

## Module 3 : Sampling and Reconstruction

### Lecture 28 : Discrete time Fourier transform and its Properties

#### Objectives:

#### Scope of this Lecture:

In the previous lecture we defined digital signal processing and understood its features. The general procedure is to convert the Continuous Time signal into Discrete Time signal. Then we try to obtain back the original signal. In this lecture we will study the concepts of Discrete time Fourier Transform and Signal Representation.

- Representation of discrete time periodic signal .
- Discrete Time Fourier Transform (DTFT) of an *aperiodic* discrete time signal .
- Another way of representing DTFT of a periodic discrete time signal.
- Properties of DTFT

#### Representation of Discrete periodic signal.

A **periodic** discrete time signal  $x[n]$  with period **N** can be represented as a Fourier series:

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/N)n} \quad (i)$$

where

$$a_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n} \quad (ii)$$

Here the summation ranges over *any consecutive N integers* of  $x[n]$ ,

where N is the period of the discrete time signal  $x[n]$ .

Here equation (i) is called the Synthesis Equation and equation (ii) is called the Analysis Equation.

Now since  $x[n]$  is periodic with period N; the Fourier series coefficients are related as;

$$a_k = a_{k+N}$$

#### Discrete Time Fourier Transform of an aperiodic discrete time signal

Given a general aperiodic signal  $x[n]$  of finite duration, that is; for some integer N,  $x[n] = 0$  if  $|n| > N$ . From this aperiodic signal we can construct a periodic signal  $\tilde{x}[n]$  for which  $x[n]$  is one period. As we chose period N to be larger than the duration of  $x[n]$ ,  $\tilde{x}[n]$  is identical to  $x[n]$ . As the period  $N \rightarrow \infty$ ,  $\tilde{x}[n] = x[n]$  for any finite value of n.

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The Fourier series representation of  $\tilde{x}[n]$  is :

$$\tilde{x}[n] = \sum_{k \in \langle N \rangle} a_k e^{jk \frac{2\pi}{N} n}$$

$$a_k = \frac{1}{N} \sum_{n \in \langle N \rangle} \tilde{x}[n] e^{-jk \frac{2\pi}{N} n}$$

Since  $\tilde{x}[n] = x[n]$  over a period that includes the interval  $|n| < N$ , it is convenient to choose the interval of summation to be this period, so that  $\tilde{x}[n]$  can be replaced by  $x[n]$  in the summation. Therefore,

$$a_k = \frac{1}{N} \sum_{n=-N}^N x[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n}$$

here we have used the fact that  $x[n]$  is zero outside the interval  $|n| < N$ .  
Now defining  $X(\omega)$  as :

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

**$X(\omega)$  is the angular representation of the Discrete Time Fourier Transform (DTFT) of the signal  $x[n]$**

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### Another way of representing DTFT of a periodic discrete signal

In continuous time, the fourier transform of  $e^{j\omega_0 t}$  is an impulse at  $\omega = \omega_0$ . However in discrete time, for signal  $x[n] = e^{j\omega_0 n}$  the discrete time fourier transform is periodic in  $\omega$  with period  $2\pi$ . The DTFT of  $x[n] = e^{j\omega_0 n}$  is a train of impulses at  $\omega = \omega_0, \omega_0 \pm 2\pi, \omega_0 \pm 4\pi$  and so on. i.e Fourier Transform can be written as :

$$X(\omega) = \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 - 2\pi l)$$

Consider a periodic sequence  $x[n]$  with period  $N$  and with fourier series representation

$$x[n] = \sum_{k \in \langle N \rangle} a_k e^{jk \frac{2\pi}{N} n}$$

Then discrete time Fourier Transform of a periodic signal  $x[n]$  with period  $N$  can be written as :

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})$$

## Properties of DTFT

### Periodicity:

DTFT is always periodic in  $\omega$  with period  $2\pi$

$$X(\omega + 2\pi) = X(\omega)$$

### Linearity:

The DTFT is linear.

If

$$x_1[n] \xrightarrow{\text{DTFT}} X_1(\omega)$$

and

$$x_2[n] \xrightarrow{\text{DTFT}} X_2(\omega)$$

then

$$ax_1[n] + bx_2[n] \xrightarrow{\text{DTFT}} aX_1(\omega) + bX_2(\omega)$$

### Stability:

The DTFT is an unstable system i.e the input  $x[n]$  gives an unbounded output.

Example :

If  $x[n] = 1$  for all  $n$

then DTFT diverges i.e Unbounded output.

### Time Shifting and Frequency Shifting:

If,

$$x[n] \xrightarrow{\text{DTFT}} X(\omega)$$

then,

$$x[n - n_0] \xrightarrow{\text{DTFT}} e^{-j\omega n_0} X(\omega)$$

and,

$$e^{j\omega_0 n} x[n] \xrightarrow{\text{DTFT}} X(\omega - \omega_0)$$

### Time and Frequency Scaling:

Time reversal

Let us find the DTFT of  $x[-n]$

$$\begin{aligned}
 x[n] &\xrightarrow{\text{DTFT}} X(\omega) \\
 \therefore x[-n] &\xrightarrow{\text{DTFT}} \sum_{n=-\infty}^{\infty} x[-n] e^{-j\omega n} \\
 &= \sum_{m=-\infty}^{\infty} x[m] e^{-j(-\omega)m} \\
 &= X(-\omega) \\
 \therefore x[-n] &\xrightarrow{\text{DTFT}} X(-\omega)
 \end{aligned}$$

#### Time expansion:

It is very difficult for us to define  $x[an]$  when  $a$  is not an integer. However if  $a$  is an integer other than 1 or -1 then the original signal is not just speeded up. Since  $n$  can take only integer values, the resulting signal consists of samples of  $x[n]$  at  **$an$** .

If  $k$  is a positive integer, and we define the signal

$$\begin{aligned}
 x_k[n] &= x\left[\frac{n}{k}\right] && \text{if } n \text{ is a multiple of } k; \\
 &= 0 && \text{if } n \text{ is not a multiple of } k.
 \end{aligned}$$

then

$$x_k[n] \xrightarrow{\text{DTFT}} X(k\omega)$$

#### Convolution Property :

Let  $h[n]$  be the impulse response of a discrete time LSI system. Then the frequency response of the LSI system is

$$h[n] \xrightarrow{\text{DTFT}} H(\omega)$$

Now

$$x[n] \xrightarrow{\text{DTFT}} X(\omega)$$

and

$$y[n] \xrightarrow{\text{DTFT}} Y(\omega)$$

If

$$y[n] = x[n] * h[n]$$

then

$$Y(\omega) = X(\omega)H(\omega)$$

#### Proof

$$y[n] = x[n] * h[n]$$

$$\begin{aligned}
 Y(\omega) &= \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} \\
 &= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right\} e^{-j\omega n}
 \end{aligned}$$

now put  $n-k = m$ , for fixed  $k$ ,  $-\infty$  to  $\infty$ ,  $m$  goes from  $-\infty$  to  $\infty$

$$\begin{aligned}
 Y(\omega) &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h[k]x[n]e^{-j\omega(k+n)} \\
 &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h[k]x[n]e^{-j\omega k}e^{-j\omega n} \\
 &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \left\{ \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \right\} \\
 &= X(\omega)H(\omega)
 \end{aligned}$$

This is a very useful result.

#### Symmetry Property:

If

$$x[n] \xrightarrow{\text{DTFT}} X(\omega)$$

then

$$\overline{x[n]} \xrightarrow{\text{DTFT}} \overline{X(-\omega)}$$

#### Proof

$$\begin{aligned}
 \overline{x[n]} &\xrightarrow{\text{DTFT}} \sum_{n=-\infty}^{\infty} \overline{x[n]}e^{-j\omega n} \\
 &= \sum_{n=-\infty}^{\infty} \overline{x[n]e^{j(-\omega)n}} \\
 &= \overline{X(-\omega)}
 \end{aligned}$$

Furthermore if  $x[n]$  is real then,

$$\begin{aligned}
 \overline{x[n]} &= x[n] \\
 \Rightarrow X(\omega) &= \overline{X(-\omega)}
 \end{aligned}$$

#### The DTFT of Cross-Correlation Sequence between $x[n]$ and $h[n]$

If the DTFT of the cross correlation sequence between  $x[n]$  and  $h[n]$  exists then,

$$R_{xh}[n] = \sum_{m=-\infty}^{\infty} x[m+n] \overline{h[m]}$$

$$R_{xh}[\cdot] \xrightarrow{\text{DTFT}} X(\omega) \overline{H(\omega)}$$

In particular,

$$R_{xx}[\cdot] \xrightarrow{\text{DTFT}} |X(\omega)|^2$$

## Conclusion:

In this lecture you have learnt:

- For a Discrete Time Periodic Signal the Fourier Coefficients are related as  $a_k = a_{k+N}$ .
- DTFT is unstable which means that for a *bounded* 'x[n]' it gives an *unbounded* output.
- We saw its *time shifting & frequency shifting* properties & also *time scaling & frequency scaling*.
- Convolution Property for an LSI system is given as, if 'x[n]' is the input to a system with transfer function 'h[n]' then the DTFT of the output 'y[n]' is the multiplication of the DTFTs of 'x[n]' and 'h[n]'.
- We saw *symmetry properties* and DTFT of **cross-correlation** between 'x[n]' and 'h[n]' .

**Congratulations, you have finished Lecture 28.**