Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 18

Analysis of an Orthotropic Ply

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Lecture Overview

- Introduction
- Engineering constants for an 2-D orthotropic lamina
- Relationship between engineering constants and compliance and stiffness matrices
- Restrictions on elastic constants

Introduction

- In previous lecture, we have developed stress-strain relationships for a two-dimensional orthotropic material.
- Also, we have defined four stiffness constants [Q], which can be used to evaluate stresses in terms of strains.
- Additionally, we have defined four compliance constants [S], which relate strains and stresses. Relationships between components of [S] and [Q] have also been defined.
- However, neither components of [S] nor those of [Q] are easy to measure. Hence, there is a need to define a third set of constants, known as “engineering constants”, which may be determined experimentally. Isotropic materials have two independent engineering constants. These are; Young’s modulus (E), and Poisson’s ratio (ν).
- Similarly, we need four, easy to measure engineering constants for two-dimensional orthotropic materials, which may be used to calculate elements of [Q] and [S] matrices.

Engineering Constants for a 2-D Orthotropic Lamina

- A two-dimensional orthotropic linearly elastic lamina has four engineering constants. These are:
  - Longitudinal modulus (E_L): If a unidirectional lamina is pulled in tension along its fiber length, then its longitudinal modulus is defined as:

    $E_L = \sigma_L/\varepsilon_L$. (Eq. 18.1)

  - Transverse modulus (E_T): If a unidirectional lamina is pulled in tension across its fiber length, i.e. transversely, then its transverse modulus is defined as:

    $E_T = \sigma_T/\varepsilon_T$. (Eq. 18.2)

  - Shear modulus (G_{LT}): If a unidirectional lamina is subjected to pure shear in L-T plane, then its shear modulus is defined as:

    $G_{LT} = \tau_{LT}/\gamma_{LT}$. (Eq. 18.3)

  - Major Poisson's ratio (ν_{LT}): It is the ratio of negative of transverse strain and longitudinal strain for a unidirectional lamina pulled in fiber direction. Mathematically:

    $\nu_{LT} = -\varepsilon_T/\varepsilon_L$. (Eq. 18.4)

Relationship between Engineering Constants and Elements of [Q] & [S] Matrices

- Consider an orthotropic lamina under arbitrary state of stress such that:
  - Stress in fiber direction is σ_L
  - Stress in transverse direction is σ_T
  - Shear stress in L-T plane is τ_{LT}

- Such a lamina will exhibit strains due these stresses. These strains are:
  - ε_L: Strain in the fiber direction
  - ε_T: Strain in transverse direction
  - γ_{LT} : Shear strain in L-T plane

- If, the lamina is stressed only in the longitudinal direction (i.e. σ_T, and τ_{LT} are zero) then, using Eq. 17.2, we can write:
  - $\sigma_L = Q_{11}\,\varepsilon_L + Q_{12}\,\varepsilon_L$
  - $\sigma_T = 0 = Q_{12}\,\varepsilon_L + Q_{22}\,\varepsilon_T$

Relationship between Engineering Constants and [Q], [S] Matrices

- Solving for strains we get:

  $\varepsilon_L = Q_{22} / (Q_{11}\, Q_{22} - Q^2_{12})\, \sigma_L$

  $\varepsilon_T = Q_{12} / (Q_{11}\, Q_{22} - Q^2_{12})\, \sigma_L$

- Rearranging these equations we get:

  $\sigma_L / \varepsilon_L = (Q_{11}\, Q_{22} - Q^2_{12}) / Q_{22}$

  $\sigma_L / \varepsilon_T = -(Q_{11}\, Q_{22} - Q^2_{12}) / Q_{12}$

- Recognizing the definitions of longitudinal modulus, and major Poisson's ratio from Eqs. 18.1, and 18.4 respectively, and comparing those with above equations, we can write:

  $E_L = (Q_{11}\, Q_{22} - Q^2_{12}) / Q_{22}$ (Eq. 18.5)

  $\nu_{LT} = Q_{12} / Q_{22}$ (Eq. 18.6)

- And similarly, we can also write:

  $E_T = (Q_{11}\, Q_{22} - Q^2_{12}) / Q_{11}$ (Eq. 18.7)

  $G_{LT} = Q_{66}$ (Eq. 18.8)

Relationship between Engineering Constants and Elements of [Q], [S] Matrices

- Similarly, engineering constants can also be used to define elements of compliance matrix. The methodology for developing these relations is very similar to one used earlier to develop Eq. 18.9 - 18.18.

- Here we directly write the resulting equations.

$S_{11} = 1/E_L$ (Eq. 18.15)

$S_{22} = 1/E_T$ (Eq. 18.16)

$S_{12} = -\nu_{LT}/E_L = -\nu_{TL}/E_T$ (Eq. 18.17)

$S_{66} = 1/G_{LT}$ (Eq. 18.18)

Constraints on Values of Elastic Constants

- An isotropic material has several elastic constants; E, ν, G, K, etc. However, out of these, only two are mutually independent. The remaining constants can be expressed in terms of other two. For instance, if E, and v, are assumed to be mutually independent, then shear and bulk moduli of the material can be expressed as:

  $G = E/[2(1+ \nu)]$, and $K = E/[3(1\text{-}2\nu)]$

- These equivalence relations have implications on the values of elastic constants. Thus; the relation $K = E/[3(1\text{-}2\nu)]$ implies that Poisson's ratio cannot exceed 0.5, for in that case, K would be negative, implying that application of an external inward pressure would cause the material to bulge outwards, which would be inconsistent with physical laws of nature.

- Similarly, there are restrictions on values of elastic constants for orthotropic, and transversely isotropic materials as well.

Constraints on Values of Elastic Constants

- For a transversely isotropic material, there are five independent elastic constants. If its natural material axes are designated as *L*, *T*, and *T'*; where T' is an axis normal to both, *L* and *T*, axes, then its five independent elastic constants are, E_L, E_T, G_{LT}, ν_{LT}, and $\nu_{TT'}$. For such a material to be transversely isotropic implies that following conditions must be satisfied.

$E_T = E_{T'}$ (Eq. 18.19a)

$G_{LT} = G_{LT'}$ (Eq. 18.19b)

$\nu_{LT} = \nu_{LT'}$ (Eq. 18.19c)

$G_{T'T} = G_{TT'} = E_T/[2(1+ \nu_{T'T})]$ (Eq. 18.19d)

- Finally, to ensure fundamental physical laws are not violated, the following equations acts as restraints on values of elastic constants.

E_L, E_T, $E_{T'}$, G_{LT}, $G_{LT'}$, and $G_{TT'} > 0$ (Eq. 18.20a)

$(1- \nu_{LT}\nu_{TL})$, $(1- \nu_{LT'}\nu_{T'L})$, $(1- \nu_{TT'}\nu_{T'T}) > 0$ (Eq. 18.20b)

$1 - \nu_{LT}\nu_{TL} - \nu_{LT'}\nu_{T'L} - \nu_{TT'}\nu_{T'T} - 2\nu_{LT}\, \nu_{T'L}\, \nu_{TT'} > 0$ (Eq. 18.20c)

- Details of above equations can be found in Lempriere' paper: Poisson's ratio in Orthotropic Materials, AIAA Journal, Nov. 1968.

What you learnt in this lecture?

- The need for engineering constants
- Definitions of engineering constants for an 2-D orthotropic lamina
- Relationship between engineering constants and compliance and stiffness matrices
- Restrictions on elastic constants