

Error control coding: An introduction to linear block codes

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Mar. 14, 2016



Lecture #3B: Problem solving session-I



Linear block code

- **Problem # 1:** Consider a linear block code, C with parity check matrix given by

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

What is (n, k) of C ?



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- **Solutions:** Rank of $\mathbf{H}$ matrix is 3. So, $n = 7$, $k = 7 - 3 = 4$.



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$$C = \{000000, 110011, 011101, 111111\}$$

Is C a linear block code? Justify your answer.



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- Sum of two codewords for a linear block code is a valid codeword.



Linear block code

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Is C a linear block code? Justify your answer.

- **Solutions:** No.
- Sum of two codewords for a linear block code is a valid codeword.
- Let $v_0 = 000000$, $v_1 = 110011$, $v_2 = 011101$, and $v_3 = 111111$, then $v_1 + v_2$, $v_1 + v_3$, $v_2 + v_3$, and $v_1 + v_2 + v_3$ must also be a valid codeword.

$$v_1 + v_2 = 101110$$

$$v_1 + v_3 = 001100$$

$$v_2 + v_3 = 100010$$

$$v_1 + v_2 + v_3 = 010001$$



Problem # 2 (contd.)

- Thus a linear block code should have the following codewords

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- This is a (6,3) linear binary code.



Problem # 2 (contd.): Generator matrix in systematic form

- How to write the generator matrix in systematic form?

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

- Row 3 $\rightarrow$ Row 3 + Row 1

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- Row 2 $\rightarrow$ Row 3 + Row 2

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$



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$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

- Similarly parity check matrix in systematic form can be written as

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$



Linear block code

- Problem # 3:** Let $\mathbf{H}$ be the parity check matrix of an (n,k) linear code C that has both odd and even-weight codewords. Construct a new linear code C_1 with the following parity-check matrix

$$\mathbf{H}_1 = \left[\begin{array}{c|c} 0 & \\ 0 & \\ \vdots & \mathbf{H} \\ 0 & \\ \dots & \dots \\ 1 & 11\dots 1 \end{array} \right]$$

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Navigation icons: back, forward, search, etc.

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- Thus all the rows of $\mathbf{H}_1$ are linearly independent. Hence the row space of $\mathbf{H}_1$ has dimension $n - k + 1$.



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- The inner product of a vector with odd weight and the all-one vector is “1”. Hence, for any odd weight vector $\mathbf{v}$,

$$\mathbf{v}\mathbf{H}_1^T \neq 0$$

and $\mathbf{v}$ cannot be a code word in C_1 .



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- Therefore, C_1 consists of only even-weight code words.



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- For $\mathbf{v}_1$ to be a vector in C_1 , we must require that

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- Note that the inner product of v_1 with any of the first $n-k$ rows of $\mathbf{H}_1$ is 0.



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- Therefore, any vector v_1 formed as above is a codeword in C_1 , there are 2^k such codewords.



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- Therefore, any vector v_1 formed as above is a codeword in C_1 , there are 2^k such codewords.
- The dimension of C_1 is k , these 2^k codewords are all the code words of C_1 .