Lecture 27: Renormalization group approach to the quantum rotor model: Finite temperature properties

In this lecture, we will develop the renormalization group equations that will allow us to understand the properties of systems at small non-zero temperature in the vicinity of a $T = 0$ quantum phase transition characterized by a zero temperature fixed point. But before we get started, we have some loose ends to tie up from the previous lecture—In the previous lecture, we derived flow equations to leading order in $\epsilon \equiv D-2$ in the vicinity of the $\mathcal{O}(\epsilon)$ critical fixed point g_c. These can be used to compute predictions for the behaviour of correlations at and near criticality to leading order in ϵ, and this is what we focus on first:

Consider the correlation function at criticality. We have the equation

$$\langle \hat{n}(x_\mu) \cdot \hat{n}(0) \rangle_{g_c} \;=\; \zeta(l) \langle \tilde{\hat{n}}(x_\mu e^{-l}) \cdot \tilde{\hat{n}}(0) \rangle_{g_c} \;, \tag{1}$$

where $\zeta(l)$ is the solution to the equation for ζ at criticality:

$$\frac{d\zeta(l)}{dl} \;=\; -\frac{N-1}{N-2}\epsilon\zeta(l) \;. \tag{2}$$

In other words, $\zeta(l)$ is given by

$$\zeta(l) = e^{-\frac{(N-1)\epsilon l}{(N-2)}} \tag{3}$$

We now choose l such that $|x_\mu| e^{-l} = 1$. Then we may deduce

$$\langle \hat{n}(x_\mu) \cdot \hat{n}(0) \rangle_{g_c} \;=\; \frac{1}{|x_\mu|^{\frac{(N-1)\epsilon}{(N-2)}}} \;. \tag{4}$$

The critical exponent η is usually defined by writing the critical correlations as

$$\langle \hat{n}(x_\mu) \cdot \hat{n}(0) \rangle_{g_c} \;=\; \frac{1}{|x_\mu|^{D-2+\eta}} \;. \tag{5}$$

Comparing this with our expression for the critical correlations, we obtain our prediction for η, correct to $\mathcal{O}(\epsilon)$:

$$\eta = \frac{\epsilon}{N-2} \;. \tag{6}$$

Next, let us consider the case in which $g = g_c + \delta_0$, with δ_0 positive and small. In this case, the system is disordered at the longest length scales, although the proximity to the critical point means that the correlation length is large, since it must actually diverge as $\delta_0 \to 0$ (in order for the exponentially decaying short-ranged correlations characteristic of the disordered phase to give way to the power-law correlators characteristic of the critical point). An important property of such critical points is the manner in which the correlation length diverges as one approaches the transition. In order to obtain a prediction for this from our RG analysis, we now work with the linearized off-critical flows we studied at the end of the previous lecture. As we saw in the previous lecture, if we start with a small initial value $\delta(0) \equiv \delta_0$, then $\delta(l)$ at scale l is given by

$$\delta(l) \quad = \quad \delta_0 e^{\epsilon l} \; . \tag{7}$$

Now, when $\delta(l)$ gets to be $\mathcal{O}(1)$, the corresponding problem will, in new units, have an order one correlation length ξ_0. Translated back to the original units, the correlation length will actually be $\xi_0 e^{l^*}$, where l^* is the value of l for which $\delta(l^*) = 1$, i.e $l^* = \epsilon^{-1} \log(1/\delta_0)$. Thus, we predict

$$\xi(\delta_0) = \frac{1}{\delta_0^{1/\epsilon}} \; . \tag{8}$$

Conventionally, one defines a correlation length exponent ν by the relation

$$\xi \sim (g - g_c)^{-\nu} \; . \tag{9}$$

Comparing with our result, we see that our RG analysis predicts

$$\nu = \frac{1}{\epsilon} \tag{10}$$

to leading order in ϵ.

Finally, let us analyse the physics on the other side of g_c: $g = g_c + \delta_0$ with δ_0 negative. In this case, g flows to zero, and the system is ordered at long length scales, albeit with a small ordered moment which goes to zero as $|\delta_0|$ goes to zero. To discuss this, we consider the off-critical flows of $g(l)$ and $\zeta(l)$ starting with a small negative value of δ. Since the linearized flow for $\delta(l)$ reads

$$\delta(l) \quad = \quad \delta_0 e^{\epsilon l} \; , \tag{11}$$

where δ_0 is now negative, we see that g renormalizes to zero within this linear approximation at l^* given by the solution to the equation:

$$\begin{aligned} |\delta_0| e^{\epsilon l^*} &= g_c \\ &= \frac{\epsilon}{(N-2)S_2} \; . \end{aligned} \tag{12}$$

The renormalized theory corresponding to RG scale l^* is expected to have essentially perfect long-range order, *i.e.* the correlation function of the $\hat{n}$ field in this theory will tend to an $\mathcal{O}(1)$ number in the long-distance limit. For the correlations in our original theory, this implies

$$\lim_{|x_\mu| \to \infty} \langle \hat{n}(x_\mu) \cdot \hat{n}(0) \rangle_{g_c + \delta_0} \;\; \sim \;\; \zeta(l^*) \; . \tag{13}$$

Since the limit on the left hand side defines $\vec{m}^2$, the square of the order parameter, this implies that

$$|\vec{m}| \;\; \sim \;\; \sqrt{\zeta(l^*)} \; . \tag{14}$$

We can work out $\zeta(l^*)$ by integrating the equation for the linearized off-critical flow of ζ derived in the last lecture. This gives

$$\zeta(l^*) \;\; \sim \;\; |\delta_0|^{\frac{N-1}{N-2}} \; , \tag{15}$$

which implies

$$|\vec{m}| \;\; \sim \;\; |\delta_0|^{\frac{N-1}{2(N-2)}} \; . \tag{16}$$

The critical exponent β is conventionally defined by the relation $|\vec{m}| \sim |\delta_0|^\beta$. Therefore, our leading order RG prediction for β reads

$$\beta \;\; = \;\; \frac{N-1}{2(N-2)} \; . \tag{17}$$

One must remember that each of these predictions for the critical exponents η, ν, and β are leading order results in an "ϵ-expansion" about $D = 2$. Using more sophisticated field-theoretical reformulations of this idea, it is possible to obtain higher order terms in this expansion, and these estimates provide a good analytical guideline for the critical behaviour in the physical $D = 3$ case.

Next, we turn our attention to the $T > 0$ properties in the vicinity of this critical point. In order to do this, we must rewind back to the last-but-one lecture and remind ourselves that the flow equations we have been using were derived at $T = 0$. As we have seen in the original derivation, this enters only in the penultimate step of the derivation. Therefore we go back to that step and now interpret the trace $\mathrm{Tr}_{\vec{q}}$ over modes being eliminated in a different way, as already indicated in that lecture:

$$\mathrm{Tr}_{\vec{q}} = \sum_{\omega_n=-\infty}^{\infty} \int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{d^d q}{(2\pi)^d} , \tag{18}$$

where $\omega_n = \frac{2\pi n}{\beta} = 2\pi n T$ as usual. Using this formulation and repeating the steps of the derivation in the last-but-one lecture, we obtain

$$\frac{1}{g'} = \frac{1}{g} - \frac{N-2}{\beta} \sum_{\omega_n} \int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{\omega_n^2 + k^2} . \tag{19}$$

Next, we redefine units of space and imaginary-time as before. The new feature now is that rescaling units of imaginary-time implies that the new theory after renormalization has a different value of renormalized temperature $\tilde{T}$, since the inverse temperature is the "size" of the system in the imaginary time direction. We therefore have the two equations

$$\begin{aligned} \frac{1}{\tilde{g}} &= \frac{e^{(D-2)\delta l}}{g} - \frac{(N-2)}{\beta} \sum_{\omega_n} \int_{\Lambda-\delta\Lambda}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{\omega_n^2 + k^2} , \\ \tilde{T} &= e^{\delta l} T , \end{aligned} \tag{20}$$

where $D = d + 1$ is the space-time dimension as usual. Doing the ω_n summation explicitly (this can be done using contour integration techniques that you must have studied in a mathematical methods course, and I urge you to check my answer as a homework exercise), we obtain

$$\begin{aligned} \frac{1}{\tilde{g}} &= \frac{e^{(D-2)\delta l}}{g} - (N-2) \int_{\Lambda e^{-\delta l}}^{\Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{2k} + \frac{1}{k} \frac{1}{e^{k/T} - 1} \right) , \\ \tilde{T} &= e^{\delta l} T . \end{aligned} \tag{21}$$

Taking the limit of small δl as usual and iterating, we obtain the flow equations

$$\frac{dg}{dl} = -(d-1)g + \frac{(N-2) S_d \Lambda^{d-1}}{2} \coth\left(\frac{\Lambda}{2T} \right) ,$$

$$\frac{dT}{dl} = T . \tag{22}$$

As an example of the use of these flow equations, let us try and answer the following question in $d = 2$: From the Mermin-Wagner theorem, we know that there can be no real long-range order and breaking of continuous symmetry at $T > 0$ in spatial dimension $d = 2$. However, the ground state can and does have long-range order for $g < g_c$. The interesting question then arises—What is the correlation length at very small but non-zero temperature $T > 0$ above the ordered ground state? Clearly, this correlation length $\xi(T)$ must diverge as $T \to 0$, since we are going from short-ranged correlations at small $T > 0$ to long-range order at $T = 0$. And the question we are really asking is: How does $\xi(T)$ diverge as we approach the ordered ground state by lowering the temperature?

To answer this to leading order, we note that for $g < g_c$, the flow of g is dominated by the linear term that sends $g(l)$ to zero exponentially quickly in l. Therefore, we have

$$g(l) \approx g e^{-l} . \tag{23}$$

On the other hand, we also have

$$T(l) = T e^{l} . \tag{24}$$

Thus, to leading order we have the approximate "constant of motion"

$$g(l)T(l) \approx gT . \tag{25}$$

To see what this implies, let us follow the flows from the initial small value of T up to a scale l^* at which $T(l^*) = \mathcal{C}$, some $\mathcal{O}(1)$ number. In this new theory, all $\omega_n \neq 0$ modes play a negligible role because their contribution is suppressed in the path integral by a factor $\sim \exp(-(2\pi\mathcal{C})^2)$. Therefore, the physics of this renormalized theory is controlled by the behaviour of the $\omega_n = 0$ mode. Put another way, only configurations of $\hat{n}(\tilde{x}, \tilde{\tau})$ that are independent of $\tilde{\tau}$ matter for understanding the renormalized theory at scale l^*. If we denote this $\omega_n = 0$ mode of the renormalized quantum statistical mechanics problem by $\hat{n}(\tilde{x})$ and write the path integral for $\hat{n}(\tilde{x})$, we see that the problem is effectively classical, with the correlations of $\hat{n}(\tilde{x})$ being controlled by the classical effective action

$$S_{\text{eff}} = \frac{1}{2g(l^*)T(l^*)} \int d^2\tilde{x} (\nabla_{\tilde{x}} \hat{n})^2 , \tag{26}$$

where it is understood that $T(l^*)$ takes on the value $\mathcal{C}$. But we already know that $g(l^*)T(l^*) = gT$ within our approximate analysis of the flows. Therefore, this action can be written as

$$S_{\text{eff}} = \frac{1}{2gT}\int d^2\tilde{x}(\nabla_{\tilde{x}}\hat{n})^2 \; . \tag{27}$$

We can now interpret this as the zero temperature theory for a one-dimensional quantum problem, and use the results of the previous lecture that tell us that this theory has a correlation length $\tilde{\xi}$ given by

$$\tilde{\xi} = \xi_0 \exp(2\pi/(N-2)gT) \; . \tag{28}$$

Translating this result back to the original units, we obtain the following answer to the question we posed above:

$$\xi(T) = e^{l^*}\xi_0 \exp(2\pi/(N-2)gT) \; , \tag{29}$$

where we need to remember that e^{l^*} is fixed by the requirement that $Te^{l^*} = \mathcal{C}$. Using this, we have the final result

$$\xi(T) = \frac{\mathcal{A}}{T}\exp(2\pi/(N-2)gT) \; , \tag{30}$$

where $\mathcal{A}$ is some order one constant that cannot be predicted by this analysis.

Now, the basic assumption that went into this result (apart from our approximate treatment of the RG flows) is that it is legitimate to neglect the $\omega_n \neq 0$ modes once the temperature becomes some $\mathcal{O}(1)$ number $\mathcal{C}$. In other words, we are assuming that the destruction of long-range order at finite-temperature is an essentially classical phenomenon, although we are looking at very small temperatures above the ground state of a quantum problem. Is this correct? To reassure ourselves that this is indeed correct, we can estimate the mean occupation of spin wave modes of wavevector $q \sim \xi^{-1}$ [since the system looks ordered up to $x \sim \xi$, and we know from our earlier discussion of the Mermin-Wagner theorem that it is these spin-wave modes that are responsible for the destruction of long-range order in systems with continuous symmetries]. If this mean occupation is large, then the corresponding mode can indeed be treated classically, and this would then suggest that our RG analysis above is correct.

Now, the energy of a spin-wave with wavevector $q \sim \xi^{-1}$ is $\epsilon(\xi^{-1}) \sim c\xi^{-1}$. The mean occupation is simply the Bose-function evaluated at this energy:

$$\bar{n}(\xi^{-1}) = \frac{1}{e^{c/\xi T} - 1} \; . \tag{31}$$

To check if our RG results are consistent, we plug in the RG prediction for $T\xi(T)$ into this formula to get

$$\begin{aligned}\bar{n}(\xi^{-1}) &= \frac{1}{\exp(ce^{-(2\pi/(N-2)gT)}/\mathcal{A})-1} \\ &\sim \exp(2\pi/(N-2)gT)\ ,\end{aligned} \tag{32}$$

which is indeed very large in the $T \to 0$ limit. Therefore, our RG argument and results are internally consistent and correct. This regime of behaviour is sometimes called the "renormalized classical" regime, since the physics at large length scales, of order the exponentially large correlation $\xi(T)$, is effectively classical. Finally, we note for completeness that although our overall logic is correct, our crude analysis of the flows causes us to miss a multiplicative power-law prefactor to the exponential form of $T\xi(T)$. The full answer (which one can derive by a more elaborate analysis along similar lines) is

$$T\xi(T) = \mathcal{A}\left(\frac{(N-2)gT}{2\pi}\right)^{\frac{1}{N-2}}\exp(2\pi/(N-2)gT)\ . \tag{33}$$

This concludes our discussion of the renormalization group theory for the N-vector model, and its applications to quantum antiferromagnets. In the next lecture, we switch gears and focus on the $N = D = 2$ case appropriate for a study of superfluids at $T > 0$ in spatial dimension $d = 2$, or a study of quantum liquids in their superfluid phase at $T = 0$ in $d = 1$.