Module 2 : Signals in Frequency Domain
Lecture 17 : Fourier Transform of periodic signals and some Basic Properties of Fourier Transform

Objectives

In this lecture you will learn the following

- Fourier Transform of **Periodic** signals
- Fourier transform of **x(-t)**
- Fourier transform of conjugate of **x(t)**
- The Fourier transform of an **even** signal
- The Fourier transform of a **real** signal

Fourier Transform of Periodic signals.

We know the Fourier transform of the signal that assumes the value 1 identically is the dirac-delta function.

$$\boxed{1 \xrightarrow{F.T.} \delta(f)}$$

By the property of translation in the frequency domain, we get:

$$\boxed{e^{j2\pi f_0 t} \xrightarrow{F.T.} \delta(f - f_0)}$$

This is the result we will make use of in this section.

Suppose **x(t)** is a periodic signal with the period **T**, which admits a Fourier Series representation. Then,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$$

$$where\ c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j2\pi k f_0 t}\, dt$$

Now since the Fourier transformation is linear, the above result can be used to obtain the Fourier Transform of the periodic signal x(t):

$$X(f) = \Sigma\ c_k\ (\text{ Fourier transform of } e^{j(2\pi/T)kt}\ )$$

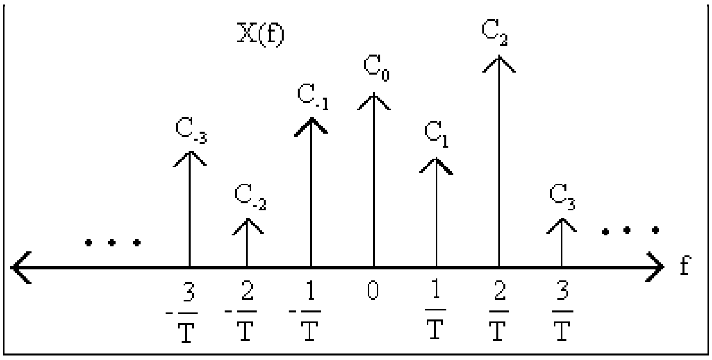
Therefore, $$\boxed{X(f) = \sum_{-\infty}^{\infty} c_k\ \delta(f - \frac{k}{T})}$$

By putting this transform in inverse Fourier transform equation, one can indeed confirm that one obtains back the Fourier series representation of **x(t)**.

$$\begin{aligned}\int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df &= \int_{-\infty}^{\infty} \sum_{-\infty}^{\infty} c_k \delta(f - \frac{k}{T})\ e^{j2\pi f t}\ df \\ &= \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_k \delta(f - \frac{k}{T})\ e^{j2\pi f t}\ df \\ &= \sum_{-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T} t}\end{aligned}$$

Thus, the Fourier transform of a periodic signal having the Fourier series coefficients c_k is a train of impulses, occurring at multiples of the fundamental frequency, the strength of the impulse at $\frac{k}{T}$ being c_k.

This looks like:



Basic Properties of Fourier Transform

Consider a signal **x(t)** with Fourier transform **X(f)**. We'll see what happens to the Fourier transform of x(t) on **time-reversal** and **conjugation**. i.e:

$$x(t) \xrightarrow{\text{F.T.}} X(f)$$

$$x(-t) \xrightarrow{\text{F.T.}} ?$$

$$\overline{x(t)} \xrightarrow{\text{F.T.}} ?$$

Now, we are aware that $\mathbf{X(f)} = \int_{-\infty}^{\infty} \mathbf{x(t)\, e^{-j2\pi f t}\, dt}$

Transform **X'(f)** of **x(-t)** is: $\boxed{X'(f) = \int_{-\infty}^{\infty} x(-t)\, e^{j2\pi f t}\, dt}$.

Substitute $\boxed{t = -\lambda,}$

Therefore, $\boxed{X'(f) = \int_{-\infty}^{\infty} x(\lambda)\, e^{-j2\pi f \lambda}\, d\lambda = X(-f)}$.

Therefore, $\boxed{x(-t) \xrightarrow{\text{F.T.}} X(-f)}$

Applying this result to periodic signals (we have just seen their Fourier transform), you see that if c_k is the k^{th} Fourier Series co-efficient of a periodic signal x(t), c_{-k} is the k^{th} Fourier series co-efficient of x(-t).

Now lets see how the Fourier Transform of $\overline{x(t)}$ is related to that of x(t).

Starting with $\mathbf{X(f)} = \int_{-\infty}^{\infty} \mathbf{x(t)\, e^{-j2\pi f t}\, dt}$

taking conjugates, we get : $\overline{\mathbf{X(f)}} = \int_{-\infty}^{\infty} \overline{\mathbf{x(t)}}\ \mathbf{e^{\,j2\pi f t}\ dt}$

Thus, $\overline{\mathbf{X(-f)}} = \int_{-\infty}^{\infty} \overline{\mathbf{x(t)}}\ \mathbf{e^{-j2\pi f t}\ dt}$

And, therefore,

$$\boxed{\overline{x(t)} \xrightarrow{\text{F.T.}} \overline{X(-f)}}$$

Applying this in the context of periodic signals, we see that if c_k is the k^{th} Fourier Series co-efficient of a periodic signal **x(t)**, then $\overline{c_{-k}}$ is the k^{th} Fourier series co-efficient of $\overline{x(t)}$.

Let us look at some simple consequences of these properties:

a) What can we say about the Fourier transform of an **even signal x(t)** (with Fourier transform X(f)) ?

x(-t) has Fourier transform **X(-f)**. As x(t) is real, **x(t) = x(-t)**, implying, **X(f) = X(-f)**.

Thus, the Fourier transform of an even signal is even. Similarly, you can show the Fourier transform of an odd signal is odd.

b) What can we say about the Fourier transform of a **real signal x(t)**, with Fourier transform X(f) ?

If x(t) is real,

$$x(t) = \overline{x(t)} \quad \text{for every } t, \qquad \text{implies } X(f) = \overline{X(-f)} \quad \text{for every } f.$$

Thus the Fourier transform of a real signal is **Conjugate Symmetric**.

Conclusion:

In this lecture you have learnt:

- Fourier transformation is linear .
- Fourier transform of **x(-t)** is **X(-f)**.
- Fourier transform of conjugate of x(t) is conjugate of X(-f).
- The Fourier transform of an even signal is even
- The Fourier transform of a real signal is Conjugate Symmetric .

Congratulations, you have finished Lecture 17.