Module 2 : Electrostatics
Lecture 9 : Electrostatic Potential

Objectives

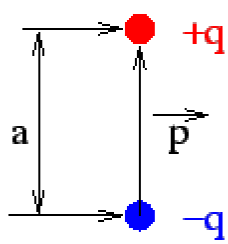
In this lecture you will learn the following

- Electric Dipole and field due to a dipole
- Torque on a dipole in an inhomogeneous electric field
- Potential Energy of a dipole
- Energy of a system of charges - discrete and continuous

Potential and Field due to an Electric Dipole

An electric dipole consists of two equal and opposite charges $+q$ and $-q$ separated by a small distance a.

The *Electric Dipole Moment* $\vec{p}$ is defined as a vector of magnitude qa with a direction from the negative charge to the positive charge. In many molecules, though the net charge is zero, the nature of chemical bonds is such that the positive and negative charges do not cancel at every point. There is a small separation between the positive charge centres and negative charge centres. Such molecules are said to be *polar molecules* as they have a non-zero dipole moment. The figure below shows an asymmetric molecule like water which has a dipole moment 6.2×10^{-30} C-m.



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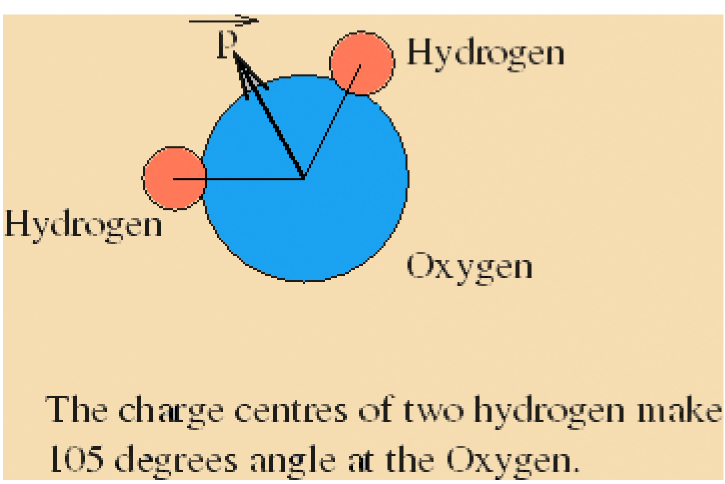

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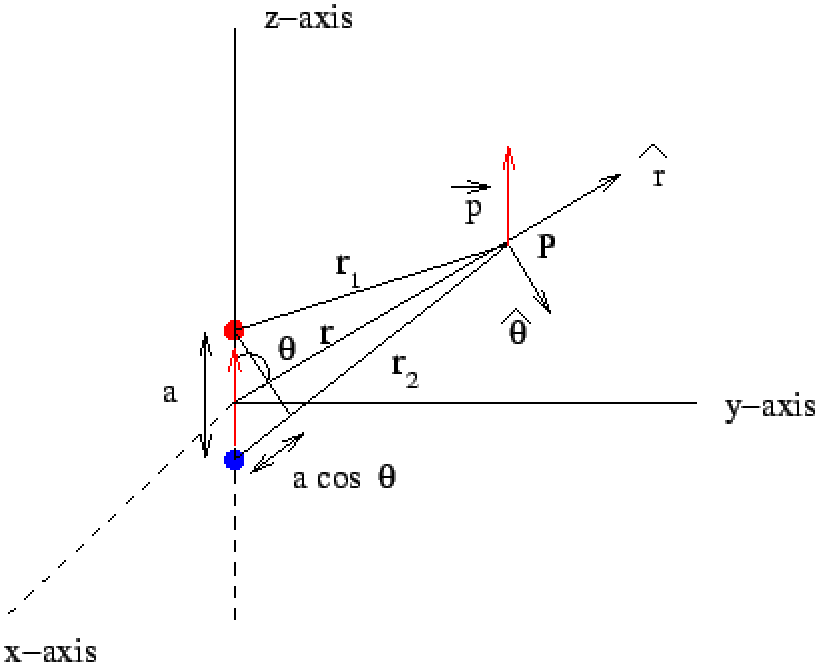
In the polar $r-\theta$ coordinates shown in the figure

$$\vec{p}=p\cos\theta\hat{r}-p\sin\theta\hat{\theta}$$

where $\hat{r}$ and $\hat{\theta}$ are unit vectors in the radial and tangential directions, taken respectively, in the direction of increasing r and increasing θ.

The electric potential at a point P with a position vector $\vec{r}$ is

$$\phi(\vec{r})=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{r_1}-\frac{q}{r_2}\right]=\frac{q}{4\pi\epsilon_0}\frac{r_2-r_1}{r_1r_2}$$



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If the distance a is small compared to r (i.e., if the point P is far away from the dipole), we may use

$$r_2-r_1\approx a\cos\theta \quad r_1r_2\approx r^2$$

where θ is the angle between $\vec{r}$ and the dipole moment vector $\vec{p}$. This gives

$$\phi(\vec{r})\approx\frac{qa\cos\theta}{4\pi\epsilon_0r^2}=\frac{p\cos\theta}{4\pi\epsilon_0r^2}$$

Electric Field of a Dipole

A.CARTESIAN COORDINATES

It is convenient to define the cartesian axes in the following way. Let the dipole moment vector be taken along the z-axis and position vector $\vec{r}$ of P in the y-z plane (We have denoted the point where the electric field is calculated by the letter P and the electric dipole moment vector as $\vec{p}$). We then have $\cos\theta=z/r$ with $r=\sqrt{y^2+z^2}$. Thus

$$\phi(x,y,z)=\frac{p\cos\theta}{4\pi\epsilon_0r^2}=\frac{pz}{(y^2+z^2)^{3/2}}$$

Since ϕ is independent of x, $E_x = 0$. The y and z components are

$$\begin{aligned} E_y &= -\frac{\partial}{\partial y}\left[\frac{pz}{4\pi\epsilon_0(y^2+z^2)}\right] \\ &= \frac{3pzy}{4\pi\epsilon_0(y^2+z^2)^{5/2}} = \frac{3p}{4\pi\epsilon_0}\frac{yz}{r^5} \end{aligned}$$

and

$$\begin{aligned} E_z &= -\frac{\partial}{\partial z}\left[\frac{pz}{4\pi\epsilon_0(y^2+z^2)}\right] \\ &= -\frac{p}{4\pi\epsilon_0}\frac{1}{(y^2+z^2)} + \frac{3pz^2}{4\pi\epsilon_0(y^2+z^2)^{5/2}} \\ &= \frac{p}{4\pi\epsilon_0}\frac{2z^2-y^2}{r^5} \end{aligned}$$

B. POLAR COORDINATES

In polar ($r-\theta$) coordinates, the radial and tangential components of the field are as follows :

$$\begin{aligned} E_r &= -\frac{\partial\phi}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3} \\ E_\theta &= -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3} \end{aligned}$$

GENERAL EXPRESSION

A respresentation independent form for the dipole field can be obtained from the above

We have

$$\begin{aligned} \vec{E} &= E_r\hat{r} + E_\theta\hat{\theta} \\ &= \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right] \end{aligned}$$

Using $\vec{p} = p\cos\theta\hat{r} - p\sin\theta\hat{\theta}$, we get

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}\left[(3\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

This form does not depend on any particular coordinate system. Note that, at large distances, the dipole field decreases with distance as $1/r^3$ where as monopole field (i.e. field due to a point charge) decreases as $1/r^2$

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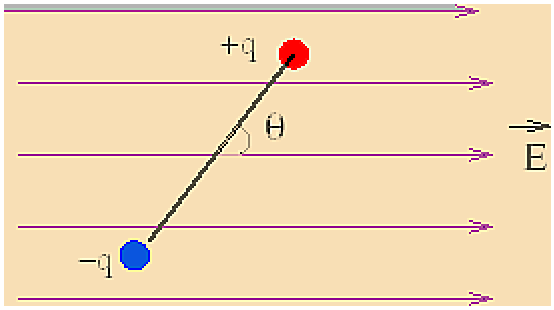
Dipole in a uniform Electric Field

The net force on the dipole is zero. There is a net torque acting on the dipole. If a is the length of the dipole, the torque is



$$\tau = (qE) \times a\sin\theta = pE\sin\theta$$

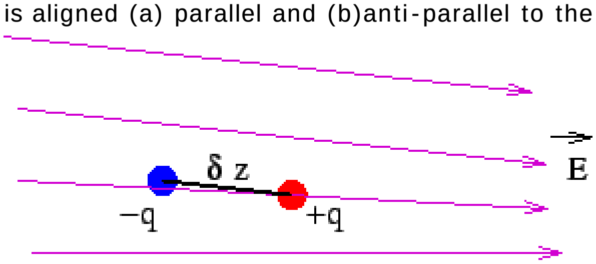
Expressing in vector form,

$$\vec{\tau} = \vec{p} \times \vec{E}$$

If $\theta = 0^\circ$ or 180°, (i.e. when the dipole is aligned parallel or antiparallel to the field) the torque vanishes and the dipole is in equilibrium. The equilibrium is stable if $\theta = 0^\circ$ and unstable if $\theta = 180^\circ$.

Example 16

The net electric force on a dipole is zero only if the field is uniform. In a non-uniform field, the dipole experiences a net force. Consider a dipole consisting of charges $\pm q$ separated by a distance d in an electric field $E_0(1 + \alpha z)\hat{k}$, where E_0 and α are constants. Determine the net force on the dipole when the dipole is aligned (a) parallel and (b)anti-parallel to the field.



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Solution :

Consider the dipole shown above, where the charges are separated by a distance δz, so that the dipole moment is $p = q\delta z$. Let the field at the charge $-q$ be E and that at be $E + \delta E$. The force on the charge $-q$ is $-qE$ and on $+q$ is $q(E + \delta E)$. There is a net force $q\delta E$ to the right (z-direction). We can write the force as

$$\text{Force} = q\delta E = q\delta z\frac{\delta E}{\delta z} = p\frac{dE}{dz}$$

where the last equality is valid for an ideal dipole for which $\delta z \to 0$.
In our case $p = qd$ and

$$\frac{dE}{dz} = E_0\alpha$$

so that the net force is $qdE_0\alpha$.

Work done in turning a dipole from equilibrium

If the dipole is twisted by an angle θ from its stable equilibrium position, work has to be done by the external agency

$$\begin{aligned} W = \int_0^\theta \tau d\theta &= \int_0^\theta pE\sin\theta d\theta \\ &= pE(1-\cos\theta) \end{aligned}$$

This work becomes the potential energy of the dipole in this position.

Exercise 4

An electric dipole consisting of two charges $\pm 3.2 \times 10^{-19}$ C separated by a distance of 2×10^{-9} m is in an equilibrium position in a uniform electric field of strength 5×10^5 N/C. Calculate the work done in rotating the dipole to a position in which the dipole is perpendicular to the field.

(Ans. 3.2×10^{-32} J)

Energy of a Dipole

To calculate energy of a dipole oriented at an angle θ in the electric field, we have to add to the work done above, the energy of the dipole in the equilibrium position. This is equal to the work done in bringing the dipole from infinity to the equilibrium position. The dipole may be aligned in the direction of the field at infinity without any cost of energy. We may now displace the dipole parallel to the field to bring to the equilibrium position. As the negative charge is displaced along the field by an additional distance a, the work done is $-qEa = -pE$, which is the potential energy of the dipole in equilibrium.

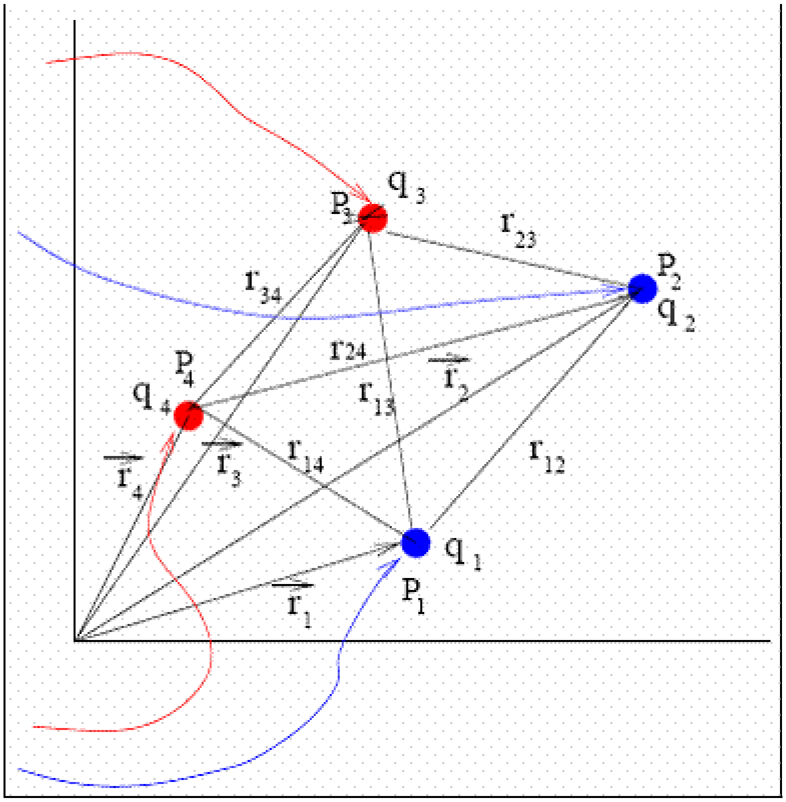
The potential energy of the dipole at position θ is

$$\mathcal{E} = -pE + pE(1-\cos\theta) = -pE\cos\theta = -\vec{p}\cdot\vec{E}$$

The energy is positive if θ is acute and is negative if θ is obtuse.

Potential Energy of a System of Charges

Assume all charges to be initially at infinity. We assemble the charges by bringing the charges one by one and fix them in their positions. There is no energy cost in bringing the first charge q_1 and putting it at P_1, as there is no force field. Thus $W_1 = 0$.



We now bring the second charge and take it to point P $_2$. Since this charge moves in the potential field of the first charge, the work done in bringing this charge is

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \frac{q_1}{r_{12}} = q_2 \phi_1(P_2)$$

where $\phi_1(P_2)$ is the potential at P $_2$ due to the charge at P $_1$. The third charge q_3 is to be brought to P $_3$ under the force exerted by bth q_1 and q_2 and is

$$W_3 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right) = q_3\left(\phi_1(P_3) + \phi_2(P_3)\right)$$

and so on.

The work done in assembling N charges $q_1, q_2, \ldots, q_N$, located respectively at $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_N}$ is

$$\begin{aligned} W &= W_1 + W_2 + \ldots + W_N \\ &= \frac{1}{4\pi\epsilon_0} \sum_{i<j} \sum_{j=2}^{N} \frac{q_i q_j}{r_{ij}} \\ &= \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{q_i q_j}{r_{ij}} \end{aligned}$$

The extra factor of $\frac{1}{2}$ in the last expression is to ensure that each pair (i, j) is counted only once. The sum excludes the terms $i = j$. Since the potential at the i- th position due to all other charges is

$$\phi(r_i) = \frac{1}{4\pi\epsilon_0}\sum_{j\neq i}\frac{q_i}{r_{ij}}$$

we get

$$W = \frac{1}{2}\sum_{i=1}^{N} q_i\phi(i)$$

Energy of a continuous charge distribution

If $\rho(\vec{r})$ is the density of charge distribution at $\vec{r}$, we can generalize the above result

$$\begin{aligned} W &= \frac{1}{8\pi\epsilon_0}\int\int\frac{\rho(\vec{r})\rho(\vec{r'})}{\mid \vec{r}-\vec{r}' \mid}d\tau d\tau' \\ &= \frac{1}{2}\int \rho(\vec{r})\phi(\vec{r})d\tau \end{aligned}$$

(In case of a line charge or a surface charge distribution, the integration is over the appropriate dimension). Since the integral is over the charge distribution, it may be extended over all space by defining the charge density to be zero outside the distribution, so that the contribution to the integral comes only from the region of space where the charge density is non-zero. Writing

$$W = \int_{\text{allspace}} \rho(\vec{r})\phi(\vec{r})d\tau$$

From the differential form of Gauss's law, we have

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

With this

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}}(\vec{\nabla}\cdot\vec{E})\phi d\tau$$

On using the vector identity

$$\vec{\nabla}\cdot(\phi\vec{E}) = \phi\vec{\nabla}\cdot\vec{E} + \vec{E}\cdot\nabla\phi$$

we get, using $\vec{E} = -\nabla\phi$,

$$W = \frac{\epsilon_0}{2}\int_{\text{surface}}\vec{\nabla}\cdot(\phi\vec{E})d\tau + \frac{\epsilon_0}{2}\int_{\text{all space}}\mid E\mid^2 d\tau$$

The first integral can be converted to a surface integral by using divergence theorem and the surface can be taken at infinite distances, where the electric field is zero. As a result the first integral vanishes and we have

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}}\mid E\mid^2 d\tau$$

Recap

In this lecture you have learnt the following

- An electric dipole consist of a pair of equal and opposite charges separated by a small distance. The dipolemoment is a vector whose magnitude is equal to product of the charge with the distance and has a direction from the negative charge to the positive charge.

- At large distances, the electric field due to an electric dipole varies as the inverse cube of distance from the dipole.

- A dipole does not experience a force in a uniform electric field. However, it experiences a torque in such a field. In an inhomogeneous field, the dipole experiences a force.

- The potential energy of a dipole in an electric field was calculated. The method was extended to compute the potential energy of a general charge distribution. Expressions for both discrete and continuous charge distribution were obtained.