

Module 1 : Signals in Natural Domain

Lecture 6 : Basic Signals in Detail

Objectives

In this lecture you will learn the following

- We shall look at some of the basic signals namely .
- Unit impulse function
- Unit step function
- Their relation in both continuous and discrete domain
- We shall even look at the Sifting property of the unit impulse function.

Basic Signals in detail

We now introduce formally some of the basic signals namely

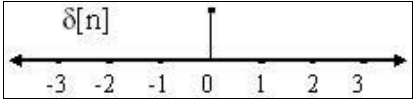
1) The Unit Impulse function.

2) The Unit Step function

These signals are of considerable importance in signals and systems analysis. Later in the course we will see these signals as the building blocks for representation of other signals. We will cover both signals in continuous and discrete time. However, these concepts are easily comprehended in Discrete Time domain, so we begin with Discrete Time Unit Impulse and Unit Step function.

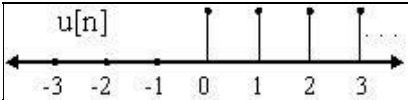
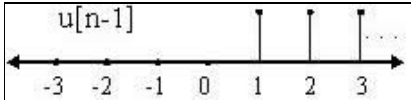
The Discrete Time Unit Impulse Function: This is the simplest discrete time signal and is defined as

$$\delta[n] = 0 \quad \forall n \neq 0$$

$$= 1 \quad \text{for } n = 0$$


The Discrete Time Unit Step Function $u[n]$: It is defined as

$$u[n] = 0 \quad \forall n < 0$$

$$= 1 \quad \forall n \geq 0$$



Unit step in terms of unit impulse function

Having studied the basic signal operations namely **Time Shifting**, **Time Scaling** and **Time Inversion** it is easy to see that

$$\delta[n] = u[n] - u[n-1]$$

similarly,

$$\delta[n-1] = u[n-1] - u[n-2]$$

$$\delta[n-2] = u[n-2] - u[n-3]$$

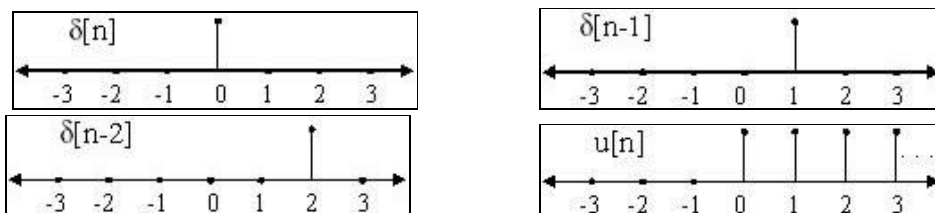
$$\delta[n-3] = u[n-3] - u[n-4]$$

$$\delta[n-4] = u[n-4] - u[n-5]$$

Summing over we get

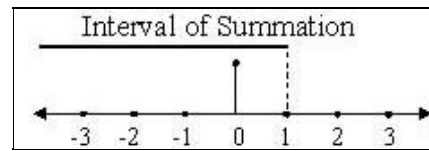
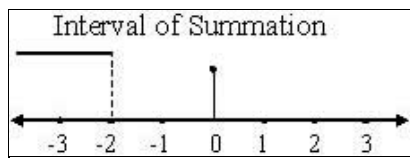
$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

Looking directly at the Unit Step Function we observe that it can be constructed as a sum of shifted Unit Impulse Functions



The unit function can also be expressed as a running sum of the Unit Impulse Function

$$u[n] = \sum_{k=-\infty}^n \delta(k)$$



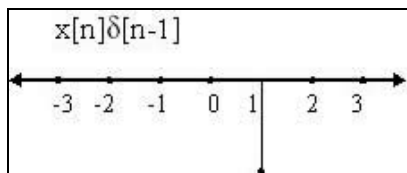
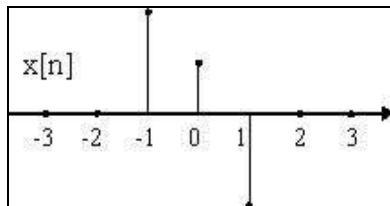
We see that the running sum is **0** for $n < 0$ and equal to **1** for $n \geq 0$ thus defining the Unit Step Function $u[n]$.

Sifting property

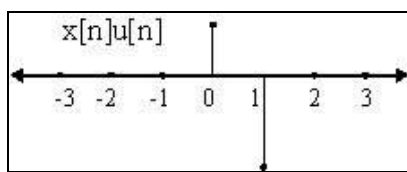
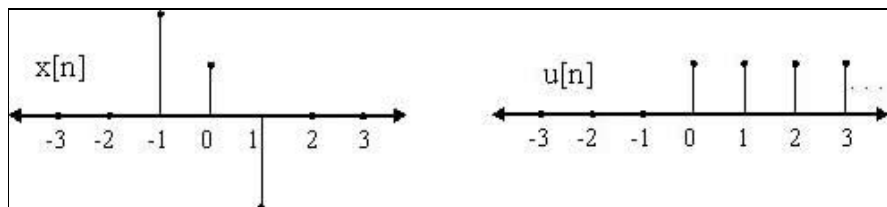
Consider the product $x[n]\delta[n]$. The delta function is non zero only at the origin so it follows the signal is the same as $x[0]\delta[n]$.

More generally $x[n]\delta[n-k] = x[k]\delta[n-k]$

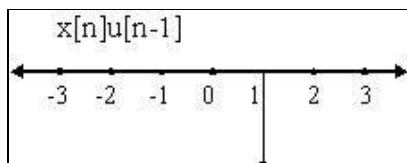
It is important to understand the above expression. It means the product of a given signal $x[n]$ with the shifted Unit Impulse Function is equal to the time shifted Unit Impulse Function multiplied by $x[k]$. Thus the signal is *0* at time *not equal to k* and at time **k** the amplitude is **$x[k]$** . So we see that the unit impulse sequence can be used to obtain the value of the signal at any time k. This is called the Sampling Property of the Unit Impulse Function. This property will be used in the discussion of LTI systems. For example consider the product $x[n]\delta[n-1]$. It gives $x[1]\delta[n-1]$.



Likewise, the product $x[n]u[n]$ i.e. the product of the signal $u[n]$ with $x[n]$ truncates the signal for $n < 0$ since $u[n] = 0$ for $n < 0$



Similarly, the product $x[n]u[n-1]$ will truncate the signal for $n < 1$.



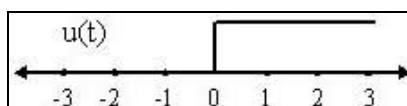
Now we move to the Continuous Time domain. We now introduce the **Continuous Time Unit Impulse Function and Unit Step Function**.

Continuous time unit step and unit impulse functions

The Continuous Time Unit Step Function: The definition is analogous to its Discrete Time counterpart i.e.

$$u(t) = 0, t < 0$$

$$= 1, t \geq 0$$



The unit step function is discontinuous at the origin.

The Continuous Time Unit Impulse Function: The unit impulse function also known as the Dirac Delta Function, was first defined by Dirac as

$$\delta(t) = 0 \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

In the strict mathematical sense the impulse function is a rather delicate concept. The Impulse function is not an ordinary function. An ordinary function is defined at all values of t. The impulse function is 0 everywhere except at t = 0 where it is undefined. This difficulty is resolved by defining the function as a GENERALIZED FUNCTION. **A generalized function is one which is defined by its effect on other functions instead of its value at every instant of time.**

Analogy from discrete domain

We will see that the impulse function is defined by its sampling property. We shall develop the theory by drawing analogy from the Discrete Time domain. Consider the equation

$$u[n] = \sum_{m=-\infty}^{\infty} \delta[m]$$

The discrete time unit step function is a running sum of the delta function. The continuous time unit impulse and unit step function are then related by

$$u(t) = \int_{-\infty}^t \delta(t) dt$$

The continuous time unit step function is a running integral of the delta function. It follows that the continuous time unit impulse can be thought of as the derivative of the continuous time unit step function.

$$\delta(t) = \frac{du(t)}{dt}$$

Now here arises the difficulty. The unit Step function is not differentiable at the origin. We take a different approach. Consider the signal whose value increases from 0 to 1 in a short interval of time say delta. The function u(t) can be seen as the limit of the above signal as delta tends to 0. Given this definition of Unit Step function we look into its derivative. The unit impulse function can be regarded as a rectangular pulse with a width of Δ and height $(1 / \Delta)$. As Δ tends to 0 the function approaches the Unit Impulse function and its derivative becomes narrower and higher and eventually a pulse of infinitesimal width of infinite height. All throughout the area under the pulse is maintained at unity no matter the value of Δ . In effect the delta function has no duration but unit area. Graphically the function is denoted as spear like symbol at t = 0 and the "1" next to the arrow indicates the area of the impulse. After this discussion we have still not cleared the ambiguity regarding the value or the shape of the Unit Impulse Function at t = 0. We were only able to derive that the effective duration of the pulse approaches zero while maintaining its area at unity. As we said earlier an Impulse Function is a Generalized Function and is defined by its effect on other functions and not by its value at every instant of time. Consider the product of an impulse function and a more well behaved continuous function. We will take the impulse function as the limiting case of a rectangular pulse of width Δ and height $(1/\Delta)$ as earlier. As evident from the figure the product function is 0 everywhere except in the small interval. In this interval the value of x(t) can be assumed to be constant and equal to x(0). Thus the product function is equal to the function scaled by a value equal to x(0). Now as Δ tends to 0 the product tends to x(0) times the impulse function.

$$\int_{-\infty}^{\infty} x(t) \delta(t) dt = x(0) \int_{-\infty}^{\infty} \delta(t) dt$$

$$= x(0)$$

i.e. The area under the product of the signal and the unit impulse function is equal to the value of the signal at the point of impulse. This is called the Sampling Property of the Delta function and defines the impulse function in the generalized function approach. As in discrete time

$$x(t) \delta(t) = x(0) \delta(t)$$

Or more generally,

$$x(t) \delta(t - t_0) = x(t_0) \delta(t - t_0)$$

Also the product **x(t)u(t)** truncates the signal for **t < 0**.

Conclusion:

In this lecture you have learnt:

- The unit impulse function is defined as:

$$\begin{aligned}\delta(n) &= 0 \quad n \neq 0 \\ &= 1 \quad n = 0\end{aligned}$$

- The unit step function is defined as:

$$\begin{aligned}u(n) &= 0 \quad n < 0 \\ &= 1 \quad n \geq 0\end{aligned}$$

$$\delta[n] = u[n] - u[n-1]$$

- **Sifting Property:** The product of a given signal $x[n]$ with the shifted Unit Impulse Function is equal to the time shifted unit Impulse Function multiplied by $x[k]$.

$$x[n]\delta[n-k] = x[k]\delta[n-k]$$

- Remember generalized functions.

Congratulations, you have finished Lecture 6.