

Error control coding: An introduction to linear block codes

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Lecture #6A: Some simple linear block codes -I



Outline of the lecture

- Dual code.

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- Examples of linear block codes

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 - Repetition code



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 - Single parity check code
 - Hamming code



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- This implies $\lambda \mathbf{x} + \mu \mathbf{y} \in C_d$



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- Thus the generator matrix $\mathbf{G}$ of a linear (n, k) block code, is the parity check matrix $\mathbf{H}$ of its dual code and vice-versa.

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- $(24, 12)$ Golay code is a self-dual code.

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- The generator matrix for the single parity check code in systematic form is given by

$$\mathbf{G} = \left[\begin{array}{c|cccc\cdots} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & \\ 1 & 0 & 0 & 0 & 0 & \cdots & 1 \end{array} \right]$$

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- The $(n, n - 1)$ SPC code and $(n, 1)$ repetition code are dual to each other.

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Code length:	$n = 2^m - 1$
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- The parity check matrix in systematic form

$$\mathbf{H} = [\mathbf{I}_m : \mathbf{P}^T],$$

where the $2^m - m - 1$ columns of $\mathbf{P}^T$ consists of all m-tuples of weight 2 or more.



Hamming code

- For $m = 3$, the Hamming code is of length $n = 2^3 - 1 = 7$, $k = 2^3 - 3 - 1 = 4$, that has parity check matrix $\mathbf{H}$,

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and generator matrix $\mathbf{G}$,

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- Here the column $(x, y, z)^T$ represents the number $x(2^0) + y(2^1) + z(2^2)$.

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- If atmost one error has occurred, the syndrome would be either the zero vector or a column of $\mathbf{H}$.
- When one error has happened, the number represented by the column of the calculated syndrome is the position in codeword which is in error, and since we considered binary code, it can be corrected.

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- The number represented by syndrome is 2, hence the error is in second bit position. Hence estimated codeword is 0 1 0 1 0 1 0.

Shortened Hamming code

- If we delete any l columns from the parity check matrix of a Hamming code, we get shortened Hamming code with following parameters

$$\begin{array}{ll} \text{Code length:} & n = 2^m - l - 1 \\ \text{Information bits:} & k = 2^m - m - l - 1 \\ \text{Parity bits:} & n - k = m \\ \text{Minimum distance:} & d_{\min} \geq 3 \end{array}$$

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- Shortened (6,3) Hamming code has a parity check matrix

$$H_1 = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$



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- Thus all the rows of $\mathbf{H}_1$ are linearly independent. Hence the row space of $\mathbf{H}_1$ has dimension $(n - k + 1)$.



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- The expurgated parity check matrix defines a code with minimum distance four.



Expurgated Hamming code: Example

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- Distance-4 expurgated Hamming code has a parity check matrix H_1 given by

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Extended Hamming code

- Let C be a (n, k) Hamming code with parity check matrix $\mathbf{H}$. Let us define a new code C_1 with parity check matrix $\mathbf{H}_1$. (all one vector as the last row.)

$$\mathbf{H}_1 = \left(\begin{array}{c|c} \mathbf{H} & \mathbf{0} \\ \hdashline \dots\dots\dots & \dots \\ 1 \dots 1 & 1 \end{array} \right)$$

- Since the parity check matrix of Hamming code doesn't have an all one vector in any of the rows, any linear combination including the last row of $\mathbf{H}_1$ will never yield a zero vector.
- Thus all the rows of $\mathbf{H}_1$ are linearly independent. Hence the row space of $\mathbf{H}_1$ has dimension $(n - k + 1)$.

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- Now, since the last row of $\mathbf{H}_1$ is an all-one vector, the inner product of any odd weight vector $\mathbf{v}$ and all-one vector is 1. Hence for any odd weight vector $\mathbf{v}$,

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- The extended parity check matrix defines a code with minimum distance four.



Extended Hamming code: Example

- H matrix of Hamming code given by

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

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- Distance-4 extended hamming code has a parity check matrix H_1 given by

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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