Lecture 24: Renormalization group approach to the breakdown of spinwave theory: Basic ideas and formalism

In the last couple of lectures, we have understood in a couple of very different ways that the $\hat{n}$ field in the $O(3)$ quantum rotor model cannot develop long range order that spontaneously breaks $O(3)$ symmetry in spatial dimension $d = 1$, no matter how low the temperature, and no matter how strong the exchange coupling between neighbouring $\hat{n}$. In the first approach, we found that low-energy long-wavelength spin-wave modes give a divergent correction to the order parameter expectation value in $d = 1$ even at $T = 0$, suggesting that the system is not ordered in the first place. While the second approach succeeded in proving a theorem ruling out long-range order.

This result leads naturally to the following interesting question: When $J_{\text{eff}}I$ is large and $T = 0$ or small, it is clear that the system prefers to align the $\hat{n}$ field at short distance and time scales. However, the results of the last two lectures tell us that the system looks disordered at the longest length scales due to the dramatic effects of very long wavelength fluctuations. How does this happen? What sets the large length scale beyond which the system "realizes" it cannot order, and ends up having exponentially decaying correlations of the $\hat{n}$ vector? These are the questions we wish to answer.

And to answer them, it is clear that a caricature of the type we used in the previous lecture to discuss the disordered phase will not suffice. There, we simply noted that if $J_{\text{eff}}I$ is very small, the kinetic energy of each rotor dominates, and each $\hat{n}$ is spread out uniformly over the corresponding unit sphere, so that even nearest neighbour rotors are uncorrelated. Further, we noted that this cartoon of the disordered state can of course be used as a starting point for a systematic expansion in which we turn on the effects of the exchange coupling J order by order in conventional perturbation theory. This results in two rotors $\sim k$ neighbours away developing some correlation at k^{th} order in $J_{\text{eff}}I$ (I urge you to check that the leading correction to the ground state wavefunction couples nearest neighbour $\hat{n}$).

This kind of strong coupling picture gives a reasonable description of the extremely disordered state with a very short correlation length, but clearly cannot be used to answer our question about the nature of the disordered state with very *large* correlation length that occurs when $J_{\text{eff}}I$ is very *large* but not infinity. In this lecture, we will use this question as our motivation to

introduce a way of thinking that goes by the name of the "renormalization group approach". As we will see in this and the next few lectures, it not only answers subtle questions of this type, but also provides an over-arching framework which one can use to discuss many diverse phenomena including classical and quantum phase transitions.

We start with the following expression for the partition function as a path integral

$$\begin{aligned} Z &= \int_{\hat{n}^2(x)=1} \mathcal{D}\hat{n}(x) e^{-S} \\ S &= \frac{1}{2g} \int_{\Lambda} d^D x \sum_{\mu} (\partial_\mu \hat{n})^2 \end{aligned} \tag{1}$$

Some comments about this formulation are necessary before we proceed, so here we go: The action is written in $D = d + 1$ dimensional "spacetime", where the "+1" refers to the imaginary time direction, and we have rescaled the time direction by the velocity c (please refer to our derivation of the quantum rotor action as the low energy theory of quantum antiferromagnets) to make all directions look isotropic—when needed, we will restore factors of c in final expressions. The coupling constant g can be readily seen to be related to $J_{\rm eff}$ and I as $g^{-1} \sim \sqrt{J_{\rm eff} I}$—we prefer this notation since it emphasizes that there is only one real coupling constant, while the other combination of $J_{\rm eff}$ and I is simply a spinwave velocity which provides us a way of converting between space and imaginary time units. We do not explicitly make a distinction between the space and the imaginary time direction as long as we are working at $T = 0$, *i.e.* concerned with the long-distance properties of the ground state correlations. Later we will have occasion to study $T > 0$ crossovers, and will then make a distinction between d infinite directions and one direction of length β (strictly speaking βc in rescaled variables, but we will drop the c except in some crucial final expressions). The subscript Λ reminds us that although we have written the action in continuum language, a momentum cutoff Λ is always present. Again, as long as we are working at $T = 0$, we use the same short-distance cutoff for all D directions—this means that the imaginary time direction is no longer strictly continuous, *i.e.* at the shortest times, the results from this formulation will not match those obtained from a Hamiltonian formulation of the quantum rotor model. However, this is not a problem at all, since the quantum rotor Hamiltonian we start with is anyway an effective Hamiltonian valid only at low-energies or

long times, whose short-time high energy properties anyway have little to do with the original quantum antiferromagnet. When we have occasion to study $T > 0$ crossovers, we will find it convenient to keep the imaginary time direction continuous and of length βc, while imposing a momentum cutoff Λ for the d spatial directions.

With all that fine-print out of the way, we proceed by noting that the difficulty with spin-wave theory arose from modes at very small momenta, which made very large divergent contributions in spite of g being very small. Therefore, what is needed is a way of treating these contributions more systematically. To this end, we decompose the $\hat{n}$ field into two pieces, one a slowly varying "background field" which is a unit vector, and the other a piece with rapid fluctuations on top of this slowly varying background

$$\hat{n}(\vec{x}) \;=\; \tilde{\hat{n}}(\vec{x})\sqrt{1-\vec{\phi}^2(\vec{x})} + \vec{\phi}(\vec{x}) \; . \tag{2}$$

Here, $\tilde{\hat{n}}$ is a unit vector like the original $\hat{n}$, while $\vec{\phi}$ has no such constraint; this implies that $\tilde{\hat{n}} \cdot \vec{\phi} = 0$ at each point $\vec{x}$.

To understand this decomposition better, note that one could more straightforwardly have written

$$\hat{n}(\vec{x}) = \vec{\phi}_< + \vec{\phi}_> \tag{3}$$

where $\vec{\phi}_<$ is made up of all the Fourier modes of $\hat{n}$ with momentum $|\vec{q}| < \Lambda - \delta\Lambda$ while $\vec{\phi}_>$ is composed of all the Fourier modes of $\hat{n}$ with $\Lambda \geq |\vec{q}| \geq \Lambda - \delta\Lambda$. However, neither are $\vec{\phi}_<$ nor $\vec{\phi}_>$ guaranteed to be unit vectors, nor are they perpendicular to each other. To remedy this, we write

$$\begin{aligned} \vec{\phi} \;&=\; \vec{\phi}_> - \frac{\vec{\phi}_> \cdot \vec{\phi}_<}{|\vec{\phi}_<|} \\ \tilde{\hat{n}} \;&=\; \frac{\vec{\phi}_<}{|\vec{\phi}_<|} \end{aligned} \tag{4}$$

and this guarantees that $\tilde{\hat{n}}$ is a slowly-varying unit vector while $\vec{\phi}$ is orthogonal to $\tilde{\hat{n}}$ and rapidly-varying.

Now the basic idea is as follows: We rewrite the partition function as a path integral over the $\tilde{\hat{n}}$ and $\vec{\phi}$ fields, and note that a small starting value of g implies that the integral over $\vec{\phi}$ can be safely and systematically approximated by the leading term in a small g expansion.

Such a small g expansion basically amounts to keeping ony quadratic terms in an expansion in $\vec{\phi}$ and is equivalent to the leading term in the spin-wave expansion we performed earlier. The difference is that in the original spin-wave expansion, we naively allowed $\vec{\phi}$ to contain all Fourier modes down to $\vec{q} \to 0$, and this caused uncontrollable divergences in the leading order spinwave expansion results (remember, our expression for the mean-square deviations from perfect ordering within spin-wave theory was proportional to $1/\sqrt{J_{\rm eff} I} \sim g$ and would have been controlled by the smallness of g were it not for the fact that it consisted of an integral that diverged as $\vec{q} \to 0$).

In our new, more careful approach, $\vec{\phi}$ only consists of Fourier modes in a thin shell in momentum space between $\Lambda - \delta\Lambda$ and Λ, and there is no longer any problem with this expansion as long as g is small enough. So the idea now is to do the path integral in steps:

First, we integrate out ϕ in an expansion that is well-controlled by the smallness of g. The result is a path integral over $\tilde{\hat{n}}$, which now has a slightly different action S' due to contributions arising from the modes we have integrated out. In order to make the book-keeping easier and more natural, we let

$$\Lambda - \delta\Lambda \equiv \Lambda e^{-\delta l} \tag{5}$$

and define

$$\tilde{x} = e^{-\delta l} x \tag{6}$$

where we have dropped the vector sign on x for notational convenience. The idea now is to write the modified action S' for $\tilde{\hat{n}}$ as an integral over $\tilde{x}$ of some functional of $\tilde{\hat{n}}(\tilde{x})$. Let us call the modified action written in this way as $\tilde{S}$. We will see below that to leading order in g, $\tilde{S}$ has the same form as S, with $\hat{n}(x)$ replaced by $\tilde{\hat{n}}(\tilde{x})$ and g replaced by $\tilde{g}$ whose value we will work out explicitly.

With this book-keeping convention, the remaining path integral over $\tilde{\hat{n}}$ again defines a theory of a unit vector field with cutoff Λ in new units of momentum, *i.e.* with $\tilde{q} \leq \Lambda$, where

$$\tilde{q} = e^{\delta l} q \, . \tag{7}$$

Therefore, we can again decompose $\tilde{\hat{n}}$ into a slowly varying background field $\tilde{\tilde{\hat{n}}}$ and a fast piece $\vec{\phi}'$. In the new units, $\vec{\phi}'$ again has Fourier modes in the

thin shell between $\Lambda e^{-\delta l}$ and Λ. This is of course merely a convenient bookkeeping choice—in original units $\vec{\phi}'$ actually has Fourier modes in the shell between $\Lambda e^{-2\delta l}$ and $\Lambda e^{-\delta l}$. Again, the integral over $\vec{\phi}'$ can be done in a controlled expansion that relies on the smallness of $\tilde{g}$, to obtain an effective action $\tilde{\tilde{S}}$ written as an integral over $\tilde{\tilde{x}}$ of some functional of $\tilde{\tilde{\hat{n}}}(\tilde{\tilde{x}})$ with cutoff for $\tilde{\tilde{q}}$ again being Λ ($\tilde{\tilde{x}} = e^{-2\delta l}x$ and $\tilde{\tilde{q}} = e^{2\delta l}q$ by our bookkeeping convention). As before $\tilde{\tilde{S}}$ has the same form as S, but with g now replaced by $\tilde{\tilde{g}}$.

These new values of g are referred to as "renormalized" values of the coupling constant. The idea is that the coupling constant g "flows" as we continue this process and reduce the cutoff, so that it takes on the value $g(l)$ in the "renormalized theory" (written in rescaled units) when the cutoff in original units has become Λe^{-l}. The function $g(l)$ is obtained as the solution to a "RG flow equation" of the form

$$\frac{dg}{dl} = \beta(g) \tag{8}$$

with initial condition $g(l = 0) = g$, the original coupling constant of our starting theory. The term on the right hand side is the so-called "beta function" of the theory (not to be confused with the inverse temperature β) and plays a central role in the renormalization group analysis of any theory.

In our example, since we are relying on a small g approximation to the β-function, we may reliably continue this process only as long as $g(l)$ remains small compared to 1. In our first application of this procedure in spatial dimension $d = 1$ at $T = 0$, we will find that $g(l)$ grows under renormalization and steadily flows to increasing values. This makes our procedure suspect once $g(l)$ reaches $g(l) \sim 1$, which happens at a value of l that depends on the original value of g, and which we denote as $l^*(g)$. However, in this new problem, we have no difficulty in understanding what happens: When $g(l^*) \sim 1$, a reasonable approximation that will give qualitatively correct results is to start with the strong-coupling caricature developed earlier for the quantum paramagnetic phase and imagine correcting this caricature order by order in $1/g(l^*)$. As explained earlier, this will convert the zero correlation length of the $g \to \infty$ caricature to some $O(1)$ length scale ξ_0, as successive terms in the expansion introduce weaker and weaker correlations between further and further away sites.

However, and this is key, this is the correlation length in new, rescaled

units. In original units, the actual correlation length is

$$\xi = e^{l^*(g)}\xi_0 \tag{9}$$

Thus, if we know l^* as a function of g, this renormalization group approach will allow us to answer our original question and give us the g dependence of the correlation length scale beyond which correlations of the original theory start looking like those of a quantum paramagnet. Further, as we will see below, this answer becomes more and more accurate as the starting value of g is made smaller and smaller—indeed, it is an asymptotically exact result for small enough g, apart from an unimportant, non-universal order one length scale ξ_0 that cannot be predicted in this way.

As you might imagine, the renormalization group idea has much wider applicability than this specific example. In the last few lectures, we will also study the effect of vortices on the superfluid density of two dimensional superfluid films using this approach, and will sketch its use in developing a theory for classical and quantum phase transitions with a diverging correlation length.

In its most general form, the idea can be stated in Hamiltonian language as follows: In order to understand the low energy physics of a problem, one should eliminate all terms in the Hamiltonian that mix states below some energy cutoff with those above this cutoff. This can be done in principle by a suitable unitary transformation of the Hamiltonian, and can be thought of as a way of putting the Hamiltonian matrix in a block-diagonal form. In doing this, the form of the Hamiltonian in the low-energy block changes. So in effect, we have lowered the energy cutoff and replaced our original Hamiltonian H by a new one H'. One can now block-diagonalize H' again into a low-energy and high-energy piece, to get a new low energy Hamiltonian H'', whose form is again different. In this way, we can systematically zoom in on the low energy physics using a description that has been specifically adapted for its study, allowing us to obtain reliable results since the successive block-diagonalization steps can be done using some reliable approximation scheme even when no such scheme is available for directly approximating the low-energy physics of the original Hamiltonian.

This idea of a flow in the space of Hamiltonians, due in large part to the work of K. G. Wilson, is a powerful philosophy that has informed many branches of theoretical physics, and the examples we study in this course represent the simplest, most tractable, and perhaps best known examples of concrete results obtained by using this approach.

With this overview in mind, we will turn in the next lecture to the task at hand, which is to expand S to quadratic order in $\vec{\phi}$ and obtain an effective action for $\tilde{\hat{n}}$ in the first step of the renormalization group procedure for the quantum rotor model, and then use this to derive an expression for the β function of the quantum rotor model.