Module 2 : Signals in Frequency Domain
Lecture 13 : Fourier Series Representation of Periodic Signals

Objectives

In this lecture you will learn the following

- Fourier Series representation of Periodic Signals
- Set of periodic signals as a vector space
- Orthogonality of $e^{j2\pi f_0 kt}$, $(k \in \mathbb{Z})$
- Frequency Domain Representation.

Fourier Series representation of Periodic Signals

Consider a periodic signal **x(t)** with fundamental period **T**, i.e. $x(t+T) = x(t) \quad \forall t$ Then the fundamental frequency of this signal is defined as the reciprocal of the fundamental period, so that $f_0 = \frac{1}{T}$ Under certain conditions, a periodic signal x(t) with period T can be expressed as a **linear combination of sinusoidal signals of discrete frequencies, which are multiples of the fundamental frequency of x(t)**. Further, sinusoidal signals are conveniently represented in terms of complex exponential signals. Hence, we can express the periodic signal in terms of complex exponentials, i.e.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi f_0 kt}$$

Such a representation of a periodic signal as a combination of complex exponentials of discrete frequencies, which are multiples of the fundamental frequency of the signal, is known as the **Fourier Series Representation of the signal.**

Inner product

The set of periodic signals with period T form a **vector space**.

We define the following inner product: $\langle x_1(t), x_2(t) \rangle = \int_0^T x_1(t)\overline{x_2(t)}dt$

And the norm or magnitude of the signal is defined as:- $\left|x(t)\right| = \langle x(t), x(t) \rangle^{\frac{1}{2}}$

Now we consider the set of vectors $e^{j2\pi f_0 kt}$, $(k \in \mathbb{Z})$, that belong to this vector space. (note $f_0 = \frac{1}{T}$)

We shall first show these vectors are **mutually orthogonal**. In other words we show that:-

$$\langle e^{j2\pi f_0 kt}, e^{j2\pi f_0 lt} \rangle = 0 \qquad \forall k \neq l$$

$$\begin{aligned}
\langle e^{j2\pi f_0 kt}, e^{j2\pi f_0 lt} \rangle &= \int_0^T e^{j2\pi f_0 kt} e^{-j2\pi f_0 lt} dt \\
&= \int_0^T e^{j2\pi f_0 (k-l)t} dt \\
&= \frac{\left[e^{j2\pi f_0 (k-l)t}\right]_0^T}{j2\pi f_0 (k-l)} \\
&= 0 \quad \forall k \neq l
\end{aligned}$$

Further, you may verify : $\langle e^{j2\pi f_0 kt}, e^{j2\pi f_0 kt} \rangle = \left|e^{j2\pi f_0 kt}\right|^2 = \mathrm{T}$

Thus, we have shown that this set of complex exponentials forms an **orthogonal set** in the vector space of all periodic signals with period T. Indeed, if we restrict ourselves to a certain class of signals in this vector space (those that satisfy the **Dirichlet Conditions**, which will be discussed in the next lecture), one can show that the above set of complex exponentials forms a basis for this class. i.e.: signals in this class can be expressed as a linear combination of these complex exponentials. In other words, such signals permit a Fourier Series representation.

Assuming the Fourier Series representation of a signal x(t), with period T exists, it is easy to find the Fourier Series coefficients, using the orthogonality of the basis set of complex exponentials.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi f_0 kt}$$

Taking inner product with $e^{j2\pi f_0 kt}$ on both sides

$$\begin{aligned} < x(t), e^{j2\pi f_0 kt} > &= < \sum_{k=-\infty}^{+\infty} c_k e^{j2\pi f_0 kt}, e^{j2\pi f_0 kt} > \\ &= \sum_{k=-\infty}^{+\infty} c_k < e^{j2\pi f_0 kt}, e^{j2\pi f_0 kt} > \\ &= c_k T \qquad (\text{all other terms drop out}) \\ \therefore\ c_k &= \frac{1}{T}\int_0^T x(t) e^{-j2\pi f_0 kt} dt \end{aligned}$$

Frequency Domain Representation

From the above discussion, we can say that a periodic signal whose Fourier Series Expansion exists, can be represented uniquely in terms of it's Fourier co-efficients. These co-efficients correspond to a particular multiples of the fundamental frequency of the signal. Thus, the signal may be equivalently represented as a discrete signal on the frequency axis:

This is called the **Frequency domain representation** of the signal.

We next discuss the conditions under which the Fourier Expansion is valid.

Conclusion:

In this lecture you have learnt:

- A representation of a periodic signal as a combination of complex exponentials of discrete frequencies, which are multiples of the fundamental frequency of the signal, is known as the Fourier Series Representation of the signal.
- Set of periodic signals, with period **T** are a vector space
- Orthogonality of $e^{j2\pi f_0 kt}, (k \in \mathbb{Z})$
- Calculating the Fourier series coefficients for a periodic signal.
- Frequency Domain Representation of a periodic signal

Congratulations, you have finished Lecture 14.