Module 2 : Signals in Frequency Domain
Lecture 14 : Convergence of Fourier Series and Gibb's Phenomenon

Objectives

In this lecture you will learn the following

- To study convergence in 2 different contexts.
- Dirichlet Conditions For Pointwise Convergence .
- Condition for convergence in squared norm .
- To understand Gibb's Phenomenon

Let $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi f_0 kt}$ be the Fourier series expansion corresponding to the periodic signal **x(t)** (i.e: the c_k's are as calculated by the formula in the previous lecture). Then the above summation may or may not converge to the actual signal **x(t)**.

We shall discuss the convergence of the Fourier series representation of a periodic signal in two contexts, namely **Pointwise convergence** and **Convergence in squared norm**. We will first see what each of these terms means and then discuss the conditions under which each kind of convergence takes place.

For the subsequent discussion let,

$$\tilde{x}_N(t) = \sum_{k=-N}^{N} c_k e^{j2\pi f_0 kt} \qquad \boxed{\Rightarrow \quad \tilde{x}(t) = \lim_{N \to \infty} \tilde{x}_N(t)}$$

Pointwise Convergence

Pointwise convergence implies the series converges to the original function at any point, i.e: the Fourier Series representation of a signal x(t) is said to converge pointwise to the signal x(t) if:

$$\lim_{N \to \infty} \tilde{x}_N(t) = x(t) \qquad \forall t$$

i.e to say $\tilde{x}(t) \equiv x(t)$.

Convergence in squared norm

The Fourier Series representation is said to converge in the sense of squared norm to the signal x(t) if

$$\lim_{N \to \infty} \int_0^T \left| x(t) - x_N(t) \right|^2 dt = 0$$

Pointwise convergence implies convergence in squared norm. As convergence in squared norm is a more relaxed condition than pointwise convergence, convergence in the squared norm sense covers a much larger domain of signals than pointwise convergence.

Finally, we now move on to the conditions for these forms of convergence.

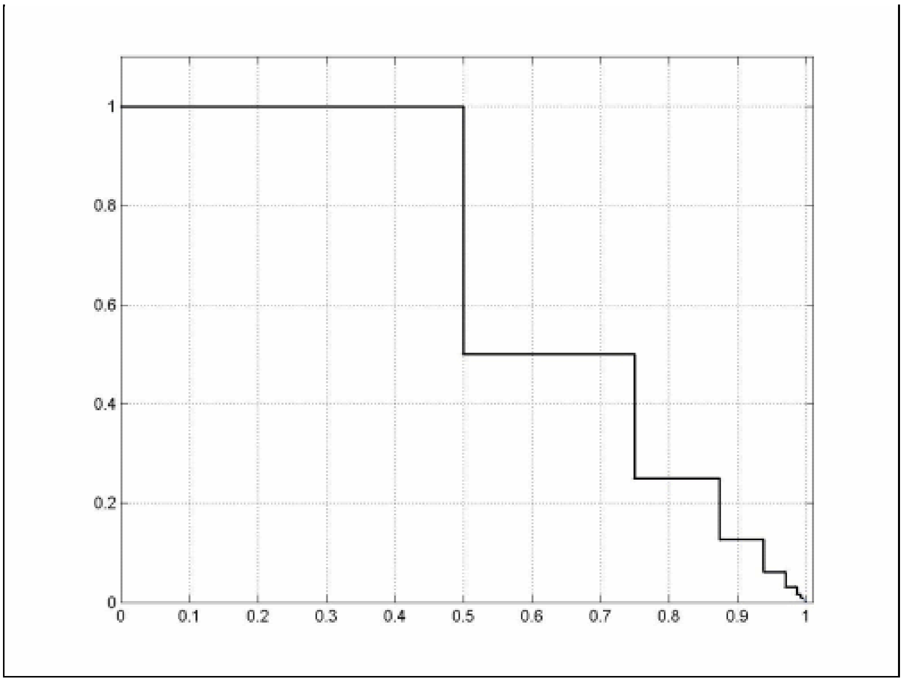
Dirichlet Conditions For Pointwise Convergence

Consider the following 3 conditions that may be imposed on a periodic signal **x(t)** :

1) x(t) should be **absolutely integrable** over a period.

A signal that does not satisfy this condition is **x(t) = tan(t)** as:- $\boxed{\int_{-\pi/2}^{\pi/2} |tan(x)|\, dx \text{ does not exist}}$

2) x(t) should have only a **finite number of discontinuities** over one period. Furthermore, each of these discontinuities must be finite. An example of a function which has infinite number of discontinuities is illustrated below. The function is shown over one of the periods.



3) The signal **x(t)** should have only a **finite number of maxima and minima** in one period. An example of a function which has infinite number of maxima and minima is: a periodic signal with period 1, defined on (0,1] as: $\boxed{x(t) = \sin(\frac{1}{t})}$.

If the signal satisfies the above conditions, then at all points where the signal is continuous, the Fourier Series converges to the signal. However, at points where the signal is discontinuous (Dirichlet conditions allow finite number of discontinuities in a period), the Fourier Series converges to the average of the left and the right hand limits of the signal. Mathematically, at a point of discontinuity $\boxed{t_o}$,

$$\boxed{\tilde{x}(t_o) = \frac{\lim_{x \to t_o^+} x(t) + \lim_{x \to t_o^-} x(t)}{2}}$$

In practice, the restrictions imposed on signals by the Dirichlet conditions are not very severe, and most of the signals we will deal with satisfy these conditions.

Condition for convergence in squared norm sense

If, for a periodic signal **x(t)** with period **T**, $\boxed{\int_0^T | x(t) |^2 \, dt}$ converges, then its Fourier Series converges to it in the **squared norm** sense.

As is expected, this is a far more relaxed constraint than the Dirichlet conditions.

At this point let us define some terms which will be of use to us later in the course:

$\frac{1}{T}\int_0^T | x(t) |^2 \, \mathrm{dt}$ is called the instantaneous power or energy density of the signal **x(t)**.

If **x(t)** is **periodic** with period **T**, and $\boxed{\frac{1}{T}\int_0^T |x(t)|^2 dt \text{ converges \& is finite}}$, **x(t)** is called a finite power signal, and the value of the integral is called the power of the signal.

(Thus we can say, if a periodic signal has finite power, we are guaranteed of convergence in squared norm of its Fourier Series)

If **x(t)** is **non-periodic**, and $\boxed{\int_{-\infty}^{\infty} |x(t)|^2 dt \text{ converges}}$, **x(t)** is said to be a finite energy signal, and the value of the integral is called the energy of the signal.

We now discuss another aspect of the convergence of the Fourier series, the Gibb's Phenomenon

Gibb's Phenomenon

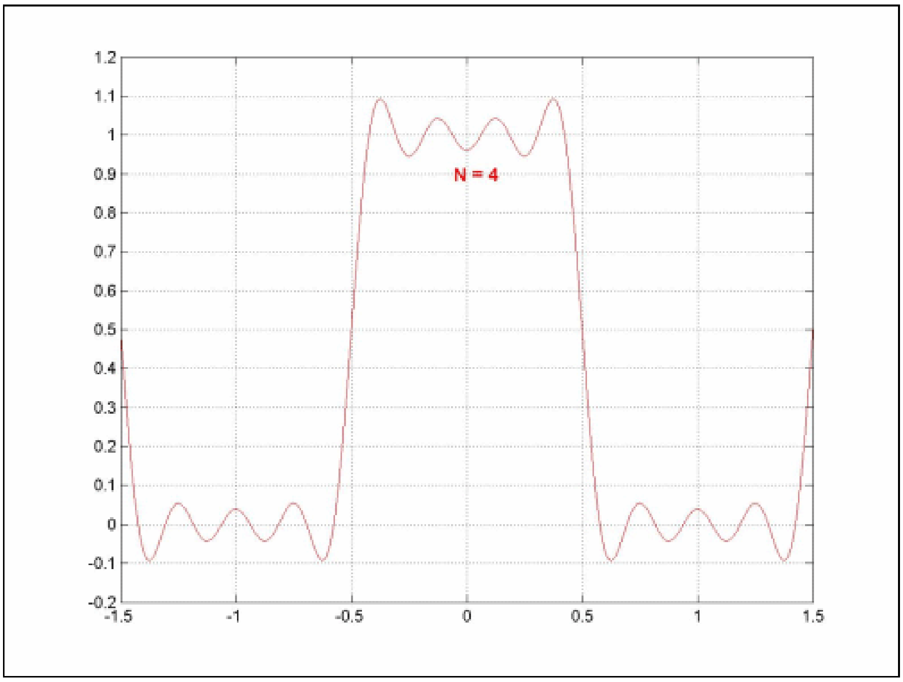
We can approximate a signal having a Fourier Series expansion by taking a finite number of terms of the expansion.

i.e: $\tilde{x}_N(t) = \sum_{k=-N}^{N} c_k e^{j2\pi f_0 kt}$ is an approximation to the periodic signal **x(t)**.

$\tilde{x}_N(t)$ is also called a Partial Sum. We would obviously expect that as the number of terms taken is increased, this summation would become a better and better approximation to **x(t)**, i.e $\tilde{x}_N(t)$ would approach **x(t)** uniformly.

Indeed this happens in regions of continuity of the original signal. However, at the points of discontinuity in the original signal, an interesting phenomenon is observed. The partial sum oscillates near the point of discontinuity. We might expect these oscillations to decrease as the number of terms taken is increased. But surprisingly, as the number of terms taken is increased, although these oscillations get closer and closer to the point of discontinuity, their amplitude does not decrease to zero, but tends to a non zero limit. This phenomenon is known as the **Gibb's Phenomenon**, after the mathematician who accounted for these oscillations.

The illustration below shows the various Fourier approximations of a periodic square wave.



Mathematically, this means if the periodic signal has discontinuities, its Fourier Series does not converge uniformly.

Conclusion:

In this lecture you have learnt:

- We have discussed the convergence of the Fourier series representation of a periodic signal in two contexts, namely Pointwise convergence and Convergence in squared norm .
- Dirichlet Conditions For Pointwise Convergence are: (a)absolute integrabilty (b) finite number of discontinuities over one period (c) finite number of extremas over one period
- If a signal is a finite power signal then it is convergent in squared norm
- The partial sum oscillates near the point of discontinuity. These oscillations do not decrease as the number of terms taken is increased. But in reality, as the number of terms taken is increased, although these oscillations get closer and closer to the point of discontinuity, their amplitude does not decrease to zero, but tends to a non zero limit. This phenomenon is known as the Gibb's Phenomenon.

Congratulations, you have finished Lecture 15.