Module 3 : MAGNETIC FIELD
Lecture 15 : Biot- Savarts' Law

Objectives

In this lecture you will learn the following

- Study Biot-Savart's law
- Calculate magnetic field of induction due to some simple current configurations.
- Define magnetic moment of a current loop.
- Find force between two current carrying conductors.

Biot- Savarts' Law

Biot-Savarts' law provides an expression for the magnetic field due to a current segment. The field $d\vec{B}$ at a position $\vec{r}$ due to a current segment $I\vec{dl}$ is experimentally found to be perpendicular to $\vec{dl}$ and $\vec{r}$. The magnitude of the field is proportional to the length $|dl|$ and to the current I and to the sine of the angle between $\vec{r}$ and $\vec{dl}$.

inversely proportional to the square of the distance r of the point P from the current element.
Mathematically,

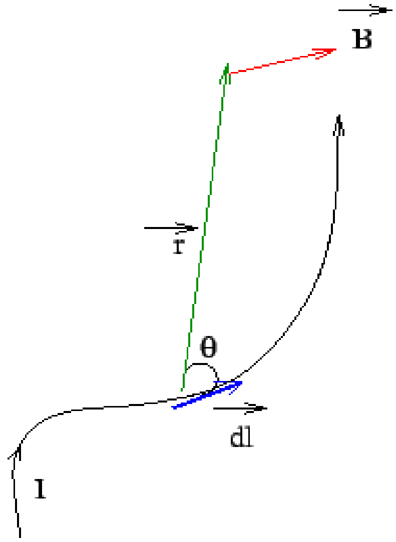
$$d\vec{B} \propto I\frac{\vec{dl} \times \hat{r}}{r^2}$$

In SI units the constant of proportionality is $\mu_0/4\pi$, where μ_0 is the permeability of the free space. The value of μ_0 is

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/amp^2}$$

The expression for field at a point P having a position vector $\vec{r}$ with respect to the current element is

$$d\vec{B} = \frac{\mu_0}{4\pi}I\frac{\vec{dl} \times \hat{r}}{r^2}$$

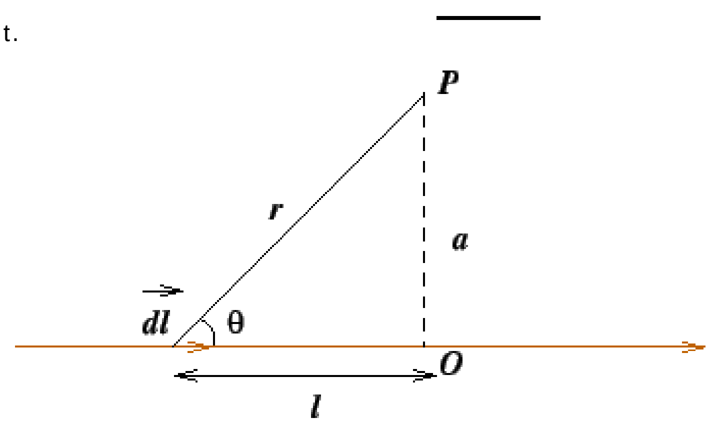


For a conducting wire of arbitrary shape, the field is obtained by vectorially adding the contributions due to such current elements as per superposition principle, $\vec{B}(P) = \frac{\mu_0}{4\pi} I \int \frac{\vec{dl} \times \hat{r}}{r^2}$ where the integration is along the path of the current flow.

Example 5

Field due to a straight wire carrying current

The direction of the field at P due to a current element $\vec{dl}$ is along $\vec{dl} \times \vec{r}$, which is a vector normal to the page (figure on the left) and coming out of it.



We have,

$$\frac{\vec{dl} \times \hat{r}}{r^2} = \frac{|\, dl \sin\theta \,|}{r^2} \hat{k}$$

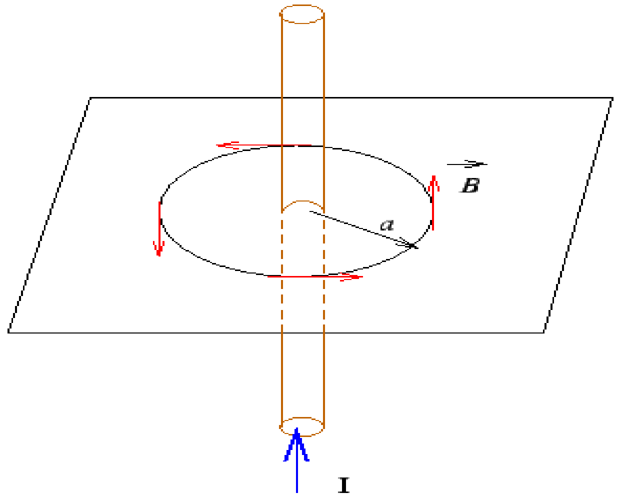
where the plane of the figure is taken as the x-y plane and the direction of outward normal is parallel to z-axis. If a be the distance of the oint P from the wire,

we have
$$\begin{aligned} r &= a/\sin\theta \\ l &= a\cot\theta \\ dl &= -(a/\sin^2\theta)d\theta \end{aligned}$$

Thus

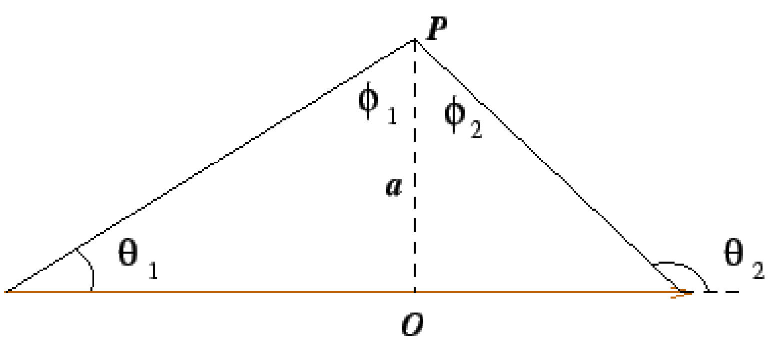
$$\frac{\vec{dl} \times \hat{r}}{r^2} = \frac{\sin\theta}{a} d\theta \hat{k}$$

The direction of the magnetic field at a distance a from the wire is tangential to a circle of radius a, as shown.



Since the magnetic field due to all current elements at P are parallel to the z-direction, the field at P due to a wire, the ends of which make angles ϕ_1 and ϕ_2 at P is given by a straightforward integration

$$\begin{aligned} \vec{B} &= \frac{\mu_0 I}{4\pi a} \int_{\theta_1}^{\theta_2} \sin\theta d\theta \hat{k} \\ &= \frac{\mu_0 I}{4\pi a} \left[-\cos\theta\right]_{\theta_1}^{\theta_2} \hat{k} \\ &= \frac{\mu_0 I}{4\pi a} \left[-\cos\theta_1 - \cos\theta_2\right] \hat{k} \\ &= \frac{\mu_0 I}{4\pi a} \left[\sin\phi_1 + \sin\phi_2\right] \hat{k} \end{aligned}$$



Note that both the angles ϕ_1 and ϕ_2 are acute angles.

If we consider an infinite wire (also called long straight wire), we have $\phi_1 = \phi_2 = \pi/2$, so that the field due to such a wire is

$$\vec{B} = \frac{\mu_0 I}{2\pi a} \hat{k}$$

where the direction of the field is given by the Right hand rule.

Exercise 1

A conductor in the shape of an n-sided polygon of side a carries current I. Calculate the magnitude of the magnetic field at the centre of the polygon.

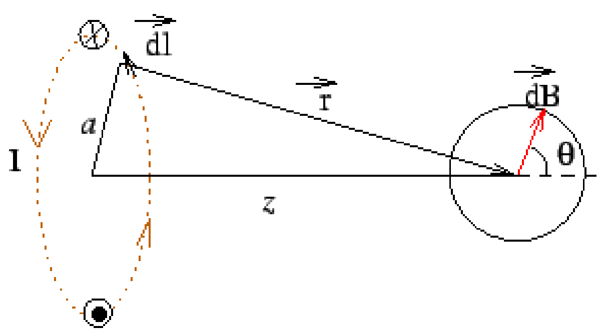
[Ans. $(\mu_0 I n/\pi a)\sin(\pi/n)$.]

Example 6

Field due to a circular coil on its axis
Consider the current loop to be in the x-y plane, which is taken perpendicular to the plane of the paper in which the axis to the loop (z-axis) lies. Since all length elements on the circumference of the ring are perpendicular to $\underline{\vec{r}}$, the magnitude of the field at a point P is given by

$$dB = \frac{\mu_0 I}{4\pi}\frac{dl}{r^2}$$

The direction of the field due to every element is in the plane of the paper and perpendicular to $\underline{\vec{r}}$, as shown. Corresponding to every element $d\vec{l}$ on the circumference of the circle, there is a diametrically opposite element which gives a magnetic field $d\vec{B}$ in a direction such that the component of $d\vec{B}$ perpendicular to the axis cancel out in pairs.



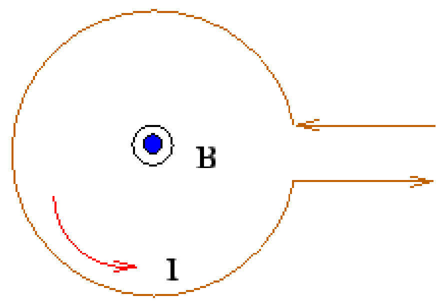
The resultant field is parallel to the axis, its direction being along the positive z-axis for the current direction shown in the figure. The net field is

$$\begin{aligned}\vec{B} &= \frac{\mu_0 I}{4\pi}\int \frac{dl}{r^2}\cos\theta \\ &= \frac{\mu_0 I}{4\pi}\frac{\cos\theta}{r^2}\int dl \\ &= \frac{\mu_0 I\cos\theta}{4\pi r^2}2\pi a = \frac{\mu_0 I a\cos\theta}{2r^2}\end{aligned}$$

In terms of the distance z of the point P and the radius a, we have

$$B = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}}$$

The direction of the magnetic field is determined by the following Right Hand Rule.



If the palm of the right hand is curled in the direction of the current, the direction in which the thumb points gives the direction of the magnetic field at the centre of the loop. The field is, therefore, outward in the figure shown.

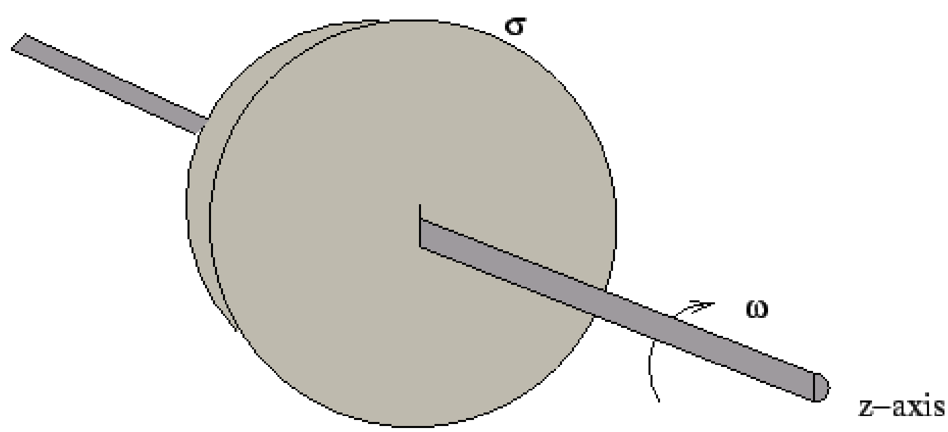
Note that for $z \gg a$, i.e. the field due to circular loop at large distances is given by

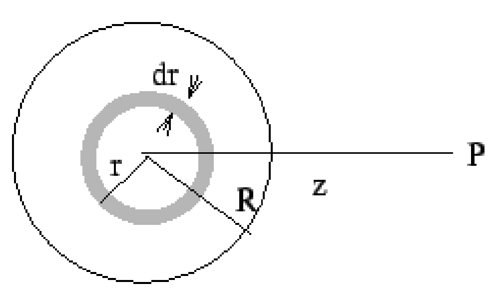
$$B = \frac{\mu_0 I a^2}{2z^3} = \frac{\mu_0 \mu}{2\pi z^3}$$

where $\mu = I\pi a^2$ is the magnetic moment of the loop. The formula is very similar to the field of an electric dipole. Thus a current loop behaves like a magnetic dipole.

Example 7

A thin plastic disk of radius R has a uniform surface charge density σ. The disk is rotating about its own axis with an angular velocity ω. Find the field at a distance z along the axis from the centre of the disk.





The current on the disk can be calculated by assuming the rotating disk to be equivalent to a collection of concentric current loops. Consider a ring of radius r and of width dr. As the disk is rotating with an angular speed ω, the rotating charge on the ring essentially behaves like a current loop carrying current $\sigma \cdot 2\pi r dr \cdot \omega / 2\pi = \sigma \omega r dr$.

The field at a distance z due to this ring is

$$dB = \frac{\mu_0(\sigma\omega r dr)}{2}\frac{r^2}{(r^2+z^2)^{3/2}}$$

The net field is obtained by integrating the above from $r = 0$ to $r = R$

$$\begin{aligned} B &= \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^3}{(r^2+z^2)^{3/2}}dr \\ &= \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^2+z^2-z^2}{(r^2+z^2)^{3/2}} r dr \end{aligned}$$

The integral above may easily be evaluated by a substitution $x = r^2 + z^2$. The result is

$$B = \mu_0\sigma\omega\left[\frac{R^2+2z^2}{(R^2+z^2)^{3/2}} - 2z\right]$$

The field at the centre of the disk ($z = 0$) is

$$B(z=0) = \mu_0\sigma\omega R$$

Exercise 2

Find the magnetic moment of the rotating disk of Example 7.

[Ans. $\pi\omega R^4/4$]

Example 8

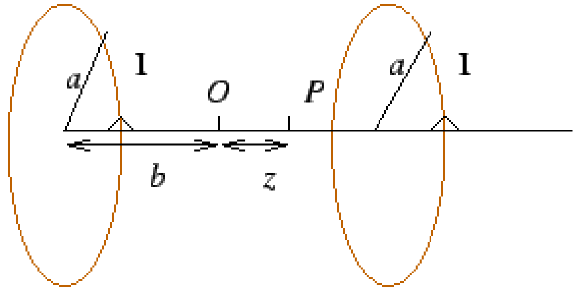
Two coaxial circular coils of radius a each carry current I each in the same sense. The centres of the coils are separated by a distance $2b$. Determine the field along the axis. The set up is called ``Helmholtz coil" when the distance $2b$ between the centres of the coils equals the radius a of each of the loop. The field in the region between the coils of such a coil is nearly uniform.

If the distance z along the axis is measured from the mid points of the line joining the centres of the two coils, the field strength due to the left coil at P is

$$B_1 = \frac{\mu_0 I}{2}\frac{a^2}{[a^2+(b+z)^2]^{3/2}}$$

and that due to the right coil is

$$B_2 = \frac{\mu_0 I}{2}\frac{a^2}{[a^2+(b-z)^2]^{3/2}}$$



The net field at P, due to both coils add up and is given by $B(z)$

$$
\begin{aligned}
&= \frac{\mu_0 I a^2}{2}\left[\frac{1}{[a^2+(b+z)^2]^{3/2}}+\frac{1}{[a^2+(b-z)^2]^{3/2}}\right] \\
&= \frac{\mu_0 I a^2}{2(a^2+b^2)^{3/2}}\left[\frac{1}{[1+\frac{z^2+2zb}{a^2+b^2}]^{3/2}}+\frac{1}{[1+\frac{z^2-2zb}{a^2+b^2}]^{3/2}}\right]
\end{aligned}
$$

We can express the above in a power series using a binomial expansion. Up to z^4, the terms in the expansion may be written as

$$
\left[\frac{1}{[1+\frac{z^2+2zb}{a^2+b^2}]^{3/2}}\right] = 1-\frac{3}{2}\left(\frac{z^2+2zb}{(a^2+b^2)}\right)+\frac{15}{8}\left(\frac{z^2+2zb}{(a^2+b^2)}\right)^2-\frac{35}{16}\left(\frac{z^2+2zb}{(a^2+b^2)}\right)^3+\frac{315}{128}\left(\frac{z^2+2zb}{(a^2+b^2)}\right)^4
$$

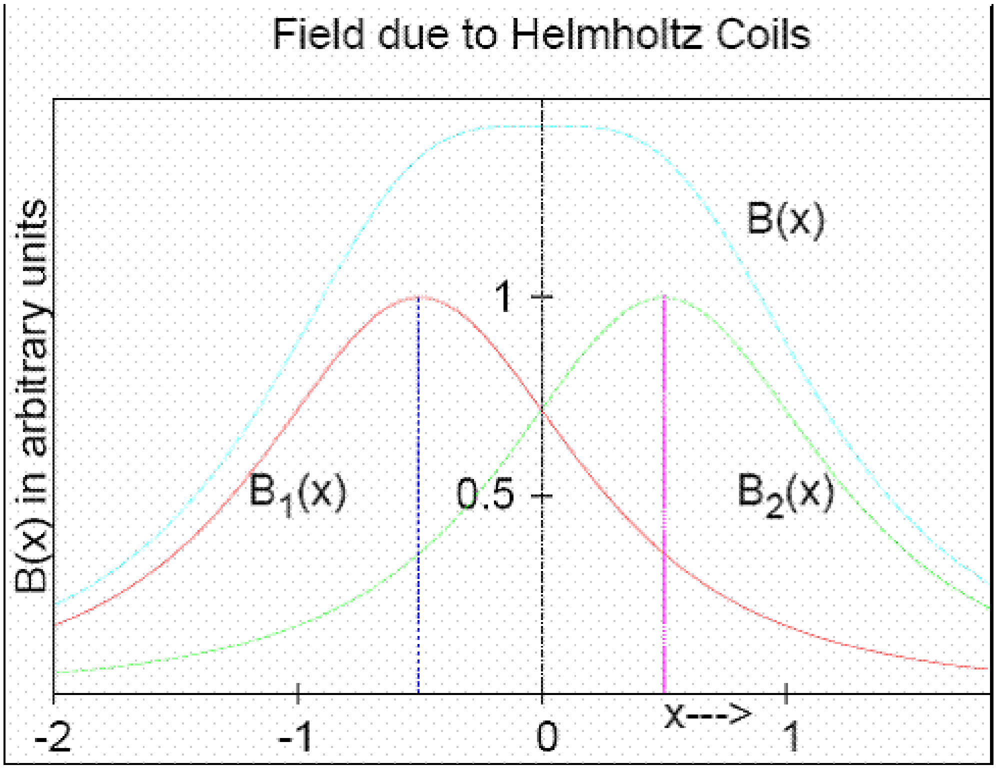
and a similar expression for the second term with $-2zb$replacing $2zb$. Adding the terms and retaining only up to powers of z^4we get

$$
B(z) = \frac{\mu_0 I a^2}{(a^2+b^2)^{3/2}}\left[1-\frac{3}{2}\frac{a^2-4b^2}{(a^2+b^2)^2}z^2+\frac{15}{8}\frac{a^4+8b^4-12a^2b^2}{(a^2+b^2)^4}z^4\right]
$$

For the case of a Helmholtz coil, $2b = a$, and the expression for field is independent of $z$up to its third power, and is given by,

$$
B(z) = \frac{\mu_0 I}{a}(4/5)^{3/2}\left[1-\frac{144}{125}\frac{z^4}{a^4}\right]
$$

It can be seen that the field along the axis is nearly uniform in the region between the coils.



Example 9

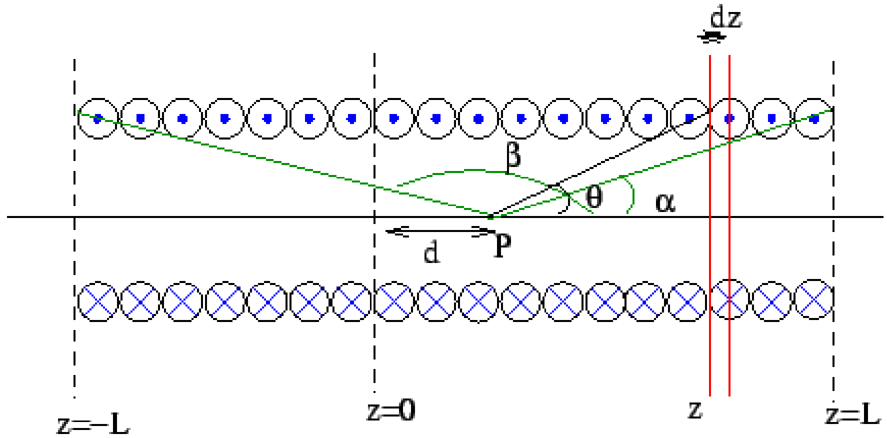
Consider a solenoid of N turns. The solenoid can be considered as stacked up circular coils. The field on the axis of the solenoid can be found by superposition of fields due to all circular coils. Consider the field at P due to the circular turns between z and $z + dz$ from the origin, which is taken at the centre of the solenoid. The point P is at $z = d$. If L is the length of the solenoid, the number of turns within z and $z + dz$ is $Ndz/L = ndz$, where n is the number of turns per unit length.

The magnitude of the field at P due to these turns is given by

$$dB = \frac{\mu_0 N I dz}{2L} \frac{a^2}{\left[a^2 + (z-d)^2\right]^{3/2}}$$

The field due to each turn is along $\hat{k}$; hence the fields due to all turns simply add up. The net field is

$$\vec{B} = \frac{\mu_0 N I a^2}{2L} \int_{-L/2}^{L/2} \frac{dz}{\left[a^2 + (z-d)^2\right]^{3/2}} \hat{k}$$



The integral above is easily evaluated by substituting

$$
\begin{aligned}
z - d &= a \cot\theta \\
dz &= -a\mathrm{cosec}^2\theta d\theta
\end{aligned}
$$

The limits of integration on θ are α and β as shown in the figure. With the above substitution

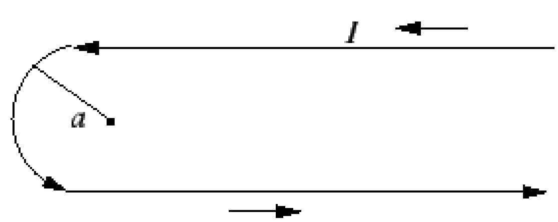
$$\vec{B} = -\frac{\mu_0 n I}{2}\int_\alpha^\beta \sin\theta d\theta = \frac{\mu_0 n I}{2}(\cos\alpha - \cos\beta)\hat{k}$$

For a long solenoid, the field on the axis at points far removed from the ends of the solenoid may be obtained by substituting $\alpha = 0^\circ$ and $\beta = 180^\circ$, so that, $\vec{B} = \mu_0 n I \hat{k}$

The field is very nearly constant. For points on the axis far removed from the ends but outside the solenoid, $\alpha \approx \beta$ so that the field is nearly zero.

Example 10

Determine the field at the point located at the centre P of the semi-circular section of the hairpin bend shown in the figure.



Solution :
The field at P may be determined by superposition of fields due to the two straight line sections and the semicircular arc. The contribution due to all three sections add up as the field due to each is into the plane of the paper.

The field due to each straight line section is obtained by putting $\phi_1 = 90^\circ$ and $\phi_2 = 0^\circ$ in the expression obtained in Example 5 above. The field due to each wire is $\mu_0 I/4\pi a$.

For the semi-circular arc, each length element on the circumference is perpendicular to $\vec{r}$, the vector from the length element to the point P. Thus

$$\begin{aligned} B_{arc} &= \frac{\mu_0 I}{4\pi}\int \frac{\vec{dl}\times\hat{r}}{r^2} \\ &= \frac{\mu_0 I}{4\pi a^2}\int dl = \frac{\mu_0 I}{4\pi a^2}\cdot \pi a = \frac{\mu_0 I}{4a} \end{aligned}$$

The net field due to the current in the hairpin bend at P is

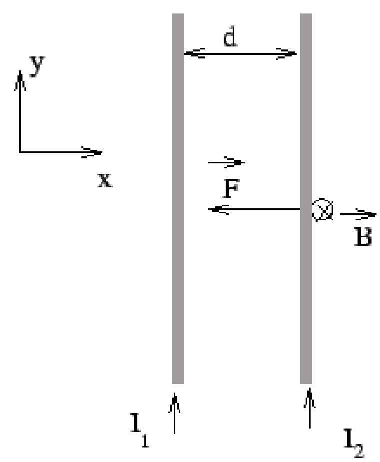
$$B = \frac{\mu_0 I}{2\pi a} + \frac{\mu_0 I}{4a}$$

Example 11

Force due to the first wire at the position of the second wire is given by

$\vec{B} = -\frac{\mu_0 I_1}{2\pi d}\hat{k}$ where $\hat{k}$ is a unit vecor out of the page. The force experienced by the second wire in this field is

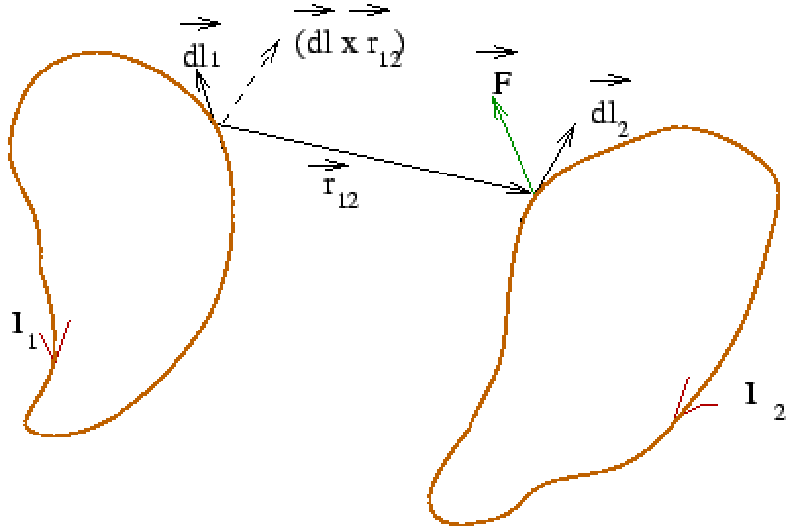
$$\begin{aligned} \vec{F} &= \int (\vec{I_2}\times\vec{B})dl \\ &= -\frac{\mu_0 I_1 I_2}{2\pi d}(\hat{\jmath}\times\hat{k})\int dl \\ &= -\frac{\mu_0 I_1 I_2}{2\pi d}\hat{\imath}\int dl \end{aligned}$$



Thus the force between the wires carrying current in the same direction is attractive and is $\mu_0 I_1 I_2/2\pi d$ per unit length. A generalization of the above is given by the mathematical expression for the force between two arbitrary current loops.

$$\vec{F} = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{\vec{dl_2}\times(\vec{dl_1}\times\vec{r_{12}})}{r_{12}^3}$$

where $\vec{r_{12}}$ is the position vector of the element $\vec{dl_2}$ with respect to $\vec{dl_1}$.



Once the integration is carried out, the expression above can be shown to be symmetrical between the two circuits. To show, we express the vector triple product

$$\vec{dl_2} \times (\vec{dl_1} \times \vec{r_{12}}) = \vec{dl_1}(\vec{dl_2} \cdot \vec{r_{12}}) - \vec{r_{12}}(\vec{dl_1} \cdot \vec{dl_2})$$

so that

$$\vec{F} = \frac{\mu_0}{4\pi} I_1 I_2 \left[\oint \oint \frac{\vec{dl_1}(\vec{dl_2} \cdot \vec{r_{12}})}{r_{12}^3} - \frac{\vec{r_{12}}(\vec{dl_1} \cdot \vec{dl_2})}{r_{12}^3} \right]$$

The integrand in the first integral is an exact differential with respect to the integral over $\vec{dl_2}$ as

$$\oint \oint \frac{\vec{dl_1}(\vec{dl_2} \cdot \vec{r_{12}})}{r_{12}^3} = \oint \vec{dl_1} \oint \nabla(\frac{1}{r_{12}}) \cdot \vec{dl_2}$$

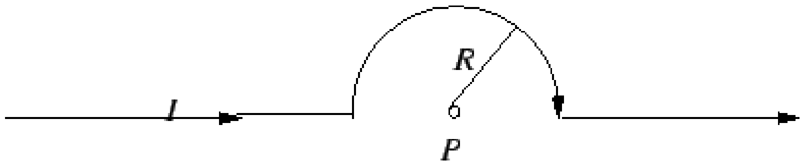
The integral above being an integral of a gradient over a closed path vanishes. Thus

$$\vec{F} = \frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{\vec{r_{12}}(\vec{dl_1} \cdot \vec{dl_2})}{r_{12}^3}$$

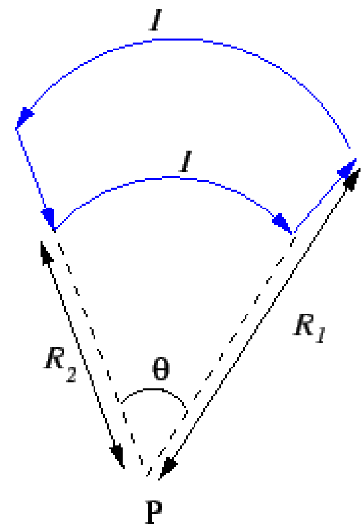
which is explicitly symmetric between the two circuits, confirming the validity of Newton 's third law.

Exercise 3

Determine the magnetic field at the point P for the two geometries shown in the figures below.



(a)



(b)

[Ans . (a) $\mu_0 I/4R$ (b) $\dfrac{\mu_0 I(R_1 - R_2)\theta}{4\pi R_1 R_2}$]

Recap

In this lecture you have learnt the following

- The magnetic field due to a current element is determined by Biot-Savart's law.
- The magnetic field due to some configurations like a line segment, a circular coil, a disk etc. was calculated using Biot-savart's law.
- A Helmholtz coil is used to produce a uniform magnetic field over a limited region of space.
- A force is exerted on a current element placed in a magnetic field.
- Two current carrying circuits exert force on each other because the magnetic field due to one circuit exerts force on the current elements of the other circuit.