Module 1 : Introduction and Background Material

Lecture 4 : Optical Waves In Anisotropic Media

Objectives

In this lecture we will look at

- Optical response of anisotropic media.
- Wave propagation in uniaxial media.

Many media behave anisotropically in their response to waves. Examples of such media are

- Non-cubic Crystals.
- Flowing gas/liquid: flow direction different from other directions.
- Isotropic media subjected to an Electric or a magnetic field or a stress.
- Liquid crystals.
- Layered or textured composites.

In isotropic media the electromagnetic fields follow the wave equation

$$\nabla \times \nabla \times \vec{E}(\omega) - \omega^2 \mu(\omega)\varepsilon(\omega)\vec{E}(\omega) = 0 \qquad (4.1)$$

where $\varepsilon(\omega)$ is the dielectric function and $\mu(\omega)$ the magnetic permittivity.

In general, μ and ε are tensors. We now explore the effects of this tensorial nature. To avoid too many complications we consider anisotropic nonmagnetic $(\mu = \mu_0)$ dielectric material.

The corresponding relation between $\vec{D}(\omega)$ and $\vec{E}(\omega)$ now become

$$D_i = \sum_{j=1}^{3} \varepsilon_{ij} E_j \qquad (4.2)$$

where i, j denote Cartesians components

In matrix form, this read as

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \qquad (4.3)$$

This relation between $\vec{D}(\omega)$ and $\vec{E}(\omega)$ can also be written as

$$\vec{D} = \vec{\vec{\varepsilon}}.\vec{E} \qquad (4.4)$$

For any non absorbing material $\varepsilon_{ij}(\omega)$ is real symmetric tensor.

Thus,

$$\varepsilon_{ij}(\omega) = \varepsilon_{ji}(\omega) \qquad (4.5)$$

a way from resonances i.e. in the transparency region. It can then always be made diagonal by appropriate choice of axes with maximum number of independent components = 3

If $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz}$ the medium is optically active

If $\varepsilon_{xx} = \varepsilon_{yy} \neq \varepsilon_{zz}$ it is uniaxial

If $\varepsilon_{xx} \neq \varepsilon_{yy} \neq \varepsilon_{zz}$ it is biaxial

For uniaxial media, there is one direction called the optic axis. For propagation along that direction, the refraction index is independent of polarization. For biaxial crystals, there are two such axes.

For $\vec{E}$ oriented along any of the principle axes $\vec{D} \parallel \vec{E}$.

$$D_x = \varepsilon_{xx} E_x ; D_y = \varepsilon_{yy} E_y ; D_z = \varepsilon_{zz} E_z \qquad (4.6)$$

For the harmonic plane waves with the electric field of the form

$$\vec{E}(\omega) e^{-i\omega t + i\vec{k}.\vec{r}} \; + cc. \qquad (4.7)$$

and similarly for $\vec{P}$ and $\vec{D}$. $\vec{E}(\omega)$, $\vec{D}(\omega)$, $\vec{P}(\omega)$ and $\vec{H}(\omega)$ are related by

$$\vec{k}\vec{D} = 0 \qquad (4.8)$$

$$\vec{k}\vec{B} = 0 \qquad (4.9)$$

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} \qquad (4.10)$$

$$\vec{D} = -\frac{\vec{k} \times \vec{H}}{\omega} \qquad (4.11)$$

Thus, $\vec{D}$ and $\vec{B}$ are perpendicular to $\vec{k}$, $\vec{B}$ is also perpendicular $\vec{E}$. Therefore, $\vec{E}\vec{D}\vec{k}$ all lie in same plane. It is important to note that $\vec{D}$ and $\vec{E}$ are no longer collinear.

$\vec{D}$ can be written as follows by eliminating $\vec{H}$ using eq(4.10) and $\vec{B} = \mu_0 \vec{H}$.

$$\vec{D} = \frac{k^2}{\mu_0 \omega^2} (\vec{E} - \hat{s}(\hat{s}.\vec{E})) = \varepsilon_0 n^2 (\vec{E} - \hat{s}(\hat{s}.\vec{E})) \qquad (4.12)$$

The Poynting vector is given by

$$(\vec{E} \times \vec{H}) = \frac{k}{\mu_0 \omega} E^2 (\hat{s} - \hat{e}(\hat{s}.\vec{E})) \qquad (4.13)$$

where $\hat{s}$ is the unit vector $\vec{k}/k$. The Poynting vector makes the same angle α with $\hat{k}$ that $\vec{E}$ makes with $\vec{D}$. The relationship of all these vectors are shown in the figure (4.1)

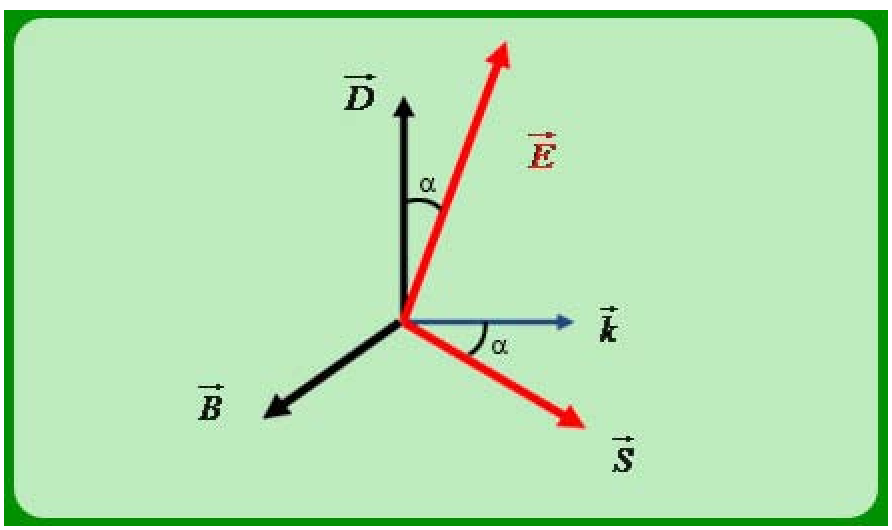

Figure 4.1

To summarize,

$\vec{D}, \vec{k}, \vec{B}$ are mutually orthogonal

$\vec{E}, \vec{S}, \vec{B}$ are mutually orthogonal

Since $\vec{E} \perp \vec{B}, \vec{E}$ has to lie in the same plane as $\vec{k}$ and $\vec{D}$

Now

$$D_x = \varepsilon_{xx} E_x;\ D_y = \varepsilon_{yy} E_y;\ D_z = \varepsilon_{zz} E_z$$
$$\Rightarrow D_i = \varepsilon_0 n^2 \left(E_i - s_i \left(\hat{s}.\vec{E} \right) \right) = \varepsilon_{ii} E_i \tag{4.14}$$
$$\Rightarrow E_i = \frac{n^2 s_i \left(\hat{s}.\vec{E} \right)}{n^2 - \left(\varepsilon_{ii} / \varepsilon_0 \right)} = \frac{n^2 s_i \left(\hat{s}.\vec{E} \right)}{n^2 - n_i^2}$$

where n_i = $\sqrt{\varepsilon_{ii} / \varepsilon_0}$

If we multiply both sides by s_i and sum over *i*, we get

$$\hat{s}.\vec{E} = \sum_i \frac{n^2 s_i^2 (\hat{s}.\vec{E})}{n^2 - n_i^2} \quad \text{or} \quad \sum_i \frac{n^2 s_i^2}{n^2 - n_i^2} = 1 \tag{4.15}$$

Thus,

$$\frac{1}{n^2} = \frac{s_x^2}{n^2 - n_x^2} + \frac{s_y^2}{n^2 - n_y^2} + \frac{s_z^2}{n^2 - n_z^2} \tag{4.16}$$

or,

$$\begin{aligned} & n^2 \{ s_x^2 (n^2 - n_y^2)(n^2 - n_z^2) + \\ & s_y^2 (n^2 - n_x^2)(n^2 - n_z^2) + \\ & s_z^2 (n^2 - n_x^2)(n^2 - n_y^2) \} = (n^2 - n_x^2)(n^2 - n_y^2)(n^2 - n_z^2) \end{aligned} \tag{4.17}$$

This is quadratic equation in n^2 as the n^6 term cancels. It has two roots which implies that two possible values of n exist for a given direction (s_x, s_y, s_z). For each such solution we can then find the electric field direction. We do this for the simpler case of uniaxial media.

UNIAXIAL MEDIUM

$$n_x = n_y \neq n_z \tag{4.18}$$

Since there is isotropy in the x, y plane, i.e. all polarization direction in this plane are equivalent, we can chose the axes such that propagation vector lies in the Oyz plane and $\vec{k}$ makes an angle θ with the z axis; Then,

$$\hat{s} = (0, \sin\theta, \cos\theta) \tag{4.19}$$

Equation (4.17) now Becomes

$$\begin{aligned} & n^2 \{ s_y^2 (n^2 - n_x^2)(n^2 - n_z^2) + s_z^2 (n^2 - n_x^2)(n^2 - n_y^2) \} \\ & = (n^2 - n_x^2)(n^2 - n_y^2)(n^2 - n_z^2) \quad , \\ & (n^2 - n_x^2)[n^2 \{ s_y^2 (n^2 - n_z^2) + s_z^2 (n^2 - n_y^2) \} - (n^2 - n_y^2)(n^2 - n_z^2)] = 0 \end{aligned} \tag{4.20}$$

$$\text{either } n = n_x \text{ or } [...] = 0 \tag{4.21}$$

Wave corresponding to $n = n_x$ is called the ORDINARY wave $n = n_x$ is a solution for all values of θ or the refractive index is same for all directions of propagation n.

For the other solution, called the EXTRA ORDINARY wave

$$n^2\{s_y^2(n^2-n_z^2)+s_z^2(n^2-n_y^2)\}=(n^2-n_y^2)(n^2-n_z^2)$$
$$n^2(n^2-s_y^2n_z^2-s_z^2n_y^2)=(n^2-n_y^2)(n^2-n_z^2)$$
$$\Rightarrow\ n^2(n_y^2+n_z^2-s_y^2n_z^2-s_z^2n_y^2)-n_y^2n_z^2=0$$
$$\Rightarrow n^2(s_y^2n_y^2+s_z^2n_z^2)+n_y^2n_z^2=0$$

Or,

$$\frac{1}{n^2}=\frac{s_z^2}{n_y^2}+\frac{s_y^2}{n_z^2}=\frac{\cos^2\theta}{n_y^2}+\frac{\sin^2\theta}{n_z^2} \tag{4.22}$$

The refractive index for this solution depends on the angle between the propagation vector and the optic axis (z - axis). For propagation along the z- axis, $\theta=0$ and $n=n_x$ for both solutions.

To get polarization for the ordinary wave we put $s_x=0, n_x=n=n_y$ in eq. 4.12 to obtain

$$(-e_x)n^2+n_x^2e_x=0 \tag{4.23}$$
$$((s_ye_y+s_ze_z)s_y-e_y)n^2+n_y^2e_y=0 \tag{4.24}$$
$$((s_ye_y+s_ze_z)s_z-e_z)n^2+n_z^2e_z=0 \tag{4.25}$$

For $n=n_x=n_y\neq n_z$, the solution is $\hat{e}=(1,0,0)$

Thus, the ordinary wave is polarized perpendicular to the optic axis $\left(\because \hat{e}.\hat{z}=0\right)$ as well as the direction of propagation $\left(\because \hat{S}.\hat{e}=0\right)$. Similarly, we find that for the polarization of the extraordinary wave lies in the plane containing the optic axis and the direction of propagation.

For $n_x^2\neq n^2$ (true for extraordinary wave), the first of equations (4.25) gives $e_x=0$.

Then from the other two equations:

$$\frac{e_y}{e_z}=\frac{-s_zs_y}{(s_y^2-1)+n_y^2/n^2}=\frac{s_zs_y}{s_z^2-n_y^2/n^2}=-\frac{n_z^2}{n_y^2}.\frac{s_z}{s_y} \tag{4.26}$$
$$\left(\text{using } \frac{1}{n^2}=\frac{s_z^2}{n_y^2}+\frac{s_y^2}{n_z^2}\right)$$

RECAP

In this lecture we have learned about the

- Wave propagation in anisotropic media described.
- For uniaxial media $n_x^2=n_y^2\neq n_z^2$

$$\varepsilon(\omega)/\varepsilon_0=\begin{pmatrix} n_0^2 & 0 & 0\\ 0 & n_0^2 & 0\\ 0 & 0 & n_e^2\end{pmatrix}$$

- For the Ordinary wave:

$$n=n_0, \vec{E}\perp\vec{k}, \vec{E}\perp \text{Optic axis}$$

- For the Extra ordinary wave:

$$\frac{1}{n^2}=\frac{\cos^2\theta}{n_0^2}+\frac{\sin^2\theta}{n_e^2}$$

$\vec{E}$ in the plane containing the optic axis and the direction n with

$$\frac{e_y}{e_z} = -\frac{n_z^2}{n_y^2} \cdot \frac{s_z}{s_y}$$