Lecture 31: Kosterlitz Thouless theory

As we have seen in the last lecture, the measured value of the superfluid stiffness ρ_s^{measured} can be expressed as

$$\rho_s^{\text{measured}} = \rho_s - \frac{\rho_s^2}{T}(2\pi)^2 C_1 \ . \tag{1}$$

Here, C_1 is defined by the relation

$$\langle m(\vec{q}) m(-\vec{q}) \rangle_{0s} = C_1 \vec{q}^2 + \cdots \ , \tag{2}$$

where the expectation value is computed in the "Coulomb-gas" of vortices, with partition function

$$Z_{0s} = \sum_{\{m_i\}} e^{-\frac{1}{T}\left(\pi\rho_s\left(\sum_i m_i\right)^2 \log\left(\frac{R}{a}\right) - \pi\rho_s \sum_{i\neq j} m_i m_j \log\frac{|x_i - x_j|}{a} + \pi\rho_s\epsilon_c \sum_i m_i^2\right)} \tag{3}$$

To use this result, let us now try and calculate C_1 to leading order in the vortex fugacity parameter $y = \exp(-\pi\rho_s\epsilon_c/T)$ (note that although the bare value of y is related to the bare value of ρ_s, this relationship will not be preserved under coarse-graining, so y should be thought of as an independent variable). This is conveniently done by starting with

$$\langle m(\vec{q}) m(-\vec{q}) \rangle_{0s} = \frac{1}{L_x L_y} \int d^2r d^2r' e^{-iq(\vec{r}-\vec{r}')} \langle m(\vec{r}) m(\vec{r}') \rangle_{0s} \tag{4}$$

and taking two derivatives with respect to $\vec{q}$ to obtain

$$\nabla_q^2 \langle m(\vec{q}) m(-\vec{q}) \rangle = -\frac{1}{L_x L_y} \int d^2r d^2r' e^{-iq(\vec{r}-\vec{r}')} \langle m(\vec{r}) m(\vec{r}')(\vec{r}-\vec{r}')^2 \rangle_{0s} \tag{5}$$

Therefore, we may write

$$\begin{aligned} C_1 &= -\frac{1}{2L_x L_y} \int d^2r d^2r' e^{-iq(\vec{r}-\vec{r}')} \langle m(\vec{r}) m(\vec{r}')(\vec{r}-\vec{r}')^2 \rangle_{0s} \\ &= -\frac{1}{2L_x L_y a^4} \sum_{ij} \langle m_i |\vec{r}_i - \vec{r}_j|^2 m_j \rangle_{0s} \ , \end{aligned} \tag{6}$$

where in the last line we have reverted to the discrete language with vortices of vorticity m_i at positions $\vec{r}_i$ rather than a continuous vortex-density field $m(\vec{r})$.

The first non-zero contribution is a single vortex-antivortex pair at two points $\vec{x}_1$ and $\vec{x}_2$:

$$\begin{aligned} C_1 &= +\frac{1}{L_x L_y a^4} y^2 \int d^2x_1 d^2x_2 |\vec{x}_1 - \vec{x}_2|^2 e^{-\frac{2\pi\rho_s}{T}\log\frac{|x_1 - x_2|}{a}} \\ &= \frac{y^2}{a^2} \int d^2r r^2 \frac{1}{(r/a)^{2\pi\rho_s/T}} \end{aligned} \tag{7}$$

Therefore, we obtain to leading order in y the equation:

$$\rho_s^{\text{measured}} = \rho_s - \frac{\rho_s^2}{T}(2\pi)^2 y^2 \int \frac{d^2r}{a^2} r^2 \frac{1}{(r/a)^{2\pi\rho_s/T}} \ , \tag{8}$$

Defining

$$\frac{\rho_s}{T} = \frac{1}{g} \ , \tag{9}$$

we can rewrite this to the same leading order in y as

$$\frac{1}{g^{\text{measured}}} = \frac{1}{g} - \left(\frac{1}{g}\right)^2 8\pi^3 y^2 a^2 \int_1^{R/a} dr r^{3-\frac{2\pi}{g}} \tag{10}$$

where we have switched to a dimensionless coordinate r.

Further analysis splits naturally into two cases: If the integral converges in the large R/a limit, we expect to have only a finite renormalization of the superfluid stiffness, and the system remains a superfluid at the longest distances, corresponding to a $\rho_s^{\text{measured}} > 0$. On the other hand, if the integral diverges, it strongly suggests that our perturbative treatment in powers of y was inadequate, and that superfluidity is destroyed due to the proliferation of defects, *i.e.* $\rho_s^{\text{measured}} = 0$ in actual fact.

To put the latter conclusion on a firmer footing, we note that we could follow a more careful approach instead of trying to perturbatively treat all vortex-effects in one calculation: Instead of doing this, one could incorporate the effects of vortex-pairs with separations less than some distance-scale $ae^{\delta l}$ as a first step, where δl is a small dimensionless quantity. This would amount to doing the same calculation as above, but with the upper-limit

R/a replaced by $e^{\delta l}$. This would define a "renormalized" or scale-dependent $g(\delta l)$ as follows:

$$\begin{aligned}\frac{1}{g(\delta l)} &= \frac{1}{g(0)} - \left(\frac{1}{g(0)}\right)^2 8\pi^3 y^2(0) a^2 \int_1^{e^{\delta l}} dx x^{3-2\pi/g(0)} \\ &= \frac{1}{g(0)} - \left(\frac{1}{g(0)}\right)^2 8\pi^3 a^2 y^2(0)\delta l \ . \end{aligned} \tag{11}$$

Having done this, we are left with a system with coupling $g(\delta l)$ in which the minimum dimensionless separation between vortices is $e^{\delta l}$. We may now repeat this procedure to obtain

$$\begin{aligned}\frac{1}{g(2\delta l)} &= \frac{1}{g(\delta l)} - \left(\frac{1}{g(\delta l)}\right)^2 8\pi^3 y^2(0) a^2 \int_{e^{\delta l}}^{e^{2\delta l}} dx x^{3-2\pi/g(\delta l)} \\ &= \frac{1}{g(\delta l)} - \left(\frac{1}{g(\delta l)}\right)^2 8\pi^3 a^2 y^2(\delta l) \int_1^{e^{\delta \ell}} d\tilde{x}\tilde{x}^{3-2\pi/g(\delta l)} \end{aligned} \tag{12}$$

where

$$\begin{aligned}\tilde{x} &= x e^{-\delta l} \\ y^2(\delta l) &= y^2(0) e^{\left(4-\frac{2\pi}{g(\delta l)}\right)\delta l}\end{aligned}$$

This last transformation in effect restores the minimum length in the problem to a again, at the expense of a renormalized value of y^2. Clearly this can be repeated systematically. The result is a flow of renormalized parameters $g(l)$ and $y(l)$, with $g^{\text{measured}} \equiv T/\rho_s^{\text{measured}}$ given in this renormalization group language as $g(l \to \infty)$. The corresponding flow equations are readily seen to be:

$$\begin{aligned}\frac{dg}{dl} &= 8\pi^3 a^2 y^2(l) \\ \frac{dy}{dl} &= \left(2 - \frac{\pi}{g(l)}\right) y(l) \ , \end{aligned} \tag{13}$$

where the coefficient of y^2 on the right-hand side of the first equation is obviously non-universal and depends on the precise lattice regularization used.

These equations are the well-known Kosterlitz Thouless RG equations which control the long-distance physics of the O(2) rotor model in space-time dimension $D = 2$. They place on a firmer footing our earlier intuitive idea that there is a well-defined superfluid-insulator transition driven by the proliferation of vortices. To see this, let us study fixed points of these equations. Clearly, $y = 0$ is a fixed-line for any value of g. This simply says that the system with no vortices has power-law correlations controlled by the bare value of g (as we have seen earlier). The important question then is the *stability* of this fixed-line.

From the second equation, we readily see that an infinitesimal perturbation in y is a relevant perturbation when $g > \pi/2$. This means that vorticity dominates at the longest length-scales, driving the system far away from the regime of validity of our perturbative analysis in y. On the other hand, when $g < \pi/2$, a small starting value of y renormalizes to 0 as l increases. This means that vorticity plays no role in the long-distance physics at the longest length-scales In other words, there is a phase transition from a superfluid to an insulator when $g(l \to \infty)$ increases to beyond $\pi/2$. Since $g(l \to \infty) \equiv T/\rho_s^{\text{measured}}$, we have demonstrated that

$$\frac{\rho_s^{\text{measured}}(T_c)}{T_c} = \frac{2}{\pi} \, . \tag{14}$$

It is possible to study behaviour near the transition in some more detail by linearizing in deviations from this critical fixed point. The analysis is quite straightforward and is left as an exercise for you to carry out. By doing this linearized analysis, you should be able to show that the correlation length of the system has a very strong exponential divergence as one approaches the transition from above T_c. In addition, you should be able to readily see that the critical system (exactly at T_c) has a very peculiar slow transient in the effective value of g:

$$g(l) \;=\; \frac{\pi}{2} - \frac{1}{\text{const.} + \frac{4l}{\pi}} \tag{15}$$

This has an interesting consequence for correlation functions. To see this, we need to first state without proof the precise connection between the long-distance correlations and the renormalized value of g. This was hinted at earlier but never established. Since the end-result is quite intuitive, we state this without proof [a detailed derivation can be found in any of the references given at the begining of the course]. Let $C(r) = \langle \hat{n}(\vec{r}) \cdot \hat{n}(0) \rangle$. Then, it can be shown that

$$-\log(C(r)) \;=\; \int \frac{d^2q}{(2\pi)^2} \frac{(1 - e^{i\vec{q}\cdot\vec{r}})}{q^2} g\left(\log(\Lambda/q)\right) \tag{16}$$

In the superfluid phase $g(l)$ approaches $g(\infty)$ exponentially, and the long-distance correlators are simply power-law in form, with power $g(\infty)/2\pi$ as expected:

$$\langle \hat{n}(x)\hat{n}(0)\rangle \sim \frac{1}{|x|^{\frac{g(\infty)}{2\pi}}}.$$

However, when there is a slow transient, as is the case at criticality, one obtains *logarithmic corrections* to the power law form, which can be computed using the procedure outlined above.

This concludes our discussion of Kosterlitz-Thouless theory, and our course of lectures as well. There are many more applications of these ideas, and I hope you will be motivated to learn about some of these from the references listed at the outset.