Module 4 : Laplace and Z Transform
Lecture 36 : Analysis of LTI Systems with Rational System Functions

Objectives

Scope of this Lecture:

Previously we understood the meaning of causal systems, stable systems and instable systems using the concept of ROC. Here we will delve deeper into the previously established concepts by studying several theorems and other properties.

- We shall derive necessary and sufficient conditions for causality and stability of both discrete and continuous rational systems.
- We shall look at plotting of poles and zeros.
- We shall even prove some theorems.
- By taking inverse transform of the rational system function, one can arrive at linear constant coefficient difference/differential equations.

Continuous Rational System

Causality

For a causal LTI system, the impulse response is zero for t=0 (and thus it is right sided!)
h(t) = 0 for all t<0

$$H(s) = \int_0^{\infty} h(t)e^{-st}dt$$

where H(s) is the system function (assuming system has a system function

$s = \sigma + j\Omega$ with $\sigma \to \infty$

$e^{-st} = e^{-\sigma t}e^{-j\Omega t}$

for t > 0, $e^{-\sigma t} \to 0$ as $\sigma \to \infty$,

Thus, if the region of convergence is non null, $\mathrm{Re}(s) \to \infty$ must be included in the **ROC** for the system to be causal.

Proof: As region of convergence is not null there exist an $s_0 \in ROC$

$$s_0 = \sigma_0 + j\omega_0$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt \ = \int_0^{\infty} h(t)e^{-st}dt$$

$$\left|\int_0^{\infty} h(t)e^{-st}dt\right| = \int_0^{\infty}\left|h(t)e^{-st}\right|dt = \int_0^{\infty}\left|h(t)e^{-s_0 t}e^{-(s-s_0)t}\right|dt$$

$$= \int_0^{\infty}\left|h(t)e^{-\sigma_0 t}e^{-(\sigma-\sigma_0)t}\right|dt$$

as $\left|e^{-(\sigma-\sigma_0)t}\right| \le 1$ for $\sigma > \sigma_0 \le \int_0^{\infty}\left|h(t)e^{\sigma_0 t}\right|dt$ which is given to be convergent

Hence **H(S)** is convergent for all **Re(S)>Re(So)**

$\mathrm{Re}(s) \to \infty$ is inthe region and belongs to **ROC**

Necessary and sufficient condition for causality in a rational system:

The region of convergence must include $\mathrm{Re}(s) \to \infty$

DISCRETE RATIONAL SYSTEM :

A Causal Discrete time LSI system has an impulse response h[n], this is zero for n<0 and thus it is right sided. **h[n]=0** for all

n< 0 for causality system function **H(z)** = $\sum_{n=0}^{\infty} h[n]z^{-n}$.

Assuming H(z) has a non null **ROC** ,we require,

$z^{-n} \rightarrow 0$ for n > 0,

thus $|z| \rightarrow \infty$ must be included in the ROC.

Example:

Let $h[n] = 3^n u[n] \quad \Rightarrow \quad H[z] = \frac{1}{1-3z^{-1}}$ for $|z| > 3$ is causal,

but consider $h[n] = 3^n u[n] + 3\delta[n+1] \quad \Rightarrow \quad H[z] = \frac{1}{1-3z^{-1}} + 3z$ for $\infty > |z| > 3$ causal.

STABILITY OF RATIONAL SYSTEMS:

A continous LSI system is stable if and only if its impulse response is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

.

Exploring the convergence of Laplace transform of impulse response of a stable LSI system, we find that

$$H(s)\Big|_{\operatorname{Re} s=0} = \int_{-\infty}^{\infty} h(t) e^{-(0+j\Omega)t} dt$$

$$\left| \int_{-\infty}^{\infty} h(t) e^{-j\Omega t} dt \right| \le \int_{-\infty}^{\infty} |h(t) e^{-j\Omega t}|\, dt \text{ as } (|a| + |b| \ge |a+b|)$$

$$= \int_{-\infty}^{\infty} |h(t)|\, dt < \infty \text{ (Stability))}$$

Thus **H(s)** converges on imaginary axis (**Re(s)=0**)

So, **Re(s)=0** or imaginary axis is contained in **ROC** of system function for all stable **LSI** systems.

We can also look at this from a different point of view.Impulse response being absolutely integrable implies Fourier transform converges as $L\{x(t)\}\,|_{\operatorname{Re} s=0}$ is nothing but Fourier transform it is also bound to converge for **Re{s} = 0** $\Rightarrow$ **Re{s} = 0** is included in its **ROC.**

In general, **Re{s} = 0** lies in **ROC** is not sufficent condition to imply stability. But for rational systems Re{s} = 0 lies in ROC $\Leftrightarrow$ system is stable.

Now, we will prove the above result .

Proof for sufficiency condition :-

For any system to be stable, poles can not lie in **ROC**.Thus, there should not be any poles on the (imaginary axis) Re(s)=0.

Suppose α and β are the poles of the system function H(s) where Re(α)<0 and Re(β)>0.

Now consider, inverse transform of $\frac{1}{(s-\alpha)}$, there are two choices

$$\frac{1}{(s-\alpha)} \rightarrow \left| \begin{array}{ll} e^{\alpha t}u(t) & \operatorname{Re} s > \operatorname{Re} \alpha \\ -e^{\alpha t}u(-t) & \operatorname{Re} s < \operatorname{Re} \alpha \end{array} \right.$$

As **Re{s}=0** is contained in the **ROC** and $\operatorname{Re} \alpha < 0$, the only possible option is $e^{\alpha t}u(t)$ (to have a non-empty **ROC**).

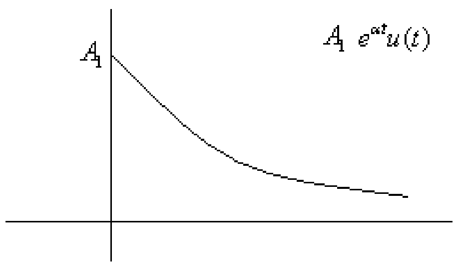
Looking at inverse transform of $\dfrac{1}{(s-\beta)}$

$$\frac{1}{(s-\beta)} \rightarrow \left| \begin{array}{ll} e^{\beta t}u(t) & \operatorname{Re} s > \operatorname{Re} \beta \\ -e^{\beta t}u(-t) & \operatorname{Re} s < \operatorname{Re} \beta \end{array} \right.$$

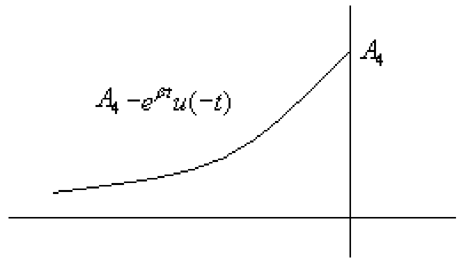
As **Re{s} = 0** lies in **ROC** , we will have to take $-e^{\beta t}u(-t)$ to be the inverse.

Thus, in a rational system, with ROC of the system function including Re(s)=0, the poles to the left of imaginary axis contribute right-sided exponentially decaying term and poles to the right of the imaginary axis contribute left-sided exponentially decaying term.

α contributes a right-sided decaying exponential



β contributes a left handed decaying exponent.



Poles to the right of imaginary axis contribute $-P_\beta(t)e^{\beta t}u(-t)$, where $P_\beta(t)$ is a polynomial of degree k-1

k = order of pole at β in H(s)

Similarly poles to the left of imaginary axis contribute $P_\alpha(t)e^{\alpha t}u(t)$

The absolute integral $\int_{-\infty}^{\infty} |h(t)|\,dt$

$$= \int_{-\infty}^{\infty} \left| P_\alpha(t)e^{\alpha t}u(t) + P_\beta(t)e^{\beta t}u(t) + \ldots\ldots \right| dt$$

$$\leq \int_{-\infty}^{\infty} \left| P_\alpha(t)e^{\alpha t}u(t) \right| + \left| P_\beta(t)e^{\beta t}u(t) \right| + \ldots\ldots\ dt$$

Thus the absolute integral $\int_{-\infty}^{\infty} |h(t)|\,dt \leq$ sum of the absolute integrals of these terms (finite number because the system function is rational) $< \infty$

Therefore, the system is stable.

Later, we shall prove the theorem ,that irrespective of the polynomial **p(t),** $\int_{T_1}^{T_2} p(t)e^{\alpha t}\,dt$ converges if and only if $\int_{T_1}^{T_2} e^{\alpha t}\,dt$ converges, in order to justify the convergence of each absolute integral.

Rational continuous system functions

Let the system function

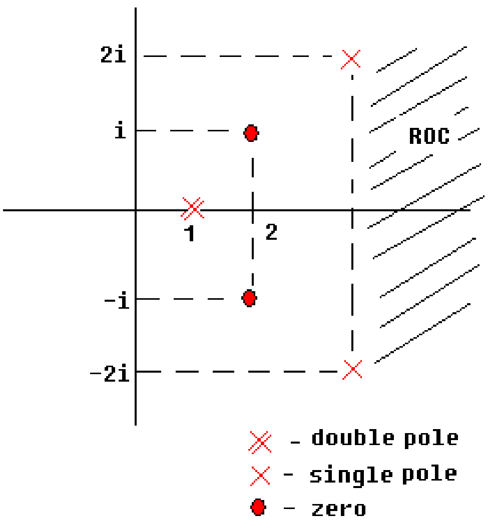
$$H(s) = \left(\frac{N(s)}{D(s)}\right)$$

$$\textit{where } N(s) \textit{ and } D(s) \textit{ are polynomials in } s$$

We can represent graphically the system function by showing all poles(zero's of **D(s)**) and zero's (zero's of **H(s)**) and it's **ROC** in s-plane.

eg: $H(s) = \left(\dfrac{2(s^2 - 4s + 5)}{(s-1)^2(s^2 - 16s + 13)}\right)$

Complete pictorial representation of the above system function **H(s)** in s-plane.



From this graph we can write the system function as

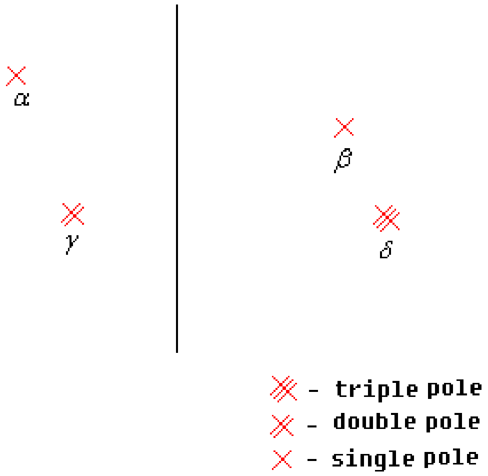
$$H(s) = \frac{A(s-(2+i))(s-(2-i))}{(s-1)^2(s-(4+2i))(s-(4-2i))}$$

$$\textit{where } \; Re(s) > 4$$

Now for stability, **Re{s} = 0** should lie in **ROC**.

Representation of poles and zeros

Consider representation of the system function



$\alpha, \gamma \rightarrow$ poles to left of imaginary axis

$\beta, \delta \rightarrow$ poles to right of imaginary axis

α, β -- poles of order 1

γ --pole of order 2

δ -- pole of order 3

Thus H(s) can be represented as $\dfrac{N(s)}{(s-\gamma)^2(s-\alpha)(s-\delta)^3(s-\beta)}$.

On expansion of H(s) in terms of partial fraction we would get

$$H(s) = Q(s) + \frac{A_1}{(s-\alpha)} + \frac{A_2}{(s-\gamma)} + \frac{A_3}{(s-\gamma)^2} + \frac{A_4}{(s-\beta)} + \frac{A_5}{(s-\delta)}$$
$$+ \frac{A_6}{(s-\delta)^2} + \frac{A_7}{(s-\delta)^3}$$
$$\text{where } Q(s) \text{ is a polynomial in } s.$$
$$A_1, A_2, A_3, A_4, A_5, A_6, A_7 \text{ are constants}$$

Recall that in a rational system, with ROC of the system function including Re(s)=0, the poles to the left of imaginary axis contribute right-sided exponentially decaying term and poles to the right of the imaginary axis contribute left-sided exponentially decaying term.

Thus, as we have seen earlier, α contributes a right handed decaying exponential and β contributes a left handed decaying exponential and the contributions of following terms in the denominator are

$$\frac{1}{(s-\gamma)^2} \rightarrow (A_2 + A_3 t)e^{\gamma t}u(t)$$
$$\frac{1}{(s-\delta)^3} \rightarrow (A_5 + A_6 t + A_7 t^2)e^{\delta t}u(-t)$$

Theorem

Irrespective of the polynomial **p(t),** $\int_{T_1}^{T_2} p(t)e^{\alpha t}dt$ converges if and only if $\int_{T_1}^{T_2} e^{\alpha t}dt$ converges.

Proof by induction:

Mathematical Induction on degree of polynomial

$$p(t) = a_0 + a_1 t + \ldots\ldots + a_n t^n$$

Base case: Suppose the statement is true for n=1 case we prove it is true for n=2 case.

$$\int_{T_1}^{T_2} (a_0 + a_1 t) e^{\alpha t} dt = \left. \frac{(a_0 + a_1 t) e^{\alpha t}}{\alpha} \right|_{T_1}^{T_2} - \int_{T_1}^{T_2} \frac{a_1 e^{\alpha t}}{\alpha} dt \quad (by\ parts)$$

Induction step: We assume $\int_{T_1}^{T_2} p(t) e^{\alpha t} dt$ converges , for any polynomial of degree (k-1), We proceed to prove $\int_{T_1}^{T_2} p(t) e^{\alpha t} dt$ converges for apolynomial of degree k

$$\int_{T_1}^{T_2} (a_0 + a_1 t + ... + a_k t^k) e^{\alpha t} dt = \left. \frac{(a_0 + a_1 t .... + a_k t^k) e^{\alpha t}}{\alpha} \right|_{T_1}^{T_2}$$

$$- \int_{T_1}^{T_2} \frac{(a_1 + 2a_2 t .... + k a_k t^{k-1}) e^{\alpha t}}{\alpha} dt \quad (by\ parts)$$

$a_1 + 2a_2 t .... + k a_k t^{k-1}$ is polynomial of degree (k-1) , by the assumption , we know $\int_{T_1}^{T_2} \frac{(a_1 + 2a_2 t .... + k a_k t^{k-1}) e^{\alpha t}}{\alpha} dt$ converges there by $\int_{T_1}^{T_2} (a_0 + a_1 t + ... + a_k t^k) e^{\alpha t} dt$ also converges.

Hence, proved.

Theorem 2

For a discrete rational system stability implies and is implied by the unit circle in the z plane belonging to the **ROC** of the system function.

Proof :-

(a) For the stability of the system function

If the discrete rational system is stable then

$$\sum | h(x) | < \infty$$

$$\begin{aligned} and \quad \sum | h(x) | &= \sum | h[x] e^{-j\omega x} | \\ &\geq | \sum h[x] e^{-j\omega x} | \end{aligned}$$

The z transform of the impulse response (or the system function) converges for | z | = 1.

(b) For a stability to be implied by | z | =1 (the unit circle) belonging to the **ROC** of the system function

A pole cannot lie on the unit circle | z | = 1 in a stable system.

Rational discrete system functions.

Considering the function

$$H(z) = Q(z^{-1}) + \left(\frac{A_1}{(1 - \alpha z^{-1})} \right) + \left(\frac{A_2}{(1 - \beta z^{-1})} \right)$$

when α is the pole of order 1 { (α < 1) is the assumption } , β is a pole of order 1 (β >1) .

Now consideringthe Inverse transform of $\left(\frac{1}{(1 - \alpha z^{-1})} \right)$ we have,

$$\left(\frac{1}{(1 - \alpha z^{-1})} \right) \xrightarrow{Z^{-1}} \begin{cases} \alpha^n u[n] & |z| > \alpha \\ -\alpha^n u[-n-1] & |z| < \alpha \end{cases}$$

as | z | =1 is contained in the **ROC** and α <1, hence the only possible option for inverse is $\alpha^n u[n]$.

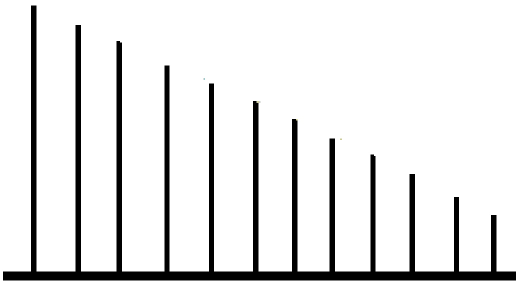
Similarly for the function

$$\left(\frac{1}{(1-\beta z^{-1})}\right) \xrightarrow{\mathcal{L}^{-1}} \left(-\beta^n\ u[-n-1]\right) \qquad |z| < \beta$$

(since β >1and | z | =1 is contained in the **ROC** of the function)

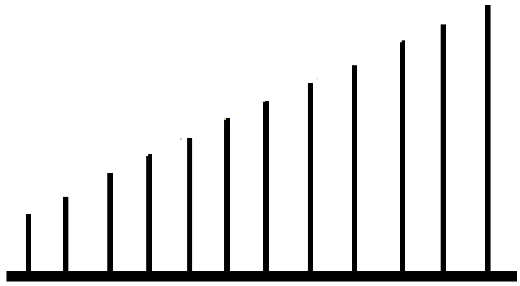
Therefore,

The contribution of α is a right sided exponentially decaying term (possibly multiplied by a polynomial in **n** if the order of pole >1)



$\alpha^n u[n]$

The contribution of β is a left sided exponentially decaying term (possibly multipled by a polynomial in **n** if the order of the pole >1)



$-(\beta^n u[-n-1])$

Proof for stability of rational discrete systems

Similar to proof for stability of rational continuous systems, the absolute sum must be convergent.

The absolute sum $\sum\left| h(x) \right|$ is

$$\sum\left| h(x) \right| = \sum\left|P_\alpha(n)\cdot \alpha^n u[n]\right| + \sum\left|P_\beta(n)\cdot \beta^n u[-n-1]\right|$$

(Assuming two poles α (α <1) and β (β >1) of the order >1)

Increasing the number of poles would not make any difference to the proof . P_α *and* P_β are polynomials in **n**.

Now we know that $\left(P_\alpha(n)\cdot \alpha^n u[n]\right)$ and $\left(P_\beta(n)\cdot \beta^n u[-n-1]\right)$ are absolutely summable.

Finite number of such terms is absolutely summable and hence the Impulse response is absolutely summable.

Therefore ,the system is stable.

The absolute summability of one sided terms of $\beta^n p(n)$ (where p(n) is a polynomial) depends only on $|\beta|$ and not on the polynomial.

Theorem 4

We prove summability of $\sum_{n=n_1}^{n} \beta^n p(n)$ depends on summability of $\sum_{n=n_1}^{n} \beta^n$.

Proof by Induction:

Induction on degree of polynomial

Base case: (k=1) $p(n) = a_0 + a_1 n$

$$\sum_{n=n_1}^{n_2} \beta^n (a_0 + a_1 n) - \beta\left(\sum_{n=n_1}^{n_2} \beta^n (a_0 + a_1 n)\right)$$

$$= \sum_{n=n_1}^{n_2} \beta^n (a_0 + a_1 n) - \sum_{n=n_1+1}^{n_2+1} \beta^n (a_0 + a_1(n-1))$$

$$= \sum_{n=n_1}^{n_2} a_1 \beta^n - \beta^{n_2+1}(a_0 + a_1 n_2)$$

$$\sum_{n=n_1}^{n_2} \beta^n (a_0 + a_1 n) = \frac{\sum_{n=n_1}^{n_2} a_1 \beta^n - \beta^{n_2+1}(a_0 + a_1 n_2) + \beta^{n_1}(a_0 + a_1(n-1))}{1-\beta}$$

Induction step: Assume $\sum_{n=n_1}^{n} \beta^n p(n)$ is summable for (k-1) case .we proceed to prove it for k case.

$$\sum_{n=n_1}^{n_2} \beta^n (a_0 + a_1 n + \ldots + a_k n^k) - \beta\left(\sum_{n=n_1}^{n_2} \beta^n (a_0 + a_1 n + \ldots + a_k n^k)\right)$$

$$= \sum_{n=n_1}^{n_2} \beta^n (a_0 + a_1 n + \ldots + a_k n^k) - \sum_{n=n_1+1}^{n_2+1} \beta^n (a_0 + a_1(n-1) + \ldots + a_k (n-1)^k)$$

$$= \sum_{n=n_1}^{n_2} (a_1 + a_2(2n-1) + \ldots + a_k(kn^{k-1} + {}^k C_2 n^{k-2} + \ldots + (-1)k))\beta^n$$

$$+ \text{ constant terms}$$

$$a_1 + a_2(2n-1) + \ldots + a_k(kn^{k-1} + {}^k C_2 n^{k-2} + \ldots + (-1)k) \textit{ is polynomial}$$

by our assumption $\sum_{n=n_1}^{n} \beta^n p(n)$ is summable for polynomial of degree (k-1).

THEOREM :

A neccesary and sufficient conditon for a continuos rational system to be a **Causal** and **Stable** is that all the poles must lie in the left half plane, i.e. **Re** (**s**)< 0.

THEOREM :

A neccesary and sufficient conditon for a discrete rational system to be a **Causal** and **Stable** is that all the poles must lie inside the unit circle, i.e. **|z| < 1** .

System Defination of Causal Rational System and Linear Constant Coefficient Difference equation

(a) Continuos system :-

The system function can be written as ,

$$H(s) = \left(\frac{s^2 + 1}{s^2 + 2s + 1}\right) = \frac{Y(s)}{X(s)}$$

$$\Rightarrow (s^2 + s)X(s) = (s^2 + 2s + 1)Y(s)$$

Taking the inverse Laplace transform we have ,

$$\left(\frac{d^2x}{dt^2}\right) + \left(\frac{dx}{dt}\right) = \left(\frac{d^2y}{dt^2}\right) + 2\left(\frac{dy}{dt}\right) + y$$

(b) Discrete system :-

$$H(s) = \left(\frac{b_o + b_1 z^{-1} + \ldots\ldots\ldots\ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots\ldots\ldots\ldots\ldots + a_n z^{-n}} \right)$$

It is always possible to write the system function this way for a Causal rational discrete system .

Taking the inverse z-transform of the above equation we have ,

$$y[n] + \sum_{l=1}^{N} a_l\, y[n-l] = \sum_{k=0}^{M} b_k\, x[n-k]$$

Conclusion:

In this lecture you have learnt:

- Necessary and sufficient condition for causality in a continuous rational system : The region of convergence must include $\operatorname{Re}(s) \to \infty$.
- Necessary and sufficient condition for causality in a discrete rational system: The region of convergence must include $|z| \to \infty$.
- In general, Re{s} = 0 lies in ROC is not sufficent condition to imply stability. But for rational systems Re{s} = 0 lies in ROC $\Rightarrow$ system is stable.
- In a rational system, with ROC of the system function including Re(s)=0, the poles to the left of imaginary axis contribute right-sided exponentially decaying term and poles to the right of the imaginary axis contribute left-sided exponentially decaying term.
- For a discrete rational system stability implies and is implied by the unit circle in the z plane belonging to the ROC of the system function.

Congratulations, you have finished Lecture 36.