

# Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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## Lecture 22

# Analysis of a Laminated Composite



# References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.
2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill
3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic
4. Nonlinear Analysis of Plates, Chia, C., McGraw-Hill International Book Company



# Lecture Overview

- Introduction
- Stiffness Matrices for Laminate
- Discussion on Stiffness Matrix Elements
- Special Lamination Sequences



# Introduction

- Earlier, resultant stress and moments have been defined as integrals of stress, and stress times distance from a plate's mid-plane, respectively.
- Substituting definitions of stress from Eq. 20.7 in Eq. 21.5, we get:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{bmatrix} \underline{Q}_{11} & \underline{Q}_{12} & \underline{Q}_{16} \\ \underline{Q}_{12} & \underline{Q}_{22} & \underline{Q}_{26} \\ \underline{Q}_{16} & \underline{Q}_{26} & \underline{Q}_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \right\} dz$$

- This can be simplified as:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{bmatrix} \underline{Q}_{11} & \underline{Q}_{12} & \underline{Q}_{16} \\ \underline{Q}_{12} & \underline{Q}_{22} & \underline{Q}_{26} \\ \underline{Q}_{16} & \underline{Q}_{26} & \underline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} dz + \sum_{k=1}^n \int_{z_{k-1}}^{z_k} \begin{bmatrix} \underline{Q}_{11} & \underline{Q}_{12} & \underline{Q}_{16} \\ \underline{Q}_{12} & \underline{Q}_{22} & \underline{Q}_{26} \\ \underline{Q}_{16} & \underline{Q}_{26} & \underline{Q}_{66} \end{bmatrix} z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} dz$$

- Since we know that mid-plane strains and curvatures are independent of  $z$  we can take them out of the integral and summation operations in the above equation.



# Laminate Stiffness Matrices

- Also,  $[Q]$  remains constant over the thickness of each layer, but its value changes from one layer to the other. Thus, the relation for resultant stresses becomes:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \left( \sum_{k=1}^n (z_k - z_{k-1}) \begin{bmatrix} \underline{Q}_{11} & \underline{Q}_{12} & \underline{Q}_{16} \\ \underline{Q}_{12} & \underline{Q}_{22} & \underline{Q}_{26} \\ \underline{Q}_{16} & \underline{Q}_{26} & \underline{Q}_{66} \end{bmatrix} \right) \begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} \\ + \left( \sum_{k=1}^n \left( \frac{z_k^2 - z_{k-1}^2}{2} \right) \begin{bmatrix} \underline{Q}_{11} & \underline{Q}_{12} & \underline{Q}_{16} \\ \underline{Q}_{12} & \underline{Q}_{22} & \underline{Q}_{26} \\ \underline{Q}_{16} & \underline{Q}_{26} & \underline{Q}_{66} \end{bmatrix} \right) \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}$$



# Laminate Stiffness Matrices

- This equation can be rewritten as:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad (\text{Eq. 22.1})$$

where,

$$A_{ij} = \sum_{k=1}^n (z_k - z_{k-1}) (\underline{Q}_{ij}^k)$$
$$B_{ij} = \sum_{k=1}^n \left( \frac{z_k^2 - z_{k-1}^2}{2} \right) (\underline{Q}_{ij}^k)$$

(Eq. 22.2)



# Laminate Stiffness Matrices

- Similarly, the relation for moment resultants can be developed as follows:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad (\text{Eq.22.3})$$

where,

$$D_{ij} = \sum_{k=1}^n \left( \frac{z_k^3 - z_{k-1}^3}{3} \right) (\underline{Q}_{ij}^k) \quad (\text{Eq. 22.4})$$

- Matrix [A] is called *extensional stiffness matrix*.
- Matrix [B] is called *bending-extension coupling matrix*.
- Matrix [D] is called *bending stiffness matrix*.



# Laminate Stiffness Matrices

- Extension Stiffness Matrix: This matrix influences extensional strains in the laminate.
  - For a given resultant force, mid-plane strains decrease as elements of this matrix increase in magnitude.
  - Magnitude of extensional stiffness increases directly in proportion to the thickness of each layer, since  $(z_k - z_{k-1})$  equals thickness of  $k_{th}$  lamina.
  - Terms  $A_{16}$  and  $A_{26}$  couple shear and normal responses of the laminate. If either of these terms is non-zero, then:
    - An extensional resultant force will generate not only extensional strain, but shear strain as well.
    - Similarly, shear force resultant  $N_{xy}$  will generate not only shear strain in the laminate, but extensional strain as well if these terms are non-zero.



# Laminate Stiffness Matrices

- Extension-Bending Coupling Matrix [B]: This matrix couples extensional response to the bending response of the laminate.
  - The magnitude of terms in [B] matrix depends on the square of the distance of each lamina's surface from the mid-plane.
  - If the magnitude of this matrix is non-zero, then:
    - A composite laminate will exhibit bending and twisting, even if external moment on it is perfectly zero.
    - A composite laminate will exhibit extensional and shear strains, even if external resultant forces on it are zero.
  - In very large number of applications the coupling of bending and extensional responses is cautiously avoided, as it can trigger early material failure. In such cases, laminates are carefully engineered to ensure that all elements of [B] are zero.



# Laminate Stiffness Matrices

- Bending Matrix [D]: This matrix influences the bending response of a laminate.
  - Magnitude of terms in [D] matrix depends on the cube of the distance of each lamina's surface from the mid-plane.
  - A [D] matrix with large magnitude implies that a unit bending moment will generate very little curvature in the laminate and vice-versa.
  - Terms  $D_{16}$  and  $D_{26}$  couple bending and twisting responses of the laminate. If either of these terms is non-zero, then:
    - A pure bending moment will generate not only bending curvature  $k_x$ , and  $k_y$ , but twist curvature,  $k_{xy}$ , as well.
    - Similarly, a pure torque will generate not only twist curvature in the laminate, but bending curvatures as well.



# Laminate Stiffness Matrices

- Equations 22.1 and 22.3 can also be expressed in one single set of equations expressed in matrix format. This is shown below.

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^o \\ \boldsymbol{\kappa} \end{Bmatrix} \quad (\text{Eq. 22.5})$$

where,

$\mathbf{N}$  is a vector for resultant stresses.

$\mathbf{M}$  is a vector for resultant moments,

$\boldsymbol{\varepsilon}^o$  is a vector for mid-plane strains, and,

$\boldsymbol{\kappa}$  is a vector for mid-plane curvatures.



# Special Lamination Sequences

- Here, we list certain special lamination sequences and associated terminologies, which are frequently used in composite industry, to meet specific functional requirements.
  - *Symmetric Laminate*: Such laminates are symmetric in lamination sequence across their mid-planes. : Such laminates:
    - Exhibit no coupling between bending and extensional responses because the value of all elements for such laminates' bending matrix are zero.
    - Do not bend and twist when cooled after fabrication.
  - *Cross-ply laminates*: Here, all layers in the laminate have an orientation of either 0 or 90 degrees. For such laminates:
    - Values of terms  $A_{16}$ ,  $A_{26}$ ,  $B_{16}$ ,  $B_{26}$ ,  $D_{16}$ , and  $D_{26}$  are zero.
    - Shear and extensional responses are not coupled.
    - Bending and twisting responses are not coupled.



# Special Lamination Sequences

- *Angle-Ply Laminates*: All layers in such laminates have an orientation of either  $\theta$ , or  $-\theta$ . For such laminates, if:
  - The laminate has an equal number of plies oriented in either directions, then terms  $A_{16}$ ,  $A_{26}$  are zero.
  - The laminate has a large number of plies totaling an odd number of plies, and the ply orientation alternates between  $\theta$ , or  $-\theta$ , then, coupling matrix  $[B]$  is zero, and terms  $D_{16}$ , and  $D_{26}$  are very small.
- *Anti-Symmetric Laminates*: Here, the material orientation of  $k_{th}$  layer above mid-plane is negative of  $k_{th}$  layer below the mid-plane. For such laminates:
  - Terms  $A_{16}$ ,  $A_{26}$ ,  $D_{16}$ , and  $D_{26}$  are zero.
  - However, matrix  $[B]$  is non-zero.
- *Balanced-Ply Laminates*: For every ply with an orientation  $\theta$ , the laminate also has a corresponding ply of same thickness, but orientation  $-\theta$  in the laminate. For such cases, value of terms  $A_{16}$  and  $A_{26}$  is zero.



# Special Lamination Sequences

- *Quasi-isotropic Laminate*: Such a laminate is widely used in the industry. An important property of such a laminate is that its extensional stiffness matrix is similar to that for isotropic materials. Thus, for such materials:
  - $A_{11} = A_{22}$
  - $A_{66} = (A_{11} - A_{12})/2$
  - $A_{16} = A_{26} = 0$ .

Such a laminate can be constructed if following conditions are met.

- The laminate should have more than two layers.
- All layers should have identical stiffness matrices  $[Q]$ , and thicknesses.
- The difference in orientation angle of two adjacent layers must be equal. Thus, the angle between two adjacent layers should be  $2\pi/n$ , where  $n$  is total number of layers in the composite plate.
- Two examples of quasi-isotropic laminates are:
  - $[0/\pm 60]$  is a quasi-isotropic three-layer laminate.
  - $[0/\pm 45/90]$  is a quasi-isotropic four-layer laminate.