Lecture 26: Renormalization group approach to the quantum rotor model: Phases and phase transitions

As we have already argued in the last but one lecture, to characterize the g dependence of the large correlation length scale ξ beyond which a $d = 1$ dimensional quantum antiferromagnet "realizes" that it cannot order even at $T = 0$ no matter how small the value of g, one simply need to find l^* such that the renormalized coupling $g(l^*)$ becomes of order one starting from an initial condition $g(l = 0) = g$. Once we do this, the correlation length of the original theory with very small g is proportional to $e^{l^*(g)}$.

Therefore, as a first application of the renormalization group flow equation derived in the previous lecture, let us compute $l^*(g)$ now. Setting $D = d + 1 = 2$, the flow equation becomes

$$\frac{dg}{dl} = (N-2)S_2 g^2 \ , \tag{1}$$

where $S_2 = \frac{1}{2\pi}$. Integrating this from $l = 0$ to $l = l^*$, with initial condition $g(l = 0) = g$ gives

$$\frac{1}{g} - \frac{1}{g(l^*)} = \frac{N-2}{2\pi} l^* \ . \tag{2}$$

For e^{l^*} this gives

$$e^{l^*} = e^{-\frac{2\pi}{(N-2)g(l^*)}} \times e^{\frac{2\pi}{(N-2)g}} \ . \tag{3}$$

The prefactor in the above is clearly some order one constant, and the physical correlation length is therefore

$$\xi(g \to 0) = \xi_0 e^{\frac{2\pi}{(N-2)g}} \ , \tag{4}$$

where ξ_0 is a non-universal order one prefactor that cannot be predicted by the foregoing analysis.

Thus, no matter how small one makes g, the correlation length remains large but finite and the system is a quantum paramagnet at the largest length scales as long as $g > 0$, although it looks ordered on small scales. Note that the g dependence of ξ derived above contains an essential singularity, and is

not something that can be obtained by naive expansions in powers of g. It is a perfect illustration of the power of the renormalization group approach that such a non-trivial result can be obtained by a simple *perturbative* calculation of the β function to *second order* in g. (The point of course is that we are only using perturbation theory in g to compute the right-hand side of the flow equation, and perturbation theory in g is quite reliable for this purpose, although it fails spectacularly when we try and *directly compute* long-distance, low energy properties in perturbation theory.)

For the original system, this leads to the following really interesting result: Integer spin Heisenberg antiferromagnetic chains, whose low energy physics is described by the $d = 1$ quantum rotor model *without any additional Berry phase effects*, develop a finite correlation length and corresponding gap in the spectrum no matter how large an antiferromagnetic exchange interaction they possess, as long as it is a short-ranged interaction. Going back to our derivation of the effective rotor model for a one-dimensional chain with a nearest-neighbour exchange interaction, we see that $g = \mathcal{C}S$ in $d = 1$, where $\mathcal{C} = 1$ if we take seriously all the prefactors in our derivation of the long-wavelength coupling constant for the case of the one dimensional chain geometry with nearest neighbour interactions.

Therefore, we predict from this renormalization group analysis that such a spin chain has finite correlation length

$$\xi \sim e^{2\pi \mathcal{C} S} \ , \tag{5}$$

where $\mathcal{C}$ is an order one constant which can be estimated given microscopic details of the model Hamiltonian. Note that the S dependence of this result can be tested by doing a series of experiments on different spin-chain compounds made up of magnetic ions with different S. Such experiments have been done, and the results do indeed agree quite well with this renormalization group prediction. The corresponding gap in the energy spectrum of the one dimensional integer spin Heisenberg antiferromagnet is generally known as the "Haldane gap" after F. D. M. Haldane, who first made the connection to the quantum rotor model and used it to predict the finite correlation length and non-zero gap for integer spin chains.

With this application in hand, we return to the general framework and add another ingredient: In the analysis of flows in the previous lecture, and in the application discussed in the preceeding paragraphs, we have not needed to worry too much about the precise connection between the $\tilde{\hat{n}}$ field of the

new theory (after one RG step) and the $\hat{n}$ field of the original theory. In other applications, this connection plays an important role, and it is therefore useful to keep track of it.

If we go back to our description of one step of the RG procedure in Lecture 24, it is clear that we can write the following correspondence between correlation functions of the new theory after one RG step, and correlation functions of the old theory:

$$\langle \hat{n}(x_\mu) \cdot \hat{n}(0) \rangle_{g(0)} = \zeta(\delta l) \langle \tilde{\hat{n}}(x_\mu e^{-\delta l}) \cdot \tilde{\hat{n}}(0) \rangle_{g(\delta l)} \; , \tag{6}$$

where $\zeta(\delta l)$ keeps track of the magnitude of the slowly-varying piece of $\hat{n}$ (whose direction is represented by $\tilde{\hat{n}}$):

$$\zeta(\delta l) = 1 - \langle \vec{\phi}^2 \rangle_> \; , \tag{7}$$

where the average is over the "fast-modes" $\vec{\phi}$ that are being integrated out at this step of the RG procedure, and we have expanded to quadratic order in $\vec{\phi}$ since, as we have seen earlier, this suffices to get leading order results for the right hand side of the differential equation for flows of various quantities.

Doing this average explicitly, we have for the first RG step

$$\begin{aligned} \zeta(\delta l) &= 1 - g(0)(N-1) \int_{\Lambda e^{-\delta l}}^{\Lambda} \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \\ &= 1 - g(0)(N-1) S_D \int_{\Lambda e^{-\delta l}}^{\Lambda} dk \frac{k^{D-1}}{k^2} \\ &= 1 - g(0) S D (N-1) \Lambda^{D-2} \delta l \; . \end{aligned} \tag{8}$$

When we repeat procedure at successive RG steps, and take the limit of very small δl as before, we obtain a flow equation that keeps track of $\zeta(l)$ as a function of the RG scale l:

$$\frac{d\zeta(l)}{dl} = -g(l) S_D (N-1) \Lambda^{D-2} \zeta(l) \; . \tag{9}$$

Solutions to this equation need to be supplemented by the obvious boundary condition $\zeta(l=0) = 1$.

In $D = 2$, i.e in spatial dimension $d = 1$ at $T = 0$, we may integrate this equation by using the explicit formula for $g(l)$ derived earlier:

$$\log\left(\zeta(l)\right) = -(N-1) S_2 \int_0^l dl' \frac{1}{g(0)^{-1} - (N-2) S_2 l'} \; . \tag{10}$$

Doing the integral on the right hand side for $N > 2$ gives

$$\log\left(\zeta(l)\right) \;=\; \frac{N-1}{N-2}\log\left(\frac{1}{(N-2)S_2 l+1}\right)\;. \tag{11}$$

This allows us to obtain the scale factor at $l = l^*$ at which $g(l^*) = 1$:

$$\zeta(l^*) \;=\; \left(g(0)\right)^{(N-1)/(N-2)}\;. \tag{12}$$

Clearly, this is an innocuous prefactor that does not invalidate our earlier identification of the correlation length with e^{l^*} as long as $N > 2$. The situation is quite different when $N = 2$ in $D = 2$. As we will see later in this course, this describes the $T = 0$ physics of one dimensional quantum liquids in their superfluid phase in an approximation that ignores the role of space-time vortices.

For $N = 2$ and $D = 2$, we have the flow equation

$$\frac{dg}{dl} \;=\; 0\;, \tag{13}$$

and this gives

$$\log\left(\zeta(l)\right) \;=\; -S_2 g(0) l\;. \tag{14}$$

In other words,

$$\zeta(l) \;=\; e^{-\frac{g(0)l}{2\pi}}\;. \tag{15}$$

Since $g(0) \equiv g$, the bare value of the couping constant, and since $g(l) = g(0)$, this implies

$$\langle \hat{n}(x_\mu)\cdot\hat{n}(0)\rangle_g \;=\; e^{-\frac{gl}{2\pi}}\langle \tilde{\hat{n}}(x_\mu e^{-l})\cdot\tilde{\hat{n}}(0)\rangle_g\;. \tag{16}$$

To see what this means, let choose a l such that $|x_\mu|e^{-l} = 1$. Then, this implies

$$\langle \hat{n}(x_\mu)\cdot\hat{n}(0)\rangle_g \;=\; \frac{1}{|x_\mu|^{\frac{g}{2\pi}}}\;. \tag{17}$$

This is really interesting: It says that when $N = D = 2$, the system is *critical* for all bare values of g, *i.e.* the correlators are neither long-ranged,

nor are they decaying exponentially beyond some scale ξ. Instead, they are power-law in form, with the power law exponent $\eta(g)$ depending continuously on g. Our leading order prediction for $\eta(g)$ is of course read off from the above:

$$\eta(g) \;=\; \frac{g}{2\pi} + \mathcal{O}(g^2)\;. \tag{18}$$

As will be clear by the end of this lecture, this is an extremely unusual situation. Usually, power-law correlators arise only at special *critical points* and are characteristic of a system on the verge of a phase transition from an ordered phase with long-range correlations to a disordered phase with a finite correlation length and short-ranged correlations. In these more usual instances of power-law behaviour, the power law exponent η takes on a value that depends only on the symmetries and the spatial dimensionality of the problem, and is usually independent of microscopic details such as the bare cutoff and the bare value of the coupling constants.

Another way of saying this is to say that the $d = 1$ $T = 0$ physics of the superfluid phase of quantum liquids is controlled by what is known as a "line of fixed points". Here, by a "fixed point", we mean a value of g for which $\beta(g) = 0$. And our result above tells us that the $N = 2$, $D = 2$ case has a line of fixed points since $\beta(g)$ is identically zero for any value of g, at least within our treatment that leaves out space-time vortices from our description. As we will see in the last few lectures of this course, this phase survives the destabilizing effects of vortices below some critical g_c ($g < g_c$), and for $g < g_c$, our analysis is essentially correct, although the dependence of η on the bare g is different. For $g > g_c$ vortices disorder the system leading to a disordered phase with short-range correlations and finite correlation length ξ.

Leaving a more detailed discussion of this to the last part of this lecture course, let us switch gears now and study the behaviour near the quantum critical point in this N-vector model. As we already know, there is no ordered phase even at $T = 0$ in $d = 1$—equivalently, in space-time dimension $D = 2$, there is no ordered state possible. However, for $D = 3$ and higher, we expect an ordered state with long-range correlations of the $\hat{n}$ vector below some critical value g_c ($g < g_c$) and a disordered phase with finite correlation length for $g > g_c$. The question then is: Can we use our RG approach and characterize exactly how order is lost as g approaches g_c from below, or characterize exactly how the correlation length grows and becomes infinite as g approaches g_c from above?

To answer these questions, let us return to the flow equation we have derived to $\mathcal{O}(g^2)$ in the previous lecture and use it for $D > 2$ and general $N > 2$. Our $T = 0$ flow equation reads

$$\frac{dg}{dl} = -(D-2)g + (N-2)S_D\Lambda^{D-2}g^2 \, . \tag{19}$$

From the structure of the right hand side, we see that g flows to 0 for small starting values, since the first linear term dominates over the second quadratic term. On the other hand, for large values of g, the second term dominates and drives g to still larger values. Strictly speaking, such large values of g are outside of the realm of validity of our derivation of the β-function, and we should not trust the right hand side of the flow equation when g gets much larger than $\mathcal{O}(1)$. Nevertheless, we expect that once g has renormalized to an $\mathcal{O}(1)$ value, the fate of the system is already determined, in the sense that the new Hamiltonian with this $\mathcal{O}(1)$ coupling constant has a disordered ground state with some $\mathcal{O}(1)$ correlation length (in new units).

These two regimes are separated by a *critical fixed point* $g = g_c$ at which the β-function vanishes. To understand the properties of the quantum phase transition separating ordered and disordered ground states, we need to analyze the flows in the vicinity of this critical point and use properties of these flows to deduce the behaviour of correlators.

Looking at the flow equation, it is clear that g_c is given by the equation

$$g_c = \frac{(D-2)}{(N-2)S_D\Lambda^{D-2}} \, . \tag{20}$$

However, we know that our flow equations cannot be trusted for g that is too large, and the question therefore is: How reliable is this estimate of g_c? The answer of course depends on the value of D. Indeed, if $D-2$ is small, then g_c is small, and we will be in a position to trust our analysis. This leads us to the idea that we can learn about critical properties *in an expansion in space-time dimensionality about the value* $D = 2$.

To implement this expansion, we set $D - 2 = \epsilon$, and rewrite the result for g_c to leading order in ϵ:

$$g_c = \frac{\epsilon}{(N-2)S_2} \, . \tag{21}$$

A system that starts with bare value $g(0) = g_c$ will be on the verge of a transition from ordered to disordered phases. As we have seen earlier in this

lecture, the long-distance behaviour of the correlation function at criticality is controlled, within our RG approach, by the flow of the scale-factor $\zeta(l)$. At this $\mathcal{O}(\epsilon)$ critical fixed-point, we may obtain this flow-equation by using the above $\mathcal{O}(\epsilon)$ result for g_c in our earlier general flow equation for ζ to get

$$\frac{d\zeta(l)}{dl} = -\frac{N-1}{N-2}\epsilon\zeta(l) \; . \tag{22}$$

For $g < g_c$, $g(l)$ will flow to $g = 0$ as $l \to \infty$, and this implies that the system will be ordered at the largest length-scales. On the other hand, for $g > g_c$, $g(l)$ flows to ever-increasing values of g and the simplest interpretation is that the system looks disordered at the largest length scales. If we represent the deviation of g from g_c by δ, we can linearize the flow equation around $g = g_c$, and obtain to leading order in ϵ and δ

$$\frac{d\delta}{dl} = \epsilon\delta \; . \tag{23}$$

This can be solved to give the near-critical flow of δ as

$$\delta(l) = \delta(0)e^{\epsilon l} \; . \tag{24}$$

The corresponding near-critical flow for $\zeta(l)$ then obeys the equation

$$\frac{d\zeta(l)}{dl} = -\left(\frac{\epsilon}{(N-2)S_2} + \delta(0)e^{\epsilon l}\right) S_2(N-1)\zeta(l) \; . \tag{25}$$

These equations for ζ at and near criticality, and the linearized flow around the critical point contain all the information needed to deduce properties of the system in the critical region. This is what we will turn to in the first part of the next lecture. Once we understand this, we will turn our attention to using the corresponding flow equations at non-zero temperature $T > 0$ to learn about the low-temperature properties of near-critical systems.