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NPTEL ONLINE CERTIFICATION COURSES

NPTEL

WEEK 3: ROBOTICS

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Topic 2: Robot Kinematics

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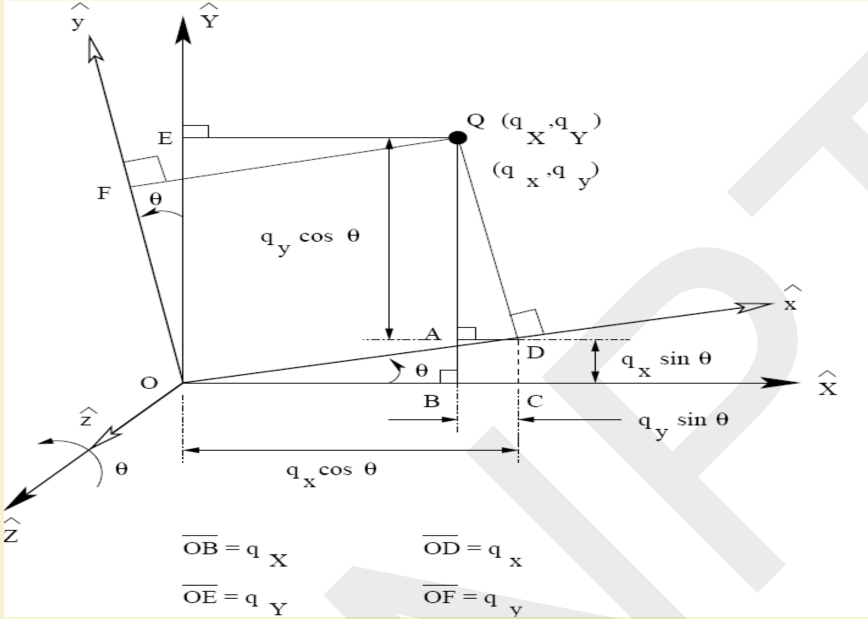
Rotational Operator

Rot $(\hat{Z}, \theta)$: Rotation about $\hat{Z}$ axis by an angle θ (anticlockwise sense)

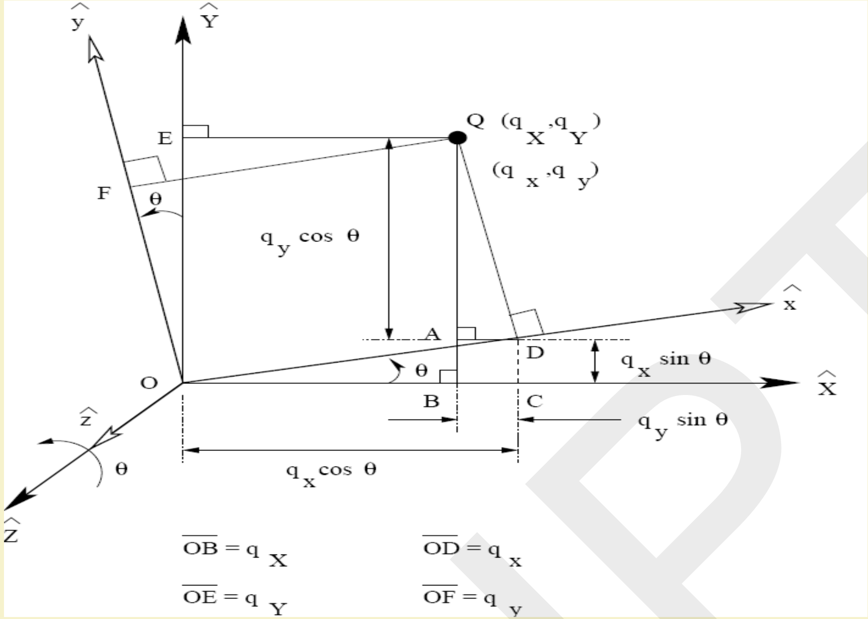


$$\overline{DC} = q_x Sin\theta$$

$$\overline{AQ} = q_y Cos\theta$$

$$\overline{OC} = q_x Cos\theta$$

$$\overline{AD} = BC = q_y Sin\theta$$



$$q_X = q_x Cos\theta - q_y Sin\theta + q_z X0$$

$$q_Y = q_x Sin\theta + q_y Cos\theta + q_z X0$$

$$q_Z = q_x X0 + q_y X0 + q_z X1$$

In matrix form:

$$\begin{bmatrix} q_X \\ q_Y \\ q_Z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}$$

$$\mathrm{Rot}\left(\hat{Z}, \theta\right) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, we get

$$\mathrm{Rot}\left(\hat{X},\theta\right)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$$

$$\mathrm{Rot}\left(\hat{Y},\theta\right)=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$

Properties of Rotation Matrix

- Each row/column of a rotation matrix is a unit vector
- Inner (dot) product of each row of a rotation matrix with each other row becomes equal to 0. The same is true for each column also.
- Rotation matrices are not commutative in nature

$$\boldsymbol{ROT}(\hat{X}, \theta_1)\boldsymbol{ROT}(\hat{Y}, \theta_2) \neq \boldsymbol{ROT}(\hat{Y}, \theta_2)\boldsymbol{ROT}(\hat{X}, \theta_1)$$

- Inverse of a rotation matrix is nothing but its transpose

$$\boldsymbol{ROT}^{-1}(\hat{X}, \theta) = \boldsymbol{ROT}^{T}(\hat{X}, \theta)$$

- ${}^{A}_{B}T = {}^{B}_{A}T^{-1}$

A Numerical Example

A frame {B} is rotated about $\widehat{X_U}$ axis of the universal coordinate system by 45 degrees and translated along $\widehat{X_U}$, $\widehat{Y_U}$ and $\widehat{Z_U}$ by 1, 2, and 3 units, respectively. Let the position of a point Q in {B} is given by [3.0 2.0 1.0]^T . Determine ${}^{U}\overline{Q}$.

Solution:

$${}^{U}\overline{Q} = {}^{U}_{B}T \times {}^{B}Q$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \cos 45 & -\sin 45 & 2 \\ 0 & \sin 45 & \cos 45 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 2\cos 45 - \sin 45 + 2 \\ 2\sin 45 + \cos 45 + 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2.707 \\ 5.121 \\ 1 \end{bmatrix}$$

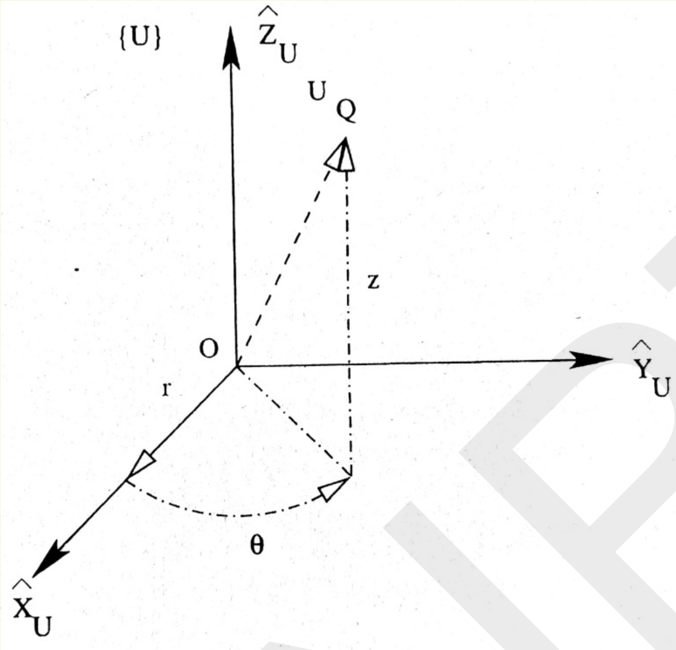
Composite Rotation Matrix

Composite rotation matrix representing a rotation of α angle about $\hat{Z}$, followed by a rotation of β angle about $\hat{Y}$ axis , followed by a rotation of γ angle about $\hat{X}$ axis.

$$ROT_{composite} = ROT(\hat{X}, \gamma) ROT(\hat{Y}, \beta) ROT(\hat{Z}, \alpha)$$

Representations of Position in Other Than Cartesian Coordinate System

Cylindrical Coordinate System



Steps:

1. Starting from the origin O, translate by r units along $\hat{X}_U$ axis
2. Rotate in anti-clockwise sense about $\hat{Z}_U$ axis by an angle θ
3. Translate along $\hat{Z}_U$ axis by z units

$$[T]_{composite} = TRANS(\hat{Z}_U, z)ROT(\hat{Z}_U, \theta)TRANS(\hat{X}_U, r)$$

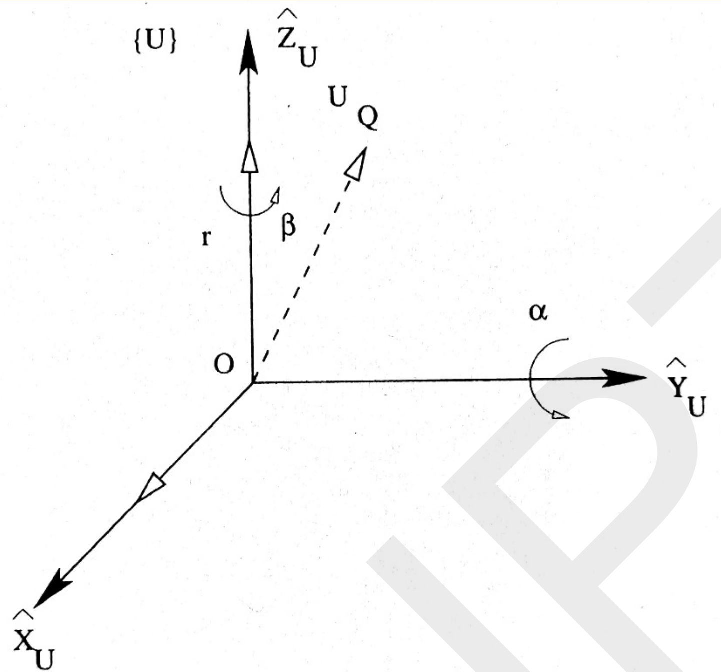
$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & r\cos\theta \\ \sin\theta & \cos\theta & 0 & r\sin\theta \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We get $q_x = r\cos\theta$

$q_y = r\sin\theta$

$q_z = z$

Spherical Coordinate System



Steps:

1. Starting from the origin O, translate along $\hat{Z}_U$ axis by r units
2. Rotate in anti-clockwise sense about $\hat{Y}_U$ axis by an angle α
3. Rotate in anti-clockwise sense about $\hat{Z}_U$ axis by an angle β

$$[T]_{composite} = ROT(\hat{Z}_U, \beta) ROT(\hat{Y}_U, \alpha) TRANS(\hat{Z}_U, r)$$

$$= \begin{bmatrix} \cos\alpha\cos\beta & -\sin\beta & \sin\alpha\cos\beta & r\sin\alpha\cos\beta \\ \cos\alpha\sin\beta & \cos\beta & \sin\alpha\sin\beta & r\sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha & r\cos\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

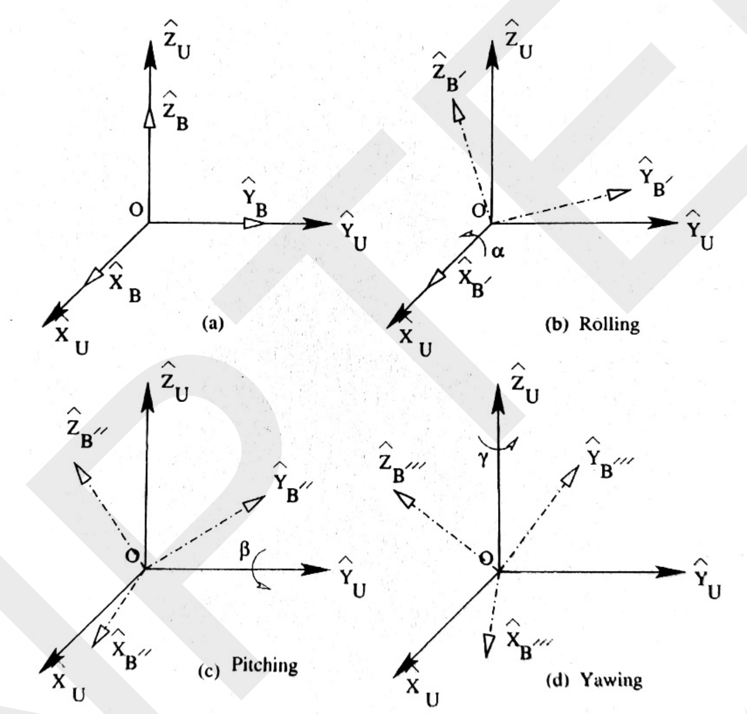
We get $q_x = r\sin\alpha\cos\beta$

$q_y = r\sin\alpha\sin\beta$

$q_z = r\cos\alpha$

Representations of Orientation in Other Than Cartesian Coordinate System

Roll, Pitch and Yaw Angles



Steps:

1. Rotate {B} about $\hat{X}_U$ by an angle α $\Longrightarrow$ rolling
2. Rotate {B$^{|}$} about $\hat{Y}_U$ by an angle β $\Longrightarrow$ pitching
3. Rotate {B$^{||}$} about $\hat{Z}_U$ by an angle γ $\Longrightarrow$ yawing

$$ {}^{U}_{B}R_{composite:\,rpy} = ROT(\hat{Z}_U,\gamma)ROT(\hat{Y}_U,\beta)ROT(\hat{X}_U,\alpha) $$

$$ = \begin{bmatrix} c\beta c\gamma & -c\alpha s\gamma + s\alpha s\beta c\gamma & s\alpha s\gamma + c\alpha s\beta c\gamma \\ c\beta s\gamma & c\alpha c\gamma + s\alpha s\beta s\gamma & -s\alpha c\gamma + c\alpha s\beta s\gamma \\ -s\beta & c\beta s\alpha & c\alpha c\beta \end{bmatrix} $$

We compare with

$$ {}^{U}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

We get

$$\alpha = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right)$$

$$\beta = \tan^{-1}\left(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}\right)$$

$$\gamma = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right)$$

A Numerical Example

The concept of roll, pitch and yaw angles has been used to represent the rotation of a frame {B} with respect to the reference frame {U}, that is ${}^{U}_{B}R$. Let us suppose that the above rotation can also be expressed by a 3X3 rotation matrix as given below.

$$ {}^{U}_{B}R = \begin{bmatrix} -0.250 & 0.433 & -0.866 \\ 0.433 & -0.750 & -0.500 \\ -0.866 & -0.500 & 0.000 \end{bmatrix} $$

Determine the angles of rolling, pitching and yawing.

Solution:

Angle of rolling $\alpha = \tan^{-1}\frac{r_{32}}{r_{33}} = \tan^{-1}\frac{-0.500}{0.000} = 90°$

Angle of pitching $\beta = \tan^{-1}\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}$

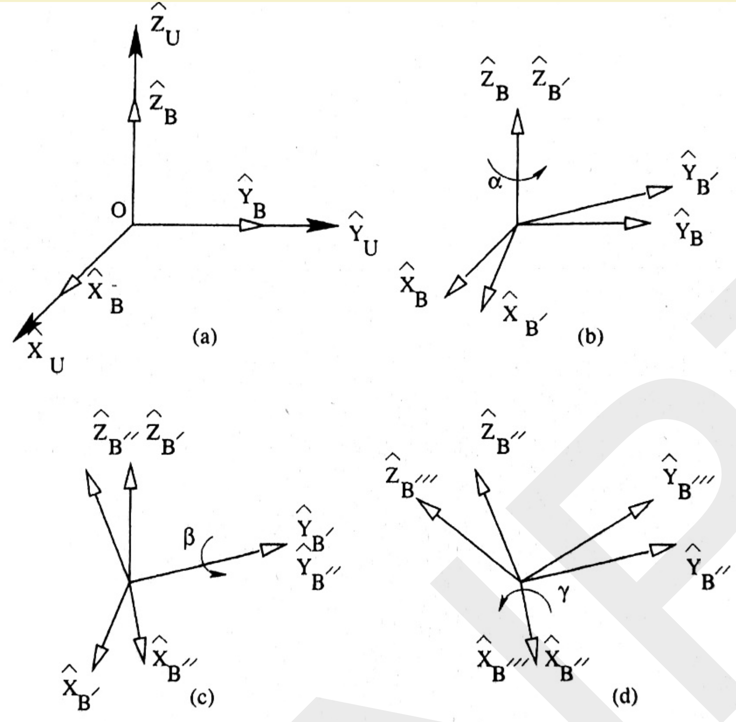
$= \tan^{-1}\frac{0.866}{\sqrt{(-0.250)^2 + (0.433)^2}}$

$= 40.89°$

Angle of yawing $\gamma = \tan^{-1}\frac{-r_{21}}{r_{11}} = \tan^{-1}\frac{0.433}{-0.250}$

$= -59.99 \approx -60°$

Using Euler Angles



$$ {}^{U}_{B}R = {}^{B}_{U}R^{-1} $$

Steps:

1.Rotate {B} about $\hat{Z}_B$ by an angle α in anti-clockwise sense

2.Rotate {B} about $\hat{Y}_{B'}$ by an angle β in anti-clockwise sense

3.Rotate {B} about $\hat{X}_{B''}$ by an angle γ in anti-clockwise sense

$$ {}^{B}_{U}R_{Eulerangles} = ROT(\hat{X}_{B''}, -\gamma)ROT(\hat{Y}_{B'}, -\beta)ROT(\hat{Z}_{B}, -\alpha) $$

$$ {}^{U}_{B}R = \begin{bmatrix} c\alpha c\beta & s\beta s\gamma c\alpha - s\alpha c\gamma & s\beta c\gamma c\alpha + s\alpha s\gamma \\ s\alpha c\beta & s\beta s\gamma s\alpha + c\alpha c\gamma & s\beta c\gamma s\alpha - s\gamma c\alpha \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

We compare with

$$ {}^{U}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

$$\alpha = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right)$$

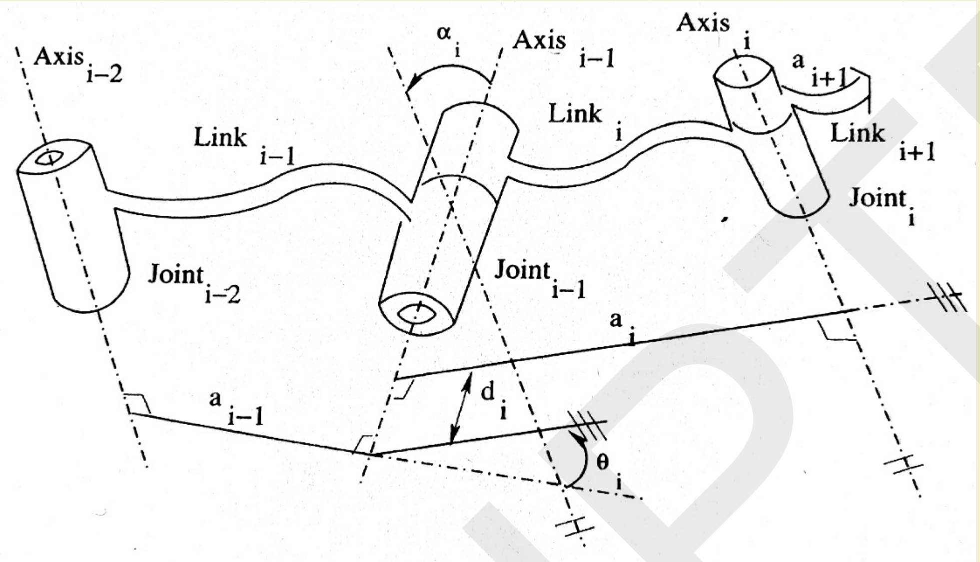
$$\beta = \tan^{-1}\left(\frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}}\right)$$

$$\gamma = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right)$$

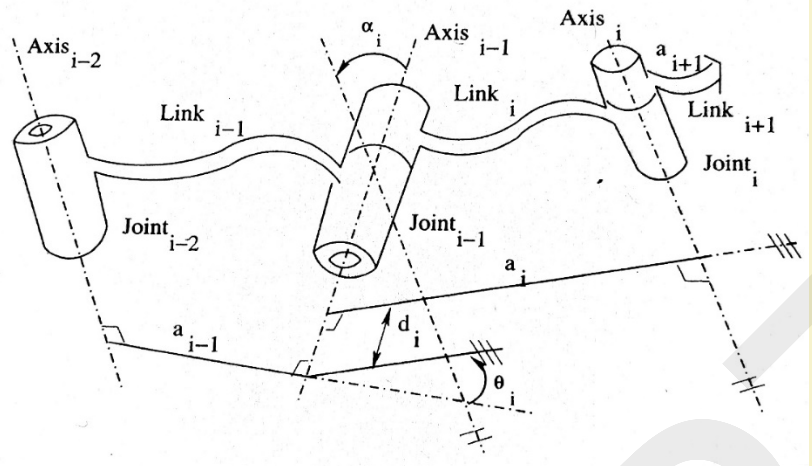
Denavit-Hartenberg Notations

- Proposed in the year 1955

Link and Joint Parameters



- **Length of link$_i$ (a_i):** It is the mutual perpendicular distance between Axis$_{i-1}$ and Axis$_i$
- **Angle of twist of link$_i$ (α_i):** It is defined as the angle between Axis$_{i-1}$ and Axis$_i$



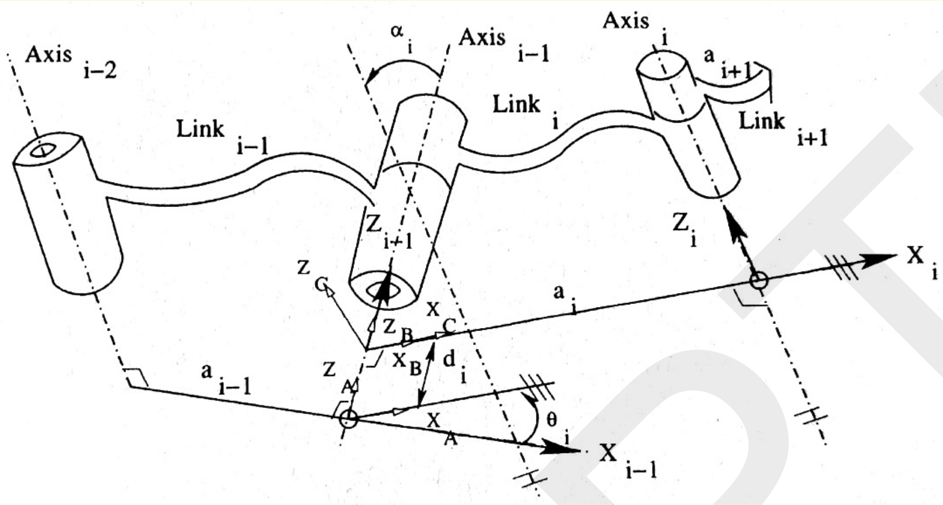
- **Offset of link$_i$ (d_i):** It is the distance measured from a point where a_{i-1} intersects the Axis$_{i-1}$ to the point where a_i intersects the Axis$_{i-1}$ measured along the said axis

- **Joint Angle (θ_i):** It is defined as the angle between the extension of a_{i-1} and a_i measured about the Axis$_{i-1}$

Notes:

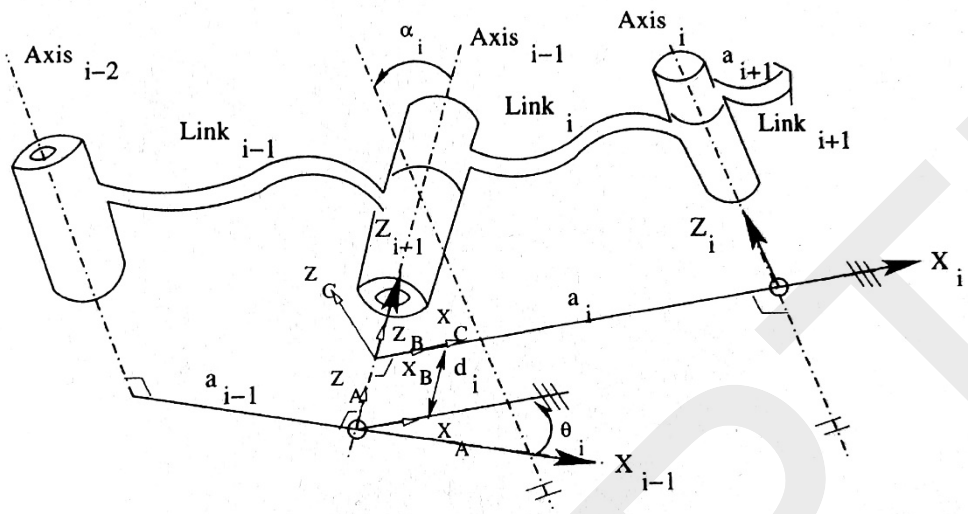
- **Revolute joint:** θ_i is variable
- **Prismatic joint:** d_i is variable

Rules for Coordinate Assignment



Note: Y axis is perpendicular to the plane of the paper and moving inside it

- Z_i is an axis about which the rotation is considered or along which the translation takes place
- If Z_{i-1} and Z_i axes are parallel to each other, X axis will be directed from Z_{i-1} to Z_i along their common normal



Note: Y axis is perpendicular to the plane of the paper and moving inside it

- If Z_{i-1} and Z_i axes intersect each other, X axis can be selected along either of two remaining directions

- If Z_{i-1} and Z_i axes act along a straight line, X axis can be selected anywhere in a plane perpendicular to them

- Y axis is decided as Y = ZxX

We have

$$ {}^{i-1}_{i}T = {}^{i-1}_{A}T\,{}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}_{i}T $$

$$ = ROT(Z,\theta_i)TRANS(Z,d_i)ROT(X,\alpha_i)TRANS(X,a_i) $$

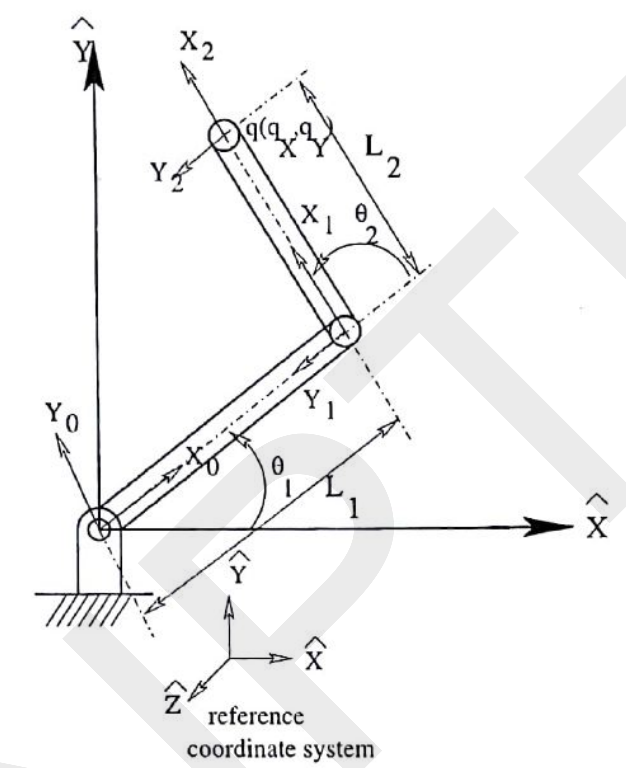
$$ = Screw_Z Screw_X $$

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Now, ${}^{i}_{i-1}T = \left[{}^{i-1}_{i}T\right]^{-1}$

$$ = \begin{bmatrix} c\theta_i & s\theta_i & 0 & -a_i \\ -s\theta_i c\alpha_i & c\theta_i c\alpha_i & s\alpha_i & -d_i s\alpha_i \\ s\theta_i s\alpha_i & -c\theta_i s\alpha_i & c\alpha_i & -d_i c\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Example 1



Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	0	L_1
2	θ_2	0	0	L_2

Forward Kinematics

$$ {}^{Base}_{2}T = {}^{Base}_{1}T\,{}^{1}_{2}T $$

$$
\begin{aligned}
{}^{Base}_{1}T &= ROT(\hat{Z},\theta_1)TRANS(\hat{X},L_1) \\
&= \begin{bmatrix} c_1 & -s_1 & 0 & L_1c_1 \\ s_1 & c_1 & 0 & L_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}_{2}T &= ROT(\hat{Z}, \theta_2)TRANS(\hat{X}, L_2) \\
&= \begin{bmatrix} c_2 & -s_2 & 0 & L_2c_2 \\ s_2 & c_2 & 0 & L_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{Base}_{2}T &= {}^{Base}_{1}T\,{}^{1}_{2}T \\
&= \begin{bmatrix} c_{12} & -s_{12} & 0 & L_1c_1 + L_2c_{12} \\ s_{12} & c_{12} & 0 & L_1s_1 + L_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Inverse Kinematics

$$
{}^{Base}_{2}T = \begin{bmatrix} c\phi & -s\phi & 0 & q_x \\ s\phi & c\phi & 0 & q_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

$$q_x = L_1 c_1 + L_2 c_{12}$$

$$q_y = L_1 s_1 + L_2 s_{12}$$

$$q_x^2 + q_y^2 = L_1^2 + L_2^2 + 2L_1 L_2 c_{12} c_1 + 2L_1 L_2 s_{12} s_1,$$

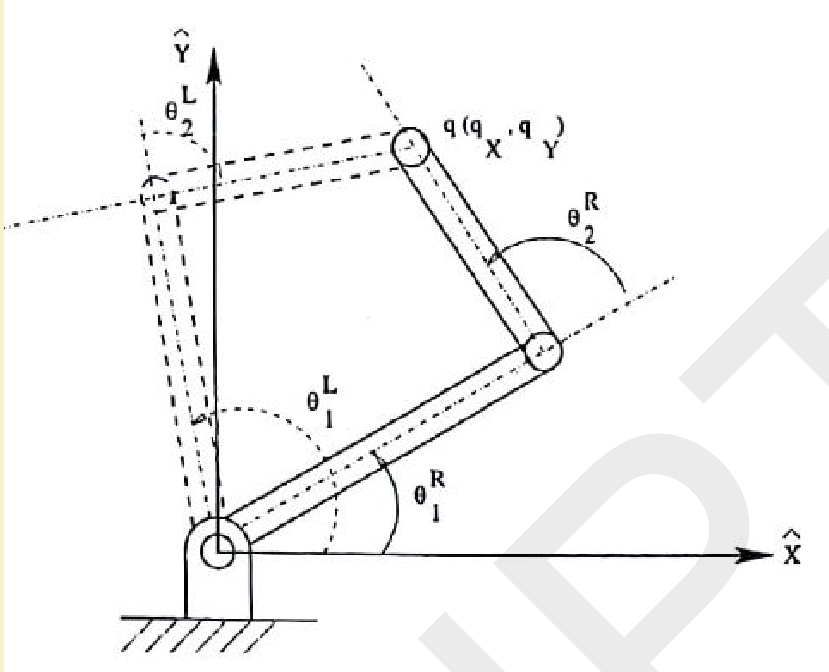
$$c_2 = \frac{q_x^2 + q_y^2 - L_1^2 - L_2^2}{2L_1 L_2}$$

$$\theta_2 = \arccos\left(\frac{q_x^2+q_y^2-L_1^2-L_2^2}{2L_1L_2}\right).$$

$$\begin{aligned} q_x &= L_1c_1 + L_2c_1c_2 - L_2s_1s_2 \\ &= c_1(L_1 + L_2c_2) - s_1(L_2s_2) \end{aligned}$$

$$\rho = \sqrt{(L_1 + L_2c_2)^2 + (L_2s_2)^2},$$
$$\psi = \arctan(\frac{L_1+L_2c_2}{L_2s_2}).$$

$$q_x = \rho c_1 sin\psi - \rho s_1 cos\psi = \rho sin(\psi - \theta_1)$$



$$q_y = \rho cos(\psi - \theta_1)$$

$$\theta_1 = \psi - \arctan(\frac{q_x}{q_y})$$