

## Module 1 : Introduction and Background Material

### Lecture 3 : Wave Propagation In Linear, Homogenous Isotropic Media

#### Objectives

##### In this lecture we will look at

- General solution of wave equation.
- Dispersion and absorption.

We now consider propagation of an electromagnetic wave in an infinite, homogeneous medium without free charges

$$\nabla \cdot \vec{B} = 0 \quad (3.1)$$

$$\nabla \cdot \vec{D} = 0 \quad (3.2)$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad (3.3)$$

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0 \quad (3.4)$$

As already observed in the previous lecture, inclusion of free electron can be easily done when we use complex dielectric function.

For the complex amplitude of harmonic waves these become

$$\vec{\nabla} \cdot \vec{B}(\omega) = 0 \quad (3.5)$$

$$\vec{\nabla} \cdot \vec{D}(\omega) = 0 \quad (3.6)$$

$$\vec{\nabla} \times \vec{E}(\omega) - i\omega \vec{B}(\omega) = 0 \quad (3.7)$$

$$\vec{\nabla} \times \vec{H}(\omega) + i\omega \vec{D}(\omega) = 0 \quad (3.8)$$

For a uniform isotropic medium, we know that  $\vec{\epsilon}(\omega)$  and  $\vec{\mu}(\omega)$  are scalar

$$\vec{D}(\omega) = \epsilon(\omega) \vec{E}(\omega) \quad (3.9)$$

$$\vec{B}(\omega) = \mu(\omega) \vec{H}(\omega) \quad (3.10)$$

For  $\mu(\omega)$  there is no simple model.

In the following the  $\omega$  dependence of fields and  $\epsilon(\omega)$  and  $\mu(\omega)$  is not written explicitly.

From eqs (3.7) and eqs (3.8) we obtain

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E}(\omega) - \epsilon \mu \vec{E}(\omega) = 0 \quad (3.11)$$

Now

$$\begin{aligned} \vec{\nabla} \times \vec{\nabla} \times \vec{E}(\omega) &= \vec{\nabla} \left( \vec{\nabla} \cdot \vec{E}(\omega) \right) - \nabla^2 \vec{E}(\omega) = -\nabla^2 \vec{E}(\omega) \quad \text{if } \epsilon(\omega) \neq 0 \\ \therefore \epsilon(\omega) \vec{\nabla} \cdot \vec{E} &= 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = 0 \quad \text{if } \epsilon(\omega) \neq 0 \end{aligned}$$

we may write thus in an isotropic medium the Electric field obey the eqs.

$$\left( \nabla^2 + \epsilon \mu \omega^2 \right) \vec{E} = 0 \quad (3.12)$$

Similarly for magnetic induction  $\vec{B}$ , we obtain

$$\left( \nabla^2 + \epsilon \mu \omega^2 \right) \vec{B} = 0 \quad (3.13)$$

The wave equation for  $\vec{E}$  and  $\vec{B}$  have the same form as in vacuum. The only difference is that

$\epsilon_0 \mu_0 = 1/c^2$  is replaced by  $\epsilon \mu$ .

The ratio  $\frac{\epsilon \mu}{\epsilon_0 \mu_0} = \frac{c^2}{v^2}$  where  $v$  is speed of light in the medium.

From eqs 3.7 to 3.10, we observe that  $\vec{E}$ ,  $\vec{H}$  and  $\vec{k}$  form a set of orthogonal vectors. Such that

$$\vec{E} \times \vec{H} \parallel \vec{k}, \vec{H} \times \vec{k} \parallel \vec{E} \text{ and } \vec{k} \times \vec{E} \parallel \vec{H}.$$

However, as we saw in lecture 2 in a medium it is also possible that  $\epsilon(\omega) = 0$  at some frequency. Then  $\vec{\nabla} \cdot \vec{D} = 0$  is satisfied even if  $\vec{\nabla} \cdot \vec{E} \neq 0$ . On the other hand for  $\epsilon(\omega) = 0$  the wave equation is

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \epsilon \mu \omega^2 \vec{E} = 0$$

implies that if  $\epsilon(\omega) = 0$ , then

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = 0$$

Or

$$\vec{k} \times (\vec{k} \times \vec{E}) = 0$$

$$\vec{k}(\vec{k} \cdot \vec{E}) = k^2 \vec{E}$$

Or

$$\left( \vec{k} \cdot \vec{E} \right)^2 = k^2 \left| \vec{E} \right|^2$$

$$\Rightarrow \vec{k} \parallel \vec{E}$$

for  $\epsilon(\omega) = 0$ ;  $\vec{k} \parallel \vec{E}$

For transverse waves ( $\epsilon(\omega) \neq 0$ )

$$\vec{k} \times \vec{k} \times \vec{E} + \frac{\omega^2}{c^2} \left( \frac{\epsilon \mu}{\epsilon_0 \mu_0} \right) \vec{E} = 0$$

becomes

$$k^2 = \frac{\omega^2}{c^2} \left( \frac{\epsilon \mu}{\epsilon_0 \mu_0} \right) \quad (3.15)$$

Since  $\epsilon(\omega)$  can be complex, in general  $k$  will also be complex.

We write

$$\vec{k} = \hat{k} (n + i\kappa) \frac{\omega}{c} = 0 \quad (3.17)$$

where  $\hat{k}$  is unit vector along  $\vec{k}$ .

$$\therefore (n + i\kappa) = \sqrt{\frac{\epsilon \mu}{\epsilon_0 \mu_0}} \quad (3.18)$$

$n$  is called the refractive index and  $\kappa$  is the attenuation coefficient.

This corresponds to an electric field variation in space that is given by

$$\vec{E} \sim e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$= e^{i\left(\frac{\omega}{c} n \hat{k} \cdot \vec{r} - \omega t\right) - \frac{\omega}{c} \kappa \hat{k} \cdot \vec{r}}$$

$\frac{\omega}{c} n \hat{k} \cdot \vec{r} - \omega t$  is the time and space dependent phase. Since a wave front is the phase constant surface

is for the present case the wave front is an infinite plane and it moves with a velocity  $(c/n)\hat{k}$ . The amplitude of this wave falls exponentially as

$$e^{-\frac{\omega}{c} \kappa \left( \hat{k} \cdot \vec{r} \right)}$$

$\kappa$  is therefore called the attenuation coefficient.

$(n + i\kappa)$  is often called the complex refractive index.

## METALS, DIELECTRICS AND SEMICONDUCTORS

Earlier, we had shown that for a damped harmonic oscillator

$$\chi(\omega) = \left( \frac{Ne^2}{m\epsilon_0} \right) \frac{1}{(\omega_0^2 - \omega^2) - i\omega\gamma}$$

In general a material will have not only one resonance frequency  $\omega_0^2 - \omega^2$  but several, each corresponding to transition between two states of the system. Thus, in general the linear susceptibility of a system has the form

$$\chi(\omega) = \frac{Ne^2}{m\epsilon_0} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\omega\gamma_j)} \quad (3.19)$$

where,  $\omega_j$  is the resonana frequency of the  $j^{\text{th}}$  transition.

For the dielectric function

$$\epsilon(\omega) = \epsilon_0(1 + \chi(\omega)) \quad (3.20)$$

We get

$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{Ne^2}{m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma} = 1 + \sum_j \frac{\omega_p^2 f_j}{\omega_j^2 - \omega^2 - i\omega\gamma} \quad (3.21)$$

where  $\omega_p^2 = \frac{Ne^2}{m\epsilon_0}$  is called the plasma frequency.

We note:

- near a resonance, say  $\omega \simeq \omega_j$ , we need to retain only the resonant term in eqs (3.21). Then

$$\begin{aligned} \text{Im} \left( \frac{\epsilon(\omega)}{\epsilon_0} \right) &= \omega_p^2 f_j \text{Im} \frac{1}{\omega_j^2 - \omega^2 - i\omega\gamma} \\ &= \frac{\omega_p^2}{2\omega_j} f_j \text{Im} \frac{1}{(\omega_j - \omega) - i\gamma/2} \\ &= \frac{\omega_p^2}{2\omega_j} f_j \frac{\gamma/2}{(\omega_j - \omega)^2 + (\gamma/2)^2} \\ &= \frac{\omega_p^2}{2\omega_j} f_j \pi \delta(\omega_j - \omega) \text{ for } \gamma \rightarrow 0 \end{aligned} \quad (3.22)$$

Thus,  $f_j$  determines the integrated absorption over the line width of the transition at  $\omega_j$ . It is important

to note that  $\int \frac{d\omega \gamma/2}{(\omega_j - \omega)^2 + (\gamma/2)^2}$  is independent of  $\gamma$ .

- For  $\omega \gg \omega_j$ , the contribution of  $j^{\text{th}}$  resonance goes as  $-\omega_p^2 (f_j/\omega^2)$  which decrease rapidly as  $\omega \rightarrow \infty$ . Thus, a resonance does not contribute much to the polarization at frequencies much larger than the resonance frequency. However, as  $\omega \gg \omega_j$  the contribution of the resonance becomes  $\omega_p^2/\omega_j^2$  i.e. at zero frequency all resonances contribute to the polarization of the medium.

Generally  $\omega_{\text{electronics-excitation}} \gg \omega_{\text{vibrational-excitation}} \gg \omega_{\text{rotational-excitation}}$

So, in the frequency range where vibrational rotational excitation occur, the dielectric function can be written as

$$\varepsilon(\omega) = \varepsilon_{\infty} + \sum_{\omega_j = \text{vibrational}} \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma}$$

where  $\varepsilon_{\infty}$  is the background dielectric function. In this region since  $\omega \ll \omega_{\text{electron}}$ ,  $\varepsilon_{\infty}$  is the low frequency limit of the electronic dielectric function. It is written as  $\varepsilon_{\infty}$  is the dielectric function at frequency far above the vibration rotation excitation frequency.

In contrast, if we are studying the optical response of a material in the electric frequency range ( $\hbar\omega \approx 1\text{eV}$ ) the low-frequency excitation contribute negligibly to the dielectric function and we may write:

$$\varepsilon(\omega) = 1 + \sum_{\omega_j = \text{electronic}} \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma}$$

## RECAP

### In this lecture we have learned about the

- Solutions of the wave equation are obtained to identify dispersion and absorption with real and imaginary parts of the complex refractive index
- For metals, the real part of the dielectric function is negative below the plasma frequency.
- Oscillator strength determines the integrated absorption of a transition.