Lecture 29: Vortices and their interactions

In the last lecture on linearized superfluid hydrodynamics, we saw that a phase difference once established remains static according to the equations of motion of the linearized (harmonic) theory and drives a steady super-current as long as there is no chemical potential gradient. This is perfect superfluidity, with no possibility for the current to degrade. How can the current degrade in a real superfluid? The answer involves a study of excitations that are not captured by this harmonic theory, which only captures the superfluid sound waves. These additional excitations are *vortices*, which we now discuss.

Vortices are the most dramatic manifestation of the fact that ϕ is an angle, *i.e.* physical observables are written in terms of $\exp(i\phi)$ and only care about the value of ϕ modulo 2π. This is something we left out of our linearized (harmonic) treatment in the last lecture. As a result, our treatment only took into account the effects of small-amplitude, long-wavelength oscillations in ϕ, not topological defects in which the O(2) vector $\hat{n}$

$$n_x + i n_y \;=\; \exp(i\phi) \tag{1}$$

winds around by 2π.

In $d = 1$ at $T = 0$, these are space-time vortices in the time-evolution of ϕ. In $d = 2$, they are point-like vortices centered at a particular spatial point. They can be thought of as particles with their own classical or quantum dynamics. In $d = 3$, we can have vortex lines or rings in space, with their own time-evolution. In the remainder of this course, we will consider finite-temperature properties in $d = 2$. As discussed in the last-but-one lecture, this reduces to a two dimensional classical theory for the zero matsubara frequency mode of the full theory. Interpreting one of these two dimensions as imaginary time, all our calculations can also be used to describe the physics of space-time vortices in a $d = 1$ theory at $T = 0$. From now on, we will use the language of the finite-temperature $d = 2$ classical theory, with effective energy

$$H_{\text{eff}} \;=\; \frac{\rho_s}{2}\int d^2x (\nabla\hat{n})^2 \tag{2}$$

and partition function

$$Z \;=\; \int \mathcal{D}\hat{n} \exp\left(-\frac{\rho_s}{2T}\int_\Lambda d^2x(\nabla\hat{n})^2\right) \;, \tag{3}$$

where the path integral is over all configurations of the O(2) vector field $\hat{n}(\mathbf{x})$ and the subscript on the spatial integral reminds us that the theory is defined with an upper cutoff in momentum space given by $\Lambda \sim a^{-1}$.

One example of a configuration with a unit vortex centered at the origin in this $d = 2$ theory is

$$\hat{n}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} . \tag{4}$$

Clearly, such vortex configurations are singular at the core of the vortex. This is where the lattice spacing a (or upper-cutoff in momentum space Λ) enter our discussion: Since we are actually modeling the long-wavelength behaviour of a lattice system, this short-distance singularity is cutoff at the lattice-scale a, and such a vortex configuration has a finite "core energy" that depends on this lattice-regularization.

With this in mind, let us calculate the energy of such vortex configurations. To do this for an m-fold vortex, it is convenient to rewrite the vortex configuration in terms of the polar angle $\theta(\mathbf{x})$ corresponding to the spatial position vector $\mathbf{x}$:

$$\hat{n}(\mathbf{x}) = (\cos(m\theta), \sin(m\theta)) . \tag{5}$$

and note that

$$(\nabla\hat{n})^2 = \frac{m^2}{|\mathbf{x}|^2} . \tag{6}$$

This gives

$$\begin{aligned} H_{\text{eff}} &= \pi\rho_s m^2 \int_a^R dr \frac{1}{r} \\ &= \pi\rho_s m^2 \log(R/a) , \end{aligned} \tag{7}$$

where R is the linear size of the sample (assumed for convenience to be a disk). In other words, the energy cost of a single vortex is infinite in the thermodynamic limit. Note that this divergence is not due to the singular nature of the vortex at short distances. As we have already discussed, this is not a problem since it is regulated by the non-zero lattice spacing a, and merely leads to a finite core energy E_v. Instead, this divergence arises due to the far-field contribution of the kinetic energy of circulating supercurrents that exist even very far away from the vortex core (remember, $\rho_s \nabla\phi$ is the supercurrent).

This is the energy, "calculated by hand", of a single vortex at the origin. A general configuration can be thought of as a "gas" of vortices of various strength at various positions. Clearly, it will be cumbersome to repeat this calculation in such a general case. To address the energetics of such general configurations, we need to think in somewhat more general terms. To this end, we first make a more general definition of the vorticity m enclosed by a countour C:

$$\oint_C \epsilon^{ab} \hat{n}^a \vec{\nabla} \hat{n}^b \cdot d\vec{l} \;=\; 2\pi m \,. \tag{8}$$

A key role is thus played by the so-called "superfluid velocity"

$$v_\alpha \;=\; \epsilon^{ab} \hat{n}^a \partial_\alpha \hat{n}^b \,. \tag{9}$$

An advantage of working with the superfluid velocity is that it has no singularity across any "cuts" at which the phase angle ϕ jumps from 2π back to zero, since such cuts are a matter of our coordinate convention and have no invariant meaning. Of course, the superfluid velocity does get very large as one approaches the core of the vortex, but that is a physical effect that is cut off at the lattice scale or short-distance cutoff in the theory. The above equation is of course equivalent to the statement that $\vec{\nabla} \times \vec{v}$ is concentrated at the positions of the vortices:

$$\vec{\nabla} \times \vec{v} \;=\; 2\pi \hat{z} \sum_i m_i \delta^2(\vec{r} - \vec{r}_i) \tag{10}$$

Next, we note that $H_{\rm eff}$ can be written in terms of $\vec{v}$ as

$$H_{\rm eff} \;=\; \frac{\rho_s}{2} \int d^2x \vec{v}^2 \,, \tag{11}$$

so long as we split up $\vec{v}$ into a curl-free contribution from sound-wave like fluctuations and a singular part that arises due to vortices, and has a curl but no divergence:

$$\vec{v} \;=\; \vec{v}_{a(\rm nalytic)} + \vec{v}_{s(\rm ingular)} . \tag{12}$$

with

$$\begin{aligned} \vec{\nabla} \cdot \vec{v}_s &= 0 \,, \\ \vec{\nabla} \times \vec{v}_a &= 0 \,, \end{aligned} \tag{13}$$

and

$$\begin{aligned}\vec{\nabla} \times \vec{v}_s &= \hat{z} 2\pi \sum_i m_i \delta^2(\vec{r} - \vec{r}_i) \\ &= 2\pi m(\vec{r}) \hat{z} \ , \end{aligned} \tag{14}$$

where m is the *vortex-density field.* Now, since

$$\vec{\nabla} \cdot \vec{v}_s = 0 \ , \tag{15}$$

we may write

$$\vec{v}_s = \vec{\nabla} \times a\hat{z} \ . \tag{16}$$

This implies

$$\vec{\nabla} \times (\vec{\nabla} \times a\hat{z}) = 2\pi m(\vec{r}) \hat{z} \ , \tag{17}$$

or, equivalently,

$$-\nabla^2 a = 2\pi m(\vec{r}) \ . \tag{18}$$

On the other hand, since

$$\vec{\nabla} \times \vec{v}_a = 0 \ , \tag{19}$$

we may write

$$\vec{v}_a = \nabla \phi_a \ , \tag{20}$$

with

$$\nabla^2 \phi_a = \vec{\nabla} \cdot \vec{v} \ . \tag{21}$$

Thus, we have split $\vec{v}$ into pure vortex part v_s and pure phase part v_a specified by vector and scalar potentials which can be determined from the gradient and curl of the original velocity field $\vec{v}$.

We are now in a position to calculate the energy E of a general configuration including the effects of both the spinwave modes and the vortices. To do this, we first note that contributions of $\vec{v}_s$ and $\vec{v}_a$ decouple completely at the level of our discussion since

$$\begin{aligned}\int d^2x \vec{v}_s \cdot \vec{v}_a &= \int d^2x (\vec{\nabla} \times a\hat{z}) \cdot \nabla \phi_a \\ &= 0 \ \text{ by integration by parts.} \end{aligned} \tag{22}$$

The role of the spinwave contribution has already been discussed at length, and it leads of course to the destruction of true long-range order, but allows for quasi-long-range order (as we have seen in previous lectures). Since the two decouple, we focus exclusively on the vortex contribution for the remainder of this lecture, and write

$$\begin{aligned} E_{\text{vortex}} &= \frac{\rho_s}{2}\int d^2x(\vec{\nabla}\times a\hat{z})^2 \\ &= \frac{\rho_s}{2}\int d^2x(\vec{\nabla}a)^2 \text{ by integration by parts.} \end{aligned} \tag{23}$$

We now use

$$-\nabla^2 a = 2\pi m(x) \tag{24}$$

and Fourier transform to rewrite this in Fourier space as

$$\begin{aligned} E_{\text{vortex}} &= \frac{(2\pi)^2\rho_S}{2}\int \frac{d^2q}{(2\pi)^2}\frac{m(\vec{q})m(-\vec{q})}{\vec{q}^2} \\ &= \pi\rho_s\int d^2xd^2x' m(\vec{x})G(\vec{x}-\vec{x}')m(\vec{x}')\ , \end{aligned} \tag{25}$$

where $G(\vec{x}-\vec{x}')$ is the Green function of two dimensional electrostatics:

$$\begin{aligned} G(\vec{x}) &= 2\pi\int \frac{d^2q}{(2\pi)^2}\frac{-e^{i\vec{q}\cdot\vec{x}}}{q^2} \\ &= \frac{1}{2\pi}\int \frac{qdqd\theta}{q^2}e^{iqx\cos(\theta)} \\ &= \int \frac{dq}{q}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{iqx\cos(\theta)} \\ &= \int_{\frac{x}{R}}^{\frac{x}{a}}\frac{d\bar{q}}{\bar{q}}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{i\bar{q}\cos(\theta)}\ . \end{aligned} \tag{26}$$

which gives

$$G(\vec{x}) = \log(\frac{R}{a}) - \log(\frac{|\vec{x}|}{a}) + C + \mathcal{O}\left[\left(\frac{|\vec{x}|}{a}\right)^{-3/2}\right]\ , \tag{27}$$

where C depends on the choice of short-distance cutoff a and R is the linear scale that sets the size of the system.

This allows us to write the vortex contribution to the energy as

$$E_{\text{vortex}} = \pi\rho_s \sum_{ij} m_i m_j \left[\log(\frac{R}{a}) - \log(\frac{|\vec{x}_i - \vec{x}_j|}{a}) + C\right] . \tag{28}$$

Of course, this expression has a problem for all the terms with $i = j$. This is related to the fact that we are not really in the continuum, and this divergence is actually cut off by the lattice scale to give a finite core energy $E_c(m_i)$ for vortex i with vorticity m_i. Therefore, we may write

$$E_{\text{vortex}} = \pi\rho_s \left(\sum_i m_i\right)^2 \log\left(\frac{R}{a}\right) - \pi\rho_s \sum_{i\neq j} m_i m_j \log\frac{|x_i - x_j|}{a} + \pi\rho_s \sum_i E_c(m_i) . \tag{29}$$

So for a system of vortices with "charge" $\{m_i\}$, the energy cost scales as $\log\frac{R}{a}$ unless $\sum_i m_i = 0$, implying that this "plasma" must obey a global charge-neutrality condition in the thermodynamic limit. Finally, we note from our earlier single-vortex calculation that $E_c(m_i)$ must scale as m_i^2. Therefore, we may write

$$E_{\text{vortex}} = \pi\rho_s \left(\sum_i m_i\right)^2 \log\left(\frac{R}{a}\right) - \pi\rho_s \sum_{i\neq j} m_i m_j \log\frac{|x_i - x_j|}{a} + \pi\rho_s\epsilon_c \sum_i m_i^2 . \tag{30}$$

In the next couple of lectures, we will study the statistical mechanics of this gas of logarithmically interacting vortices using renormalization group ideas. But here, it is worth making one qualitative point about superfluidity and vortices before concluding this lecture: In the absence of vortices, a superfluid current can never degrade (as we saw in the last lecture). Now imagine such a superfluid current set-up in the y direction. And ask what happens if a vortex "crosses its path", moving from the left edge of the sample to the right edge of the sample. Clearly, this process changes the phase difference between the top and the bottom of the sample by 2π (draw a picture and this will be clear!). This means that a vortex current from left to right will induce a steady change per unit time in the phase difference between the top and the bottom of the sample. But this rate of change of phase difference is related to the voltage dropped from the top to the bottom of the sample. Thus, mobile vortices are responsible for producing a finite voltage drop, *i.e.* a non-zero resistance!

Therefore, if vortices proliferate and move around freely, we expect superfluidity to be destroyed. Whereas if they remain bound in pairs with zero net vorticity, the system remains superfluid. The understanding of the vortex plasma that we develop in the next two lectures will therefore lead naturally to a theory for superfluid-normal fluid transitions in two dimensional superfluids.