Lecture 23: The Mermin-Wagner theorem

As we mentioned in the previous lecture, it is actually possible to rule out, in a mathematically rigorous way, the existence of a long-range ordered state that spontaneously breaks a continuous symmetry in low dimensions at nonzero temperature. Since the other way of reaching this conclusion involves assuming the existence of such a long-range ordered state and arriving at an internal inconsistency (this is basically what we did in the previous lecture), this alternate, rigorous route to the same result has much to recommend it, particularly since rigorous statements of such significance are not so common in physics. Therefore, we will devote this lecture to proving the so-called "Mermin Wagner Theorem" for the quantum rotor model (the original proof was developed for some other systems, but it is instructive to use the same procedure to construct a proof for the quantum rotor model, so that is what we will do here). However, before we begin, an important disclaimer is needed: The Mermin-Wagner theorem provides a rigorous route to the $T > 0$ part of last lecture's conclusions. For the $T = 0$ part, the corresponding rigorous statements are due to Pitaevskii and Stringari (Journal of Low Temperature Physics, Vol. 85, Nos. 5/6, 1991), and outside the scope of this course of lectures.

We begin by adding a "magnetic field" term to the quantum rotor model

$$H = -J\sum_{\langle jk\rangle} \hat{n}_j \cdot \hat{n}_k + \sum_j \frac{\vec{L}_j^2}{2I} - hn^z(\vec{q}=0) \tag{1}$$

where

$$n^z(\vec{q}=0) = \sum_j n_j^z \tag{2}$$

Let $\{E_n\}$ be the spectrum of exact eigenstates of H, and define the following scalar product of two operators A and B:

$$\begin{aligned}(A,B) &= \frac{1}{Z}\sum_{n,m \text{ such that } E_n\neq E_m} \langle n|A^\dagger|m\rangle\langle m|B|n\rangle \frac{e^{-\beta E_m}-e^{-\beta E_n}}{E_n-E_m} \\ &\quad +\frac{1}{Z}\sum_{n,m \text{ such that } E_n=E_m} \frac{1}{T}\langle n|A^\dagger|m\rangle\langle m|B|n\rangle \end{aligned} \tag{3}$$

where Z is the exact partition function at temperature T, and $\beta = 1/T$. In the rest of this lecture, we will denote the double summation involved in the above as $\sum'_{n,m}$ in order to not write out explicitly each time the limiting procedure that is used in this definition.

To get some feeling for this scalar product, it is useful to note that (A, A) is precisely Re $\chi_{AA}(\omega = 0) \equiv \chi_{AA}$, where $\chi_{AA}(\omega)$ is the usual frequency dependent linear response function we have studied in previous lectures. Also, we see that (A, A) is ≥ 0. Further, for (A, A) to be zero, *all* A_{mn} have to be zero for $E_n \neq E_m$. In addition, this definition of scalar product satisfies all the other properties one expects of a scalar product. In summary, we have:

$$\begin{aligned} (A, B + C) &= (A, B) + (A, C) \\ (A, \alpha B) &= \alpha (A, B) \\ (A, B) &= (B, A)^\star \\ (A, A) \geq 0. & \end{aligned} \tag{4}$$

Now, as you all know well, any scalar product with these properties always satisfies the following Cauchy-Schwarz inequality

$$|(A, B)|^2 \leq (A, A)(B, B) \tag{5}$$

For completeness, and as warmup, let us prove this first before using it: Start with the definitions

$$\begin{aligned} x &= (A, A) \\ z &= (B, B) \\ y &= |(A, B)| \\ (A, B) &= \alpha^\star y \end{aligned} \tag{6}$$

so that α is the phase of $y^\star$. Now, observe that for any real number r, we have

$$(A - r\alpha B, A - r\alpha B) \geq 0 \tag{7}$$

Expanding the left-hand side, we get

$$x + r^2 z - 2ry \geq 0 \tag{8}$$

Now, if $z = 0$, this would be false in the limit $r \to \infty$ unless $y = 0$. Therefore $z = 0$ implies $y = 0$. If this is not the case, one can choose $r = y/z$ to get the sharpest possible inequality

$$y^2 \leq xz \tag{9}$$

as required.

Also, since

$$\frac{\sinh(w)}{w} \quad \leq \quad \cosh w \tag{10}$$

for any real w, (A, A) clearly satisfies the inequality

$$(A, A) \quad \leq \quad \frac{1}{Z} \sum_{n,m} \frac{|\langle n|A^\dagger|m\rangle|^2}{2T} \left(e^{-E_m/T} + e^{-E_n/T}\right) \tag{11}$$

In other words

$$(A, A) \quad \leq \quad \frac{1}{2T} \langle A^\dagger A + AA^\dagger \rangle_T. \tag{12}$$

where the subscript T indicates the equilibrium expectation value at temperature T.

Now, let B such that $B = [C^\dagger, H]$. Then

$$\begin{aligned}
(A, B) &= \frac{1}{Z} {\sum_{n,m}}' \langle n|A^\dagger|m\rangle \langle m|[C^\dagger, H]|n\rangle \frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m} \\
&= \frac{1}{Z} \sum_{n,m} \langle n|A^\dagger|m\rangle \langle m|C^\dagger|n\rangle (e^{-\beta E_m} - e^{-\beta E_n}) \\
&= \langle [C^\dagger, A^\dagger] \rangle_T
\end{aligned}$$

Also

$$\begin{aligned}
(B, B) &= \frac{1}{Z} {\sum_{n,m}}' \langle n|[H, C]|m\rangle \langle m|[C^\dagger, H]|n\rangle \frac{e^{-\beta E_m} - e^{-\beta E_n}}{E_n - E_m} \\
&= \frac{1}{Z} \sum_{n,m} \langle n|[H, C]|m\rangle \langle m|C^\dagger|n\rangle (e^{-\beta E_m} - e^{-\beta E_n}) \\
&= \langle [C^\dagger, [H, C]] \rangle_T \ .
\end{aligned} \tag{13}$$

Using the Cauchy-Schwarz inequality in conjunction with these two results gives the so-called Bogoliubov's inequality:

$$|\langle [C^\dagger, A^\dagger] \rangle_T|^2 \quad \leq \quad \frac{1}{2T} \langle A^\dagger A + AA^\dagger \rangle_T \langle [C^\dagger, [H, C]] \rangle_T \tag{14}$$

This is the basic result that we will use to prove the Mermin-Wagner theorem. The idea of the proof is simple, and therefore easy to state explicitly right at the outset: For each system we are interested in, we should try and find pairs $C(\vec{q})$ and $A(\vec{q})$ such that $[C^\dagger(\vec{q}), A^\dagger(\vec{q})]$ is proportional to the extensive order parameter (the quantity that scales as the volume of the system when symmetry is spontaneously broken) for *each choice of the arbitrary wavevector* $\vec{q}$. If $C(\vec{q})$ and $A(\vec{q})$ are judiciously chosen in this manner, it then becomes possible to use the Bogoliubov inequality in conjunction with the fact that $\vec{q}$ can be chosen arbitrarily to bound the order parameter from above by a quantity that vanishes in the limit of $h \to 0$, thereby establishing the absence of spontaneous symmetry breaking.

For the quantum rotor model, we make the choices

$$\begin{aligned} A(\vec{q}) &= n^y(-\vec{q}) \\ &\equiv \sum_j e^{i\vec{q}\cdot\vec{x}_j} n_j^y \end{aligned} \tag{15}$$

and

$$\begin{aligned} C(\vec{q}) &= L^x(\vec{q}) \\ &\equiv \sum_j e^{-i\vec{q}\cdot\vec{x}_j} L_j^x \end{aligned} \tag{16}$$

Next, we work out the explicit form of the various factors that appear in the Bogoliubov inequality for this choice of $A(\vec{q})$ and $C(\vec{q})$. In order to avoid notational clutter, we leave out mention of the $\vec{q}$ dependence of A and C unless this causes ambiguity.

We have

$$\begin{aligned} [C^\dagger, A^\dagger] &= \sum_{jj'} e^{i\vec{q}\cdot\vec{x}_j}[L_j^x, n_{j'}^y] e^{-i\vec{q}\cdot\vec{x}_{j'}} \\ &= \sum_{jj'} \delta_{jj'} i n_j^z e^{i\vec{q}\cdot(\vec{x}_j - \vec{x}_{j'})} \\ &= \sum_j n_j^z \\ &= i n^z(\vec{q} = 0) \\ &\equiv i N_{\text{sites}} m \end{aligned} \tag{17}$$

where m is the usual *intensive* Néel order parameter.

Next, we note that

$$\begin{aligned}\frac{1}{2}\langle A^\dagger A + AA^\dagger\rangle_T &= \frac{1}{2}\langle n^y(\vec{q})n^y(-\vec{q})\rangle_T + \langle n^y(-\vec{q})n^y(\vec{q})\rangle_T \\ &= \langle n^y(\vec{q})n^y(-\vec{q})\rangle_T \\ &\equiv N_{\text{sites}} C^{yy}(\vec{q}) \end{aligned} \tag{18}$$

where

$$C^{yy}(\vec{q}) = \sum_j e^{-i\vec{q}\cdot\vec{x}_j}\langle n_j^y n_0^y\rangle_T \tag{19}$$

is the usual equilibrium correlation function at wavevector $\vec{q}$.

Next, we note that

$$\langle [C^\dagger, [H, C]]\rangle_T = \left(hN_{\text{sites}} m + \sum_{\langle jk\rangle}(1-\cos(\vec{q}\cdot\Delta\vec{x}_{jk}))\langle \hat{n}_j^\perp \cdot \hat{n}_k^\perp\rangle_T \right) \tag{20}$$

where $\langle jk\rangle$ denotes nearest neighbour links of the hypercubic lattice, $\Delta\vec{x}_{jk}$ denotes the corresponding spatial separation between nearest-neighbours, and $\hat{n}^\perp$ denotes the vector made up of the two components (n^x, n^y) which are perpendicular to the applied field h. Therefore, the Bogoliubov inequality reads

$$N_{\text{sites}}^2 m^2 \le \left(hN_{\text{sites}} m + J\sum_{\langle jk\rangle}(1-\cos(\vec{q}\cdot\Delta\vec{x}_{jk}))\langle \hat{n}_j^\perp \cdot \hat{n}_k^\perp\rangle_T \right) N_{\text{sites}}\frac{C^{yy}(\vec{q})}{T} \tag{21}$$

Since

$$\langle \hat{n}_j^\perp \cdot \hat{n}_k^\perp\rangle_T \le 1 , \tag{22}$$

and since

$$1-\cos(\alpha) \le \frac{\alpha^2}{2} , \tag{23}$$

we may freely rewrite this as

$$m^2 \le \frac{C^{yy}(\vec{q})}{T}\left(hm + \frac{J\vec{q}^2}{2N_{\text{sites}}}\sum_{\langle jk\rangle}(\Delta\vec{x}_{jk})^2 \right) \tag{24}$$

In other words

$$\frac{Tm^2}{hm + \frac{J\vec{q}^2}{2N_{\rm sites}} \sum_{\langle jk\rangle} (\Delta\vec{x}_{jk})^2} \leq C^{yy}(\vec{q}) \tag{25}$$

Next, we note that

$$\frac{J\vec{q}^2}{2N_{\rm sites}} \sum_{\langle jk\rangle} (\Delta\vec{x}_{jk})^2 = J\mathcal{C}\vec{q}^2 a^2 \tag{26}$$

where a is the lattice spacing and $\mathcal{C}$ is some lattice dependent $O(1)$ positive constant.

We are now in a position to exploit the arbitrariness of $\vec{q}$ and sum over $\vec{q}$ in the Brillouin zone to obtain:

$$\frac{1}{N_{\rm sites}} \sum_{\vec{q}} \frac{Tm^2}{hm + J\mathcal{C}\vec{q}^2 a^2} \leq \frac{1}{N_{\rm sites}} \sum_{\vec{q}} C^{yy}(\vec{q}) \tag{27}$$

Finally, we note that

$$\begin{aligned} \frac{1}{N_{\rm sites}} \sum_{\vec{q}} C^{yy}(\vec{q}) &= \frac{1}{N_{\rm sites}} \sum_{j} \langle (n_j^y)^2 \rangle_T \\ &\leq 1 \, , \end{aligned} \tag{28}$$

to obtain

$$\frac{1}{N_{\rm sites}} \sum_{\vec{q}} \frac{Tm^2}{hm + J\mathcal{C}\vec{q}^2 a^2} \leq 1 \tag{29}$$

We may now take the thermodynamic limit to obtain

$$\int_{BZ} \frac{d^d q}{(2\pi)^2} \frac{Tm^2}{hm + J\mathcal{C}\vec{q}^2 a^2} \leq 1 \tag{30}$$

In other words, we have concluded that

$$m^2 \leq \frac{1}{\int_{BZ} \frac{d^d q}{(2\pi)^d} \frac{T}{hm + J\mathcal{C}\vec{q}^2 a^2}} \tag{31}$$

Curiously, in the $h \to 0$ limit, the integral in the denominator of the right hand side is the same as the integral we analyzed to decide if spin-wave theory was internally consistent for $T > 0$! In that analysis, the divergence of this $h = 0$ integral in dimensions $d \leq 2$ due to contributions near $\vec{q} = 0$ was a signal of internal inconsistency in spinwave theory. Whereas in this alternate rigorous approach, the divergent integral sits in the denominator and rigorously bounds m^2 from above by 0, thereby demonstrating that there can be no long-range order of the $\hat{n}$ in the quantum rotor model in dimensions $d \leq 2$.

This concludes our discussion of the ordered and disordered phases of the quantum rotor model, as well as conditions for their realization. In the next four lectures, we will introduce the renormalization group approach to this physics.