

Module 3 : MAGNETIC FIELD

Lecture 16 : Ampere's Law

Objectives

In this lecture you will learn the following

- Establish Ampere' law in integral form.
- Calculate the magnetic field for certain current configuration using Ampere's law.
- Derive the differential form of Ampere's law.

Ampere's Law

Biot-Savart's law for magnetic field due to a current element is difficult to visualize physically as such elements cannot be isolated from the circuit which they are part of. Andre Ampere formulated a law based on Oersted's as well as his own experimental studies. Ampere's law states that *the line integral of magnetic field around any closed path equals μ_0 times the current which threads the surface bounded by such closed path.* . Mathematically,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enclosed}} \quad (1)$$

In spite of its apparent simplicity, Ampere's law can be used to calculate magnetic field of a current distribution in cases where a lot of information exists on the behaviour of \vec{B} . The field must have enough symmetry in space so as to enable us to express the left hand side of (1) in a functional form. The simplest application of Ampere' s law consists of applying the law to the case of an infinitely long straight and thin wire.

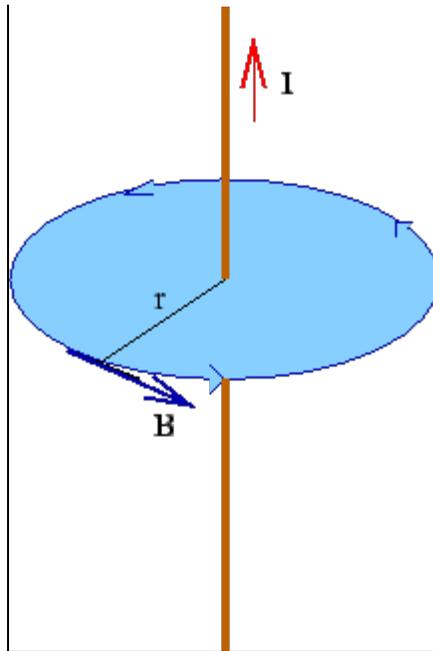
Example 12

By symmetry of the problem we know that the magnitude of the field at a point can depend only on the distance of the point from the wire. Further, the field is tangential to the circle of radius r , its direction being given by the right hand rule.

Thus the integral around the circle is $\oint \vec{B} \cdot d\vec{l} = B \oint dl = B \cdot 2\pi r$

Equating this to $\mu_0 I$, we get $B = \frac{\mu_0 I}{2\pi r}$

which is consistent with the result obtained from Biot-Savart's law.



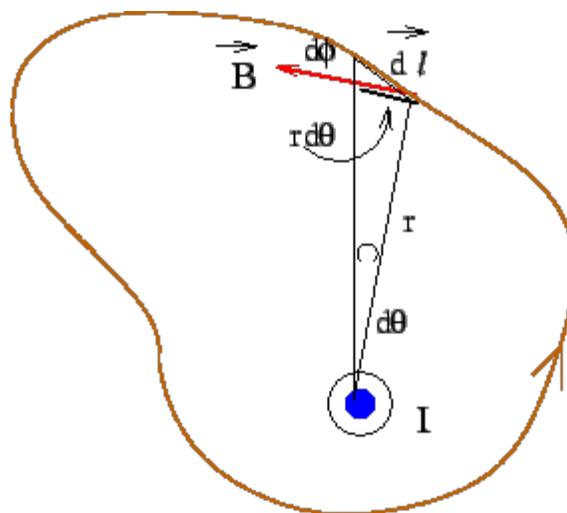
Let us see if the result above is consistent with a path which is not circular, as shown in the figure. The field at every element \vec{dl} of the path is perpendicular to \vec{r} . From geometry, it can be seen that

$$\vec{B} \cdot \vec{dl} = B dl \cos \phi = Br d\theta$$

Thus

$$\begin{aligned} \oint \vec{B} \cdot \vec{dl} &= \oint \frac{\mu_0 I}{2\pi r} r d\theta \\ &= \frac{\mu_0 I}{2\pi} \oint d\theta = \mu_0 I \end{aligned}$$

We need to specify the direction along which the path is traversed. This is done by Right Hand Rule. If we curl the fingers of our right hand along the path of integration, the direction along which the thumb points is the direction of current flow.

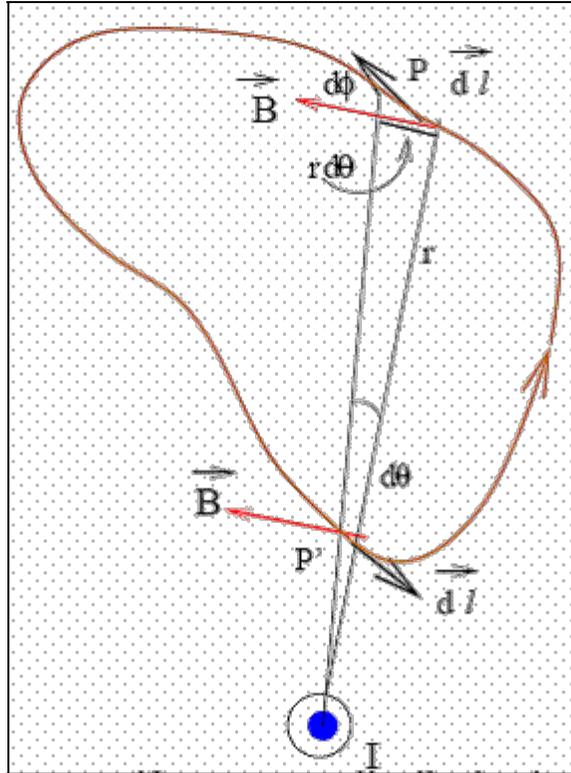


For the case where the path of integration lies totally outside the path of the current, for every element \vec{dl} at

P, there exists another element at P' for which $\vec{B} \cdot d\vec{l}$ has opposite sign. Thus when complete line integral is taken, the contributions from such pairs add to zero

$$\oint \vec{B} \cdot d\vec{l} = 0$$

Combining these, we get Ampere's law in the form of Eqn. (1)



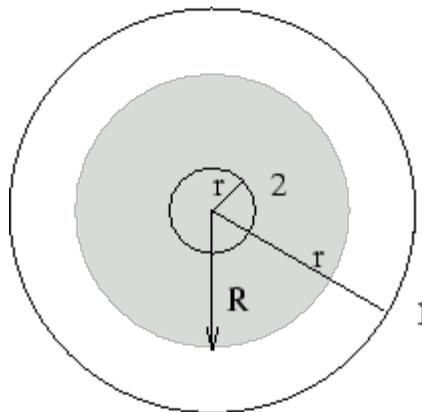
Example 13

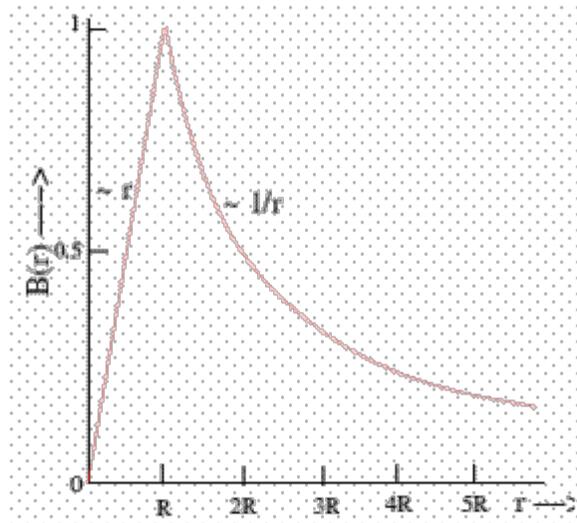
Calculate the field due to a uniform current distribution in an infinite wire of cross sectional radius R .

Solution :

Let the cross section of the wire be circular with a radius R . Take the current direction to be perpendicular to the page and coming out of it. Symmetry of the problem demands that the magnitude of the field at a point is dependent only on the distance of the point from the axis of the wire. Consider an amperian loop of radius r . As before we have

$$\oint \vec{B} \cdot d\vec{l} = 2\pi r B$$





If $r > R$ (as in loop 1), the entire current is enclosed by the loop. Hence $I_{\text{enclosed}} = I$ so that

$$B = \frac{\mu_0 I}{2\pi r}$$

If $r < R$ (loop 2), the current enclosed is proportional to the area, i.e.

$$I_{\text{enclosed}} = I \frac{\pi r^2}{\pi R^2} = I \frac{r^2}{R^2}$$

so that

$$B = \frac{\mu_0 I}{2\pi R^2} r$$

The field distribution with distance is as shown.

Exercise 1

A long wire of cross sectional radius R carries a current I . The current density varies as the square of the distance from the axis of the wire. Find the magnetic field for $r < R$ and for $r > R$.

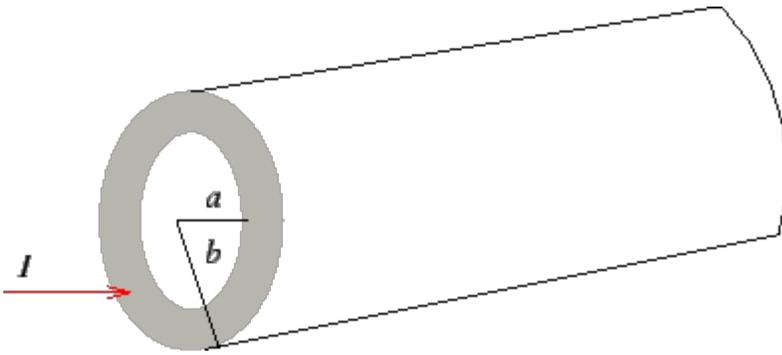
(Hint : First show that the current density $J = 2I r^2 / \pi R^4$ and obtain an expression for current enclosed

for $r < R$. Answer : $B = \mu_0 I / 2\pi r$ for $r > R$ and $B = \mu_0 I r^3 / 2\pi R^4$ for $r < R$.)

Exercise 2

A hollow cylindrical conductor of infinite length carries uniformly distributed current I from $a < r < b$.

Determine magnetic field for all r .

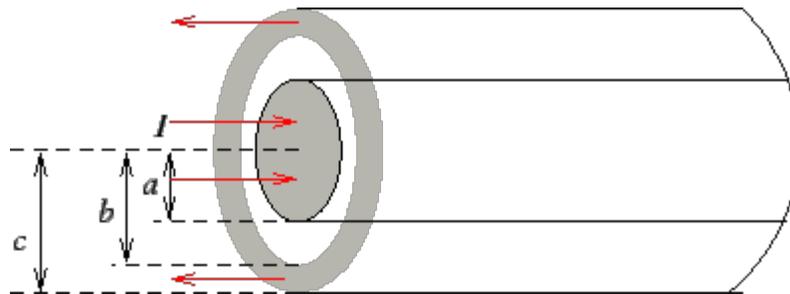


(Answer : Field is zero for $r < a$, $B = \frac{\mu_0 I r^2 - a^2}{2\pi r b^2 - a^2}$ for $a < r < b$ and $B = \mu_0 I / 2\pi r$ for $r > b$

.)

Exercise 3

A coaxial cable consists of a solid conductor of radius a with a concentric shell of inner radius b and outer radius c . The space between the solid conductor and the shell is supported by an insulating material.



A current I goes into the inner conductor and is returned by the outer shell. Assume the current densities to be uniform both in the shell and in the inner conductor. Calculate magnetic field everywhere.

(Ans. $B = \mu_0 I r / 2\pi b^2$ inside the inner conductor, $B = \mu_0 I / 2\pi r$ between the shell and the inner

conductor, $B = \frac{\mu_0 I c^2 - r^2}{2\pi r c^2 - b^2}$)

Exercise 4

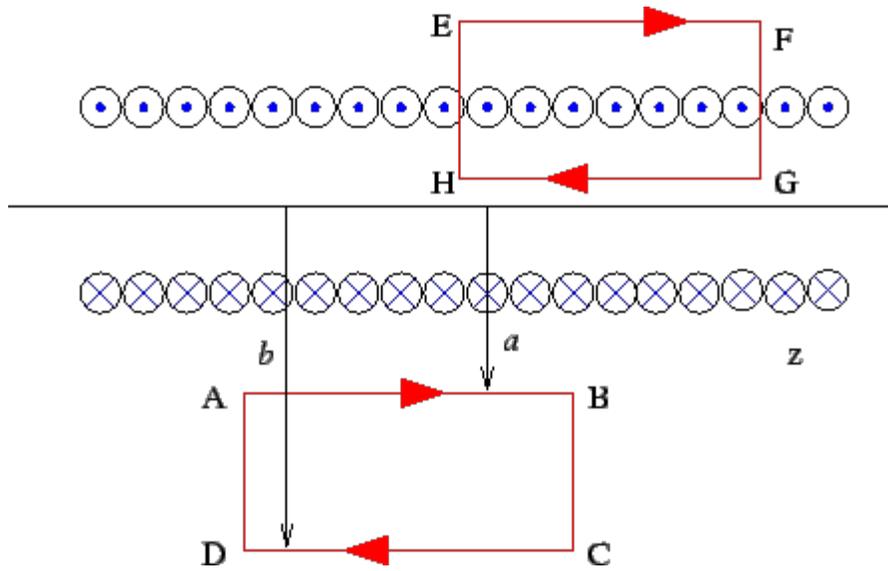
Determine the magnetic field in a cylindrical hole of radius a inside a cylindrical conductor of radius b . The cylinders are of infinite length and their axes are parallel, being separated by a distance d . The conductor carries a current I of uniform density.

(Hint : The problem is conveniently solved by imagining currents of equal and opposite densities flowing in the hole and using superposition principle to calculate the field. Answer : The field inside the hole is constant

$B = \mu_0 I d / 2\pi (b^2 - a^2)$)

Example 14

We take the solenoid to be closely wound so that each turn can be considered to be circular. We can prove that the field due to such a solenoid is entirely confined to its interior, i.e. the field outside is zero, To see this consider a rectangular amperian loop parallel to the axis of the solenoid.



Field everywhere on AB is constant and is $B(a)$. Likewise the field everywhere on CD is $B(b)$. By Right hand rule, the field on AB is directed along the loop while that on CD is oppositely directed. On the sides AD and BC, the magnetic field direction is perpendicular to the length element and hence $\vec{B} \cdot d\vec{l}$ is zero everywhere on these two sides. Thus

$$\oint \vec{B} \cdot d\vec{l} = L(B(a) - B(b))$$

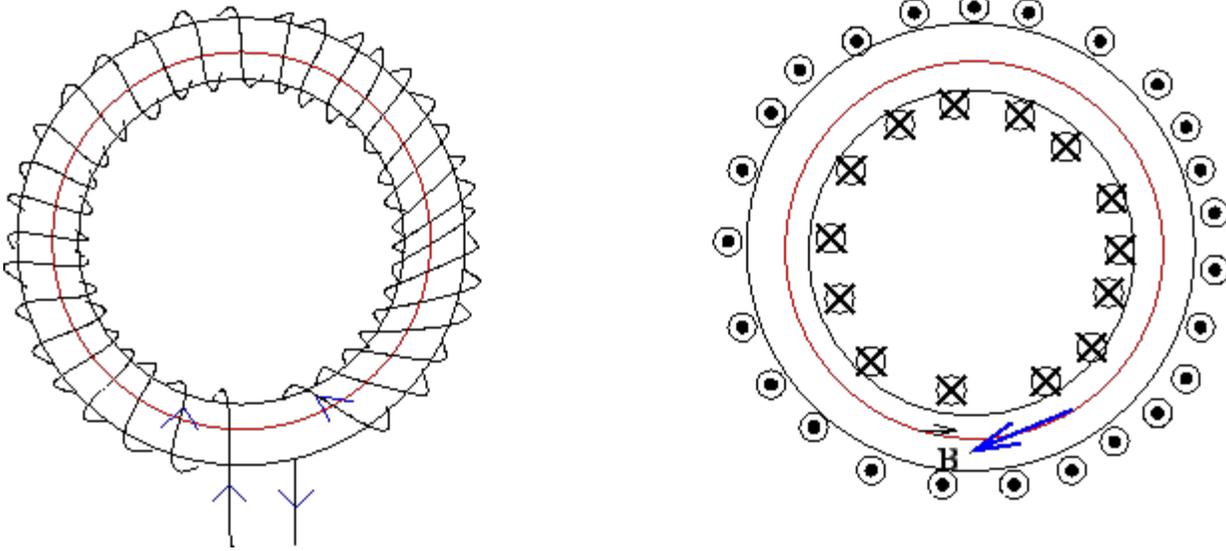
By Ampere's law, the value of the integral is zero as no current is enclosed by the loop. Thus $B(a) = B(b)$. The field outside the solenoid is, therefore, independent of the distance from the axis of the solenoid. However, from physical point of view, we expect the field to vanish at large distances. Thus $B(a) = B(b) = 0$.

To find the field inside, take an amperian loop EFGH with its length parallel to the axis as before, but with one of the sides inside the solenoid while the other is outside. The only contribution to $\oint \vec{B} \cdot d\vec{l}$ comes from the side GH. Thus, $\oint \vec{B} \cdot d\vec{l} = BL = \mu_0 I_{\text{enclosed}} = \mu_0 nLI$

where I is the current through each turn and n is number of turns per unit length. $I_{\text{enclosed}} = nLI$ because the number of turns threading the loop is nL . Hence, $B = \mu_0 nI$ is independent of the distance from the axis.

Exercise 5

A toroid is essentially a hollow tube bent in the form of a circle. Current carrying coils are wound over it. Use an amperian loop shown in the figure to show that the field within the toroid is $\mu_0 NI/L$, where N is the number of turns and L the circumference of the circular path.

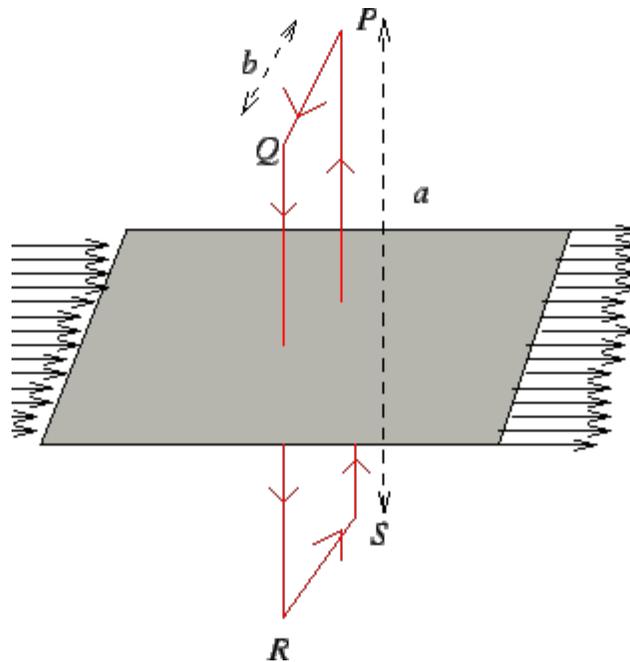


Note that as the circumference of the circular path varies with the distance of the amperian loop from the toroid axis, the magnetic field in the toroid varies over its cross section. Take the inner radius of the toroid to be 20cm and the outer radius as 21cm. Find the percentage variation of the field over the cross section of the toroid.

(Ans. 2.9%)

Example 15

An infinite conducting sheet carries a current such that the current density is λ per unit length. Take an amperian loop as shown.



The contribution to the line integral of \vec{B} from the sides QR and SP are zero as \vec{B} is perpendicular to $d\vec{l}$. For PQ and RS the direction of \vec{B} is parallel to the path. Hence $\oint \vec{B} \cdot d\vec{l} = 2bB = \mu_0\lambda b$ giving

$$B = \mu_0\lambda/2.$$

Exercise 6

Calculate the force per unit area between two parallel infinite current sheets with current densities λ_1 and λ_2 in the same direction.

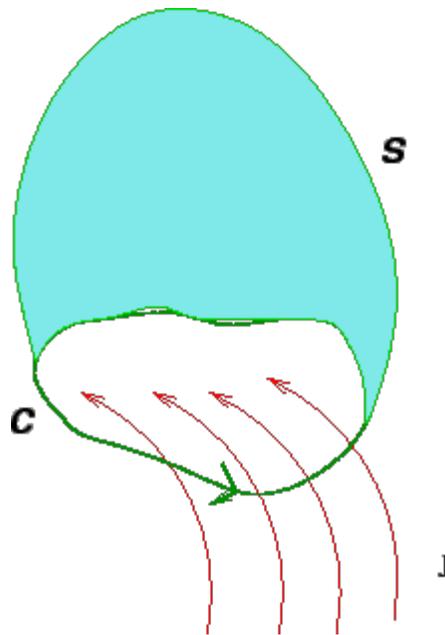
(Ans. $\mu_0 \lambda_1 \lambda_2 / 2$)

Ampere's Law in Differential Form

We may express Ampere's law in a differential form by use of Stoke's theorem, according to which the line integral of a vector field is equal to the surface integral of the curl of the field,

$$\oint \vec{B} \cdot d\vec{l} = \int_S \text{curl } \vec{B} \cdot d\vec{S}$$

The surface S is any surface whose boundary is the closed path of integration of the line integral.



In terms of the current density \vec{J} , we have, $\int_S \vec{J} \cdot d\vec{S} = I_{\text{encl}}$

where I_{encl} is the total current through the surface S . Thus, Ampere's law $\oint \vec{B} \cdot d\vec{l} = I_{\text{encl}}$ is equivalent to

$$\int \text{curl } \vec{B} \cdot d\vec{S} = \mu_0 \int \vec{J} \cdot d\vec{S}$$

which gives $\text{curl } \vec{B} = \mu_0 \vec{J}$

You may recall that in the case of electric field, we had shown that the divergence of the field to be given by $\nabla \cdot \vec{E} = \rho / \epsilon_0$. In the case of magnetic field there are no free sources (monopoles). As a result the divergence of the magnetic field is zero

$$\nabla \cdot \vec{B} = 0$$

The integral form of above is obtained by application of the divergence theorem

$$\int_S \vec{B} \cdot d\vec{S} = \int_V \nabla \cdot \vec{B} dV = 0$$

Thus the flux of the magnetic field through a closed surface is zero.

Recap

In this lecture you have learnt the following

- Ampere's law was stated in integral form and used to calculate magnetic field in symmetric situations.
- Calculation of magnetic field was done due to a long straight wire, a coaxial cable, a solenoid, a toroid and a current sheet done using Ampere's law.
- Using Stoke's theorem Ampere's law can be expressed in a differential form.