# Lecture 28: Low energy rotor description of the superfluid state and transition to insulating behaviour

In Lectures 19 and 20, we have seen how to describe a system of bosons in a lattice potential, with integer density per lattice-site in terms of an effective field theory for a complex field $\psi$. Long-range order for correlations of this complex field $\psi$ correspond to the superfluid phase of this system of bosons, while short ranged correlations correspond to the Mott insulating phase which is more naturally described in terms of a decoupled product wavefunction diagonal in the number (Fock) basis.

In this lecture, we will develop a coarse-grained low-energy description of the superfluid state, which will also allow us to discuss the long-wavelength properties of the transition at which superfluidity is lost. To this end, we first note that our Hamiltonian $\mathcal{H}$ (discussed in Lecture 19) conserves the total number of particles

$$N_{tot} = \sum_j N_j \tag{1}$$

(in that lecture, we used $n_j$ to denote the particle number operator at site $j$, while here we use $N_j$ since we wish to reserve $n_i$ for a different quantity). The corresponding global symmetry is the invariance of the Hamiltonian $\mathcal{H}$ under global phase rotations

$$c_j \;\; \rightarrow \;\; e^{i\theta} c_j \tag{2}$$

This corresponds, in the effective field theory, to a global (space-independent) rotation of the complex field $\psi(x)$ by a constant phase-factor $e^{i\theta}$.

Now, if $\bar{n}$, the average number of particles per site, is quite large, then the dynamical fluctuations in $N_j$ can be treated approximately without worrying about the fact that there is a lower bound

$$N_j \;\; \geq \;\; 0 \tag{3}$$

that must be satisfied by $N_j$ (here I am intentionally slurring over the distinction between the operator $N_j$ and its eigenvalue to avoid cluttering the notation). Henceforth we assume $\bar{n}$ is an integer. With this assumption, we may define a new operator

$$n_j = N_j - \bar{n} \; , \tag{4}$$

We can approximate the spectrum of this operator by saying $n_j$ can take on all values 0, $\pm 1$, $\pm 2$, $\cdots$, $\pm\infty$. In other words, $n_j$ can be thought of as the angular momentum of a planar rotor (or a unit-mass particle on a circle of radius $2\pi$). Let us denote the canonically conjugate variable by $\phi_j$ (which can be thought of as the angular coordinate of the particle on a circle), with $[\phi_j, n_{j'}] = i\delta_{jj'}$.

If this is to be a useful description, we will also need to approximate the creation operator $c_j^\dagger$ in terms of $\phi_j$ and $n_j$. To do this, we note that

$$c_j^\dagger |N_j\rangle = \sqrt{N_j + 1}|N_j + 1\rangle \ . \tag{5}$$

If $\bar{n}$ is much larger than 1, then $\sqrt{N_j+1}$ in the above formula can be approximated by $\sqrt{\bar{n}}$. In other words, we may write

$$c_j^\dagger \approx \sqrt{\bar{n}} e^{i\phi_j} \ . \tag{6}$$

In this new language, our original Hamiltonian $\mathcal{H}$ from Lecture 19 becomes

$$H = -t \sum_{\langle jj' \rangle} \cos(\phi_j - \phi_{j'}) + \frac{u}{2} \sum_i n_i^2 \ , \tag{7}$$

where the parameters that appear above are related to the original parameters as follows:

$$\begin{aligned} u &= U \ , \\ t &= 2w\bar{n} \ , \end{aligned} \tag{8}$$

This $H$ is nothing but the $N = 2$ version of the O(N) quantum rotor model $H_{\text{rotor}}$ we have been studying starting Lecture 21 (see Eqn. 1 of Lecture 21), with the identifications

$$\begin{aligned} u &= I^{-1} \ , \\ t &= J_{\text{eff}} \ . \end{aligned} \tag{9}$$

What are the possible phases of this model? To answer this, we need to analyze various limits just like we did for the case of the O(3) rotor model in the earlier lectures. Like in the O(3) case, the limit $u \gg t$ defines one stable phase. This is a phase in which the simple wavefunction

$$|\psi_{\text{gnd}}\rangle \approx \prod_j |n_j = 0\rangle \tag{10}$$

remains a qualitatively correct description of the ground state even when $t$ is increased from zero. In original variables, this is a phase in which the particle-number at each site remains pinned to the mean density $\bar{n}$. This is why we limited ourselves to situations in which $\bar{n}$ is integer. If it is not an integer, then the limit $u \gg t$ can also be subtle and depend on the nature of the spatial lattice defined by the external periodic potential. In the present case, there are no such subtleties and the density at each site is pinned to the integer value $\bar{n}$. In effect, this means that it is impossible to drive a particle-current through the system in this phase, because any such motion of particles would necessarily involve deviations of the particle number at some sites from $\bar{n}$, and this is energetically unfavourable.

To see this more formally, we can ask: What are the lowest lying excitations above this ground state? The answer is clear: The lowest lying excitations involve adding (or removing) a single particle at a chosen site. Since this site can be anywhere in the system, this quasiparticle (or quasihole) excitation has a huge degeneracy of order the number of sites in the system. Of course, this degeneracy is an artifact of the strict $t = 0$ limit. As soon as we turn on $t$, this degeneracy is lifted, since the hopping term allows the quasiparticle (or quasihole) excitation to hop to neighbouring sites with amplitude $-t$. This gives a quasiparticle (or quasihole) *band*, with excitation energies depending on the wavevector $\mathbf{k}$ of the quasiparticle (or quasihole) excitation. On a two-dimensional square lattice for instance, we will get

$$\epsilon_{p/n}(\mathbf{k}) \quad = \quad \frac{u}{2} - 2t\cos(k_x) - 2t\cos(k_y) \; . \tag{11}$$

This phase therefore has a energy-gap of order $u$ to adding or removing particles, and is therefore an *incompressible* state. Clearly, passing a current through the system involves creating particle-hole pairs, and therefore, this phase is an insulating phase with no linear-response current possible for small driving potentials.

One can also estimate the correlation function of the boson field

$$\langle c_j^\dagger c_{j'} \rangle \quad \sim \quad \langle \cos(\phi_j - \phi_{j'}) \rangle \tag{12}$$

in this phase as follows: To zeroth order in $t/u$, each site is decoupled from other sites, and this correlator is zero except when $j = j'$. At first order in perturbation theory, the $\mathcal{O}(t/u)$ piece of the wavefunction correlates the phase at neighbouring sites, and therefore, this correlator now becomes non-zero at nearest-neighbour sites. Reasoning in this way, we see that the first contribution to this correlator at distance $r$ is roughly of order $(t/u)^r$. In other words, the boson field has an exponentially decaying correlation function

$$\langle \cos(\phi_j - \phi_{j'}) \rangle \quad \sim \quad \exp(-r/\xi) \ , \tag{13}$$

with an $\mathcal{O}(1)$ correlation length $\xi$. Thus, this "Mott insulating" phase has short-ranged correlations for $\phi$. This is not suprising since the insulating phase is naturally described in the number (Fock) basis, and specifying the number precisely naturally leaves the phase free to fluctuate since the two are canonically conjugate variables.

Next, we contrast this behaviour with the physics of the opposite limit $t \gg u$. In this limit, it is useful to abandon the Fock (number) basis and work in the eigenbasis of $\phi_j$ at each site. In this language, it is clear that the low-energy physics is dominated by configurations of $\phi_j$ that are nearly uniform in space, with all the O(2) rotors lined up parallel to each other. In the rotor language, this is a ferromagnetic phase that spontaneously breaks the O(2) symmetry. In bosonic language, it is a superfluid phase with long-range phase coherence for the phase of the boson wavefunction. Low-lying excitations in this regime can be obtained by doing a harmonic spin-wave analysis like in the O(3) case we have already discussed in earlier lectures. This is particularly straightforward in the O(2) case, since it simply amounts to expanding the cosine coupling between neighbouring sites to quadratic order in its argument. This gives the harmonic spinwave Hamiltonian

$$H_{\text{spinwave}} \quad = \quad \frac{t}{2} \sum_{\langle jj' \rangle} (\phi_j - \phi_{j'})^2 + \frac{u}{2} \sum_j n_j^2 \tag{14}$$

As in our spinwave analysis of the O(3) case, this spinwave Hamiltonian $H_{\text{spinwave}}$ can be readily diagonalized by defining Fourier transformed operators $\phi_{\mathbf{k}}$ and $n_{\mathbf{k}}$ and constructing creation and annihilation operators $a^\dagger_{\mathbf{k}}$ and $a_{\mathbf{k}}$ corresponding to the spinwave mode with wavenumber $\mathbf{k}$.

Rather than repeat this analysis for the present O(2) case and then focus on the low-energy long-wavelength excitations corresponding to small wavenumbers $\mathbf{k}$, it is more instructive to go directly from $H_{\text{spinwave}}$ to a coarse-grained continuum Hamiltonian designed to capture the universal physics of the small $\mathbf{k}$ excitations. To do this, we assume that $n$ and $\phi$ vary slowly, and view $H_{\text{spinwave}}$ as the Riemann approximation to an integral. This allows us to replace the finite-differences in $H_{\text{spinwave}}$ with derivatives, and summations with integration, to give the continuum theory

$$H_{\text{eff}} = \frac{\rho_s}{2}\int d^dx(\nabla\tilde{\phi})^2 + \frac{1}{2\kappa}\int d^dx\tilde{n}^2(x) \tag{15}$$

where $\tilde{n}(x_j) \equiv n_j/a^d$ is the density of particles, $a$ is the lattice spacing, the canonical commutation relations now read

$$[\phi(\mathbf{x}), \tilde{n}(\mathbf{x}')] = i\delta^d(\mathbf{x}-\mathbf{x}')\ , \tag{16}$$

and

$$\begin{aligned} \rho_s &= ta^{2-d}\ , \\ \kappa^{-1} &= ua^d\ . \end{aligned} \tag{17}$$

This quadratic Hamiltonian is of course diagonalized in terms of the spin-wave eigenmodes discussed earlier. In bosonic language, these waves are the superfluid sound modes. Their existence becomes apparent if we write down the Heisenberg equations of motion corresponding to this continuum Hamiltonian. These follow from the commutation relations and read:

$$\begin{aligned} \frac{\partial n}{\partial t} &= i[H_{\text{eff}}, n] \\ &= \rho_s\nabla^2\phi \\ &\equiv -\vec{\nabla}\cdot\vec{j}_s\ , \\ \frac{\partial\phi}{\partial t} &= +i[H,\phi] \\ &= \frac{n(\mathbf{x})}{\kappa} \\ &\equiv \mu(\mathbf{x}) \end{aligned} \tag{18}$$

where we have dropped the tilde on $n$ for notational convenience and the last lines of the two equations define the "superfluid current" $\vec{j}_s$ and the local chemical potential $\mu$ as

$$\begin{aligned} \vec{j}_s &= -\rho_s\vec{\nabla}\phi \\ \mu(\mathbf{x}) &= \frac{n(\mathbf{x})}{\kappa} \end{aligned} \tag{19}$$

Thus, we see that the superfluid current can be non-zero even in the absence of a gradient of the chemical potential, unlike in a normal system. This is because the superfluid current arises due to gradients in the "condensate phase" $\phi$ rather than chemical potential gradients. Indeed, it is easy to see that these equations of motion imply that

$$\begin{aligned} \frac{\partial \phi(\mathbf{x})}{\partial t} &= \mu(\mathbf{x}) \\ \frac{\partial j_s}{\partial t} &= -\rho_S \nabla \mu \ . \end{aligned} \tag{20}$$

Thus, a gradient of chemical potential is actually related to the time derivative of the superfluid current. And the local chemical potential determines the rate of change of the local condensate phase.

Now, it is easy to obtain the low-lying sound wave modes from these equations of motion as follows:

$$\begin{aligned} \frac{\partial n}{\partial t} &= \rho_s \nabla^2 \phi \\ \frac{\partial \phi}{\partial t} &= \frac{n}{\kappa} \\ \Rightarrow \quad \frac{\partial^2 n}{\partial t^2} &= \frac{\rho_s}{\kappa} \nabla^2 n \end{aligned} \tag{21}$$

As expected, they have a linear dispersion $\omega = ck$, with $c = \sqrt{\rho_s/\kappa}$ and no gap, similar to our earlier analysis of spinwave modes in the O(3) case—in both cases, the gapless nature of the dispersion is of course guaranteed by Goldstone's argument that we described in the context of the O(3) rotor model. This argument can fail if the system has long-range interactions. This is of relevance in the present context, since our bosonic fluid, if made up of charged bosons, will have long-range Coulomb interactions between the bosons apart from the short-ranged repulsion that we have already taken into account. It is therefore interesting to see just how the spectrum of low-lying excitations is modified by these long range interactions. To answer this question, we note that if bosons interact via a Coulomb repulsion, there is a term

$$\int\int d^d x d^d x' n(\mathbf{x}) V(\mathbf{x} - \mathbf{x}') n(\mathbf{x}') \tag{22}$$

in $H_{\text{eff}}$ apart from the local repulsion energy

$$\int d^d x \frac{n^2(\mathbf{x})}{2\kappa} \tag{23}$$

that we have already included. Here $V$ captures the effects of the long-range Coulomb interaction, and it is understood that $V$ is cut-off at short distances since the short-distance part can be included by simply adjusting the value of $\kappa$.

With this additional term, the equations of motion are modified:

$$\begin{aligned} \frac{\partial \phi(\mathbf{x})}{\partial t} &= \mu(\mathbf{x}) + W(\mathbf{x}) \ , \\ \frac{\partial n(\mathbf{x})}{\partial t} &= \rho_s \nabla^2 \phi \end{aligned} \tag{24}$$

where $W$ is the Coulomb potential due to all the other particles, and obeys

$$\nabla^2 W = 4\pi e n(\mathbf{x}) \ . \tag{25}$$

This gives the wave-equation

$$\frac{\partial^2 n}{dt^2} = c^2 \nabla^2 n - 4\pi \rho_s e^2 n \tag{26}$$

which implies that the superfluid sound waves no longer have a gapless dispersion, but instead obey the following relationship between their frequency and wavenumber:

$$\omega = \sqrt{c^2 k^2 + 4\pi \rho_s e^2} \tag{27}$$

These are gapped plasma oscillations characteristic of a charged fluid, and represent the simplest example of the "Higgs" mechanism that you may have learnt about in a course on particle-physics or quantum field theory.