

## Module 4 : Laplace and Z Transform

### Lecture 30 : Laplace Transform

#### Objectives:

Scope of this lecture:

**Laplace Transform** is a powerful tool for analysis and design of Continuous Time signals and systems. The Laplace Transform differs from Fourier Transform because it covers a broader class of CT signals and systems which may or may not be stable. It can also be used for obtaining solutions to integro-differential equations of C.T. systems.

- First ,we shall look at the definition of Laplace Transform .
- Then we will understand the meaning of **ROC** (Region of Convergence) and the need to consider it.
- We will solve sufficient examples for an in depth understanding of concepts covered.

#### Introduction

Till now we have been dealing with continuous and discrete domains . Then we studied the relationships involved using the transform domains. A system actually operates in a **natural domain** but it can be well understood in transform domains . The advantage of transform domains is that a few of the properties which may not be observed in natural domains are clear in transform domains. Most of the LTI-Systems act in time domain but they are more clearly described in the frequency domain instead.

Till now ,we have seen the importance of Fourier analysis in solving many problems involving signals and LTI systems. Now, we shall deal with signals and systems which do not have a Fourier transform.

But what was so special about Fourier transform in case of LSI systems?

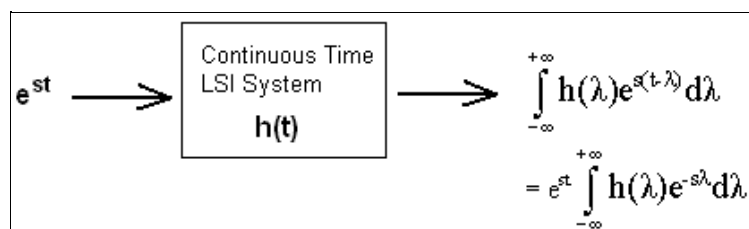
We found that continuous-time Fourier transform (F.T.) is a tool to represent signals as linear combinations of complex exponentials. The exponentials are of the form  $e^{st}$  with  $s=j\omega$  and  $e^{j\omega t}$  is an **eigen** function of the LSI system. Also , we note that the Fourier Transform only exists for signals which can absolutely integrated and have a finite energy.

This observation leads to generalization of continuous-time Fourier transform by considering a broader class of signals using the powerful tool of "**Laplace transform**". It will be trivial to note that the L.T can be used to get the discrete-time representation using relevant substitutions. This leads to a link with the **Z-Transform** and is very handy for a digital filter realization/designing. Also it will be helpful to note that, the properties of Laplace Transform and Z-Transform are quite similar.

With this introduction let us go on to formally defining both Laplace and Z-transform.

#### Definition of Laplace transform:

The response of a Linear Time Invariant system with impulse response  $h(t)$  to a complex exponential input of the form  $e^{st}$  can be represented in the following way :



Let

$$H(s) = \int_{-\infty}^{+\infty} h(\lambda) e^{-s\lambda} d\lambda$$

Where  $H(s)$  is known as the Laplace Transform of  $h(t)$ . We notice that the limits are from  $[-\infty$  to  $+\infty]$  and hence this transform is also referred to as **Bilateral or Double sided** Laplace Transform. There exists a one-to-one correspondence between the  $h(t)$  and  $H(s)$  i.e. the original domain and the transformed domain. Therefore L.T. is a unique transformation and the 'Inverse Laplace Transform' also exists.

Note that  $e^{st}$  is also an **eigen function** of the LSI system only if  $H(s)$  **converges**. The range of values for which the expression described above is finite is called as the Region of Convergence (**ROC**). In this case, the region of convergence is  $\text{Re}(s) > 0$ .

Thus, the Laplace transform has two parts which are , the expression and region of convergence respectively. The region of convergence of the Laplace transform is essentially determined by  $\text{Re}(s)$ . Here onwards we will consider trivial examples for a better understanding of the ROC.

## Example of Laplace Transform

Consider that the impulse response  $\mathbf{h(t) = e^t u(t)}$ .

Thus we notice that by multiplying by the term  $\mathbf{u(t)}$  we are effectively considering the unilateral Laplace Transform whereby the limits tend from 0 to  $+\infty$

Also we notice that  $\mathbf{h(t)}$  is **not Fourier transformable** as it is not absolutely integrable.

Consider the Laplace transform of  $\mathbf{h(t)}$  as shown below:

$$\begin{aligned} H(s) &= \int_0^{+\infty} e^{\lambda} e^{-s\lambda} d\lambda \\ &= \int_0^{+\infty} e^{(1-s)\lambda} d\lambda \end{aligned}$$

As stated earlier the symbol  $\mathbf{s}$  is a complex number and is defined as  $\mathbf{s = \sigma + j\Omega}$ .

Substituting  $\mathbf{s}$  in the above equation we get:

$$\Rightarrow H(s) = \int_0^{+\infty} e^{(1-\sigma)\lambda - j\Omega\lambda} d\lambda$$

Observing the above equation closely, we realize that firstly  $\mathbf{H(s)}$  converges if and only if

$$e^{(1-\sigma)\lambda} \text{ converges in the given range } \quad \left( \text{as } \left| e^{j\Omega\lambda} \right| = 1 \right)$$

i.e.  $e^{(1-\sigma)\lambda}$  must be a decaying exponential in the given limits

We analyze the part  $\mathbf{(1-\sigma)}$  as follows:

For a decaying  $e^{(1-\sigma)\lambda}$  it is essential that  $\mathbf{(1-\sigma) < 0}$ . This implies that  $\mathbf{(\sigma > 1)}$  which means that the Real part of 's' is greater than '1' which is also denoted as  $\mathbf{Re(s) > 1}$ .

This is what defines the " **Region of Convergence** " in an S-Complex Plane. The ROC of the Laplace Transform is always determined by the  $\mathbf{Re(s)}$ . The ROC in general gives us an idea of the stability of a system and is also a representation of the poles-zero plot of a system. It is essential to note that the ROC never includes poles.

Evaluation of the integral yields:

$$H(s) = \int_0^{+\infty} e^{(1-s)\lambda} d\lambda = 1/(s-1)$$

We observe that there is a single pole at  $\mathbf{s=1}$ . Since the Region of Convergence cannot contain poles therefore ROC start from '1' and tends outwards to infinity.

$e^{st}$  in physical systems:

We consider the real part of  $e^{st}$ , where  $\mathbf{s = \sigma + j\Omega}$ .

$$\mathbf{Re(e^{\sigma t} e^{j\Omega t}) = e^{\sigma t} \cos(\Omega t)}$$

Such a response is visible in RLC (Resistance-Inductance and Capacitance) systems. It is not only visible in the electrical field but also in other disciplines like mechanical field. In such cases the above expression is multiplied by a polynomial or a combination of such expressions.

## What is the need to consider region of convergence while determining the Laplace transform?

If we consider the signals  $\mathbf{e^{-at}u(t)}$  and  $\mathbf{-e^{-at}u(-t)}$  we note that although the signals are differing, their Laplace Transforms are identical which is  $\mathbf{1/(s+a)}$ . Thus we conclude that to distinguish L.T's uniquely their *ROC's must be specified*. Further from the ROC we can define many important conclusions which

A few important properties of the ROC are listed below:

- **The ROC of F(S) consists of strips parallel to  $j\Omega$ -axis in complex variable plane (S-plane).**

We know that the ROC is dependent only on the real parts of 'S' which is ' $\sigma$ ', therefore the property.

- **The ROC does not contain any poles.**

Since if this happens then the Laplace Transform becomes infinity. For example, if the  $\mathbf{F(S) = 1/(s-1)}$  then the ROC cannot contain  $\mathbf{s = 1}$  because at this point the L.T becomes infinity.

- If  $h(t)$  is a time limited signal and is Laplace Transformable, then its ROC will be the entire S-plane.

For example the ROC for  $\delta(t)$  is the entire S-plane.

- The region of convergence is always between two vertical lines in s-plane. These vertical lines need not be in finite region. But note that the ROC is always simply-connected but not multiply-connected in the s-plane.
- This fact can be explained by the following illustration:

$$h(t) = e^t u(t) + e^{-t} u(-t)$$

Let  $H_1(s)$  and  $H_2(s)$  be the respective Laplace transforms of the first and second terms.  $H_1(s)$  and  $H_2(s)$  converge in the region  $\text{Re}(s) > 1$  and  $\text{Re}(s) < 1$  respectively. But,  $h(t)$  doesn't have any Laplace transform due to no common ROC where both  $H_1(s)$  and  $H_2(s)$  converge.

#### Consider another example on ROC:

Let us consider another example which illustrates the need to specify ROC for completely defining the Laplace transform of a given function:

$$h_1(t) = e^t u(t) \quad h_2(t) = -e^t u(-t)$$

We will use the concepts gathered till now to determine the ROC's of the above signals after computing the respective Laplace Transforms.

$$H_1(s) = \frac{1}{s-1} \quad [\text{ROC for this is given by } \text{Re}(s) > 1]$$

$$\begin{aligned} H_2(s) &= \int_{-\infty}^{\infty} -e^t u(-t) e^{st} dt \\ &= \int_{-\infty}^0 -e^{(1-s)t} dt \\ &= - \left. \frac{e^{(1-s)t}}{1-s} \right|_{-\infty}^0 \\ &= \left( \frac{1}{-1+s} \right) \\ &= \frac{1}{s-1} \end{aligned}$$

[ Thus the ROC of  $H_2(s)$  is given by  $\text{Re}(s) < 1$ ; provided  $\text{Re}(1-s) > 0$  ]

**Thus, two different functions may have same expressions but correspond to different ROC.**

ROC's are given as:

Expression	ROC	Function
$\frac{1}{s-1}$	$\left\{ \begin{array}{l} \text{Re}(s) > 1 \\ \text{Re}(s) < 1 \end{array} \right.$	$\begin{array}{l} e^t u(t) \\ -e^t u(-t) \end{array}$

#### Conclusion:

In lecture you have learnt:

$$H(s) = \int_{-\infty}^{+\infty} h(\lambda) e^{-s\lambda} d\lambda$$

- is called the Laplace Transform of  $h(t)$  and the Region of Convergence (ROC) of the Laplace transform is essentially determined by the real part of the complex number ' $s$ ' denoted as  $\text{Re}(s)$ .
- Two different functions may have the same Laplace Transform so the only way to uniquely describe them is by the means of ROC.
- The ROC consists of strips parallel to the  $j\omega$ -axis, it does not include poles and is always simply-connected.

**Congratulations, you have finished Lecture 30.**