

Error control coding: An introduction to linear block codes

Adrish Banerjee

Department of Electrical Engineering
Indian Institute of Technology Kanpur
Kanpur, Uttar Pradesh
India

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Lecture #5C: Problem solving session



Linear block code

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- Therefore no column in the code array contains only zeros.



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- **Solution:** To prove that each column of this array has 2^{k-1} zeros and 2^{k-1} ones, we will show that the number of codewords that “1” at the l -th position is same as number of codewords that have “0” at the l -th position.

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- In the code array, each column contains at least one nonzero entry. Consider the l -th column of the code array.
- Let S_0 be the codewords with a “0” at the l -th position and S_1 be the codewords with a “1” at the l -th position.



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- Let $\mathbf{x}$ be a codeword from S_1 . Adding $\mathbf{x}$ to each vector in S_0 , we obtain a set S'_1 of codewords with a “1” at the l -th position.

$$|S'_1| = |S_0| \quad \text{and} \quad S'_1 \subseteq S_1$$



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- From (1) and (2), we get $|S_0| = |S_1|$. Therefore l -th column contains 2^{k-1} zeros and 2^{k-1} ones.



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- Hence, the ν errors in the unerased positions can be corrected. As a result the shortened code with e components erased can be recovered.
- Finally, since $d_{\min} \geq e + 1$, there is only one and only one codeword in the original code that agrees with the unerased components. Hence, the entire codeword can be recovered.



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- Suppose that x and y are in the same coset. Then $x + y$ is a nonzero code word. The weight of this code word satisfies

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- This is impossible since the minimum weight of the code is $d_{\min}$. Hence x and y are in different cosets. As a result, when x occurs, it will not be mistaken as y . Therefore x is detectable.



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- **Problem # 5** Let C_i be the binary (n, k_i) linear code with generator matrix G_i and minimum distance d_i , respectively. Let C be the binary $(2n, k_1 + k_2)$ linear code with generator matrix

$$G = \begin{bmatrix} G_1 & G_1 \\ \mathbf{0} & G_2 \end{bmatrix}$$

where $\mathbf{0}$ is a $k_2 \times n$ zero matrix. Calculate the minimum distance of C . Prove your result.



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- **Solution:** Let $\mathbf{u} = (u_0, u_1, \dots, u_{n-1})$ and $\mathbf{v} = (v_0, v_1, \dots, v_{n-1})$ be two binary n -tuples. We form $2n$ -tuple from $\mathbf{u}$ and $\mathbf{v}$ as follows

$$|\mathbf{u}|\mathbf{u} + \mathbf{v}| = (u_0, u_1, \dots, u_{n-1}, u_0 + v_0, u_1 + v_1, \dots, u_{n-1} + v_{n-1})$$



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- The linear block code C is

$$\begin{aligned} C &= |C_1|C_1 + C_2| \\ &= \{|\mathbf{u}|\mathbf{u} + \mathbf{v}| : \mathbf{u} \in C_1, \text{ and } \mathbf{v} \in C_2\} \end{aligned}$$

Navigation icons: back, forward, search, etc.

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- Let $\mathbf{x} = [\mathbf{u} | \mathbf{u} + \mathbf{v}]$ and $\mathbf{y} = [\mathbf{u}' | \mathbf{u}' + \mathbf{v}']$ be two distinct codewords in C .

$$d(\mathbf{x}, \mathbf{y}) = w(\mathbf{u} + \mathbf{u}') + w(\mathbf{u} + \mathbf{u}' + \mathbf{v} + \mathbf{v}')$$

where $w(\mathbf{z})$ is the Hamming weight of $\mathbf{z}$.



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- Consider two cases $\mathbf{v} = \mathbf{v}'$ and $\mathbf{v} \neq \mathbf{v}'$. If $\mathbf{v} = \mathbf{v}'$, since $\mathbf{x} \neq \mathbf{y}$, we must have $\mathbf{u} \neq \mathbf{u}'$. In this case

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- Since $\mathbf{u} + \mathbf{u}'$ is a nonzero codeword in C_1 , $w(\mathbf{u} + \mathbf{u}') \geq d_1$. Therefore

$$d(\mathbf{x}, \mathbf{y}) \geq 2d_1 \quad (3)$$



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$$d(\mathbf{x}, \mathbf{y}) = w(\mathbf{u} + \mathbf{u}') + w(\mathbf{u} + \mathbf{u}')$$

- Since $\mathbf{u} + \mathbf{u}'$ is a nonzero codeword in C_1 , $w(\mathbf{u} + \mathbf{u}') \geq d_1$. Therefore

$$d(\mathbf{x}, \mathbf{y}) \geq 2d_1 \quad (3)$$

- From triangle inequality, we have

$$d(\mathbf{x}, \mathbf{y}) \geq d(\mathbf{x}, \mathbf{z}) - d(\mathbf{y}, \mathbf{z})$$

$$w(\mathbf{x} + \mathbf{y}) \geq wt(\mathbf{x} + \mathbf{z}) - wt(\mathbf{y} + \mathbf{z})$$



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- If $\mathbf{v} \neq \mathbf{v}'$, we have

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) &\geq w(\mathbf{u} + \mathbf{u}') + w(\mathbf{v} + \mathbf{v}') - w(\mathbf{u} + \mathbf{u}') \\ &= w(\mathbf{v} + \mathbf{v}') \end{aligned}$$



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- Since $\mathbf{v} + \mathbf{v}'$ is a nonzero codeword in C_2 , $w(\mathbf{v} + \mathbf{v}') \geq d_2$, we have

$$d(\mathbf{x}, \mathbf{y}) \geq d_2 \quad (4)$$



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- **Problem #5 (contd.):** Let $\mathbf{x} + \mathbf{z} = \mathbf{v} + \mathbf{v}'$ and $\mathbf{y} + \mathbf{z} = \mathbf{u} + \mathbf{u}'$, then we get

$$w(\mathbf{u} + \mathbf{u}' + \mathbf{v} + \mathbf{v}') \geq w(\mathbf{v} + \mathbf{v}') - w(\mathbf{u} + \mathbf{u}')$$

- If $\mathbf{v} \neq \mathbf{v}'$, we have

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) &\geq w(\mathbf{u} + \mathbf{u}') + w(\mathbf{v} + \mathbf{v}') - w(\mathbf{u} + \mathbf{u}') \\ &= w(\mathbf{v} + \mathbf{v}') \end{aligned}$$

- Since $\mathbf{v} + \mathbf{v}'$ is a nonzero codeword in C_2 , $w(\mathbf{v} + \mathbf{v}') \geq d_2$, we have

$$d(\mathbf{x}, \mathbf{y}) \geq d_2 \quad (4)$$

- From (3) and (4) we have

$$d(\mathbf{x}, \mathbf{y}) \geq \min \{2d_1, d_2\}$$



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- The vector $|u_0|u_0|$ is a codeword in C with weight $2d_1$.
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- Therefore

$$d(\mathbf{x}, \mathbf{y}) = \min \{2d_1, d_2\}$$