Module 1 : A Crash Course in Vectors
Lecture 2 : Coordinate Systems

Objectives

In this lecture you will learn the following

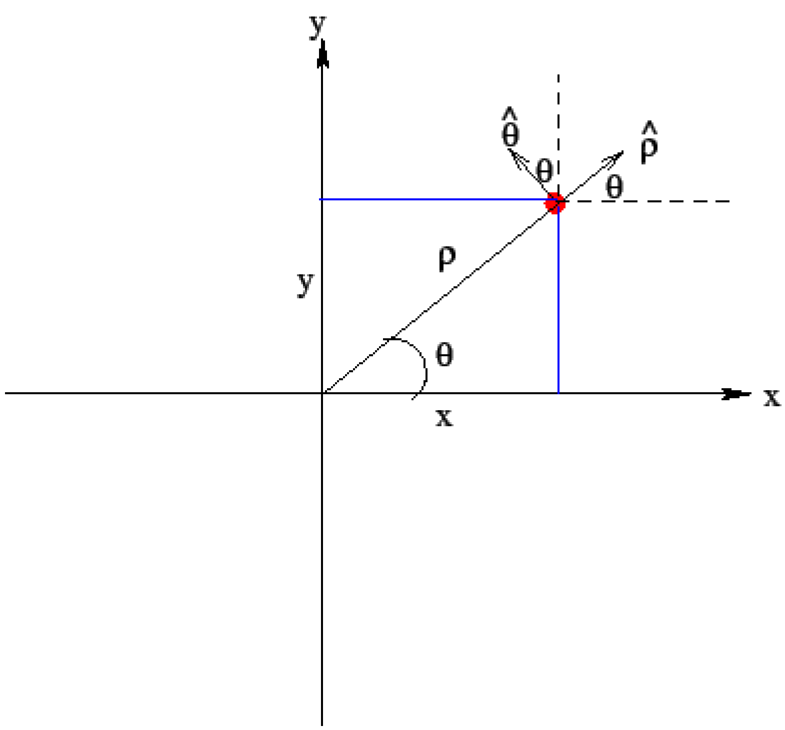
- Define different coordinate systems like spherical polar and cylindrical coordinates
- How to transform from one coordinate system to another and define Jacobian

Coordinate Systems :

We are familiar with cartesian coordinate system. For systems exhibiting cylindrical or spherical symmetry, it is convenient to use respectively the cylindrical and spherical coordinate systems.

Polar Coordinates :

In two dimensions one defines the polar coordinate (ρ, θ) of a point by defining ρ as the radial distance from the origin O and θ as the angle made by the radial vector with a reference line (usually chosen to coincide with the x-axis of the cartesian system). The radial unit vector $\hat{\rho}$ and the tangential (or angular) unit vector $\hat{\theta}$ are taken respectively along the direction of increasing distance ρ and that of increasing angle θ respectively, as shown in the figure.



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Relationship with the cartesian components are

$$
\begin{aligned}
x &= \rho \cos\theta \\
y &= \rho \sin\theta
\end{aligned}
$$

so that the inverse relationships are

$$\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\frac{y}{x} \end{aligned}$$

By definition, the distance $\rho > 0$. we will take the range of angles θ to be $0 \le \theta < 2\pi$ (It is possible to define the range to be $-\pi \le \theta < +\pi$). One has to be careful in using the inverse tangent as the arc-tan function is defined in $0 \le \theta < \pi$. If y is negative, one has to add π to the principal value of θ calculated by the arc - tan function so that the point is in proper quadrant.

Example 2

A vector $\vec{A}$ has cartesian components A_x and A_y. Write the vector in terms of its radial and tangential components.

Solution :

Let us write

$$\vec{A} = A_\rho\hat{\rho} + A_\theta\hat{\theta}$$

Since $\hat{\rho}$ and $\hat{\theta}$ are basis vectors $\hat{\rho}\cdot\hat{\rho} = \hat{\theta}\cdot\hat{\theta} = 1$ and $\hat{\rho}\cdot\hat{\theta} = 0$. Thus

$$A_r = \vec{A}\cdot\hat{\rho}, \quad A_\theta = \vec{A}\cdot\hat{\theta}$$

Note that (see figure) the angle that $\hat{\rho}$ makes an angle θ with the x-axis ($\hat{i}$) and $\pi/2 - \theta$ with the y-axis ($\hat{j}$). Similarly, the unit vector $\hat{\theta}$ makes $\pi/2 + \theta$ with the x-axis and θ with the y-axis. Thus

$$\begin{aligned} A_\rho = \vec{A}\cdot\hat{\rho} &= A_x\hat{i}\cdot\hat{\rho} + A_y\hat{j}\cdot\hat{\rho} \\ &= A_x\cos\theta + A_y\cos(\pi/2-\theta) \\ &= A_x\cos\theta + A_y\sin\theta \\ A_\theta = \vec{A}\cdot\hat{\theta} &= A_x\hat{i}\hat{\theta} + A_y\hat{j}\cdot\hat{\theta} \\ &= A_x\cos(\pi/2+\theta) + A_y\cos\theta \\ &= -A_x\sin\theta + A_y\cos\theta \end{aligned}$$

The Jacobian :

When we transform from one coordinate system to another, the differential element also transform.

For instance, in 2 dimension the element of an area is $dxdy$ but in polar coordinates the element is not $d\theta d\rho$ but $(\rho d\theta)d\rho$. This extra factor ρ is important when we wish to integrate a function using a different coordinate system.

If $f(x,y)$ is a function of x,y we may express the function in polar coordinates and write it as $g(\rho,\theta)$. However, when we evaluate the integral $\int f(x,y)dxdy$ in polar coordinates, the corresponding integral is $\int g(r,\theta)\rho d\theta d\rho$. In general, if $x=f(u,v)$ and $y=g(u,v)$, then, in going from (x,y) to (u,v), the differential element $dxdy \rightarrow \mid J \mid dudv$ where J is given by the determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The differentiations are partial, i.e., while differentiating $\partial x/\partial u = \partial f(u,v)/\partial u$, the variable v is treated as constant. An useful fact is that the Jacobian of the inverse transformation is $1/J$ because the detrminant of the inverse of a matrix is equal to the inverse of the determinant of the original matrix.

Example 3

Show that the Jacobian of the transformation from cartesian to polar coordinates is ρ.

Solution :

We have

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

Using $x=\rho\cos\theta$ and $y=\rho\sin\theta$, we have

$$J = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho$$

Exercise 1

Show that the Jacobian of the inverse transformation from polar to cartesian is $1/\rho = 1/\sqrt{x^2+y^2}$

Example 4

Find the area of a circle of radius R.

Solution :

Take the origin to be at the centre of the circle and the plane of the circle to be the $\rho-\theta$ plane. Since the area element in the polar coordinates is $\rho d\theta dr$, the area of the circle is

$$\int_0^{2\pi} d\theta \int_0^R \rho d\rho = 2\pi\left[\frac{\rho^2}{2}\right]_0^R = \pi R^2$$

a very well known result !

Example 5

Find the integral $\int e^{-(x^2+y^2)}dxdy$ where the region of integration is a unit circle about the origin.

Using polar coordinates the integrand becomes $e^{-\rho^2}$. The range of i integration for ρ is from 0 to 1 and for θ is from 0 to 2π. The integral is given by

$$I = \int_0^{2\pi} d\theta \int_0^1 e^{-\rho^2}\rho d\rho = 2\pi \int_0^1 e^{-\rho^2}\rho d\rho$$

The radial integral is evaluated by substitution $w = \rho^2$ so that $\rho d\rho = dw/2$. The value of the integral is

$$I = 2\pi \int_0^1 e^{-w} dw/2 = \pi[-e^{-w}]_0^1 = \pi(1 - \frac{1}{e})$$

Exercise 2

Evaluate $\int\int xydxdy$ where the region of integration is the part of the area between circles of radii 1 and 2 that lies in the first quadrant. (Ans. $15/8$)

Exercise 3

Evaluate the Gaussian integral $I = \int_0^\infty e^{-x^2}dx$

[Hint : The integration cannot be done using cartesian coordinates but is relatively easy using polar coordinates and properties of definite integrals. By changing the dummy variable x to y, one can write $I = \int_0^\infty e^{-y^2}dy$, so that we can write $I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$

Transform this to polar. Range of integration for ρ is from 0 to ∞ and that of θ is from 0 to $\pi/2$ (why ?)]

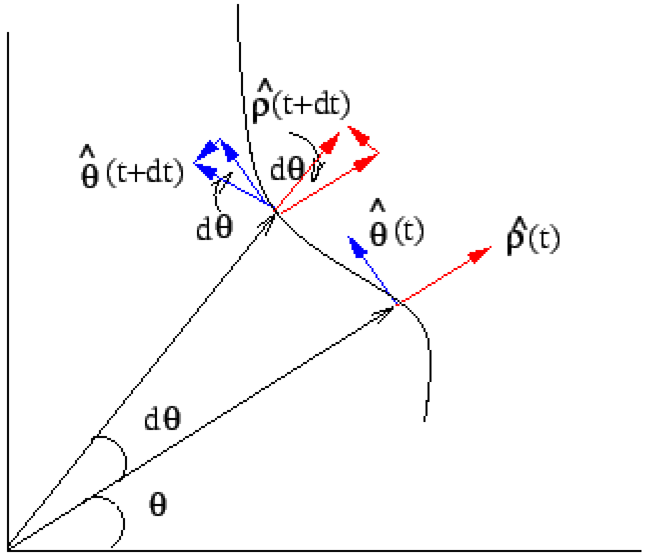
[Answer : $\sqrt{\pi}/2$]

Differentiation of polar unit vectors with respect to time :

It may be noted that the basis vectors $\hat{\rho}$ and $\hat{\theta}$, unlike $\hat{\imath}$ and $\hat{\jmath}$ are not constant vectors but depend on the position of the point. The time derivative of the unit vectors are defined as follows

$$\begin{aligned}\frac{d\hat{\rho}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\hat{\rho}}{\Delta t} &= \lim_{\Delta t\to 0}\frac{\hat{\rho}(t+\Delta t)-\hat{\rho}(t)}{\Delta t}\\ \frac{d\hat{\theta}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta\hat{\theta}}{\Delta t} &= \lim_{\Delta t\to 0}\frac{\hat{\theta}(t+\Delta t)-\hat{\theta}(t)}{\Delta t}\end{aligned}$$

One can evaluate the derivatives by laborious process of expressing the unit vectors $\hat{\rho}$ and $\hat{\theta}$ in terms of constant unit vectors of cartesian system, differentiating the resulting expressions and finally transform back to the polar form. Alternatively, we can look at the problem geometrically, as shown in the following figure.



In the figure, the positions of a particle are shown at time t and $t+dt$. The unit vectors $\hat{\rho}$ is shown in red while the unit vector $\hat{\theta}$ is shown in blue. It can be easily seen by triangle law of addition of vectors that the magnitude of $\Delta\hat{\rho}$ and $\Delta\hat{\theta}$ is $1.d\theta = d\theta$. However, as the limit $dt \to 0$, the direction of $\Delta\hat{\rho}$ is in the direction of $\hat{\theta}$ while that of $\Delta\hat{\theta}$ is in the direction of $-\hat{\rho}$. Thus

$$\begin{aligned} \frac{d\hat{\rho}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\hat{\rho}}{\Delta t} &= \frac{d\theta}{dt}\hat{\theta} \\ \frac{d\hat{\theta}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\hat{\theta}}{\Delta t} &= -\frac{d\theta}{dt}\hat{\rho} \end{aligned}$$

Now, $d\theta/dt$ is the angular velocity of the point, which is usually denoted by ω, Thus we have,

$$\begin{aligned} \frac{d\hat{\rho}}{dt} &= \omega\hat{\theta} \\ \frac{d\hat{\theta}}{dt} &= -\omega\hat{\rho} \end{aligned}$$

Cylindrical coordinates :

Cylnidrical coordinate system is obtained by extending the polar coordinates by adding a z-axis along the height of a right circular cylinder. The z-axis of the coordinate system is same as that in a cartesian system.

In the figure ρ is the distance of the foot of the perpendicular drawn from the point to the $x-y(\rho,\theta)$ plane.

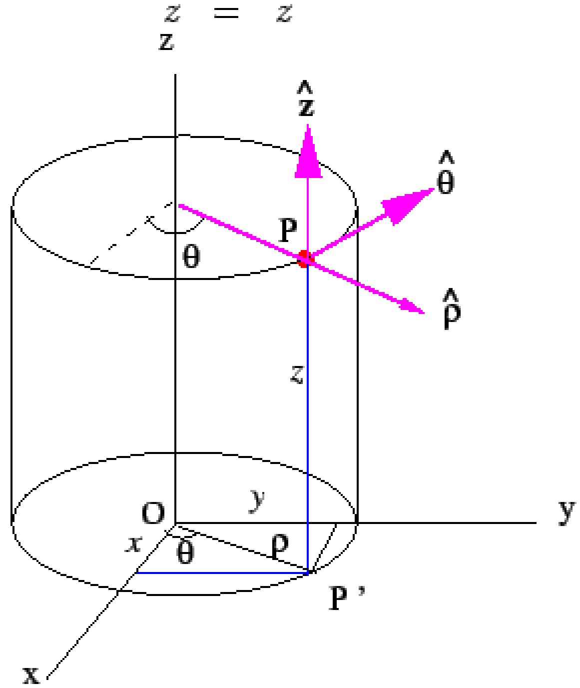
Note that ρ here is *not* the distance of the point P from the origin, as is the case in polar coordinate systems.

(Some texts use r to denote what we are calling as ρ here. However, we use ρ to denote the distance from the origin to the foot of the perpendicular to avoid confusion.) In terms of cartesian coordinates

$$\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \\ z &= z \end{aligned}$$

so that the inverse relationships are

$$\rho = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}\frac{y}{x}$$
$$z = z$$



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Exercise 4

Find the cylindrical coordinate of the point $3\hat{\imath}+4\hat{\jmath}+\hat{k}$.

[Hint : Determine ρ and $\tan\theta$ using above transformation]

(Ans. $5\hat{\rho}+\tan^{-1}(4/3)\hat{\theta}+\hat{k}$)

The line element in the system is given by

$$\vec{dl} = \hat{\rho}d\rho + \rho d\theta\hat{\theta} + d\hat{z}$$

and the volume element is

$$dV = \rho d\theta d\rho dz$$

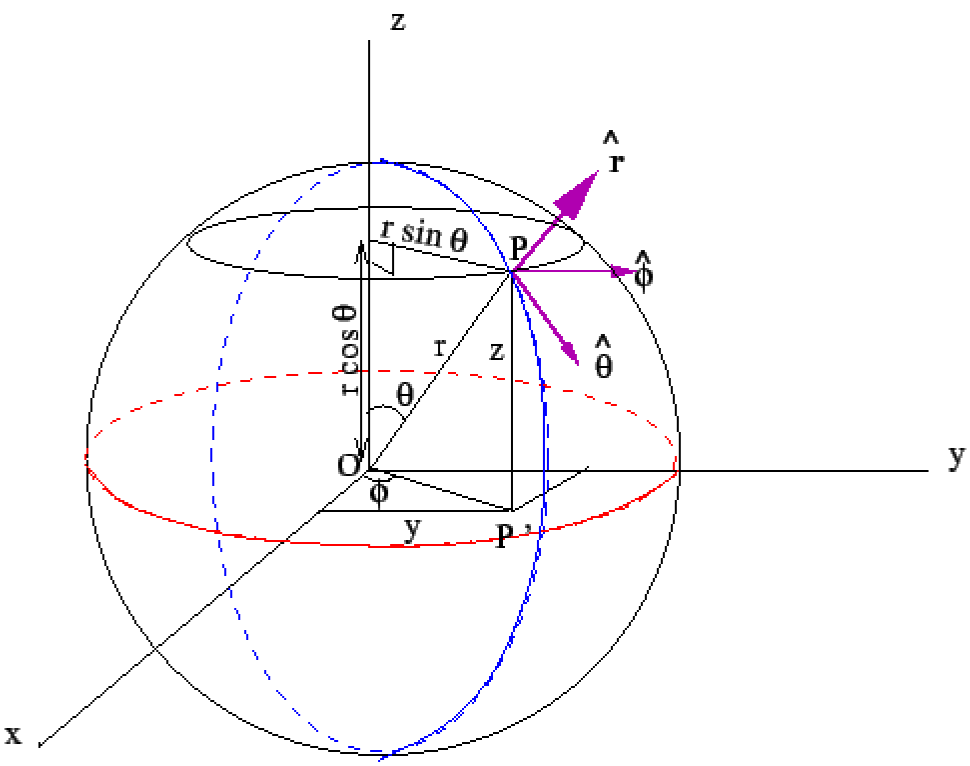
The Jacobian of transformation from cartesian to cylindrical is ρ as in the polar coordinates since z coordinate remains the same.

Spherical Polar Coordinates :

Spherical coordinates are useful in dealing with problems which possess spherical symmetry. The independent variables of the system are (r,θ,ϕ). Here r is the distance of the point P from the origin. Angles θ and ϕ

are similar to latitudes and longitudes.
Two mutually perpendicular lines are chosen, taken to coincide with the x-axis and z-axis of the cartesian system. We take angle θ to be the angle made by the radius vector (i.e. the vector connecting the origin to P) with the z-axis (the angle θ is actually complementary to the latitude). The angle ϕ is the angle between the x-axis and the line joining the origin to P', the foot of the perpendicular from P to the x-y plane.

The unit vectors $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ are respectively along the directions of increasing r, θ and ϕ.



The surface of constant r are spheres of radius r about the centre. the surface of constant θ is a cone of semi-angle θ about the z-axis. The reference for measuring ϕ is the x-z plane of the cartesian system. A surface of constant ϕ is a plane containing the z-axis which makes an angle ϕ with the reference plane.

Example 6

Express unit vectors of spherical coordinate system in terms of unit vectors of cartesian system.

Solution :

From the point P drop a perpendicular on to the x-y plane. Denote $\vec{OP'}$ by $\vec{\rho}$. The figure below shows the unit vectors in both the systems. By triangle law of vector addition,

$$\vec{r} = \vec{OP} = \vec{OP'} + \vec{P'P} = r\sin\theta\hat{\rho} + r\cos\theta\hat{k}$$

However, $\hat{\rho} = \cos\phi\hat{\imath} + \sin\phi\hat{\jmath}$. Substituting this in the expression for $\underline{\vec{r}}$, we get on dividing both sides by the magnitude of r

$$\hat{r} = \sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + r\cos\theta\hat{k}$$

Transformation from spherical to cartesian :

Using the expression for $\underline{\vec{r}}$ in terms of cartesian basis, it is seen that

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$

and the inverse transformation

$$
\begin{aligned}
r &= \sqrt{x^2+y^2+z^2} \\
\theta &= \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \cos^{-1}\frac{z}{r} \\
\phi &= \tan^{-1}\frac{y}{x}
\end{aligned}
$$

Range of the variables are as follows :

$$0 \le r < \infty \quad 0 \le \theta \le \pi \quad 0 \le \phi \le 2\pi$$

Exercise 5

A particle moves along a spherical helix. its position coordinate at time t is given by

$$x = \frac{\cos t}{\sqrt{1+t^2}}, \quad y = \frac{\sin t}{\sqrt{1+t^2}}, \quad z = \frac{t}{\sqrt{1+t^2}}$$

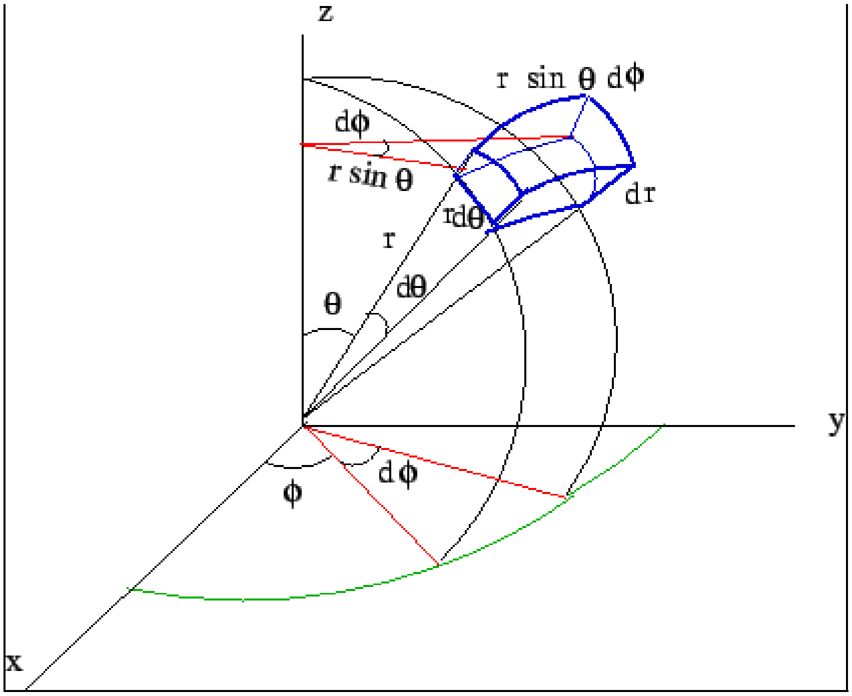
Express the equation of the path in spherical coordinates.

(Ans. $r = 1, \quad \cos\theta = t/\sqrt{1+t^2} \quad \phi(t) = t$)

The differential element of volume is obtained by constructing a closed volume by extending r, θ and ϕ respectively by $dr, d\theta$ and $d\phi$. The length elements in the direction of $\hat{r}$ is dr, that along $\hat{\theta}$ $rd\theta$ while that along $\hat{\phi}$ is $r\sin\theta d\phi$ (see figure). The volume element, therefore, is

$$dV = dr \cdot rd\theta \cdot r\sin\theta d\phi = r^2\sin\theta d\theta d\phi dr$$

Thus the Jacobian of transformation is $r^2\sin\theta$.



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Example 7

Find the volume of a solid region in the first octant that is bounded from above by the sphere $x^2+y^2+z^2=9$ and from below by the cone $x^2+y^2=3z^2$.

Solution :

Because of obvious spherical symmetry, the problem is best solved in spherical polar coordinates. The equation to sphere is $r=3$ so that the range of r vaiable for our solid is from 0 to 3.

The equation to the cone $x^2+y^2=3z^2$ becomes $r^2\sin^2\theta=3r^2\cos^2\theta$. Solving, the semi-angle of cone is $\tan\theta=\sqrt{3}$ i.e. $\theta=\pi/3$. Since the solid is restricted to the first octant, i.e., ($x,y,z\geq 0$), the range of the azimuthal angle ϕ is from 0 to $\pi/2$.

Exercise 7

Using direct integration find the volume of the first octant bounded by a sphere $x^2+y^2+z^2=9$

Recap

In this lecture you have learnt the following

- In addition to the cartesian coordinates, two other coordinate systems, viz., spherical polar coordinates and cylindrical coordinates were introduced. The relationship of the components of a vector in various coordinates was studied.
- When we transform from one coordinate to another, length, area and volume elements also change. Jacobian provides the transformation of such elements in different coordinate systems.
- While the unit vectors in cartesian coordinates are fixed, the unit vectors associated with the position of a moving particle changes as the particle moves, and are therefore, time dependent.
- The differentiation of such time dependent unit vectors with respect to time was discussed.

- Integration techniques to find out volumes of objects having different types of symmetry was studied.