Module 1 : A Crash Course in Vectors
Lecture 5 : Curl of a Vector - Stoke's Theorem

Objectives

In this lecture you will learn the following

- Curl of a Vector field
- Expression for curl in cartesian cylindrical and spherical coordinate
- Dirac and Function

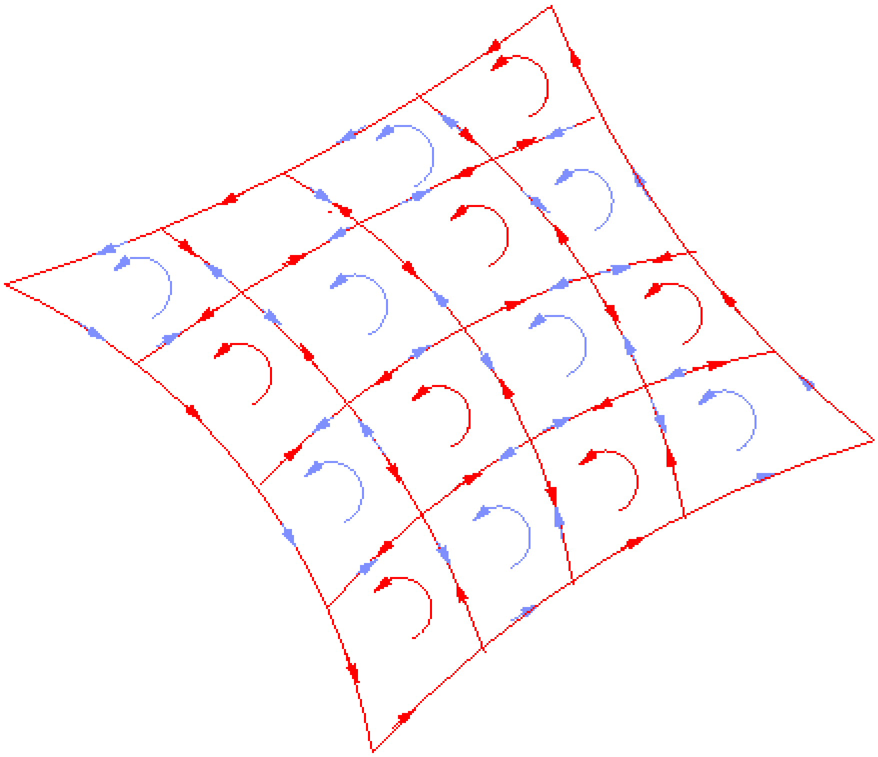
Curl of a Vector - Stoke's Theorem

We have seen that the line integral of a vector field $\int \vec{F} \cdot \vec{dl}$ is essentially a sum of the component of $\vec{F}$ along the curve. If the line integral is taken over a closed path, we represent it as $\oint \vec{F} \cdot \vec{dl}$. If the vector field is consevative, i.e., if there exists a scalar function V such that one can write $\vec{F}$ as ∇V, the contour integral is zero. In other cases, it is, in general, non-zero.

Consider a contour C enclosing a surface S. We may split the contour into a large number of elementary surface areas defined by a mesh of closed contours.

Since adjacent contours are traversed in opposite directions, the only non-vanishing contribution to the integral comes from the boundary of the contour C. If the surface area enclosed by the $i-th$ cell is ΔS_i, then

$$\begin{aligned} \oint_C &= \sum_i \int_{C_i} \vec{F} \cdot \vec{dl} \\ &= \sum_i \frac{\int_{C_i} \vec{F} \cdot \vec{dl}}{\Delta S_i} \Delta S_i \end{aligned}$$



We define the quantity

$$\lim_{\Delta S_i \to 0} \frac{\int_{C_i} \vec{F} \cdot \vec{dl}}{\Delta S_i} \hat{n}_i$$

as the curl of the vector $\vec{F}$ at a point P which lies on the surface ΔS_i. Since the area ΔS_i is infinitisimal it is a point relationship. The direction of $\hat{n}_i$ is, as usual, along the outward normal to the area element ΔS_i. For instance, the x-component of the curl is given by

$$(\text{curl})_x = \lim_{\Delta y, \Delta z \to 0} \frac{\int_{C_i} \vec{F} \cdot \vec{dl}}{\Delta y \Delta z} \hat{n}_i$$

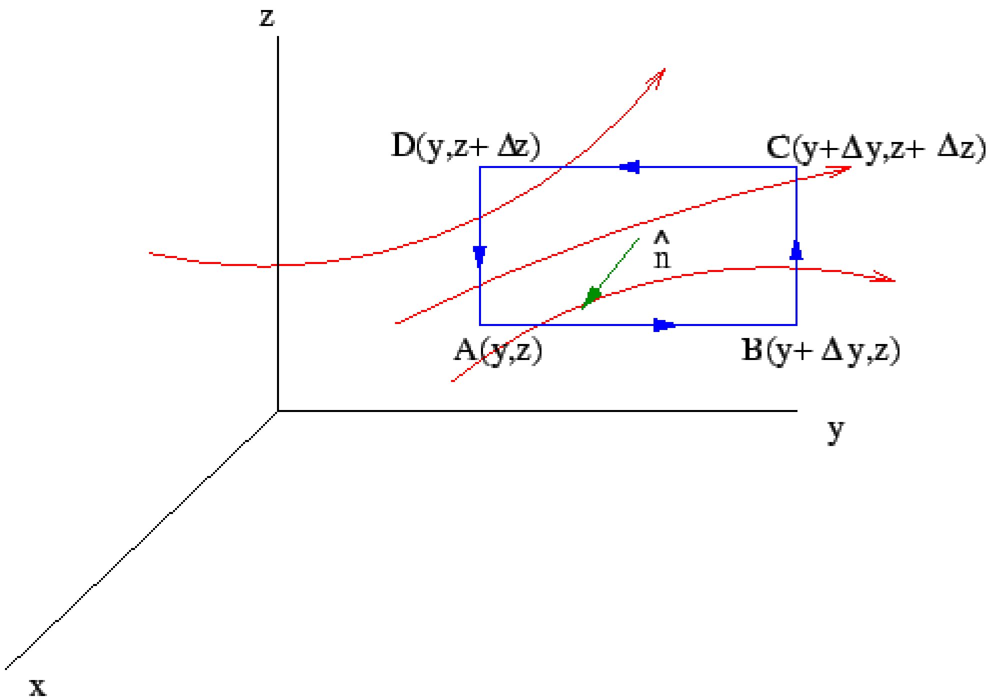
Thus

$$\boxed{\oint_C \vec{F} \cdot \vec{dl} = \int_S \text{curl} \vec{F} \cdot dS}$$

This is **Stoke's Theorem** which relates the surface integral of a curl of a vector to the line integral of the vector itself. The direction of $\vec{dl}$ and $\vec{dS}$ are fixed by the right hand rule, i.e. when the fingers of the right hand are curled to point in the diurection of $\vec{dl}$, the thumb points in the direction of $\vec{dS}$.

Curl in Cartesian Coordinates :

We will obtain an expression for the curl in the cartesian coordinates. Let us consider a rectangular contour ABCD in the y-z plane having dimensions $\Delta y \times \Delta z$. The rectangle is oriented with its edges parallel to the axes and one of the corners is located at (y, z). We will calculate the line integral of a vector field $\vec{F}$ along this contour. We assume the field to vary slowly over the length (or the bredth) so that we may retain only the first term in a Taylor expansion in computing the field variation.



Contribution to the line integral from the two sides AB and CD are computed as follows.

On AB : $\int \vec{F} \cdot \vec{dl} = \int \vec{F} \cdot \hat{\jmath} dy = \int F_y dy$

ON CD : $\int \vec{F} \cdot \vec{dl} = \int \vec{F} \cdot (-\hat{\jmath}) dy = \int F_y dy$

Using Taylor expansion (retaining only the first order term), we can write

$$F_y \mid_{CD} = F_y \mid_{AB} + \frac{\partial F_y}{\partial z} \Delta z$$

Thus the line integral from the pair of sides AB and CD is

$$-\int \frac{\partial F_y}{\partial z} \Delta z dy \approx -\frac{\partial F_y}{\partial z} \Delta z \Delta y$$

In a similar way one can calculate the contributions from the sides BC and DA and show it to be

$$\int \frac{\partial F_z}{\partial y} \Delta y dz \approx \frac{\partial F_z}{\partial y} \Delta y \Delta z$$

Adding up we get

$$(\text{curl } F)_x = \lim_{\Delta y, \Delta z \to 0} \frac{\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) \Delta y \Delta z}{\Delta y \Delta z} = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$$

In a very similar way, one can obtain expressions for the y and z components

$$\begin{aligned} (\text{curl } F)_y &= \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ (\text{curl } F)_z &= \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{aligned}$$

One can write the expression for the curl of $\vec{F}$ by using the del operator as

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Expression for curl in Cylidrical and Spherical Coordinates :

In the cylindrical coordinates the curl is given by

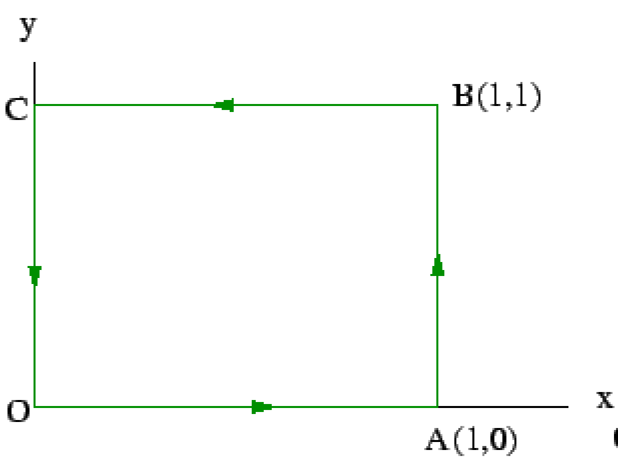
$$\begin{aligned} \nabla \times \vec{F} &= \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{\rho} \\ &+ \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\hat{\theta} + \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\theta) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \theta}\right)\hat{k} \end{aligned}$$

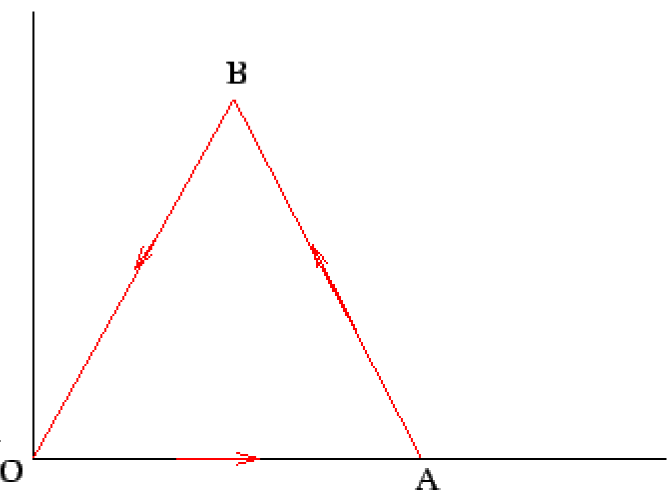
In the spherical coordinates the corresponding expression for the curl is

$$\begin{aligned} \nabla \times \vec{F} &= \frac{1}{r \sin\theta}\left(\frac{\partial}{\partial \theta}(F_\phi \sin\theta) - \frac{\partial F_\theta}{\partial \phi}\right)\hat{r} \\ &+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r}(rF_\phi)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right)\hat{\phi} \end{aligned}$$

Example 21

Verify Stoke's theorem for $\vec{F} = x^2\hat{\imath} + 2x\hat{\jmath} + z^2\hat{k}$ for the rectangle shown below, defined by sides $x = 0, y = 0, x = 1$ and $y = 1$.





The line integrals along the four sides are

$$\begin{aligned} \oint \vec{F} \cdot \vec{dl} &= \int_0^1 F_x \,|_{y=0}\, dx + \int_0^1 F_y \,|_{x=0}\, dy + \int_1^0 F_x \,|_{y=1}\, dx + \int_0^1 F_y \,|_{x=1}\, dy \\ &= \int_0^1 x^2 dx + 0 + + \int_1^0 x^2 dx + \int_0^1 2dy \\ &= \frac{1}{3} + 0 - \frac{1}{3} + 2 = 2 \end{aligned}$$

Since the normal to the plane is along $\hat{k}$, we only need $z-$ component of $\nabla \times \vec{F}$ to calculate the surface integral.

It can be checked that

$$(\nabla \times \vec{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 2 - 0 = 2$$

Thus

$$\int_S (\nabla \times \vec{F}) \cdot \hat{k} dS = \int_0^1 \int_0^1 2dxdy = 2$$

which agrees with the line integral calculated.

Exercise 1

A vector field is given by $\vec{F} = -y\hat{\imath} + x\hat{\jmath} + x^2\hat{k}$. Calculate the line integral of the field along the triangular path shown above. Verify your result by Stoke's theorem.

(Ans. 1)
(Hint : To calculate the line integral along a straightline, you need the equation to the line. For instance, the equation to the line BO is $y = 2x$. Check that $\int_{AB} \vec{F} \cdot \vec{dl} = -\int ydx + \int xdy = 1$.)

Example 22

A vector field is given by $\vec{F} = -y\hat{\imath} + z\hat{\jmath} + x^2\hat{k}$. Calculate the line integral of the field along a circular path of radius R in the x-y plane with its centre at the origin. Verify Stoke's theorem by considering the circle to define (i) the plane of the circle and (ii) a cylinder of height $z = h$.

Solution :

The curl of $\vec{F}$ may be calculated as

$$\nabla \times \vec{F} = -\hat{\imath} + 2x\hat{\jmath} + \hat{k}$$

Because of symmetry, we use cylindrical (polar) coordinates. The transformations are $x = \rho\cos\theta, \ \ y = \rho\sin\theta, \ \ z = z$. The unit vectors are

$$\begin{aligned} \hat{\imath} &= \hat{\rho}\cos\theta - \hat{\theta}\sin\theta \\ \hat{\jmath} &= \hat{\rho}\sin\theta - \hat{\theta}\cos\theta \\ \hat{k} &= \hat{k} \end{aligned}$$

Substituting the above, the field $\vec{F}$ and its curl are given by

$$\begin{aligned} \vec{F} &= \hat{\rho}(-\rho\sin\theta\cos\theta + z\sin\theta) \\ &+ \hat{\theta}(\rho\sin^2\theta + z\cos\theta) + \hat{k}\rho^2\cos^2\theta \\ \nabla \times \vec{F} &= \hat{\rho}(-\cos\theta + 2\rho\cos\theta\sin\theta) \\ &+ \hat{\theta}(\sin\theta + 2\rho\cos^2\theta) + \hat{k} \end{aligned}$$

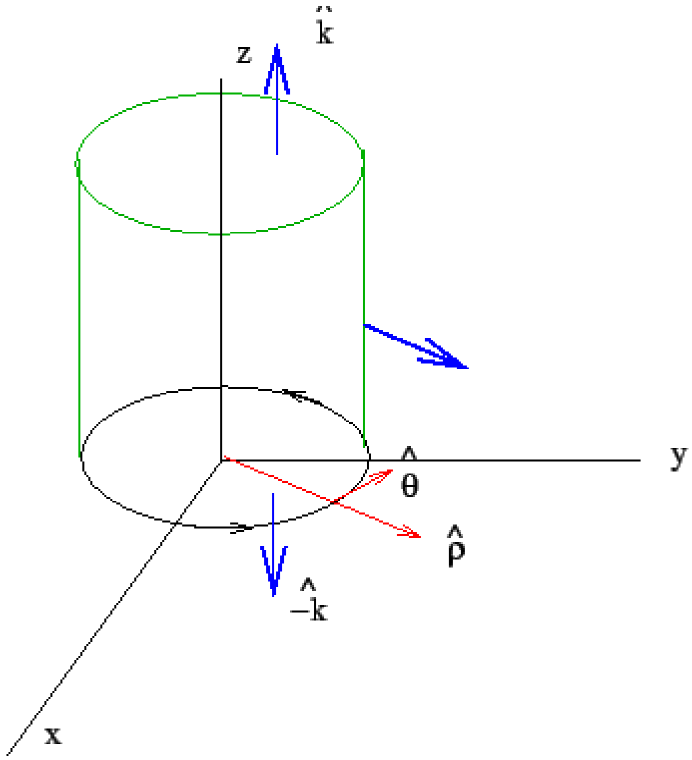
The line integral of $\vec{F}$ around the circular loop :

Since the line element is $\vec{dl} = Rd\theta\hat{\theta}$,

$$\oint \vec{F} \cdot \vec{dl} = \int_0^{2\pi} (R\sin^2\theta + z\cos\theta)Rd\theta$$

On the circle $z = 0$. The integral over $\sin^2\theta$ gives 1/2. Hence

$$\oint \vec{F}\cdot\vec{dl} = \pi R^2$$



Let us calculate the surface integral of the curl of the field over two surfaces bound by the circular curve.

(i) On the circular surface bound by the curve in the x-y plane, the outward normal is along $\hat{k}$ (right hand rule). Thus

$$\int_S (\nabla\times\vec{F})\cdot(\hat{k}dS) = \int_S dS = \pi R^2$$

(ii) For the cylindrical cup, we have two surfaces : the curved face of the cylinder on which $\hat{n} = \hat{\rho}$ and the top circular face on which $\hat{n} = \hat{k}$. The contribution from the top circular cap is πR^2, as before because the two caps only differ in their z values (the z-component of the curl is independent of z). The surface integral from the curved surface is (the area element is $Rd\theta dz\hat{\rho}$)

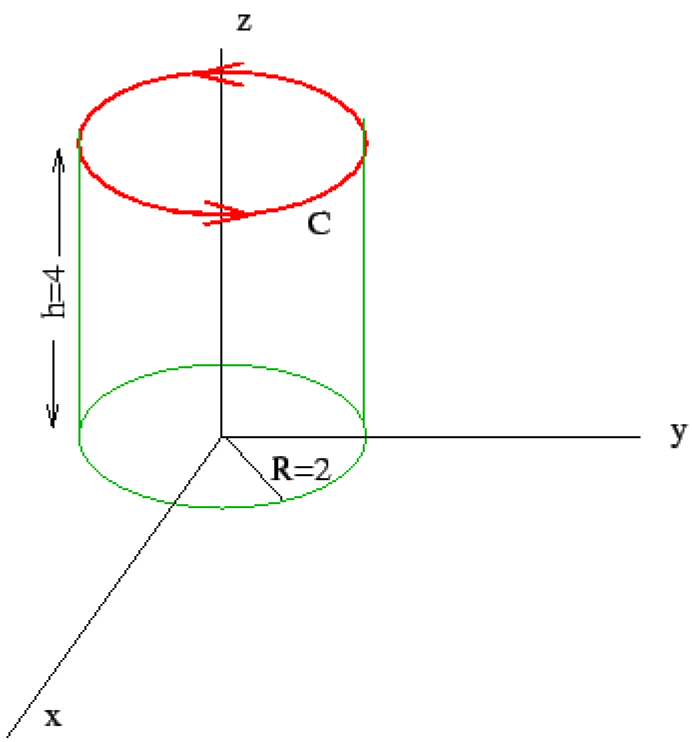
$$\int_0^{2\pi} Rd\theta \int_0^h dz(-R\sin\theta\cos\theta + z\cos\theta)$$

For both the terms of the above integral, the angle integration gives zero. Thus the net surface integral is πR^2, as expected.

Exercise 2

A vector field is given by $\vec{F} = k\rho^3 z\hat{\theta}$. Check the validity of the Stoke's theorm

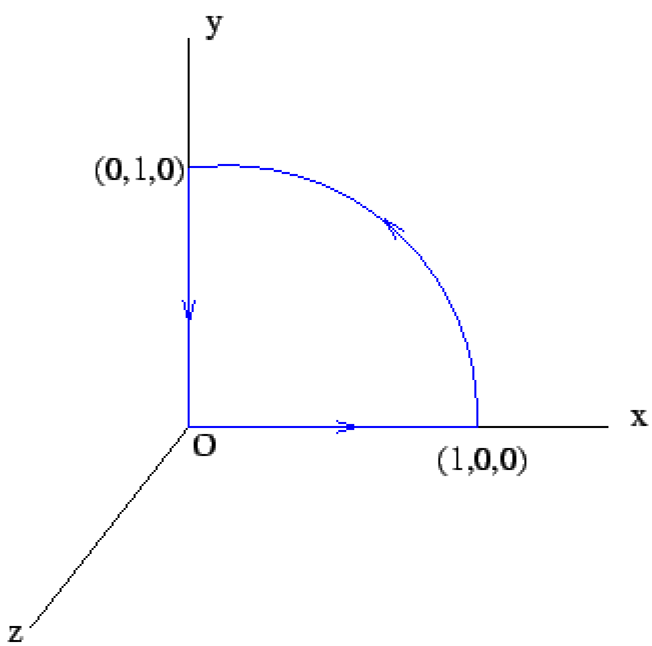
by calculating the line integral about the closed contour in the form of a circle at $z = 4$ and also calculating the surface integral of the open surface of the cylinder below it, as shown. (Hint : Express the curl in cylindrical coordinates and take care of the signs of the surface elements from the curved surface and the bottom cap. Ans. $64k\pi$)



Exercise 3

Let C be a closed curve in the x-y plane in the shape of a quadrant of a circle of radius R.

If $\vec{F} = \hat{i}y + \hat{j}z + \hat{k}x$. calculate the line integral of the field along the contour shown in a direction which is anticlockwise when looked from above the plane ($z > 0$). Take the surface of the quadrant enclosed by the curve as the open surface bounded by the curve and verify Stoke's theorem.



(Ans. $-\pi R^2/4$)

Example 23

A vector field is given by $\vec{F}(r,\theta,\phi) = f(r)\hat{\phi}$ where ϕ is the azimuthal angle variable of a spherical coordinate system. Calculate the line integral over a circle of radius R in the x-y plane centered at the origin. Consider an open surface in the form of a hemispherical bowl in the northern hemisphere bounded by the circle.

Solution :

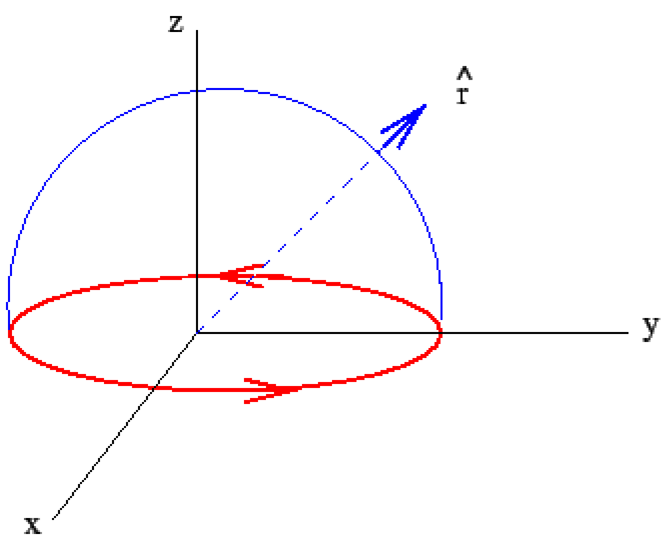
On the equatorial circle $\vec{dl} = Rd\phi\hat{\phi}$. Hence,

$$\oint \vec{F}\cdot\vec{dl} = \int_0^{2\pi} f(R)Rd\phi = 2\pi R f(R)$$

The expression for curl in spherical coordinates may be used to calculate the curl of $\vec{F}$. Since the field only has azimuthal component, the curl has radial and polar (θ) components.

$$\begin{aligned} \nabla\times\vec{F} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(f(r)\sin\theta)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rf(r))\hat{\theta} \\ &= \frac{f(r)\cos\theta}{r\sin\theta}\hat{r} - \left(\frac{f(r)}{r} + \frac{\partial f(r)}{\partial r}\right)\hat{\theta} \end{aligned}$$

The area element on the surface of the northern hemisphere is $R^2\sin\theta d\theta d\phi\hat{r}$.



Hence the surface integral is

$$\begin{aligned} \int_0^{2\pi} d\phi\int_0^{\pi/2}\frac{f(r)\cos\theta}{R\sin\theta}R^2\sin\theta d\theta &= 2\pi R f(R)\int_0^{\pi/2}\cos\theta d\theta \\ &= 2\pi R f(R) \end{aligned}$$

Exercise 4

Verify Stoke's theorem for a vector field $2z\hat{\imath} + 3x\hat{\jmath} + 5y\hat{k}$ where the contour is an equatorial circle of radius R and is anticlockwise when viewed from above and the surface is the hemisphere shown in the preceding example.

(Ans. $3R^2\pi$)

Laplacian :

Since gradient of a scalar field gives a vector field, we may compute the divergence of the resulting vector field to obtain yet another scalar field. The operator div(grad) $= \nabla \cdot \nabla$ is called the **Laplacian** and is written as ∇^2.
If V is a scalar, then,

$$\begin{aligned}
\nabla^2 V &= \nabla \cdot (\nabla V) \\
&= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{\imath}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right) \\
&= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}
\end{aligned}$$

Example 24

Calculate the Laplacian of $1/r = 1/\sqrt{x^2+y^2+z^2}$.

Solution :

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\frac{1}{\sqrt{x^2+y^2+z^2}} &= \frac{\partial}{\partial x}\frac{-x}{(x^2+y62+z^2)^{3/2}} \\
&= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}} \\
&= \frac{3x^2-r^2}{r^5}
\end{aligned}$$

Adding similar contributions from $\partial^2/\partial y^2$ and $\partial^2/partial z^2$, we get

$$\nabla^2\frac{1}{r} = \frac{3(x^2+y^2+z^2)-3r^2}{r^5} = \frac{3r62-3r^2}{r^5} = 0$$

Laplacian in cylindrical and spherical coordinates:

In cylindrical :

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

In spherical :

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Frequently the Laplacian of a vector field is used. It is simply a short hand notation for the componentwise Laplacian

$$\nabla^2\vec{F} = \hat{\imath}\nabla^2 F_x + \hat{\jmath}\nabla^2 F_y + \hat{k}\nabla^2 F_z$$

Exercise 5

Show that

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

Dirac- Delta Function :

In electromagnetism, we often come across use of a function known as Dirac- δ function. The peculiarity of the function is that though the value of the function is zero everywhere, other than at one point, the integral of the function over any region which includes this singular point is finite. We define

$$\delta(x-a) = 0 \text{ if } x \neq a$$

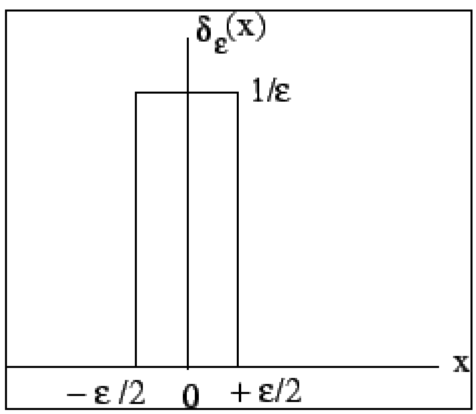
with

$$\int f(x)\delta(x-a)dx = f(a)$$

where $f(x)$ is any function that is continuous at $x = a$, provided that the range of integration includes the point $x = a$. Strictly speaking, $\delta(x-a)$ is not a function in the usual sense as Riemann integral of any function which is zero everywhere, excepting at discrete set of points should be zero. However, one can look at the δ function as a limit of a sequence of functions. For instance, if we define a function $\delta_\epsilon(x)$ such that

$$\begin{aligned} \delta_\epsilon(x) &= \frac{1}{\epsilon} \text{ for } -\frac{\epsilon}{2} < x < +\frac{\epsilon}{2} \\ &= 0 \text{ for } \mid x \mid > \frac{\epsilon}{2} \end{aligned}$$

Then $\delta(x)$ can be thought of as the limit of $\delta_\epsilon(x)$ as $\epsilon \longrightarrow 0$.



One can easily extend the definition to three dimensions

$$\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

which has the property

$$\int f(\vec{r})\delta(\vec{r} - \vec{a}) = f(\vec{a})$$

provided, of course, the range of integration includes the point $\vec{r} = \vec{a}$.

Example :

Evaluate $\int_0^5 \cos x\delta(x-\pi)dx$.

Solution :

The range of integration includes the point $x = \pi$ at which the argument of the delta function vanishes. Thus, the value of the integral is $\cos\pi = -1$.

Exercise :

Evaluate $\int \vec{r} \cdot (\vec{a} - \vec{r})\delta(\vec{r} - \vec{b})d^r$, where $\vec{a} = (1,2,3), \vec{b} = (3,2,1)$ and the integration is over a sphere of radius 1.5 centered at (2,2,2)(Ans. -4) A physical example is the volume density of charge in a region which contains a point charge q. The charge density is zero everywhere except at the point where the charge is located. However, the volume integral of the density in any region which includes this point is equal to q itself. Thus if q is located at the point $\vec{r} = \vec{a}$, we can write

$$\rho(\vec{r}) = q\delta(\vec{r} - \vec{a})$$

Example :

Show that $\nabla^2(1/r)$ is a delta function.

Solution :

As $r = \sqrt{x^2 + y^2 + z^2}$, we have

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

using this it is easy to show that

$$\frac{\partial^2}{\partial x^2}(\frac{1}{r}) = \frac{3x^2 - r^2}{r^5}$$

Thus

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\frac{1}{r}) = \frac{3(x^2 + y^2 + z^2) - 3r^2}{r^5} = 0$$

However, the above is not true at the origin as $1/r$ diverges at $r = 0$ and is not differentiable at that point. Interestingly, however, the integral of $\nabla^2(1/r)$ over any volume which includes the point $r = 0$ is not zero. As the value of the integrand is zero everywhere excepting at the origin, the point $r = 0$ has to be treated with care.
Consider an infinitisimally small sphere of radius r_0 with the centre at the origin. Using divergence theorem, we have,

$$\int_V \nabla^2\left(\frac{1}{r}\right) d^3r = \int_V \nabla \cdot \nabla\left(\frac{1}{r}\right) d^3r = \int_S \nabla\left(\frac{1}{r}i\right) \cdot dS$$

where the last integral is over the surface x', y', z' of the sphere defined above. As the gradient is taken at points on the surface for which $r \neq 0$, we may replace $\nabla(1/r)$ with $-1/r_0^2$ at all points on the surface. Thus the value of the integral is

$$-\frac{1}{r_0^2}\int_S dS = -\frac{1}{r_0^2} \cdot 4\pi r_0^2 = -4\pi$$

Hence

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$$

Recap

In this lecture you have learnt the following

- Curl of a vector field was defined.
- Stokes theorem was established. According to Stoke's theorem the surface integral of the curl of a vector through a surface is equal to the line integral of the field over any curve which binds the surface.
- Stokes theorem was verified by calculating the curl for several cases of vector field.
- Laplacian was defined and its expression in spherical polar and cylindrical coordinates was obtained.
- We defined a generalized function called Dirac- Delta function which has the property that it vanishes everywhere except at a point where its argument vanishes. Even so, the integral of the function over any region of space which includes the point at which the argument of the delta function vanishes, is, in general, non-zero.