Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

Bishakh Bhattacharya & Nachiketa Tiwari
Indian Institute of Technology Kanpur

Lecture 30

The Galerkin Method

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Example 3: Simply-Supported Plate on All Sides

- Consider a plate of dimensions *a* X *b* in *x* and *y* directions respectively, which is simply supported on all four sides.

- Further, we assume that this plate has a symmetric and specially orthotropic lamination sequence. Thus,
  - [B] = [0]   due to symmetry
  - $A_{16} = A_{26} = D_{16} = D_{26} = 0$ due to special orthotropy.

- Finally we assume that the plate is normally loaded as shown in Fig. 30.1 with a constant load intensity of q_0.

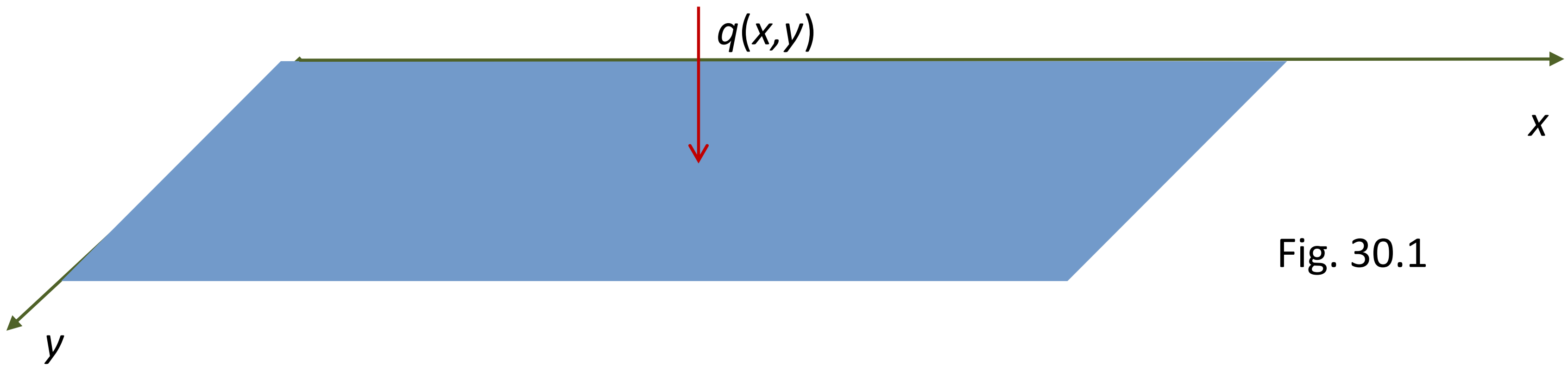

Fig. 30.1

Example 3: Simply-Supported Plate on All Sides

- For such a plate, the out-of-plane boundary conditions are:
  - $w^{\pm} = 0$ on all four sides.
  - $M_x^{\pm} = 0$ at $x = 0, a$.
  - $M_y^{\pm} = 0$ at $y = 0, b$.

- As explained earlier, the governing equation for equilibrium for out-of-plane direction for such a plate is decoupled with in-plane equations, because the plate's lamination sequence is symmetric. This equation is reproduced below.

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

- Expressing this equation in terms of derivatives of w^o, we get:

$$q(x,y) = D_{11}\frac{\partial^4 w^o}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w^o}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w^o}{\partial y^4} \qquad \text{(Eq. 30.1)}$$

Example 3: Simply-Supported Plate on All Sides

- For such a plate, we have already developed an exact solution. Here we develop Galerkin solution and compare it with the exact solution.

- For this, we assume that the solution is of form:

$$w^o(x,y) = w_{11}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

- Such an assumed solution form satisfies all the kinematic BCs for the problem. Using this, we compute the error in PDE.

$$E(x,y) = \pi^4\left[D_{22}\left(\frac{1}{a}\right)^4 + 2(D_{12}+2D_{66})\left(\frac{1}{ab}\right)^2 + D_{22}\left(\frac{1}{b}\right)^4\right]w_{11}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} - q_o$$

- Multiplying this error with virtual displacement and integrating the product over plate's area we get:

$$\int_0^b\int_0^a\left[\pi^4\left\{D_{22}\left(\frac{1}{a}\right)^4 + 2(D_{12}+2D_{66})\left(\frac{1}{ab}\right)^2 + D_{22}\left(\frac{1}{b}\right)^4\right\}w_{11}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} - q_o\right]\epsilon w_{11_1}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,dxdy = 0 \quad \text{(Eq. 30.1)}$$

Example 3: Simply-Supported Plate on All Sides

- From Eq. 30.1, we get:

$$w_{11} = \frac{16q_o}{\mathbb{D}\pi^4}$$

where,

$$\mathbb{D} = \left[D_{22}\left(\frac{\pi}{a}\right)^4 + 2(D_{12} + 2D_{66})\left(\frac{\pi^2}{ab}\right)^2 + D_{22}\left(\frac{\pi}{b}\right)^4 \right]$$

- This value of w_{11} is exactly the same as that determined in the exact solution. In this case, we get same results because our choice of $w(x,y)$ coincides with that of the exact solution.

- In this Example, we have assumed the plate to be specially orthotropic. Thus, the role of terms D_{16} and D_{26} was not present. Later, we will explore this role by solving another problem which is slightly different than Example 3.

Another Interpretation of Galerkin Method

- Consider the 1-D beam equation as discussed earlier. For such a beam, the error in force equilibrium equation using an assumed solution is:

  $E[w(x)] = EI\cdot(d^2w/dx^2) - q_0$

- The value of such an error changes with position, both in magnitude and sign.

- Also, the integral of square of this error over the domain (i.e. beam length) can be expressed as:

  $\int_\Omega \{E[w(x)]\}^2 \, dx$,

  or,

  $\int_\Omega [EI\cdot(d^2w/dx^2) - q_0]^2 \, dx$

Another Interpretation of Galerkin Method

- A "good enough" solution for $w(x)$, would be when the integral of this squared error is minimized.

- The condition for a minima of a function is when its 1st derivative is zero. Using this, we get the condition for minima of integral of square-error as:

$$\frac{\partial}{\partial w}\int_{-\frac{L}{2}}^{\frac{L}{2}}\left(EI\frac{d^4w}{dx^4}-q_o\right)^2 dx=0$$

- At this stage, we assume a form for w(x), and introduce it in above equation. We assume,

$$w(x) = Acos\frac{\pi x}{L}$$

Putting this in expression for integral of square-error, we get:

$$\frac{\partial}{\partial w}\int_{-\frac{L}{2}}^{\frac{L}{2}}\left(AEI\left(\frac{\pi}{L}\right)^4 cos\frac{\pi x}{L}-q_o\right)^2 dx=0$$

Another Interpretation of Galerkin Method

- Now the integral of square error involves *E*, *I*, *A*, *L*, q_o, but *not x*. Thus, when this error is minimized we have to differentiate it w.r.t. *A*, because *E*, *I*, and *L* are known constants.

$$\frac{\partial}{\partial w}\int_{-\frac{L}{2}}^{\frac{L}{2}}\left(AEI\left(\frac{\pi}{L}\right)^4 cos\frac{\pi x}{L}-q_o\right)^2 dx=\frac{\partial}{\partial A}\int_{-\frac{L}{2}}^{\frac{L}{2}}\left(AEI\left(\frac{\pi}{L}\right)^4 cos\frac{\pi x}{L}-q_o\right)^2 dx=0$$

- Thus, the condition for minima of integral of square error is:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{\partial}{\partial A}\left(AEI\left(\frac{\pi}{L}\right)^4 cos\frac{\pi x}{L}-q_o\right)^2 dx=\int_{-\frac{L}{2}}^{\frac{L}{2}}2\left(AEI\left(\frac{\pi}{L}\right)^4 cos\frac{\pi x}{L}-q_o\right)AEI\left(\frac{\pi}{L}\right)^4 cos\frac{\pi x}{L}\,dx=0$$

Another Interpretation of Galerkin Method

- However, since the integral is over *x*, we can rewrite above expression as:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\partial}{\partial A}\left(AEI\left(\frac{\pi}{L}\right)^4 cos\frac{\pi x}{L} - q_o\right)^2 dx = \int_{-\frac{L}{2}}^{\frac{L}{2}} 2\left(AEI\left(\frac{\pi}{L}\right)^4 cos\frac{\pi x}{L} - q_o\right)AEI\left(\frac{\pi}{L}\right)^4 cos\frac{\pi x}{L} dx = 0$$

- This could be re-written as:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \left(AEI\left(\frac{\pi}{L}\right)^4 cos\frac{\pi x}{L} - q_o\right) cos\frac{\pi x}{L} dx = 0$$

or,

$$\int_{-L/2}^{L/2} E[w\{x)]w_1(x)dx = 0$$

Another Interpretation of Galerkin Method

- Thus we see that such an approach is equivalent to Special Galerkin method, where mathematical expressions for virtual displacement and actual displacement expressions are very much the same.

- As Galerkin method involves a least squares approach, multiple terms increase the flexibility of the system so that its energy is reduced.

- Hence, the accuracy of Galerkin method increases with number of terms used in the solution.

- Such an interpretation can also be generalized in context of partial differential equations.