Module 3 : Sampling & Reconstruction
Lecture 24 : Realistic sampling of signals

Objective:

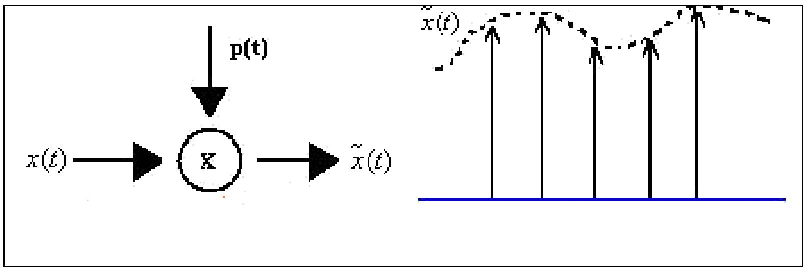
Scope of this lecture:

In the previous lecture we have seen the application of a train of pulses in obtaining a sampled version of the Continuous Time signal 'x(t)'. Here you will see in reality that how a train of pulses gets multiplied with the C.T. signal and thus resulting in the sampled signal.

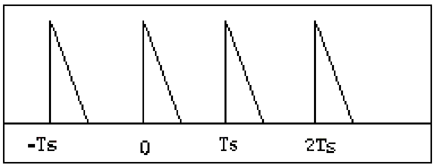
- Realistic sampling of signals by train of pulses.
- Method of generating train of pulses.
- Fourier transform of sampled signal $x_s(t)$.
- To find condition for the train of pulses used for sampling.

Realistic sampling of signals:

Our goal of achieving a sampled signal is possible by the multiplication of the original C.T. signal with the generated **train of pulses**. Now these two signals are multiplied practically with the help of a multiplier as shown in the schematic below. In our analysis so far, this is how we imagined sampling of a signal.



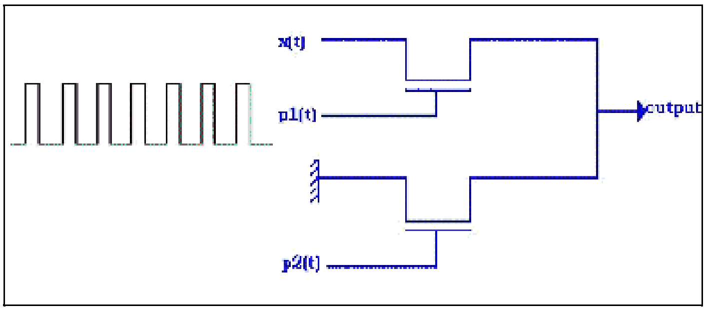
But impulses are a mathematical concept and they cannot be realized in a real system. In practice we can best obtain a train of pulses called a saw-tooth pulse. These pulses are generally used for creating a time-base for the operation of many electronic devices like the CRO (Cathode Ray Oscilloscope).



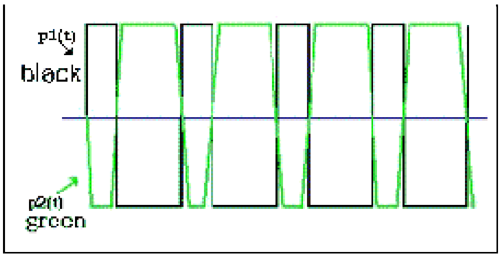
Practical Implementation:

Lets see how the train of pulses of the following kind can be multiplied by a signal 'x(t)'.

Consider the circuit below.

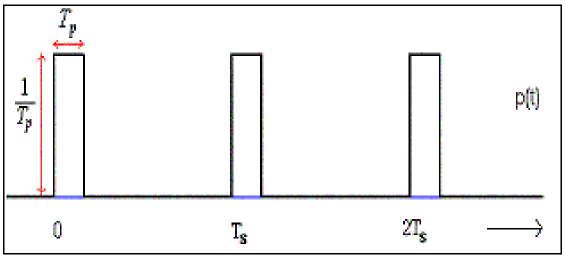


The two pulse trains $\mathbf{p_1(t)}$ and $\mathbf{p_2(t)}$ are synchronized so that when one is high the other is low and vice verse as shown in the figure below:



In the circuit when **x(t)** and **p_1(t)** are multiplied we get the output. Thus we get the output when **p_1(t)** is **ON** and it is **zero** when **p_2(t)** is **ON**.

You have just seen how we can multiply a signal **x(t)** with the following periodic pulse train **p(t)** to obtain the **sampled** signal $x_s(t)$. Now the train of pulses that we had used is shown below with respect to its amplitude and period.



Fourier series representation of p(t)

Now the Fourier Series Representation of 'p(t)' is given as:

$$p(t) = \sum_k C_k e^{j\frac{2\pi}{T_s}kt}$$

Where the Fourier Coefficients of the series are defined as:

$$C_k = \frac{1}{T_s}\int_{T_s} p(t) e^{-j\frac{2\pi}{T_s}kt}\,dt$$

For the constant term (**k = 0**) in the Fourier Series expansion is:

$$C_0 = \frac{1}{T_s} \text{ x } \frac{1}{T_p} \text{ x } T_p = \frac{1}{T_s}$$

In general we can represent k^{th} coefficient as:

$$C_k = \left.\frac{e^{-j\frac{2\pi}{T_s}kt}}{T_p T_s\left(-jk\frac{2\pi}{T_s}\right)}\right|_0^{T_p}$$

Simplifying the above term we get the envelope of the coefficients as a sinc function:

$$C_k = \frac{e^{-j\pi k\left(\frac{T_p}{T_s}\right)}}{T_s} \text{ x } \mathrm{Sinc}\left(k\left(\frac{T_p}{T_s}\right)\right)$$

$$\Rightarrow \qquad |C_k| = \frac{1}{T_s}\left|\mathrm{Sinc}\left(k\left(\frac{T_p}{T_s}\right)\right)\right|$$

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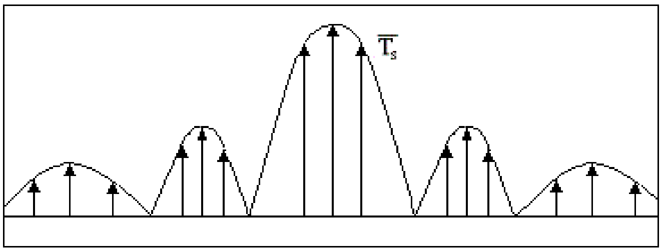
Lets have a look at the envelope of

$$|C_k|$$

which is shown as below:



Looking at the expression for the coefficients of the Fourier Series Expansion we observe that:

If $\frac{T_p}{T_s}$ is large then there are few samples in the main lobe.

As $\frac{T_p}{T_s}$ increases then the main lobe broadens.

As $T_p \rightarrow 0$, coefficients become constant (they tend to $\frac{1}{T_s}$) as the central lobe tends to infinity.

As $T_p \rightarrow 0$, 'p(t)' tends to the train of impulses we had started our discussion on sampling with. Notice then that the observations above are consistent with this. The Fourier coefficients of the periodic train of impulses are indeed all constant and equal to the reciprocal of the period of the impulse train.

The Fourier Transform of the Sampled Signal $x_s(t)$.

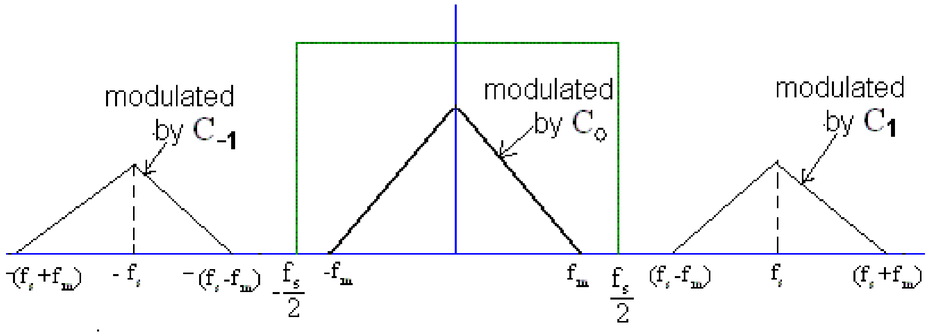
We now see what happens to the spectrum of continuous time signal on multiplication with the train of pulses. Having obtained the Fourier Series Expansion for the train of periodic pulses the expression for the sampled signal can be written as:

$$x_s(t) = x(t)\sum_k C_k e^{j\frac{2\pi}{T_s}kt}$$
$$= \sum_k C_k e^{j\frac{2\pi}{T_s}kt}\, x(t)$$

Taking Fourier transform on both sides and using the property of the Fourier transform with respect to translations in the frequency domain we get:

$$X_s(f) = \sum_k C_k X\left(f - \frac{k}{T_s}\right)$$

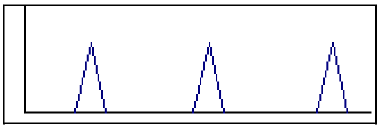
This is essentially the sum of displaced copies of the original spectrum modulated by the Fourier series coefficients of the pulse train. If 'x(t)' is **Band-limited** so long as the the displaced copies in the spectrum do not overlap. For this the condition that 'f_s' is greater than twice the bandwidth of the signal must be satisfied. The reconstruction is possible theoretically, using an I**deal low-pass filter** as shown below:



Thus the condition for faithful reconstruction of the original continuous time signal is : $f_s - f_m > f_m \Rightarrow f_s > 2f_m$ where f_m is the bandwidth of the original band-limited signal.

A General Case for the train of pulses.

Till now we have studied sampling using a rectangular train of pulses which permits the faithful reconstruction of the original signal. This might lead us to question whether the train of pulses needs to be rectangular. Will, say a train of triangular pulses have the same effect as the periodic rectangular train of pulses?

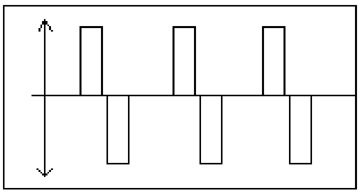


The answer is **YES.**

Let us look more closely into our analysis of sampling using a rectangular train of pulses. This signal had a Fourier series representation and multiplication of the band-limited signal with it gave rise to a signal $\mathbf{x_s(t)}$. The spectrum of this signal had periodic repetitions of the original spectrum modulated by the Fourier series coefficients of the train of pulses. But this much would hold even if the rectangular pulse train were replaced by any periodic signal (whose Fourier series exists) with the same period.

The Fourier series coefficients would definitely change but we are interested only in the central copy. As long as that is non-zero we can still reconstruct the signal by passing it through an ideal low-pass filter. The constant Fourier series co-efficient is proportional to the average value of the periodic signal. Thus, any periodic signal, whose Fourier series exists, and has a non-zero average, with fundamental frequency greater than twice the bandwidth of the band-limited signal can be used to sample it; and the original signal can be reconstructed using an ideal low-pass filter.

Of course, if the periodic signal used has a zero average, like the one shown below, an ideal low-pass filter cannot be used for reconstruction.



Conclusion:

In this lecture you have learnt:

- In practice a train of pulses is used for sampling a signal instead of a train of impulses.
- Train of pulses **p(t)** is periodic and obeys **Dirichlet's conditions.** It can be represented as a Fourier series and is used in deriving the condition for reconstruction of the original **band-limited** signal.
- Any periodic signal whose Fourier series exists and has a non-zero average with fundamental frequency greater than twice the bandwidth of the band-limited signal can be used to sample it and the original signal can be reconstructed using an **ideal low-pass filter**

Congratulations, you have finished Lecture 24.