MODULE 2: LORENTZ AND POINCARE GROUPS

Group: It is a set of objects following specific rules.

(1) Composition: $\forall\, g_1, g_2 \in G$, then $(g_1 * g_2) \in G$.

(2) Associative: $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$

(Product may or may not be commutative)

(3) Identity: $\exists\, \mathbb{1} \in G$, such that $g * \mathbb{1} = \mathbb{1} * g = g, \forall g \in G$

(4) Inverse: $\forall\, g \in G$, $\exists\, g^{-1} \in G$, such that $g * g^{-1} = g^{-1} * g = \mathbb{1}$.

In physical problems, group elements refer to certain transformations of an object.

Symmetry transformations change some property of the object, while leaving some other property invariant.

They define "symmetry groups".

Useful for studying dynamical properties of the object, when invariance is for L or H.

Group symmetries lead to conservation laws in physical problems. The object on which transformations act is described by a "state". Such a state is convenient, but may not be completely physical.

Representations : Mapping between abstract group elements and square matrices.

$$(g_1 * g_2) * g_3 \iff (M_1 \times M_2) \times M_3$$

Dimension of the matrix is the dimensionality of the representation. $1 \iff$ Identity matrix.

States are constructed as column vectors, on which the transformation matrices act.

For a given group, no. of elements may be finite, countably infinite, continuously infinite. Representation dimension can be finite or infinite.

Trivial rep. : $\forall g_i \iff 1$ (one-dimensional)

If a basis can be chosen, such that the matrices become block-diagonal, then the representation is called reducible. Otherwise it is irreducible.

Examples : (1) Smallest nontrivial group is $Z_2 = \{1, -1\}$.
Occurs in reflection, parity, time reversal.
Two irreducible reps. : Trivial and $\{1, -1\}$.

(2) Crystal groups for rotations and reflections.

(3) Lattice translation groups.

(4) Rotations and translations in continuum.

(5) Gauge groups for field theories

Lorentz group: Describes transformations of space-time in special relativity.

They leave $dx^\mu dx_\mu = \eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - d\vec{x}^2$; invariant when changing inertial frames.

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \text{ is held fixed.}$$

Assume : Familiarity with tensors and angular momentum algebra.

Invariance of $\eta_{\mu\nu}$ implies that the transformations are linear : $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, $(\det \Lambda)^2 = 1$.

$\det \Lambda = 1$: Continuously connected to identity.

$\det \Lambda = -1$: Disjoint from identity (e.g. P, T).

The full group is called inhomogeneous Lorentz group or Poincaré group.

If $a^\mu = 0$ (no translation), the group is called homogeneous Lorentz group. It breaks up into four disconnected subsets, labeled by symmetries of parity and time-reversal.

$$L^{\uparrow\downarrow}_{\pm}$$

$\uparrow\downarrow$ $\leftarrow$ Arrow of time

$\pm$ $\leftarrow$ Parity

$L^{\uparrow}_{+}$ is called proper orthochronous Lorentz group.

It has $\det \Lambda = +1$, $\Lambda^0{}_0 \geq +1$, includes identity.

It is an exact symmetry of all QFT.

Note: Transformations in general relativity leave $dx_\mu dx^\mu = g_{\mu\nu} dx^\mu dx^\nu$ invariant. But $g_{\mu\nu}$ changes. Lorentz group describes tangent space of curved space-time at any point (Locally inertial frame).

Symmetry operations in quantum theory preserve the norm of physical states. That implies that the corresponding transformation is unitary. Physical states are classified according to unitary reps. of the symmetry group.
Field operators may not be Hermitian, and then do not belong to unitary representations.
For continuous groups, it is convenient to talk in terms of parameters that cover the group manifold. The elements in the neighbourhood of identity can be described using Taylor expansion.
Generators of the group define the various directions in which one can move away from identity.

$$g = 1 + \sum_i \epsilon_i T_i + \dots$$

T_i form a vector space, and describe the tangent space to the group manifold at identity.
For unitary reps., generators are taken to be Hermitian.

Generators and their commutation rules define the algebra of the group. The commutation rules give a combination of generators.
This algebra completely defines the tangent space of the group manifold, but not its topology.
(Lie groups and Lie algebras with their unitary reps are heavily used in High Energy Physics.)
When norms of the generators have the same sign, the group is called compact. If not, then the group is called non-compact.
The Lorentz group, acting on space-time coordinates x^μ, is non-compact, and is denoted by $SO(3,1)$.

Translations: They form a commutative group, whose composition rule is addition.

Representations multiply, and hence are exponentials of the coordinates, i.e. $g(k) = e^{ikx}$.

$\{e^{ikx}\}$ defines a unitary rep. for translation, labeled by the real parameter k. All reps. are one-dimensional, because the group is commutative.

Generalise to arbitrary dimension by $kx \longrightarrow \vec{k}\cdot\vec{x}$.

$\{x\}$ is integers, then $\{k\}$ is the Brillouin zone.

$\{x\}$ is real line, then $\{k\}$ is also the real line.

$$f(x+a) = \left[\exp\left(a\frac{d}{dx}\right) f(x)\right]_x$$

Generators are $P^\mu \equiv i\partial^\mu$. Group elements can then be expressed as $\exp(-i\,a_\mu P^\mu)$, together with commutation rule $[P^\mu, P^\nu] = 0$.

Rotations and boosts: The group generators are

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \quad : \text{Antisymmetric}$$

The group algebra is given by

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}).$$

In the defining (adjoint) representation,

$$x'^\alpha = \left[\exp\left(\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}\right)\right]^\alpha{}_\beta x^\beta$$

$$(J^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha).$$

Rotation around z-axis by $\theta = \omega_{12}$: $\Lambda^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

This is an orthogonal (real unitary) matrix.
Generators $J^i = \epsilon^{ijk} J^{jk}$ are Hermitian.

Boost along z-axis by $\eta = \omega_{03}$ $\left(\tanh\eta = \frac{v}{c}\right)$: $\Lambda^\alpha{}_\beta = \begin{pmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}$

$\Lambda^\alpha{}_\beta$ is not unitary, but has $\det = 1$.

Rapidity η is additive for successive boosts.

Generators $K^i = J^{i0}$ are Hermitian.

In total, Poincaré group has 10 generators.

$$\{P^\mu, J^i, K^i\}.$$

The four coordinates can also be arranged as 2×2 complex matrix: $x^\mu \sigma_\mu = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}$

It is Hermitian. The Lorentz transformations take the form $x'^\mu \sigma_\mu = Q\,(x^\mu \sigma_\mu)\,Q^\dagger$.

$\det Q = 1$ preserves the norm $t^2 - x^2 - y^2 - z^2$.

Rotation about z-axis by angle θ : $Q = \exp(\frac{i}{2}\theta\sigma_3)$. Unitary matrix

Leaves t, z components unchanged.

x, y components get rotated by $e^{i\theta}$.

Boost along z-axis by rapidity η : $Q = \exp(\frac{1}{2}\eta\sigma_3)$ Hermitian matrix

Leaves x, y components unchanged.

t, z components get changed by e^{η}.

The 2×2 complex matrices with $\det = 1$, form the group $SL(2, \mathbb{C})$.

Spinors are objects transforming according to

$$\psi' = Q\psi \quad \text{(Linear in } Q\text{)}$$

Arbitrary Q can be broken up into rotations and boosts by polar decomposition: $Q = U e^H$.

$$\det Q = 1 \Rightarrow \det U = 1, \ \mathrm{Tr}(H) = 0.$$

$SL(2,C)$ is topologically equivalent to $S_3 \times R_3$.

$Q = -1$ produces no change for $x^\mu \sigma_\mu$, but it takes $\psi \longrightarrow -\psi$. It can be produced by rotation angle 2π. Further rotation to angle 4π, brings back ψ to itself.

This behaviour is not seen for point objects or rigid bodies. It can be seen in rotations of deformable objects.

Coordinate representation: Q and $-Q$ are identified.

$$SO(3,1) \simeq SL(2,C)/Z_2 \quad : \text{Doubly connected.}$$

Topological superselection rule: Boson states cannot be superposed with fermion states.

Spin-statistics theorem: Follows from Lorentz group, together with discrete P, C, T symmetries.

Exchange effect can be undone by 2π rotation of one object (belonging to a pair of identical objects).

For a general representation, we use $J^{\mu\nu}$ as differential operators.

$$\vec{J} = \{J^{23}, J^{31}, J^{12}\}, \quad \vec{K} = \{J^{10}, J^{20}, J^{30}\}$$

Their Lie algebra is:

$$[J^i, J^j] = i\epsilon^{ijk} J^k \qquad (\text{or } \vec{J}\times\vec{J} = i\vec{J})$$

$\{J^i\}$ form the rotation subgroup $SO(3)$ or $SU(2)$.

$$[J^i, K^j] = i\epsilon^{ijk} K^k$$

$$[K^i, K^j] = -i\epsilon^{ijk} J^k \quad : \quad \{K^i\} \text{ do not form subgroup}$$

These also obey parity and time-reversal properties.
P, T labels can be assigned to Lorentz group reps.

Let $\vec{N} = \frac{1}{2}(\vec{J} + i\vec{K})$, $\vec{N}^\dagger = \frac{1}{2}(\vec{J} - i\vec{K})$.

$\vec{N}, \vec{N}^\dagger$ are not Hermitian, but they obey the angular momentum algebra individually, while commuting with each other.

$$[N^i, N^j] = \frac{1}{4}\{[J^i, J^j] + i[J^i, K^j] + i[K^i, J^j] - [K^i, K^j]\}$$

$$= \frac{1}{2}\{i\epsilon^{ijk}J^k + i^2\epsilon^{ijk}K^k\}$$

$$= \frac{1}{2} i\epsilon^{ijk}(J^k + iK^k)$$

$$= i\epsilon^{ijk}N^k$$

$$[N^i, N^{\dagger j}] = 0, \quad [N^{i\dagger}, N^{j\dagger}] = i\epsilon^{ijk}N^{k\dagger}.$$

One can construct the raising and lowering operators for these algebras: $N^\pm$, $N^{\pm\dagger}$.

Remember: $J^\pm|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$

This gives finite dim. representations when $m = -j, \ldots, j$ in steps of 1.

Finite dimensional representations are then labeled by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$, with dimensionality $2j+1$. These are unitary for rotation group. The corresponding representations are not unitary for $N, N^\dagger$. But there are no finite dimensional unitary representations of the Lorentz group.

These representations of the Lorentz group are denoted by a pair of numbers (n, m), with $n, m \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$.

Since $\vec{J} = \vec{N} + \vec{N}^\dagger$, the total spin of the representation is $n+m$. Also parity and charge conjugation switch $\vec{N} \leftrightarrow \vec{N}^\dagger$. So $(n, m) \xrightarrow[\text{or } C]{P} (m, n)$.

When $n \neq m$, the reps. are not parity or charge conjugation eigenstates.

Note that T does not interchange N and $N^\dagger$, and keeps the helicities the same.

$(0,0)$: Scalar (e.g. Klein Gordon field).

$(\frac{1}{2},0)$ and $(0,\frac{1}{2})$: Weyl spinors (left and right handed)

$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$: Dirac spinor (reducible rep.)

$(1,0)$ and $(0,1)$: Vectors with specific chirality
(E.g. $\vec{E}\pm i\vec{B}$)

$(1,0)\oplus(0,1)$: Electromagnetic field $F_{\mu\nu}$.

$(\frac{1}{2},\frac{1}{2})$: 4-vector (e.g. x^μ, A^μ)

The total dimensionality is $(2n+1)\times(2m+1)$.

The fields do not belong to unitary reps.

Gravitons cover the tangent space of $g_{\mu\nu}$.

$$(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2}) = (1_S\oplus 0_A,\ 1_S\oplus 0_A)$$

sym: $(1,1)\oplus(0,0)$ antisym: $(1,0)\oplus(0,1)$

Gravitons thus belong to the representation $(1,1)$.

In Euclidean time, $x^4 = it$, and the invariance of $(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$ is described by the group $SO(4)$. It is equivalent to $SU(2) \otimes SU(2)$ with all generators Hermitian, and all finite dimensional representations unitary. The representations are again labeled by (n, m). The Casimir operators $\vec{N} \cdot \vec{N}$ or $\vec{N}^\dagger \cdot \vec{N}^\dagger$ can be used to obtain eigenvalues $n(n+1)$ and $m(m+1)$.

In Poincaré group, $[P^\mu, J^{\rho\sigma}] = i(g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho)$. Only a commuting set of operators constructed from $\{P^\mu, J^{\rho\sigma}\}$ can be used to specify quantum numbers of physical quantum states. Commonly used set includes P^μ and Casimir operators for $\vec{N}$, $\vec{N}^\dagger$ and $\vec{J}$.

· Galilean group : This is the symmetry group of non-relativistic space-time transformations. The 10 generators are the same as in Poincaré group, but their commutation rules simplify for $v \ll c$. This limiting technique is called Inönü-Wigner contraction.

$\vec{P} \sim mv$, $P^0 = H = M + W$ with $M \sim mc^2$ and $W \sim mv^2$.

$\vec{J} \sim 1$, but $\vec{K} \sim \frac{1}{v}$.

$$\left.\begin{array}{l}[P_i , J_j] = i\,\epsilon_{ijk} P_k \;,\; [P_i , K_j] = -i P_0 \delta_{ij} \\ [P_0 , J_j] = 0 \;,\; [P_0 , K_i] = -i P_i \,.\end{array}\right\} \text{Inhomogeneous Lorentz group}$$

$$\left.\begin{array}{l}[K_i , K_j] = 0 \;,\; [P_i , K_j] = -i M \delta_{ij} \\ [W , K_i] = -i P_i \;,\; [M, \text{anything}] = 0 \,.\end{array}\right\} \text{Galilean group}$$

These differ from the Poincaré group relations.

"M" appearing in the commutator is an example of central charge in the algebra.

Superselection rule: States with different values of M cannot be superposed.

Projective representation: $U(T_2)U(T_1) = e^{i\phi(T_2,T_1)} U(T_2 T_1)$.

Alternatively, one can interpret M as an extra generator (Abelian). It will have an eigenvalue for each physical state.

<u>Classification of 1-particle states:</u>

We need a set of mutually commuting generators. $\{P^\mu\}$ commute and are conventionally taken to be part of this set. Casimir operator $P^\mu P_\mu = m^2$ is an invariant for this classification.

Pauli-Lubanski vector : $W^\mu = -\frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} P_\nu J_{\lambda\sigma}$

$[W^\mu, P^\alpha] = 0$ and $W^\mu P_\mu = 0$.

For fixed P_ν, W^μ are linear combinations of $J_{\lambda\sigma}$.

W^μ do not commute with each other.

$[W^\mu, W^\nu] = i \epsilon^{\mu\nu\lambda\sigma} W_\lambda P_\sigma$: Group property for fixed P_σ.

$W^\mu W_\mu$ Casimir operator quantifies spin of particle.

Group produced by $\{W^\mu\}$ depends on value of m^2.

Unitary representations of Poincaré group are infinite dimensional. (Momenta are unbounded)

Little group : These are Lorentz transformations that leave P^μ invariant. (i.e. $W^\mu{}_\nu p^\nu = p^\mu$)

These identify the spin of the representation.

Various possibilities for p^μ:

$p^\mu = (0,0,0,0)$ Vacuum $W \in SO(3,1)$

$p^2 = m^2 > 0 : p^\mu = (m,0,0,0)$ Massive particle in rest frame $W \in SO(3)$

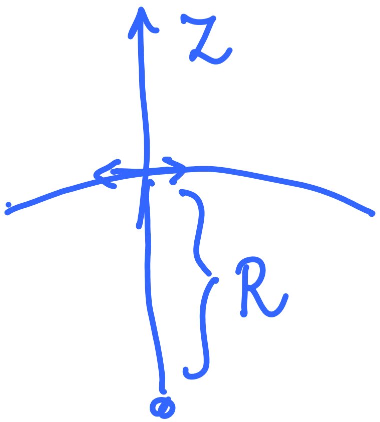
$p^2 = 0 : p^\mu = (k,0,0,k)$ Massless particle moving along z-axis $W \in E(2)$.

$p^2 = -n^2 < 0 : p^\mu = (0,0,0,n)$ Tachyon (Unphysical) $W \in SO(2,1)$

The little groups $SO(3)$ and $E(2)$ have finite dimensional unitary representations. (Labeled by $W_\alpha W^\alpha$)

$E(2)$ can be visualised as the limit of rotation group, with radius of sphere $R \to \infty$, and examining the neighbourhood of the rotation axis. (Inönü-Wigner contraction)

$R \to \infty$: $J_3 \sim 1$, J_1 and $J_2 \sim R$ ($x_3 \sim R$, x_1 and $x_2 \sim 1$)



Let $J_1 = RA$ and $J_2 = RB$.

Then $[A, B] = 0$ (Correspond to translations)

$[J_3, A] = iB$, $[J_3, B] = -iA$

Light-cone coordinates: $x^{\pm} = x^0 \pm x^3$.

For massless particles, $W^- = 0$. The algebra satisfied by W^1, W^2, W^+ is that of $E(2)$.

Other scalar Casimir operators constructed from Poincaré generators either vanish or reduce to those for Lorentz group. (e.g. $P^\alpha P^\beta J_{\alpha\beta} = 0$, $J^{\alpha\beta} J^{\gamma\delta} \epsilon_{\alpha\beta\gamma\delta}$.)

$[P^\mu, J^{\alpha\beta}] \neq 0$ for $\mu = \alpha$ or $\mu = \beta$.
The four $J^{\alpha\beta}$ that commute with P^μ are rearranged as the W^ν components.
With $P_\mu W^\mu = 0$, the little group has 3 parameters.

$m^2 > 0$: In the rest frame, $W^0 = 0$

$$W^i = -\frac{1}{2}\epsilon^{i0jk} P_0 J_{jk} = m\,\epsilon^{ijk} J_{jk}$$

So the Casimir $W^\mu W_\mu = -m^2 s(s+1)$.

Let Lorentz boost be $L(p)(m,0,0,0) = p^\mu$: $\Lambda(p)^\alpha{}_\beta = \begin{pmatrix} E/m & -p_j/m \\ p^i/m & \delta^i_j - \frac{p^i p_j}{m(E+m)} \end{pmatrix}$

Wigner rotation : Rotation induced by boosts.

$W(\Lambda, p) \equiv L^{-1}(\Lambda p)\,\Lambda\, L(p)$: Belongs to the little group

$$(m,0,0,0) \xrightarrow{L(p)} p^\mu \xrightarrow{\Lambda} \Lambda p^\mu \xrightarrow{L^{-1}(\Lambda p)} (m,0,0,0).$$

When Λ is a rotation, then $W(R,p) = R$.
So the states of a moving massive particle have the same transformation rules as the particle at rest.

The machinery of rotation group in non-relativistic QM can be fully carried over to the relativistic case. (e.g. spherical harmonics, Clebsch-Gordon coeffs.)

<u>$m=0$</u>: $P^\mu W_\mu = 0 \Rightarrow kW_3 = -kW_0$.

Casimir $W^\mu W_\mu = -(W_1)^2 - (W_2)^2$

$W_1^2 + W_2^2$ is not quantised in $E(2)$.

But in the real world, all physical states obey

$$W_1 \Phi_{phys} = 0 = W_2 \Phi_{phys}.$$

Then $W^\mu = -\sigma P^\mu$ defines the helicity σ.

$$\sigma = \frac{\vec{J}\cdot\vec{P}}{|\vec{P}|} \quad \text{and} \quad S = |\sigma|.$$

Rotational invariance (around direction of $\vec{P}$) quantises σ. So σ = integer or half-integer.

σ is rotationally invariant in general, and Lorentz invariant for $m=0$.

A state with a fixed σ is allowed by Lorentz group structure. (e.g. left-handed neutrino)

If parity is a good symmetry, states of $\pm\sigma$ appear together in the theory (e.g. L and R photons).

Generic quantum states of a single particle are labeled by $|[m,s], \vec{p}, \sigma\rangle$.

Arbitrary states can be constructed by superpositions.

Normalisation: Define the states in the little group, and then transform it to a generic form.

Let k^μ be in little group, p^μ in the Lorentz group

$\Psi_{p,\sigma} \equiv N(p)\, U(L(p))\, \Psi_{k,\sigma}$ defines $N(p)$.

In the little group, $U(W)\Psi_{k,\sigma} = \sum_{\sigma'} D^{(s)}_{\sigma'\sigma}(W)\Psi_{k,\sigma'}$.

This defines Wigner's D-matrices ($D^\dagger = D^{-1}$).

$$\langle \Psi_{k',\sigma'} | \Psi_{k,\sigma} \rangle = \delta^3(\vec{k}' - \vec{k})\,\delta_{\sigma'\sigma}$$

$$\Rightarrow \langle \Psi_{p',\sigma'} | \Psi_{p,\sigma} \rangle = |N(p)|^2\,\delta^3(\vec{k}' - k)\,\delta_{\sigma'\sigma}$$

With $p^0\delta^3(\vec{p}' - \vec{p}) = k^0\delta^3(\vec{k}' - \vec{k})$, convenient choice of normalisation is $N(p) = \sqrt{\frac{k^0}{p^0}}$.

Thus for $m \neq 0$: $U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\sigma'} D^{(s)}_{\sigma'\sigma}(W(\Lambda,p))\Psi_{\Lambda p,\sigma'}$

For $m=0$: Different σ-values do not mix.

So $U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}} \exp(i\sigma\theta(\Lambda,p))\Psi_{\Lambda p,\sigma}$.

This completes classification of states under the proper, orthochronous Lorentz group.

The analysis in the full group can be obtained by adding rules for parity and time reversal.

P and T : Action of P and T on the generators p^μ and $J^{\alpha\beta}$ is enough to specify transformation rules for the states. (e.g. $P(iH)P^{-1} = iH$, $\tau(iH)\tau^{-1} = -iH$)

$$P\vec{J}P^{-1} = \vec{J} \,, \quad \tau\vec{J}\tau^{-1} = -\vec{J}$$

: Sufficient to work out transformations of eigenstates

$m \neq 0$: States are $\Psi_{p,\sigma}$, together with m, s.

Then $P\Psi_{p,\sigma} = \eta_\sigma \Psi_{\mathcal{P}p,\sigma}$, with η_σ independent of σ because P commutes with $J_\pm$.

Also $\tau\Psi_{p,\sigma} = \zeta_\sigma \Psi_{\mathcal{P}p,-\sigma}$, with ζ_σ changing by a factor of (-1) when σ changes by ± 1 due to action of $J_\pm$. So $\zeta_\sigma = \zeta(-1)^{s-\sigma}$.

$m=0$: Here the reference state in the little group changes under P and $\mathcal{T}$. The state can be brought back to its starting value, by a rotation of π around 2-axis (convention). Then depending on the sign of p_2,

$$P\,\Psi_{p,\sigma} = \eta_\sigma \exp(\mp i\pi\sigma)\,\Psi_{\mathcal{P}p,-\sigma} \quad \text{and}$$

$$\mathcal{T}\,\Psi_{p,\sigma} = \zeta_\sigma \exp(\pm i\pi\sigma)\,\Psi_{\mathcal{P}p,\sigma}$$

The intrinsic phases (η and ζ) are chosen by some convention. They can be modified by combining Lorentz group with other symmetries, e.g. by factors of type $e^{i\alpha B + i\beta L + i\gamma Q}$, in general.

For neutral particles, absolute phases can be defined (without specific convention).

$\mathcal{T}^2 \Psi_{p,\sigma} = (-1)^{2s} \Psi_{p,\sigma}$ for both $m \neq 0$ and $m=0$.

(note that $e^{\pm 2\pi i\sigma} = (-1)^{2|\sigma|}$ for all $\sigma = 0, \pm\frac{1}{2}, \pm 1, \ldots$)

For half-integer spin states, the change of sign under $\mathcal{T}^2$ implies Kramer's doublet degeneracy. (Even without enforcing rotational invariance.)

Charge conjugation operation can also be included, and its action on Lorentz group eigenstates is

$$\mathcal{C}\, \Psi_{p,\sigma,n} = \xi_n \Psi_{p,\sigma,n^c}$$

Convention is to choose $\xi \eta \zeta = 1$ for CPT transformation.

Particle and antiparticle states are thus related by: $\mathcal{C}\mathcal{P}\mathcal{T}\, \Psi_{p,\sigma,n} = (-1)^{s-\sigma} \Psi_{p,-\sigma,n^c}$.

Klein-Gordon equation: Spin $=0$. Lorentz group gives only the standard space-time transformations.

Dirac equation: Spin $=\frac{1}{2}$. Lorentz group also transforms the spinor components, in addition to space and time.

Lorentz covariance of spinors:

Dirac spinors form a reducible representation.

$$x' = \Lambda x \,, \quad \psi'(x') = S(\Lambda)\psi(x) \,, \quad S^{-1}(\Lambda) = S(\Lambda^{-1}) .$$

Dirac equation transforms to:

$$\left[i\hbar\, S(\Lambda)\gamma^{\mu} S^{-1}(\Lambda)\, \Lambda^{\nu}{}_{\mu} \frac{\partial}{\partial x'^{\nu}} - mc \right] \psi'(x') = 0 .$$

Covariance requires: $\Lambda^{\nu}{}_{\mu}\gamma^{\mu} = S^{-1}(\Lambda)\gamma^{\nu} S(\Lambda)$.

Consider infinitesimal transformation:

$$\Lambda^{\nu}{}_{\mu} = \delta^{\nu}{}_{\mu} + \omega^{\nu}{}_{\mu} \,, \quad S = 1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} .$$

Then $\omega^{\nu}{}_{\mu}\gamma^{\mu} = -\frac{i}{4}\omega^{\alpha\beta}[\gamma^{\nu}, \sigma_{\alpha\beta}]$.

$\omega^{\alpha\beta}$ is antisymmetric for the Lorentz group.
The solution is : $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.
Finite transformations are : $S = \exp[-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}]$.
For boosts : $\sigma^{0i} = i\alpha^i$, $S_B^\dagger = S_B$.
For rotations : $\sigma^{ij} = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$, $S_R^\dagger = S_R^{-1}$.

$$\left.\begin{aligned} S_B &= \cosh\frac{\omega}{2} + \hat{\omega}\cdot\vec{\alpha}\sinh\frac{\omega}{2} \\ S_R &= \cos\frac{\omega}{2} - i\,\hat{\omega}\cdot\vec{\sigma}\sin\frac{\omega}{2} \end{aligned}\right\} \text{Half-angles appear here.}$$

In all cases, $S^{-1} = \gamma_0 S^\dagger \gamma_0$.

$\psi \longrightarrow S\psi$, $\bar{\psi} \longrightarrow \bar{\psi}S^{-1}$: Adjoint spinor
Then $\bar{\psi} = \psi^\dagger\gamma_0$ is the required form.
$x \longrightarrow \Lambda x$ leads to $\gamma^\mu \longrightarrow S(\Lambda^\mu{}_\nu\gamma^\nu)S^{-1}$.
Thus $\bar{\psi}\psi$ is Lorentz invariant, $\bar{\psi}\gamma^\mu\psi$ is a vector etc. $\bar{\psi}\gamma^\mu\psi$ is the current, and its time component $\psi^\dagger\psi$ is the density.

The space spanned by γ-matrices has 16 independent components (complex). A 16-component basis can be specified, with simple Lorentz transformation properties. Parity and time reversal properties can be added (operators γ^0, $i\gamma^1\gamma^3$ with complex conjugation respectively). Operators are thus classified by their J^P properties.

Conventional orthogonal basis (each operator squares to ± 1) is described by (form $\bar{\Psi}\Gamma\Psi$):

$$\Gamma : \quad 1 \;, \quad \gamma^\mu \;, \quad \sigma^{\mu\nu} \;, \quad \gamma^{[\mu\nu\rho]} \sim \gamma^\tau\gamma^5 \;, \quad \gamma^5 .$$

$$\text{No:} \quad 1 \;+\; 4 \;+\; 6 \;+\; 4 \;+\; 1 \;=\; 16$$

$$J^P : \quad \underset{S}{0^+} \;, \quad \underset{V}{1^-} \;, \quad \underset{T}{\text{tensors}} \;, \quad \underset{A}{1^+} \;, \quad \underset{P}{0^-} .$$

General form of spinor wavefunctions:

Free particle solutions are $\psi^r(x) = \omega^r(\vec{p})\, e^{-i\epsilon_r p_\mu x^\mu/\hbar}$,

with $p^0 > 0$, $\epsilon_r = \pm 1$ and $r = 1, 2, 3, 4$.

In the rest frame,

$$\omega^r(0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$ is orthornormal set of spinors.

In an arbitrary frame, $\omega^r(\vec{p}) = S\,\omega^r(0)$.

$$(\not{p} - \epsilon_r mc)\,\omega^r(\vec{p}) = 0, \quad \bar{\omega}^r(\vec{p})\,(\not{p} - \epsilon_r mc) = 0.$$

Orthogonality and completeness relations are:

$$\bar{\omega}^r(\vec{p})\,\omega^{r'}(\vec{p}) = \delta_{rr'}\,\epsilon_r.$$

$$\sum_{r=1}^{4} \epsilon_r\, \omega^r_\alpha(\vec{p})\, \bar{\omega}^r_\beta(\vec{p}) = \sum_{r=1}^{4} \epsilon_r\, S_\alpha{}^\gamma\, \omega^r_\gamma(0)\, \bar{\omega}^r_\delta(0)\, (S^{-1})^\delta{}_\beta$$

$$= S_\alpha{}^\gamma\, \delta_{\gamma\delta}\, (S^{-1})^\delta{}_\beta = \delta_{\alpha\beta}$$

Related forms are :

$$\omega^{r\dagger}(\epsilon_r \vec{p})\, \omega^{r'}(\epsilon_{r'} \vec{p}) = \frac{E}{mc^2} \delta_{rr'}$$

$$\sum_{r=1}^{4} \omega^{r}_{\alpha}(\epsilon_r \vec{p})\, \omega^{r\dagger}_{\beta}(\epsilon_r \vec{p}) = \frac{E}{mc^2} \delta_{\alpha\beta}$$

Any given wavefunction can be projected onto components of this basis. Conventionally,

$$\left.\begin{aligned} \omega^{1,2}(\vec{p}) &= u(p, \pm S^{\mu}_{z}) \\ \omega^{3,4}(\vec{p}) &= v(p, \mp S^{\mu}_{z}) \end{aligned}\right\}$$

S^{μ}_{z} is Lorentz transformation of $(0,0,0,1)$ in the rest frame

Energy value picks up ϵ_r (or u, v) and spin value picks up up/down along z-axis in the rest frame.

Orthogonality and completeness relations can be used to construct appropriate projection operators.

Projection operators: $\sum_i P_i = 1$, $P_i P_j = \delta_{ij} P_i$.

They can be easily picked out in the rest frame, and then transformed to arbitrary frame using Lorentz covariance.

Energy: $\dfrac{\not{p}+m}{2m}$ and $\dfrac{-\not{p}+m}{2m}$

(Generalised from rest frame: $\dfrac{1\pm\gamma^0}{2} \longrightarrow \dfrac{m(1\pm\gamma^0)}{2m}$)

Spin: $\dfrac{1+\gamma_5 \not{s}}{2}$ and $\dfrac{1-\gamma_5\not{s}}{2}$ $\left(\begin{array}{l} s^\mu p_\mu = 0 \\ s^\mu s_\mu = -1 \end{array}\right)$

(Generalised from rest frame: $\dfrac{1\pm\sigma_z}{2} \longrightarrow \dfrac{1\pm\gamma_5\gamma_z}{2}$)