Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 26

Finite Rectangular Plates

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Lecture Overview

- Introduction
- Commonly used simplification approaches while solving problems related to finite plates
  - Stiffness matrix elements
  - Boundary conditions
- Out-of-plane BCs for different end conditions.

Introduction

- Consider a rectangular composite plate with dimensions *a* and *b*. Solving for displacement field for such a plate is significantly different than that for a semi-infinite plate.

- In the semi-infinite plate, we had assumed that all partial derivatives in one direction (e.g. *y* direction) were zero. Such a simplification renders equilibrium equations, which are originally Partial Differential Equations (PDEs), to Ordinary Differential Equations (ODEs).

- Such a simplification helped us develop closed form solutions for the problem at hand.

- The same simplification can not be used for finite plates, since the displacement field exhibits variation in both, *x*, and *y* directions.

Introduction

- Hence, only in very special cases, closed form solutions for finite plates are possible. In general, finite plate problems are solved using approximate methods. Some of the more popular methods used to address finite plate problems are:
  - Galerkin method
  - Rayleigh-Ritz method
  - Finite element method
  - Series methods

- The last approach, i.e. the series method, works well in some special cases, and yields solutions which are of "*semi-closed*" nature.

- The overall strategy to solve a finite plate problem starts with simplification of the problem to the maximum possible extent, without compromising on the physics of the problem. Such simplifications can be applied to governing equations, kinematic relations, material properties, and BCs.

Common Simplification Strategies

- Kinematics: Look out for specific displacements which may be constant or zero throughout the field.

- Equilibrium: Some force resultants or moment resultants may be known or zero throughout the field.

- Stiffness matrices:
  - Symmetric laminates: $B_{ij} = 0$.

    Such a simplification decouples resultant force {*N*} and moment vectors {*M*}. Hence the third equilibrium equation for *z* direction involving w^o, gets decoupled with those for inplane displacement variables, i.e. u^o and v^o, when infinitesimal strains are considered.

  - Balanced and symmetric laminates: $B_{ij} = 0$, and $A_{16} = A_{26} = 0$.

    Such a simplification not only decouples the out-of-plane response of the plate with in-plane response, but also extensional strains with shear strains.

Common Simplification Strategies

- Stiffness matrices (contd.):
  - Symmetric and specially orthotropic laminates: $B_{ij} = A_{16} = A_{26} = D_{16} = D_{26} = 0$
    Such a simplification not only decouples the out-of-plane response of the plate with in-plane response, but also extensional strains do not cause shear strains. Further, there is no twist curvature in the system, unless it is caused by externally applied twisting moments. In such a case, we can solve the inplane part of the problem independently vis-à-vis out-of-plane part of the problem.
  - Non-symmetric but specially orthotropic laminates: $A_{16}=A_{26} = B_{16}=B_{26} =D_{16}=D_{26} =0$
  - Balanced and anti-symmetric laminates
  - Cross-ply and anti-symmetric laminates

Common Simplification Strategies

- Boundary Conditions: As seen earlier, there are four sets of boundary conditions for each edge of the plate. These are relisted below for edge $x = a$.

(i) $\boldsymbol{(N_x - N_x^+)\delta u^o = 0,}$

where δu^o implies variation in midplane displacement u_o.
Thus, either $N_x = N_x^+$ must be known, or u_o must be known.

(ii) $\boldsymbol{\left(N_{xy} - N_{xy}^+\right)\delta v^o = 0,}$

where δv^o implies variation in midplane displacement v_o.
Thus, either $N_{xy} = N_{xy}^+$ must be known, or v_o must be known.

(iii) $\boldsymbol{\left\{\left(\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y}\right) - \left(Q_x^+ + \frac{\partial M_{xy}^+}{\partial y}\right)\right\}\delta w^o = 0}$

where δw^o implies variation in midplane displacement w_o.
Thus, either $\left(\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y}\right) = \left(Q_x^+ + \frac{\partial M_{xy}^+}{\partial y}\right)$ or, or w_o must be known.

(iv) $\boldsymbol{(M_x - M_x^+)\delta(\frac{\partial w^o}{\partial x}) = 0}$

Thus, either $M_x = M_x^+$ or $\frac{\partial w^o}{\partial x}$ must be known.

Common Simplification Strategies

- Boundary Conditions (contd.): Out of these four BCs listed earlier:
  - Conditions (i) and (ii) relate to inplane variables.
  - Conditions (iii) and (iv) relate to out-of-plane variables.
  - If the laminate is symmetric and we are only interested in out-of-plane response of the plate [i.e. only $w^o(x,y)$], then only BCs (iii) and (iv) may be considered, and appropriate simplifications may be made.
- Next, we look at different edge conditions for rectangular plates, and their associated boundary conditions. Here we are addressing only out-of-plane boundary conditions.

Different Edge Conditions for Plates

- Simply Supported Condition: For a plate simply supported along edge $x = a/2$, the out-of-plane BCs along this edge will be:
  - $w^o = 0$ and $M^+_x = 0$.
  - The condition for moment can be further elaborated as shown below.

$$M_x^+ = D_{11}\frac{\partial^2 w^o}{\partial x^2} + D_{12}\frac{\partial^2 w^o}{\partial y^2} + 2D_{16}\frac{\partial^2 w^o}{\partial x \partial y} = 0$$

But on edge, x=a/2, $w^o = \frac{\partial w^o}{\partial y} = \frac{\partial^2 w^o}{\partial y^2} = 0$ because w^o is continuously zero along the entire edge.

Thus,

$$M_x^+ = D_{11}\frac{\partial^2 w^o}{\partial x^2} + 2D_{16}\frac{\partial^2 w^o}{\partial x \partial y} = 0$$

It may be noted here that the condition $w^o = \frac{\partial w^o}{\partial y} = \frac{\partial^2 w^o}{\partial y^2} = 0$ does not imply that the term $\frac{\partial^2 w^o}{\partial x \partial y}$ is zero along the same edge, because this term equals the rate of change of slope $\frac{\partial w^o}{\partial x}$ in *y* direction. When D_{16} does not equal zero, this term causes twist in the plate along its edge.

- Clamped Support: Here, displacement and slope are zero along the edge.

Different Edge Conditions for Plates

- Traction Free Edge: For a plate with traction free edge at $x = a/2$, the relevant BCs will be $Q^+_x = M^+_x = M^+_{xy} = 0$. Substituting this in BC (iii), we get,

  $$M_x + 2\frac{\partial M_{xy}}{\partial y} = 0$$ (Equivalent BC for traction free edge)

  since Q^+_x and partial derivative of M_x w.r.t. to y are zero along edge $x = a/2$. Also from Eq. iv, we set the other BC for traction free edge as:

  $M_x = 0$.

- Now for a symmetric laminate,

$$M_x = D_{11}\frac{\partial^2 w^o}{\partial x^2} + D_{12}\frac{\partial^2 w^o}{\partial y^2} + 2D_{16}\frac{\partial^2 w^o}{\partial x \partial y} = 0$$

$$M_{xy} = D_{16}\frac{\partial^2 w^o}{\partial x^2} + D_{26}\frac{\partial^2 w^o}{\partial y^2} + 2D_{66}\frac{\partial^2 w^o}{\partial x \partial y} = 0$$

Different Edge Conditions for Plates

- Traction Free Edge (contd.): Thus, out-of-plane BCs for such plates at x=a/2 are:

$$\left(D_{11}\frac{\partial^3 w^o}{\partial x^3} + D_{12}\frac{\partial^3 w^o}{\partial x \partial y^2} + 2D_{16}\frac{\partial^3 w^o}{\partial x^2 \partial y}\right) + 2\left(D_{16}\frac{\partial^3 w^o}{\partial x^2 \partial y} + D_{26}\frac{\partial^3 w^o}{\partial y^3} + 2D_{66}\frac{\partial^3 w^o}{\partial x \partial y^2}\right) = 0$$

and

$$D_{11}\frac{\partial^2 w^o}{\partial x^2} + D_{12}\frac{\partial^2 w^o}{\partial y^2} + 2D_{16}\frac{\partial^2 w^o}{\partial x \partial y} = 0$$

- For specially orthotropic case, when $D_{16} = D_{26} = 0$, we get:

$$\left(D_{11}\frac{\partial^3 w^o}{\partial x^3} + D_{12}\frac{\partial^3 w^o}{\partial x \partial y^2}\right) + 4D_{66}\frac{\partial^3 w^o}{\partial x \partial y^2} = 0$$

and

$$D_{11}\frac{\partial^2 w^o}{\partial x^2} + D_{12}\frac{\partial^2 w^o}{\partial y^2} = 0$$