Module 2 : Electrostatics

Lecture 7 : Electric Flux

Objectives

In this lecture you will learn the following

- Concept of flux and calculation of eletric flux throught simple geometrical objects

- Gauss's Law of electrostatics

- Applications of Gauss's Law to Calculate electric field due to a few symmetric charge distributions.

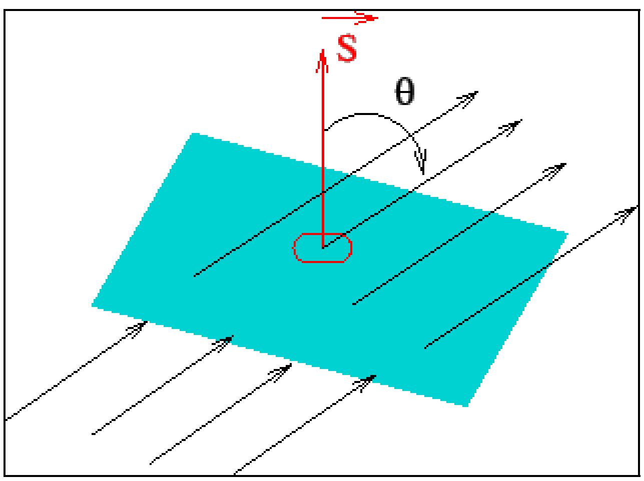
Electric Flux

The concept of *flux* is borrowed from flow of water through a surface. The amount of water flowing through a surface depends on the velocity of water, the area of the surface and the orientation of the surface with respect to the direction of velocity of water.
Though an area is generally considered as a scalar, an element of area may be considered to be a vector because

- It has magnitude (measured in m ^{2}).

- If the area is infinitisimally small, it can be considered to be in a plane. We can then associate a direction with it.

For instance, if the area element lies in the x-y plane, it can be considered to be directed along the z-direction. (Conventionally, the direction of the area is taken to be along the outward normal.)



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In the figure above, the length of the vector $\vec{S}$ is chosen to represent the area in some convenient unit and its direction is taken to be along the outward normal to the area.

We define the flux of the electric field through an area $d\vec{S}$ to be given by the scalar product

$d\phi = \vec{E}\cdot d\vec{S}$ If θ is the angle between the electric field and the area vector $d\phi = | E || dS | \cos\theta$

For an arbitrary surface S, the flux is obtainted by integrating over all the surface elements

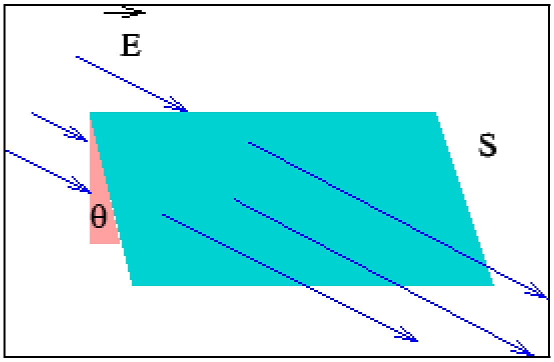
$$\phi = \int d\phi = \int_S \vec{E}\cdot d\vec{S}$$

θ

If the electric field is uniform, the angle is constant and we have

$$\phi = ES\cos\theta = E(S\cos\theta)$$

Thus the flux is equal to the product of magnitude of the electric field and the projection of area perpendicular to the field.



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Unit of flux is N-m ^{2}/C. Flux is positive if the field lines come out of the surface and is negative if they go into it.

Solid Angle :

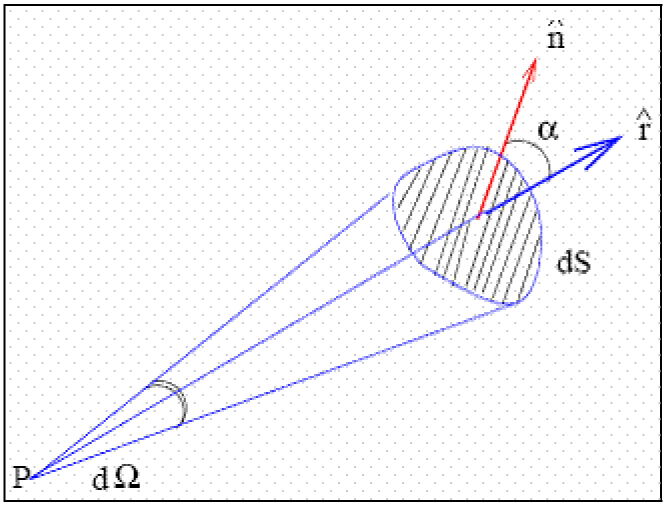
The concept of solid angle is a natural extension of a plane angle to three dimensions. Consider an area element dS at a distance r from a point P. Let be the unit vector along the outward normal to .

The element of the solid angle subtended by the area element at P is defined as

$$d\Omega = \frac{dS_\perp}{r^2}$$

where is the projection of along a direction perpendicular to . If is the angle between $\hat{r}$ and , then,

$$d\Omega = \frac{dS\cos\alpha}{r^2}$$



Solid angle is dimensionless. However, for practical reasons it is measured in terms of a unit called **steradian** (much like the way a planar angle is measured in terms of degrees).

The maximum possible value of solid angle is , which is the angle subtended by an area which encloses the point P completely.

Example

A right circular cone has a semi-vertical angle α. Calculate the solid angle at the apex P of the cone.

Solution :

The cap on the cone is a part of a sphere of radius R, the slant length of the cone. Using spherical polar coordinates, an area element on the cap is R^2 sin $\theta d\theta d\phi$, where θ is the polar angle and ϕ is the azimuthal angle. Here, goes from 0 to 2π while θ goes from 0 to α.

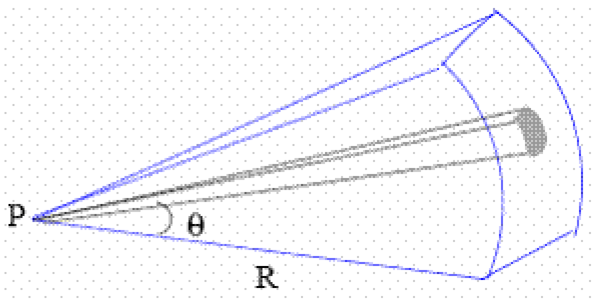
ϕ

Thus the area of the cap is

$$\begin{aligned} dA &= 2\pi R^2 \int_0^{\alpha} \sin\theta d\theta \\ &= 2\pi R^2 (1 - \cos\alpha) \end{aligned}$$

Thus the solid angle at P is

$$d\Omega = \frac{dA}{R^2} = 2\pi(1 - \cos\alpha)$$



Exercise

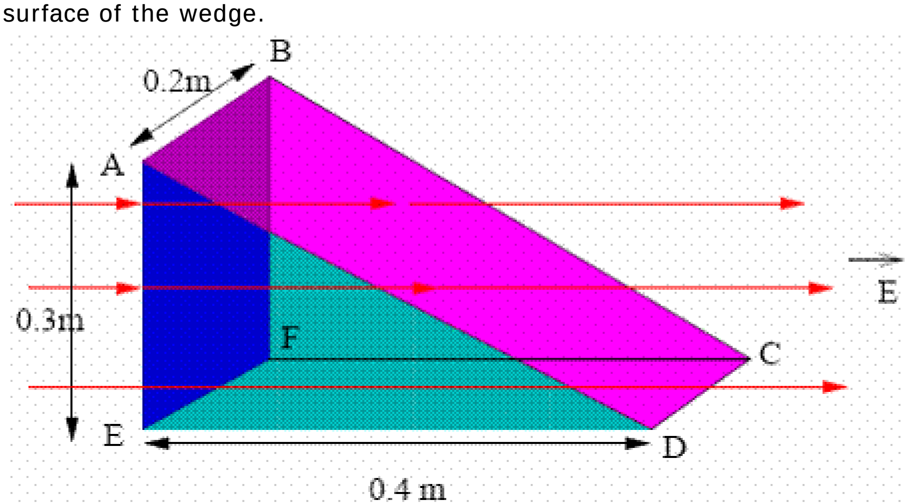
Calculate the solid angle subtended by an octant of a sphere at the centre of the sphere.

(Ans. $\pi/2$)

The flux per unit solid angle is known as the **intensity** .

Example 3

An wedge in the shape of a rectangular box is kept on a horizontal floor. The two triangular faces and the rectangular face ABFE are in the vertical plane. The electric field is horizontal, has a magnitude 8×10^4 N/C and enters the wedge through the face ABFE, as shown. Calculate the flux through each of the faces and through the entire surface of the wedge.



The outward normals to the triangular faces AED, BFC, as well as the normal to the base are perpendicular to $\vec{E}$. Hence the flux through each of these faces is zero. The vertical rectangular face ABFE has an area 0.06 m ^{2}. The outward normal to this face is perpendicular to the electric field. The flux is entering through this face and is negative. Thus flux through ABFE is

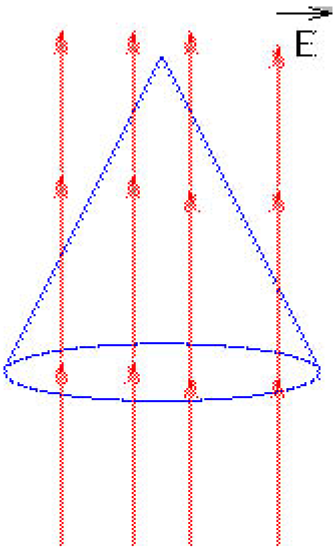
$$\phi_1 = -0.06 \times 8 \times 10^4 = -.48 \times 10^4 \ \ \mathrm{N-m^2/C}$$

To find the flux through the slanted face, we need the angle that the normal to this face makes with the horizontal electric field. Since the electric field is perpendicular to the side ABFE, this angle is equal to the angle between AE and AD, which is $\cos^{-1}(.3/.5)$. The area of the slanted face ABCD is 0.1 m ^{2}. Thus the flux through ABCD is $\phi_2 = 0.1 \times 8 \times 10^4 \times (.3/.5) = +0.48 \times 10^4 \ \ \mathrm{N-m^2/C}$

The flux through the entire surface of the wedge is $\phi_1 + \phi_2 = 0$

Example 4

Calculate the flux through the base of the cone of radius R.



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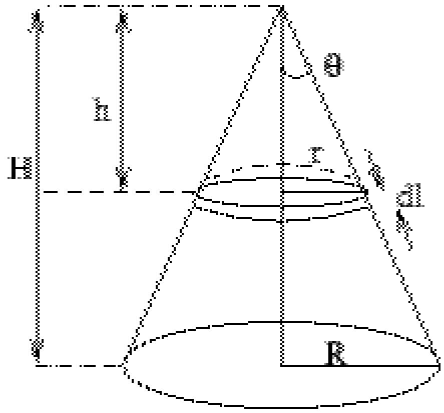
Solution :
The flux entering is perpendicular to the base. Since the outward normal to the circular base is in the opposite sense, the flux is negative and is equal to the product of the magnitude of the field and the area of the base, The flux, therefore is, $\pi R^2 E$.

Example 5

Calculate the flux coming out through the curved surface of the cone in the above example.

Solution :



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Consider a circular strip of radius r at a depth h from the apex of the cone. The angle between the electric field through the strip and the vector $\vec{dS}$ is $\pi - \theta$, where θ is the semi-angle of the cone. If dl is the length element along the slope, the area of the strip is $2\pi r dl$. Thus,

$$\vec{E} \cdot \vec{dS} = 2\pi r dl \mid E \mid \sin\theta$$

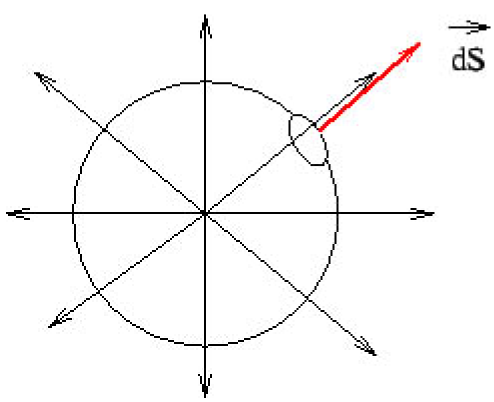
We have, $l = h/\cos\theta$, so that $dl = dh/\cos\theta$. Further, **r = h tan** θ Substituting, we get

$$\vec{E} \cdot \vec{dS} = 2\pi h \tan^2\theta \mid E \mid dh$$

Integrating from $h = 0$ to $h = H$, the height of the cone, the outward flux is $|e| H^2 \tan^2\theta = \pi R^2 |E|$.

Example 6

A charge Q is located at the center of a sphere of radius R. Calculate the flux going out through the surface of the sphere.



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By Coulomb's law, the field due to the charge Q is radial and is given on the surface of the sphere by,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{r}$$

The direction of the area vector $d\vec{S}$, is also radial at each point of the surface $d\vec{S} = dS\hat{r}$. The flux

$$\begin{aligned}\phi &= \int \vec{E}\cdot d\vec{S} \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\int dS\end{aligned}$$

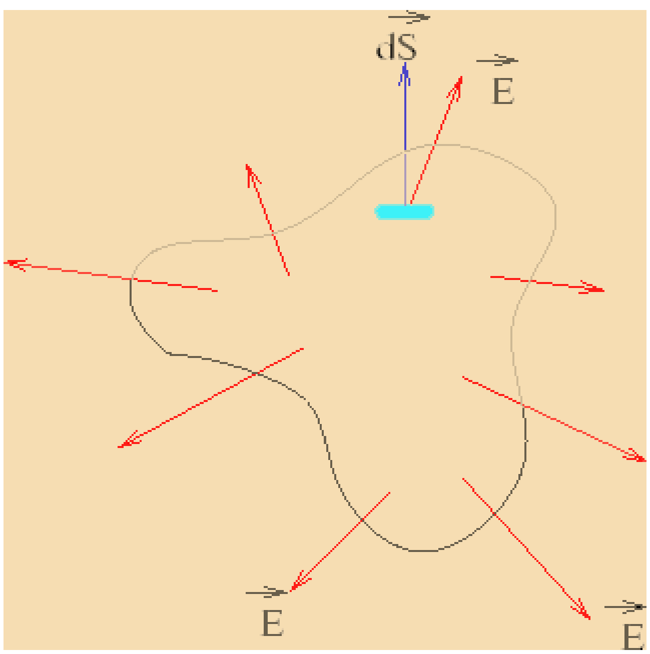
The integral over dS is equal to the surface area of the sphere, which is, $4\pi R^2$. Thus the flux out of the surface of the sphere is

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\cdot 4\pi R^2 = \frac{Q}{\epsilon_0}$$

GAUSS'S LAW - Integral form

The flux calculation done in Example 4 above is a general result for flux out of any closed surface, known as Gauss's law.

Total outward electric flux ϕ through a closed surface S is equal to $1/\epsilon_0$ times the charge enclosed by the volume defined by the surface S



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Mathematicaly, the surface integral of the electric field over any closed surface is equal to the net charge enclosed divided by ϵ_0

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q_{enclosed}}{\epsilon_0}$$

- The law is valid for arbitry shaped surface, real or imaginary.

- Its physical content is the same as that of Coulomb's law.

- In practice, it allows evaluation of electric field in many practical situations by forming imagined surfaceswhich exploit symmetry of the problem. Such surfaces are called *Gaussian surfaces* .

GAUSS'S LAW - Differential form

The integral form of Gauss's law can be converted to a differential form by using the divergence theorem. If V is the volume enclosed by the surface S,

$$\oint_S \vec{E} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{E} dv \qquad (A)$$

If ρ is the volume charge density,

$$Q = \int_V \rho dv \qquad (B)$$

Thus we have

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Applications of Gauss's Law

Field due to a uniformly charged sphere of radius with a charge λ

Gaussian surface is a cylinder of radius r and length L.

By symmetry, the field is radial. Gaussian surface is a concentric sphere of radius r. The outward normals to the Gaussian surface is parallel to the field at every point. Hence For ,

so that

The field outside the sphere is what it would be if all the charge is concentrated at the origin of the sphere. For , a fraction of the total charge is enclosed within the gaussian surface, so that

The field inside is

Exercise

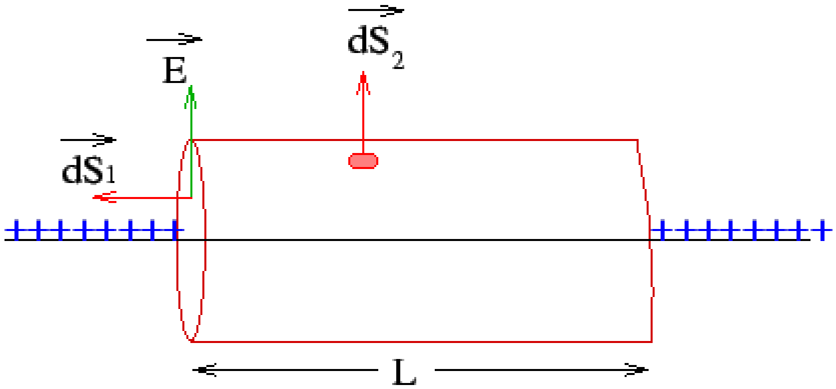
Find the electric field both inside and outside a spherical shell of radius carrying a uniform charge .

Example

Find the electric field inside a sphere of radius which carries a charge density where r is the distance from the origin and is a constant.

Solution :

By symmetry the field is radial. Take the gaussian surface to be a sphere of radius . The flus is . The charge enclosed by the gaussian surface is



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$$\oint \vec{E}\cdot d\vec{S} = |E|.2\pi r L$$
$$= \frac{Q}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$$

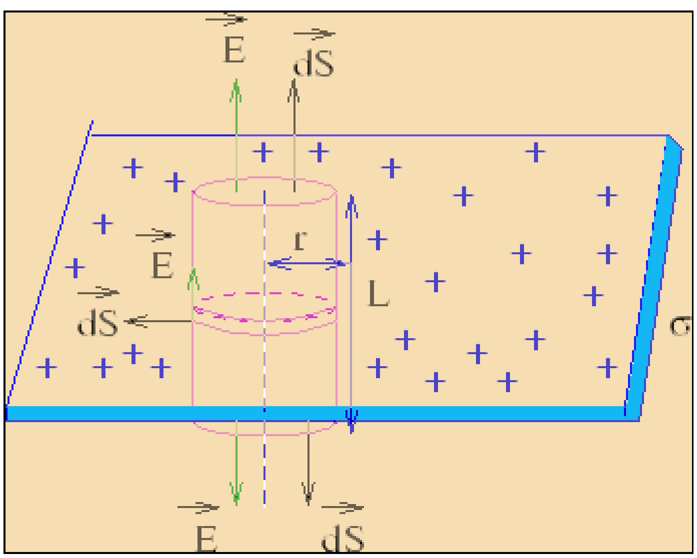
Thus

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\rho}$$

where $\hat{\rho}$ is a unit vector perpendicular to the line,directed outward for positive line charge and inward for negative line charge.

Field due to an infinite charged sheet with surface charge density σ

Choose a cylindrical *Gaussian pillbox* of height h (with $h/2$ above the sheet and $h/2$ below the sheet) and radius r.



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The amount of charge enclosed is area times the surface charge density, i.e., $Q = \pi r^2\sigma$. By symmetry, the field is directed perpendicular to the sheet, upward at points above the sheet and downward for points below. There is no contribution to the flux from the curved surface. The flux from the two end faces is $\pi r^2 \mid E \mid$ each, i.e. a total outward flux of $2\pi r^2 \mid E \mid$. Hence

$$2\pi r^2 \mid E \mid = \frac{Q}{\epsilon_0} = \frac{\pi r^2 \sigma}{\epsilon_0}$$

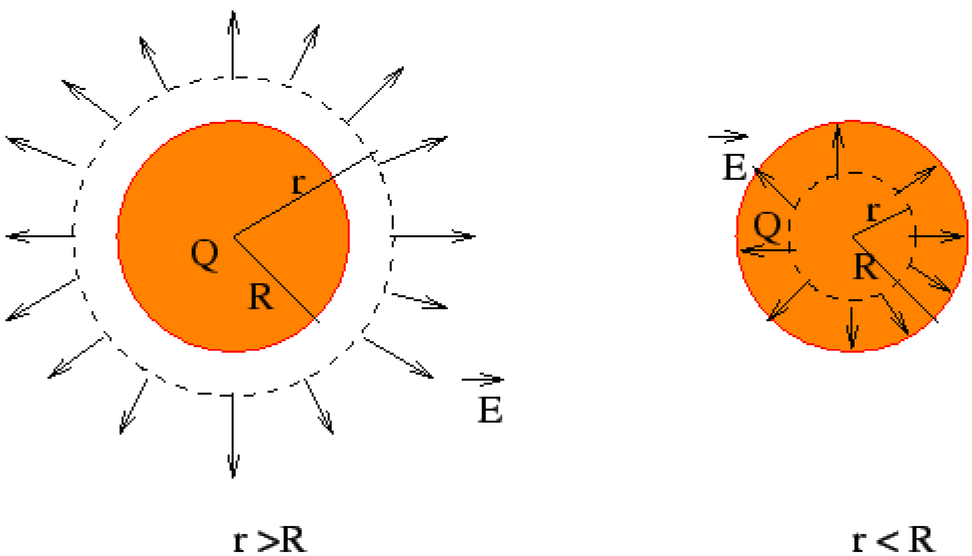
so that

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the sheet, directed upward for points above and downwards for points below (opposite, if the charge density is negative).

Field due to a uniformly charged sphere of radius R with a charge Q

By symmetry, the field is radial. Gaussian surface is a concentric sphere of radius r. The outward normals to the Gaussian surface is parallel to the field $\vec{E}$ at every point. Hence $\int \vec{E}\cdot d\vec{S} = 4\pi r^2 \mid E \mid$



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For $r > R$,

$$4\pi r^2 \mid E \mid = \frac{Q}{r^2}$$

so that

$$\vec{E} = \frac{Q}{4\pi r^2}\hat{r}$$

The field outside the sphere is what it would be if all the charge is concentrated at the origin of the sphere.

For $r < R$, a fraction r^3/R^3 of the total charge is enclosed within the gaussian surface, so that

$$4\pi r^2 \mid E \mid = \frac{1}{\epsilon_0}\frac{Qr^3}{R^3}$$

The field inside is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{r}$$

Exercise 1

Find the electric field in the region between two infinite parallel planes carrying charge densities $+\sigma$ and $-\sigma$.

Exercise 2

Find the electric field both inside and outside a spherical shell of radius R carrying a uniform charge Q.

Exercise 3

Find the electric field both inside and outside a long cylinder of radius R carrying a uniform volume charge density ρ.

(Hint : Take the gaussian surface to be a finite concentric cylinder of radius r (with $r < R$ and $r > R$), as shown)

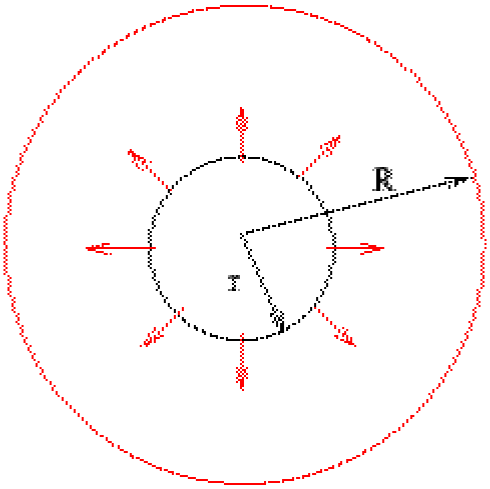
Example 7

Find the electric field inside a sphere of radius R which carries a charge density $\rho = kr$ where r is the distance from the origin and k is a constant.

Solution :

By symmetry the field is radial. Take the gaussian surface to be a sphere of radius R. The flus is

$$4\pi r^2 \mid E \mid$$



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The charge enclosed by the gaussian surface is

$$\begin{aligned} Q &= \int_0^r \rho(r) d^3r = \int_0^r \rho(r) 4\pi r^2 dr \\ &= 4\pi k \int_0^r r^3 dr \\ &= \pi k r^4 \end{aligned}$$

Thus

$$\vec{E} = \frac{1}{\epsilon_0} \frac{kr^2}{4} \hat{r}$$

(what is the dimension of k?)

Exercise 4

A very long cylinder carries a charge density $\rho = kr$, where r is the distance from the axis of the cylinder.

Find the electric field at a distance $r < R$. (Ans. $(1/3\epsilon_0) kr^2 \hat{r}$)

Exercise 5

A charge Q is located at the center of a cube of side a. Find the flux through any of the sides.

(Ans. $Q/6\epsilon_0$)

Example 8

Find the flux through the curved surface of a right circular cone of base radius R in an external electric field $\vec{E}$. The cone has no charge and the electric field is normal to the base.

Solution :

In Example 4, we calculated the flux through the base to be $-\pi R^2 E$. As the cone does not contain any charge, by Gauss's law, the flux through the curved surface must be $+\pi R^2 E$,

which is the result we obtained in Example 4.

Recap

In this lecture you have learnt the following

- Electric flux through a surface is surface integral of normalcomponent of electric field through the surface.
- Gauuss's of electrostatics states that the flux of electric field from any closed surface is proportional to the charge enclosed by the surface. Both the integral and differential form of Gauss's law were studied.
- Gauss's law helps us to deternmine electric field due to charge distributions having spatial symmetry. Fields due to distributions showing spherical and cylidrical symmetry were studied.
- Electric field due a charged sheet was obtained using Gauss's law.