Lecture 25: Renormalization group for the quantum rotor model: Details and the flow equation

In order to implement the renormalization group approach outlined in the last lecture, it is useful to discuss some "geometric" preliminaries first. Since the calculations involved are no harder for the general $O(N)$ model compared to the $O(3)$ case we have been discussing thus far, we choose to work in this more general setting for completeness.

We write the $\hat{n}$ field of our original action as

$$\hat{n}(\vec{x}) = \tilde{\hat{n}}(\vec{x})\sqrt{1-\vec{\phi}^2(\vec{x})} + \vec{\phi}(\vec{x}) \tag{1}$$

and let $\{\hat{e}_1(\vec{x}), \hat{e}_2(\vec{x}) \ldots \hat{e}_{N-1}(\vec{x})\}$ be a local frame of unit vectors perpendicular to $\tilde{\hat{n}}$. $\phi(\vec{x})$ can be expanded in terms of this frame of unit vectors as

$$\vec{\phi}(\vec{x}) = \sum_{a=1}^{N-1} \hat{e}_a(\vec{x})\phi_a(\vec{x}) \tag{2}$$

Further, since $\tilde{\hat{n}}$ is a unit vector, any derivative $\partial_\mu \tilde{\hat{n}}$ is perpendicular to $\tilde{\hat{n}}$, and also admits an expansion in terms of this frame. We write

$$\partial_\mu \tilde{\hat{n}} = \sum_{a=1}^{N-1} Q^a_\mu \hat{e}_a \tag{3}$$

Conversely, derivatives of any of the $\hat{e}_a$ admit an expansion in a basis that now includes $\tilde{\hat{n}}$:

$$\partial_\mu \hat{e}_a = \sum_{b=1}^{N-1} A^{ab}_\mu \hat{e}_b - Q^a_\mu \tilde{\hat{n}} \tag{4}$$

where the A are obviously antisymmetric in their upper indices and the same object Q^a_μ, makes its appearance in this equation and the equation for the derivative of $\tilde{\hat{n}}$, with the sign in front being different (this is a reflection of the fact that $\hat{e}_a$ is a unit vector orthogonal to the unit vector $\tilde{\hat{n}}$ at each point in spacetime). A^{ab}_μ and Q^a_μ are geometric properties of the way different frames are connected to each other at neighbouring points in spacetime. Their specific values depend on the choice of frame perpendicular to $\tilde{\hat{n}}$ at each point in spacetime. Although the field $\tilde{\hat{n}}$ is given to us, the frame of unit vectors in the hyperplane perpendicular to $\tilde{\hat{n}}$ is ours to choose, and this freedom leads to a kind of "gauge-invariance" principle that we will be able to exploit to drastically simplify our calculations.

To understand this gauge-invariance principle, we begin by noting that any *locally chosen* (spatially varying) $N-1$ dimensional rotation matrix that rotates in the $N-1$ directions perpendicular to $\tilde{\hat{n}}$ will give us a new and equally valid $N-1$ dimensional frame. Denote this rotation matrix by $\mathbf{R}$. Clearly, both ϕ^a and $\hat{e}_a$ transform as a column vector under $\mathbf{R}$. In other words

$$
\begin{aligned}
\phi_a &\rightarrow \sum_{b=1}^{N-1} R^{ab}\phi_b \\
\hat{e}_a &\rightarrow \sum_{b=1}^{N-1} R^{ab}\hat{e}_b
\end{aligned}
\tag{5}
$$

and of course, this leaves the physical $\vec{\phi} = \sum_{a=1}^{N-1} \phi_a \hat{e}_a$ unchanged. Further, it is easy to check from their definitions that Q transforms quite simply as a column vector, while A has a much more complex transformation law:

$$
\begin{aligned}
Q_\mu^a &\rightarrow \sum_{b=1}^{N-1} R^{ab} Q_\mu^b \\
A_\mu^{ab} &\rightarrow [(\partial_\mu \mathbf{R})\mathbf{R}^T + \mathbf{R}\mathbf{A}_\mu \mathbf{R}^T]^{ab}
\end{aligned}
\tag{6}
$$

With this in mind, we ask the following question: When we expand S to second order in $\vec{\phi}$ and do the $\vec{\phi}$ path integral to obtain an effective action for $\tilde{\hat{n}}$, what are the possible terms in this effective action? The answer of course is that all "gauge-invariant" objects that respect translation and rotation symmetry are possible. From Q, clearly the simplest term one can make is

$$
\sum_{a=1}^{N-1} \sum_{\mu=1}^{D} (Q_\mu^a)^2 \ . \tag{7}
$$

If we rewrite it in more familiar language, this is nothing but

$$
\sum_{\mu=1}^{D} (\partial_\mu \tilde{\hat{n}})^2 \tag{8}
$$

But what can we construct from the A? Instead of answering this up front, let us instead ask: Can we exploit the gauge-invariance and set A to zero for a given background field configuration $\tilde{\hat{n}}$? In other words, can we find a R such that

$$(\partial_\mu \mathbf{R})\mathbf{R}^T + \mathbf{R}\mathbf{A}_\mu \mathbf{R}^T \;=\; 0 \;\forall \vec{x} \;? \tag{9}$$

This translates to looking for solutions of the equation

$$\partial_\mu \mathbf{R} \;=\; -\mathbf{R}\mathbf{A}_\mu \;. \tag{10}$$

Now, since

$$\partial_\mu \partial_\nu \mathbf{R} \;=\; \partial_\nu \partial_\mu \mathbf{R} \;, \tag{11}$$

a necessary condition for finding such a $\mathbf{R}$ clearly is

$$\partial_\mu(\mathbf{R}\mathbf{A}_\nu) - \partial_\nu(\mathbf{R}\mathbf{A}_\mu) \;=\; 0 \;. \tag{12}$$

This translates to the requirement

$$\partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu] \;=\; 0 \;. \tag{13}$$

The quantity on the left hand side of the above equation can be thought of as a (non-abelian) "magnetic field" $\mathbf{B}_{\mu\nu}$, defined as

$$\mathbf{B}_{\mu\nu} \;=\; \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - [\mathbf{A}_\mu, \mathbf{A}_\nu] \;, \tag{14}$$

and this condition then translates to the requirement that $\mathbf{B}_{\mu\nu}$ be zero.

Conversely, if this magnetic field is not zero, then one can construct gauge and rotationally invariant terms out of this field, and these terms can then appear in our expression for the effective action for $\tilde{\hat{n}}$. To construct such terms, we note that $\mathbf{B}_{\mu\nu}$ transforms as a tensor under $\mathbf{R}$:

$$\mathbf{B}_{\mu\nu} \;\to\; \mathbf{R}\mathbf{B}_{\mu\nu}\mathbf{R}^T \;. \tag{15}$$

The simplest rotationally and gauge invariant quantity one can make out of this is therefore

$$\sum_{\mu\nu} \mathrm{Tr}(\mathbf{B}^T_{\mu\nu}\mathbf{B}_{\mu\nu}) \tag{16}$$

Now since $\mathbf{B}_{\mu\nu}$ itself involves two derivatives, this quantity is a term with four derivatives.

After that digression, let us go back to the task at hand, which is integrating out $\vec{\phi}$ to leading order in g and obtaining an effective action for $\tilde{\hat{n}}$. We begin by writing $\vec{\phi}$ in terms of components along the frame $\hat{e}_a$ and expanding the square-root to quadratic order to obtain an expression in terms of the fields Q^a_μ and A^{ab}_μ. Next, we note that the smallness of g implies that terms with additional derivatives and additional powers of various fields, are both going to be strongly suppressed. Therefore, we use our analysis above to leave out all terms involving A at this stage itself—the logic of course is that these terms can only give rise to invariants composed of the magnetic field $\mathbf{B}$, and the lowest order such invariant that we have identified above has four derivatives and will not matter for our leading order calculations at small g. This leaves us with:

$$\begin{aligned}\sum_{\mu=1}^{D}(\partial_\mu \hat{n})^2 &= \sum_{\mu=1}^{D}(\sum_{a=1}^{N-1}(\partial_\mu \phi_a)^2 + \\ &+ \sum_{a,b=1}^{N-1}(Q^a_\mu Q^b_\mu - (\sum_{c=1}^{N-1}(Q^c_\mu)^2)\delta_{ab})\phi_a\phi_b + \\ &+2\sum_{a=1}^{N-1} Q^a_\mu \partial_\mu \phi_a + \sum_{a=1}^{N-1}(Q^a_\mu)^2) \end{aligned} \tag{17}$$

Since

$$\sum_{a=1}^{N-1}(Q^a_\mu)^2 = (\partial_\mu \tilde{\hat{n}})^2 \ , \tag{18}$$

we may now write

$$e^{-S_{\text{eff}}[\tilde{\hat{n}}]} = e^{-\frac{1}{2g}\int_{\Lambda e^{-\delta l}} d^D x \sum_{\mu=1}^{D}(\partial_\mu \tilde{\hat{n}})^2} \int_{(\Lambda e^{-\delta l},\Lambda)} \mathcal{D}\vec{\phi} e^{-S_1[\vec{\phi},Q]} \tag{19}$$

where the subscript in the spatial integral reminds us that $\tilde{\hat{n}}$ is slowly varying, and the subscript on the path integral reminds us that $\vec{\phi}$ only has Fourier content in a thin shell below the cutoff Λ. Since the Q have Fourier content only at low momenta $|\vec{q}| < \Lambda e^{-\delta l}$, one can do this path integral over $\vec{\phi}$ in an approximation that treats the factors of Q as being *constants*. This is because corrections that go beyond this approximation will involve derivatives of Q, and will therefore be subleading terms in our small g expansion. Likewise, we can ignore the bilinear term that couples Q to a derivative of ϕ, since this will be subdominant at small g due to the additional explicit derivative. With these approximations, we can write the following expression for S_1 in q space

$$S_1 = \frac{1}{2g}\sum_{\mu=1}^{D}\int_{\Lambda e^{-\delta l}}^{\Lambda}\frac{d^Dq}{(2\pi)^D}\phi_a((q_\mu^2 - \sum_c \overline{(Q_\mu^c)^2})\delta_{ab} + \overline{(Q_\mu^a Q_\mu^b)})\phi_b \tag{20}$$

where the overline on the expressions involving Q reminds us that we need to average these expressions over all space since they are being treated as being independent of $\vec{x}$ as far as doing the $\vec{\phi}$ path integral is concerned.

We may now use the results on Gaussian integrals derived earlier in the context of coherent state path integrals to do the $\vec{\phi}$ path integral to obtain S_{eff}. We obtain

$$S_{\text{eff}} = \frac{1}{2}\text{Tr}[\log(\mathbf{P})] + \frac{1}{2g}\int_{\Lambda e^{-\delta l}} d^Dx \sum_{\mu=1}^{D}(\partial_\mu \tilde{\hat{n}})^2 \tag{21}$$

where the operator $\mathbf{P}$ is diagonal in q space but has non-trivial structure in the $N-1$ dimensional space of transverse components of the $O(N)$ model:

$$P_{ab}(\vec{q}) = \frac{1}{g}\sum_{\mu=1}^{D}\left(\delta_{ab}(q_\mu^2 - -\sum_{c=1}^{N-1}\overline{(Q_\mu^c)^2}) + \overline{(Q_\mu^a Q_\mu^b)}\right) \tag{22}$$

We now expand $\mathbf{P}$ to quadratic order in the Q and then take the trace, throwing out constant (Q independent terms) that only affect the overall normalization of the path integral which is anyway ill-defined:

$$S_{\text{eff}} = \frac{1}{2g}\int_{\Lambda e^{-\delta l}} d^Dx \sum_{\mu=1}^{D}(\partial_\mu \tilde{\hat{n}})^2 + \frac{(1-(N-1))}{2}\text{Tr}_{\vec{q}}\sum_{c=1}^{N-1}\sum_{\mu=1}^{D}\overline{(Q_\mu^c)^2}\ , \tag{23}$$

where we have carried out the trace in the $N-1$ dimensional space of transverse components but left explict the trace over the D dimensional $\vec{q}$.

At this step, we have a choice: We could either work, as outlined in the previous lecture, at $T = 0$ and treat all $D = d + 1$ directions in euclidean spacetime as equivalent, imposing the same cutoff Λ isotropically in D dimensional spacetime, or we could work at $T > 0$, and allow all possible imaginary frequencies $\omega_n = 2\pi nT$ without any cutoff, but use a cutoff Λ in the d spatial directions. This only changes what we mean by $\mathrm{Tr}_{\vec{q}}$ in the above formula. In the $T = 0$ formulation with isotropic cutoff in D dimensions, we must use

$$\begin{aligned} \mathrm{Tr}_{\vec{q}} &= \int_{\Lambda e^{-\delta l}}^{\Lambda} \frac{d^D q}{(2\pi)^D} \\ &= S_D \int_{\Lambda e^{-\delta l}}^{\Lambda} dq q^{D-1} \end{aligned} \tag{24}$$

where by S_D we mean the area of the $D-1$ dimensional unit sphere in D dimensions, written in units of $(2\pi)^D$. While in the $T > 0$ formulation with strictly continuous imaginary time, we should use

$$\mathrm{Tr}_{\vec{q}} = \sum_{\omega_n=-\infty}^{\infty} \int_{\Lambda e^{-\delta l}}^{\Lambda} \frac{d^d q}{(2\pi)^d} \tag{25}$$

Now, taking the trace is trivial in the limit of small δl, since

$$\overline{(Q_\mu^c)^2} \equiv \int_{\Lambda e^{-\delta l}} d^D x (Q_\mu^c)^2 \tag{26}$$

has no $\vec{q}$ dependence. Using the $T = 0$ formulation appropriate for answering questions about the ground state properties, we therefore obtain

$$\begin{aligned} S_{\mathrm{eff}} &= \frac{1}{2g'} \int_{\Lambda e^{-\delta l}} d^d x \sum_{\mu=1}^{D} (\partial_\mu \tilde{\hat{n}})^2 \\ \frac{1}{g'} &= \frac{1}{g} - (N-2) S_D \Lambda^{D-2} \delta l \end{aligned} \tag{27}$$

Now, we must of course change units and rewrite this effective action in terms of an integral over $\tilde{x} = e^{-\delta l} x$, to finally obtain

$$\begin{aligned} \tilde{S}[\tilde{\hat{n}}] &= \frac{1}{2\tilde{g}} \int_{\Lambda} d^d \tilde{x} \sum_{\mu=1}^{D} (\frac{\partial \tilde{\hat{n}}}{\partial \tilde{x}_\mu})^2 \\ \frac{1}{\tilde{g}} &= \frac{1}{g}(1 + (D-2)\delta l) - (N-2) S_D \Lambda^{D-2} \delta l \end{aligned} \tag{28}$$

to leading order in δl

Taking the limit of $\delta l \to 0$ and iterating this procedure, we obtain the promised flow equation for $g(l)$:

$$\frac{dg}{dl} = -(D-2)g + (N-2)S_D \Lambda^{D-2} g^2 \qquad (29)$$

In the next two lectures, we will use this and its $T > 0$ analog to answer some of the questions we have raised in the last two lectures.