Module 2 : Electrostatics
Lecture 11 : Capacitance

Objectives

In this lecture you will learn the following

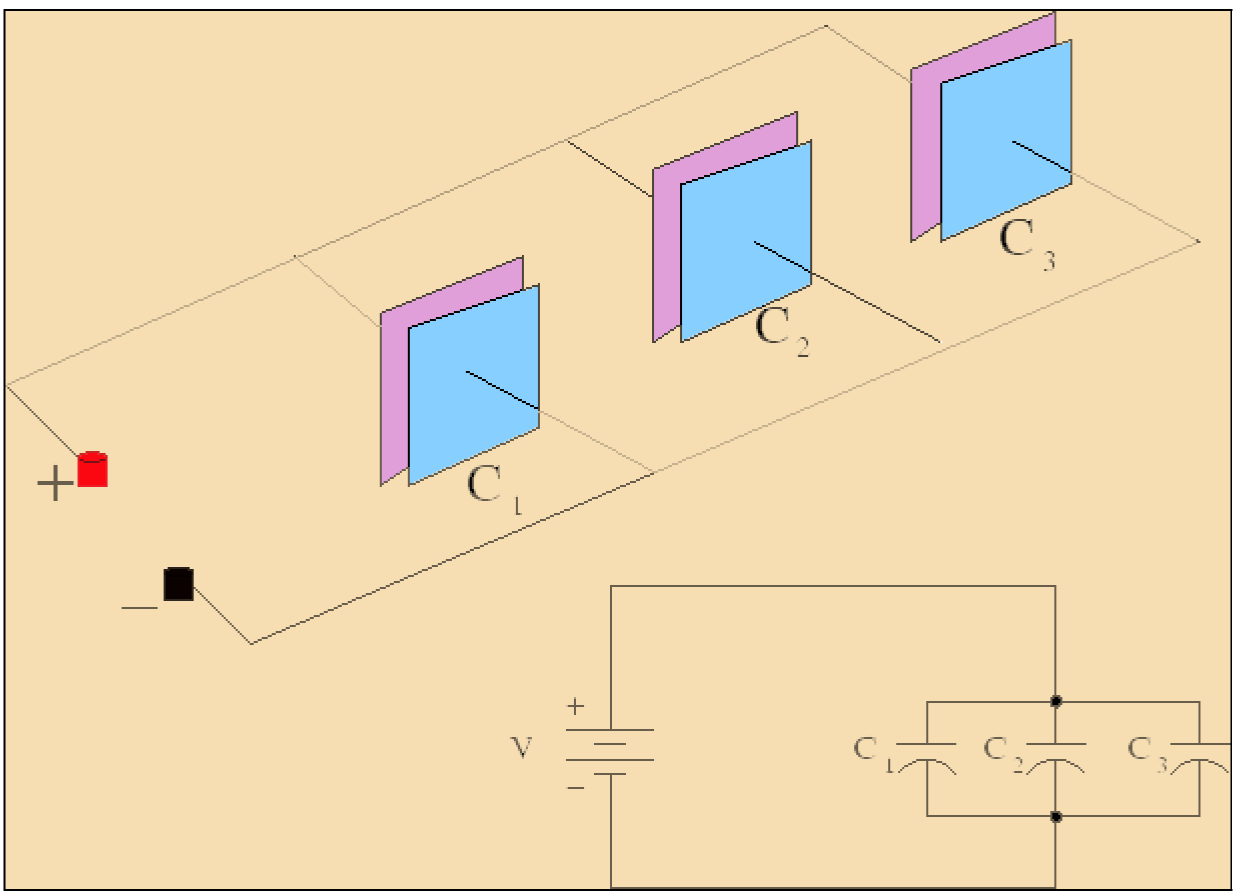
- Capacitors in series and in parallel
- Properties of dielectric
- Conductor and dielectric in an electric field.
- Polarization and bound charges
- Gauss's Law for dielectrics

Capacitors in Combination :

Capacitors can be combined in series or parallel combinations in a circuit.

Parallel Combination

When they are in parallel, the potential difference across each capacitor is the same.



The charge on each capacitor is obtained by multiplying V with the capacitance, i.e.

.

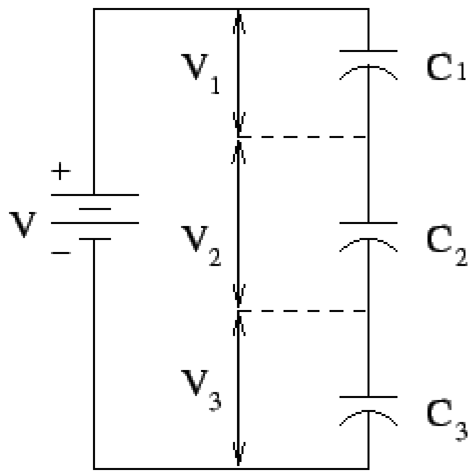
$$Q_1 = C_1 V \ \ Q_2 = C_2 V \dots$$

Since total charge in the capacitors is sum of all the charges, the effective capacitance of the combination is

$$C = \frac{Q_1 + Q_2 + Q_3 + \dots}{V} = \frac{Q_1}{V} + \frac{Q_2}{V} + \frac{Q_3}{V} + \dots = C_1 + C_2 + C_3 + \dots$$

Series Combination :

When capacitors are joined end to end in series, the first capacitor gets charged and induces an equal charge on the second capacitor which is connected to it. This in turn induces an equal charge on the third capacitor, and so on.



The net potential difference between the positive plate of the first capacitor and the negative plate of the last capacitor in series is

$$V = V_1 + V_2 + V_3 + \dots$$

The individual voltage drops are

$$V_1 = \frac{Q_1}{C_1} \ \ V_2 = \frac{Q_2}{C_2} \ \ \dots$$

so that

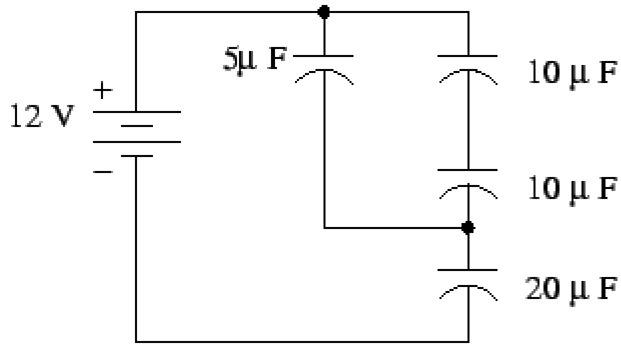
$$V = Q \left(\frac{1}{C_1} + \frac{1}{Q_2} + \dots \right)$$

The effective capacitance is, therefore, given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \dots$$

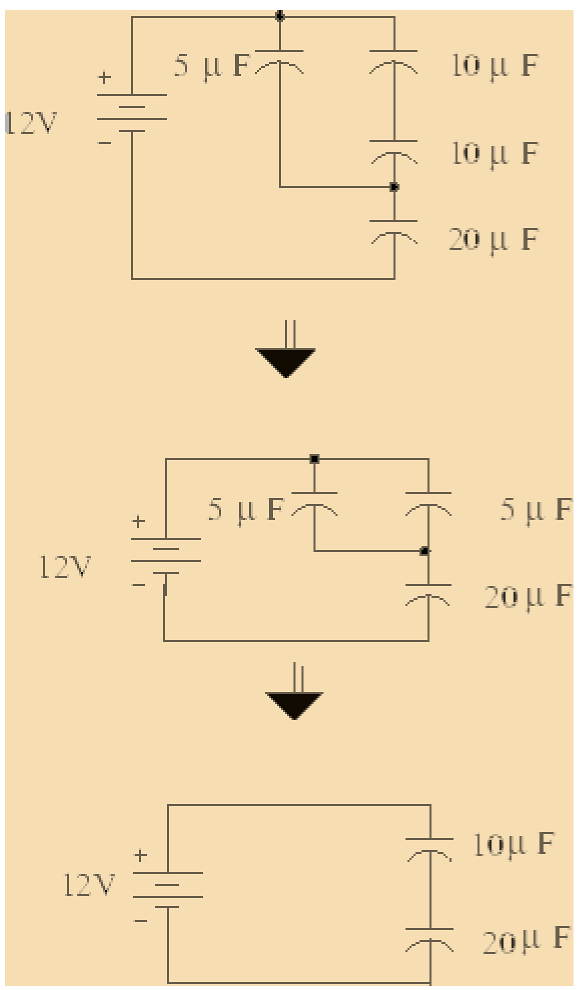
Example 19

Calculate the voltage across the 5 μF capacitor in the following circuit.



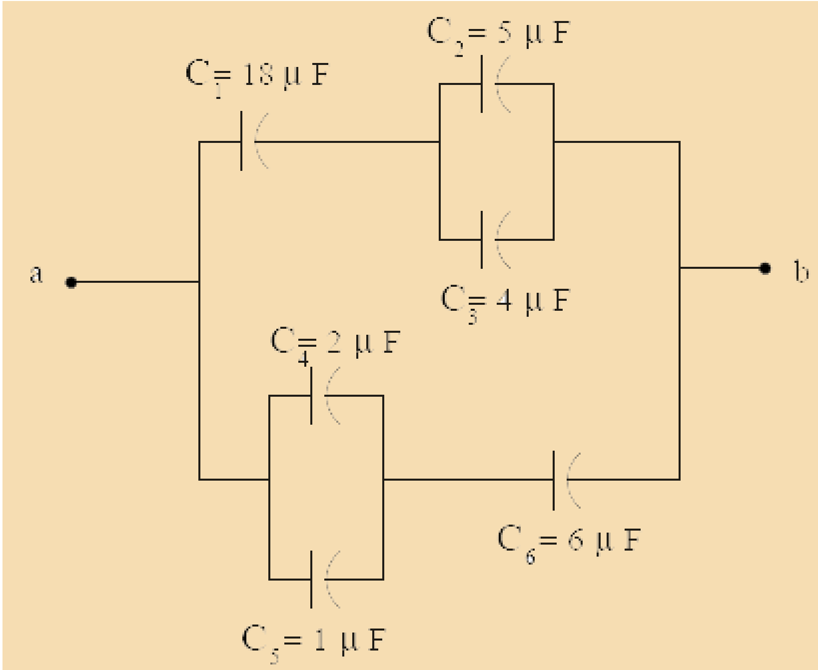
Solution :

The equivalent circuit is shown above.The two 10 μF capacitors in series is equivalent to a 5 μF capacitor.5 μF in parallel with this equivalent capacitor gives 10 μF as the next equivalent.The circuit therefore consists of a 10 μF in series with the 20 μF capacitor. Since charge remains constant in a series combination, the potential drop across the 10 μF capacitor is twice as much as that across 20 μF capacitor. The voltage drop across the 10 μF (and hence across the given 5 μF) is $(2/3) \times 12 = 8$v.



Exercise 2

Determine the effective capacitance of the following capacitance circuit and find the voltage across each capacitance if the voltage across the points a and b is 300 V.



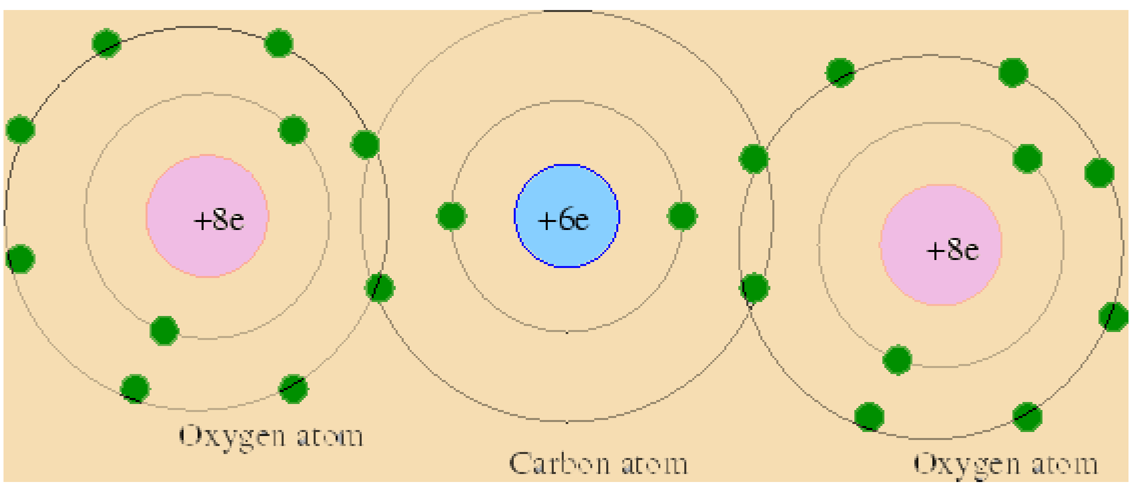
[Ans. 8 μF., 100V,200V,200V,200V,200V,100V]

Conductors and Dielectric

A conductor is characterized by existence of *free electrons* . These are electrons in the outermost shells of atoms (the valence electrons) which get detatched from the parent atoms during the formation of metallic bonds and move freely in the entire medium in such way that the conductor becomes an equipotential volume. In contrast, in dielectrics (insulators), the outer electrons remain bound to the atoms or molecules to which they belong. Both conductors and dielectric, on the whole, are charge neutral. However, in case of dielectrics, the charge neutrality is satisfied over much smaller regions (e.g. at molecular level).

Polar and non-polar molecules :

A dielectric consists of molecules which remain locally charge neutral. The molecules may be polar or non-polar. In non-polar molecules, the charge centres of positive and negative charges coincide so that the net dipole moment of each molecule is zero. Carbon dioxide molecule is an example of a non-polar molecule.



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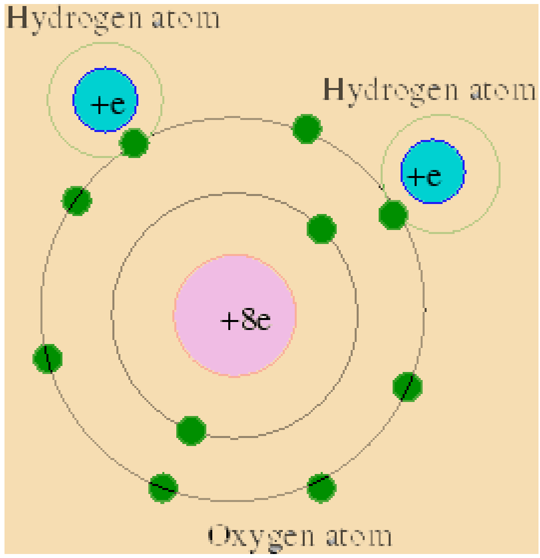

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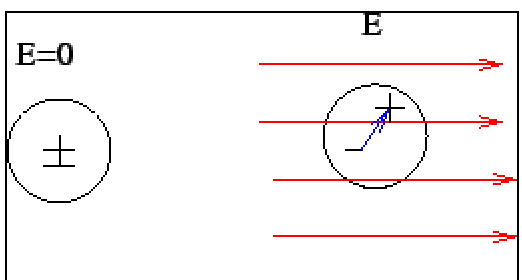
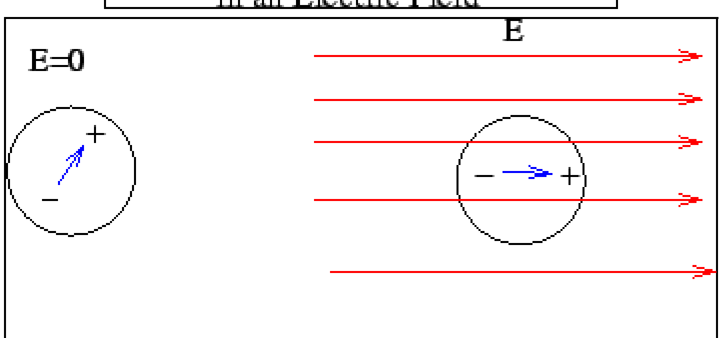
In a polar molecules, the arrangement of atoms is such that the molecule has a permanent dipole moment because of charge separation. Water molecule is an example of a polar molecule.

When a non-polar molecule is put in an electric field, the electric forces cause a small separation of the charges. The molecule thereby acquires an induced dipole moment.
A polar molecule, which has a dipole moment in the absence of the electric field, gets its dipole moment aligned in the direction of the field. In addition, the magnitude of the dipole moment may also increase because of increased separation of the charges.



A non–polar molecule in an Electric Field



A polar molecule in an Electric Field

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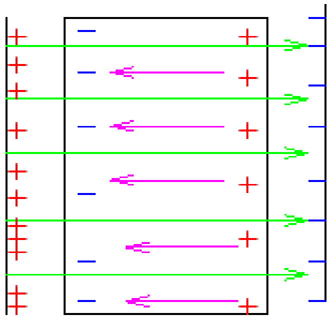
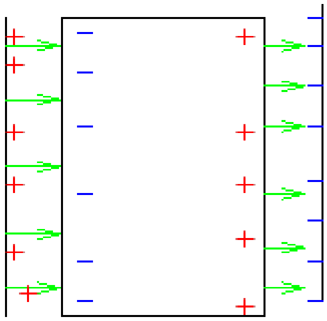
Conductor in an Electric Field

Consider what happens when a conductor is placed in an electric field, say, between the plates of a parallel plate capacitor.



Induced and external electric fields in a conductor



Net field inside a conductor is zero.

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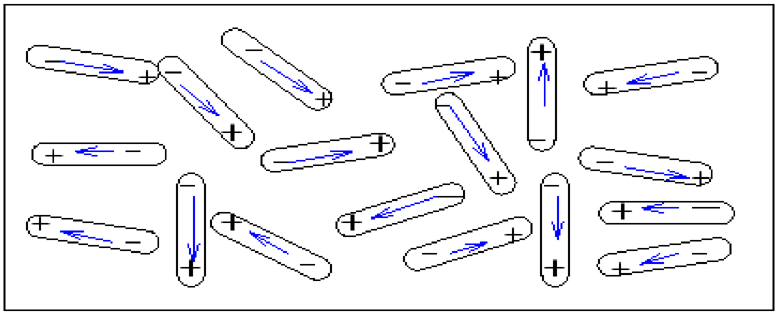
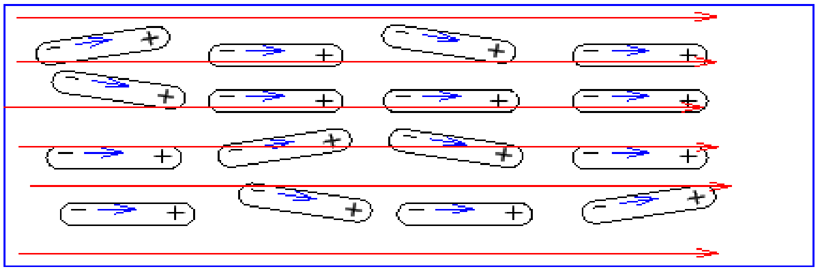
As the conductor contains free charges (electrons), these move towards the positive plate, making the surface of the conductor closer to the positive plate of the capacitor negatively charged. These are called *induced charges.* Consequently, the surface of the conductor at the end closer to the negative plate is positively charged. The motion of charges continue till the internal electric field created by induced charges cancel the external field, thereby making the field inside the conductor zero.

Dielectric in an Electric Field

A dielectric consists of molecules which may (polar) or may not (non-polar) have permanent dipole moment. Even in the former case, the dipoles in a dielectric are randomly oriented because dipole energies are at best comparable to thermal energy.



Randomly oriented dipole in a dielectric (E=0)



Polarised Dipoles in an electric field

Click here for Animation 1   **Click here for Animation 2**

When a dielectric is placed in an electric field the dipoles get partially aligned in the direction of the field. The charge separation is opposed by a restoring force due to attaraction between the charges until the forces are balanced. Since the dipoles are partially aligned, there is a net dipole moment of the dielectric which opposes the electric field. However, unlike in the case of the conductors, the net field is not zero. The opposing dipolar field reduces the electric field inside the dielectric.

Dielectric Polarization

Electric polarization is defined as the dipole moment per unit volume in a dielectric medium. Since the distribution of dipole moment in the medium is not uniform, the polarization $\vec{P}$ is a function of position. If $\vec{p}(\vec{r})$ is the sum of the dipole moment vectors in a volume element $d\tau$ located at the position $\vec{r}$,

$$\vec{p}(\vec{r}) = \vec{P}(\vec{r})d\tau$$

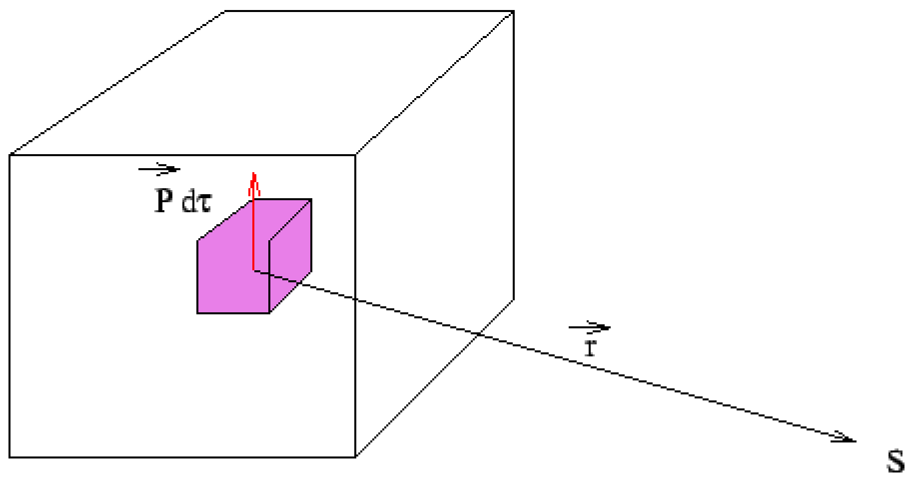
It can be checked that the dimension of $\vec{P}$ is same as that of electric field divided by permittivity ϵ_0. Thus the source of polarization field is also electric charge, except that the charges involved in producing polarization are *bound charges* .

Potential due to a dielectric

Consider the dielectric to be built up of volume elements $d\tau$. The dipole moment of the volume element is $\vec{P}d\tau$

The potential at a point S, whose position vector with respect to the volume element is $\vec{r}$ is

$$d\phi = \frac{1}{4\pi\epsilon_o}\frac{\vec{P}\cdot\hat{r}}{r^2}d\tau$$



The potential due to the whole volume is

$$\phi = \frac{1}{4\pi\epsilon_0}\int_{volume}\frac{\vec{P}\cdot\hat{r}}{r^2}d\tau = \frac{1}{4\pi\epsilon_0}\int_{volume}\vec{P}\cdot\nabla(\frac{1}{r})d\tau$$

where, we have used

$$\nabla(\frac{1}{r}) = \frac{\hat{r}}{r^2}$$

Use the vector identity

$$\vec{\nabla}\cdot(\vec{A}f(r)) = \vec{A}\cdot\nabla f(r) + f(r)\vec{\nabla}\cdot\vec{A}$$

Substituting $\vec{A} = \vec{P}$ and $f(r) = 1/r$,

$$\vec{\nabla}\cdot(\frac{\vec{P}}{r}) = \vec{P}\cdot\nabla(\frac{1}{r}) + \frac{1}{r}\vec{\nabla}\cdot\vec{P}$$

we get

$$\phi = \frac{1}{4\pi\epsilon_o}\int_{vol}\vec{\nabla}\cdot(\frac{\vec{P}}{r})d\tau - \frac{1}{4\pi\epsilon_o}\int_{vol}\frac{1}{r}\vec{\nabla}\cdot\vec{P}d\tau$$

The first integral can be converted to a surface integral using the divergence theorem giving,

$$\phi = \frac{1}{4\pi\epsilon_o}\int_{surface}\frac{\vec{P}}{r}\cdot d\vec{S} - \frac{1}{4\pi\epsilon_0}\int_{vol}\frac{1}{r}\vec{\nabla}\cdot\vec{P}d\tau$$

The first term is the potential that one would expect for a surface charge density σ_b where

$$\sigma_b = \vec{P}\cdot\hat{n}$$

where $\hat{n}$ is the unit vector along outward normal to the surface. The second term is the potential due to a volume charge density ρ_b given by

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

The potential due to the dielectric is, therefore, given by

$$\phi = \frac{1}{4\pi\epsilon_o}\int_{surface}\frac{\sigma_b dS}{r} + \frac{1}{4\pi\epsilon_0}\int_{vol}\frac{\rho_b d\tau}{r}$$

and the electric field

$$\begin{aligned}\vec{E} &= -\nabla\phi \\ &= \frac{1}{4\pi\epsilon_0}\int_{surface}\frac{\sigma_b\hat{r}}{r^2}dS + \frac{1}{4\pi\epsilon_0}\int_{vol}\frac{\rho_b\hat{r}}{r^2}d\tau\end{aligned}$$

Gauss's Law in a Dielectric :

We have seen that the effect of polarization of a dielectric is to produce bound charges of volume density ρ_b and surface density σ_b, given by

$$\begin{aligned}\rho_b &= -\vec{\nabla}\cdot\vec{P} \\ \sigma_b &= \vec{P}\cdot\hat{n}\end{aligned}$$

The total electric field of a system which includes dielectrics is due to these polarization charge densities and other charges which may be present in the system. The latter are denoted as *free charges* to distinguish them from charges attributable to polarization effect. For instance, the valence charges in a metal or charges of ions embedded in a dielectric are considered as *free* charges.

The total charge density of a medium is a sum of free and bound charges

$$\rho = \rho_f + \rho_b$$

We can now formulate Gauss's law in the presence of a dielectric. Gauss's Law takes the form

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_f+\rho_b}{\epsilon_0}$$

Substituting $\rho_b = -\vec{\nabla}\cdot\vec{P}$, we get

$$\vec{\nabla}\cdot(\epsilon\vec{E}+\vec{P}) = \rho_f$$

The electri displacement vector $\vec{D}$ is defined by

$$\vec{D} = \epsilon_0\vec{E}+\vec{P}$$

which has the same dimension as that of $\vec{P}$. The equation satisfied by $\vec{D}$ is thus,

$$\vec{\nabla}\cdot\vec{D} = \rho_f$$

which is the differential form of Gauss's law for a dielectric medium.
Integrating over the dielectric volume,

$$\int_{volume}\vec{\nabla}\cdot\vec{D}d\tau = \int_{volume}\rho_f d\tau = Q_f$$

where Q_f is the free charge *enclosed* in the volume. The volume integral can be converted to a surface integral using the divergence theorem, which gives

$$\int_{surface}\vec{D}\cdot d\vec{S} = Q_f$$

Thus the flux over the vector $\vec{D}$ over a closed surface is equal to the free charged enclosed by the surface.

The above formulatons of Gauss's law for dielectric medium is useful because they refer to only free charges for which we may have prior knowledge.

Constitutive Relation

Electric displacement vector $\vec{D}$ helps us to calculate fields in the presence of a dielectric. This is possible only if a relationship between $\vec{E}$ and $\vec{D}$ is known.

For a weak to moderate field strength, the electric polarization $\vec{P}$ is found to be directly proportional to the external electric field $\vec{E}$. We define *Electric Susceptibility* through

$$\chi$$

$$\vec{P} = \epsilon_0 \chi \vec{E}$$

so that

$$\begin{aligned} \vec{D} &= \epsilon_0 \vec{E} + \vec{P} \\ &= \epsilon_o (1+\chi) \vec{E} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E} \end{aligned}$$

where $\kappa \equiv \epsilon_r = 1 + \chi$ is called the *relative permittivity* or the dielectric constant and ϵ is the permittivity of the medium. Using differential form of Gauss's law for $\vec{D}$, we get

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon} \vec{\nabla} \cdot \vec{D} = \frac{\rho_f}{\epsilon}$$

Thus the electric field produced in the medium has the same form as that in free space, except that the field strength is reduced by a factor equal to the dielectric constant κ.

Recap

In this lecture you have learnt the following

- How to calculate effective capaciance when capacitors are in series and in parallel.

- Dielectrics are material in which, unlike in conductors, the valence electrons are not detached from the parent atoms.

- In the absence of an electric field the dipoles in a dielectric have their moments directed randomly. In the presence of an electric field, these dipoles get partially aligned in the direction of the field as a result of which a dielectric acquires a net dipole moment.

- The polarization effect can be attributed as arising out of bound charges in the dielectric. The charge density in the medium consists of free and bound charges.

- The net electric field in a dielectric is due to both polarization effect and the field produced by external charges. One can define an electric displacement vector as a vector sum of fields due to polarization and the electric field vector. Gauss's law can be modified appropriately for such a situation in a way that the flux of the dsplacement vector is given by only the enclosed free charges.