Module 4 : Laplace and Z Transform
Lecture 35 : Inverse Laplace and Z Transform of Rational Functions

Objectives

Scope of this Lecture:

In the previous lecture we have studied the concepts of poles, zeroes and rational systems. In this lecture we will continue the same rythm and dig deeper concepts.

- We shall look at the Inverse Laplace and Z-transform of rational functions
- We shall solve numericals for a better understanding.

After looking at inverse Laplace and Z - transforms with multiple poles case , we now proceed in a step by step manner towards finding the inverse Laplace and z - Transform of a given function. We will focus only on rational system functions as in earlier cases.

Inverse Laplace transform : Rational functions

Consider an arbitrary rational polynomial in Laplace Transform

$$H(s) = (\text{polynomial in } s) + \frac{N(s)}{D(s)}$$

$$Deg\ N(s) < Deg\ D(s)$$

$\frac{N(s)}{D(s)}$ can be decomposed into partial fractions and

then the inverse laplace transform can be taken.

Examples:

1) Let us consider the function in s:

$$H(s) = A + Bs;^{-1}$$

$$h(t) = A\delta(t) + Bu(t) \text{ for } t > 0$$

2) Let us consider an LTI system with system function:

$$H(S) = \frac{S-1}{(S+1)(S-2)} \qquad \text{i.e.} \qquad H(s) = \frac{2/3}{s+1} + \frac{1/3}{s-2}$$

As the ROC has not been specified. there are several different ROCs and correspondingly, several different system impulses.

Possible ROCs for the system with poles at s = -1and s = 2 and a zero at s = 1

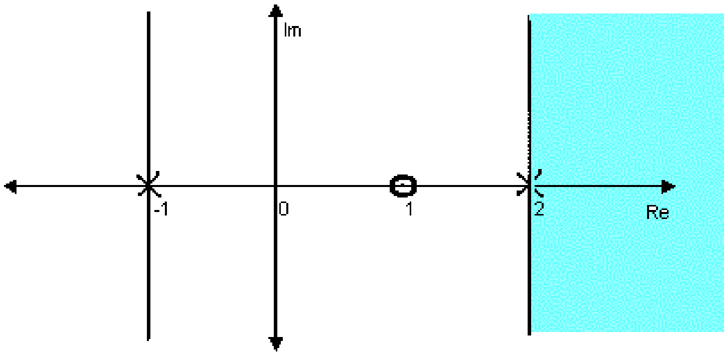

Fig.a Causal, unstable system.

$$h(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t) \qquad \text{ROC: } \Re e\{s\} > 2$$

For this case(fig. a) the system is *causal* and *unstable* since $h(t)$ is not absolutely integrable (it does not include j axis)

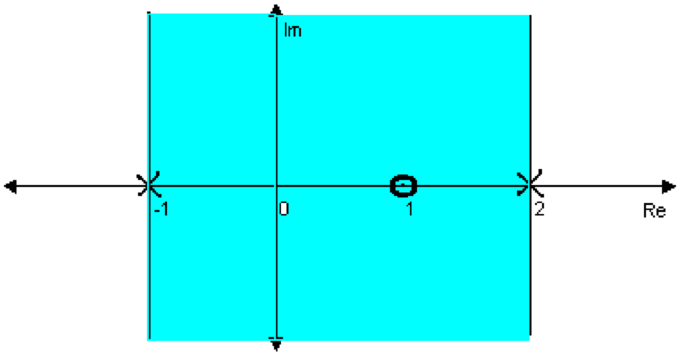

Fig.b noncausal, stable system.

$$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t) \qquad \text{ROC: } -1 < \Re e\{s\} < 2$$

For this case (fig. b)the system is *stable* since $h(t)$ is absolutely integrable .

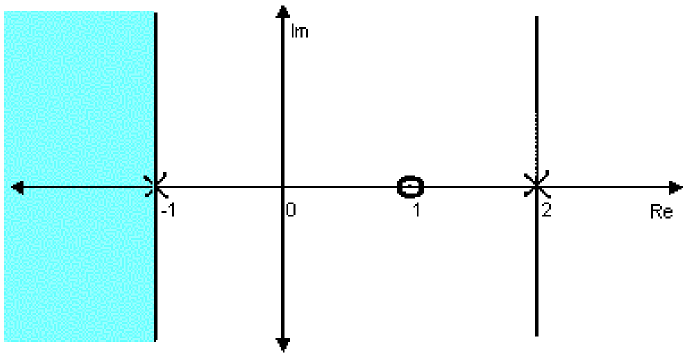

Fig.c Noncausal, unstable system.

$$h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(-t) \qquad \text{ROC: } \Re e\{s\} < -1$$

For this case (fig. c)the system is *noncausal* and *unstable*.

Conclusions:

Properties of certain class of systems can be explained simply in terms of the locations of the poles. Particularly, consider a causal LTI system with a rational system function H(s). Since the system is causal, the ROC is to the right of the right most pole. Consequently, for this system to be stable (i.e. for the ROC to include the j-axis), the right most pole of H(s) must be to the left of the j-axis. i.e.

A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane – i.e., all of the poles have negative real parts.

Inverse Z - transform:

Consider an arbitrary rational z-transform:

$$H(z) = (\text{finite series in powers of } z) + \frac{N(z^{-1})}{D(z^{-1})}$$

$$Deg\, N(z^{-1}) < Deg\, D(z^{-1})$$

$\frac{N(z^{-1})}{D(z^{-1})}$ can be decomposed into partial fractions and then inverse z-transform can be taken using the formula for $Z^{-1}\frac{1}{(1-\beta z^{-1})^M}$.

Examples:

Example 1:

Consider the z transform

$$H(z) = 3z^2 + \frac{1}{2}z + 2 + \frac{1}{5}z^{-1} + 6z^{-4}$$

We know that, $1 \xrightarrow{Z^{-1}} \delta[n]$

$$z^{-n_0} \xrightarrow{Z^{-1}} \delta[n-n_0]$$

$$\therefore h[n] = 3\delta[n+2] + \frac{1}{2}\delta[n+1] + 2\delta[n] + \frac{1}{5}\delta[n-1] + 6\delta[n-4]$$

Example :

Consider the z transform

$$X(z) = \frac{3-z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$$

There are two poles one at z=1/4 and at z=1/3. The partial fraction expansion, expressed in polynomials in 1/z, is

$$X(z) = \frac{1}{(1-\frac{1}{4}z^{-1})} + \frac{2}{(1-\frac{1}{3}z^{-1})}$$

Thus, x[n] is the sum of 2 terms, one with z - transform 1/[1-(1/4z)] and the other with z - transform 2/[1-(1/3z)]. Thus,

$$x[n] = x_1[n] + x_2[n]$$

As the ROC is not mentioned, we get different inverses for different possible ROCs. We do not discuss causality and stability as this may not be a system function. One possible inverse is worked out, the other two left as an exercise to the reader.

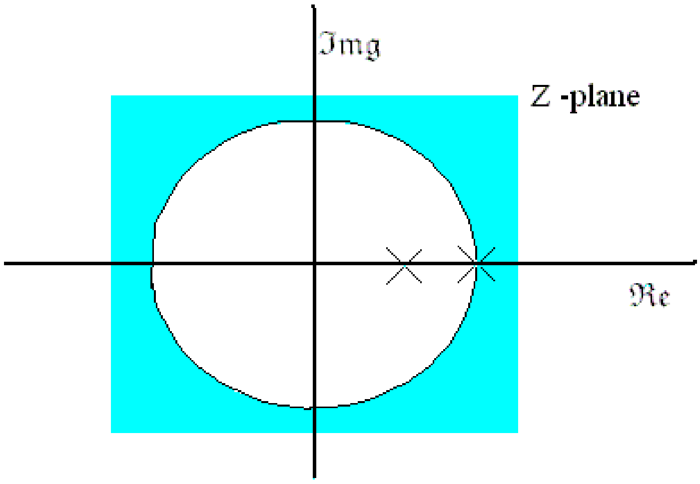

fig d: Pole pattern when ROC is right sided, i.e. |z|>1/3

We can identify by inspection ,

$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n] \qquad \text{ROC}: |z| > \frac{1}{3}$$

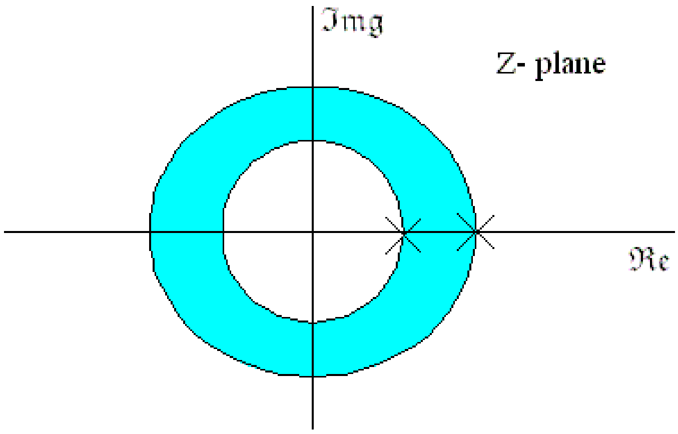

fig e: When the ROC is two-sided, i.e. 1/4<|z|<1/3

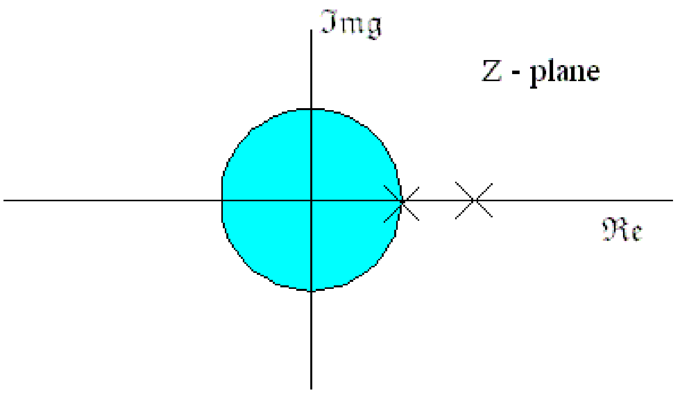

fig f: When the ROC is left sided,i.e. |z|<1/4

Conclusion:

In this lecture you have learnt:

- If the system is causal then the ROC extends from the **right most pole** to infinity.
- A system is stable if the ROC includes the imaginary axis and therefore the right most pole of 'H(s)' must be to the **left** of the imaginary axis
- A causal system with a rational function 'H(s)' is stable if and only if all poles of H(s) lie in the left-half of the *s-plane* and must include the **unit radius circle** in the *z-plane*.

Congratulations, you have finished Lecture 35.