MODULE 3: QUANTUM ELECTRODYNAMICS

<u>Propagator theory</u> :   (Henceforth $\hbar = c = 1$)

Linear partial differential equations with a source term can be solved in terms of 2-point Green's function (propagator).

$$\mathcal{L} u = v$$
$$\mathcal{L}(x) \, G(x', x) = \delta(x' - x)$$
$$u(x) = \int G(x', x) \, v(x) \, dx$$

$\left. \right\}$ Superposition principle (wave equation)

In problems with translationally invariant $\mathcal{L}(x)$, $G(x', x) \equiv G(x' - x)$.

Perturbative framework : Interactions are treated as source terms. The solution is expanded as a power series in the interaction terms.

Scattering analysis : Interactions are localised in space. Asymptotic states $(t \to \pm\infty)$ are defined in co-ordinate space. Often they are taken to be the plane-wave states.

Calculations are easier to carry out in momentum space, with Poincaré group eigenstates.

Non-relativistic case:

$$\Psi(x', t') = i \int d^3x \; G(x', t'; x, t) \Psi(x, t)$$

with $t' > t$ (causality). Huygen's principle (first order evolution)

The evolution equation is:

$$\left(i \frac{\partial}{\partial t} - H\right) \Psi(x, t) = 0 \implies \left(i \frac{\partial}{\partial t} - H_0\right) \Psi(x, t) = V(x, t) \Psi(x, t).$$

Asymptotic initial state is the plane wave

$$\phi_i(x, t) \text{ with } t \to -\infty.$$

We want $\Psi(x', t')$ as the final state for $t' \to +\infty$.
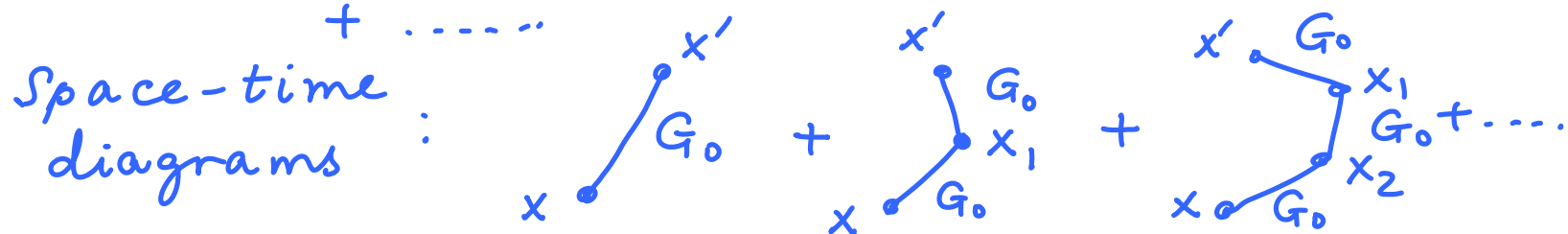
Time ordering is here Galilean invariant.

$G(x', t'; x, t)$ is obtained perturbatively in terms of $G_0(x', t'; x, t)$ as a power series.

The formal general solution is

$$G(x'; x) = G_0(x'; x) + \int d^4x_1 \, G_0(x', x_1) V(x_1) G(x_1, x).$$

Iterative expansion gives

$$G(x', x) = G_0(x'; x) + \int d^4x_1 \, G_0(x', x_1) V(x_1) G_0(x_1, x)$$

$$+ \int d^4x_1 \, d^4x_2 \, G_0(x', x_1) V(x_1) G_0(x_1, x_2) V(x_2) G_0(x_2, x)$$

$$+ \ldots\ldots$$

Space-time diagrams :

$$\Psi(x', t') = \phi_i(x', t') + \int d^4x_1 \, G_0(x', t'; x_1, t_1) V(x_1, t_1) \Psi(x_1, t_1)$$

Can also be expanded iteratively.

Scattering matrix elements are defined as

$$S_{fi} = \lim_{t' \to \infty} \int \phi_f^*(x', t') \Psi^{(+)}(x', t') \, d^3x'$$

← Forward propagating wave

$$= i \lim_{t' \to \infty} \lim_{t \to -\infty} \int d^3x' \, d^3x \, \phi_f^*(x') G(x', x) \phi_i(x)$$

$$= \delta^3(p_f - p_i) + \lim_{t' \to \infty} \int d^3x' \, d^4x \, \phi_f^*(x', t') G_0(x', t'; x, t) V(x, t)$$
$$\times \Psi^{(+)}(x, t).$$

A Hermitian Hamiltonian will have a complete set of orthogonal eigenstates, and a unitary S-matrix. (The completeness sum includes any bound states if they exist.)

Causality is included using the step function

$$\Theta(\tau) = \begin{cases} 1, & \tau > 0 \\ 0, & \tau < 0 \end{cases}, \qquad \frac{d\Theta(\tau)}{d\tau} = \delta(\tau).$$

For example,

$$\Theta(t'-t)\,\Psi(x') = i \int d^3x \; G(x',x)\,\Psi(x).$$

Using contour integration, we find that

$$\Theta(\tau) = \lim_{\epsilon \to 0} -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \cdot \frac{e^{-i\omega\tau}}{\omega + i\epsilon}$$

Pole is slightly below the real axis.

$\tau > 0$: Contour closed as : Pole inside

$\tau < 0$: Contour closed as : Pole outside

Thus, $\left[i\frac{\partial}{\partial t'} - H(x') \right] G(x'; x) = \delta^4(x'-x)$

with non-relativistic $H_0 = -\frac{\nabla^2}{2m}$,

leads to the non-relativistic propagator

$$G_0(p, \omega) = \frac{1}{\omega - \frac{p^2}{2m} + i\epsilon} .$$

Relativistic case: The propagator $G(x'; x)$ must vanish outside the light cone (causality).

Time-like separation: $t' - t$ has definite sign

Space-like separation: $t' - t$ does not have unique sign.

This restriction has to be imposed by carefully chosen $i\epsilon$-prescription.

$$\underbrace{\langle x' | e^{-i H_{free} t} | x \rangle}_{\text{Propagation amplitude}} = \langle x' | e^{-i \frac{p^2}{2m} t} | x \rangle$$

$$= \frac{1}{(2\pi)^3} \int d^3p \; e^{-i \frac{p^2}{2m} t} \; e^{i p \cdot (x' - x)}$$

$$= \left(\frac{m}{2\pi i t} \right)^{3/2} e^{i m (x' - x)^2 / 2t}$$

This describes the dispersion of a point particle. The wavefunction broadens with time. For finite t, the tail for $|x' - x|$ large is non-zero.

Try the propagation in relativistic case as

$$\langle x' | e^{-it\sqrt{p^2+m^2}} | x \rangle = \frac{1}{(2\pi)^3} \int d^3p \, e^{-it\sqrt{p^2+m^2}} e^{ip\cdot(x'-x)}$$

$$= \frac{1}{2\pi^2 |x'-x|} \int_0^\infty dp \cdot p \sin(p|x'-x|) e^{-it\sqrt{p^2+m^2}}$$

For $|x'-x|^2 \gg t^2$ (outside the light cone), the integral can be approximated by the method of stationary phase.

$$p_{stationary} = \frac{i\,m\,|x'-x|}{\sqrt{|x'-x|^2-t^2}}$$ gives the behaviour

of propagation amplitude as $e^{-m\sqrt{|x'-x|^2-t^2}}$.

It falls off as $|x'-x| \to \infty$, but it is not exactly zero outside the light cone.

Causality is exactly implented in quantum theory by cancellation between particle propagation and antiparticle propagation in the opposite direction.

Equivalent description of causality in terms of field operators is that $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$. A measurement at one point cannot influence anything at spacelike separation from it. Simplest measurement is that of the field $\phi(x)$ itself.

For a free Dirac particle, the propagator satisfies

$$(i \not\partial' - m) S_F(x'; x) = \delta^4(x' - x)$$

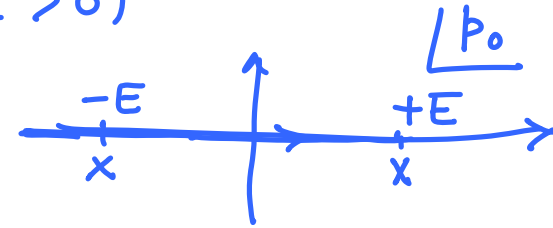$$S_F(x'; x) = S_F(x' - x) = \int \frac{d^4 p}{(2\pi)^4} e^{-i p \cdot (x' - x)} S_F(p).$$

$$\therefore (\not p - m)_{\alpha\beta} [S_F(p)]_{\beta\gamma} = \delta_{\alpha\gamma}$$

$$\therefore S_F(p) = \frac{1}{\not p - m} = \frac{\not p + m}{p^2 - m^2} \qquad (\not p \not p = p^2)$$

$i\epsilon$-prescription has to define the behaviour of propagator near the on-shell singularities at $p^2 = m^2$. ($p^2 \neq m^2$ states are virtual off-shell states)

Singularities of the propagator are simple poles
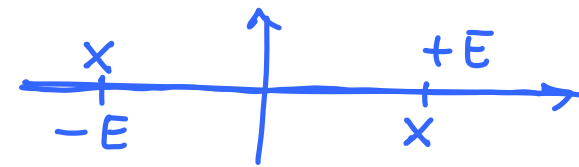in the complex p_0-plane. $(E > 0)$

Retarded prescription propagates
both solutions forward in time :

Advanced prescription propagates :
both solutions backward in time
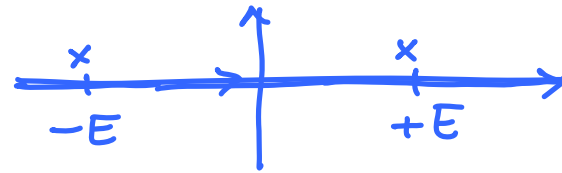
Feynman prescription propagates
positive energy component forward
in time, and negative energy
component backward in time.

This is consistent with hole-theory interpretation.

Implementation : $p^2 - m^2 \longrightarrow p^2 - m^2 + i\epsilon$ $(\epsilon \rightarrow 0_+)$.

$$S_F(x'-x) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}'-\vec{x})} \int_C \frac{dp_0}{2\pi} e^{-ip_0(t'-t)} \frac{\not{p}+m}{p^2-m^2+i\epsilon}$$

$$= -i \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}'-\vec{x})} \times \begin{cases} e^{-iE(t'-t)} \dfrac{E\gamma_0 - \vec{p}\cdot\vec{\gamma} + m}{2E}; t'>t \\[2ex] e^{+iE(t'-t)} \dfrac{-E\gamma_0 - \vec{p}\cdot\vec{\gamma} + m}{2E}; t'<t \end{cases}$$

Projection operators can be used to select solutions with specific sign of energy $\left(\Lambda_{\pm}(p) = \dfrac{\pm\not{p}+m}{2m} \right)$.

Change $\vec{p}$ to $-\vec{p}$ in the negative energy part, to get

$$S_F(x'-x) = -i \int \frac{d^3p}{(2\pi)^3} \cdot \frac{m}{E} \left[\Lambda_+(p) e^{-ip\cdot(x'-x)} \Theta(t'-t) \right.$$
$$\left. + \Lambda_-(p) e^{ip\cdot(x'-x)} \Theta(t-t') \right] \begin{array}{l} \text{with} \\ p_0 \equiv E > 0 \end{array}$$

With the normalised plane-waves,
$$\psi^r(x) = \sqrt{\frac{m}{E}} (2\pi)^{-3/2} \omega^r(p) e^{-i\epsilon_r p\cdot x},$$
this allows solutions of scattering problems.

Stückelberg –Feynman interpretation :

Particles go forward in time.

Antiparticles go backward in time (opposite p, s).

$$\Psi = \Psi^{(+)}_{E>0} + \Psi^{(-)}_{E<0}$$

The two components evolve as :

$$\Theta(t'-t)\,\Psi^{(+)}(x') = i \int S_F(x'-x)\,\gamma_0\,\Psi^{(+)}(x)\,d^3x ,$$

$$\Theta(t-t')\,\Psi^{(-)}(x') = -i \int S_F(x'-x)\,\gamma_0\,\Psi^{(-)}(x)\,d^3x .$$

In presence of interactions, the full propagator $\tilde{S}_F(x';x)$ satisfies :

$$(i\slashed{\partial}'-m)\,\tilde{S}_F(x';x) = \delta^4(x'-x) + (\text{Interaction}) \times \tilde{S}_F$$

$$= \int d^4y\,\,\delta^4(x'-y)\left[\delta^4(y-x) + e\slashed{A}(y)\tilde{S}_F(y;x)\right]$$

This can be related to $S_F(x';x)$, with the solution

$$\tilde{S}_F(x';x) = S_F(x';x) + e \int d^4y\,S_F(x'-y)\,\slashed{A}(y)\,\tilde{S}_F(y;x) .$$

This is a formal result. But it can be easily iterated in a perturbative framework.

The solution of the Dirac equation
$$(i\not\partial - m)\,\tilde{\psi}(x) = e\not A\,\tilde{\psi}(x)$$
thus becomes
$$\tilde{\psi}(x) = \psi(x) + e\int d^4y\; S_F(x-y)\,\not A(y)\,\tilde{\psi}(y)\,.$$
The iterative procedure provides results for
scattering problems as power series in coupling "e".
The components of wavefunction propagate from
$$t = -\epsilon_f\,\infty \quad \text{to} \quad t = +\epsilon_f\,\infty\,.$$
In this convention,
$$S_{fi} = \lim_{t'\to\epsilon_f\infty}\int d^3x'\;\overline{\Psi}_f(x',t')\,\tilde{\psi}(x',t')$$
$$= \delta_{fi} - i\epsilon_f\int d^4y\;\overline{\Psi}_f(y)\,e\not A\,\tilde{\psi}(y)$$
Covariant expressions for physical observables
result only when both particle and antiparticle
modes are included in analysis.

Feynman diagrams : Space-time pictures representing
interactions in a perturbative framework.
In quantum electrodynamics, they are an
alternating sequence of propagators and interaction
vertices. The fundamental structure is:

(The arrows describe evolution in time.)
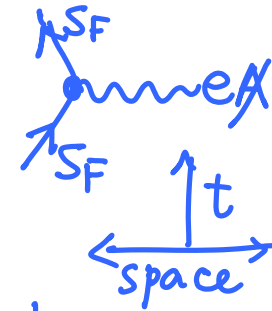
The processes involving antiparticles can be
understood in two different ways.
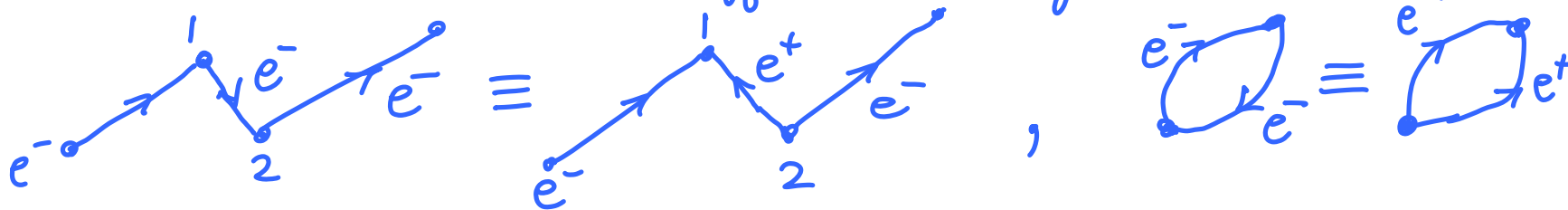
Negative energy,   Only positive energy
  all e^- both e^- and e^+.

In pair production,
$$\langle e^-_{E>0} | H_{int} | e^-_{E<0} \rangle \sim \bar{u}(p_-) \, e \slashed{A} \, v(p_+)$$
$$\sim \left(e^{-ip_- \cdot x} \right)^{\dagger} e^{ip_+ \cdot x} \quad \longleftarrow \text{Agree}$$

$$\langle e^-_{E>0}\, e^+_{E>0}|\, H_{int}\,|0\rangle \sim \left(e^{-ip_-\cdot x}\ e^{-ip_+\cdot x}\right)^\dagger \hookleftarrow$$

This equivalence is an example of crossing symmetry.
Generic procedure in calculations is to draw all allowed Feynman diagrams and sum the results. When time-orderings are combined, the total result is of the form:

That combines both pole contributions for the propagator between 1 and 2.

$$\frac{1}{p^2 - m^2} = \frac{1}{2E}\left(\frac{1}{E - \sqrt{\vec{p}^2 + m^2}} + \frac{1}{E + \sqrt{\vec{p}^2 + m^2}} \right) = $$

In these diagrams, fermion lines do not end. They either go from initial state to final state, or form closed loops. (Photons can appear or disappear singly, since they are bosons.)
Two differently ordered fermion contributions show a sign difference (arising from exclusion principle).

Trace theorems for products of γ-matrices:

Perturbative framework of S-matrix calculation produces alternating sequence of propagators and vertices ($\not{p}$ and $\not{A}$). The amplitudes are thus products of these factors (and projection operators too). The transition probabilities involve $|\text{amplitude}|^2$, and lead to trace of the values in internal degrees of freedom.

Simplification of calculation is achieved using the commutation algebra of γ-matrices, and the cyclic nature of trace. These properties are independent of the choice of representation of γ-matrices. The results are automatically Lorentz covariant.

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \quad : \text{ Dimension independent}$$

$$\text{Tr}(\mathbf{1}) = 4 \quad, \quad g_\mu{}^\mu = 4 \quad : \text{ Dimension dependent}$$

(More generally, $2^{[d/2]}$ $\xleftarrow{}$ d. These can be easily programmed in a computer. Symbolic manipulation packages.)

(a) $\text{Tr}(\not{\alpha}_1 \not{\alpha}_2 \cdots \not{\alpha}_n) = 0$ for n odd.

Using γ_5, which anticommutes with all γ_μ,

$$\text{Tr}(\not{\alpha}_1 \cdots \not{\alpha}_n) = \text{Tr}(\gamma_5 \cdot \gamma_5 \not{\alpha}_1 \cdots \not{\alpha}_n)$$
$$= \text{Tr}(\gamma_5 \not{\alpha}_1 \cdots \not{\alpha}_n \gamma_5)$$
$$= (-1)^n \text{Tr}(\not{\alpha}_1 \cdots \not{\alpha}_n \gamma_5 \gamma_5)$$
$$= (-1)^n \text{Tr}(\not{\alpha}_1 \cdots \not{\alpha}_n)$$

This is nonzero only for even n.

[Result is not valid when γ_5 does not exist.]

(b) $\mathrm{Tr}\,(\not{a}\,\not{b}) = a^\mu b^\nu\,\mathrm{Tr}\,(\gamma_\mu\gamma_\nu) = a^\mu b^\nu\,\mathrm{Tr}\,(1\cdot g_{\mu\nu}) = 4\,a\cdot b$

$\not{a}\not{b} = a\cdot b - i\,\sigma_{\mu\nu}\,a^\mu b^\nu$

(c) $\mathrm{Tr}\,(\not{a}_1 \dots \not{a}_n)$ for n even is evaluated by reducing it to Trace of $(n-2)$ γ-matrix products, and then using induction.

$\mathrm{Tr}\,(\not{a}_1 \not{a}_2 \dots \not{a}_n) = 2\,a_1\cdot a_2\,\mathrm{Tr}\,(\not{a}_3 \dots \dots \not{a}_n)$
$\qquad\qquad\qquad\qquad - \mathrm{Tr}\,(\not{a}_2 \not{a}_1 \not{a}_3 \dots \not{a}_n)$

$\qquad\qquad\qquad = 2\,a_1\cdot a_2\,\mathrm{Tr}\,(\not{a}_3 \dots \dots \not{a}_n)$
$\qquad\qquad\qquad\quad - 2\,a_1\cdot a_3\,\mathrm{Tr}\,(\not{a}_2 \dots \not{a}_n)$
$\qquad\qquad\qquad\quad + \mathrm{Tr}\,(\not{a}_2 \not{a}_3 \not{a}_1 \dots \not{a}_n)$

$\qquad\qquad\qquad = 2\,a_1\cdot a_2\,\mathrm{Tr}\,(\not{a}_3 \dots \dots \not{a}_n)$
$\qquad\qquad\qquad\quad - 2\,a_1\cdot a_3\,\mathrm{Tr}\,(\not{a}_2 \dots \not{a}_n)$
$\qquad\qquad\qquad\quad + 2\,a_1\cdot a_n\,\mathrm{Tr}\,(\not{a}_2 \dots \not{a}_{n-1})$
$\qquad\qquad\qquad\quad - \mathrm{Tr}\,(\not{a}_2 \dots \not{a}_n \not{a}_1)$

$\therefore\ \mathrm{Tr}\,(\not{a}_1 \not{a}_2 \dots \not{a}_n) = a_1\cdot a_2\,\mathrm{Tr}\,(\not{a}_3 \dots \not{a}_n)$
$\qquad\qquad\qquad\quad - a_1\cdot a_3\,\mathrm{Tr}\,(\not{a}_2 \dots \not{a}_n) + \dots + a_1\cdot a_n\,\mathrm{Tr}\,(\not{a}_2 \dots \not{a}_{n-1})$.

(d) Other operators (for measurement or specific states) can involve γ_5. (Outside the set of γ_μ.)

$\text{Tr}(\gamma_5) = 0$. γ_5 is the product of all four γ_μ's. After rewriting γ_5 as product of four γ_μ's, only nonzero contribution comes from the parts where each of the γ_μ's is squared to identity.

It follows that $\text{Tr}(\gamma_5 \not{a} \not{b}) = 0$,

$$\text{Tr}(\gamma_5 \not{a} \not{b} \not{c} \not{d}) = 4i\,\epsilon_{\alpha\beta\gamma\delta}\, a^\alpha b^\beta c^\gamma d^\delta.$$

(e) Contractions of Lorentz indices can also be simplified using Clifford algebra rules.

$$\gamma_\mu \gamma^\mu = 4.$$

$$\gamma_\mu \not{a} \gamma^\mu = a^\alpha \gamma_\mu \gamma_\alpha \gamma^\mu = a^\alpha 2g_{\mu\alpha} \gamma^\mu - a^\alpha \gamma_\alpha \gamma_\mu \gamma^\mu$$

$$= 2\not{a} - 4\not{a} = -2\not{a}$$

$$\gamma_\mu \not{a} \not{b} \gamma^\mu = a^\alpha (2g_{\mu\alpha} - \gamma_\alpha \gamma^\mu) \not{b} \gamma^\mu$$

$$= 2\not{b}\not{a} - \not{a}(-2\not{b})$$

$$= 2(\not{b}\not{a} + \not{a}\not{b}) = 4a\cdot b$$

$$\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu = -2 \not{c} \not{b} \not{a}$$
$$\gamma_\mu \not{a} \not{b} \not{c} \not{d} \gamma^\mu = 2(\not{d} \not{a} \not{b} \not{c} + \not{c} \not{b} \not{a} \not{d}).$$

(f) $\text{Tr}(\not{a}_1 \cdots \not{a}_n) = \text{Tr}(\not{a}_n \cdots \not{a}_1)$, for n even

This follows from charge conjugation invariance. The matrix C exists, such that $C \gamma_\mu C^{-1} = -\gamma_\mu^T$.

So, $\text{Tr}(\not{a}_1 \cdots \not{a}_n)$

$= \text{Tr}(C \not{a}_1 C^{-1} \cdot C \not{a}_2 C^{-1} \cdots \cdots C \not{a}_n C^{-1})$

$= (-1)^n \text{Tr}(\not{a}_1^T \not{a}_2^T \cdots \not{a}_n^T)$

$= (-1)^n \text{Tr}(\not{a}_n \cdots \not{a}_1)^T$

$= \text{Tr}(\not{a}_n \cdots \not{a}_1)$, with n even.

In Feynman diagrams, this relates particle and antiparticle descriptions with oppositely pointing arrows.

The machinery described so far allows calculations of electrons (fermions) evolving in a background electromagnetic field.

To go further, we need a framework where the electromagnetic field also evolves due to its interaction with charged particles.

Photons : Fully relativistic particles.

Lorentz transformations were discovered as transformations that change Maxwell's equations in a covariant manner. There is no rest frame for photons. The position uncertainty $\Delta x \sim \frac{\hbar}{p}$ is the same as the de Broglie wavelength, so a photon cannot be localised better than its wavelength. There is no photon wavefunction, that can be used to define its probability density. Particularly, concepts of conservation laws remain, e.g. momentum, energy, charge.

There is no non-relativistic limit for photons.
But there is a classical limit refered to as
geometric optics. It refers to situations where the
photon wavelength is much smaller than the
characteristic dimensions of the system.

Maxwell's equations :

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho \quad , \quad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \iff \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad , \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \iff \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0$$

Covariant EM tensor is defined by (antisymmetric)

$$F^{io} = E^i \quad , \quad F^{ij} = -\epsilon^{ijk} B_k .$$

Homogeneous equations (Bianchi identity) can be
written as, $\partial_\mu \tilde{F}^{\mu\nu} = 0$, $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\beta} F_{\lambda\beta}$.

Antisymmetry of $F^{\mu\nu}$ implies $\partial_\mu \partial_\nu F^{\mu\nu} = \frac{4\pi}{c} \partial_\nu j^\nu = 0$.

So the current is automatically conserved.

The homogeneous equations are solved by

$$\vec{B} = \vec{\nabla} \times \vec{A} \; , \quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \iff F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

This solution introduces vector potential A^μ.
It turns out to be the more fundamental variable for describing gauge fields, compared to $F^{\mu\nu}$.
Photons have only two physical degrees of freedom, i.e. transverse polarisations. The other two degrees of freedom for A^μ (temporal and longitudinal) are not physical in describing photons. They are kept in calculational frameworks to explicitly maintain Lorentz covariance, but drop out from the final physical result.
This behaviour is a consequence of the powerful underlying symmetry, called gauge symmetry.

<u>Gauge symmetry</u>: Under a shift of variables

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda, \quad \phi \rightarrow \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t} \Longleftrightarrow A_\mu \rightarrow A_\mu - \partial_\mu \Lambda,$$

the equations above remain unchanged. $(F_{\mu\nu} \rightarrow F_{\mu\nu})$.

The freedom to choose arbitrary Λ is called the gauge degree of freedom. By choosing appropriate Λ (gauge fixing), the degrees of freedom of A^μ can be reduced to the physical ones only.

Many prescriptions for gauge fixing Λ exist.

A popular Lorentz invariant prescription is $\partial_\mu A^\mu = 0$.

The potentials than satisfy $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$.

The inhomogeneous Maxwell's equations then become

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{4\pi}{c} j^\nu \Rightarrow \partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} j^\nu.$$

This is the wave equation for photons.

But $\partial_\mu A^\mu = 0$ does not perform complete gauge fixing.

There is residual gauge freedom with $\partial_\mu \partial^\mu \Lambda = 0$.

Complete gauge fixing can be achieved by imposing
$\phi = 0$, $\vec{\nabla} \cdot \vec{A} = 0$ (radiation gauge fixing).
That leaves only two physical degrees of freedom behind.
The change in $k_\mu A^\mu$ is $k^2 \Lambda$ under gauge
transformation. So $k^2 \Lambda$ must vanish from all
physical observables, which demands $k^2 = 0$.
So photons are massless (physical degrees of freedom)
Actually, gauge invariance, coservation of
electric charge and masslessness of photon are
all interrelated and equivalent properties.

Photons are massless particles, with helicity $= \pm 1$.
The theory can be described in a covariant form,
using the gauge field A^μ. Constraints of gauge fixing
then remove the unphysical degrees of freedom.

Local symmetry:

Global symmetries have parameters of transformation that are independent of position. (e.g. Poincaré group symmetries). They often correspond to choice of basis in a theory, and then change to a new basis without changing any observable.

In local symmetries, parameters of transformation depend on the location in space-time.

This extension introduces new degrees of freedom, whose physical meaning is then explored.

Weyl introduced the concept of local symmetry as extension of Einstein's general theory of relativity. $ds^2 = -d\tau^2$ is an invariant in GTR, but what if the measuring rods have location dependent lengths? Einstein pointed out that such variation of measuring rods is inconsistent with postulates of GTR, but the name "gauge theory" survived.

The concept was resurrected in quantum mechanics.
The overall phase of the wavefunction is unobservable.
This can be called a global phase symmetry.
It can be made local, with the appearance of new
gauge degrees of freedom.

Let $\Psi(x) \longrightarrow e^{ie\Lambda(x)/\hbar c}\Psi(x)$ be the gauge
trasformation for the "charged" particle.
The derivatives should also transform the same
way, to keep the equations independent of $\Lambda(x)$.
$\partial_\mu \Psi(x)$ does not transform in this manner.
But a new type of "covariant derivative" can be
defined with the desired transformation property.
$D_\mu \equiv \partial_\mu + \frac{ie}{\hbar c} A_\mu(x)$ implies that $D_\mu \Psi(x) \longrightarrow e^{ie\Lambda(x)/\hbar c} D_\mu \Psi(x)$,
provided that $A_\mu(x) \longrightarrow A_\mu(x) - \partial_\mu \Lambda(x)$.

This structure is conveniently described in the language of differential geometry. The phase degree of freedom at each point is called a fibre bundle. The symmetry group here is $U(1)$. Covariant derivative describes "parallel transport". The components of D_μ do not commute.

$$[D^\mu, D^\nu] = \frac{ie}{\hbar c}(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{ie}{\hbar c} F^{\mu\nu}.$$

Thus $F^{\mu\nu}$ measures the "curvature" of the gauge space. $F^{\mu\nu}$ is invariant under gauge transformation. So a gauge invariant theory can be constructed from suitable combinations of $\Psi, D_\mu \Psi, F_{\mu\nu}$.

Let $V(x) = e^{ie\Lambda(x)/\hbar c}$, such that $\Psi(x) \longrightarrow V(x)\Psi(x)$.

Then the link $U(x_1, x_2) = \exp\left[-\frac{ie}{\hbar c}\int_{x_1}^{x_2} A_\mu(x) dx^\mu\right]$ transforms to $V(x_2) U(x_1, x_2) V^{-1}(x_1)$.

This is a covariant transformation.

This transformation shows that

(1) $\bar{\Psi}(x_2) U(x_1, x_2) \Psi(x_1)$ is gauge invariant.

(2) $\text{Tr}\left[\exp\left(-\frac{ie}{\hbar c} \oint A_\mu \, dx^\mu \right) \right]$ is gauge invariant.
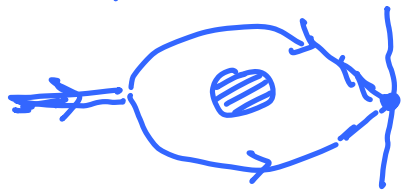
   (It is called a Wilson loop.)

An infinitesimal Wilson loop can be used as a definition of $F_{\mu\nu}$. (It is nonperturbative.)

Covariant derivative gives the minimal coupling formulation of the gauge theory.

It can happen that in some regions $A_\mu \neq 0$, but $F_{\mu\nu} = 0$. But there can be an influence on $\Psi(x)$ in this region, because $\Psi(x)$ couples to $A_\mu(x)$.
A particular case is the Aharanov-Bohm effect.

When the Wilson loop is non-trivial, interference effect is observed.
(Flux is enclosed in the loop.)
In a sense, A_μ is more fundamental than $F_{\mu\nu}$.

$A_\mu \longrightarrow A_\mu - \partial_\mu \Lambda \; : \; \Lambda \in [-\infty, \infty]$ Non-compact formulation, $\mathbb{R}$

Phase $e^{ie\Lambda/\hbar c} \; : \; \frac{e\Lambda}{\hbar c} \in [0, 2\pi]$ Compact formulation, $U(1)$

If there is a cyclic path, which changes "Λ" by 2π, then the phase of the wavefunction is unchanged. (Trajectories with winding numbers.)

Then considering particles with different charges, all phases will be invariant, when "Λ" for the smallest charge changes by 2π, and all other charges are integral multiples of the smallest charge. This is "charge quantisation", arising from the compact nature of the gauge symmetry group.

In our world, the smallest charge is that of an electron. (We can choose units with $e=1$.)

Photon propagator: This is the 2-point Green's function for Maxwell's inhomogeneous equations. We choose the covariant Lorentz gauge, $\partial_\mu A^\mu = 0$. The equations to be solved are,

$$\partial^2 A^\mu(x) = j^\mu(x) \quad , \qquad \partial^2 \equiv \Box \ .$$

The propagator is defined by

$$\Box \, D_F(x;y) = \delta^4(x-y) \ ,$$

$$D_F(x-y) = \int \frac{d^4 q}{(2\pi)^4} \, e^{-iq\cdot(x-y)} \, D_F(q^2) \ .$$

Then the Feynman propagator is $D_F(q^2) = \dfrac{-1}{q^2 + i\epsilon}$, which respects causality.

$$A^\mu(x) = \int d^4 y \, D_F(x-y) \, j^\mu(y)$$

$$= \int d^4 y \, g^{\mu\nu} D_F(x-y) \, j_\nu(y) \ .$$

Positive frequency solutions propagate forward in time.

Current conservation: $\partial_\mu j^\mu = 0$ or $k_\mu j^\mu = 0$.
The photon propagator is generically sandwiched
between two currents (which produce end-points).
Then the photon propagator can be changed by

$$g_{\mu\nu} \longrightarrow \left(g_{\mu\nu} - \alpha \frac{q_\mu q_\nu}{q^2} \right), \text{ without altering any}$$

physical observables. The free parameter α
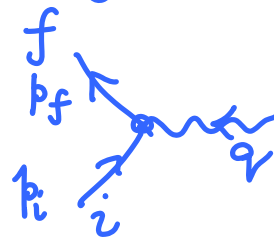represents the gauge degree of freedom. The choice
$\alpha = 0$ is refered to as the Feynman gauge.
Gauge transformations shift A^μ by multiples of q^μ.
Then $j^\mu A_\mu$ shifts by $\overline{\Psi}_f \not{q} \Psi_i$, which should vanish.
Equations of motion for Dirac fermion give

$$\overline{\Psi}_f \not{q} \Psi_i = \overline{\Psi}_f (\not{p}_f - \not{p}_i) \Psi_i = \overline{\Psi}_f (m - m) \Psi_i = 0.$$

This relation is called the "Ward identity".

For a conserved current, $\int j^\mu A_\mu \, d^4x$, is gauge invariant, and becomes a valid term for describing interaction between photons and charged particles. It is the basis of formulating an interacting theory of dynamical charged particles and dynamical photons. That gives the theory of Quantum Electrodynamics.

Free photons are described by the plane waves

$$A^\mu(x; k) = \frac{\epsilon^\mu}{\sqrt{2kV}} \left(e^{-ik\cdot x} + e^{+ik\cdot x} \right)$$

Arises from $\underset{\longleftarrow}{\underline{\phantom{\sqrt{2kV}}}}$ Normalisation per unit volume
integrating $\delta(k^2)$.

Only two transverse components of ϵ^μ are physical. Gauge invariance implies that when factors of ϵ^μ are replaced by factors of k^μ, the amplitude for interaction vanishes.

Polarisations are chosen so as to satisfy

$$\epsilon^\mu \epsilon_\mu = -1 , \quad \epsilon^\mu k_\mu = 0 .$$

In addion, $\epsilon^0 = 0$ in the radiation gauge.
Without fixing the gauge completely,

$$\sum_{\lambda=1}^{4} \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda)} = - g_{\mu\nu} . \quad \left(\begin{array}{c} \text{Without transversality} \\ \text{constraint} \end{array} \right)$$

In the final result, only contribution of the physical transverse degrees of freedom survives.

All this machinery can now be put together to construct arbitrary S-matrix amplitudes, to any desired order in perturbation theory.

Feynman rules: These allow us to quickly write
down the amplitude for a given process
(represented by Feynman diagrams).

Electrons: $S_{fi} = \delta_{fi} - i\, \epsilon_f \int d^4y\; \overline{\Psi}_f(y)\, e\!\!\!/A\, \tilde{\Psi}(y)$.

Photons: $A^\mu(x) = \int d^4y\; D_F(x-y)\, j^\mu(y)$

Putting these together,

$$S_{fi} = \delta_{fi} - i\, e\, \epsilon_f \int d^4x\, d^4y\; \left[\overline{\Psi}_f(x)\, \gamma_\mu\, \tilde{\Psi}(x)\right] D_F(x-y)\, j^\mu(y).$$

This is formally exact, and can be easily iterated
by expanding $\tilde{\Psi}(x)$ and $j^\mu(y)$ to desired order.

The procedure to quickly write down S_{fi} is
to first draw all the diagrams contributing
to the desired order in perturbation theory,
and then convert them to algebraic expressions.

The rules can be easily obtained for the tree diagrams, and with quantum field theory methods can be generalised to loop diagrams also.

(1) Draw all diagrams to desired order in e. (These are space-time pictures.)

(2) There are vertices, propagators (internal lines) and external lines. For tree diagrams, no. of vertices = no. of propagators + 1. Each vertex provides the factor $-ie\gamma_\mu \int_{-\infty}^{\infty} d^4x$.

(3) Each propagator gives the factor $iS_F(x-y)$ (for Dirac fermion) or $iD_F(x-y)g_{\mu\nu}$ (for photon).

(4) Each external line gives the wavefunction factor : $u, v, \bar{u}, \bar{v}, \epsilon_\mu$.

There are overall signs for fermions:

(a) For positrons in initial state, the sign is $(-1)^{n_i(e^+)}$ from $\in_f$.

(b) For every closed fermion loop, and for each exchange of external identical fermion lines, there is a factor of (-1) from statistics.

The overall sign can be convention dependent, but the relative signs have to be fixed correctly.

These rules (and the associated description) is often used in momentum space. The Fourier transforms are easily carried out.

At every vertex, factors of type $e^{ip \cdot x}$ accumulate. Then integral over position of the vertex (i.e. $\int d^4x$) produces a δ-function corresponding to momentum conservation. (i.e. $\delta^4(\sum_i p_i)$.)

So the momentum space diagrams will have a momentum (with direction) for every line. Extra momentum variables are eliminated by imposing momentum conservation at every vertex. For tree diagrams, all internal line momenta are uniquely assigned ($V = P+1$), and an overall δ-function for momentum conservation remains, i.e. $\delta^4(p_f - p_i)$.

In case of diagrams with loops, there remain undetermined momenta going around, and they have to be integrated over. (These integrals have to be often handled with care.)

Typically, each loop gives $\int_{-\infty}^{\infty} \frac{d^4 p}{(2\pi)^4}$, and some of these may diverge (and hence requires regularisation).

Normalisation of external lines are chosen as per the plane-wave convention:

$$\Psi(x) = \sqrt{\frac{m}{EV}} \, u(p,s) \, e^{-ip\cdot x} \quad \text{etc. for fermions.}$$

$$A_\mu(x) = \sqrt{\frac{1}{2kV}} \, \epsilon_\mu \left[e^{-ik\cdot x} + e^{ik\cdot x} \right].$$

Nature of perturbative expansion:

(1) The expansion contains denominators of type $p^2 - m^2 + i\epsilon$. They vanish on-shell. In the virtual quantum states (off-shell propagators appearing as internal lines), they do not vanish. (Uncertainty principle.)

In non-relativistic QM, there are energy denominators in perturbative expansions. ($\vec{p}$ is conserved, while E is not.) Here, in relativistic case, p^μ is conserved, but dispersion relation is violated.

If different time orderings of propagators are separated, then $p^2 - m^2 + i\epsilon$ denominators get converted to energy denominators, and E is not conserved for individual terms.

(2) The series expansion is in powers of e. For charge conjugation symmetric observables, the series expansion is in powers of $\alpha = \frac{e^2}{4\pi}$. The convergence property can then be analysed in the complex α-plane, around the expansion point $\alpha = 0$. The result is that the series is not convergent for any $\alpha \neq 0$ (zero radius of convergence), but it is asymptotic. The theory is unstable, when analytically continued to $\alpha < 0$. (i.e. branch cut for $\alpha < 0$).

Dyson's argument: $\alpha < 0$ corresponds to a theory where like charges attract and unlike charges repel. This situation is similar to that of gravitational interaction. The attraction collapses the system to a situation where the energy is unbounded from below. Quantum uncertainty allows this to happen only when the no. of mutually attracting particles is large enough. (With $\frac{1}{r}$ potential and no screening, a massive star can collapse to a black hole.) In electrodynamics with $\alpha < 0$, quantum fluctuations produce pairs, which tunnel to a state where all e^-'s collect in one region and all e^+'s collect in another region far away. This is a state of spontaneous polarisation, with energy

$$2 \sum_i \left(mc^2 + K.E. \right) + \sum_{i,j=1}^{N}{}' \frac{e^2}{r_{ij}} + \left(\begin{array}{c} \text{Interaction} \\ \text{between} \\ \text{groups} \end{array} \right).$$

N pairs: $\quad\quad O(N) \quad\quad\quad\quad\quad O(\alpha N^2) \quad\quad\quad$ Negligible

This can be negative for sufficiently large N, for $\alpha < 0$.

So the theory is singular in the complex α-plane, along the whole negative real axis.

The perturbative expansion is not convergent, but still useful as an asymptotic series. (Asymptotic series diverges for fixed α and no. of terms going to infinity. But at a fixed order, it gives a result that approaches the correct value as $\alpha \to 0$. Typically, the successive terms of such a series first decrease in value and then start diverging.)

The number of terms that should be kept in the asymptotic series to keep it meaningful can be infered from the exact solution of simple cases. For $\alpha > 0$, the hydrogen atom solution gives such an answer. The energy of a positronium is $O(-\alpha^2 mc^2)$, which indicates instability for $\alpha \gtrsim 1$. (This is instability for a single pair production.)

In practice, such a behaviour can be produced by a series with terms behaving as $n! \alpha^n$, and asymptotic series is useful up to order $\frac{1}{\alpha}$.

In real QED, the best calculations are only upto $O(\alpha^5)$ or $O(\alpha^6)$, and the series appears well-behaved. (But in QCD, one has $\alpha_s \sim O(1)$, and ill-behaved nature of the asymptotic series is observed.)

(3) Generally, momentum conservation constraints keep the intermediate states (propagators) off-shell. But this can change, when new states appear in the theory. A single propagator becoming on-shell represents a bound state in the theory (which shows up as a pole in the S-matrix).
More than one propagators becoming on-shell, when they share a common vertex, represent production thresholds (branch cuts in the S-matrix).

New effects can be assigned to such singularities of the S-matrix, which generically lead to expansion of the theory. These features are properly handled by extending the S-matrix into complex momenta planes. (There are no singularities for Euclidean momenta.)

This technique of analytical continuation provides many constraints on the properties of the S-matrix.

QED processes : Now we can easily use

Feynman rules to write down the amplitudes of various tree-level processes, and calculate their rates (or cross-sections).

Examples:

(1) Coulomb scattering: $\bar{e}^- \gamma^* \to \bar{e}^-$
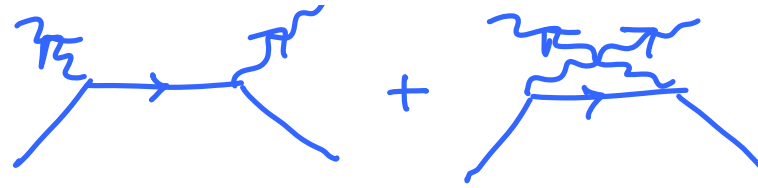(virtual photon from a static
nucleus provides the field)
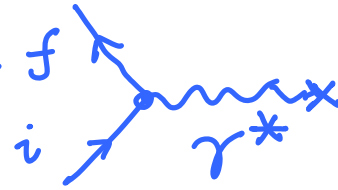
(2) Compton scattering:
$$e^- \gamma \to e^- \gamma$$
(e^- is often bound in atoms)
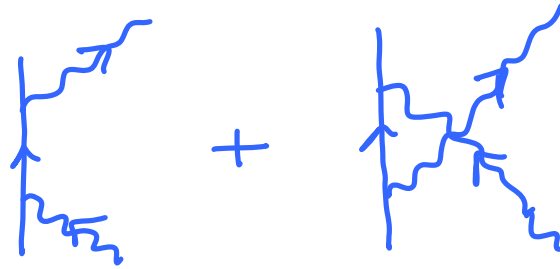
(3) Bremsstrahlung:
$$e^- \gamma^* \to e^- \gamma \quad \text{in external}$$
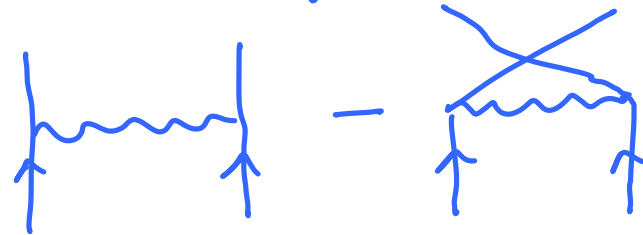electric field
(Initial/final state radiation)
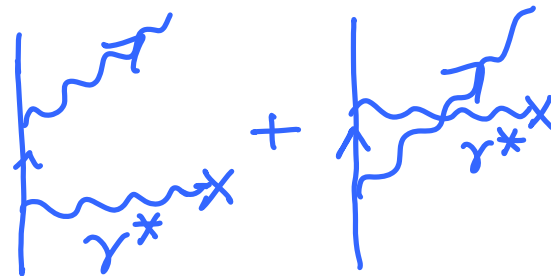
(4) Pair annihilation
    $e^+ e^- \longrightarrow \gamma \gamma$
    (e^- is often in materials)
In calculation of cross-section, Bose statistics of two indistinguishable photons gives a factor of $\frac{1}{2!}$.

(5) Pair production is the time reversed process.
    $\gamma \gamma \longrightarrow e^+ e^-$ (often one photon is virtual γ^*, i.e. contribution of atomic electric field)

(6) Møller scattering:
    $e^- e^- \longrightarrow e^- e^-$

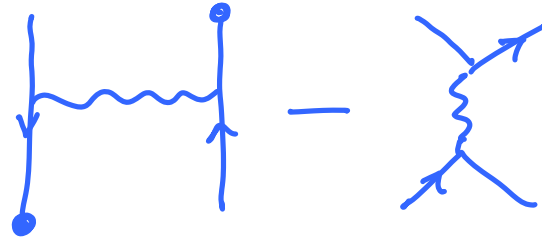In calculation of cross-section, Fermi statistics of two indistinguishable electrons gives a factor of $\frac{1}{2!}$. The two diagrams have a relative minus sign due to Fermi statistics of electrons.

(7) Bhabha scattering :

$$e^+ e^- \rightarrow e^+ e^-$$

(used for calibrating luminosity
  in accelerators)

Fermi statistics applies between incoming states.

In general, amplitudes for processes $A + B \rightarrow C + D$ are related to those for $A + \bar{C} \rightarrow B + \bar{D}$ by crossing symmetry : Reverse the arrows, flip the momentum sign, replace $u \leftrightarrow v$ for spinors.
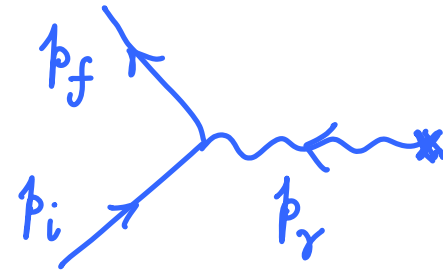
In calculation of cross-sections, it is convenient to use the Transition matrix (T) defined by :

$$S_{fi} = \delta_{fi} + i T_{fi} (2\pi)^4 \delta^4 (p_f - p_i),$$

with cross-sections depending on $|T_{fi}|^2$.

Coulomb scattering :

e^- is scattered from a stationary target of charge Q.

The non-relativistic target absorbs momentum, but not energy (elastic scattering).

$$S_{fi} = \bar{u}(p_f, s_f)(-ie\gamma^\mu) u(p_i, s_i) A_\mu(p_\gamma)(2\pi)^4 \delta^4(p_f - p_i - p_\gamma)$$

The photon wavefunction is :

$$A_o(x) = \frac{Q}{4\pi |\vec{r}|} \quad , \quad \vec{A} = 0 \quad \Rightarrow \quad A_o(p) = \frac{Q}{|\vec{p}|^2} 2\pi \delta(E).$$

If photon is interpreted as an internal line,

$$\frac{-1}{p^2} \longrightarrow \frac{1}{|\vec{p}|^2} \quad , \quad j^o(x) = Q\, \delta(\vec{x}), \quad \vec{j} = 0.$$

Either way, $\quad iT_{fi} = -ie\, \bar{u}_f \gamma^o u_i \dfrac{Q}{|\vec{p}_f - \vec{p}_i|^2}$.

The cross-section is defined as the transition probability per unit volume of space-time, and per unit incident flux.

The volume of space-time is just $(2\pi)^4 \delta^4(0)$.

No. of final states in the momentum interval $d^3 p_f$ is $V \frac{d^3 p_f}{(2\pi)^3}$ for each particle.

All the unobserved states must be summed over (or integrated over), e.g. the virtual photon γ^*.

$|S_{fi}|^2$ contains square of $(2\pi)^4 \delta^4(p_f - p_i)$, which is equivalent to $(2\pi)^4 \delta^4(p_f - p_i) \times (2\pi)^4 \delta^4(0)$.

$$\therefore d\sigma_{fi} = V \frac{d^3 p_f}{(2\pi)^3} \cdot \int V \frac{d^3 p_\gamma}{(2\pi)^3} \cdot \frac{(2\pi)^4 \delta^4(p_f - p_i - p_\gamma)}{(E_f V/m)(E_i V/m)} |T_{fi}|^2$$

$$= \frac{2\pi \, \delta(E_f - E_i) \, m^2}{E_f E_i} |T_{fi}|^2 \frac{d^3 p_f}{(2\pi)^3}$$

$$d^3 p_f = p_f^2 \, dp_f \, d\Omega \;, \quad E_f = E_i \implies |\vec{p_f}| = |\vec{p_i}| \;.$$

$$p_f \, dp_f = E_f \, dE_f \;, \quad \text{incident flux is } |\vec{v_i}| = \frac{|\vec{p_i}|}{E_i} \;.$$

$$\therefore \quad \frac{d\sigma}{d\Omega} = \frac{e^2 Q^2}{4\pi^2} \cdot \frac{m^2}{|\vec{p_f} - \vec{p_i}|^4} \, |\bar{u}_f \gamma^0 u_i|^2 \;.$$

$$\vec{p_f} \cdot \vec{p_i} = p_i^2 \cos\theta = \beta^2 E^2 \cos\theta$$

$$|\vec{p_f} - \vec{p_i}|^2 = p_f^2 + p_i^2 - 2\vec{p_f} \cdot \vec{p_i} = 2E^2 - 2m^2 - 2\beta^2 E^2 \cos\theta$$

$$= 2\beta^2 E^2 - 2\beta^2 E^2 \cos\theta = 2\beta^2 E^2 (1 - \cos\theta) = 4\beta^2 E^2 \sin^2\frac{\theta}{2} \;.$$

Generically, $|\bar{u}_f \Gamma u_i|^2 = (\bar{u}_f \Gamma u_i)(\bar{u}_i \bar{\Gamma} u_f)$,

where $\bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$. Explicitly,

$$\overline{\gamma_\mu} = \gamma_\mu \;, \quad \overline{\sigma_{\mu\nu}} = \sigma_{\mu\nu} \;, \quad \overline{\gamma_\mu \gamma_5} = \gamma_\mu \gamma_5 \;, \quad \overline{i\gamma_5} = i\gamma_5 \;.$$

The electron wavefunctions appearing in the cross-section can be arranged in the form $u\bar{u}$ (or $v\bar{v}$), and then replaced by the corresponding projection operators.

Consider unpolarised scattering. Then we sum over s_f (unobserved), and average over s_i (equiprobable). The required $|\text{matrix element}|^2$ becomes:

$$\frac{1}{2}\sum_{s_i,s_f} |\bar{u}_f \gamma^0 u_i|^2 = \frac{1}{2}\sum_{s_i s_f} \bar{u}_f \gamma^0 \underbrace{u_i \bar{u}_i} \gamma^0 u_f$$

$$= \frac{1}{2}\sum_{s_f} \bar{u}_f \gamma^0 \frac{\not{p}_i + m}{2m} \gamma^0 u_f$$

$$= \frac{1}{2}\,\text{Tr}\left[\gamma^0 \frac{\not{p}_i + m}{2m} \gamma^0 \frac{\not{p}_f + m}{2m}\right]$$

$$= \frac{1}{8m^2}\,\text{Tr}\left[\gamma^0 \not{p}_i \gamma^0 \not{p}_f + m^2 \gamma^0 \gamma^0\right]$$

$$= \frac{1}{8m^2}\,\text{Tr}\left[E_i \gamma^0 \not{p}_f - \not{p}_i \not{p}_f + E_f \not{p}_i \gamma^0 + m^2\right]$$

$$= \frac{1}{2m^2}\left[2 E_i E_f - p_i \cdot p_f + m^2\right]$$

$$p_i \cdot p_f = E_i E_f - |\vec{p_i}| \cdot |\vec{p_f}| \cos\theta = E_i E_f - \beta_i \beta_f E_i E_f \cos\theta$$

$$2E_i E_f - p_i \cdot p_f + m^2 = E^2 + \beta^2 E^2 \cos\theta + m^2$$

$$= E^2 + |\vec{p}|^2 - 2\beta^2 E^2 \sin^2\frac{\theta}{2} + m^2$$

$$= 2E^2 \left(1 - \beta^2 \sin^2\frac{\theta}{2}\right)$$

Putting all the above factors together,

$$\frac{d\sigma}{d\Omega} = \frac{e^2 Q^2}{64\pi^2} \cdot \frac{(1 - \beta^2 \sin^2\theta/2)}{\beta^4 E^2 \sin^4\theta/2} \quad : \text{Mott cross-section}$$

Features:

(1) It is independent of m, and also signs of e and Q. (Higher order calculations alter these properties, e.g. at order $e^3 Q^3$.)

(2) $\beta^2 \sin^2\frac{\theta}{2}$ is due to magnetic moment of electron interacting with the magnetic field it sees in its own rest frame (where the target charge Q is moving).

(3) Backward scattering ($\theta = \pi$) vanishes as $\beta \to 1$. This is due to helicity conservation of the electron. (Helicity cannot flip.)

For scattering of positrons from the Coulomb field,

$$i T_{fi} = \bar{v}(p_i, s_i)(+ie\gamma^\mu) v(p_f, s_f) A_\mu(p_\gamma).$$

The projection operators give $v\bar{v} = \dfrac{\not{p} - m}{2m}$ for unpolarised scattering.

The leading order cross-section is the same as that for electrons.

Compton scattering :

Initial electron is essentially at rest.
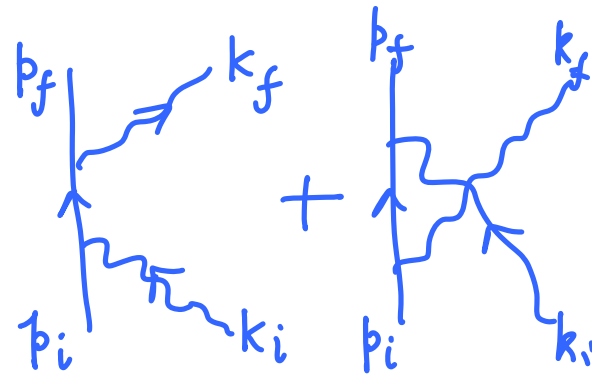
Final electron is not observed.

Incoming and outgoing photons are plane waves.

$$A^\mu(k) = \frac{\epsilon^\mu}{\sqrt{2\omega V}} \, , \quad \omega = k^0 = |\vec{k}| \, , \quad \epsilon^\mu k_\mu = 0, \; k^2 = 0.$$

The normalisation of photon wavefunction is chosen such that the energy of the plane wave is

$$\frac{1}{2} \int d^3x \, (\vec{E}^2 + \vec{B}^2) = \omega.$$

The sum of the two diagrams gives the matrix element,

$$i T_{fi} = (-ie)^2 \, \bar{u}_f \times$$

$$\times \left[\not{\epsilon}_f \frac{i}{\not{p}_i + \not{k}_i - m + i\epsilon} \not{\epsilon}_i + \not{\epsilon}_i \frac{i}{\not{p}_i - \not{k}_f - m + i\epsilon} \not{\epsilon}_f \right] u_i$$

$$= -i e^2 \bar{u}_f \left[\not{\epsilon}_f \frac{\not{p}_i + \not{k}_i + m}{2 p_i \cdot k_i} \not{\epsilon}_i + \not{\epsilon}_i \frac{\not{p}_i - \not{k}_f + m}{-2 p_i \cdot k_f} \not{\epsilon}_f \right] u_i$$

$$= i e^2 \bar{u}_f \left[\frac{\not{\epsilon}_f \not{\epsilon}_i \not{k}_i}{2 p_i \cdot k_i} + \frac{\not{\epsilon}_i \not{\epsilon}_f \not{k}_f}{2 p_i \cdot k_f} \right] u_i$$

$$p_i \approx (m, 0, 0, 0) \; : \; \text{At rest} \; , \quad (\not{p}_i - m) u_i = 0$$

Transverse polarisations : $\epsilon_i \cdot p_i = 0, \; \epsilon_f \cdot p_i = 0, \; \epsilon \cdot k = 0.$

Crossing symmetry for photons is: $k_i, \epsilon_i \longleftrightarrow -k_f, \epsilon_f$

This is manifest in the structure of iT_{fi}.

The probability of scattering is then,

$$d\sigma_{fi} = \frac{(2\pi)^4 \delta^4(p_f + k_f - p_i - k_i)}{(E_i/m) \, 2\omega_i} |T_{fi}|^2 \cdot \frac{d^3 p_f}{(2\pi)^3 (E_f/m)} \cdot \frac{d^3 k_f}{(2\pi)^3 \cdot 2\omega_f}$$

$$\times \left(\frac{1}{flux} \right)$$

For photons, scattering off stationary electrons,

$flux = |\vec{v}_\gamma| = C = 1$. All factors of V (box normalisation) have cancelled out.

Integrating out all unobserved electron degrees of freedom,

$$d\sigma_{fi} = \frac{m^2}{16\pi^2 \, m\omega_i \, E_f \omega_f} \delta(E_f + \omega_f - E_i - \omega_i) \, k_f^2 \, dk_f \, d\Omega_f |T_{fi}|^2.$$

To integrate over dk_f, we have to find $\frac{\partial E_f}{\partial k_f}$, using momentum (and not energy) conservation.

$$\frac{\partial E_f}{\partial k_f} = \frac{\partial \left(\sqrt{(\vec{k_f} - \vec{k_i})^2 + m^2} \right)}{\partial k_f} = \frac{\partial \left(\sqrt{k_f^2 + k_i^2 - 2k_f k_i \cos\theta + m^2} \right)}{\partial k_f}$$

$$= \frac{k_f - k_i \cos\theta}{E_f} = \frac{(k_f - k_i) + k_i(1 - \cos\theta)}{E_f}$$

Energy conservation : $(p_f - p_i)^2 = (k_i - k_f)^2$

$$\Rightarrow 2m^2 - 2m E_f = -2k_i k_f (1 - \cos\theta)$$

$$\Rightarrow m(k_i - k_f) = k_i k_f (1 - \cos\theta)$$

$\therefore k_i > k_f$, $\frac{1}{k_f} - \frac{1}{k_i} = \frac{1}{m}(1 - \cos\theta)$: Compton's formula for change in photon wavelength

Also, $\frac{\partial E_f}{\partial k_f} = \frac{(k_f - k_i) + \frac{m}{k_f}(k_i - k_f)}{E_f} = \frac{m - E_f + \frac{m}{k_f}(k_i - k_f)}{E_f}$

$$= \frac{m}{E_f}\left(1 + \frac{1}{k_f}(k_i - k_f) \right) - 1 = \frac{m k_i}{E_f k_f} - 1$$

Integration over dk_f gives $\frac{1}{\left| 1 + \frac{\partial E_f}{\partial k_f} \right|}$ in $d\sigma_{fi}$.

Putting all the factors together,

$$d\sigma_{fi} = \frac{1}{16\pi^2}\left(\frac{k_f}{k_i}\right)^2 d\Omega_f \, |T_{fi}|^2.$$

The cross-section for unpolarised electrons is

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2}\left(\frac{k_f}{k_i}\right)^2 \text{Tr}\left[\frac{\slashed{p}_f+m}{2m}\left(\frac{\slashed{\epsilon}_f \slashed{\epsilon}_i \slashed{k}_i}{2mk_i} + \frac{\slashed{\epsilon}_i \slashed{\epsilon}_f \slashed{k}_f}{2mk_f}\right)\cdot\frac{\slashed{p}_i+m}{2m}\right.$$
$$\left. \times \left(\frac{\slashed{k}_i \slashed{\epsilon}_i \slashed{\epsilon}_f}{2mk_i} + \frac{\slashed{k}_f \slashed{\epsilon}_f \slashed{\epsilon}_i}{2mk_f}\right)\right]$$

Trace involves products of upto 8 γ-matrices.

Look at the four terms individually.

Use $\slashed{a}\slashed{b} = 2a\cdot b - \slashed{b}\slashed{a}$ extensively.

Many dot products vanish: $\epsilon\cdot k = 0, \slashed{k}\slashed{k}=k^2=0$
$$\epsilon\cdot p_i = 0, \quad \epsilon\cdot\epsilon = -1$$

p_f is written in terms of p_i, k_i, k_f by conservation.

First term: $\text{Tr}\left[(\slashed{p}_f+m)\slashed{\epsilon}_f\slashed{\epsilon}_i\slashed{k}_i(\slashed{p}_i+m)\slashed{k}_i\slashed{\epsilon}_i\slashed{\epsilon}_f\right]$
$$= \text{Tr}\left[\slashed{p}_f\slashed{\epsilon}_f\slashed{\epsilon}_i\slashed{k}_i\slashed{p}_i\slashed{k}_i\slashed{\epsilon}_i\slashed{\epsilon}_f\right] \binom{\text{mass terms vanish}}{\text{because } k_i^2 = 0}$$

$$= 2k_i\not p_i\, \mathrm{Tr}\left[\not k_f \not\epsilon_f \not\epsilon_i \not k_i \not\epsilon_i \not\epsilon_f\right]$$

$$= 2\, k_i\cdot p_i\, \mathrm{Tr}\left[\not k_f \not\epsilon_f \not k_i \not\epsilon_f\right]$$

$$= 8\, k_i\cdot p_i\left(2 p_f\cdot\epsilon_f\; k_i\cdot\epsilon_f + p_f\cdot k_i\right)$$

$$= 8\, k_i\cdot p_i\left(2\left(k_i\cdot\epsilon_f\right)^2 + p_i\cdot k_f\right)$$

Last term is related to this by crossing symmetry.

$\epsilon_i, k_i \leftrightarrow \epsilon_f, -k_f$ substitutions give the result

$$8\, k_f\cdot p_i\left(-2\left(k_f\cdot\epsilon_i\right)^2 + p_i\cdot k_i\right)$$

The two other cross-terms are equal by crossing symmetry. Each gives the contribution

$$\mathrm{Tr}\left[(\not p_f + m)\, \not\epsilon_f \not\epsilon_i \not k_i\, (\not k_i + m)\, \not k_f \not\epsilon_f \not\epsilon_i\right]$$

$$= \mathrm{Tr}\left[(\not p_i + m)\, \not\epsilon_f \not\epsilon_i \not k_i\, (\not k_i + m)\, \not k_f \not\epsilon_f \not\epsilon_i\right]$$

$$\quad + \mathrm{Tr}\left[(\not k_f - \not k_i)\, \not\epsilon_f \not\epsilon_i \not k_i \not k_i \not k_f \not\epsilon_f \not\epsilon_i\right]$$

$$= 8\,(k_i\, p_i)(k_f\cdot p_i)\left[2(\epsilon_f\cdot\epsilon_i)^2 - 1\right] - 8\,(k_i\cdot\epsilon_f)^2\, k_f\cdot p_i$$

$$+ 8\,(k_f\cdot\epsilon_i)^2\, k_i\cdot p_i$$

Combining all the four terms together,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m^2} \left(\frac{k_f}{k_i}\right)^2 \left[\frac{k_f}{k_i} + \frac{k_i}{k_f} + 4(\epsilon_f \cdot \epsilon_i)^2 - 2\right]$$

$$= \frac{\alpha^2}{m^2} \left(\frac{k_f}{k_i}\right)^2 \left[\frac{k_f k_i \sin^4\frac{\theta}{2}}{m^2} + (\epsilon_f \cdot \epsilon_i)^2\right]$$

This is the Klein - Nishina result.

$\frac{\alpha}{m}$ is called the classical radius of electron.
It is about 2.8 fm in magnitude.
In the low energy limit, or forward scattering,
the scattering becomes elastic, the differential
cross-section reduces to Thomson cross-section

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{m^2} (\epsilon_f \cdot \epsilon_i)^2.$$

When the photons are unpolarised as well, we need to sum over ϵ_f and average over ϵ_i.

In 3-dim. space, $\epsilon^{(1)}$, $\epsilon^{(2)}$ and $\hat{k}$ form a complete orthonormal basis. Therefore,

$$\sum_{\lambda=1}^{2} \epsilon_i^{(\lambda)} \epsilon_j^{(\lambda)} = \delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}.$$

We have to perform: $\frac{1}{2} \sum_{\lambda_i \lambda_f} \left(\frac{d\sigma}{d\Omega} \right)$.

$$\sum_{\lambda_i \lambda_f} (\epsilon_f \cdot \epsilon_i)^2 = \left(\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2} \right)_f \left(\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2} \right)_i$$

$$= 3 - 1 - 1 + \cos^2\theta$$

$$= 1 + \cos^2\theta$$

$$\therefore \frac{d\bar{\sigma}}{d\Omega} = \frac{\alpha^2}{2m^2} \left(\frac{k_f}{k_i} \right)^2 \left[\frac{k_f}{k_i} + \frac{k_i}{k_f} + 1 + \cos^2\theta - 2 \right]$$

$$= \frac{\alpha^2}{2m^2} \left(\frac{k_f}{k_i} \right)^2 \left[\frac{k_f}{k_i} + \frac{k_i}{k_f} - \sin^2\theta \right]$$

Alternate derivation uses $\sum_{\lambda=1}^{4} \epsilon_\mu^{(\lambda)} \epsilon_\nu^{(\lambda)} = -g_{\mu\nu}$, which is equivalent in presence of current conservation. But in that scheme $\epsilon \cdot p_i = 0$ cannot be used.

Then expression for $|T_{fi}|^2$ is different.
The total cross-section is obtained using,

$$k_f = \frac{k_i}{1 + \frac{k_i}{m}(1-\cos\theta)} \quad \text{and} \quad z = \cos\theta :$$

$$\bar{\sigma} = \frac{\pi\alpha^2}{m^2} \int_{-1}^{1} dz \left[\frac{1}{\left[1 + \frac{k_i}{m}(1-z)\right]^3} + \frac{1}{\left[1 + \frac{k_i}{m}(1-z)\right]} - \frac{1-z^2}{\left[1 + \frac{k_i}{m}(1-z)\right]^2} \right].$$

$$= \begin{cases} \frac{8\pi}{3} \frac{\alpha^2}{m^2}, & \text{for } \frac{k_i}{m} \longrightarrow 0 \\ \frac{\pi\alpha^2}{k_i m} \left(\ln \frac{2k_i}{m} + \frac{1}{2} + O\left(\frac{m}{k}\ln\frac{k}{m}\right) \right), & \text{for } k_i \gg m. \end{cases}$$

Pair production through annihilation:

Typical situation is illustrated by $e^+e^- \longrightarrow \mu^+\mu^-$.
This is a benchmark process for colliders, to
produce new types of particles.
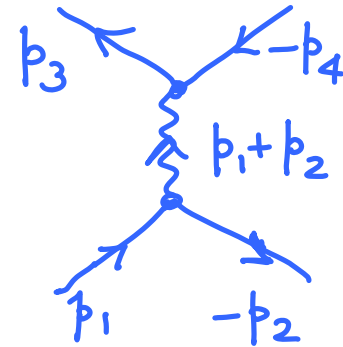It is clean, and conservation laws are simple.
(Electromagnetic interaction is needed.)

It is convenient to analyse the process in the CM frame.

$$S = (p_1 + p_2)^2 = (E_1 + E_2)^2 = 4 E_{beam}^2$$

The scattering matrix element is

$$i T_{fi} = \bar{V}_e(p_2)(-ie\gamma_\mu) U_e(p_1) \cdot \frac{-i}{(p_1 + p_2)^2} \cdot \bar{U}_\mu(p_3)(-ie\gamma^\mu) V_\mu(p_4)$$

Current conservation follows from

$$\bar{V}(\not{p_1} + \not{p_2}) U = \bar{V}(m - m) U = 0.$$

The probability of scattering is given by

$$d\sigma_{fi} = \frac{(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)}{(E_1/m_e)(E_2/m_e)} |T_{fi}|^2 \frac{d^3 p_3}{(2\pi)^3 (E_3/m_\mu)} \cdot \frac{d^3 p_4}{(2\pi)^3 (E_4/m_\mu)}$$

$$\times \frac{1}{flux}$$

(Box normalisation factors of V have been cancelled.)

In the CM frame, collision is collinear.
Then

$$flux = |\vec{V_1} - \vec{V_2}| = \frac{|\vec{p_1}|}{E_1} + \frac{|\vec{p_2}|}{E_2} = \frac{E_2 |\vec{p_1}| + E_1 |\vec{p_2}|}{E_1 E_2}.$$

(Flux is not Lorentz invariant.)

Simpler form is: $\text{flux} = \dfrac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}$.

Note that $(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = (E_1 E_2 + |\vec{p_1}||\vec{p_2}|)^2 - m_1^2 m_2^2$

$$= E_1^2 E_2^2 + |\vec{p_1}|^2 |\vec{p_2}|^2 + 2 E_1 E_2 |\vec{p_1}||\vec{p_2}| - m_1^2 m_2^2$$

$$= E_1^2 (|\vec{p_2}|^2 + m_2^2) + |\vec{p_1}|^2 (E_2^2 - m_2^2) - m_1^2 m_2^2$$
$$\quad + 2 E_1 E_2 |\vec{p_1}||\vec{p_2}|$$

$$= E_1^2 |\vec{p_2}|^2 + |\vec{p_1}|^2 E_2^2 + m_2^2 (E_1^2 - |\vec{p_1}|^2) - m_1^2 m_2^2$$
$$\quad + 2 E_1 E_2 |\vec{p_1}||\vec{p_2}|$$

$$= (E_1 |\vec{p_2}| + E_2 |\vec{p_1}|)^2.$$

In our case, $p_1 \cdot p_2 = E_1 E_2 + |\vec{p_1}| \cdot |\vec{p_2}|$

$$= E_{beam}^2 + |\vec{p}|_{beam}^2$$

Then $(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = (E^2 + |\vec{p}|^2)^2 - m^4$

$$= E^4 + 2 E^2 |\vec{p}|^2 + |\vec{p}|^4 - m^4$$

$$= E^2 [E^2 + 2|\vec{p}|^2 + |\vec{p}|^2 - m^2]$$

$$= 4 E^2 |\vec{p}|^2 = (2 E |\vec{p}|)^2$$

Then, integrating out the δ-functions,

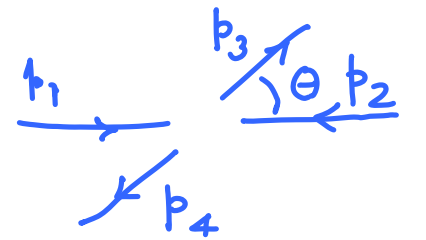$$d\sigma_{fi} = \frac{m_e^2 m_\mu^2}{4\pi^2 E_1 E_2} \delta(E_1+E_2-E_3-E_4) \frac{p_3^2 \, dp_3 \, d\Omega}{E_3 E_4} |T_{fi}|^2 \cdot \frac{E_1 E_2}{2 E_1 |\vec{p}_1|}$$

$$= \frac{m_e^2 m_\mu^2}{4\pi^2} \cdot \frac{p_\mu}{p_e} |T_{fi}|^2 \, d\Omega \cdot \underbrace{\frac{1}{2 E_4 \cdot 2 E_1}}_{= S}$$

$\dfrac{p_\mu}{p_e}$ factor arises from phase-space integral,
and describes the threshold behaviour.

For the unpolarised case,

$$\left(\frac{1}{2}\right)^2 \sum_{s_i, s_f} |T_{fi}|^2 = \frac{e^4}{4 S^2} \text{Tr}\left(\gamma^\mu \frac{\not{p}_1 + m_e}{2 m_e} \gamma^\nu \frac{\not{p}_2 - m_e}{2 m_e} \right)$$

$$\times \text{Tr}\left(\gamma_\nu \frac{\not{p}_3 + m_\mu}{2 m_\mu} \gamma_\mu \frac{\not{p}_4 - m_\mu}{2 m_\mu} \right)$$

$$= \frac{e^4}{4 m_e^2 m_\mu^2 S^2} \left(p_1^\mu p_2^\nu - g^{\mu\nu} p_1 \cdot p_2 + p_1^\nu p_2^\mu - m_e^2 g^{\mu\nu} \right)$$

$$\times \left(p_{3\mu} p_{4\nu} - g_{\mu\nu} p_3 \cdot p_4 + p_{3\nu} p_{4\mu} - m_\mu^2 g_{\mu\nu} \right)$$

$$= \frac{e^4}{4 m_e^2 m_\mu^2 S^2} \left[2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) \right.$$
$$\left. - S(p_1 \cdot p_2 + p_3 \cdot p_4) + \frac{S^2}{4} \cdot 4 \right]$$

$$= \frac{e^4}{4m_e^2 m_\mu^2 S}\left[\frac{S^2}{4} + 4\left(\frac{S}{4} - m_e^2\right)\left(\frac{S}{4} - m_\mu^2\right)\cos^2\theta + S\left(m_e^2 + m_\mu^2\right)\right]$$

Then, we get the unpolarised cross-section

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} = \frac{\alpha^2}{S^3}\sqrt{\frac{S - 4m_\mu^2}{S - 4m_e^2}}\left[\frac{S^2}{4} + S\left(m_e^2 + m_\mu^2\right) + 4\cos^2\theta\left(\frac{S}{4} - m_e^2\right)\left(\frac{S}{4} - m_\mu^2\right)\right]$$

In the high energy limit $(\sqrt{S} > 2m_\mu \gg m_e)$,

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} \approx \frac{\alpha^2}{S^3}\sqrt{1 - \frac{4m_\mu^2}{S}}\left(\frac{S^2}{4} + m_\mu^2 S + S\cos^2\theta\left(\frac{S}{4} - m_\mu^2\right)\right)$$

$$\xrightarrow{\sqrt{S} \gg 2m_\mu} \frac{\alpha^2}{4S}\left(1 + \cos^2\theta\right)$$

The total cross-section becomes

$$\sigma_{unpol} \approx \frac{\alpha^2}{S^3}\sqrt{1 - \frac{4m_\mu^2}{S}} \cdot 2\pi\left(\frac{S^2}{2} + 2m_\mu^2 S + \frac{2S}{3}\left(\frac{S}{4} - m_\mu^2\right)\right)$$

$$= \frac{4\pi\alpha^2}{3S}\sqrt{1 - \frac{4m_\mu^2}{S}}\left(1 + \frac{2m_\mu^2}{S}\right)$$

$$\xrightarrow{\sqrt{S} \gg 2m_\mu} \frac{4\pi\alpha^2}{3S}$$

Experimentally, in e^+e^- colliders, the annihilation cross-section increases every time $\sqrt{s}$ crosses the threshold for creation of new $f\bar{f}$ pair.

$$R(s) = \frac{\sigma(e^+e^- \rightarrow f\bar{f})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = Q_f^2 \sqrt{1 - \frac{4m_f^2}{s}}\left(1 + \frac{2m_f^2}{s}\right)\Theta(s - 4m_f^2)$$

(assuming $\sqrt{s} > 2m_f \gg m_\mu$)

There can be extra factors in R for other degrees of freedom (e.g. 3 for no. of quark colours).

$R(s)$ for $e^+e^- \rightarrow$ hadrons shows steps for production of each new quark flavour as $\sqrt{s}$ increases.

For $\sqrt{s}$ close to threshold for pair production, $\sqrt{s} \approx 2m_\mu$ (non-relativistic muons):

$$\left(\frac{d\sigma}{d\Omega}\right)_{unpol} \simeq \frac{\alpha^2}{2s}\sqrt{1 - \frac{4m_\mu^2}{s}} = \frac{\alpha^2}{2s}\frac{|\vec{p}_\mu|}{E_\mu} : \text{Isotropic}$$

In this limit, pair-production is in s-wave.

The muon current $\bar{u}_\mu(p_3)(-ie\,\sigma^i)V_\mu(p_4)$ is non-zero only for space components. It has no angular dependence, and no angular momentum.
The conservation of total angular momentum implies that spins of μ^+ and μ^- are aligned with e^+ and e^- helicities (i.e. the collision direction).

<u>Polarised cross-sections</u>: These can be obtained by inserting appropriate factors of $\left(\frac{1+\gamma_5\not{s}}{2}\right)$ in the Dirac trace. It is convenient to work in the helicity basis, where the projection operators are $\left(\frac{1\pm\gamma_5}{2}\right)$.

Let $\quad \frac{1+\gamma_5}{2} \longrightarrow R \quad$ and $\quad \frac{1-\gamma_5}{2} \longrightarrow L$.

The current in the helicity basis is non-zero only when the spinors are of the same type.
$$\bar{V}\gamma^\mu u = V^\dagger\gamma^0\gamma^\mu u \quad \text{can have } \frac{1\pm\gamma_5}{2}$$
factors that commute with $\gamma^0\gamma^\mu$ only, e.g.

$$v^\dagger \left(\frac{1+\gamma_5}{2}\right)\gamma^0\gamma^\mu\left(\frac{1+\gamma_5}{2}\right)u \neq 0, \text{ but } v^\dagger\left(\frac{1+\gamma_5}{2}\right)\gamma^0\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)u = 0.$$

The helicity of positron is opposite to the corresponding projection operator for v. So the process occurs only when e^+ and e^- have opposite helicities, or the same spin in the CM frame.

The angular momentum along direction of collision $=1$.

When the helicities are opposite,

$$\left(\tfrac{1}{2}, 0\right) \otimes \left(0, \tfrac{1}{2}\right) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) : \text{Vector representation}$$
allowed for $\gamma^\mu A_\mu$ interaction

When the helicities are the same,

$$\left(\tfrac{1}{2}, 0\right) \otimes \left(\tfrac{1}{2}, 0\right) = (0, 0) \oplus (1, 0)$$

$\uparrow$ Mass $\uparrow$ $\sigma_{\mu\nu} F^{\mu\nu}$ (e.g. anomalous magnetic moment)

These Lorentz group properties also produce the same helicity selection rules.

The total process has 16 terms (R and L for each fermion). Out of these, only four are non-zero. QED also obeys parity, i.e. $R \leftrightarrow L$ exchange. That leads to the relations between amplitudes

$(LR \rightarrow LR) = (RL \rightarrow RL)$ and

$(LR \rightarrow RL) = (RL \rightarrow LR)$.

Explicitly evaluating the traces:

$$\mathrm{Tr}\left(\gamma^\mu \frac{1}{2}(1+\gamma_5)\left(\frac{\not{p}_1 + m_e}{2m_e}\right)\gamma^\nu \frac{1}{2}(1+\gamma_5)\left(\frac{\not{p}_2 - m_e}{2m_e}\right)\right)$$

$$= \mathrm{Tr}\left(\gamma^\mu \frac{1}{2}(1+\gamma_5)\frac{\not{p}_1}{2m_e}\gamma^\nu \frac{\not{p}_2}{2m_e}\right)$$

$$= \frac{1}{2m_e^2}\left[p_1^\mu p_2^\nu - g^{\mu\nu} p_1 \cdot p_2 + p_1^\nu p_2^\mu + i\epsilon^{\mu\nu\varsigma\sigma}p_{1\varsigma}p_{2\sigma}\right]$$

$$= O\left(\frac{E^2}{m^2}\right) \quad : \text{Dominant contribution.}$$

$$\mathrm{Tr}\left(\gamma^\mu \frac{1}{2}(1+\gamma_5)\frac{\not{p}_1 + m_e}{2m_e}\gamma^\nu \frac{1}{2}(1-\gamma_5)\left(\frac{\not{p}_2 - m_e}{2m_e}\right)\right)$$

$$= -\mathrm{Tr}\left(\gamma^\mu \frac{1}{2}(1+\gamma_5)\frac{m_e}{2m_e}\gamma^\nu \frac{m_e}{2m_e}\right)$$

$$= -\frac{1}{2}g^{\mu\nu} = O(1) \quad : \text{Subdominant contribution.}$$

Mass term can flip helicity. But that suppresses the amplitude by $\frac{m}{E}$ (and $Tr(\cdots)$ by $\frac{m^2}{E^2}$).

We neglect mass terms ($\sqrt{s} \gg m_\mu$), and use

$$\epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\lambda\tau} = -2 \left(\delta_\lambda^\gamma \delta_\tau^\delta - \delta_\tau^\gamma \delta_\lambda^\delta \right).$$

Then the polarised contributions are:

$$\left(\frac{d\sigma}{d\Omega} \right)_{pol} = \frac{\alpha^2}{s^3} \left[2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) \right.$$
$$\left. \pm 2(p_1 \cdot p_3)(p_2 \cdot p_4) \mp 2(p_1 \cdot p_4)(p_2 \cdot p_3) \right]$$

$$= \frac{\alpha^2}{4s} \times \begin{cases} (1 + \cos\Theta)^2 & : \quad LR \to LR, \quad RL \to RL \\ (1 - \cos\Theta)^2 & : \quad LR \to RL, \quad RL \to LR \end{cases}$$

Adding all four possibilities, and divide by 4 to average over initial helicities, gives the earlier result for $\left(\frac{d\sigma}{d\Omega} \right)_{unpol}$.

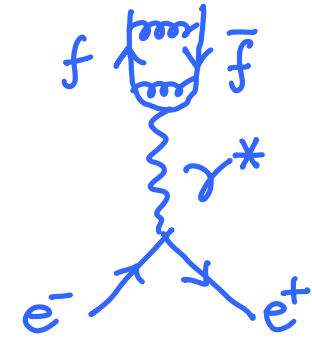More detailed calculation shows that the factors of $(1 \pm \cos\Theta)$ appear in T_{fi} itself.

Conservation of angular momentum implies that $(LR \rightarrow LR)$ vanishes at $\Theta = \pi$, and $(LR \rightarrow RL)$ vanishes at $\Theta = 0$.

In experiments, polarisations are not perfect, and separate contributions are combined with appropriate weight factors (partial polarisation).

Bound states: For $e^+ e^- \rightarrow f\bar{f}$, bound states appear below the threshold $\sqrt{s} = 2m_f$.

Most of these bound states are non-relativistic.

So initial relativistic $f\bar{f}$ production can be combined with subsequent non-relativistic evolution to a reasonable approximation. (Without this strategy, bound states lie outside perturbative framework.)

The $f\bar{f}$ system is in $\ell = 0$, $s = 1$ state. f $\bar{f}$
(We still assume $m_f \gg m_e$.)
These are massive vector boson states.
Let the mass be M ($\approx 2m_f$), and
the wavefunction $\Psi(\vec{r}) = \Psi(r)$.
We have $\vec{r} = \vec{r_3} - \vec{r_4}$, $\vec{k} = \frac{1}{2}(\vec{P_3} - \vec{P_4})$ and
$$\tilde{\Psi}(\vec{k}) = \int d^3x \; e^{i\vec{k}\cdot\vec{r}} \Psi(\vec{r}).$$
With non-relativistic normalisation,
$$\int \frac{d^3k}{(2\pi)^3} |\tilde{\Psi}(\vec{k})|^2 = 1 .$$

Then, $iT_{fi}(e^+e^- \to B) = \int \frac{d^3k}{(2\pi)^3} \tilde{\Psi}^*(\vec{k}) \cdot iT_{fi}(e^+e^- \to f\bar{f})$

$$= \Psi^*(\vec{r}=0) \cdot \underbrace{iT_{fi}(e^+e^- \to f\bar{f})}_{\substack{\text{Independent of } \vec{k} \\ \text{for } M \approx 2m_f}}$$

$\Psi^*(0)$ must be obtained from non-relativistic
analysis of $f\bar{f}$ bound state. In s-wave, this
is the only parameter needed for annihilation process.

Then the unpolarised vector boson production cross-section is :

$$\sigma(e^+e^- \to B) = \frac{(2\pi)^4 \delta^4(p_1 + p_2 - p_B)}{(E_1/m_e)(E_2/m_e)} |T_{fi}|^2 \cdot \frac{d^3p_B}{(2\pi)^3} \cdot \frac{1}{flux}$$

$$= \frac{(2\pi)\, \delta(E_1 + E_2 - M)}{E_1 E_2} m_e^2 \cdot \frac{E_1 E_2}{2E_1 |\vec{p_1}|} |T_{fi}|^2$$

$$= \frac{\pi m_e^2}{E_1 |\vec{p_1}|} |T_{fi}|^2 \delta(\sqrt{s} - M).$$

Assuming $\sqrt{s} = M \approx 2m_f \gg m_e$,

$$\left(\frac{1}{2}\right)^2 \sum_{s_i, s_f} |T_{fi}|^2 = \frac{e^4}{2m_e^2 M^2} |\Psi(0)|^2 \quad \text{for unpolarised case.}$$

$$\therefore \sigma_{unpol}(e^+e^- \to B) = \frac{32 \pi^3 \alpha^2}{M^4} |\Psi(0)|^2 \delta(\sqrt{s} - M)$$

The actual bound state decays (back to e^+e^- or to some other states), and so the δ-function gets broadened into a resonance peak. The area under the peak is preserved, however.

Bound state decay: Assume only decay is to e^+e^-.

The transition matrix element for $B \to e^+ e^-$ is the complex conjugate of that for $e^+ e^- \to B$, by time reversal invariance.

The decay rate (in the rest frame of B) is

$$d\Gamma(B \to e^+ e^-) = \frac{(2\pi)^4 \, \delta^4(p_B - p_1 - p_2)}{(E_1/m_e)(E_2/m_e)} |T_{fi}|^2 \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3}$$

$$= \frac{m_e^2}{(2\pi)^2 E_1 E_2} \delta(M - E_1 - E_2) \, p_1 E_1 \, dE_1 \, d\Omega \, |T_{fi}|^2$$

$$= \frac{m_e^2}{4\pi^2} \cdot \frac{p_1}{2 E_2} |T_{fi}|^2 \, d\Omega$$

For unpolarised case, sum over $e^+ e^-$ spin directions and average over spin directions of B.

B is a $S = 1$ vector particle with 3 spin states.

$$\therefore \Gamma_{unpol}(B \to e^+ e^-) \underset{\sqrt{s} \gg m_e}{\approx} \frac{m_e^2}{8\pi^2} \frac{e^4 |\psi(0)|^2}{m_e^2 M^2} \cdot \frac{2}{3} \cdot 4\pi$$

$$= \frac{16\pi \alpha^2}{3} \cdot \frac{|\psi(0)|^2}{M^2}$$

The relation between $\sigma(e^+e^- \to B)$ and $\Gamma(B \to e^+e^-)$ is independent of $|\psi(0)|^2$. Explicitly,

$$\sigma(e^+e^- \to B) = \frac{2\pi^2}{M^2} \cdot 3\Gamma(B \to e^+e^-)\, \delta(\sqrt{s} - M).$$

This can be experimentally tested.

For quarkonia (spin=1 bound states of $q\bar{q}$), both sides are multiplied by $3Q_f^2$.

Also, $|\psi(0)|^2 \sim (1\,\mathrm{fm})^{-3}$ in this case.

Because of helicity conservation, the decay helicities of B are restricted to:

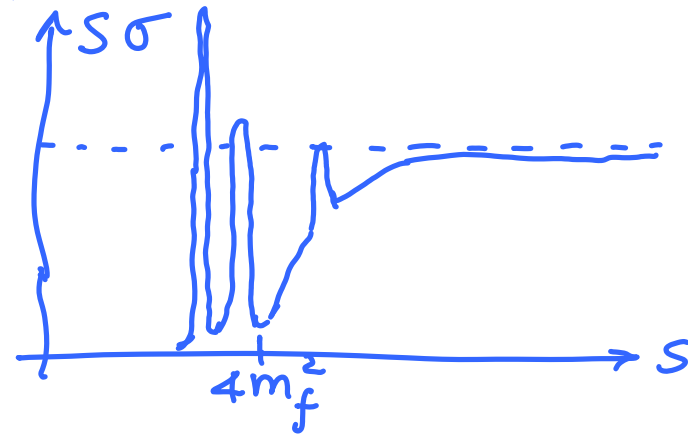$$+1 \longrightarrow RL \;,\; -1 \longrightarrow LR \;,\; 0 \longrightarrow \text{No decay}.$$

A good parametrisation of the resonance peak is replacement of $\delta(\sqrt{s} - M)$ by the Lorentzian Breit-Wigner form $\dfrac{\Gamma/2\pi}{(\sqrt{s}-M)^2 + \Gamma^2/4}$.

The singularity is a pole in the complex plane located at $E = M - \dfrac{i\Gamma}{2}$.

The peak height is $\frac{2}{\pi \Gamma}$ here. So the peak in $\sigma(e^+ e^- \to B)$ has height $\frac{12\pi}{M^2}$.

Actual experimental data would show several peaks before showing asymptotic behaviour.

Non-relativistic potentials:

These can be extracted from the relativistic formalism, by looking at situations where $\frac{|\vec{p}|}{m}$ is small and no pair creation/annihilation occurs. The particles are separated over distances much larger than their Compton wavelengths, and so they become distinguishable.

Interactions are produced by exchange of virtual particles.

In case of two charges e_1 and e_2,

$$i T_{fi} = (-i e_1 \, \bar{u}_3 \gamma^\mu u_1) \frac{-i}{q^2 + i\epsilon} (-i e_2 \, \bar{u}_4 \gamma_\mu u_2)$$
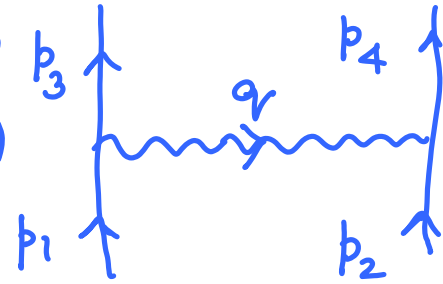
In the non-relativistic limit, the lower components of u are $O(\frac{p}{m})$.

So only $\mu = 0$ term dominates the photon exchange.

Also $\bar{u} \gamma^0 u = \frac{E}{m} \longrightarrow 1$. Then we have:

$$i T_{fi} = +i \frac{e_1 e_2}{q^2} = -i \frac{e_1 e_2}{|\vec{q}|^2} \qquad \left[q^0 = O\left(\frac{p^2}{m}\right), \; q^i = O(p) \right]$$

The potential is just $V = -T_{fi} = \frac{e_1 e_2}{|\vec{q}|^2}$.

Fourier transforming, $V(\vec{r}) = \frac{e_1 e_2}{4\pi r}$.

At higher order, this gets corrected by processes depicted by the diagrams (all 1-loop and $O(\alpha^2)$):

Lagrangian formulation of QED:

$S = \int d^4x \, \mathcal{L}$. In quantum theory, S has units of $\hbar$.

$\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}F_{\mu\nu}^2$, $D_\mu = \partial_\mu + ieA_\mu$

It is Lorentz and gauge invariant.
It also has symmetries of P, C, T.
Equations of motion are obtained by variational
Euler-Lagrange procedure, taking $\Psi, \bar{\Psi}$ and A_μ
as independent variables.

Vary $\bar{\Psi}$: Dirac equation $(i\not{D} - m)\Psi = 0$

Vary Ψ: Conjugate of Dirac equation
$$\bar{\Psi}(-i\overleftarrow{\not{\partial}} - e\not{A} - m) = 0$$

Vary A_μ: Maxwell's equations (inhomogeneous)
$$\partial_\mu F^{\mu\nu} = e\bar{\Psi}\gamma^\nu\Psi = j^\nu$$

Solutions of these equations give the classical behaviour, and fluctuations around them provide the quantum corrections.

Formal manipulations often use integration by parts, and ignore boundary terms (at infinity).

In topological effects, more careful treatment of boundary terms becomes necessary. (Not in QED)

Given the Lagrangian, one can choose a perturbative vacuum (i.e. no particles), and then expand around it to obtain desired correlation functions (i.e. Green's functions).

The Feynman rules follow from the Lagrangian in a straightforward manner.

In calculation of Feynman diagrams of arbitrary order, divergences may appear in momentum integrals. We need to develop a procedure to deal with that.

<u>Divergences</u> : These appear either at $k \to \infty$ (UV),
or at $k \to 0$ (IR) in theories with massless particles.

Ultraviolet : $\int_{-\infty}^{\infty} \dfrac{d\,k}{k^{-D+1}}$ behaviour as $k \to \infty$.

D is called superficial degree of divergence.

$D = 0$ is logarithmic divergence.

$D > 0$ is powerlaw divergence.

In QED with 3+1 dim, coupling is dimensionless,
and D depends only on propagator factors.

A given diagram, there are vertices V, internal
propagator lines I, external state lines E, and
closed loops (irreducible and independent) L.

The total diagram behaves as $\lim_{k \to \infty} k^{D}$.

(masses are negligible in this limit)

Count the independent momenta involved. These are constrained by momentum conservation at every vertex. (No free momenta left in tree diagrams.)

Undetermined no. of momenta = No. of loops L

$$= I - V + 1 = I_e + I_\gamma - V + 1$$

Count the powers of momenta in propagators.

$$D = 4L - 2I_\gamma - I_e$$

Look at the structure of interaction vertex
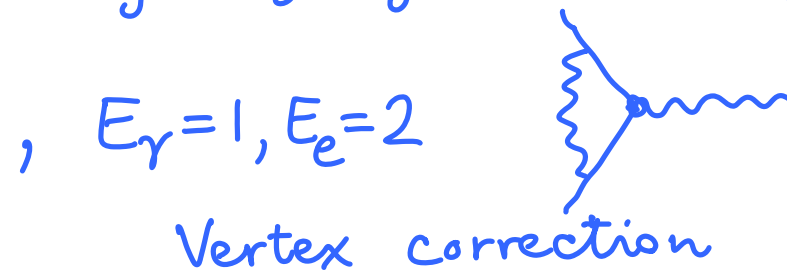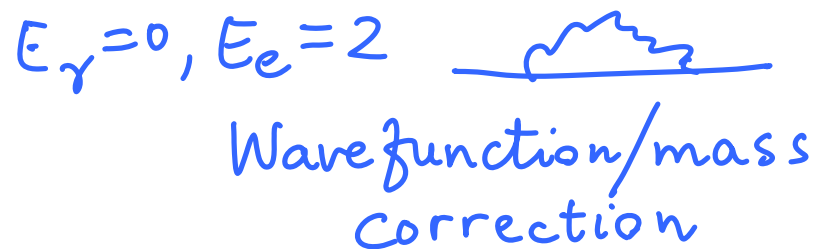
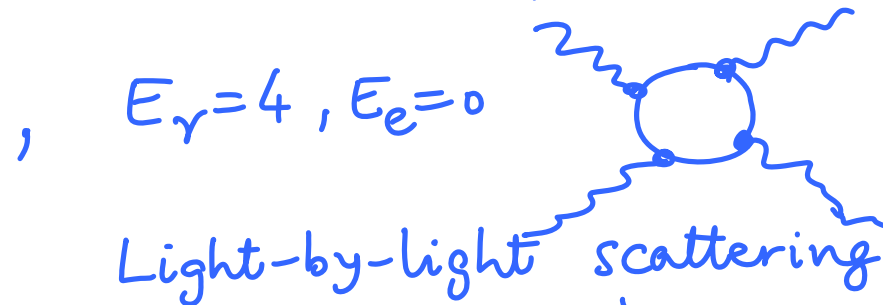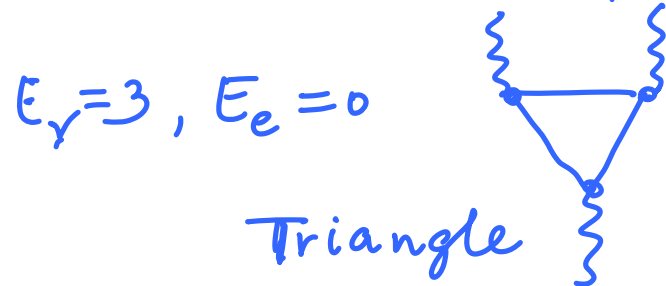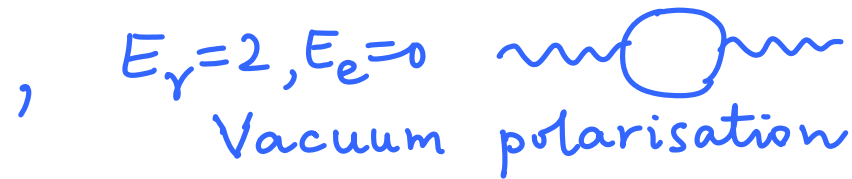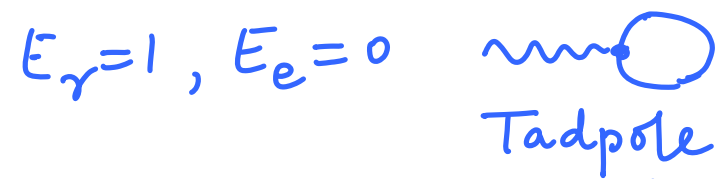$$V = 2I_\gamma + E_\gamma \;, \quad 2V = 2I_e + E_e \;.$$

Then eliminating V, I_e, I_γ:

$$D = 4 - \underbrace{4V}_{V+3V} + 2I_\gamma + 3I_e = 4 - E_\gamma - \tfrac{3}{2}E_e$$

Independent of internal lines and vertices.

D only depends on the type of Green's function.

Specific diagrams at 1-loop are $(D \geqslant 0)$:

$E_\gamma = 1$, $E_e = 0$

Tadpole

, $E_\gamma = 2$, $E_e = 0$

Vacuum polarisation

$E_\gamma = 3$, $E_e = 0$

Triangle

, $E_\gamma = 4$, $E_e = 0$

Light-by-light scattering

$E_\gamma = 0$, $E_e = 2$

Wave function/mass
correction

, $E_\gamma = 1$, $E_e = 2$

Vertex correction

Once we take care of these divergences at
1-loop, no further divergences at higher orders
or loops need to be tackled.
Symmetries of the theory can reduce the value
of D from the superficial value above.

The procedure to deal with these divergences require modification of the momentum integrals. This modification is called regularisation.

Momentum space cutoff : Change the upper limit of integrals from ∞ to Λ. It is analogous to introducing atomic structure in mechanics, where momenta get restricted to the Brillouin zone. In position space, spacing $\sim a$ gives $\Lambda \sim \frac{1}{a}$.

There are other regularisation procedures as well : Pauli-Villars, ζ-function, dimensional regularisation etc.

Gauge symmetry has to be handled carefully, while regulating the momentum integrals. Gauge transformations are space-time dependent, and so change the values of momenta. So gauge invariant cut-offs have to be introduced in the theory.

Popular cut-off procedure is dimensional regularisation, where momentum integrals are evaluated in $d = 4 - \epsilon$ dimensions, and the results expanded as Laurent series in ϵ. Divergences appear as poles in ϵ. It is algebraically simple, but physical meaning is not transparent.

Infrared: $\int_{\lambda}^{\Lambda} \frac{dk}{k}$ are divergent contributions as $\lambda \to 0$.

To tackle it, we need another regularisation. The simplest one is to give the photon some mass, i.e. $k^2 \longrightarrow k^2 - \lambda^2$. This is not gauge-invariant, but it is used in cases where it works. Dimensional regularisation also works in case of infrared divergences.

Regularisation isolates the divergent parts of a calculation ($\Lambda \to \infty$, $\lambda \to 0$), and keeps the mathematical expressions well-defined.

To obtain physically meaningful results, the carefully isolated divergences must be cancelled or eliminated from observable quantities.

Renormalisation:

(1) After adding several contributions to a physical process, the divergences cancel. It can also happen that the result can be broken up into distinct separately observable parts, and the part of interest may be finite.

Example: The vertex correction is logarithmically divergent. But it has two parts, $\gamma^\mu A_\mu$ and $\sigma_{\mu\nu} F^{\mu\nu}$. Former diverges, latter is finite.

Anomalous magnetic moment of the electron at 1-loop is: $g - 2 = \dfrac{\alpha}{\pi}$ (Schwinger)

(2) The cut-offs may have physical relevance in the problem. This means that the theory is valid only between certain energy scales (i.e. it is an effective theory), and we need only the corresponding contribution to the observable. Then the result with physical values of cut-offs is meaningful in comparison with experiments.

Example: Lamb shift arises from 1-loop QED corrections, and separates $2S-2p$ energy levels of Hydrogen atom. Vacuum fluctuations of electromagnetic field cause this effect. The states most affected are the ones that experience large Coulomb field of nucleus (i.e. $l=0$). The divergent integral appearing is $\int \frac{d\omega}{\omega}$.

For large wavelengths, the size of the atom acts as a cut-off. For small wavelengths (compared to Compton wavelength), the non-relativistic

nature of the problem makes mc^2 a cut-off. The inbetween contribution matches nicely with experimental data. (Bethe)

(3) The physical states appearing in the calculation can be redefined. This is necessary in case of infrared divergences. The sum over physically indistinguishable states may cancel divergences. A charged particle can undergo soft Bremsstrahlung, and the soft photons can be unobservable by a physical detector with a specific threshold. The total contribution of the charged particle and its soft photon cloud to the physical process is finite (dependent on detector threshold but independent of λ).

QED calculation: Yennie, Frautschi, Suura.

(4) Divergences remain in the results as $\Lambda \to \infty$.
In absence of effective description, we have to invent a new procedure. There is a solution for theories that are called renormalisable.
In these theories, the divergent contributions have the same algebraic structure as the terms already present in the Lagrangian.
The coefficients of terms in the Lagrangian are considered unobservable. The contribution from modification of these coefficients cancels the divergences arising from Feynman diagrams.
This "renormalisation" is not mathematically rigorous, but it works in practice.
It leaves behind observable signature, i.e. how the coefficients in the Lagrangian depend on the cut-off scale Λ. This relation is often expressed as "renormalisation group equations".

i.e. $\Lambda \frac{d}{d\Lambda}$ (coupling) = Power series in couplings.
It does not explicitly refer to/involve any
divergences, and gives observable effects.
Experimentally, scale-dependence of couplings
can be observed, e.g. $\alpha \approx \frac{1}{137}$ at scale of m_e
but $\alpha \approx \frac{1}{128}$ at scale of m_Z.
RG framework : Gell-Mann and Low, Wilson.

Symmetries in the theory can reduce the divergences
compared to the superficial value given by D.
Charge conjugation and Furry's theorem:
When there are fermion loops (closed) in the diagram,
the arrow on the line can be placed either way,
and we must get the same result.
Contributions with opposite arrows correspond to
reversing the factors in the loop trace.

Transpose operation performs the switch.
That can be accomplished by the charge conjugation
operation, which gives $C \gamma^\mu C^{-1} = (-\gamma^\mu)^T$.
The extra "$-$" sign gives $(-1)^n$ for n vertices,
and also changes $\not{p} - m$ to $(\not{p} - m)^T$ in fermion
propagators. Sign change of $\not{p}$ is necessary to
reverse the arrow on the fermion loop, but
$(-1)^n$ for vertices is extra.

In theories with charge conjugation symmetry,
therefore, only $n = $ even contributions can be nonzero.
At 1-loop, this makes tadpole and triangle
diagrams vanish in QED. The contribution
obeys $e \leftrightarrow -e$ symmetry (particle-antiparticle).

Gauge symmetry and Ward-Takahashi identities:

The amplitudes have to be gauge-covariant in terms of external fields (fermion and photon).

(1) The vector field A_μ is not gauge invariant. So when external photons are present, A_μ must be part of gauge invariant combinations. One possibility is that A_μ is combined with other charged particle fields in the form $j^\mu A_\mu$. Current conservation then ensures gauge symmetry. When no charged particles are around, then A_μ must appear as part of $F_{\mu\nu}$. Each change from A_μ to $F_{\mu\nu}$ reduces the superficial degree of divergence D by one.

For 1-loop divergences, this makes vacuum polarisation have $D=0$, and light-by-light scattering have $D=-4$. (i.e. finite)

Light-by-scattering is $O(\alpha^4)$, and is not yet experimentally observed. (Large EM fields required have not been achieved in lasers.)

(2) Current conservation is an exact relation, not restricted to specific diagrams or perturbative expansion. In general, $q_\mu \bar{u}(p_2) \Gamma^\mu u(p_1) = 0$ in the complete theory. At leading order,

$$q_\mu \gamma^\mu = (\not{p}_1 - m) - (\not{p}_2 - m) = S_F^{-1}(p_1) - S_F^{-1}(p_2).$$
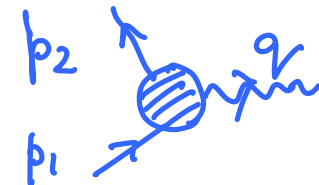
When extended to arbitrary order, we have

$$q_\mu \Gamma^\mu_{ren} = \left[S_F^{-1}(p_1) - S_F^{-1}(p_2) \right]_{ren},$$

when renormalisation respects gauge symmetry.

Equivalently, $\Gamma^\mu_{ren} = \dfrac{\partial}{\partial p_\mu} \left[S_F^{-1}(p) \right]_{ren}$, which relates vertex and wavefunction corrections.

Vertex diagram has $D = 0$, and so the wavefunction

diagram also must have $D = 0$. (Both log divergent)
In terms of renormalisation factors, we have $Z_1 = Z_2$.
Thus, from all six diagrams with $D \geq 0$, only
three are divergent, all of them logarithmically.
QED has only log divergences in 3+1 dimensions.

(3) The gauge symmetry can be broken by
physical systems spontaneously. In such a
situation, Lagrangian is gauge invariant,
but the ground state (vacuum) is not.
This breakdown of symmetry has physical signature.
Gauge symmetry can be broken by condensates
of charged particles. (Vacuum is not the
perturbative one, but has nonzero charge density.)
This occurs in superconductors due to Cooper
pairing of electrons.

When vacuum is not gauge-invariant, we have to use new fields to describe the theory. These are called effective fields/order parameters. The effective Lagrangian then describes the consequences of theory about the physical vacuum.

In superconductors, the effective field is the Cooper pair field, typically chosen as the scalar field $\rho e^{i\phi}$. (ϕ is gauge dependent)

The effective Lagrangian can have new terms that are gauge-invariant functions of $A_\mu - \partial_\mu \phi$.

In particular, $(A_\mu - \partial_\mu \phi)^2$ produces a mass for the photon. The massive photon manifests itself in superconductors as Meissner effect. No electric and magnetic fields can penetrate superconductors.

Organisation of perturbation theory:

Natural organisation is in powers of the coupling.
For any Green's function, power of coupling is the no. of vertices in the Feynman diagram.
Commonplace to describe the accuracy in terms of the no. of. loops. These two descriptions are related.

$$L = I_e + I_\gamma - V + 1 \quad, \quad V = 2I_\gamma + E_\gamma \,, 2V = 2I_e + E_e.$$

These give $V = E_e + E_\gamma + 2L - 2.$
For a specific Green's function, E_e and E_γ are fixed.
Each extra loop increases no. of vertices by two.
Transition probabilities become power series in α.

Note: (1) Valid for any QFT with a single coupling constant.
Some reorganisation needed when background fields or multiple coupling constants are present.

(2) Logarithmic divergences in QED contribute terms involving logarithms of ratios of energy scales. These logarithms are effectively summed up using RG techniques and running couplings. Non-relativistic bound states in QED have $\frac{E}{mc^2} \sim \alpha^2$, and so their perturbative analysis contains terms with powers of $\ln\alpha$.

Another reason for expressing perturbative results in terms of no. of loops is the relation of no. of loops to the powers of Planck's constant.

Fundamental scale of quantum effects is the commutator of a canonically conjugate pair of variables, e.g. $[x, p] = i\hbar$. With $L = p\dot{q} - H$, both L and H have dimensions of $\hbar/$time. The action $S = \int L \, dt$ and evolution phase $\int H \, dt$ are naturally measured in units of $\hbar$ to quantify quantum effects. Path integrals use $e^{iS/\hbar}$ as trajectory weight, and time evolution operator is $U = \exp[-\frac{i}{\hbar}\int H \, dt]$.

Each interaction vertex is proportional to $\frac{1}{\hbar}$.
Each propagator is proportional to $\hbar$.
Contributions to the Green's functions scale as $(\hbar)^{I-V} = (\hbar)^{L-1}$.
The number of loops represents the order of quantum correction to the classical result.
Classical result is represented as $e^{iS_{cl}/\hbar}$, and corresponds to the tree diagrams with $L=0$.

Precision tests of QED : Some of the most precise experimentally measured and theoretically calculated quantities are
   1. Anomalous magnetic moments of e, μ.
   2. Atomic recoils in electromagnetic transitions.
   3. Energy spectra of hydrogen, positronium, muonium.
   4. Quantum Hall effect, AC Josephson effect.
The best measurements have an accuracy less than one part in a billion. The best calculations have been carried out to four loop order (i.e. α^4 terms).

The experimental and theoretical results agree to this level of accuracy. This is a spectacular success!

Successors to QED: QED is a prototype of quantum gauge field theory. It has been followed by

1. Yang-Mills theory. Replaces abelian gauge group $U(1)$ by a non-abelian gauge group.

2. Electroweak theory. Gauge group is $SU(2)_L \otimes U(1)_Y$, which is spontaneously broken to $U(1)_{EM}$. The resultant masses for W and Z make weak interactions weak.

3. QCD: Gauge group is $SU(3)$, which describes the strong interactions.

Put together, these make up the Standard Model of particle physics. It has many parameters: Masses, Couplings, Mixing angles. Given the experimental values of the parameters, predictions of standard model agree with experiments It is a highly successful effective field theory.