

Module 7 : Laser- I

Lecture : Laser- I

Part-I

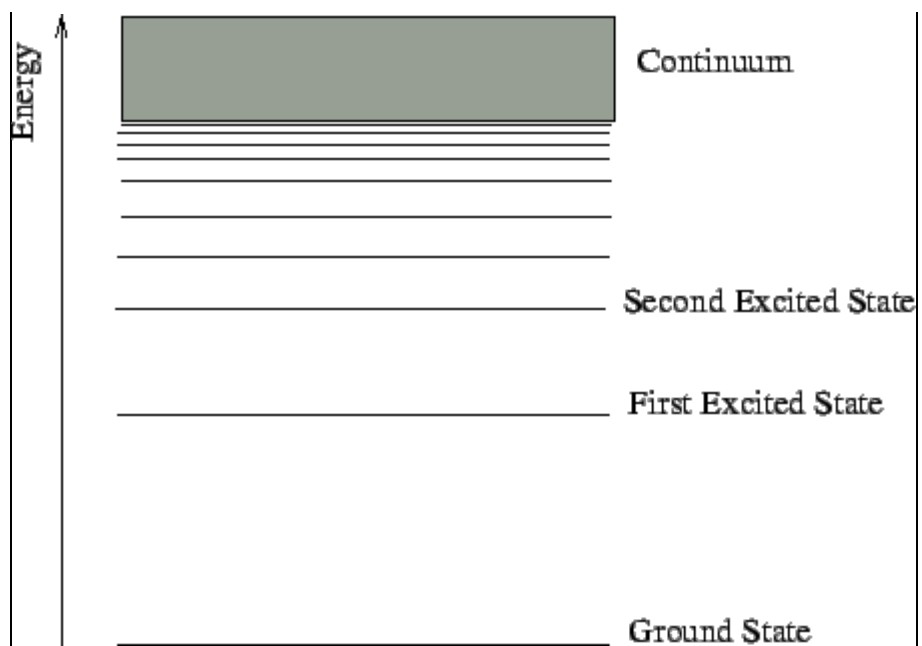
Objectives

In this lecture you will learn the following

- Introduction
- Absorption
- Spontaneous Emission
- Stimulated Emission
- LASER - Light Amplification by Stimulated Emission of Radiation
- Interaction of Radiation with Matter
- Blackbody Radiation
- Boltzmann Statistics
- Einstein Relations - A and B Coefficients
- Population Inversion

1. Introduction:

We have learnt that the energy levels in atom, ions and molecules are discrete. The lowest possible energy level is known as the **ground state** and higher energy levels are called the excited states. As the energy of the excited states increase, the separation between the adjacent energy levels become smaller and smaller until the separation becomes so small that the energy levels appear continuous. Such continuous spread of energy is called the **continuum** .



There are three ways in which an incident radiation can interact with the energy levels of atoms.

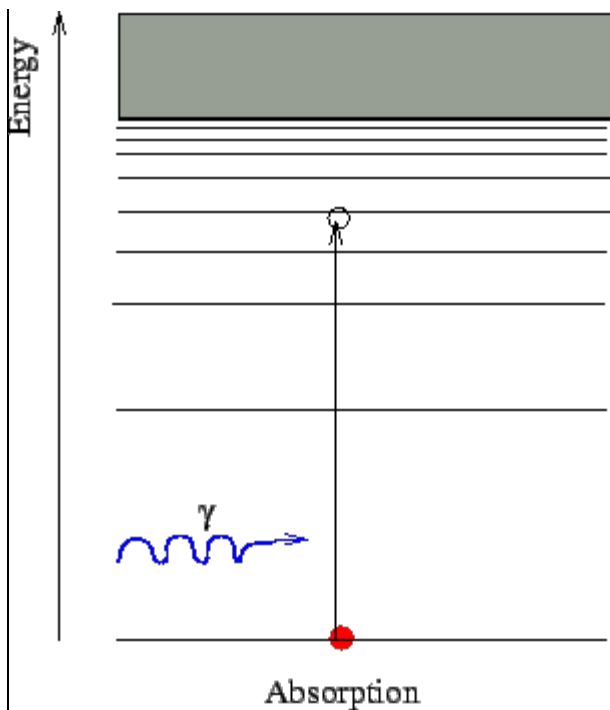
Absorption:

An electron in one of the lower level (ground state or a lower lying excited state) with energy E_i can make a transition to a higher level having energy E_f by absorbing an incident photon. Absorption can occur only when the frequency of the

incident radiation ν is given by

$$\nu = \frac{E_f - E_i}{h}$$

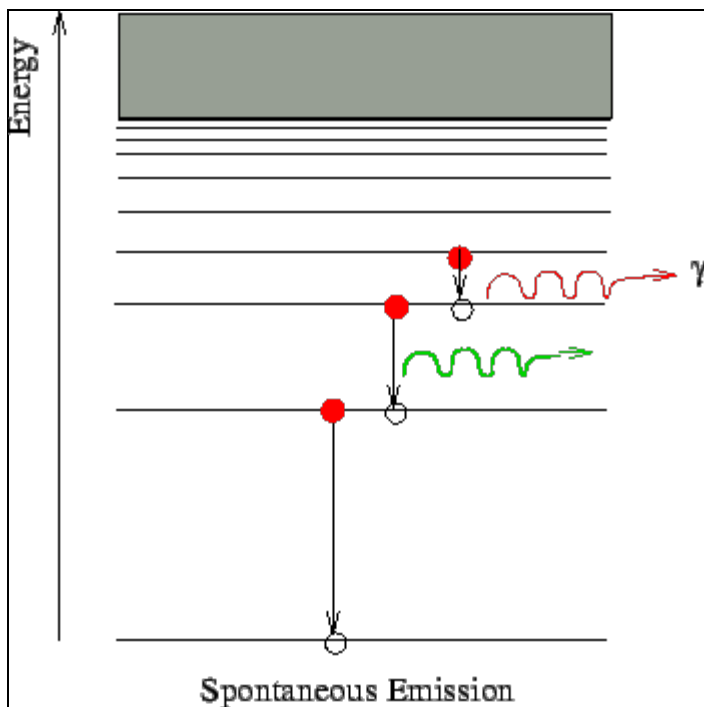
If this condition is not satisfied, the matter becomes transparent to incident radiation.



Spontaneous Emission:

Atoms which are in excited states are not in thermal equilibrium with their surroundings. Such atoms will eventually return to their ground state by emission of a photon. If E^* is the energy in the excited state and E the energy of a lower lying state (which could be the ground state), the frequency of the emitted photon is given by

$$\nu = \frac{E^* - E}{h}$$



Stimulated Emission:

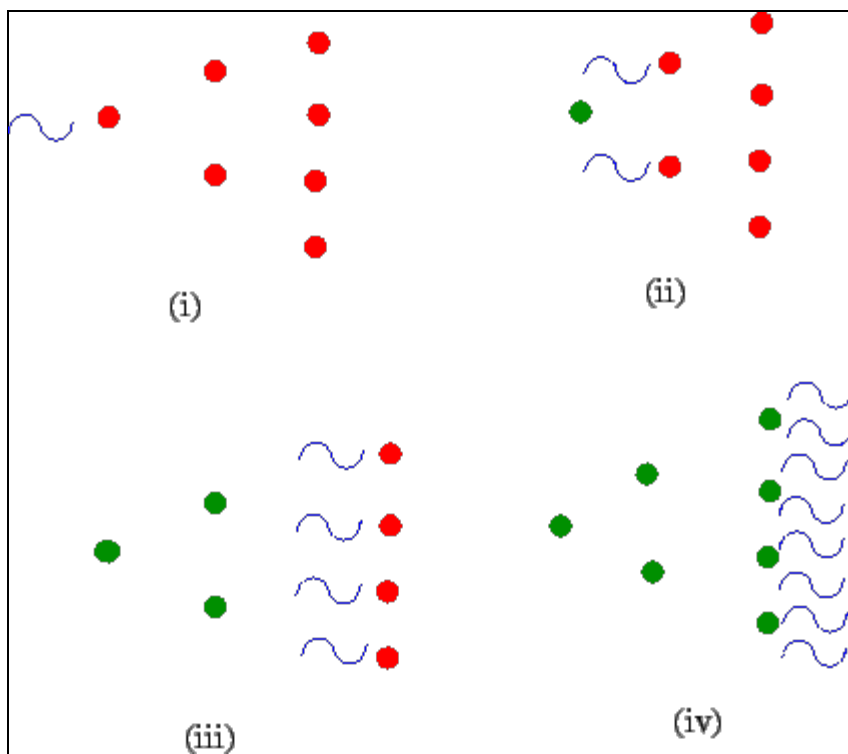
In 1917, Einstein showed that under certain conditions, emission of light may be stimulated by radiation incident on an excited atom. This happens when an electron is in an excited

state E^* and a photon whose energy is equal to the difference between E^* and the energy E of a lower lying level (could be the ground state) is incident on the atom. The incident photon induces the electron in the excited state to make a transition to the lower level by emission of a photon. The emitted photon travels in the same direction as the incident photon. Significantly, the new photon has the same energy as that of the incident photon and is **perfectly in phase** with it. When two waves travel in the same direction with a constant phase relationship, they are said to be **coherent**.

LASER - Light Amplification by Stimulated Emission of Radiation :

In 1958, Charles H. Townes and Arthur L. Schawlow showed that the effect of stimulated emission can be amplified to produce a practical source of light, which is coherent and can travel long distances without appreciable spread of the beam width. Such a light source is called LASER, an acronym for ***Light Amplification by Stimulated Emission of Radiation***.

The principle behind such amplification is simple. Suppose we start with one photon which strikes an atom in an excited state and releases a photon, we would have two photons and an atom in the ground state. These two photons, in turn, may be incident on two more atoms and give rise to four photons, and so on.



In the figure above, the excited state atoms are shown in red while those in the ground states are in green. However, the simple picture above does not work in practice because of the following :

- The time for which an electron remains in an excited state is approximately 10^{-8} seconds. Thus it is difficult to keep atoms in excited states till they are stimulated to radiate a photon. The excited atom is more likely to de-excite **spontaneously**.

Photons released through spontaneous processes are emitted in random directions and are not coherent with the incident photon.

- The photons that are incident and those which are generated may be absorbed by atoms in ground states, leading to depletion in the number of photons.

2. Interaction of Radiation with Matter:

In order to have an insight into the principle of laser, we need to understand the way radiation field interacts with matter. In the early 20th century, Planck formulated the theory of spectral distribution of thermal radiation. Einstein, by combining Planck's theory and the Boltzmann statistics gave a theory of stimulated emission which is the governing principle of lasers.

2.1 Blackbody Radiation :

Planck's formula gives the radiation of radiant intensity when electromagnetic radiation is confined to an isothermal cavity - known as the **blackbody** . In classical physics, radiation is considered as waves which form standing wave pattern in the cavity with nodes at the walls. The classical formula for the radiant energy density $u(\nu)$ at a frequency ν in the

energy interval ν to $\nu + d\nu$ is obtained by counting the number of modes of the

electromagnetic waves in this interval and multiplying it with the average energy per mode kT . The resulting formula is known as **Rayleigh- Jeans' Law** , which is given by

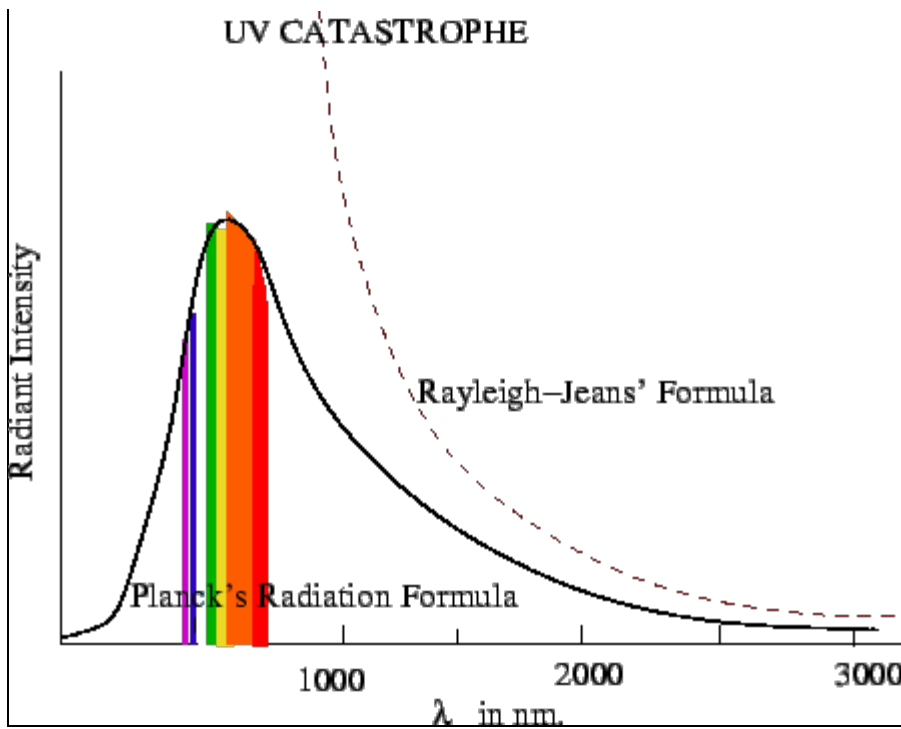
$$u(\nu)d\nu = \frac{8\pi kT}{c^3} \nu^2 d\nu$$

Rayleigh-Jeans' formula leads to unphysical result in the short wavelength region (known as **ultraviolet catastrophe**).

Planck suggested that the oscillating atoms could emit or absorb energy in tiny bursts called **quanta** , the energy of a quantum being proportional to its frequency.

Planck's formula for the radiant energy density is given by

$$u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} d\nu$$



The figure given here shows the radiant intensity distribution, $I(\lambda)d\lambda$ which is obtained by multiplying $u(\lambda)d\lambda$ by a factor $c/4$ arising out of averaging over the angles and considering the outgoing waves.

Using Planck's law one can show that the total power emitted by a blackbody at a temperature T is given by

$$U = \sigma T^4$$

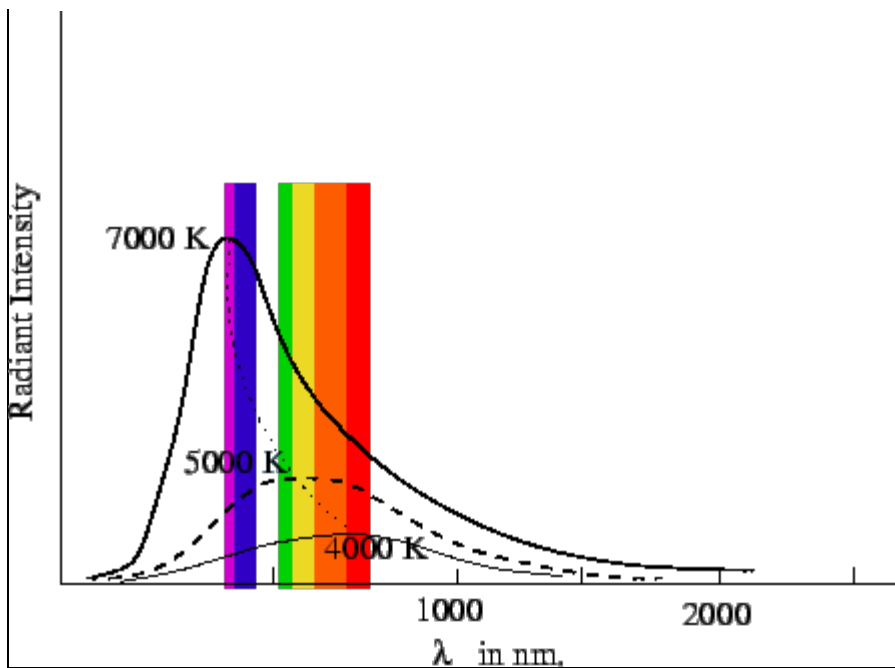
where σ is called Stefan's constant, which has a value $5.67 \times 10^{-8} \text{ J-K}^4/\text{m}^2$.

The emitted radiation has a peak at a wavelength λ_{max} give by **Wien's Displacement**

Law

$$\lambda_{\text{max}} T = 2.9 \times 10^3 \text{ m-K}$$

For instance, a blackbody at a temperature of 5000K has a radiation peak at 580 nm, which is near the middle of the visible region.



2.2 Boltzmann Statistics:

Consider a two level system. Let there be N_1 number of atoms per unit volume in the energy level E_1 and N_2 per unit volume in a higher energy level E_2 . Let $E_2 - E_1 = h\nu$.

Total population density in the system is

$$N = N_1 + N_2$$

If the atoms are in **thermal equilibrium** with the surrounding at a temperature T , the relative population in the two levels are given by **Boltzmann distribution**

$$\frac{N_2}{N_1} = e^{-h\nu/kT}$$

This equation shows that as the temperature increases, the population of excited states increase. However, the population of an excited state always lies lower than the population of the ground state, under equilibrium condition. For large energy gaps such that

$h\nu \gg kT$, the ratio above is close to zero so that very few of the atoms are in the upper energy state.

Exercise :

The ground state and the first excited state of Ruby are separated by 1.8 eV. Calculate the ratio of the number of atoms in the excited state to that in the ground state. (Ans.

$$e^{-70} \simeq 10^{-30})$$

When two or more states have the same energy, the states are said to be **degenerate**. The number of states at the same energy level is called the **multiplicity** of the energy level. As all states having the same energy have the same population, we have

$$\frac{N_2}{N_1} = \frac{g_2 e^{-h\nu/kT}}{g_1}$$

where g_1 and g_2 , respectively, are the multiplicities of the levels E_1 and E_2 .

2.3 Einstein Relations - A and B Coefficients:

The distribution of atoms in the two energy levels will change by absorption or emission of radiation. Einstein introduced three empirical coefficients to quantify the change of population of the two levels.

- **Absorption** - If B_{12} is the probability (per unit time) of absorption of radiation, the population of the upper level increases. The rate is clearly proportional to the population of atoms in the lower level and to the energy density $u(\nu)$ of radiation in the system. Thus the rate of increase of population of the excited state is given by

$$\frac{\partial N_2}{\partial t} = B_{12}u(\nu)N_1$$

where B_{12} is a constant of proportionality with dimensions $\text{m}^3/\text{s}^2\text{-J}$.

- **Spontaneous Emission** - The population of the upper level will decrease due to spontaneous transition to the lower level with emission of radiation. The rate of emission will depend on the population of the upper level. If A_{12} is the probability that an atom in the excited state will spontaneously decay to the ground state,

$$\frac{\partial N_2}{\partial t} = -A_{12}N_2$$

The equation above has the solution

$$N_2(t) = N_2(0)e^{-t/\tau}$$

where $\tau = 1/A_{12}$ gives the average **lifetime** of an atom in the excited level before the atom returns to the ground state. Thus the spontaneous emission depends on the lifetime of the atom in the excited state. The process is statistical and the emitted quanta bear no phase relationship with one another, i.e. the emission is **incoherent**.

- **Stimulated Emission** - Stimulated or induced emission depends on the number of atoms in the excited level as well as on the energy density of the incident radiation. If B_{21} be the transition probability per unit time per unit energy density

$$B_{21}$$

of radiation, the rate of decrease of the population of the excited state is $B_{21}u(\nu)N_2$.

The rate equation for the population of the upper level is

$$\frac{dN_2}{dt} = B_{12}u(\nu)N_1 - [A_{21} + B_{21}u(\nu)]N_2$$

Since $N_1 + N_2 = \text{constant}$,

$$\frac{\partial N_2}{\partial t} = -\frac{\partial N_1}{\partial t}$$

The emitted quanta under stimulated emission are coherent with the impressed field. The spontaneous emission, being incoherent, is a source of noise in lasers. When equilibrium is reached, the population of the levels remains constant, so that $dN_2/dt = 0$ and the rate of

emission equals rate of absorption, so that

$$B_{12}u(\nu)N_1 = [A_{21} + B_{21}u(\nu)]N_2$$

Using the Boltzmann factor $N_2/N_1 = (g_2/g_1) \exp(-h\nu/kT)$, and simplifying, we get

$$u(\nu) = \frac{A_{21} \frac{g_2}{g_1}}{B_{12}e^{h\nu/kT} - B_{21} \frac{g_2}{g_1}} = \frac{A_{21}/B_{21}}{\frac{g_1}{g_2} e^{h\nu/kT} \frac{B_{12}}{B_{21}} - 1}$$

If we regard the matter to be a blackbody and compare the above expression for the energy density with the corresponding energy density expression derived for the blackbody radiation, viz.,

$$u(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1}$$

we get

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3}$$

and

$$\frac{B_{21}}{B_{12}} = \frac{g_1}{g_2}$$

The last equation shows that in the absence of degeneracy, the probability of stimulated emission is equal to that of absorption. In view of this we replace the two coefficients by a single coefficient B and term them as B -coefficient. The spontaneous emission coefficient will be called the A -coefficient.

The ratio of spontaneous emission probability to the stimulated emission probability is

$$\frac{A}{B u(\nu)} = e^{h\nu/kT} - 1$$

so that for low temperatures, when $h\nu/kT \gg 1$, spontaneous emission is much more

probable than induced emission and the latter may be neglected. For high enough temperatures, stimulated emission probability can be significant though for optical frequencies, this requires very high temperature. For microwave frequencies the stimulated emission processes may be significant even at room temperatures.

Exercise :

Find the ratio of the probability of spontaneous emission to stimulated emission at 300 K for (a) microwave photons ($\nu = 10^{13}$ Hz.) and (b) optical photons ($\nu = 10^{15}$ Hz.). (Ans. (a) 0.17 (b) $\sim 10^7$.)

2.4 Population Inversion :

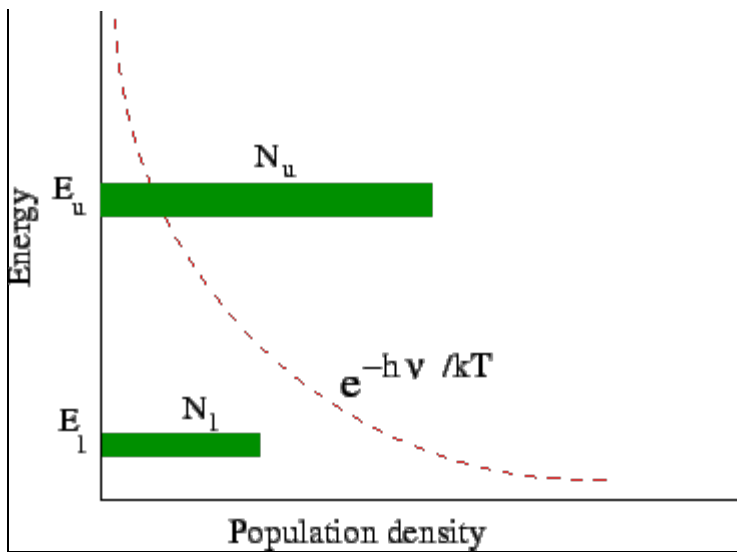
We have seen that when atoms are in equilibrium with the surrounding, the population of atoms in the ground state is more than that in any of the excited states. Population of excited states can be increased by absorption of radiation. However, the life time in the excited states being typically of the order of 10^{-8} seconds, atoms which make transitions to the excited states fall back to the ground state soon thereafter. This is also indicated by the ratios of the Einstein coefficients. It is, therefore, not possible to keep the population in the excited states higher than that in the ground state.

The basic principle involved in the operation of laser is **population inversion**, a situation in which the population of the excited state is kept higher than that of the ground state.

When $N_u < N_l$, i.e., the population in the upper level is less than that in the lower level,

the number of transitions from the lower to the upper level with absorption of radiation is more than that with emission and hence the radiation is attenuated. If, on the other hand, $N_u > N_l$, emissions are more than absorption and the radiation is amplified as it passes

through the material.



The figure shows population inversion required for light amplification with the dashed curve being the Boltzmann distribution. As atoms get deexcited, the laser action would stop unless atoms are continuously pumped into the upper level by some means.

Recap

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Part - II

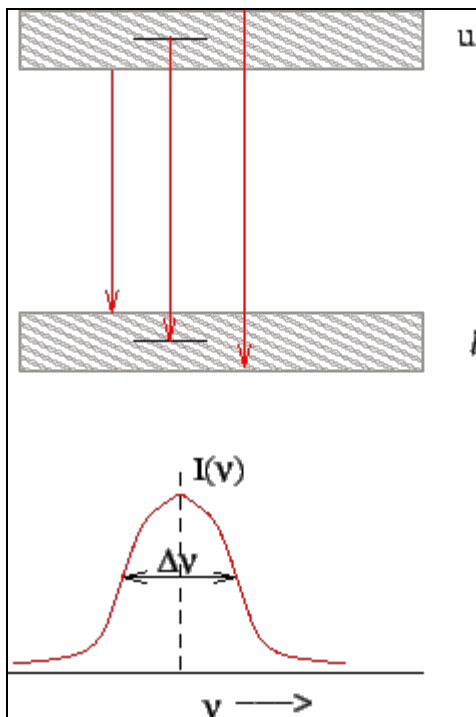
Objectives

In this lecture you will learn the following

- Level Broadening
- Lineshape Function
- Natural Linewidth
- Collision Broadening
- Doppler Broadening
- Consequence of Line Broadening

2.5 Level Broadening:

Atomic energy levels are not infinitely sharp but have some width associated with them. As a result, the spectrum of transition is not sharp lines but have some distribution. For instance, the emission spectrum for transition from E_u to E_l is a bell shaped curve.



Lineshape Function :

Define **lineshape function** $g(\nu)$ such that $g(\nu)d\nu$ gives the probability that a transition between two levels is an emission (or absorption) of photon whose frequency lies in the range ν and $\nu + d\nu$. Normalization demands

$$\int_0^{\infty} g(\nu) d\nu = 1$$

If N is the number of atoms in a given energy level, the spectral distribution of population in the level is given by $N(\nu) = g(\nu)N$, i.e., $N(\nu)d\nu$ is the number of atoms in the levels within frequency range ν and $\nu + d\nu$, so that

$$\int_0^{\infty} N(\nu) d\nu = N \int_0^{\infty} g(\nu) d\nu = N$$

Using the above, one can rewrite the equation defining Einstein's **A** - coefficient for spontaneous emission

$$\frac{\partial N_2}{\partial t} = -AN_2$$

as

$$\frac{\partial N_2(\nu)}{\partial t} = -AN_2 g(\nu) d\nu$$

$g(\nu)$ is a complicated function which depends on the transition involved and also on

external factors. The broadening of spectral lines is classified into two major categories :

Homogeneous Broadening :

In homogeneous broadening all atoms interact identically with the radiation field so that each atom makes the same transition with identical lineshape and width. Two of the prominent causes of homogeneous broadening are **natural line broadening** and **collision broadening**. Other sources of homogeneous broadening are **dipolar broadening** arising out of interactions between electric and magnetic fields of neighbouring atoms and **thermal broadening** due to effects of lattice vibrations on atomic transitions.

Inhomogeneous Broadening:

In inhomogeneous broadening, different atoms interact differently with the radiation field.

Doppler broadening due to thermal motion of atoms and broadening due to crystal inhomogeneities belong to this category.

2.5.1 Natural Linewidth :

Natural linewidth is an inherent width associated with any energy level due to quantum mechanical uncertainty principle. According to this principle, if Δt is the minimum uncertainty in the lifetime of an atom in an energy level, the uncertainty in determining the energy level itself is given by

$$\Delta E \Delta t \simeq h/2\pi$$

Using $E = h\nu$, the relation may be written as

$$\Delta \nu \Delta t \simeq 1/2\pi$$

Thus in a two level system, there is an width associated with both the upper and lower levels. This, in turn, implies that there is an inherent width in the frequency of photons emitted due to the transition. In a typical transition, the excited state lifetime is $\sim 10^{-8}$ seconds. The corresponding linewidth is about 16 MHz.

2.5.2 Collision Broadening:

Collision of atoms and molecules among themselves interrupts a radiative transition. If an atom which is emitting a radiation suddenly collides with another atom, the process of radiation is interrupted. the radiating atom starts its motion after such a collision with a completely random phase without any memory of the phase of the interrupted radiation. Linewidth in the presence of collisions is more than that in the absence of collisions. Collision broadening is also termed as **pressure broadening** as increased pressure is one of the primary causes of collision.

We can use a classical model to obtain an expression for homogeneous lineshape broadening. We take the electron to be classical particle bound to the nucleus by a spring force given by Hooke's law $F = -kx$. The equation of motion of the electron is given by

$$\frac{d^2 x}{dt^2} + \omega_0^2 x = 0$$

where $\omega_0 = \sqrt{k/m}$. The motion of the electron is oscillatory, $x(t) = x_0 e^{i\omega t}$. The total

energy of the electron $\mathcal{E} = (1/2)kx_0^2 = (1/2)m\omega^2 x_0^2$ is constant. The model does not

take into account the power that an oscillating electron radiates. In classical electrodynamics, an accelerating charge emits radiation, the emitted power being given by the **Poynting vector** $\vec{E} \times \vec{B}$. The total power radiated by a dipole is given by

$$-\frac{dE}{dt} = \frac{e^2 x_0^2 \omega_0^4}{12\pi \epsilon_0 c^3} \equiv \gamma E$$

which shows that the radiated energy is proportional to the instantaneous energy. We may model the situation by modifying our equations of motion for the electron to include a damping term proportional to the velocity ($F_d = -\alpha v$)

$$\frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

where $2\gamma = \alpha/m > 0$. (It may be noted that in classical electromagnetic theory the radiation damping term is proportional to $d^2 v / dt^2$, whereas our model assumes it to be proportional to v itself. This is actually of no consequence as the solution being sine and cosine functions, any even number of derivative would do.) The solution of the above equation for small damping ($\gamma^2 \ll \omega_0^2$) gives damped oscillations

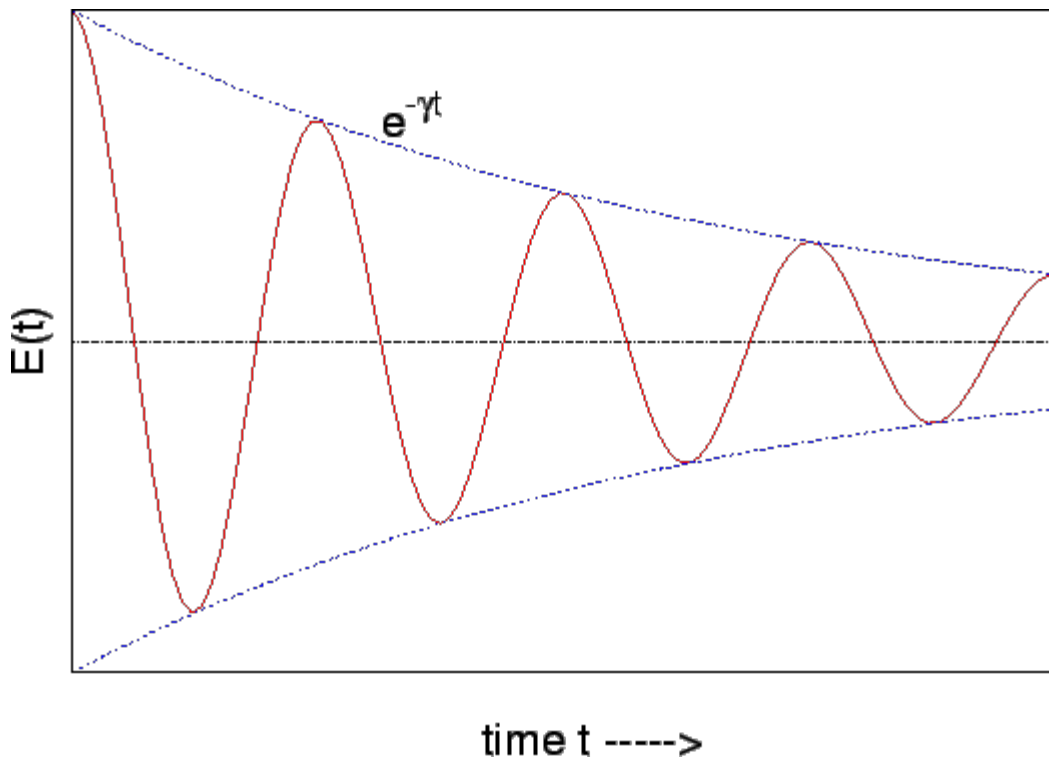
$$x = x_0 e^{-\gamma t} \sin(\omega t + \phi)$$

where $\omega = \sqrt{\omega_0^2 - \gamma^2} \simeq \omega_0$.

Since the dipole moment and hence the electric field of the radiation emitted is proportional to x , we have

$$E(t) = E_0 e^{-\gamma t} \sin \omega_0 t \text{ for } t \geq 0$$

$$= 0 \text{ for } t < 0$$



To obtain the lineshape, we take the Fourier transform of $E(t)$ which gives the distribution of frequencies in the electric field. Integration becomes simple if we write $E(t)$ as a complex function $E(t) = E_0 e^{-\gamma t} e^{i\omega_0 t}$

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty E_0 e^{-\gamma t} e^{i(\omega_0 - \omega)t} dt$$

$$= \frac{1}{\sqrt{1\pi}} \frac{1}{i(\omega - \omega_0) - \gamma}$$

The magnitude of the electric field is

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(\omega - \omega_0)^2 + \gamma^2} \right]^{1/2}$$

since the intensity $I(\omega)$ is proportional to $|E(\omega)|^2$, the normalized intensity is

$$I(\omega) = I_0(\gamma/\pi) \left[\frac{1}{(\omega - \omega_0)^2 + \gamma^2} \right]$$

where $\int_0^\infty I(\omega) d\omega = 1$. The distribution of intensity is called **Lorentzian** distribution. The maximum intensity $I_{\max} = I_0/\pi\gamma$ at $\omega = \omega_0$. The full width at half the maximum intensity is 2γ . The lineshape function can be easily inferred from the intensity distribution function, and is given by

$$g(\nu) = \left(\frac{\Delta\nu}{2\pi} \right) \frac{1}{\left[(\nu - \nu_0)^2 + \left(\frac{\Delta\nu}{2} \right)^2 \right]}$$

with

$$g(\nu_0) = \frac{2}{\pi\Delta\nu}$$

Exercise:

Verify the normalization of the lineshape function.

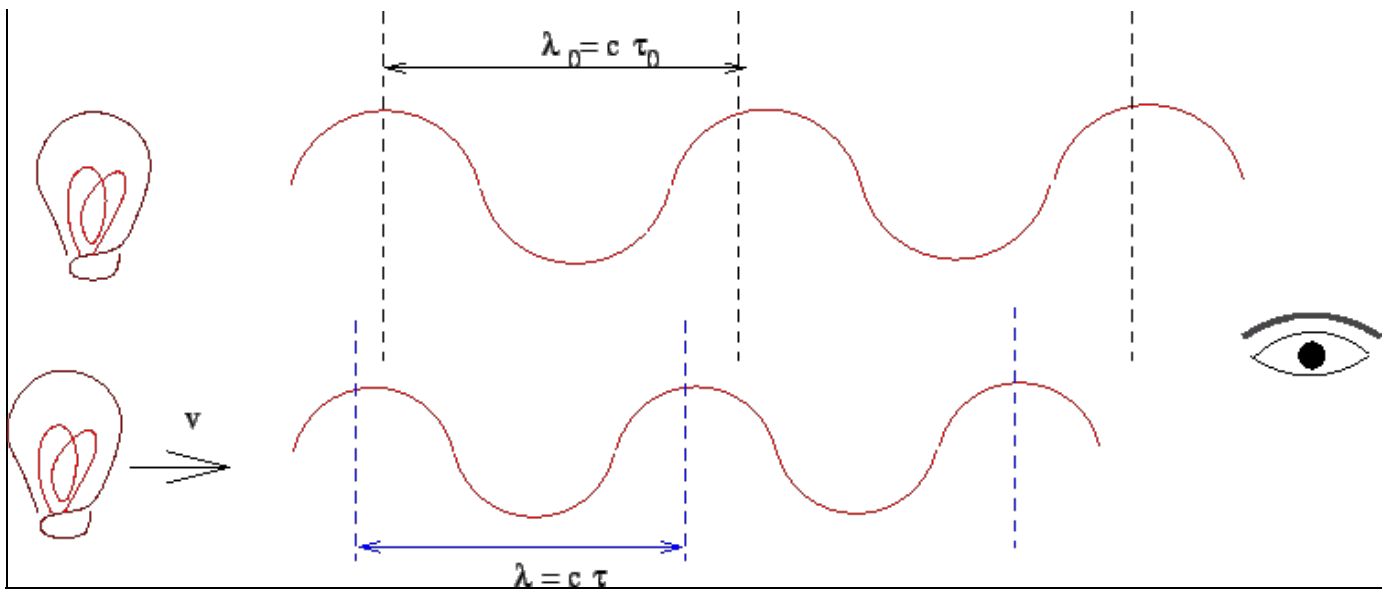
2.5.3 Doppler Broadening:

We are familiar with Doppler effect in sound by which an observer perceives a change in the frequency emitted by a source when there is relative motion between the source and the observer. The apparent frequency is more than the source frequency when the source and the observer approach each other and is less than the source frequency if they are receding from each other. Similar Doppler shift occurs in case of electromagnetic waves as well, excepting that unlike in the case of sound waves, the medium does not play any role.

When the observer and the source are stationary, the separation between two successive crests or troughs is λ_0 , where λ_0 is the wavelength emitted by the source. If c is the

velocity of light, the frequency $\nu_0 = c/\lambda_0$.

If the source is moving towards the observer with a speed v , two effects have to be considered arising out of special theory of relativity.



1. The time period in the observer's frame is dilated. If $\tau_0 = 1/\nu_0$ is the period of emission, the period as seen in the observer's frame is $\tau = \tau_0 / \sqrt{1 - v^2/c^2}$.
2. As the source moves towards the observer, the separation between the successive crests appears as

$$\lambda = c\tau - v\tau = \frac{(c - v)}{\sqrt{1 - v^2/c^2}}$$

Thus the apparent frequency is

$$\nu = \frac{c}{\lambda} = \nu_0 \sqrt{\frac{1 + v/c}{1 - v/c}}$$

For non-relativistic velocities, the Doppler shifted frequencies are

$$\nu = \nu_0 \left(1 + \frac{v}{c}\right) \quad \text{source moving towards observer}$$

$$\nu = \nu_0 \left(1 - \frac{v}{c}\right) \quad \text{source moving away from observer}$$

(It may be noted that in this case there is a Doppler effect in the transverse direction as well. However, the effect being of the order v^2/c^2 , is neglected.) Doppler shift may be

significant in gases in which atoms move randomly due to thermal motion. According to the kinetic theory of gases, the distribution of velocities of atoms (or molecules) in a gas at temperature T is given by Maxwellian velocity distribution

$$N(v)dv = 4\pi N \left(\frac{m}{2\pi kT} \right)^{3/2} \exp \left(-\frac{mv^2}{2kT} \right) v^2 dv$$

where $N(v)dv$ is the number of atoms with speeds between v and $v + dv$. All other

symbols in the equation have their standard meaning as in the kinetic theory. The average speed of the molecules is given by

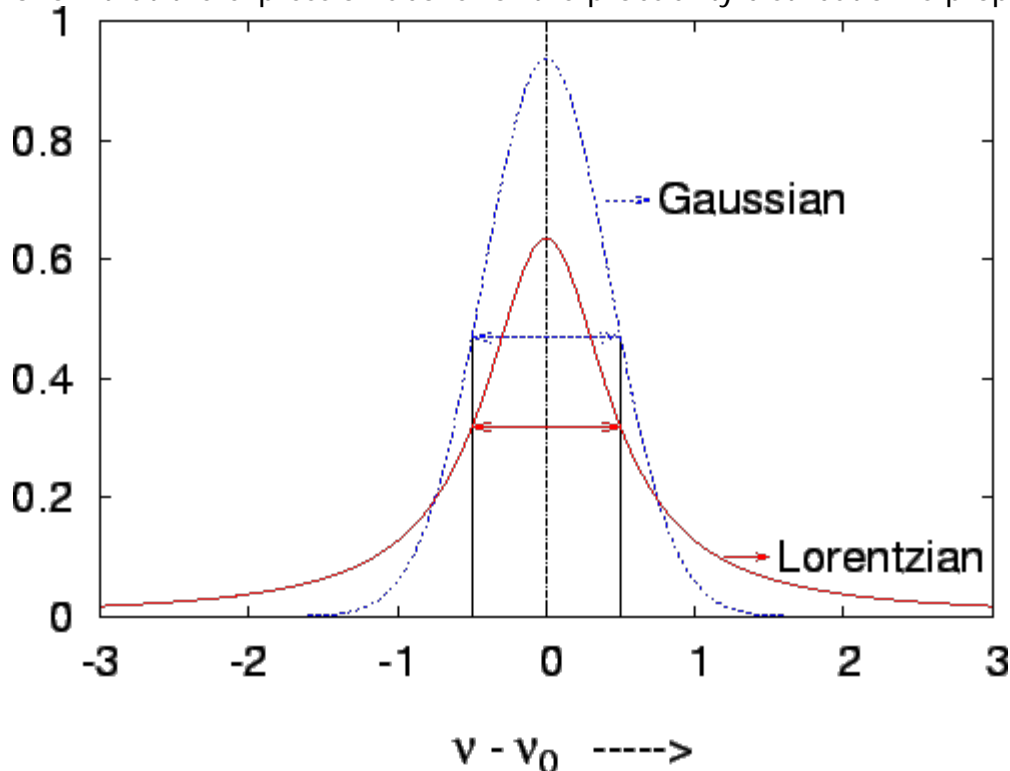
$$\bar{v} = \frac{\int_0^\infty v N(v) dv}{N} = \sqrt{\frac{8kT}{\pi m}}$$

As the distribution of velocity in thermal equilibrium is spherically symmetric, the probability distribution of velocity along any direction (say z direction, the direction) is given by

$$P(v_z)dv_z = \left(\frac{m}{2\pi kT} \right)^{1/2} \exp \left(-\frac{mv_z^2}{2kT} \right) dv_z$$

Exercise:

Show that the expression above for the probability distribution is properly normalized.



As there is a distribution of velocities, there will be a distribution of resonant frequencies. As $\nu = \nu_0(1 + v_z/c)$, we may substitute $v_z = c(\nu - \nu_0)/\nu_0$ in the expression for $P(v_z)$ and get an expression for the lineshape function $g(\nu)$

$$g(\nu)d\nu = \frac{1}{\sqrt{\pi}} \frac{1}{\Delta\nu'} \exp\left(-\frac{(\nu - \nu_0)^2}{(\Delta\nu')^2}\right) d\nu$$

where

$$\Delta\nu' = \nu_0 \sqrt{\frac{2kT}{mc^2}}$$

is the distance from the central maximum of the frequency value at which the probability is $1/e$ times the maximum value. It is often convenient to use the full width at half maximum

(FWHM), which is defined as the distance between the points on either side of the central maximum, where the probability (or the intensity) is half its maximum value. Defining this to be $\Delta\nu$, the expression for the lineshape function becomes

$$g(\nu)d\nu = \frac{1}{\sqrt{\pi}} \frac{2\sqrt{\ln 2}}{\Delta\nu} \exp\left(-\frac{(\nu - \nu_0)^2 4 \ln 2}{(\Delta\nu)^2}\right) d\nu$$

where the FWHM value $\Delta\nu$ is given by

$$\Delta\nu = 2\nu_0 \sqrt{\frac{2kT \ln 2}{mc^2}}$$

Example 1:

A He-Ne laser has a gas discharge temperature of 400 K. The atomic mass of Ne (which is responsible for lasing) is 3.35×10^{-26} kg. If the lasing wavelength is 630 nm, calculate the

lineshape broadening due to Doppler effect.

Solution:

The Doppler frequency broadening is given by

$$\Delta\nu = 2\nu_0 \sqrt{\frac{2kT \ln 2}{mc^2}}$$

using $\lambda = c/\nu$, the FWHM for wavelength is given by

$$\Delta\lambda = \frac{c}{\nu^2} \Delta\nu \simeq \frac{c}{\nu_0^2} \Delta\nu$$

Substituting the numerical values, $\Delta\lambda = 0.002\text{nm}$.

2.6

Consequence of Line Broadening:

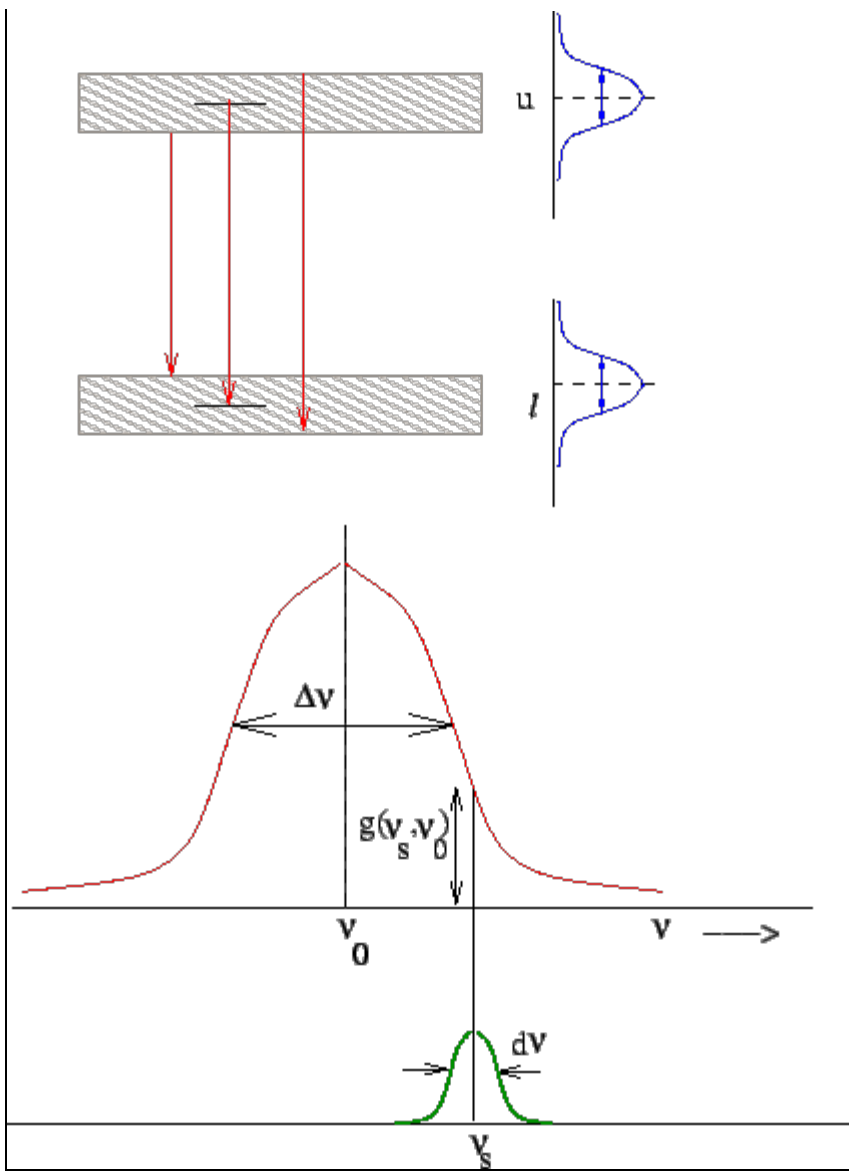
We have seen that the atomic energy levels are broadened due to both homogeneous and inhomogeneous effects. As a result of this, the energy separation between the two laser levels itself is centred around some frequency ν_0 with a spread $\Delta\nu$. The line shape

distribution corresponding to this transition will be denoted as $g(\nu, \nu_0)$.

The laser is forced to operate at a signal frequency ν_s with a width $d\nu$. In practice

$d\nu \ll \Delta\nu$. For instance for Ruby, the line width $\Delta\nu \approx 0.5\text{nm}$ while the laser output width is typically $d\nu \approx .01$ to $.001\text{ nm}$. Thus the signal may be

taken to be approximately monochromatic with a frequency ν_s . Thus the lineshape function that is relevant to our discussion is $g(\nu, \nu_0)$ with $\nu = \nu_s$. This is denoted as $g(\nu_s, \nu_0)$.



As the rate of spontaneous emission is much smaller than the rate of stimulated emission, we may neglect the former. Using Einstein's equation for the population of lower level, we have

$$\begin{aligned} \frac{\partial N_1}{\partial t} &= -B_{12}u(\nu)N_1 + B_{21}u(\nu)N_2 \\ &= +B_{21}u(\nu) \left[N_2 - \frac{g_2}{g_1}N_1 \right] \end{aligned}$$

where $\frac{B_{12}}{B_{21}} = \frac{g_2}{g_1}$

The number of atoms which interact with the laser signal at frequency is obtained by

$$\nu_s$$

multiplying the above with the prob. function $g(\nu_s, \nu_0)d\nu$. Thus

$$\frac{\partial N_1}{\partial t} = +B_{21}u(\nu_s) \left[N_2 - \frac{g_2}{g_1} N_1 \right] g(\nu_s, \nu_0)d\nu(A)$$

The equation may be rewritten using the following:

1. Define the population densities of the two levels $n_2 = \frac{N_2}{V}$ and $n_1 = \frac{N_1}{V}$.
2. The rate of decrease of population of atoms in the lower level is also equal to the rate at which incident photons

are depleted. If $u(\nu_s)$ is the energy density of the signal, the rate of depletion of number of photons N_{ph} is

$$-\frac{dN_{ph}}{dt} = -\frac{d}{dt} \left[\frac{u(\nu_s)d\nu}{h\nu_s} \right] = \frac{-d}{dt} [n_1]$$

where we have used the fact that the incident photons are nearly monochromatic.

Thus equation (A) becomes

$$\frac{\partial u(\nu_s)}{\partial t} = B_{21}u(\nu_s)g(\nu_s, \nu_0)h\nu_s \left[n_2 - \frac{g_2}{g_1} n_1 \right] (B)$$

Consider laser photons travelling through a slab of active material of thickness dx . If c is the speed of light in the material, as the signal advances from x to $x + dx$, we may use

$$\frac{\partial}{\partial t} = c \frac{\partial}{\partial x} \text{ to rewrite the above as}$$

$$\frac{\partial u(\nu_s)}{\partial x} = \frac{1}{c} B_{12}u(\nu_s)g(\nu_s, \nu_0)h\nu_s \left[n_2 - \frac{g_2}{g_1} n_1 \right]$$

Integrating

$$u(\nu_s) = u_0(\nu_s) \left[-h\nu_s g(\nu_s, \nu_0) B_{21} \left(n_2 - \frac{g_2}{g_1} n_1 \right) \frac{x}{c} \right]$$

If we define absorption coefficient by

$$\alpha(\nu_s) = \left(\frac{g_2}{g_1} n_1 - n_2 \right) \sigma_{21}$$

where the **stimulated emission cross section** σ_{21} is defined as

$$\sigma_{21} = \frac{h\nu_s g(\nu_s, \nu_0)}{c} B_{12}$$

we have

$$u(\nu_s) = u_0(\nu_s) e^{-\alpha(\nu_s)x}$$

which gives the penetration depth to be $\frac{1}{\alpha(\nu_s)}$. The signal get exponentially attenuated.

$\alpha(\nu_s)$ is maximum when all atoms are in the ground state.

Analogous to σ_{21} , we may define an absorption cross section

$$\sigma_{12} = \frac{h\nu_s g(\nu_s, \nu_0)}{c} B_{21}$$

so that

$$\frac{\sigma_{12}}{\sigma_{21}} = \frac{B_{12}}{B_{21}} = \frac{g_2}{g_1}$$

If $g_1 = g_2$ $\sigma_{12} = \sigma_{21}$

If $\alpha < 0$,

$$u(\nu_s) = u_0(\nu_s) e^{+|\alpha|x} \equiv u_0(\nu_s) e^{gx}$$

where g is known as the **gain**, which implies that the laser radiation is amplified.

Exercise:

Using Boltzmann distribution for the ratio $\frac{N_2}{N_1}$ show that for a 2 level system, α is always positive, i.e. in a two level system only absorption can take place.

Recap

In this lecture you have learnt the following

- Level Broadening
- Lineshape Function
- Natural Linewidth
- Collision Broadening
- Doppler Broadening
- Consequence of Line Broadening