Module 2 : Signals in Frequency Domain
Lecture 19 : Periodic Convolution and Auto-Correlation

Objectives

In this lecture you will learn the following

- To look at a modified definition of convolution for periodic signals
- Circular convolution
- Parseval's theorem
- Convolution theorem in the context of periodic convolution.
- Auto correlation
- Cross correlation

Periodic Convolution

We have applied the convolution theorem to convolutions involving:

(i) two aperiodic signals

(ii) one aperiodic and one periodic signal.

But, convolutions between periodic signals diverge, and hence the convolution theorem cannot be applied in this context. However a modified definition of convolution for periodic signals whose **periods are rationally related** is found useful. We look at this definition now. Later, we will prove a result similar to the Convolution theorem in the context of periodic signals.

Consider the following signals

x(t) periodic with period $\mathbf{T_1}$ and **h(t)** periodic with period $\mathbf{T_2}$ where T_1 and T_2 are rationally related.

Let $\mathbf{T_1 / T_2 = m / n}$ (where m and n are integers)

Hence, $\mathbf{m\,T_2 = n\,T_1 = T}$ is a common period for both x(t) and h(t).

Periodic convolution or **circular convolution** of **x(.)** with **h(.)** is denoted by $x \otimes h$ and is defined as :

$$x \otimes h = \frac{1}{T}\int_0^T h(\lambda)x(t-\lambda)\,d\lambda$$

Note the definition holds even if T is not the smallest common period for x(t) and h(t) due to the division by T. Thus we don't need m and n to be the smallest possible integers satisfying $\mathbf{T_1 / T_2 = m / n}$ in the process of finding T.

Also, show for yourself that the periodic convolution is commutative, i.e: $x \otimes h = h \otimes x$. Also, notice that the convolution is periodic with period $\mathbf{T_1}$ as well as $\mathbf{T_2}$. More on this later.

Fourier Transform of $\boxed{y = x \otimes h}$

Say **x(t)** is periodic with period $\mathbf{T_1}$ and **h(t)** is periodic with period $\mathbf{T_2}$ with $\mathbf{T_1 / T_2 = m / n}$ (where m and n are integers).

Thus $\mathbf{m\,T_2 = n\,T_1 = T}$ is a common period for the two.

We can expand x(t) and h(t) into Fourier Series with fundamental frequency $\frac{1}{\mathrm{T}}$:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{j2\pi \frac{k}{T} t} \quad \& \quad h(t) = \sum_{l=-\infty}^{\infty} d_l\, e^{j2\pi \frac{l}{T} t}$$

If one compares the Fourier co-efficients in these expansions with those in the expansions with the original fundamental frequencies, i.e:

$$x(t) = \sum_{k=-\infty}^{\infty} \tilde{c}_k\, e^{j2\pi \frac{k}{T_1} t} \quad \& \quad h(t) = \sum_{l=-\infty}^{\infty} \tilde{d}_l\, e^{j2\pi \frac{l}{T_2} t}$$, we find:

$$c_k = \tilde{c}_{k/n} \quad \text{when } k \text{ is a multiple of n} \qquad d_l = \tilde{d}_{l/m} \quad \text{when } l \text{ is a multiple of m}$$
$$0 \quad \text{else} \qquad\qquad 0 \quad \text{else}$$

Now,

$$x \otimes h = \int_0^T h(\lambda)x(t-\lambda)d\lambda$$

$$= \int_0^T h(\lambda)\left\{ \sum_{k=-\infty}^{\infty} c_k e^{j2\pi\frac{k}{T}(t-\lambda)} \right\}d\lambda$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi\frac{k}{T}t} \int_0^T h(\lambda)e^{-j2\pi\frac{k}{T}\lambda} d\lambda$$

$$= \sum_{k=-\infty}^{\infty} c_k d_k e^{j2\pi\frac{k}{T}t}$$

But then, we have seen that c_k can be non-zero only when k is a multiple of n, and d_k can be non-zero only when k is a multiple of m. Their product can clearly be non-zero only when k is a multiple of m and n. Thus if p is the LCM (least common multiple) of m and n, we have:

$$x \otimes h = \sum_{k=-\infty}^{\infty} c_{\mathrm{kp}} d_{\mathrm{kp}}\ e^{j2\pi\frac{\mathrm{kp}}{\mathrm{T}}t}$$

What can we make out of this?

The Fourier Transform of the circular convolution has impulses at all (common) frequencies where the Fourier transforms of x(t) and h(t) have impulses. The circular convolution therefore "picks out" common frequencies, at which the spectra of x(t) and h(t) are non-zero and the strength of the impulse at that frequency is the product of the strengths of the impulses at that frequency in the original two spectra.

This result is the equivalent of the Convolution theorem in the context of periodic convolution.

Parseval's Theorem

We now obtain the result equivalent to the Parseval's theorem we have already seen in the context of periodic signals.

Let x(t) and y(t) be periodic with a common period T.

$$x(t) = \sum_{-\infty}^{\infty} c_k e^{j2\pi\frac{k}{T}t} \quad \& \quad h(t) = \sum_{-\infty}^{\infty} d_k e^{j2\pi\frac{k}{T}t}$$

Applying the Convolution theorem equivalent we have just proved on $x(t)$ and $\overline{y(-t)}$ we get:

$$\frac{1}{T}\int_{(T)} x(\lambda)\ \overline{y(\lambda-t)}\ d\lambda = \sum_{-\infty}^{\infty} c_k\ \overline{d_k}\ e^{j2\pi\frac{k}{T}t}$$

Put t = 0, to get: $\boxed{\frac{1}{T}\int_{(T)} x(\lambda)\ \overline{y(\lambda)}\ d\lambda = \sum_{-\infty}^{\infty} c_k\ \overline{d_k}}$

Compare this equation with the Parseval's theorem we had proved earlier.

If we take x = y, then T becomes the fundamental period of x and:

$$\boxed{\frac{1}{T}\int_{(T)} \left|x(t)\right|^2 dt = \sum_{-\infty}^{+\infty} \left|c_k\right|^2}$$

Note the left-hand side of the above equation is the power of x(t).

Note also that the periodic convolution of $x(t)$ and $\overline{x(-t)}$ yields a periodic signal with Fourier coefficients that are the modulus square of the coefficients of x(t).

Another important result

If, $y(t) = x(t) \otimes h(t)$

Then $\frac{1}{T}\int_{(T)} |y(t)|^2 dt$ represents the power of y(t), where T is a period common to x(t) and h(t).

If , $x(t) = \sum_{-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T} t}$ & $h(t) = \sum_{-\infty}^{\infty} d_k e^{j2\pi \frac{k}{T} t}$

$$y = x \otimes h = \sum_{k=-\infty}^{\infty} c_k d_k e^{j2\pi \frac{k}{T} t}$$

Applying the Parseval's theorem to y,

$$\frac{1}{T}\int_{(T)} |y(t)|^2 dt = \sum_{-\infty}^{\infty} |c_k|^2 |d_k|^2$$

The Auto-correlation and the Cross-correlation.

Proceeding with our work on the Fourier transform, let us define two important functions, the Auto-correlation and the Cross-correlation.

Auto Correlation

You have seen that for a Periodic signal y(t), $y(t) \otimes \overline{y(-t)}$ has Fourier series coefficients that the modulus square of the Fourier series coefficients of y(t).

Lets look at an equivalent situation with aperiodic signals, i.e:

Assume that $x(t) \xrightarrow{FT} X(f)$

then $? = \xrightarrow{FT} |X(f)|^2$

Notice that $|X(f)|^2 = X(f)\overline{X(f)}$

Since $\overline{x(t)} \xrightarrow{FT} \overline{X(-f)}$

We have, $\overline{x(-t)} \xrightarrow{FT} \overline{X(f)}$

Using the dual of the convolution theorem,

$$\begin{aligned} |X(f)|^2 \xrightarrow{FT^{-1}} & x(t) * \overline{x(-t)} \\ &= \int_{-\infty}^{\infty} x(t-\lambda)\, \overline{x(-\lambda)}\, d\lambda \\ &= \int_{-\infty}^{\infty} x(t+\gamma)\, \overline{x(\gamma)}\, d\gamma \end{aligned}$$

The auto-correlation of x(t), denoted by R_{xx} is defined as:

$$R_{XX}(t) = \int_{-\infty}^{+\infty} x(t+\gamma)\overline{x(\gamma)}\, d\gamma$$

Its Spectrum is the modulus square of the spectrum of x(t).

It can also be interpreted as the projection of **x(t)** on its own shifted version, shifted back by an interval ‘t'.

It can be shown that $R_{xx}(t) <= R_{xx}(0)$ (note that $R_{xx}(0)$ is nothing but the energy in the signal **x(t)**)

Cross Correlation

The cross correlation between two signals x(t) and y(t) is defined as : $R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\overline{y(t)}dt$

Note that the cross-correlation R_{xy} is the convolution of $x(t)$ and $\overline{y(-t)}$.

If $y(t) = x(t-\tau_0)$ then using the fact that the auto-correlation integral peaks at 0 , the cross correlation peaks at $\tau = -\tau_0$.

It may be said that cross-correlation function gives a measure of resemblance between the shifted versions of signal x(t) and y(t). Hence it is used to in Radar and Sonar applications to measure distances . In these systems, a transmitter transmits signals which on reflection from a target are received by a receiver. Thus the received signal is a time shifted version of the transmitted signal . By seeing where the cross-correlation of these two signals peaks, one can determine the time shift and hence the distance of the target.

The Fourier transform of $R_{xy}(t)$ is of-course $\hat{R}_{xy}(f) = X(f)\ \overline{Y(F)}$

Conclusion:

In this lecture you have learnt:

- Periodic convolution or circular convolution of **x(.)** with **h(.)** is denoted by $x \otimes h$ and is defined as :

$$x \otimes h = \frac{1}{T}\int_0^T h(\lambda)x(t-\lambda)d\lambda$$

- Fourier Transform of $y = x \otimes h$
- Parseval's theorem in the context of periodic signals is $\boxed{\frac{1}{T}\int_{(T)} |x(t)|^2\, dt = \sum_{-\infty}^{+\infty} |c_k|^2}$
- Auto correlation is defined as $R_{XX}(t) = \int_{-\infty}^{+\infty} x(t+\gamma)\overline{x(\gamma)}d\gamma$
- Cross correlation is defined as $R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\overline{y(t)}dt$

Congratulations, you have finished Lecture 19.