Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 21

Analysis of a Laminated Composite

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Lecture Overview

- Introduction
- Resultant Forces and Moments
- Piece-wise integration for calculating resultant forces and moments.

Introduction

- In previous lecture, mathematical relations have been developed, which define:
  - Variation of strains over the thickness of a laminate
  - Variation of stresses over the thickness of a laminate

- Given mid-plane strains and curvature, these relations may be used to calculate stresses in a plate.

- In a large number of real applications, we may not necessarily know mid-plane strains and curvatures for a composite, and sans their knowledge, predicting stresses in composite laminates is not possible just by using equations developed earlier.

- However, in a several cases we do know the value of externally applied loads and moments on plates. Thus, there is a need to develop relations which connect mid-plane strains, mid-plane curvatures, stresses, and external forces and moments.

Resultant Forces and Moments

- Consider a small part of composite plate, which is acted upon by forces and moments on its edges as shown in Fig. 21.1 due to different stresses. Here, N_x, N_y and N_{xy} are *resultant forces* per unit length acting on the edges of the composite plate in directions as shown in Fig. 21.1. Similarly, M_x, M_y and M_{xy} are *resultant moments* per unit length acting on the edges of the composite plate in directions as shown in Fig. 21.1.

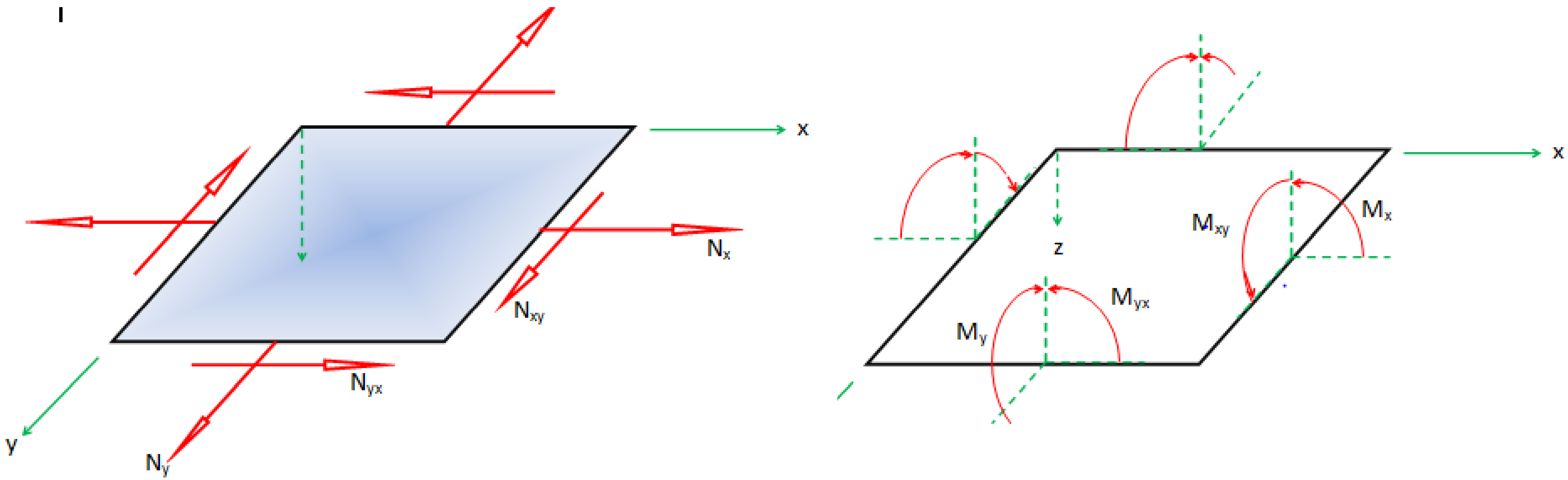


Fig. 21.1: Forces and Moments at Mid-Plane of a Plate

Resultant Forces and Moments

- Using principles of equilibrium, we can relate stresses to force resultants by integrating appropriate stress components through the plate thickness. Thus, we get.

$$N_x = \int_{-t/2}^{t/2} \sigma_{xx} dz$$

$$N_y = \int_{-t/2}^{t/2} \sigma_{yy} dz \qquad \text{(Eq. 21.1)}$$

$$N_{xy} = \int_{-t/2}^{t/2} \tau_{xy} dz$$

- In above equation, *t*, represents the thickness of composite plate.

Resultant Forces and Moments

- Similarly, using principles of equilibrium, we can relate stresses to moment resultants by integrating appropriate products of stress components and distance from mid-plane, through the plate thickness. Thus, we get.

$$M_x = \int_{-t/2}^{t/2} \sigma_{xx} z dz$$

$$M_y = \int_{-t/2}^{t/2} \sigma_{yy} z dz \qquad \text{(Eq. 21.3)}$$

$$M_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z dz$$

- In these equations, unit of force resultants (N_x, N_y and N_{xy}) is N/m, while that for moment resultant (M_x, M_y and M_{xy}) is N. Also, Fig. 21.1 depicts conventions used for positive resultant forces and resultant moments.

Resultant Forces and Moments

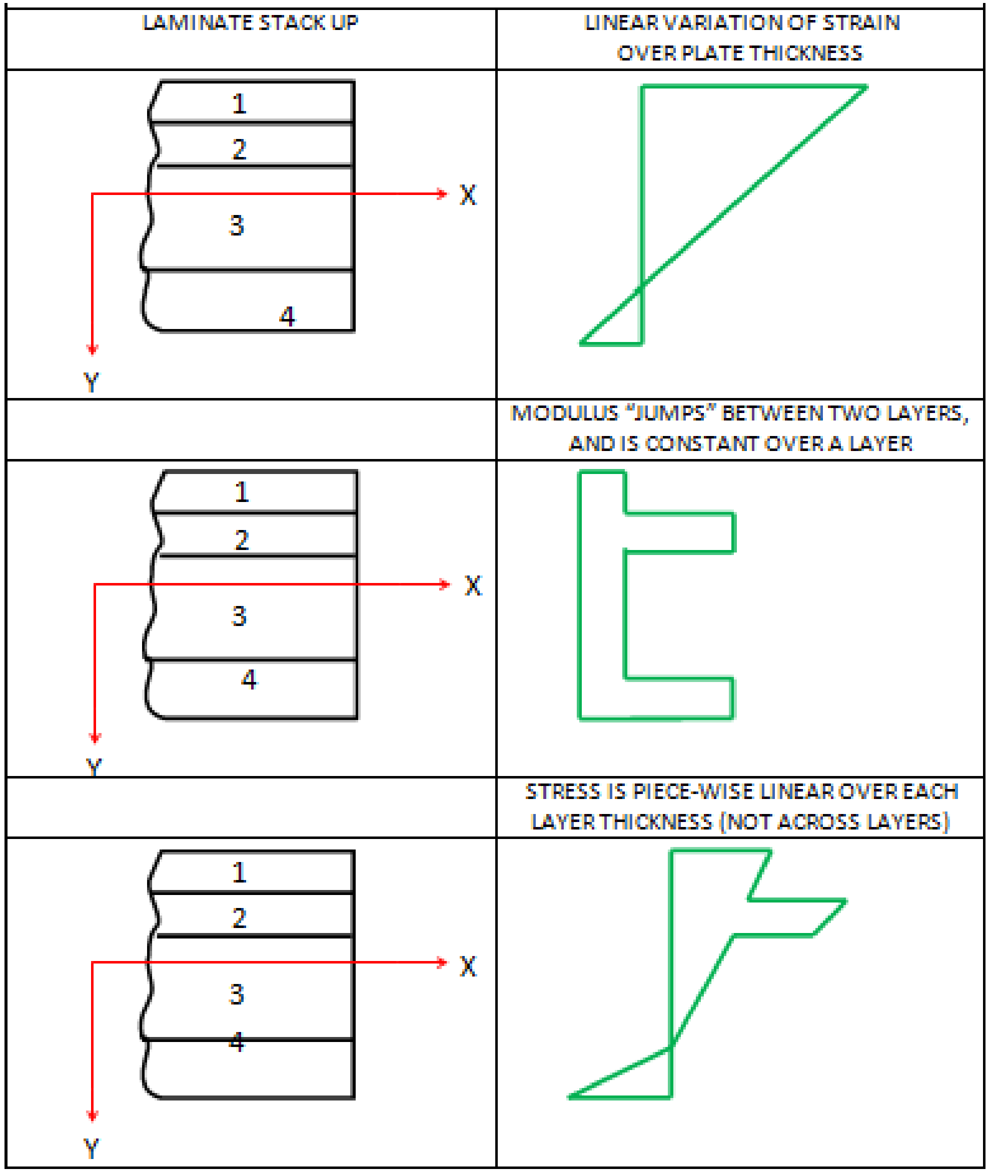


Fig. 21.2: Possible Variations in Stresses, Moduli and Strains Over the Thickness of a Hypothetical Four-Layer Laminate

Resultant Forces and Moments

- Consider Fig. 21.2. Looking at it, and also realizing from previous analysis, we infer that variation of stress is:
  - Discontinuous over the thickness of the whole laminate.
  - Linearly continuous over the thickness of a single layer. Such a variation of stresses over laminate thickness.

- Thus, the integrands for resultant forces and moments, as defined in Eqs. 21.1 and 21.2 are not continuous functions of *z*.

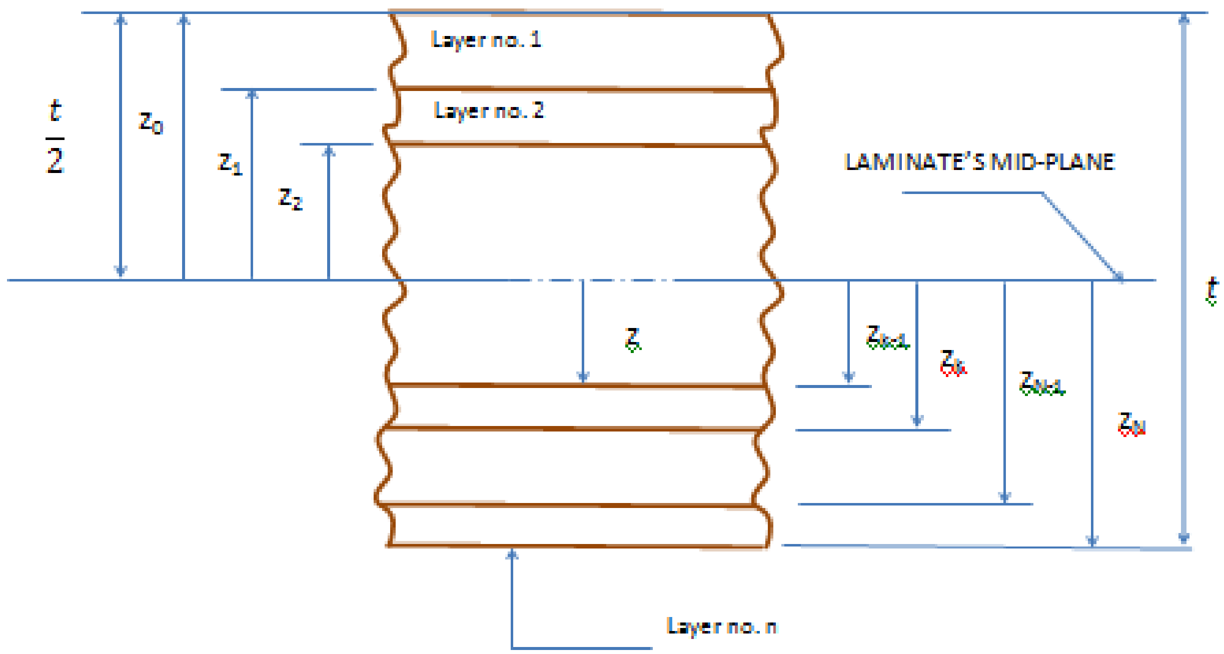
- Rather, they are piece-wise continuous over the thickness of composite plate. Hence, their integration over entire thickness requires one to:
  - Piece-wise integrate the function over each lamina's thickness.
  - Add up lamina specific integrals for all the layers.

  This is accomplished subsequently.

Resultant Forces and Moments

- Consider Fig. 21.3, which shows cross-section of the stack up of an *n*-layered orthotropic laminate. Here, the *z* coordinate of top and bottom surfaces of k^{th} laminate is z_k, and z_{k-1}. Further, as per the convention in this schematic, top most layer, with a *z*-coordinate of $t/2$ is considered the 1^{st} layer, while the bottom most layer is considered the n^{th} layer.

Fig. 21.3: Numbering Conventions for a Laminate with n Layers



Resultant Forces and Moments

- For such a schematic, the relations of resultant forces and moments, using Eqs. 21.1 and 21.2 can be written as:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_{xx}^k \\ \sigma_{yy}^k \\ \tau_{xy}^k \end{Bmatrix} dz \qquad \text{(Eq. 21.4)}$$

and,

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} z dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_{xx}^k \\ \sigma_{yy}^k \\ \tau_{xy}^k \end{Bmatrix} z dz \qquad \text{(Eq. 21.5)}$$

Resultant Forces and Moments

- Equations 21.4 and 21.5 are *summations* of integrals. If there are n layers in the composite, then there will be n summations.

- In this way, contribution of each layer is summed up while calculating resultant forces and moments.

- After integration and summation, coordinate z no longer appears in expressions for resultant forces and moments.

- These force resultants, N_x, N_y, and N_{xy}, and moment resultants, M_x, M_y, and M_{xy}, get applied on a composite plate's mid-plane, thereby generating stresses and strains in the plate.

- It should be noted here that even though these force and moment resultant do not vary with respect to the z-direction, they are indeed functions of x and y coordinates.