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NPTEL ONLINE CERTIFICATION COURSES

NPTEL

WEEK 4: ROBOTICS

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Topic 2: Robot Kinematics

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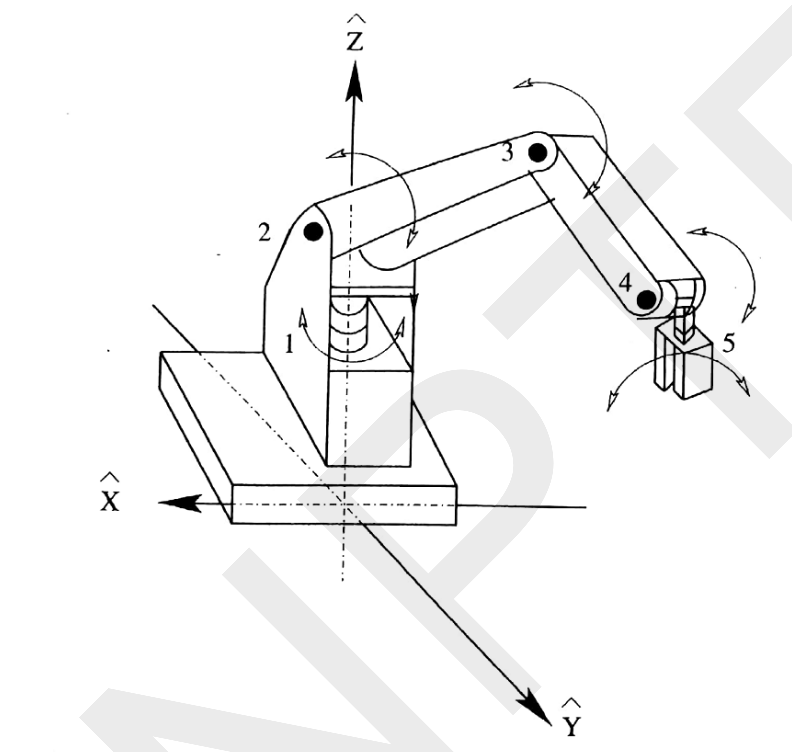
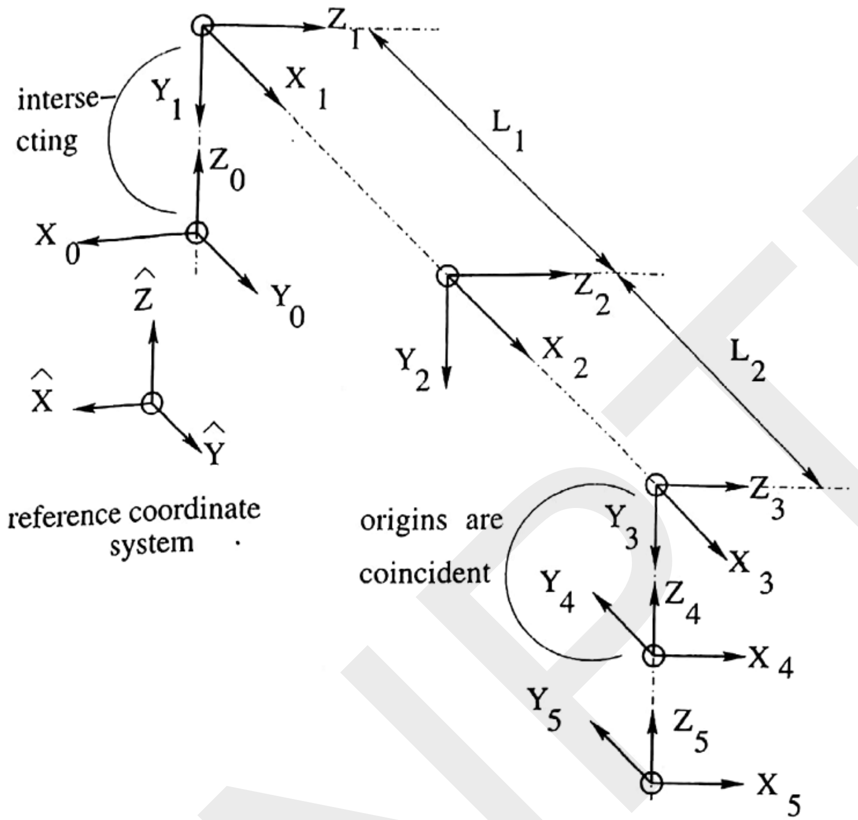
Example 2





Frame	θ_i	d_i	α_i	a_i
1	θ_1	0	−90	0
2	θ_2	0	0	L_1
3	θ_3	0	0	L_2
4	θ_4	0	90	0
5	θ_5	0	0	0

Forward Kinematics

$$ {}^0_5T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T $$

$$
\begin{aligned}
{}^0_1T &= Rot(\hat{Z},\theta_1)Rot(\hat{X},-90) \\
&= \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Here, c_1 and s_1 denote $c\theta_1$ (or, $cos\theta_1$) and $s\theta_1$ (or, $sin\theta_1$), respectively.

$$
{}^{0}_{1}T^{-1} = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}T = Rot(\hat{Z}, \theta_2)Trans(\hat{X}, L_1)
$$

$$= \begin{bmatrix} c_2 & -s_2 & 0 & L_1c_2 \\ s_2 & c_2 & 0 & L_1s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} {}^2_3T &= Rot(\hat{Z}, \theta_3)Trans(\hat{X}, L_2) \\ &= \begin{bmatrix} c_3 & -s_3 & 0 & L_2c_3 \\ s_3 & c_3 & 0 & L_2s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
{}^{3}_{4}T &= Rot(\hat{Z},\theta_4)Rot(\hat{X},90) \\
&= \begin{bmatrix} c_4 & 0 & s_4 & 0 \\ s_4 & 0 & -c_4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
{}^{4}_{5}T = Rot(\hat{Z},\theta_5)
$$

$$
= \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
{}^0_5T &= {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T \\
&= \begin{bmatrix} v_{11} & v_{12} & v_{13} & p_x \\ v_{21} & v_{22} & v_{23} & p_y \\ v_{31} & v_{32} & v_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
v_{11} &= c_1 c_{234} c_5 - s_1 s_5 \\
v_{12} &= -c_1 c_{234} s_5 - s_1 c_5 \\
v_{13} &= c_1 s_{234} \\
v_{21} &= s_1 c_{234} c_5 + c_1 s_5 \\
v_{22} &= -s_1 c_{234} s_5 + c_1 c_5 \\
v_{23} &= s_1 s_{234} \\
v_{31} &= -s_{234} c_5 \\
v_{32} &= s_{234} s_5 \\
v_{33} &= c_{234} \\
p_x &= c_1 (L_1 c_2 + L_2 c_{23}) \\
p_y &= s_1 (L_1 c_2 + L_2 c_{23}) \\
p_z &= -L_1 s_2 - L_2 s_{23}
\end{aligned}
$$

Inverse Kinematics

$$ {}^0_5T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & q_x \\ r_{21} & r_{22} & r_{23} & q_y \\ r_{31} & r_{32} & r_{33} & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \begin{aligned} {}^0_5T &= {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T \\ \Rightarrow {}^0_1T^{-1}({}^0_5T) &= {}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T \end{aligned} $$

$$\Rightarrow \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & q_x \\ r_{21} & r_{22} & r_{23} & q_y \\ r_{31} & r_{32} & r_{33} & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} c_{234}c_5 & -c_{234}s_5 & s_{234} & L_1c_2 + L_2c_{23} \\ s_{234}c_5 & -s_{234}s_5 & -c_{234} & L_1s_2 + L_2s_{23} \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} r_{11}c_1 + r_{21}s_1 & r_{12}c_1 + r_{22}s_1 & r_{13}c_1 + r_{23}s_1 & q_xc_1 + q_ys_1 \\ -r_{31} & -r_{32} & -r_{33} & -q_z \\ -r_{11}s_1 + r_{21}c_1 & -r_{12}s_1 + r_{22}c_1 & -r_{13}s_1 + r_{23}c_1 & -q_xs_1 + q_yc_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} c_{234}c_5 & -c_{234}s_5 & s_{234} & L_1c_2 + L_2c_{23} \\ s_{234}c_5 & -s_{234}s_5 & -c_{234} & L_1s_2 + L_2s_{23} \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$q_x c_1 + q_y s_1 = L_1 c_2 + L_2 c_{23}$$

$$-q_x s_1 + q_y c_1 = 0$$

$$q_z = -L_1 s_2 - L_2 s_{23}$$

$$s_{234} = r_{13} c_1 + r_{23} s_1$$

$$c_{234} = r_{33}$$

$$-r_{11} s_1 + r_{21} c_1 = s_5$$

$$-r_{12} s_1 + r_{22} c_1 = c_5$$

$$\begin{aligned} -q_x s_1 + q_y c_1 &= 0 \\ \Rightarrow \theta_1 &= \arctan\left(\frac{q_y}{q_x}\right) \end{aligned}$$

$$
\begin{aligned}
q_x^2 + q_y^2 + q_z^2 &= L_1^2 + L_2^2 + 2L_1L_2c_3 \\
\Rightarrow \theta_3 &= \arccos\left(\frac{q_x^2 + q_y^2 + q_z^2 - L_1^2 - L_2^2}{2L_1L_2}\right)
\end{aligned}
$$

$$
\begin{aligned}
L_1c_2 + L_2c_{23} &= q_xc_1 + q_ys_1 \\
\Rightarrow (L_1 + L_2c_3)c_2 - (L_2s_3)s_2 &= q_xc_1 + q_ys_1
\end{aligned}
$$

Let us assume $L_1 + L_2c_3 = \rho\sin\alpha$ and $L_2s_3 = \rho\cos\alpha$, where $\rho \neq 0$ and $\rho = \sqrt{(L_1 + L_2c_3)^2 + (L_2s_3)^2}$; $\alpha = \arctan(\frac{L_1+L_2c_3}{L_2s_3})$. Thus, the above expression can be written as follows:

$$\rho \sin \alpha c_2 - \rho \cos \alpha s_2 \quad = \quad q_x c_1 + q_y s_1$$

$$\rho \sin(\alpha - \theta_2) = q_x c_1 + q_y s_1$$

$$\rho \cos(\alpha - \theta_2) = -q_z$$

$$\tan(\alpha - \theta_2) = \frac{q_x c_1 + q_y s_1}{-q_z}$$

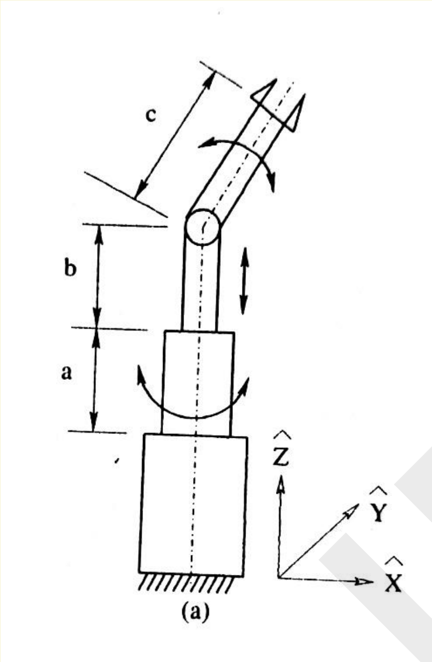
$$\Rightarrow \theta_2 = \alpha - \arctan(\frac{q_x c_1 + q_y s_1}{-q_z})$$

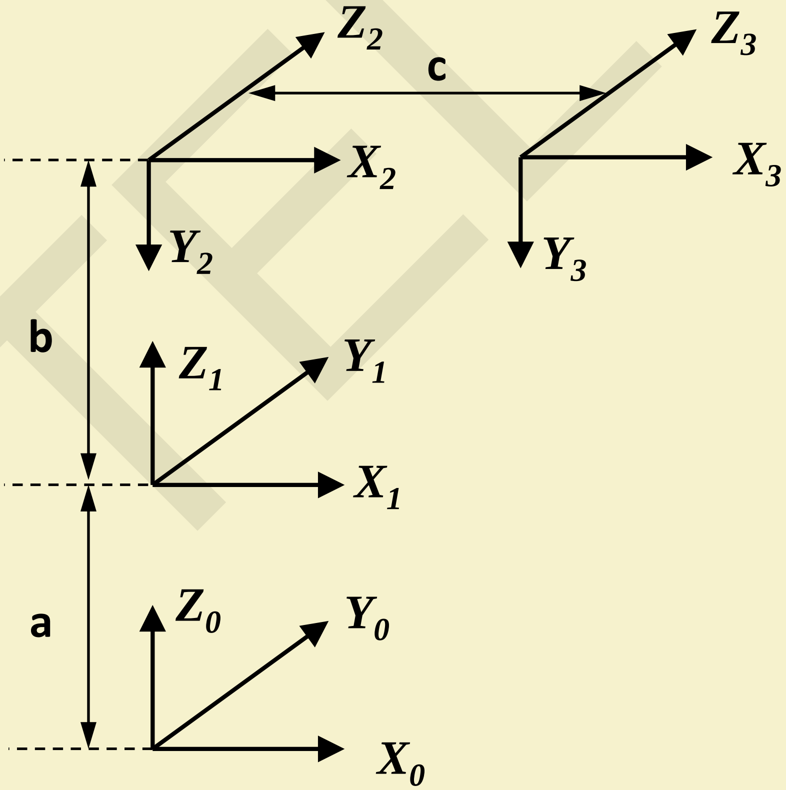
$$\theta_2 + \theta_3 + \theta_4 = \arctan(\frac{r_{13} c_1 + r_{23} s_1}{r_{33}})$$

$$\Rightarrow \theta_4 = \arctan(\frac{r_{13} c_1 + r_{23} s_1}{r_{33}}) - \theta_2 - \theta_3$$

$$\theta_5 = \arctan(\frac{-r_{11} s_1 + r_{21} c_1}{-r_{12} s_1 + r_{22} c_1})$$

An Example



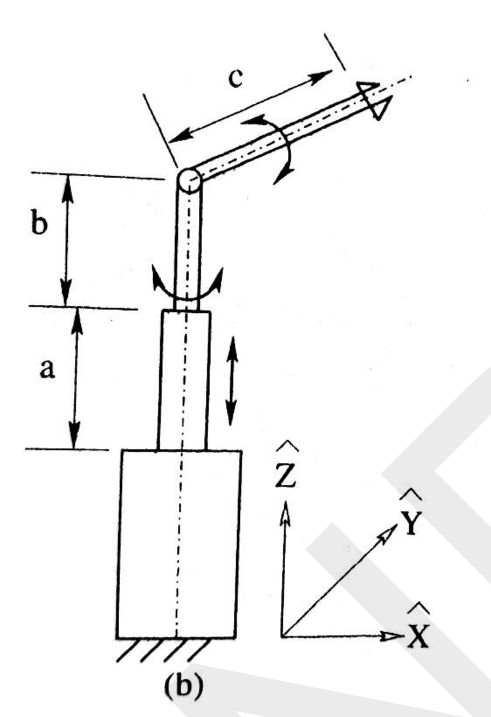


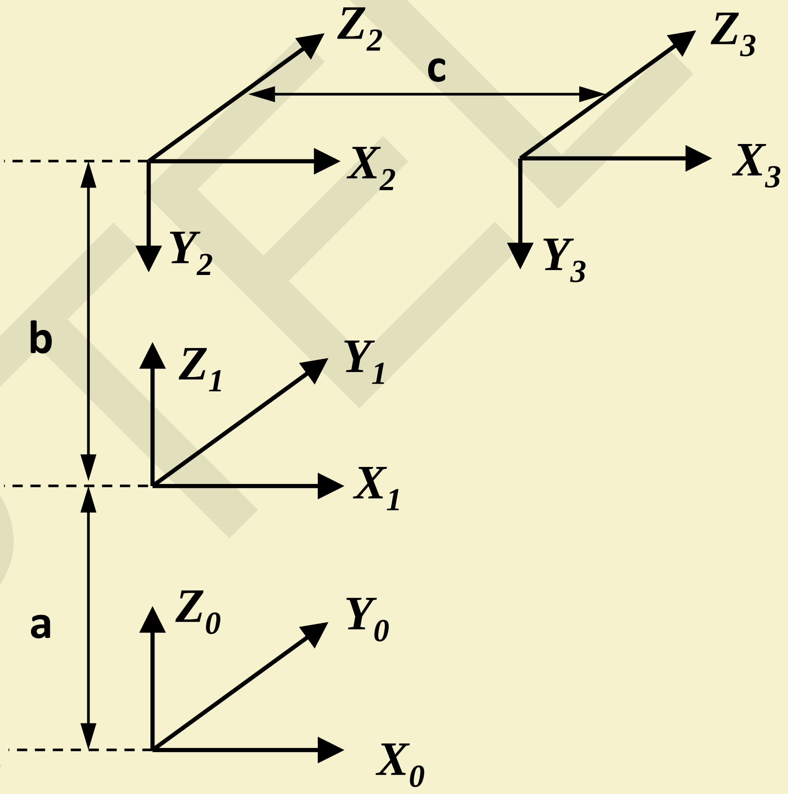
D-H parameters' table:

Frame	θ_i	d_i	α_i	a_i
1	θ_1	a*	0	0
2	0	b	-90	0
3	θ_3	0	0	c

a*: fixed offset

Another Example





D-H parameters' table:

Frame	θ_i	d_i	α_i	a_i
1	0	a	0	0
2	θ_2	b*	-90	0
3	θ_3	0	0	c

b*: fixed offset

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TOPIC 3: TRAJECTORY PLANNING

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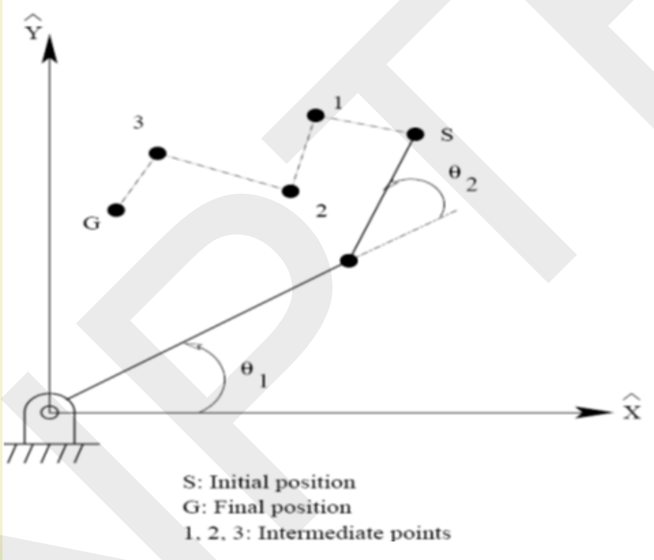
Aim: To determine time history of position, velocity and acceleration of end-effector of a manipulator, while moving from an initial position to a final position through some intermediate/via points.



Points	Cartesian scheme	Joint-space scheme
S	(X_S, Y_S)	(θ_1^S, θ_2^S)
1	(X_1, Y_1)	(θ_1^1, θ_2^1)
2	(X_2, Y_2)	(θ_1^2, θ_2^2)
3	(X_3, Y_3)	(θ_1^3, θ_2^3)
G	(X_G, Y_G)	(θ_1^G, θ_2^G)

Trajectory Planning

Cartesian scheme

- Computationally expensive, as inverse kinematics problem has to be solved, on-line

Joint-space scheme

Joint-Space Scheme

- To fit a smooth (continuous) curve through (θ_1^S, θ_1^1, θ_1^2, θ_1^3, θ_1^G)
- First and second order derivatives must be continuous.

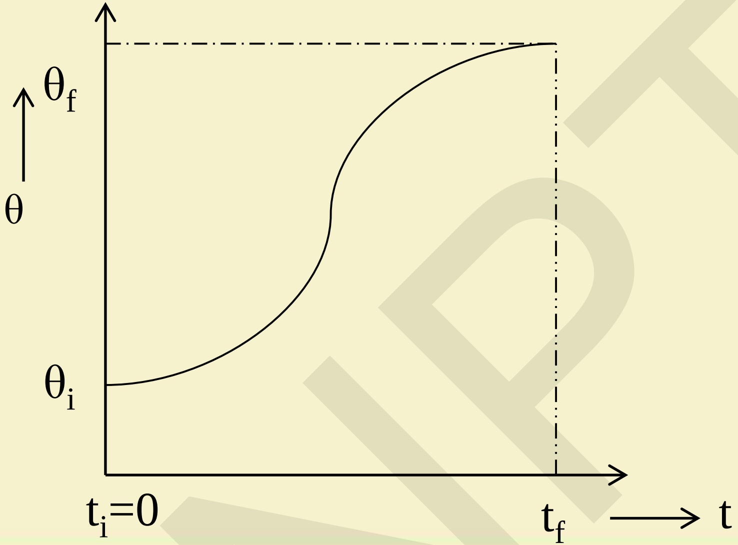
Various Trajectory Functions

- Cubic polynomial
- Fifth-order polynomial
- Linear trajectory function

POLYNOMIAL TRAJECTORY FUNCTION

Case-1

Initial and final values of joint angle are known, and angular velocities at the beginning and end of the cycle are kept equal to zero.



$$\text{At } t = t_i = 0;\ \theta = \theta_i,\ \dot{\theta} = 0$$

$$\text{At } t = t_f;\ \theta = \theta_f,\ \dot{\theta} = 0$$

Let us consider cubic polynomial

$$\theta(t) = C_0 + C_1 t + C_2 t^2 + C_3 t^3$$

where C_0, C_1, C_2, C_3 **are the coefficients.**

Differentiate $\theta(t)$ **with respect to time to get angular velocity**

$$\dot{\theta}(t) = C_1 + 2C_2 t + 3C_3 t^2$$

Apply the initial conditions to angular displacement and velocity equations. We get,

$$C_0 = \theta_i$$

$$C_1 = 0$$

$$C_0 + C_1 t_f + C_2 t_f^2 + C_3 t_f^3 = \theta_f$$

$$C_1 + 2C_2 t_f + 3C_3 t_f^2 = 0$$

Solving above equations, We get

$$\theta(\mathrm{t}) = \theta_{\mathrm{i}} + \frac{3(\theta_{\mathrm{f}} - \theta_{\mathrm{i}})}{\mathrm{t}_{\mathrm{f}}^2} t^2 - \frac{2(\theta_{\mathrm{f}} - \theta_{\mathrm{i}})}{\mathrm{t}_{\mathrm{f}}^3} t^3$$

Case-2

Initial and final values of joint angle are known and angular velocities at the beginning and end of the cycle are assumed to have non zero values.

At $t = t_i = 0;\ \theta = \theta_i,\ \dot{\theta} = \dot{\theta}_i$

At $t = t_f;\ \theta = \theta_f,\ \dot{\theta} = \dot{\theta}_f$

Let us consider a third order polynomial of the form:

$$\theta(t) = C_0 + C_1 t + C_2 t^2 + C_3 t^3$$

where C_0, C_1, C_2, C_3 **are the coefficients.**

Differentiate $\theta(t)$ with respect to time to get angular velocity

$$\dot{\theta}(t) = C_1 + 2C_2 t + 3C_3 t^2$$

Apply the initial conditions to angular displacement and velocity equations. We get,

$$C_0 = \theta_i$$

$$C_1 = \dot{\theta}_i$$

$$C_0 + C_1 t_f + C_2 t_f^2 + C_3 t_f^3 = \theta_f$$

$$C_1 + 2C_2 t_f + 3C_3 t_f^2 = \dot{\theta}_f$$

Solving above equations, We get

$$C_0 = \theta_i,$$

$$C_1 = \dot{\theta}_i,$$

$$C_2 = \frac{3(\theta_f - \theta_i)}{t_f^2} - \frac{2}{t_f}\dot{\theta}_i - \frac{1}{t_f}\dot{\theta}_f,$$

$$C_3 = -\frac{2(\theta_f - \theta_i)}{t_f^3} + \frac{1}{t_f^2}(\dot{\theta}_f + \dot{\theta}_i).$$

Case-3

Initial and final values of joint angle are known and angular velocities and accelerations at the beginning and end of the cycle are assumed to have non zero values.

At $t = t_i = 0;\ \theta = \theta_i,\ \dot{\theta} = \dot{\theta}_i;\ \ddot{\theta} = \ddot{\theta}_i$

At $t = t_f;\ \theta = \theta_f,\ \dot{\theta} = \dot{\theta}_f,\ \ddot{\theta} = \ddot{\theta}_f$

Let us consider a fifth-order polynomial as follows:

$$\theta(t) = C_0 + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4 + C_5 t^5$$

where C_0, C_1, C_2, C_3, C_4 **and** C_5 **are the coefficients.**

Differentiate $\theta(t)$ with respect to time once to get angular velocity and twice to get angular acceleration

$$\dot{\theta}(t) = c_1 + 2c_2t + 3c_3t^2 + 4c_4t^3 + 5c_5t^4$$

$$\ddot{\theta}(t) = 2c_2 + 6c_3t + 12c_4t^2 + 20c_5t^3$$

Apply the initial conditions to angular displacement, velocity and acceleration equations. We get,

$$c_0 = \theta_i$$

$$c_1 = \dot{\theta}_i$$

$$c_2 = \frac{1}{2}\ddot{\theta}_i$$

$$c_0 + c_1 t_f + c_2 t_f^2 + c_3 t_f^3 + c_4 t_f^4 + c_5 t_f^5 = \theta_f$$

$$c_1 + 2c_2 t_f + 3c_3 t_f^2 + 4c_4 t_f^3 + 5c_5 t_f^4 = \dot{\theta}_f$$

$$2c_2 + 6c_3 t_f + 12c_4 t_f^2 + 20c_5 t_f^3 = \ddot{\theta}_f$$

Solving above equations, We get

$$C_0 = \theta_i,$$

$$C_1 = \dot{\theta}_i,$$

$$C_2 = \tfrac{1}{2}\ddot{\theta}_i$$

$$C_3 = \frac{20(\theta_f - \theta_i) - (8\dot{\theta}_f + 12\dot{\theta}_i)t_f - (3\ddot{\theta}_i - \ddot{\theta}_f)t_f^2}{2t_f^3}$$

$$C_4 = \frac{30(\theta_i - \theta_f) + (14\dot{\theta}_f + 16\dot{\theta}_i)t_f + (3\ddot{\theta}_i - 2\ddot{\theta}_f)t_f^2}{2t_f^4}$$

$$C_5 = \frac{12(\theta_f - \theta_i) - 6(\dot{\theta}_f + \dot{\theta}_i)t_f - (\ddot{\theta}_i - \ddot{\theta}_f)t_f^2}{2t_f^5}$$

A Numerical Example

- A single-link robot with a revolute joint is motionless at θ = 20º. It is desired to move the joint in a smooth manner to θ = 80º in 4.0 seconds. Find a suitable cubic polynomial to generate this motion and bring the manipulator to rest at the goal.

Solution:

cubic polynomial

$$\theta(t) = C_0 + C_1 t + C_2 t^2 + C_3 t^3 \quad \text{-------} \ (1)$$

Conditions:

At time $t = t_i = 0,$

$\theta = \theta_i = 20^{\circ},$

$\dot{\theta} = 0;$

At time $t = t_f = 4.0 \text{ s},$

$\theta = \theta_f = 80^{\circ},$

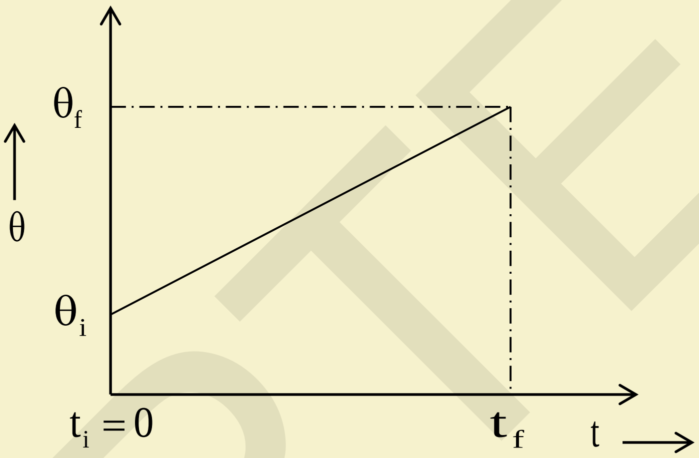
$\dot{\theta} = 0;$

$$\theta(t) = \theta_i + \frac{3(\theta_f - \theta_i)}{t_f^2} t^2 - \frac{2(\theta_f - \theta_i)}{t_f^3} t^3$$

$$= 20 + \frac{3(80-20)}{(4.0)^2} t^2 - \frac{2(80-20)}{(4.0)^3} t^3$$

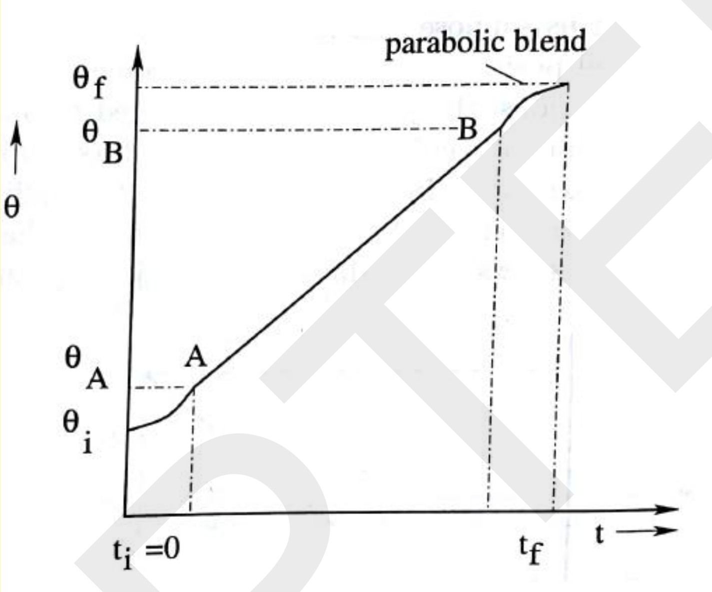
$$= 20 + 11.25t^2 - 1.875t^3$$

LINEAR TRAJECTORY FUNCTION



(Pure Linear Trajectory Function)

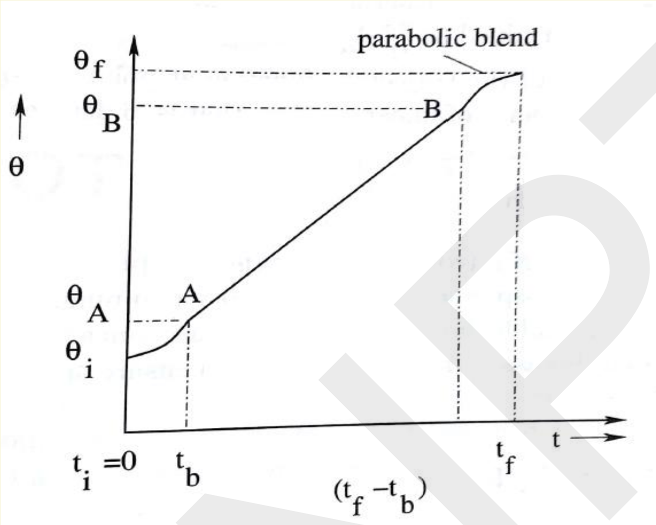
Note: Infinite acceleration and declaration.



(Modified Linear trajectory function with parabolic blends)

A Numerical Example

- **A linear trajectory function with parabolic blends at its two ends is to be obtained to satisfy the following conditions given below.**



At time $t = t_i = 0,$

$\theta = \theta_i = 20.0^{o},$

$\dot{\theta} = 0.0;$

At time $t = t_f = 12.0 \text{ s},$

$\theta = \theta_f = 74.0^{o},$

$\dot{\theta} = 0.0;$

Total cycle time $t_c = t_f - t_i = 12.0$ s

Time duration at each of the blend portion $t_b = 3.0$ s

Magnitude of acceleration/deceleration

$$\ddot{\theta} = 2.0 \text{ degree/s}^2$$

Determine angular displacement and velocity at two junctions of parabolic blends with the straight portion of trajectory function.

Solution:

At point A

Angular displacement

$$\theta_A = \theta_i + \frac{1}{2} \times \ddot{\theta} \times t_b^2 = 20.0 + \frac{1}{2} \times (2.0) \times (3.0)^2$$

$$= 29.0^{o}$$

Angular velocity

$$\dot{\theta}_A = \dot{\theta}_i + \ddot{\theta} \times t_b = 0.0 + 2.0 \times 3.0$$

$$= 6.0 \text{ degree/s}$$

At point B: **From the symmetry of the trajectory function,**

$$\theta_f - \theta_B = \theta_A - \theta_i$$

$$74.0 - \theta_B = 29.0 - 20.0$$

$$\theta_B = 65.0^{\circ}$$

Angular velocity in the linear portion of trajectory function

$$= \frac{\theta_B - \theta_A}{t_f - 2t_b}$$

$$= \frac{65.0 - 29.0}{12.0 - 2 \times 3.0} = \frac{36.0}{6.0} = 6.0 \text{ degree/s}$$

To maintain continuity of the trajectory function at point B, $\dot{\theta}_B$ Should be equal to the velocity of the linear portion, that is, 6.0 degree per second.

SINGULARITY CHECKING THROUGH JACOBIAN

Let us consider six functions and each of which is a function of six Independent variables.

$$
\begin{aligned}
y_1 &= f_1(x_1, x_2, x_3, x_4, x_5, x_6) \\
y_2 &= f_2(x_1, x_2, x_3, x_4, x_5, x_6) \\
&\vdots \\
&\vdots \\
y_6 &= f_6(x_1, x_2, x_3, x_4, x_5, x_6)
\end{aligned}
$$

In vector notation: $Y = F(X)$

Now,

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots. + \frac{\partial f_1}{\partial x_6}\delta x_6$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \ldots. + \frac{\partial f_2}{\partial x_6}\delta x_6$$

$$\vdots$$

$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \ldots. + \frac{\partial f_6}{\partial x_6}\delta x_6$$

In vector notation: $\delta Y = J(X)\delta X$

where J(X) is Jacobian.

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_6} \\ . & & & \\ . & & & \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \cdots & \frac{\partial f_6}{\partial x_6} \end{bmatrix}$$

Now,

$$\lim_{\delta t \to 0} \frac{\delta Y}{\delta t} = \lim_{\delta t \to 0} J(X) \frac{\delta X}{\delta t}$$

$$\Rightarrow \dot{Y} = J(X)\dot{X}$$

In Robotics,

$$V = J(\theta)\dot{\theta}$$

Cartesian Velocity

Jacobian

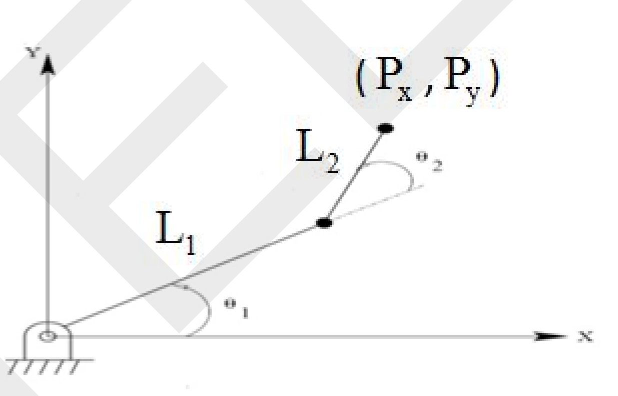
Joint Velocity

TWO DOF SERIAL MANIPULATOR

$$P_x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$

$$\frac{\partial P_x}{\partial \theta_1} = -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2)$$

$$\frac{\partial P_x}{\partial \theta_2} = -L_2 \sin(\theta_1 + \theta_2)$$



$$P_y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

$$\frac{\partial P_y}{\partial \theta_1} = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$

$$\frac{\partial P_y}{\partial \theta_2} = L_2 \cos(\theta_1 + \theta_2)$$

Jacobian $J(\theta) = \begin{pmatrix} -L_1S_1 - L_2S_{12} & -L_2S_{12} \\ L_1C_1 + L_2C_{12} & L_2C_{12} \end{pmatrix}$

Now, $\dot{\theta} = J^{-1}(\theta)\,V$

$J^{-1}(\theta)$ **should exist, that is,** $|J(\theta)| \neq 0$

For Singularity Checking

$$|J(\theta)| = 0$$

$$\Rightarrow L_1 L_2 S_2 = 0$$

Now,

$$L_1 \neq 0, L_2 \neq 0,$$

$$S_2 = 0$$

So,

$$\theta_2 = 0^0 \text{ or } 180^0$$

When

$\theta_2 = 0^0; \longrightarrow$ **Fully- Stretched**

$\theta_2 = 180^0; \longrightarrow$ **Folded-back Situation**