Module 4 : Laplace and Z Transform
Lecture 32 : Properties of Laplace and Z Transform

Objectives:

Scope of this Lecture:

In the previous lecture you have learnt about the **ROC** conditions for Laplace Transform as well as Z-Transform along with their respective plots. The LT as well as the ZT have several properties , which will be covered in detail in this lecture.

- We shall look at the properties of Laplace and Z-transform.
- The properties in general are *Linearity*, *Differentiation in Time Domain*, *Time Shift* and *Time Scaling*.
- Later, we shall use the above properties to determine Laplace transform and Z-transform.

Properties of Laplace and Z-Transform

1) Linearity

For Laplace:

If $x_1(t) \xleftrightarrow{L} X_1(s)$ with ROC R1 and $x_2(t) \xleftrightarrow{L} X_2(s)$ with ROC R2,

then

$ax_1(t)+bx_2(t) \xleftrightarrow{L} aX_1(s)+bX_2(s)$ with ROC containing $R_1 \cap R_2$.

The ROC of X(s) is at least the intersection of R1 and R2,which could be empty,in which case x(t) has no Laplace transform.

For z-transform :

If $x_1[n] \xrightarrow{Z} X_1(z)$ with ROC = R1 and $x_2[n] \xrightarrow{Z} X_2(z)$ with ROC = R2

then

$ax_1[n]+bx_2[n] \xleftrightarrow{z} aX_1(z)+bX_2(z)$ with ROC containing $R_1 \cap R_2$.

2) Differentiation in the time domain

If $x(t) \xleftrightarrow{L} X(s)$ with ROC = R

then $\dfrac{dx(t)}{dt} \xleftrightarrow{L} sX(s)$ with ROC = R.

This property follows by integration-by-parts. Specifically let

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Then

$$\int_{-\infty}^{\infty} \frac{dx(t)}{dt} e^{-st}dt = x(t)e^{-st}\Big|_{t\to-\infty}^{t\to\infty} + s\int_{-\infty}^{\infty} x(t)e^{-st}dt = 0 + sX(s)$$

and hence $\dfrac{dx(t)}{dt} \xleftrightarrow{L} sX(s)$

The ROC of *sX(s)* includes the ROC of *X(s)* and may be larger.

This property holds for z-transform as well.

3) Time Shift

For Laplace transform:

If $x(t) \xleftrightarrow{L} X(s)$ with ROC = R

then

$x(t-t_0) \xleftrightarrow{L} e^{-st_0}X(s)$ with ROC = R

For z-transform:

If $x(n) \xleftrightarrow{z} X(z)$ with ROC = R

then

Because of $x[n-n_0] \xleftrightarrow{z} z^{-n_0}X(z)$ with ROC = R except for the possible addition or deletion of the origin or infinity

the multiplication by z^{-n_0} for $n_0 > 0$ poles will be introduced at z=0, which may cancel corresponding zeroes of *X(z)* at *z=0.* In this case the ROC for $z^{-n_0}X(z)$ equals the ROC of *X(z)* but with the origin deleted. Similarly, if $n_0 < 0, z = \infty$ may get deleted.

4) Time Scaling

For Laplace transform:

If $x(t) \xleftrightarrow{L} X(s)$ with ROC=R.

then

$x(at) \xleftrightarrow{L} \frac{1}{|a|}X(\frac{s}{a})$ with . $ROC = \frac{R}{a}$ $i.e. \frac{s}{a} \in R$

Let a $\in \mathbb{R}$

$$\int_{-\infty}^{+\infty} x(at)e^{-st}dt$$
$$= \frac{1}{|a|}\int_{-\infty}^{+\infty} x(\lambda)e^{-(\frac{s}{a})\lambda}d\lambda$$
$$= \frac{1}{|a|}X(\frac{s}{a})$$

where

$$at = \lambda;$$
$$adt = d\lambda$$

A special case of time scaling is **time reversal,** when *a= -1*

For z transform:

The continuous-time concept of time scaling does not directly extend to discrete time. However, the discrete time concept of time expansion i.e. of inserting a number of zeroes between successive values of a discrete time sequence can be defined. The new sequence can be defined as

$x_{(k)}$[n] = x[n / k] if n is a multiple of k

= 0 if n is not a multiple of k

has ***k - 1*** zeroes inserted between successive values of the original sequence. This is known as **upsampling by k**. If

x[n] ↔ X(z) with **ROC = R**

then **$x_{(k)}$[n] ↔ $X(z^k)$** with **ROC = $R^{1/k}$** i.e. **z^k R**

i.e. **$X(z^k)$ = S $x[n](z^k)^{-n}$, $-\infty < n < \infty$**

= S $x[n]z^{-nk}$, $-\infty < n < \infty$

For Laplace transform

If

x(t) ↔ **X(s)** with **ROC = R**

then

$e^{\alpha t}x(t) \leftrightarrow X(s - \alpha)$ **where** $Re(s - \alpha)$ **ROC(X(.))**

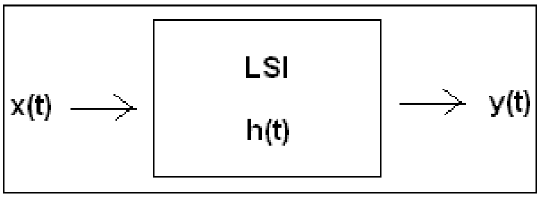
For z-transform

If **x[n]** ↔ **X(z)** with **ROC = R**

then

$\beta^{n} \leftrightarrow X(z/\beta)$ $\beta \neq 0$ where $z\beta^{-1}$ **ROC(X(.))**

Consider



$$y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} y(t)\, e^{-st}\, dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda \right) e^{-st}\, dt \end{aligned}$$

Put $\phi = t - \lambda$

$$\begin{bmatrix} \lambda \\ \phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda \\ t \end{bmatrix}$$

$$d\phi\, d\lambda = Jacobian \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} d\lambda\, dt$$

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\lambda)\, h(\phi) \right) e^{-s(\phi+\lambda)}\, d\phi\, d\lambda \\ &= \left(\int_{-\infty}^{\infty} x(\lambda)\, e^{-s\lambda}\, d\lambda \right) \left(\int_{-\infty}^{\infty} h(\phi) e^{-s\phi}\, d\phi \right) \end{aligned}$$

$$\mathbf{Y(s) = X(s)\,.\,H(s)} \quad \text{where} \quad \text{ROC of } Y(s) = \{\text{ROC of } X(s)\} \cap \{\text{ROC of } H(s)\}$$

Conclusion:

In this lecture you have learnt:

- If $x(t) \overset{L}{\longleftrightarrow} X(s)$ with ROC = R then
    1. $\frac{dx(t)}{dt} \overset{L}{\longleftrightarrow} sX(s)$ with ROC = R.
    2. $x(t-t_0) \overset{L}{\longleftrightarrow} e^{-st_0}X(s)$ with ROC = R.
    3. $x(at) \overset{L}{\longleftrightarrow} \frac{1}{|a|}X(\frac{s}{a})$ with $ROC = \frac{R}{a}$ i.e. $\frac{s}{a} \in R$
    4. 4. $e^{\alpha t}x(t) \leftrightarrow X(s - \alpha)$ where $Re(s - \alpha)$ ROC(X(.))
    5. 5. If y(t) = (x*h)(t), Y(s) = H(s).X(s) where ROC of Y(s) = ROC(X) $\cap$ ROC(H)
- If $x(n) \overset{z}{\longleftrightarrow} X(z)$ with ROC = R then
    1. $x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0}X(z)$ with ROC = R except for the possible addition or deletion of infinity from ROC.
    2. The continuous-time concept of time scaling does not directly extend to discrete time.Read upsampling for the reason.
    3. Other properties of z-transform are similar to that of Laplace transform.

Congratulations, you have finished Lecture 32.