

Module 5 : Pulse propagation through third order nonlinear optical medium

Lecture 36 : Optical pulse propagation in nonlinear medium

Objectives

In this lecture we will

1. Develop the theoretical formalism based on Maxwell equations to describe the propagation in a nonlinear medium with dispersion and losses.
2. Describe the phenomena of self-phase modulation.

Optical pulse propagation in nonlinear medium

Propagation of optical pulses through nonlinear medium gives rise to many interesting phenomena, both from the fundamental and application points of view. Some of these will be discussed in this and subsequent lectures. We have seen that the nonlinear interactions depend on the intensity and length of interaction. Due to confinement of light to very small area core ($\sim \mu m^2$) large intensities are easily available from very low power laser sources like the diode laser. Unlike media in other forms, single mode optical fibers are amenable to extremely large interaction length. Optical fibers are then the media of choice for such phenomena. In fact, it will not be an exaggeration that an optical fiber contains a full-fledged nonlinear optics lab. Since optical pulses are used to define bits in fiber optic communication system, study of these phenomena acquires special significance for the technological aspects. We will explore these phenomena from the perspective of optical fibers. For the study of propagation of optical pulses through nonlinear medium our starting point is the following wave equation derived from Maxwell's equations for charge free and non-magnetic materials.

$$\nabla^2 E = -\mu_0 \frac{\partial^2 D}{\partial t^2} \quad (36.1)$$

Where $D = \epsilon_0 E + P$ and electric polarization P is responsible for coupling dielectric medium response to the e. m. field.

For intense field P consists of both linear and nonlinear terms i.e.

$$P = P_L + P_{NL} \quad (36.2)$$

In view of these, we can write

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial^2 P_L}{\partial t^2} + \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \quad (36.3)$$

$$\nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} \quad (36.4)$$

Optical fibers are centrosymmetric (as glass is an amorphous material) and as a result the lowest order nonlinear contribution to is due to third order susceptibility. We have shown in Lecture 20 that nonlinear polarization for intensity dependent refraction effects is given as

$$P_{NL} = \frac{3\epsilon_0 \chi^{(3)}}{4} |E|^2 E \quad (36.5)$$

$$\nabla^2 E = \left(\frac{n_0^2}{c^2} + \frac{3\chi^{(3)}}{4c^2} |E|^2 \right) \frac{\partial^2 E}{\partial t^2} \quad (36.6)$$

Note: as an approximation $|E|^2$ has been taken out of the time derivative. This is valid approximation as rapid time variations at frequency ω disappear on taking $|E|^2$ and only slow time variations of $|E|^2$ remain.

Let us consider the propagating optical field of the form

$$E(x, y, z, t) = F(x, y) \tilde{u}(z, t) e^{-i(\omega_0 t - \beta_0 z)} \quad (36.7)$$

Here, $F(x, y)$ is the modal field profile or eigen function which is solution of the linear wave equation and $\tilde{u}(z, t)$ is slowly varying function of time or pulse envelope.

Let us first ignore the effects of dispersion

$$\frac{\partial^2}{\partial z^2} \tilde{u} e^{i\beta_0 z} = \frac{\partial}{\partial z} \left(\frac{\partial}{\partial z} \tilde{u} e^{i\beta_0 z} \right) = \left[-\beta_0^2 \tilde{u} - 2i\beta_0 \frac{\partial \tilde{u}}{\partial z} + \frac{\partial^2 \tilde{u}}{\partial z^2} \right] e^{i\beta_0 z} \quad (36.8)$$

As pulse envelope and instantaneous phase vary little over the pulse period, we can use slowly varying envelope approximation (SVEA), $\frac{\partial^2 \tilde{u}}{\partial z^2} \ll \beta_0 \frac{\partial \tilde{u}}{\partial z}$. Therefore,

$$\frac{\partial^2}{\partial z^2} \tilde{u} e^{i\beta_0 z} = \left[-\beta_0^2 \tilde{u} - 2i\beta_0 \frac{\partial \tilde{u}}{\partial z} \right] e^{i\beta_0 z} \quad (36.9)$$

Similarly,

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \tilde{u} e^{-i\omega_0 t} &= \left[-\omega_0^2 \tilde{u} + 2i\omega_0 \frac{\partial \tilde{u}}{\partial t} + \frac{\partial^2 \tilde{u}}{\partial t^2} \right] e^{-i\omega_0 t} \\ &= \left[-\omega_0^2 \tilde{u} + 2i\omega_0 \frac{\partial \tilde{u}}{\partial t} \right] e^{-i\omega_0 t} \end{aligned} \quad (36.10)$$

Using Equations (36.9) and (36.10), equation 36.8 can be written as

$$\tilde{u} \left[\nabla_t^2 F - \beta_0^2 F \right] - 2i\beta_0 F \frac{\partial \tilde{u}}{\partial t} = \left(\frac{n_0^2}{c^2} + \frac{3\chi^{(3)}}{4c^2} F^2 |\tilde{u}|^2 \right) \left[-\omega_0^2 \tilde{u} + 2i\omega_0 \frac{\partial \tilde{u}}{\partial t} \right] F \quad (36.11)$$

Or

$$\begin{aligned} \left[\nabla_t^2 F + \left(\frac{n_0^2 \omega_0^2}{c^2} - \beta_0^2 \right) F \right] \tilde{u} - 2i\beta_0 F \left[\frac{\partial \tilde{u}}{\partial t} + \frac{n_0^2}{c^2} \frac{\omega_0}{\beta_0} \frac{\partial \tilde{u}}{\partial t} + \frac{3\chi^{(3)}}{4c^2} \frac{\omega_0}{\beta_0} F^2 |\tilde{u}|^2 \frac{\partial \tilde{u}}{\partial t} \right] \\ = -\frac{3\chi^{(3)} \omega_0^2}{4c^2} F^3 |\tilde{u}|^2 \tilde{u} \end{aligned} \quad (36.12)$$

Further $\nabla_t^2 F + \left(\frac{n_0^2 \omega_0^2}{c^2} - \beta_0^2 \right) F = 0$ and $\chi^{(3)} \frac{\partial \tilde{u}}{\partial t} \ll \frac{\partial \tilde{u}}{\partial t}$ as $\chi^{(3)}$ is very small

$$\left[\frac{\partial \tilde{u}}{\partial z} + \frac{n_0^2}{c^2} \frac{\omega_0}{\beta_0} \frac{\partial \tilde{u}}{\partial t} \right] F = -\frac{i3\chi_3}{8c^2} \frac{\omega_0^2}{\beta_0} F^3 |\tilde{u}|^2 \tilde{u} \quad (36.13)$$

Using

$$\frac{\omega_0}{\beta_0} = \frac{c}{n_0}; \frac{\partial \omega_0}{\partial \beta_0} = \dot{\beta}_0 = \frac{n_0}{c} \text{ and } k_0 = \frac{\omega_0}{c}$$

we can write for the evolution of the pulse envelop

$$\left[\frac{\partial \tilde{u}}{\partial z} + \dot{\beta}_0 \frac{\partial \tilde{u}}{\partial t} \right] F = -i \frac{3}{8} \chi^{(3)} \frac{k_0}{n_0} F^3 |\tilde{u}|^2 \tilde{u} \quad (36.14)$$

Multiplying both sides by F and integrating over the infinite (x, y) cross-section

$$\langle F^4 \rangle = \int \int_{-\infty}^{\infty} F^4(x, y) dx dy \quad (36.15)$$

One gets

$$\frac{\partial \tilde{u}}{\partial z} + \dot{\beta}_0 \frac{\partial \tilde{u}}{\partial t} = -i \frac{3k_0 \chi^{(3)}}{8n_0} \frac{\langle F^4 \rangle}{\langle F^2 \rangle} |\tilde{u}|^2 \tilde{u} \quad (36.16)$$

Defining power carried in a mode as

$$P = |U|^2 = \frac{1}{2} n_0 \sqrt{\frac{\epsilon_0}{\mu_0}} \langle F^2 \rangle |\tilde{u}|^2 \quad (36.17)$$

and replacing \tilde{u} by U , one gets

$$\frac{\partial U}{\partial z} + \beta_0 \frac{\partial U}{\partial t} = -i \frac{3\chi^{(3)} k_0}{4c\epsilon_0 n_0^2} \frac{\langle F^4 \rangle}{\langle F^2 \rangle \langle F^2 \rangle} |U|^2 U \quad (36.18)$$

$\chi^{(3)}$ is related to the nonlinear refractive index n_2 as

$$n_2 = \frac{3\chi_3}{4c\epsilon_0 n_0^2} \quad (36.19)$$

Let us define

$$A_{eff} = \frac{\langle F^4 \rangle}{\langle F^2 \rangle \langle F^2 \rangle} = \frac{\left[\int \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy \right]^2}{\left[\int \int_{-\infty}^{\infty} |F(x, y)|^4 dx dy \right]} \quad (36.20)$$

as the effective modal area and approximate the fundamental fiber mode by a Gaussian, then $A_{eff} = \pi w^2$. Here w depends upon fiber parameters and is mode radius. Equation (36.19) can be written as

Thus,

$$\frac{\partial U}{\partial z} + \beta_0 \frac{\partial U}{\partial t} = -i \frac{3k_0 n_2}{A_{eff}} |U|^2 U \quad (36.21)$$

$$A_{eff} = \frac{\left[\int_0^{2\pi} \int_0^{\infty} \psi^2 r dr d\theta \right]^2}{\left[\int_0^{2\pi} \int_0^{\infty} \psi^4 r dr d\theta \right]} \quad (36.22)$$

$$= \pi w^2$$

$$\frac{\partial U}{\partial z} + \beta_0 \frac{\partial U}{\partial t} = -i \frac{k_0 n_2}{A_{eff}} |U|^2 U \quad (36.23)$$

In Lecture 34 we have derived wave envelope equation in a dispersive medium (see equation (34.20)) as

$$\frac{\partial U}{\partial z} + \beta_0 \frac{\partial U}{\partial t} = \frac{i}{2} \beta_0'' \frac{\partial^2 U}{\partial t^2} \quad (36.24)$$

In optical fibers both the nonlinear and dispersion effect are weak. Thus they can be considered as additive.

$$\frac{\partial U}{\partial z} + \beta_0 \frac{\partial U}{\partial t} = \frac{i}{2} \beta_0'' \frac{\partial^2 U}{\partial t^2} - i \frac{k_0 n_2}{A_{eff}} |U|^2 U \quad (36.25)$$

Loss can also be included in this equation by adding $-\frac{1}{2} \alpha U^2$ to the R.H.S. The parameter $\frac{\alpha}{2}$ is the amplitude decay constant corresponding to power loss coefficient α . Including loss term, we get

$$\frac{\partial U}{\partial z} + \beta_0 \frac{\partial U}{\partial t} = \frac{i}{2} \beta_0'' \frac{\partial^2 U}{\partial t^2} - i\gamma |U|^2 U - \frac{\alpha}{2} U \quad (36.26)$$

Where $\gamma = \frac{k_0 n_2}{A_{eff}}$

Or

$$i \left[\frac{\partial U}{\partial z} + \beta_0 \frac{\partial U}{\partial t} \right] = \gamma |U|^2 U - \frac{\alpha}{2} U - \frac{1}{2} \beta_0'' \frac{\partial^2 U}{\partial t^2} \quad (36.27)$$

Generalized nonlinear wave equation for a weakly nonlinear and dispersive medium can thus be written

as

$$i \left[\frac{\partial U}{\partial z} + \beta_0 \frac{\partial U}{\partial t} \right] = \gamma |U|^2 U - i \frac{\alpha}{2} U - \frac{1}{2} \beta_0 \frac{\partial^2 U}{\partial t^2} \quad (36.28)$$

We now transform variable to the pulse frame (a reference frame moving with the the group velocity of the pulse)

$$\text{i.e. } T = t - \beta_0 z \text{ \& } Z = z$$

$$U(t, z) \rightarrow U(T, Z)$$

Using:

$$\begin{aligned} \frac{\partial U(t, z)}{\partial z} &= \frac{\partial U(T, Z)}{\partial Z} \frac{\partial Z}{\partial z} + \frac{\partial U(T, Z)}{\partial T} \frac{\partial T}{\partial z} \\ &= \frac{\partial U(T, Z)}{\partial Z} - \beta_0 \frac{\partial U(T, Z)}{\partial T} \end{aligned} \quad (36.29)$$

$$\begin{aligned} \frac{\partial U(t, z)}{\partial z} &= \frac{\partial U(T, Z)}{\partial T} \frac{\partial T}{\partial z} + \frac{\partial U(T, Z)}{\partial Z} \frac{\partial Z}{\partial z} \\ &= \frac{\partial U(T, Z)}{\partial T} \end{aligned} \quad (36.30)$$

We get

$$i \frac{\partial U}{\partial Z} = \gamma |U|^2 U - i \frac{\alpha}{2} U - \frac{1}{2} \beta_0 \frac{\partial^2 U}{\partial T^2} \quad (36.31)$$

It is first instructive to consider lossless medium i.e. $\alpha = 0$

$$i \frac{\partial U}{\partial Z} = \gamma |U|^2 U - \frac{1}{2} \beta_0 \frac{\partial^2 U}{\partial T^2} \quad (36.32)$$

Before solving this equation let us consider the effect of nonlinearity alone i.e. $\beta_0 = 0$

$$i \frac{\partial U}{\partial Z} = \gamma |U|^2 U \text{ where } \gamma = \frac{k_0 n_2}{A_{eff}}$$

This equation can be easily solved as there is no change of temporal profile

$$U(Z, T) = U_0(T) \exp \left[-i \left(\frac{k_0 n_2}{A_{eff}} \right) |U_0(T)|^2 Z \right] \quad (36.33)$$

Where $U_0(T)$ is initial pulse. We thus see that pulse acquires a nonlinear phase as it propagates through nonlinear medium which is given by

$$\phi = - \frac{n_2 k_0}{A_{eff}} L |U|^2 = - \frac{L}{L_{NL}} \quad (36.34)$$

$$\text{where } L_{NL} = \left[\frac{A_{eff}}{n_2 k_0} |U|^2 \right] \quad (36.35)$$

is known as nonlinear length

This is not the entire phase shift as the rapidly varying phase $(\omega_0 t - \beta_0 z)$ was separated out earlier.

To include the loss in the medium, we use effective length of the medium L_{eff} rather than " L " which is given by

$$L_{eff} = \frac{1 - e^{-\alpha L}}{\alpha} \quad (36.36)$$

The phenomenon of the emergence of the time varying intensity dependent phase ϕ is known as self phase modulation (SPM) and the phase ϕ is proportional to the power in the mode. SPM grows linearly with propagation length.

$$\phi = -\frac{k_0 n_2 L}{A_{eff}} |U(T)|^2 \quad (36.37)$$

SPM, therefore, converts amplitude modulation into the phase modulation.

The temporally varying phase implies that the instantaneous frequency varies across the pulse

$$\Delta\omega = \omega - \omega_0 = \frac{\partial\phi}{\partial T} = -\frac{k_0 n_2 L}{A_{eff}} \frac{\partial |U(T)|^2}{\partial T} \quad (36.38)$$

For a Gaussian pulse, the phase delay and corresponding instantaneous frequency sweep is shown in figure 36.1

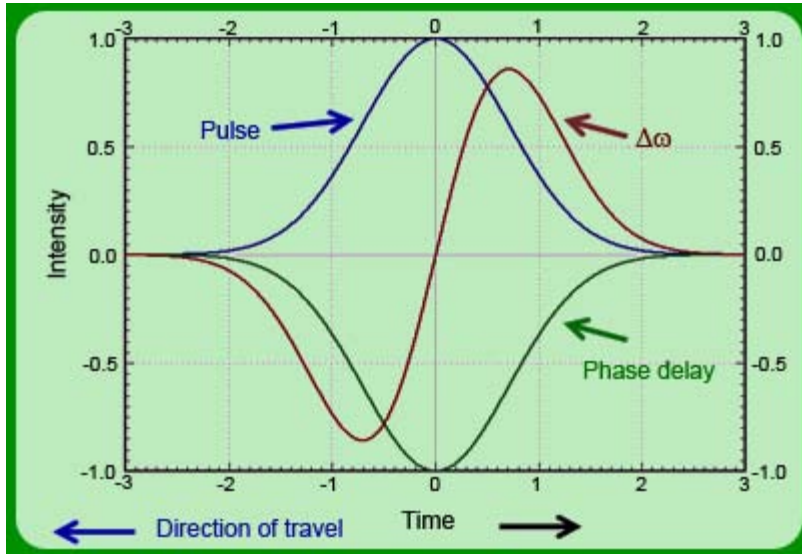


Figure 36.1

Frequency shift \propto differential of the pulse intensity profile

$$I = |U(T)|^2 / A_{eff} - \text{pulse } U(T) \sim \exp(-t^2/\tau^2)$$

Note that SPM generates new frequencies and chirps the spectrum of the optical pulse. Leading edge has lower frequency whilst trailing has higher frequency (see figure 36.2). Hence, it results in change of spectral content without altering the pulse intensity profile.

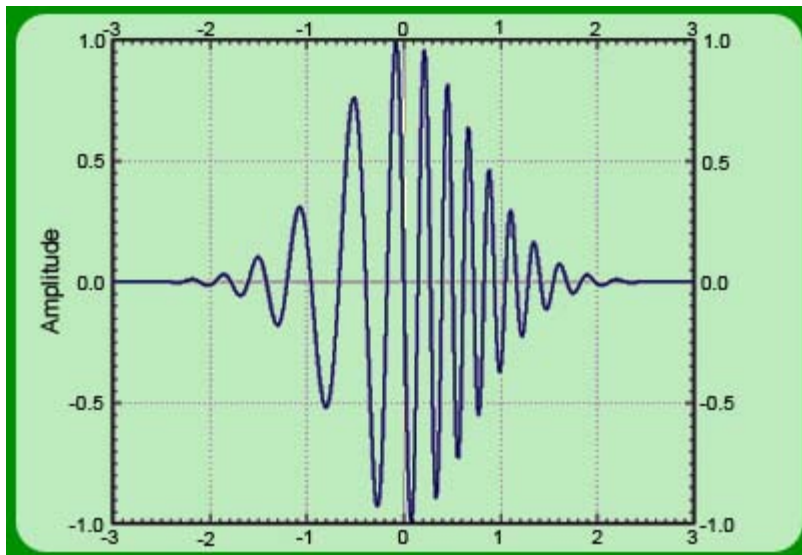


Figure 36.2

Recap:**In this lecture we have**

- Developed the theoretical formalism based on Maxwell equations to describe the propagation in a nonlinear medium with dispersion and losses and have.
- Described the phenomenon of self-phase modulation.