot product of the vector field $\vec{V}$ with the area vector $\vec{dS}$.

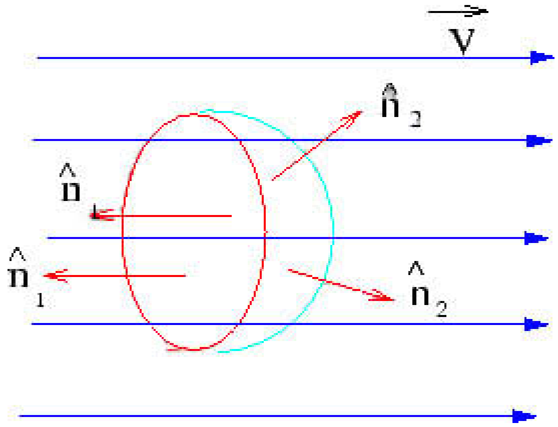
When $\vec{V}$ is parallel to $d\vec{S}$, i.e. if the surface is oriented perpendicular to the direction of flow, the flux is maximum. On the other hand, a surface oriented parallel to the flow does not contribute to the flux.

Example 10

A hemispherical bowl of radius R is oriented such that the circular base is perpendicular to direction of flow. Calculate the flux through the curved surface of the bowl, assuming the flow vector $\vec{V}$ to be constant.

Solution :

Since the curved surface makes different angles at different positions, it is somewhat difficult to calculate the flux through it. However, one can circumvent it by calculating the flux through the circular base.



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Since the flow vector is constant all over the circular face which is oriented perpendicular to the direction of flow, the flux through the base is $-\pi R^2 V$. The minus sign is a result of the fact that the direction of the surface is opposite to the direction of flow. Thus we may call the flux through the base as *inward flux* . Since there is no source or sink of flow field (i.e. there is no accumulation of water) inside the hemisphere, whatever fluid enters through the base must leave through the curved face. Thus the *outward flux* from the curved face is $+\pi R^2 V$.

We may now generalize the above for a surface over which the field is not uniform by defining the flux through an area as the sum of contribution to the flux from infinitisimal area elements which comprises the total area by treating the field to be uniform over such area elements. Since the flux is a scalar, the surface integral, defined as the limit of the sum, is also a scalar.

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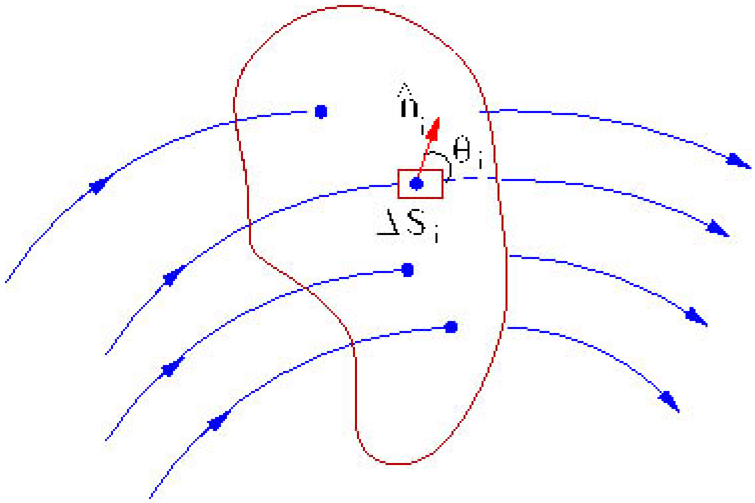
If ΔS_i is the $i-$th surface element, the normal $\hat{n}_i$ to which makes an angle θ_i with the vector field $\vec{V}_i$ at the position of the element, the total flux

$$\Phi = \sum_i \vec{V}_i \cdot \hat{n}_i \Delta S_i = \sum_i V_i \delta S_i \cos\theta_i$$

In the limit of $\Delta S_i \rightarrow 0$, the sum above becomes a surface integral

$$\Phi = \int_S \vec{V} \cdot dS$$



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If the surface is closed, it encloses a volume and we define

$$\Phi = \oint_S \vec{V} \cdot dS$$

to be the net outward flux. In terms of cartesian components

$$\Phi = \int_S (V_x dydz + V_y dxdz + V_z dxdy)$$

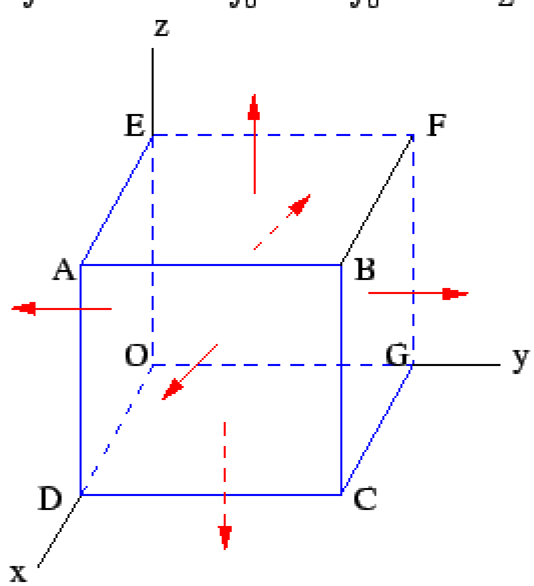
Example 11

A vector field is given by $\vec{B} = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$. Evaluate the flux through each face of a unit cube whose edges along the cartesian axes and one of the corners is at the origin.

Solution :

Consider the base of the cube (OGCD), which is the x-y plane on which $z = 0$. On this face $\vec{B} = xy\hat{\imath}$. The surface vector $\hat{n}$ is along $-\hat{k}$ direction. Thus on this surface flux $\int \vec{B} \cdot \hat{n} dxdy = \int xy\hat{\imath} \cdot \hat{k} dxdy = 0$ since $\hat{\imath} \cdot \hat{k} = 0$. For the top surface (ABFE), $z = 1$ and $\hat{n} = \hat{k}$. The flux from this surface is

$$\int V_z dxdy = \int_0^1 xdx \int_0^1 dy = \frac{1}{2}$$



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In a similar way one can show that flux from left side (AEOD) is zero while the contribution from the right side (BFGC) is 1/2. The back face (EFGO) contributes zero while the front face (ABCD) contributes 1/2. The net flux, therefore, is 3/2.

Exercise 4

Find the flux of the vector field $\vec{V} = Ax\hat{i} + By^2\hat{j}$ through a rectangular surface in the x-y plane having dimensions $a \times b$. The origin of the coordinate system is at one of the corners of the rectangle and the x-axis along its length.

(Ans. $Bab^3/3$)

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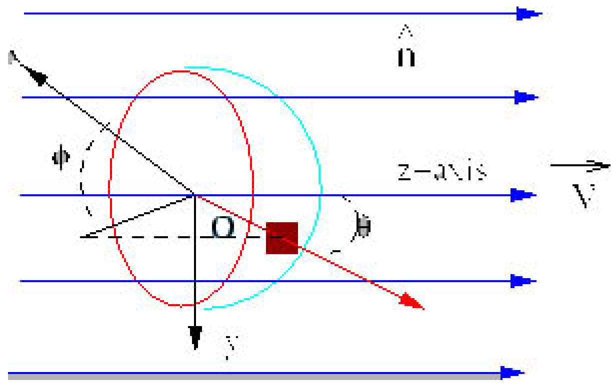
Example 12

Calculate directly the flux through the curved surface of the hemispherical bowl of Example 10.

Solution :

Use a spherical polar coordinates with the base of the hemisphere being the x-y plane and the direction of the vector field as the z-axis. We have seen that an area element on the surface is given by $dS = R^2 \sin\theta d\theta d\phi$. Thus the flux $\Phi = \int \vec{V}\cdot\hat{n}$ is given by

$$\begin{aligned}\Phi &= \mid V \mid R^2 \int_0^2 \pi d\phi \int_0^{\pi/2} \cos\theta \sin\theta d\theta \\ &= \mid V \mid R^2 . 2\pi \left[\frac{\sin^2\theta}{2}\right]_0^{2\pi} \\ &= \pi \mid V \mid R^2\end{aligned}$$



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Exercise 5

Find the flux through a hemispherical bowl with its base on the x-y plane and the origin at the centre of the base. The vector field, in spherical polar coordinates is $\vec{V} = r\sin\theta\hat{r} + \hat{\theta} + \hat{\phi}$.

(Ans. $\pi(1 + \pi/2)$)

Example 13

A cylindrical object occupies a volume defined by $x^2 + y^2 \leq R^2$ and $0 \leq z \leq h$. Find the flux through each of the surfaces when the object is in a vector field $\vec{V} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$.

Solution :

Because of cylindrical symmetry, it is convenient to work in a cylindrical (ρ, θ, z) coordinates. The vector field is given by

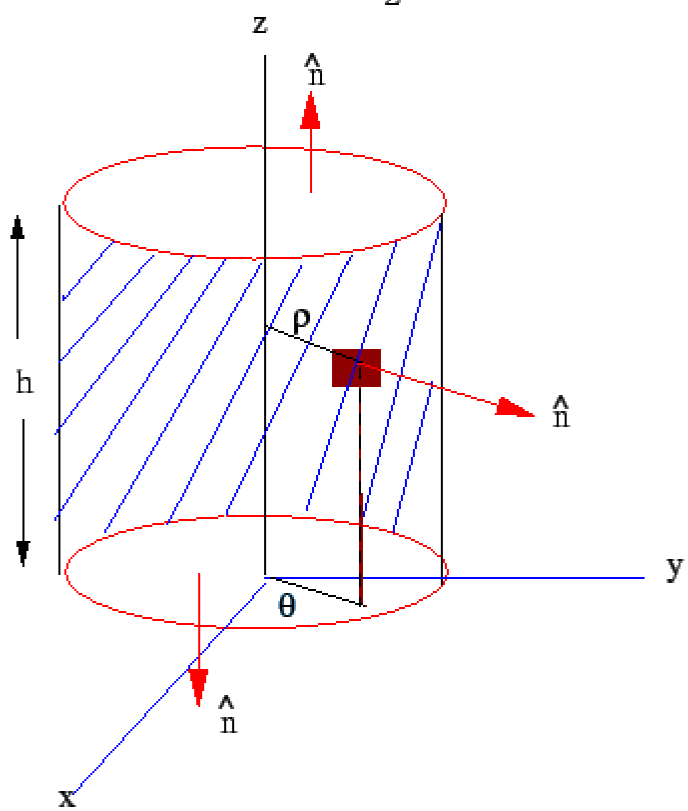
$$\vec{V} = \rho\hat{\rho} + z\hat{k}$$

Bottom face has $z = 0$ and the normal to the face points in $-\hat{k}$ direction. Thus the flux from this face

$\int \vec{V} \cdot \hat{n} = \int V_z \rho d\theta d\rho = 0$.

The top face has $z = h$ and $\hat{n} = \hat{k}$. The flux is

$$\begin{aligned} \int V_z \rho d\theta dr &= h \int_0^R \rho d\rho \int_0^{2\pi} d\theta \\ &= h\frac{R^2}{2}.2\pi = \pi R^2 h \end{aligned}$$



The normal to the curved face is along $\hat{\rho}$ direction. An area element on the curved face is $Rd\theta dz$ Thus the flux from this face is

$$\int (R\hat{\rho} + z\hat{k}) \cdot \hat{\rho} R d\theta dz = R^2 \int_0^{2\pi} d\theta \int_0^h dz = 2\pi R^2 h$$

The net flux from the faces of the object is $3\pi R^2 h$.

Exercise 6

Find the flux of the vector field $\vec{V} = 2\hat{\rho} - 3r\hat{\vartheta} + z\rho\hat{k}$ through surfaces of a right cylinder of radius 1 and height 2. The base of the cylinder is in the $z = 0$ plane with the origin at the centre of the base.

(Ans. $28\pi/3$)

Recap

In this lecture you have learnt the following

- Line integral of a vector field along an arbitrary curve was defined. This is useful in applications, for instance, in calculating work done under action of a force when a particle moves along a curve.

- Concept was flux was understood by defining surface integral of a vector field.

- Several problems involving line and surfave integrals were studied.