Module 2 : Electrostatics
Lecture 10 : Poisson Equations

Objectives

In this lecture you will learn the following

- Poisson's equation and its formal solution
- Equipotential surface
- Capacitors - calculation of capacitance for parallel plate, spherical and cylindrical capacitors
- Work done in charging a capacitor

Poisson Equation

Differential form of Gauss's law,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

Using $\vec{E} = -\nabla\phi$,

$$\vec{\nabla}\vec{E} = -\nabla\cdot(\nabla\phi) = -\nabla^2\phi$$

so that

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

This is Poisson equation. In cartesian form,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -\frac{\rho}{\epsilon_0}$$

A formal soltion to Poisson equation can be written down by using the property Dirac - function discussed earlier. It can be seen that

$$\phi(r) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(r')}{\mid\vec{r}-\vec{r}'\mid}d\tau'$$

Operating with ∇_r^2 operator on both sides (The subscrpt r indicates that ∇ here is to be taken with respect to variable r)

$$\nabla_r^2\phi(r) = \frac{1}{4\pi\epsilon_0}\int\rho(r')\nabla_r^2\frac{1}{\mid\vec{r}-\vec{r}'\mid}d\tau'$$

We had shown that

$$\nabla^2\frac{1}{\mid r-r'\mid} = -4\pi\,\delta(\vec{r}-\vec{r}')$$

substituting which the expression follows.

Equipotential surface

Equipotential surfaces are defined as surfaces over which the potential is constant

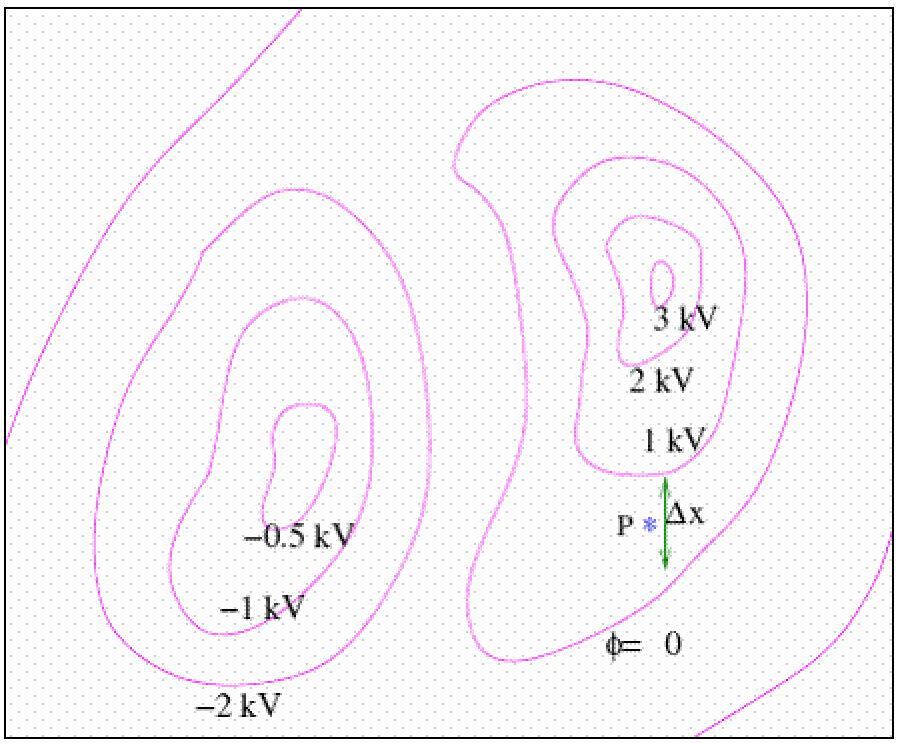
$$\phi(\vec{r}) = \text{constant}$$

At each point on the surface, the electric field is perpendicular to the surface since the electric field, being the gradient of potential, does not have component along a surface of constant potential.

- We have seen that any charge on a conductor must reside on its surface. These charges would move along the surface if there were a tangential component of the electric field. The electric field must therefore be along the normal to the surface of a conductor. The conductor surface is, therefore, an equipotential surface.

- Electric field lines are perpendicular to equipotential surfaces (or curves) and point in the direction from higher potential to lower potential.

- In the region where the electric field is strong, the equipotentials are closely packed as the gradient is large.



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The electric field strength at the point P may be found by finding the slope of the potential at the point P. If Δx is the distance between two equipotential curves close to P,

$$E = -\frac{\Delta\phi}{\Delta x}$$

where $\Delta\phi$ is the difference between the two equipotential curves near P.

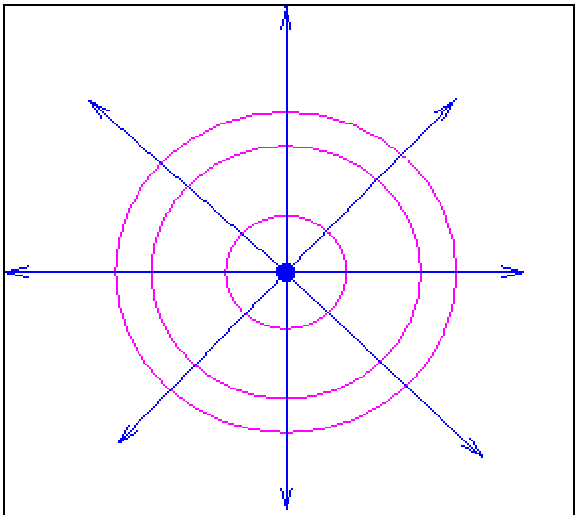
Example 17

Determine the equipotential surface for a point charge.

Solution :

Let the point charge q be located at the origin. The equation to the equipotential surface is given by

$$\phi(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+z^2}} = \phi_0 = \text{constant}$$



Equipotential surfaces (magenta) and field lines (blue) for a positive charge.

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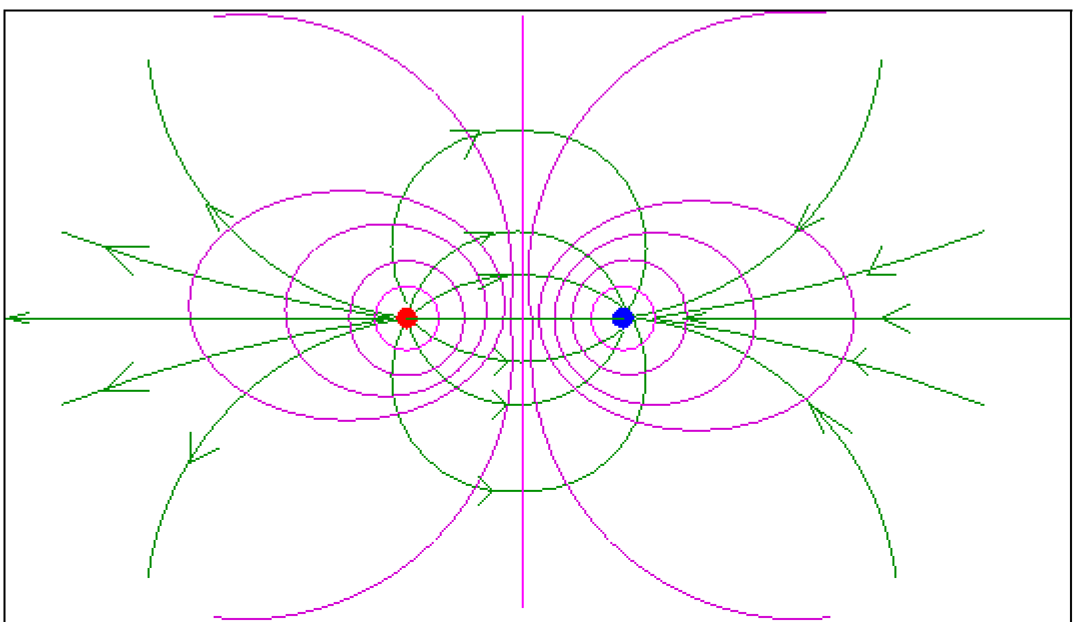
Thus the surfaces are concentric spheres with the origin (the location of the charge) as the centre and radii given by

$$R = \frac{q}{4\pi\epsilon_0\phi_0}$$

The equipotential surfaces of an electric dipole is shown below.



Electric Field and Equipotential lines for an electric dipole

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Example 18

Determine the equipotential surface of an infinite line charge carrying a positive charge density λ.

Solution :
Let the line charge be along the z- axis. The potential due to a line charge at a point P is given by

$$\phi(r) = -\frac{\lambda}{2\pi\epsilon_0}\ln r$$

where r is the distance of the point P from the line charge. Since the line charge along the z-axis, $r = \sqrt{x^2+y^2}$ so that

$$\phi(r) = -\frac{\lambda}{4\pi\epsilon_0}\ln(x^2+y^2)$$

The surface $\phi = \text{constant} = \phi_0$ is given by

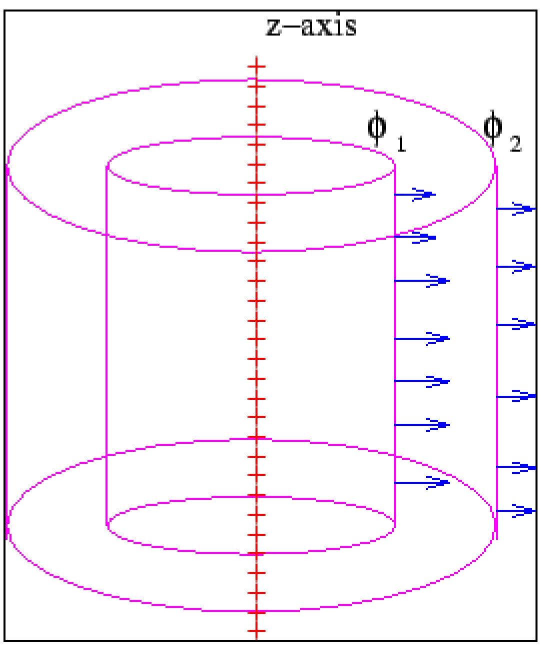
$$\ln(x^2+y^2) = -\frac{4\pi\epsilon_0\phi_o}{\lambda}$$

i.e.

$$x^2+y^2 = e^{-\frac{4\pi\epsilon_0\phi_0}{\lambda}}$$

which represent cylinders with axis along the z-axis with radii

$$r = e^{-\frac{2\pi\epsilon_0\phi_0}{\lambda}}$$



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As ϕ_0 increases, radius becomes smaller. Thus the cylinders are packed closer around the axis, showing that the field is stronger near the axis.

Exercise 1

Determine the equipotential surface of an infinite plane with charge density σ

Capacitance

Consider a spherical conductor of radius R carrying a charge Q. The potential of the sphere is given by

$$\phi = \frac{Q}{4\pi\epsilon_0 R}$$

The potential of the conductor is proportional to the charge it contains. This linear relationship is true in general, independent of the shape of the conductor,

$$Q = C\phi$$

The constant of proportionality C is called the capacitance of the conductor. For the conducting sphere the capacitance is $4\pi\epsilon_0 R$.

Unit of capacitance :

The M.K.S. unit of capacitance is Coulomb/Volt which is called a Farad. However, one Farad turns out to be very large capacitance (the capacitance of the Earth is approximately 700 micro-Farad). A more practical unit of capacitance is a micro-Farad (μF) or a pico- Farad (pF) :

$$\begin{aligned} 1\mu F &= 10^{-6} F \\ 1pF &= 10^{-12} F \end{aligned}$$

Capacitor :

A capacitor is essentially a device consisting of an arrangement of conductors for storing charges. As a consequence, it also stores electrostatic energy. The simplest capacitor consists of two conductors, one carrying a charge Q and the other a charge $-Q$. Let ϕ_1 be the potential of the first conductor and ϕ_2 that of the second. Since the conductor is an equipotential surface, the potential difference between the conductors $\phi_1 - \phi_2$ is also constant, and is given by

$$\phi_1 - \phi_2 = -\int_2^1 \vec{E} \cdot \vec{dl}$$

where the line integral is carried out along any path joining the two conductors. The electric field is proportional to the charge Q since if the charge on each conductor is multiplied by a constant α, the charge density and hence the electric field also gets multiplied by the same factor. Thus Q is proportional to the potential difference $\phi = \phi_1 - \phi_2$

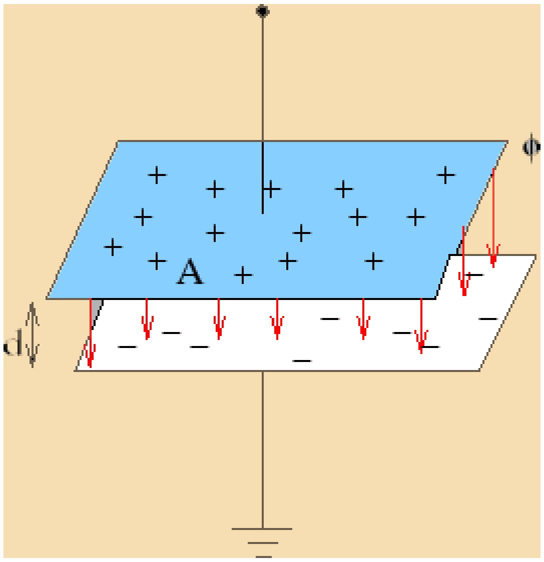
$$Q = C\phi$$

where C is the capacitance of the conductor pair.

A capacitor consisting of a single conductor (like the spherical conductor described above) may be considered to be one part of a conductor pair where the second conductor containing the opposite charge is at infinity.

Parallel Plate Capacitor :

A parallel plate capacitor consists of two parallel metal plates, each of area A separated by a distance d. A potential difference ϕ is maintained between the two plates. If the charge on the positive plate is $+Q$ and that on the negative plate is $-Q$, the electric field in the region between the two plates is $\sigma/\epsilon_0 = Q/A\epsilon_0$.



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The potential difference between the plates is

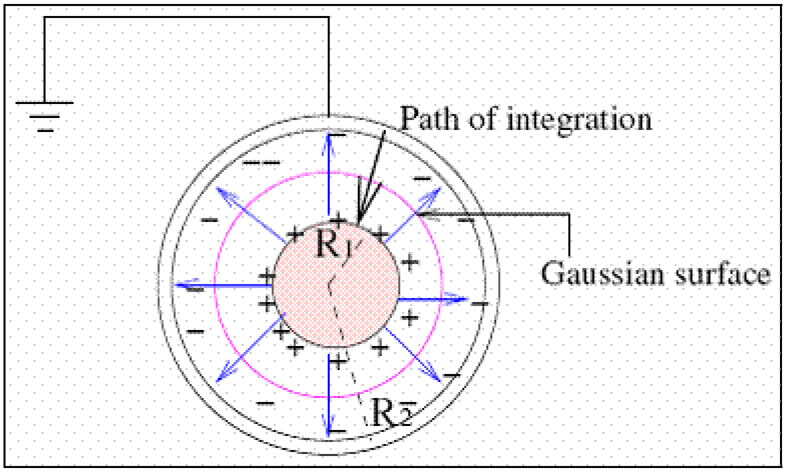
$$\phi = -\int_2^1 \vec{E}\cdot\vec{dl} = Ed = \frac{Qd}{A\epsilon_0}$$

The capacitance C is

$$C = \frac{Q}{\phi} = \epsilon_0\frac{A}{d}$$

Spherical Capacitor :

The spherical capacitor consists of two concentric spherical conducting shells of radii R_1 and R_2.



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The electric field at a distance r from the centre is calculated by using the Gaussian surface shown. The fileld is radial and is given by

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

The voltage drop between the shells is obtained by integrating the electric field along a radial path (the electric field being conservative, the path of integration is chosen as per our convenience) from the negative plate to the positrive plate.

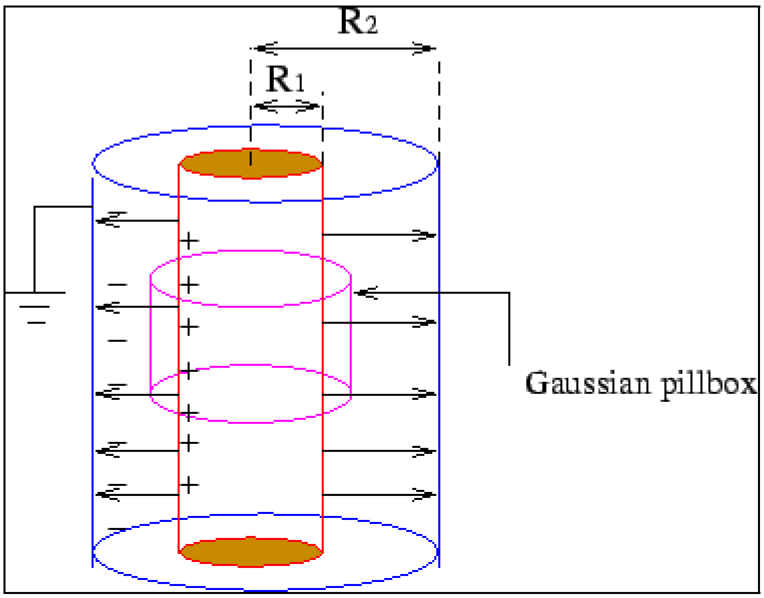
$$\begin{aligned}\phi &= -\int_{R_2}^{R_1} \vec{E}\cdot d\vec{r} = \frac{Q}{4\pi\epsilon_0}\int_{R_1}^{R_2}\frac{dr}{r^2}\\ &= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{R_1}-\frac{1}{R_2}\right]\end{aligned}$$

The capacitance is

$$C = \frac{Q}{\phi} = \frac{4\pi\epsilon_0 R_1 R_2}{R_2 - R_1}$$

Cylindrical Conductor :

A cylindrical capacitor consists of two long coaxial conducting cylinders of length L and radii R_1 and R_2. The electric field in the space between the cylinders may be calculated by Gauss Law, using a pillbox in the shape of a short coaxial cylinder of length $l \ll L$ and radius r. Neglecting edge effects, the field is in the radial direction and depends only on the distance r from the axis.



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The contribution to the flux from the end caps of the pillbox is zero as the field is along the surface. The field at a distance r is given by

$$\vec{E}\cdot d\vec{S} = 2\pi r l E = \frac{q}{\epsilon_0}$$

where q is the charged enclosed by the pillbox, which is given in terms of the surface charge density σ on the inner cylinder by

$$q = 2\pi R_1 l\sigma$$

The field at a distance r is given by

$$\vec{E} = \frac{R_1\sigma}{\epsilon_0 r}\hat{r}$$

The potential difference between the cylindrical conductors is

$$\begin{aligned}\phi &= -\int_{R_2}^{R_1} \vec{E}\cdot\vec{dr} = \frac{R_1\sigma}{\epsilon_0}\int_{R_1}^{R_2}\frac{dr}{r}\\ &= \frac{R_1\sigma}{\epsilon_0}\ln\frac{R_2}{R_1}\end{aligned}$$

Substituting

$$\sigma = \frac{Q}{2\pi R_1 L}$$

the capacitance is given by

$$C = \frac{Q}{\phi} = \frac{2\pi\epsilon_0 L}{\ln\frac{R_2}{R_1}}$$

Work Done in Charging a Capacitor :

Consider a parallel plate capacitor. The process of charging a capacitor consists of removing negative charges (electrons) from the positive plate and depositing them on the negative plate.

Suppose at a particular instant, the charge on the plates are $\pm q$, so that the potential difference between the plates is q/C. To transport an infinitisimal charge dq from the positive plate to the negative plate, the work done by an external agency is

$$dW = \frac{q}{C}dq$$

Total work done in charging the plates from $q = 0$ to $q = Q$ is

$$W = \frac{1}{C}\int_0^Q q\,dq = \frac{Q^2}{2C}$$

In terms of potential difference ϕ,

$$W = \frac{1}{2}C\phi^2$$

This is the amount of energy stored in the capacitor.

One can also get the same expression by using the expression for the energy of a charge distribution derived earlier

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$$

For a parallel plate capacitor $E = Q/A\epsilon_0$ within the volume Ad of the capacitor and zero outside. Hence

$$\begin{aligned}W &= \frac{\epsilon_0}{2}\left(\frac{Q}{A\epsilon_0}\right)^2\cdot Ad\\ &= \frac{Q^2 d}{2\epsilon_0 A} = \frac{Q^2}{2C}\end{aligned}$$

Exercise 1

Obtain an expression for the energy of a spherical capacitor of radii and containing charges .

R_1 R_2 $\pm Q$

[Ans. $\left(Q^2/8\pi\epsilon_0\right)\left[1/R_1 - 1/R_2\right]$]

Recap

In this lecture you have learnt the following

- Poisson's equation relates the potential to charge density. A formal solution to Poisson's equation was obtained.
- A equipotential surface is one on which the potential is constant. The electric field on an equipotential surface can only have component normal to the surface.
- The potential of a conductor is proportional to the charge it contains, the constant of proportionality is known as the capacitance of the conductor. A capacitor is a device to store charges and hence it also stores electrostatic energy.
- The capacitance for a parallel plate capacitor is proportional to the surface area and inversely proportional to the separation between its plates.
- Capacitance for spherical and cylindrical capacitors were calculated. The work done in charging a capacitor was also calculated.