Module 5 : MODERN PHYSICS
Lecture 26 : Wave Nature of Particle - the de Broglie Hypothesis

Objectives

In this course you will learn the following

- Matter at very small length scale behave like waves.
- de Broglie hypothesis associates a wavelength $= h/p$ with matter waves.
- Electron diffraction from crystals which confirm wave nature of electrons.
- Bohr's model of hydrogen atom and its relationship with matter waves.
- Double slit experiments performed with electrons give results similar to Young's experiment for light.
- Heisenberg uncertainty principle.

Wave Nature of Particle - the de Broglie Hypothesis

In experiments like photoelectric effect and Compton effect, radiation behaves like particles. de Broglie, a french physicist asked whether in some situations, the reverse could be true, i.e., would objects which are generally regarded as particles (e.g. electrons) behave like waves ? In 1924 de Broglie postulated that we can associate a wave with every material object. In analogy with photons, he proposed that the wavelength λ associated with such a **matter wave** is related to the particle momentum p through the relationship

$$\lambda = \frac{h}{p}$$

where h is the Planck's constant

Example 13

Calculate the wavelength associated with a cricket ball of mass 0.2 kg moving with a speed of 30 m/s.

Solution :

$$p = mv = 0.2 \times 30 = 6 \ \mathrm{kg\,m/s}$$

$$\lambda = \frac{h}{p} = \frac{6.63 \times 10^{-34}}{6} = 1.1 \times 10^{-34} \ \mathrm{m}$$

Exercise 1

Neutrons produced in a reactor are used for chain reaction after they are ``thermalized", i.e., their kinetic energies are reduced to that of the energy of air molecules at room temperature. Taking the room temperature as 300 K, estimate the de Broglie wavelength of such thermal neutrons. (mass of neutron = 1.67×10^{-27} kg.)

(Ans. 0.145 nm)

Exercise 2

Calculate de Broglie wavelength of a proton moving with a velocity of 10^4 m/s.

(Ans. 4×10^{-11} m/s)

Example 14

What is the speed of an electron if its de Broglie wavelength equals its Compton wavelength ?

Solution :

We need to use relativistic formula for momentum

$$p = mv = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the rest mass of the electron. We have

$$\frac{h}{m_0 c} = \frac{h}{p} = \frac{h\sqrt{1 - v^2/c^2}}{m_0 v}$$

Solving, $v = c/\sqrt{2}$.

Exercise 3

The resolving power of a microscope is approximately equal to the wavelength of light used to illuminate the object. In an *electron microscope* , instead of light, the object is irradiated with a beam of electron. If the resolving power of an electron microscope is 0.01 nm, find the kinetic energy of the electrons used.

(Ans. 15 keV)

Wavelike behaviour of a macroscopic object is difficult to detect as the wavelength is very small.

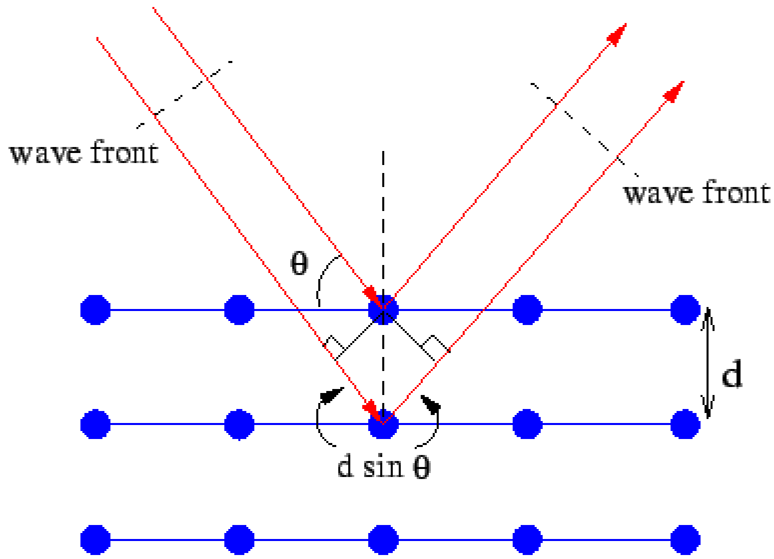
However, wave nature of particles may be detected in diffraction experiments where the dimensions of the obstacles are comparable with the wavelength of matter wave incident on the obstacle.

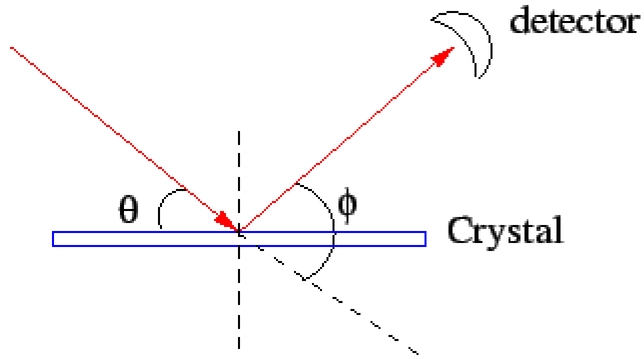
Electron Diffraction from a Crystal

Consider a beam of electron with a speed 5×10^6 m/s corresponding to a wavelength $\lambda = h/mv = 0.14$ nm. Such a wave may be diffracted by gratings with separation of similar order as that of the wavelength. Crystals provide such natural gratings.

Davisson - Germer Experiment :

Experimental confirmation of de Broglie hypothesis was provided in 1926 by Davisson and Germer, who studied diffraction of a beam of electrons from the surface of a nickel crystal.





A beam of electrons from a heated filament, accelerated through a potential difference V is made to strike the surface of a crystal of Ni. Electrons are scattered in all directions and may be detected by an array of detectors located at various angles of scattering. It is found that the intensity of scattered beam is maximum at some particular angles of incidence, in the same manner as the case when a beam of x-rays strikes the crystal

In Davisson - Germer experiment, the electron beam was accelerated through a potential difference of $V = 54$ volts. The kinetic energy E of the electron is thus 54 eV. The de Broglie wavelength associated with an electron accelerated through a potential difference V may be expressed as

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2meV}}$$

Using numerical values of h, m and e, we get a relationship of the form

$$\lambda = \sqrt{\frac{1.5}{V}}\ \text{nm}$$

The wavelength of the electrons in Davisson - Germer experiment can be calculated from the above to be 0.167 nm.

Exercise 4

Through what potential difference should an electron be accelerated to have a de Broglie wavelength of 1Å?

(Ans. 150 volts)

Exercise 5

An electron is released at a large distance from a proton. What will be the wavelength of the electron when it is at a distance of (i) 1 m (ii) 0.1 nm from the proton ? [Hint : The potential through which the electron moves is $e/4\pi\epsilon_0 r$.]

(Ans. (i) $3.24 \times 1)^{-5}$ m (ii) 3.24×10^{-10} m)

Bragg condition is satisfied when the path difference between beams scattered from two adjacent planes is a multiple of the wavelength. The path difference is $2d\sin\theta$. For constructive interference of order n we have

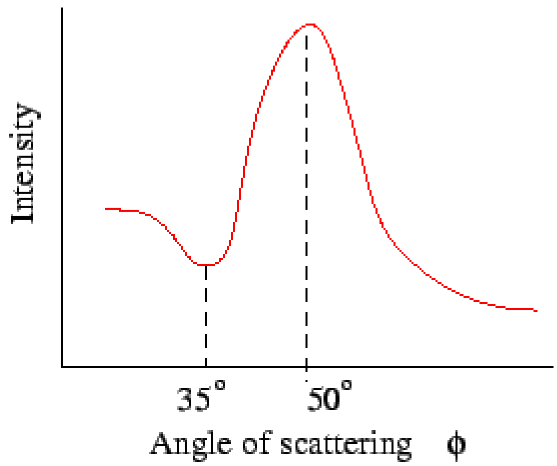
$$\boxed{2d\sin\theta = n\lambda}$$

Electron Diffraction from a Crystal

In case of Ni crystals, the interplane separation $d = 0.215$ nm, so that one expects Bragg condition to be satisfied for first order ($n = 1$) for

$$\sin\theta = \frac{\lambda}{2d} = \frac{0.167}{0.430} = 0.39$$

i.e. for $\theta = 25^\circ$. Thus we expect a strong interference effect for scattering angle $\phi = 2\theta = 50^\circ$.



Example 15

One of the diffraction peaks observed by Davisson and Germer for a 65 keV electron beam was at a direction such that the angle between the incident beam and the scattered beam is 60 °. For what value of crystal spacing is this peak seen in the first order ?

Solution :

For $\phi = 60^\circ$, the angle $\theta = 30^\circ$. A kinetic energy of 65 keV corresponds to

$K = 65 \times 10^3 \times 1.6 \times 10^{-19} = 1.04 \times 10^{-14}$ J. Equating this to $p^2/2m$, the momentum

$p = \sqrt{2mK} = 1.39 \times 10^{-22}$ kg-m/s. The wavelength corresponding to this momentum is

$h/p = 4.76 \times 10^{-12}$ m. The crystal spacing is given by

$$d = \frac{\lambda}{2\sin\theta} = 4.76 \times 10^{-12} \text{ m}$$

Exercise 6

Thermal neutrons having a wavelength of 0.145 nm are diffracted by a crystal of lattice spacing 0.29 nm. Find the angle at which the first order diffraction maximum occurs.

(Ans. 14 $^{\circ}$)

Bohr Model :

Bohr's model of an atom, which was very successful in explaining the spectra of hydrogen like atoms is based on the following postulates **:**

Electrons move in stationary orbits around the nucleus. As long as an electron is in such an orbit, it does not y

radiate. However, it emits (absorbs) radiation when it makes a *discontinuous* transition form an orbit with energy E to an orbit of lower (higher) energy E'. The frequency of emitted (absorbed) radiation is given b

$$\nu = (-)\frac{E - E'}{h}$$

The angular momentum of an electron in a stationary orbit is an integral multiple of $h/2\pi$:

$$mvr = n\frac{h}{2\pi}$$

where n is an integer.

Using Bohr model, one can show that the wavelength of the radiation emitted when an electron makes a transition from an orbit with quantum number n_i to an orbit with n_f , is given by

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

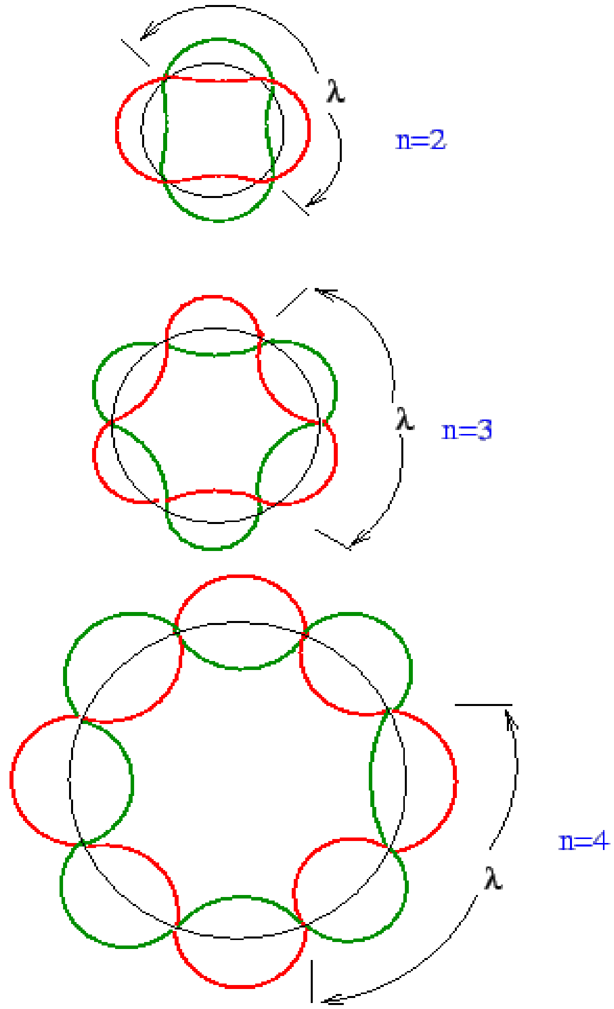
where R is called the Rydberg constant, which has a numerical value 1.097×10^7 m $^{-1}$.

The de Broglie hypothesis may be used to derive Bohr's formula by considering the electron to be a wave spread over the entire orbit, rather than as a particle which at any instant is located at a point in its orbit. **The satble orbits in an atom are those which are standing waves.** Formation of standing waves require that the circumference of the orbit is equal in length to an intergral multiple of the wavelegth. Thus, if r is the radius of the orbit

$$2\pi r = n\lambda = n\frac{h}{p}$$

which gives the **angular momentum quantization**

$$L = rp = n\frac{h}{2\pi}$$



Exercise 7

Calculate the wavelength of an electron the ground state of hydrogen atom. (First Bohr radius of hydrogen atom is 0.053 nm)

(0.33 nm)

Example 16

If an electron makes a transition from $n=4$ to $n=2$, determine (i) the wavelength of emitted radiation and (ii) the recoil speed of the electron.

Solution :

The wavelength of emitted radiation is

$$\frac{1}{\lambda}=1.097\times10^{7}\left(\frac{1}{4}-\frac{1}{16}\right)=0.2055\times10^{7}\ \mathrm{nm}^{-1}$$

The wavelength is $\lambda=486$ nm. The momentum associated with this radiation is

$$p=\frac{h}{\lambda}=\frac{6.63\times10^{-34}}{4.86\times10^{7}}=1.36\times10^{-27}\ \mathrm{kg-m/s}$$

By conservation of momentum, this is also the magnitude of the momentum imparted to the atom as a whole. The recoil speed of the electron is

$$v=\frac{p}{m_{atom}}=\frac{1.36\times10^{-27}}{1.67\times10^{-27}}=0.81\ \mathrm{m/s}$$

Double Slit Interference with electrons

We know that when a coherent source of light is incident on a Young's Double slit, an interference pattern is observed. The intensity when only the slit S_1 is open is I_1, while the intensity with S_2 open is I_2.

When both slits are open, the expression for the intensity at any position P on the screen is given by

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$$

where ϕ is the phase difference between the waves arriving at P from S_1 and S_2. Points where the phase difference $\phi = 2n\pi$, the intensity is a maximum

$$I_{max} = (\sqrt{I_1} + \sqrt{I_2})^2$$

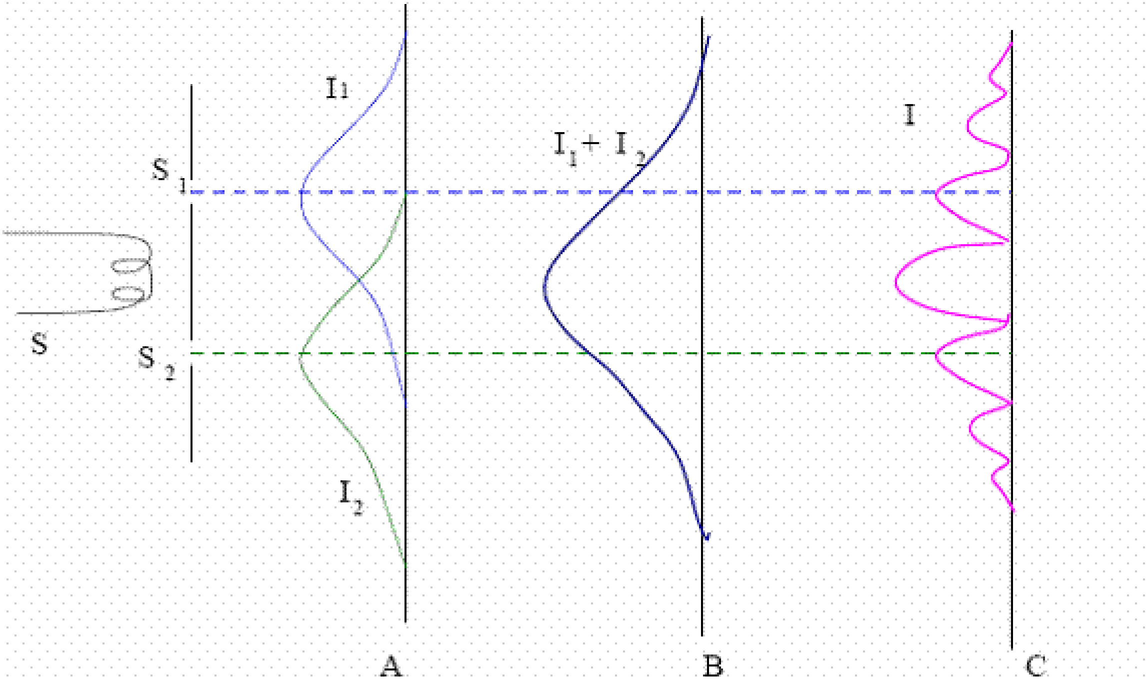
If the phase difference is an odd integral multiple of $\pi/2$,

$$I_{min} = (\sqrt{I_1} - \sqrt{I_2})^2$$

Note that at the minima less energy is received with both slits open than is received with any one slit open, which is strange.

Let us repeat the experiment with beams of particles, such as electrons, traditionally considered localized objects. The screen consists of an array of detectors which can record arrival of an electron. Experiment is done with a beam of electrons and the number of times a detector records the arrival of electron is noted. This is plotted as a histogram. The distribution of the electrons in the detectors at various points on the screen is what corresponds to the intensity pattern stated above. The plot can be looked upon as a probability distribution curve.

Let us focus our attention on a detector located at some position P. We close one of the slits and determine the probability of an electron arriving at P. This is done by simply finding out the fraction of the total number of electrons that are emitted by the source S are detected by the detector at P. When S_1 is open, let this fraction be p_1. Similarly, with S_1 closed but with S_2 open, we determine the corresponding fraction at the same position. If we open both the slits, we would expect the fraction to be $p = p_1 + p_2$, as every electron must pass either through S_1 or through S_2. However, what one finds experimentally is that $p \neq p_1 + p_2$.



A : The blue curve is obtained when only S $_1$ is open. The green curve is when only S $_2$ is open.

B : This is what is expected when both slits are open

C : The intensity distribution that is actually obtained

Clearly, the result is absurd in the sense that it says that when both slits are open, there are particles which neither goes through the slit S_1 nor goes through the slit S_2. Let us try to look at the actual situation. To keep track of which electron came through which slit is easy enough. We put a source of light near each of the slits, so that when an electron passes through one of the slits, it scatters light and we can see a flash. If we do the experiment this way, keeping track of the particle, we find that $p = p_1 + p_2$ and there is no contradiction. However, if we do not keep track of which slit each electron goes through, we get the distribution pattern shown in curve C. What is even more funny is that in curve C there are some points (minima) where the number of particles is even less than that which reach these points when only one slit is kept open.

We define a probability amplitude ψ such that $p = |\psi|^2$. In terms of probability amplitude, it turns out that if we keep track of electrons by watching them,

$$p = p_1 + p_2 = |\psi_1|^2 + |\psi_2|^2 \ (1)$$

However, if we do not observe the electrons the probability distribution is given by

$$p = |\psi_1 + \psi_2|^2 \neq p_1 + p_2 (2)$$

The reason behind this paradox is that when we observed the electron, the electron has to interact with whatever probe we use for observing, in this case, with the light source. This interaction can alter the chance of arrival of the electron at the point P.

Can we use a weaker source of light to reduce the effect ? The answer clearly is no because a weaker source does not mean photon energy is different, it simply means that there are less number of photons. If there are less number of photons, some of the electrons will escape without being detected by them. Those which are detected are distributed according to

(1) while those which escape being detected are distributed according to

(2) and the net result that we get is a weighted mean of the two distributions.

Principle of Complementarity :

The above behaviour illustates an important principle of physics called the principle of complementarity. When we tried to determine which slit the electron went through, we were investigating the particle nature of an electron. However, only a wave can simultaneously go through both the slits and the interference effect was a manifestation of the wave nature. According to the principle of complementarity, an object has *both* the particle property and wave property. However, in a given experiment either one or the other property can be determined. **It is not possible to siumultaneously get information on both the particle nature and the wave nature of an object in the same experiment.**

What the photons do to the electron is to transfer momentum so that the electrons are scattered in different directions from their original directions. Can we then reduce the effect by using light of longer wavelength which can impart less momentum causing less disturbance ? The answer once again is no because if the wavelength is too long we will not be in a position to say whether the scattered electron came from the slit S_1 or from S_2 as the position cannot be determined with a precision better than the wavelength.

Heisenberg Uncertainty Principle

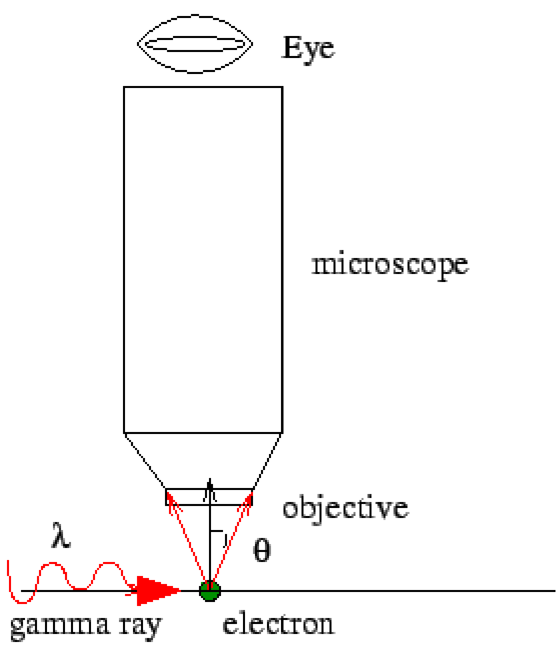
The clue to this apparent paradox is in Heisenberg's uncertainty principle, according to which, there are limitations on the accuracy with which experiments aimed at determining which hole a particular electron goes through, may be performed, while still not disturbing the interference effect.

Classically, one can simultaneously measure the position and momentum of a particle to infinite precision. In the quantum mechanical world, there is an uncertainty associated with every measurement. Whenever we attempt to measure the position of an object with a precision Δx, there is some uncertainty Δp in the momentum measurement, which we cannot get rid of, because, in order to make a measurement we must disturb the system.

Thought Experiment

In the thought experiment *gedanken* shown in the figure, the electron is assumed to be at restly below the objective of the microscope. The electron is illuminated by gamma rays, which have a short wavelength. Using a short wavelength yields high resolution. According to principles of optics, a microscope can resolve objects to within Δx, which is related to the wavelength λ through

$$\Delta x = \frac{\lambda}{2\sin\theta}$$



See the animation

However, gamma rays behaves like a particle with momentum $p = h/\lambda$ and is Compton scattered. In order to be observed by the microscope, γ ray must be scattered within an angle 2θ of the cone. Since the electron is initially at rest, the total moment of the electron- photon system is the momentum of the photon.

Consider two extreme limits when the gamma ray photon is scattered by an angle θ to the left extreme wall of the microscope. The x-component of the total momentum is the sum of the x component of the momentum of the scattered electron, p'_x and the x component of the momentum of the scattered photon, which is $-\frac{h}{\lambda'}\sin\theta,$, i.e. $p'_x - \frac{h}{\lambda'}\sin\theta$,where λ' is the wavelength of the scattered photon.

Similarly, when the photon is scattered to the extreme right, the total momentum is $p''_x + \frac{h}{\lambda''}\sin\theta$,where λ'' is the wavelength of the scattered photon. As the angle θ is small, the Compton shift $\Delta\lambda = (h/mc)(1-\cos\theta)$ is small, and we may take $\lambda' \approx \lambda'' \approx \lambda$. The total x-component of the momentum incases must be the same, each being equal to momentum of the incident photon.

Thus

$$p'_x - \frac{h}{\lambda'}\sin\theta = p''_x + \frac{h}{\lambda''}\sin\theta$$

so that the momemntum uncertainty of the electron is

$$\Delta p_x = p'_x - p''_x = \frac{2h}{\lambda}\sin\theta = \frac{h}{\Delta x}$$

giving

$$\Delta x \Delta p_x \approx h$$

It may be noted that one can always determine the momentum along the y-direction with any desired degree of accuracy when there is position uncertainty along the x-direction.

Exercise 8

The position of an electron is determined with an accuracy of 0.01 nm. Find the uncertainty in its momentum.(Ans. 6.63×10^{-23} kg m/s)

Example 17

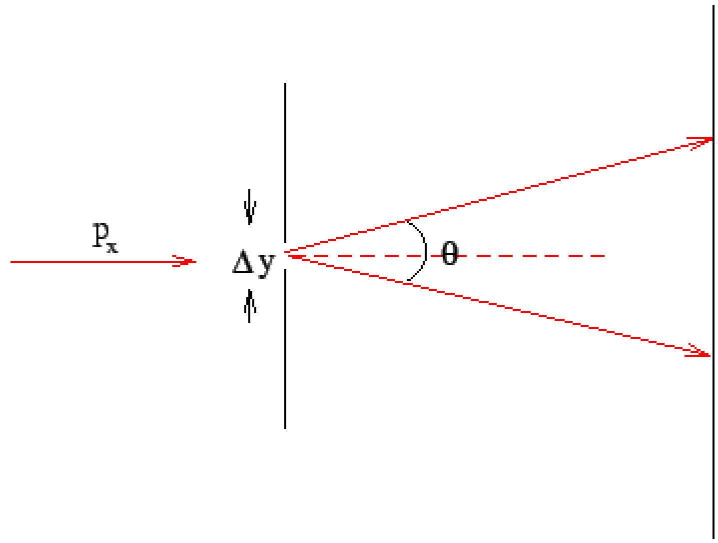
A beam of electrons with a de Broglie wavelength of 10^{-5} m passes through a slit 10^{-4} m wide. Calculate the angular spread of the beam by diffraction.

Solution :

Initial momentum of the electron is along the x-direction.

$$p_x = \frac{h}{\lambda} = \frac{6.63\times10^{-34}}{10^{-5}} = 6.63\times10^{-29}\,\mathrm{kg\,m/s}$$

Due to the uncertainty in the position of the electron along the slit direction (y-direction), there is a momentum spread along the y-direction, given by



[See the animation]

$$\Delta p_y = \frac{h}{\Delta y} = \frac{6.63 \times 10^{-34}}{10^{-4}} = 6.63 \times 10^{-30} \text{ kg m/s}$$

The angular spread of the beam is given by

$$\tan\theta = \frac{\Delta p_y}{p_x} = 0.1$$

so that the spread $\theta = 5^\circ 47'$.
Similar uncertainty relationship exists for other pairs of observables. For instance, if we wish to measure the energy E of a particle with an accuracy ΔE. According to uncertainty principle, we may determine the time at which the particle has such an energy only with an uncertainty Δt where

$$\Delta E \Delta t \approx h$$

Example 18

Find the wavelength spread of a 1 nano-second pulse from a ruby laser with a wavelength of 630 nm.

Solution :

Since $\Delta E \Delta t = h$, and $E = h\nu$, we have $\Delta \nu \Delta t = 1$. Thus $\Delta \nu = 1/\Delta t = 10^9$ Hz.

Using $\nu = c/\lambda$, we get

$$d\nu = -\frac{c}{\lambda^2} d\lambda$$

so that

$$\Delta\lambda = \frac{\lambda^2}{c} \Delta\lambda = 1.32 \times 10^{-12} \text{ m}$$

Recap

In this course you have learnt the following

- The dual nature not only is exhibited by radiation but is also associated with matter. In some experiments matter shows wave character.

- de Broglie hypothesis poastulates a wavelength of h/p with a particle having a momentum p .

- Experimental confirmation of wave nature of matter comes from experiments such as Davisson Germer experiments on electron diffraction from crystals. It is seen that the intensity of scattered beam is maximum at those points where one would expect Laue spots in x-ray diffraction assuming the electronsare waves with de Broglie wavelength.

- Bohr model can be understood by postulating that stable orbits in atoms are those which are standing waves of electrons.

- One can perform double slit experiment with electrons, similar to the way Young's double slit experiment is performed with light waves. The intensity pattern obtained on a screen is very similar in both cases.

- According to the principle of complementarity one cannot obtain information on both the wave nature and particle nature of matter or radiation in the same experiment.

- Heisenberg uncertainty principle states that one cannot precisely measure both position and momentum of a particle in the same experiment.