Lecture 30: Statistical mechanics of vortices—consequences for superfluid density

In the last lecture, we have seen that the contribution to the energy of the xy model coming from vortices can be rewritten as the energy of a plasma of charges interacting with a two-dimensional (logarithmic) analog of the Coulomb potential between charges in the real three-dimensional world. As in real plasmas, opposite charges attract and charges of like sign repel each other in this plasma.

This contribution to the energy adds on to another independent contribution that can roughly be thought of as the contribution of spin-wave modes which have no vorticity, being small-amplitude waves around an ordered state. We have already seen in the previous lectures that these small-amplitude long-wavelength oscillations about an ordered state are responsible for destroying true long-range order at any non-zero temperature in a two dimensional system. However, in the absence of vortices, the resulting quasi-long-range ordered phase with power-law spin correlations is stable at all temperatures.

Nevertheless, superfluids and $O(2)$ spin systems do undergo a finite temperature transition in two dimensions to a disordered or paramagnetic phase above a critical temperature T_c. This transition is a consequence of the proliferation of vortices, and can be understood in terms of the statistical mechanics of this "Coulomb gas" of vortices. We will devote these last two lectures of our course to this topic, which will be the most difficult part of this whole course, and one of its highlights.

Given that difficult terrain lies ahead, it is good to start with an overview of what we expect before diving into details. To this end, we ask: When does it pay to produce a single vortex in an otherwise vortex-free system? This single vortex can be placed anywhere in the two dimensional sample, and therefore there is a relative entropy of

$$\Delta S = 2\log(R/a) \tag{1}$$

associated with such configurations (as compared with the entropy of configurations without any vortex). However, all such configurations are suppressed relative to vortex-free configurations by the additional energy cost of producing a single vortex

$$\Delta E = \pi\rho_s \log(R/a)\ . \tag{2}$$

Here, as before, R is the linear-scale corresponding to the size of the sample. Therefore, the free-energy cost of producing a vortex in a vortex-free sample is

$$\Delta F = \pi\rho_s - 2T\log(R/a) \ , \tag{3}$$

in units with $k_B = 1$. This suggests that for $T < \pi\rho_s/2$, the system prefers to remain vortex-free, with power-law correlations and quasi-long-range order. Whereas for $T > \pi\rho_s/2$, the system prefers to produce vortices and destroy the quasi-long-range order to go into a high-temperature paramagnetic phase. In this picture, the transition from quasi-long-range order to paramagnetic behaviour is basically driven by a "proliferation" of "free" vortices.

In the remainder of this lecture, and in the last lecture of this course, we will go through a more detailed renormalization group analysis that will confirm this basic conclusion, with one caveat: The value of ρ_s that determines the transition temperature will *not* be the "bare" or input value of ρ_s that we start with when we write down the coarse-grained action, but the "fully-renormalized" long-distance value that would actually be measured in an experiment designed to probe the superfluid response of the system at the largest scales. In other words, we will finally conclude that

$$\frac{\rho_s^{\text{measured}}(T_c)}{T_c} \;=\; \frac{2}{\pi} \tag{4}$$

in the units we are using.

Since this is a key and oft-quoted result, it is important to understand the operational definition of ρ_s^{measured}, at least in terms of a theorist's caricature of the relelvant experimental set-up. The idea is simple: Consider a very long pipe in the x direction with finite but large lateral extent L_y and $L_x \gg L_y$ [eventually, we will be taking the thermodynamic limit by first sending L_x to infinity and then taking L_y to infinity]. Let there be a phase difference of $\Delta\phi$ between $x = 0$ and $x = L_x$. As we have seen from the equations of motion for a superfluid, this will drive a super-current which we can write as

$$j \;=\; \rho_s^{\text{measured}}\Delta\phi/L_x \ . \tag{5}$$

This supercurrent will lead to a free-energy cost, corresponding to the kinetic energy of the moving superfluid

$$\Delta F \;=\; \frac{1}{2\rho_s^{\text{measured}}}\int d^2x\vec{j}^2 \ , \tag{6}$$

which is precisely the form of the coarse-grained free-energy that we have written down earlier, except that we have now replaced the bare ρ_s by the "measured" value $\rho_s^{\rm measured}$. In effect, this *defines* $\rho_s^{\rm measured}$.

In the specific configuration described above, the right hand side simplifies to give

$$\Delta F(L_x, L_y, \Delta\phi) \;=\; \frac{1}{2\rho_s^{\rm measured}} \frac{L_y(\Delta\phi)^2}{L_x} \; . \tag{7}$$

This gives us our *definition* of $\rho_s^{\rm measured}$:

$$\rho_s^{\rm measured} = \lim_{L_y\to\infty} \lim_{L_x\to\infty} \lim_{\Delta\phi\to 0} \frac{2L_x \Delta F(L_x, L_y, \Delta\phi)}{L_y(\Delta\phi)^2} \; . \tag{8}$$

Why do we insist that $\rho_s^{\rm measured}$ defined in this way is the physical, measured value? To see that this is the case, we first note that $\rho_s^{\rm measured}$ as defined above is the density of the superfluid component that moves at a steady velocity

$$v_s \;=\; \Delta\phi/L_x \tag{9}$$

in the x-direction relative to the valls of the pipe, while the normal component of the fluid remains at rest with respect to the walls of the pipe. Now, this situation is related by Gallilean transformation to another configuration in which we move the pipe at a steady speed v_s parallel to its axis. In this latter configuration, the normal component of the fluid will move with the pipe (*i.e.* remain at rest with respect to the walls of the pipe), while the superfluid density $\rho_s^{\rm measured}$ remains at rest although the pipe is moving (*i.e.* move with speed v_s along the axis of the pipe with respect to the walls of the pipe). Therefore, $\rho_s^{\rm measured}$ is, in this latter configuration, the density of fluid that is "left behind" when the pipe is made to move at a steady speed.

This second interpretation of $\rho_s^{\rm measured}$ allows us to make contact with a standard measurement protocol for $\rho_s^{\rm measured}$, which goes by the name of "Andronikasvilli Torsion Pendulum" measurement. The basic idea of this measurement can be understood by imagining that we take our long pipe and close it on itself to create a hollow ring which can undergo periodic angular oscillations in some configuration in which the center is kept fixed and there is an angular restoring force due to some supports that get twisted during the angular motion. Then, the natural resonance frequency of this structure will be determined by the total moment of inertia of the apparatus, which will depend sensitively on how much of the fluid the pipe can "take along" with itself during its angular motion, and how much is "left behind". $\rho_s^{\mathrm{measured}}$ can then be obtained directly from the *reduction* in the measured moment of inertia below T_c. [Of course, the real measurement apparatus does not use precisely the geometry we describe above, but something equivalent and experimentally more tractable, but that is beyond the scope of our elementary discussion.]

Thus, we see that $\rho_s^{\mathrm{measured}}$ does indeed encode the result of an experiment that measures the density of the "superfluid component" of the system. In a spin-system described by the same coarse-grained action, the same quantity $\rho_s^{\mathrm{measured}}$ acquires a different operational significance, although there is no superfluid current and no torsion oscillator set-up to measure $\rho_s^{\mathrm{measured}}$. To understand this, it is useful to remember that our vortex-free theory had power-law long-distance correlations of $\hat{n}$, with the power-law exponent controlled by the bare value of ρ_s used in the coarse-grained free-energy density we started with:

$$\langle \hat{n}(\vec{x}) \cdot \hat{n}(0) \rangle \quad \sim \quad \frac{1}{r^{\frac{T}{2\pi\rho_s}}} \tag{10}$$

In our earlier analysis of rotor models (Lecture 26 Equation 17), we obtained this result from the scaling equation for the coupling constant g ($= T/\rho_s$ in our present notation) and the equation for the field-renormalization scale factor ζ, both derived without considering vortex excitations.

It is perhaps instructive to see how this emerges from a direct calculation in a vortex-free theory with partition function

$$Z \quad = \quad \int \mathcal{D}\phi \exp\left(-\frac{\rho_s}{2T}\int d^2x (\vec{\nabla}\phi)^2\right) \tag{11}$$

To see this, note that

$$\begin{aligned} \langle \hat{n}(\vec{x}) \cdot \hat{n}(0) \rangle_Z &= \langle e^{i\phi(\vec{x})} e^{-i\phi(0)} \rangle_Z \\ &= e^{-\frac{1}{2}\langle (\phi(\vec{x})-\phi(0))^2 \rangle_Z} \ . \end{aligned} \tag{12}$$

The latter is readily evaluated to give

$$\langle (\phi(\vec{x}) - \phi(0))^2 \rangle_Z \quad = \quad \frac{2T}{\rho_s}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2}\left(1 - e^{i\vec{k}\cdot\vec{x}}\right)$$

$$
\begin{aligned}
&= \frac{2}{2\pi}\frac{T}{\rho_s}\int_{R^{-1}}^{a^{-1}}\frac{dq}{q}\int_0^{2\pi}\frac{d\theta}{2\pi}\left(1-e^{ikx\cos(\theta)}\right) \\
&\sim \frac{T}{\pi\rho_s}\log(|\vec{x}|/a) \text{ as } |\vec{x}|\to\infty\ ,
\end{aligned} \tag{13}
$$

from which the claimed result follows directly.

How do vortices modify this result? The answer that will be established by the RG analysis we sketch in the remainder of this course is intuitively appealing: So long as vortices do not destroy the quasi-long-range order of the O(2) rotors $\hat{n}$, their correlation function remains a power-law at large distances, with the measured value $\rho_s^{\mathrm{measured}}$ determining the power-law exponent in place of the bare value that appears in the expression above. In other words, the RG analysis gives

$$
\langle \hat{n}(\vec{x})\cdot\hat{n}(0)\rangle \sim \frac{1}{r^{\frac{T}{2\pi\rho_s^{\mathrm{measured}}}}} \tag{14}
$$

With this long preamble in place, we are now ready to do the actual work involved in establishing these results. As warmup, we begin with the vortex-free partition function

$$
\begin{aligned}
Z_a &= \int \mathcal{D}\phi e^{-\frac{\rho_s}{2T}\int d^2x(\nabla\phi)^2} \\
\text{with} \qquad & \phi(x=L_x,y)=\phi(0,y)+\Delta\phi
\end{aligned} \tag{15}
$$

Here, the boundary condition specified on the second line enforces the twist of $\Delta\phi$ across the length L_x of the sample in the x direction. Next, we define $\tilde{\phi}(x,y) = \phi(x,y) - \frac{\Delta\phi}{L_x}x$, so that $\tilde{\phi}$ obeys conventional periodic boundary conditions: $\tilde{\phi}(L,y)=\tilde{\phi}(0,y)$. In new variables, the partition function can be expressed as

$$
\begin{aligned}
Z_a &= \int \mathcal{D}\tilde{\phi} e^{-\frac{\rho_s}{2T}\int d^2x(\nabla(\tilde{\phi}+\vec{v}_{\mathrm{ext}}))^2} \\
&= \int \mathcal{D}\tilde{\phi} e^{-\frac{\rho_s}{2T}\int d^2x\left[(\nabla\tilde{\phi})^2+\vec{v}_{\mathrm{ext}}^2\right]}
\end{aligned} \tag{16}
$$

where we have used the fact that $\vec{v}_{\mathrm{ext}} = \frac{\Delta\phi}{L}\hat{x}$ is a constant and integrated by parts to obtain the second line. Computing $F_a = -T\log(Z_a)$ from the above, we naturally find:

$$
F_a = F_{0a} + \frac{\rho_s}{2}(\delta\phi)^2 .
$$

where F_{0a} is the vortex-free free energy without the twist in the boundary conditions ($v_{\rm ext} = 0$). This simply says that our "measured" superfluid stiffness $\rho_s^{\rm measured}$ will be be the same as the "bare" superfluid stiffness ρ_s if there are no vortices.

With this in hand, we now try and incorporate vorticity into our calculations in a systematic way and see how this changes $\rho_s^{\rm measured}$.

We begin by rewriting our starting point in the language of superfluid velocities, as in the last lecture:

$$\begin{aligned} Z &= \int \mathcal{D}\hat{n} e^{-\frac{\rho_s}{2T}\int d^2x(\vec{v}+\vec{v}_{\rm ext})^2} \\ \text{with} \quad & \vec{v} = \epsilon^{ab}\hat{n}^a\nabla\hat{n}^b \\ \text{and} \quad & \vec{v}_{\rm ext} = \frac{\delta\phi}{L}\hat{x} \end{aligned} \tag{17}$$

This is merely a translation to the language of superfluid velocities of the earlier equation for the partition function with twisted boundary conditions. It allows us to separate the contribution of vortices from that of regular "spin-wave fluctuations" by splitting the superfluid velocity $\vec{v}$ into its analytical part (corresponding to spin-wave modes) and singular part (arising from vortices):

$$\vec{v} = \vec{v}_{a({\rm nalytic})} + \vec{v}_{s({\rm ingular})}. \tag{18}$$

with

$$\begin{aligned} \vec{\nabla}\cdot\vec{v}_s &= 0 , \\ \vec{\nabla}\times\vec{v}_a &= 0 , \end{aligned} \tag{19}$$

and

$$\begin{aligned} \vec{\nabla}\times\vec{v}_s &= \hat{z}2\pi\sum_i m_i\delta^2(\vec{r}-\vec{r}_i) \\ &= 2\pi m(\vec{r})\hat{z} , \end{aligned} \tag{20}$$

where m is the *vortex-density field* as in the previous lecture.

Since these conditions imply that $\int d^2x\vec{v}_a\cdot\vec{v}_{\rm ext} = 0$, the partition function with twisted boundary conditions now separates into two factors. One of them is a pure spin-wave part involving $\vec{v}_a$, and the other is a part that couples the vortex-part of the superfluid velocity to the external velocity $\vec{v}_{\text{ext}}$:

$$Z = Z_a Z_s \tag{21}$$

where

$$Z_a = Z_{0a} e^{-\frac{\rho_s}{2T}\vec{v}_{\text{ext}}^2} \tag{22}$$

and

$$Z_s = \text{Tr}(e^{-\frac{\rho_s}{2T}\int d^2x \vec{v}_s^2 + 2\vec{v}_s\cdot\vec{v}_{\text{ext}}}) \tag{23}$$

The measured value ρ_s^{measured} is now related to the coefficient of the $\mathcal{O}(\vec{v}_{\text{ext}}^2)$ term in the expansion of $F \equiv -T\log Z$ in powers of $\vec{v}_{\text{ext}}$. Expanding to second order and assembling all the pieces, we get

$$F = F_{0a} + \frac{\rho_s}{2}(\Delta\phi)^2 + F_{0s} - \frac{\rho_s^2}{2T}\frac{1}{L_x L_y}\int d^2r d^2r' \langle v_s^x(\vec{r}) v_s^x(\vec{r}')\rangle_{0s}(\Delta\phi)^2 \tag{24}$$

where $\langle \ldots \rangle_{0s}$ denotes the expectation value in an ensemble of vortex configurations with partition function

$$Z_{0s} = \text{Tr}(e^{-\frac{\rho_s}{2T}\int d^2x \vec{v}_s^2}) \tag{25}$$

and $F_{0s} = -T\log(Z_{0s})$, where

$$\vec{v}_s = \vec{\nabla} \times a\hat{z} \ , \tag{26}$$

with

$$-\nabla^2 a = 2\pi m(\vec{r}) \ , \tag{27}$$

and the trace is over different configurations of the vortex-density field $m(\vec{r})$.

Now, as noted in the previous lecture, Z_{0s} can also be written as

$$Z_{0s} = \sum_{\{m_i\}} e^{-\frac{1}{T}\left(\pi\rho_s\left(\sum_i m_i\right)^2 \log\left(\frac{R}{a}\right) - \pi\rho_s \sum_{i\neq j} m_i m_j \log\frac{|x_i - x_j|}{a} + \pi\rho_s\epsilon_c \sum_i m_i^2\right)} \tag{28}$$

Further, we have $\vec{\nabla} \cdot \vec{v}_s = 0$ and $\vec{\nabla} \times \vec{v}_s = 2\pi m(\vec{r})$, which allow us to write

$$v_x(\vec{q}) \quad = \quad \frac{-2\pi q_y m(\vec{q})}{i(\vec{q}^2)} \tag{29}$$

for the Fourier transform of $v_x(\vec{r})$.

Using this, we can re-express the required correlator

$$\frac{1}{L_x L_y} \int d^2 r d^2 r' \langle v_s^x(\vec{r}) v_s^x(\vec{r}') \rangle_{0s}$$

entirely in terms of a correlation function involving the vortex-density field $m(\vec{r})$ by writing

$$\frac{1}{L_x L_y} \int d^2 r d^2 r' \langle v_s^x(\vec{r}) v_s^x(\vec{r}') \rangle_{0s} \quad = \quad (2\pi)^2 \lim_{q_y \to 0} \lim_{q_x \to 0} \frac{q_y^2}{\vec{q}^4} \langle m(\vec{q}) m(-\vec{q}) \rangle_{0s} \ , \tag{30}$$

where we have used the fact that L_y is sent to ∞ after L_x has been sent to ∞ to carefully specify the limits in $\vec{q}$ space.

Finally, we note that

$$m(\vec{q} = 0) \quad \propto \quad 0 \tag{31}$$

in the thermodynamic limit, since the plasma is forced to obey global charge-neutrality due to the presence of a divergent energy-cost for configurations with net charge in the thermodynamic limit. Therefore, we expect

$$\langle m(\vec{q}) m(-\vec{q}) \rangle_{0s} = C_1 \vec{q}^2 + \cdots \tag{32}$$

in the small q limit.

Putting all this together, we obtain

$$\rho_s^{\text{measured}} = \rho_s - \frac{\rho_s^2}{T} (2\pi)^2 C_1 \tag{33}$$

Our goal, therefore, is to calculate C_1. We will attempt to do this in the next lecture, and this will lead us naturally to the idea of a scale-dependent stiffness ρ_s and renormalization group flow equations.