Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 25

Semi-Infinite Plates

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Lecture Overview

- Solution for Case C

- Solution for Case D

Solution for Case C

- First, we reproduce equilibrium relations (Eq. 24.2), which apply to all cases. These are:

$$N_x = c_1$$

$$N_{xy} = c_2 \qquad \text{(Eq. 24.2)}$$

$$M_x = -qx^2/2 + c_3 x + c_4$$

- Further, we also reproduce Eq. 24.1 since it represents kinematic relations for all cases.

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad \text{where,}$$

(Eq. 24.1)

$$\begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{du^0}{\partial x} \\ 0 \\ \frac{dv^0}{\partial x} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{d^2 w^0}{\partial x^2} \\ 0 \\ 0 \end{Bmatrix}$$

Solution for Case C

- For Cases C and D, the lamination sequence is $[0_2/90_2]_T$. For such a laminate, we reproduce results from Eq. 24.4.
  - $A_{16} = A_{26} = 0$
  - $B_{16} = B_{26} = B_{12} = B_{66} = 0$
  - $D_{16} = D_{26} = 0$ (Eq. 24.4)
- Finally, boundary conditions for Case C are:

BCs at x = -a/2	**BCs at x = +a/2**
$u^{o-} = 0$	$N_x^+ = 0$
$v^{o-} = 0$	$N_{xy}^+ = 0$
$M_x^- = 0$	$M_x^+ = 0$
$w^{o-} = 0$	$w^{o+} = 0$

- Since N_x is zero at $+a/2$, we get from Eq. 20.2:

  $C_1 = N_x = 0 = A_{11}(du^o/dx) + B_{11}(d^2w^o/dx^2)$, or,

  $u^o(x) = [C_1x + C_5 + B_{11}(dw^o/dx)]/A_{11}$ (Eq. 25.1)

Solution for Case C

- Since N_{xy} is zero at $+a/2$, we get from earlier results:

  $C_2 = N_{xy} = 0 = A_{66}(dv^o/dx)$, or,

  $v^o(x) = [C_2 x + C_6]/A_{66}$ (Eq. 25.2)

- Also, we know that $M_x = B_{11}(du^o/dx) + D_{11}(d^2w^o/dx^2)$. Integrating this equation after substituting the definition of M_x from Eq. 16.2, we get:

  $B_{11}u^o(x) + D_{11}(dw^o/dx) = -qx^3/6 + C_3x^2/2 + C_4x + C_7$ (Eq. 25.3)

- Equations 25.1-3 can be used to solve for u^o, v^o, and w^o. Here, the term B_{11} couples in-plane and out-of-plane displacements. Solving Eqs. 25.1-3 for displacement field yields:

  $(A_{11}-B^2_{11}/D_{11})\, u^o(x) = (B_{11}/D_{11})(qx^3/6) - (B_{11}/A_{11})\,(C_3qx^2/2) + (C_1-C_4B_{11}/A_{11})x + (C_5-C_7B_{11}/A_{11})$

  $(D_{11}-B^2_{11}/A_{11})\, w^o(x) = qx^4/24 - C_3x^3/6 - (C_1B_{11}/A_{11} - C_4)x^2/2 + (C_5B_{11}/A_{11} - C_7)x + C_8$

  (Eq. 25.4a, b)

Solution for Case C

- Also, $M_y = D_{12}(d^2w^o/dx^2)$, and $M_{xy} = 0$ (Eq. 25.5)

- Equations 25.1-25.5 are applicable to Case C and Case D, as BCs yet remain to be applied. At this stage, we apply boundary conditions for Case C.

- Since N_x and N_{xy} are zero at $x = +a/2$, we get:
  $C_1 = C_2 = 0$.
- Further, since v^o is zero at $x = +a/2$, we get from Eq. 25.2:
  $C_6 = 0$.
- Further, since moment M_x is zero at both ends, we get from Eq. 20.2:
  $C_3 = 0$ and $C_4 = qa^2/8$.
- Now, we are left with C_5, C_7, and C_8. To find them, we apply two BCs for w^o condition, and one BC for the u^o condition.

Solution for Case C

- Applying two BCs for w^o in Eq. 25.4b, and one BC for u^o in Eq. 25.4a, we get values for C_5, C_7, and C_8. Finally, we get the following relations for u^o, v^o, and w^o:

$$u^o(x) = (B_{11} q a^3)[4(x/a)^3 - 3(x/a) - 1] / [24 D_{11}(A_{11} - B^2_{11}/D_{11})] \qquad \text{(Eq. 25.6)}$$

$$v^o(x) = 0 \qquad \text{(Eq. 25.7)}$$

$$w^o(x) = (-5 q a^4)[16(x/a)^4 - 24(x/a)^2 + 5] / [384 (D_{11} - B^2_{11}/A_{11})] \qquad \text{(Eq. 25.8)}$$

- Equations 18.6-8, constitute the displacement-field for Case C.

- Next, we consider Case D. Here, the general solution as expressed through Eqs. 25.1-5 is also the valid for Case D. However, integration constants are different due to differences in boundary conditions.

Solution for Case D

- The boundary conditions for Case D are given below.

BCs at x = -a/2	BCs at x = +a/2
$u^{o-} = 0$	$u^{o+} = 0$
$v^{o-} = 0$	$v^{o+} = 0$
$M_x^- = 0$	$M_x^+ = 0$
$w^{o-} = 0$	$w^{o+} = 0$

- Using these boundary conditions we get the final solutions as:

$u^o(x) = (B_{11}qa^3)[4(x/a)^2 - 1]/ [24D_{11}(A_{11}-B^2{}_{11}/D_{11})](x/a)$ (Eq. 25.9)

$v^o(x) = 0$ (Eq. 25.10)

$w^o(x) = (-qa^4)[16(x/a)^4 + 48\{B_{11}/(3A_{11}D_{11}) - 1/2\}(x/a)^2 +$
$\{5 - 4B^2{}_{11}/(A_{11}D_{11})\}]/ [384 (D_{11}-B^2{}_{11}/A_{11})]$ (Eq. 25.11)

Comments on Cases C & D

- In Cases C and D, the direction of $u^o(x)$ depends on sign of B_{11}.
  - The sign for B_{11} for a $[0_2/90_2]_T$ laminate is negative of that for a $[90_2/0_2]_T$ laminate.
  - Thus, direction of $u^o(x)$ for a $[0_2/90_2]_T$ laminate is negative of that for a $[90_2/0_2]_T$ laminate.

- In either case, $u^o(x)$ and $w^o(x)$ are odd and even functions of x. This observation is consistent with intuition.

- The term $(D_{11}-B^2_{11}/A_{11})$ is known as *reduced bending stiffness*. It appears as denominator in expressions for $w^o(x)$. The out of plane displacement is inversely proportional to this term, and not just D_{11}. For symmetric laminates, this term is identical to D_{11}. Also, higher the value of B_{11}, lower *reduced bending stiffness.*

- The term $(A_{11}-B^2_{11}/D_{11})$ is known as *reduced extensional stiffness*. It appears as denominator in expressions for $u^o(x)$. The in-plane displacement is inversely proportional to this term, and not just A_{11}.