

Module 2 : Signals in Frequency Domain

Lecture 20 : Properties of Fourier Transform

Objectives

In this lecture you will learn the following

- Behaviour of the Fourier Transform w.r.t. differentiation and integration
- Behaviour of the Fourier Transform w.r.t. scaling of the independent variable by a real constant a.
- Behaviour of the Fourier Series w.r.t. time shifting
- Behaviour of the Fourier Series w.r.t. differentiation
- Behaviour of the Fourier Series w.r.t. scaling of the independent variable
- Behaviour of the Fourier Series w.r.t. multiplication by t

Differentiation/Integration

$$X(f) = \int x(t) e^{-j2\pi ft} dt$$

$$\frac{dX(f)}{df} = \int x(t) \frac{d(e^{-j2\pi ft})}{df} dt$$

$$\frac{dX(f)}{df} = \int [-j2\pi tx(t)] e^{-j2\pi ft} dt$$

Hence if

$$x(t) \xrightarrow{FT} X(f)$$

then

$$-j2\pi tx(t) \xrightarrow{FT} X'(f)$$

Now,

$$x(t) = \int X(f) e^{j2\pi ft} df$$

$$\frac{dx(t)}{dt} = \int [j2\pi fX(f)] e^{j2\pi ft} df$$

Hence if,

$$x(t) \xrightarrow{FT} X(f)$$

then,

$$x'(t) \xrightarrow{FT} j2\pi fX(f)$$

The inverse operation of taking the derivative is running the integral : $\int_{-\infty}^t x(\lambda) d\lambda$

$$\text{eg : } \int_{-\infty}^t \delta(\lambda) d\lambda = u(t)$$

let

$$y(t) = \int_{-\infty}^t x(\lambda) d\lambda$$

$$\frac{dy(t)}{dt} = x(t)$$

$$x(t) \xrightarrow{FT} X(f)$$

$$\int_{-\infty}^t x(\lambda) d\lambda \xrightarrow{FT} \frac{X(f)}{j2\pi f}$$

This causes problem when

$$x(t) = \delta(t)$$

$$\delta(t) \xrightarrow{FT} 1$$

$$u(t) \xrightarrow{FT} \frac{1}{j2\pi f} + \text{impulse in frequency.}$$

Example:

$$e^{-t}u(t) \xrightarrow{FT} \frac{1}{1+j2\pi f}$$

$$te^{-t}u(t) \xrightarrow{FT} \frac{j}{2\pi} \frac{d}{df} \left(\frac{1}{1+j2\pi f} \right)$$

$$te^{-t}u(t) \xrightarrow{FT} \frac{1}{(1+j2\pi f)^2}$$

Scaling of the independent variable by a real constant a

When $a > 0$ or $a < 0$

$$\begin{aligned} & \int_{-\infty}^{\infty} x(at) e^{-j2\pi f t} dt \\ &= \frac{1}{|a|} \int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f \frac{\lambda}{a}} d\lambda \quad (\text{put } at = \lambda) \\ &= \frac{1}{|a|} X\left(\frac{f}{a}\right) \end{aligned}$$

$$\text{or } |a|^{\frac{1}{2}} x(at) \xrightarrow{FT} \frac{1}{|a|^{\frac{1}{2}}} X\left(\frac{f}{a}\right)$$

Hence the scaling of the independent variable is a self-dual operation.

Consider

$$\begin{aligned} \int_{-\infty}^{\infty} \left| |a|^{\frac{1}{2}} x(at) \right|^2 dt &= |a| \times \int_{-\infty}^{\infty} \frac{|x(\lambda)|^2}{|a|} d\lambda \\ &= \int_{-\infty}^{\infty} |x(\lambda)|^2 d\lambda \end{aligned}$$

Hence, $x(t)$ and $|a|^{\frac{1}{2}} x(at)$ have the same energy. Therefore such scaling is called energy normalized scaling of the independent variable.

Properties of Fourier Series.

Using the properties we just proved for the Fourier Transform, we state now the corresponding properties for the Fourier series.

Time-shift

Recall, that if $\mathbf{x(t)}$ is periodic then $\mathbf{X(f)}$ is a train of impulses.

$$x(t) \xrightarrow{FT} X(f) \text{ where } X(f) = \sum c_k \delta(f - kf_0)$$

$$\text{We know: } x(t - t_0) \xrightarrow{FT} e^{-j2\pi f t_0} X(f)$$

Thus if $x(t)$ is periodic with period T , $x(t - t_0)$ has Fourier series coefficients $c_k e^{-j2\pi \frac{k}{T} t_0}$

Differentiation

If the periodic signal is differentiable then

$$x(t) \xrightarrow{FT} \sum_{-\infty}^{\infty} c_k \delta(f - kf_0)$$

$$\frac{dx(t)}{dt} \xrightarrow{FT} j2\pi fX(f)$$

$$\therefore \frac{dx(t)}{dt} \xrightarrow{FT} \sum_{-\infty}^{\infty} j2\pi \frac{k}{T} c_k \delta(f - kf_0)$$

Thus if $x(t)$ is periodic with period T , $x'(t)$ has Fourier Series coefficients $2\pi j \frac{k}{T} c_k$.

Scaling of the independent variable

$$x(t) = \sum_{-\infty}^{\infty} c_k e^{j\frac{2\pi}{T}kt}$$

$$x(at) = \sum_{-\infty}^{\infty} c_k e^{j\frac{2\pi}{T}kat}$$

If $a > 0$, $x(at)$ is periodic with period (T/a) and now c_k becomes Fourier coefficient corresponding to frequency $\frac{k}{T/a}$.

If $a < 0$, $x(at)$ is periodic with period $(T/|a|)$ and now c_k becomes Fourier coefficient corresponding to frequency $\frac{-k}{T/|a|}$.

Multiplication by t

Multiplication by t of-course will not leave a periodic signal periodic. But what we can do is, multiply by t in one period, and then consider a periodic extension. i.e: $x(t)$ is periodic with period T , we see what the Fourier series coefficients of $y(t)$, defined as follows is:

$$y(t) = tx(t) \quad \text{in} \quad 0 \leq t \leq T \quad \text{and} \quad y(t+T) = y(t)$$

$$\text{Let } \tilde{x}(t) = x(t) \quad 0 \leq t \leq T$$

$$\tilde{x}(t) = 0 \quad \text{otherwise}$$

$$\text{Then } \tilde{X}(f) = \int_0^T x(t) e^{-j2\pi ft} dt$$

$$\text{Note the } k^{\text{th}} \text{ Fourier series co-efficient of } x(t) \text{ is } c_k = \frac{1}{T} \tilde{X}\left(\frac{k}{T}\right)$$

$$\text{Similarly, let } \tilde{y}(t) = y(t)\{u(t) - u(t-T)\}$$

$$\tilde{y}(t) = t\tilde{x}(t)$$

$$\tilde{Y}(f) = \frac{j}{2\pi} \frac{d\tilde{X}(f)}{df}$$

$$\text{Therefore, } k^{\text{th}} \text{ Fourier series coefficient of } y = \frac{1}{T} \tilde{Y}\left(\frac{k}{T}\right)$$

This idea is not of much use without knowledge of $\tilde{X}(f)$

Conclusion:

In this lecture you have learnt:

- Properties of the Fourier Transform w.r.t. differentiation and integration
- Properties of the Fourier Transform w.r.t. scaling of the independent variable by a real constant a .
- Properties of the Fourier Series w.r.t. time shifting
- Properties of the Fourier Series w.r.t. differentiation
- Properties of the Fourier Series w.r.t. scaling of the independent variable
- Properties of the Fourier Series w.r.t. multiplication by t

Congratulations, you have finished Lecture 20.