Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 19

Analysis of an Orthotropic Ply

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Lecture Overview

- Transformation of stresses and strains
- Stress-strain relations for a lamina with any orientation
- Strength of an orthotropic lamina

Introduction

- Earlier, while discussing the stress state in 2-D orthotropic materials, it was assumed that reference axes for measuring stresses and strains were coincident with material axes. In reality, that may not be the case.

- Hence, there is a need to develop stress-strain relations in a 2-D orthotropic lamina oriented arbitrarily. Towards this goal, as a first step, we have to transform stresses and strains from material axes to arbitrary axes, and vice-versa.

- Consider a tetrahedron with vertices ABCP. Its face ABC, with surface area A and, normal $\boldsymbol{n}$ (with direction cosines, n_x, n_y, and n_z), experiences is subjected to stress vector $\boldsymbol{T}$, such that the total external force on face ABC is $\boldsymbol{T} \cdot \boldsymbol{A}$. Further, the x, y, and z components of $\boldsymbol{T}$ are T_x, T_y, and T_z, respectively.

- Further, we assume that the length of normal to face ABC passing through P is h. Such a tetrahedron is shown in Figure 19.1.

Engineering Constants for a 2D Lamina

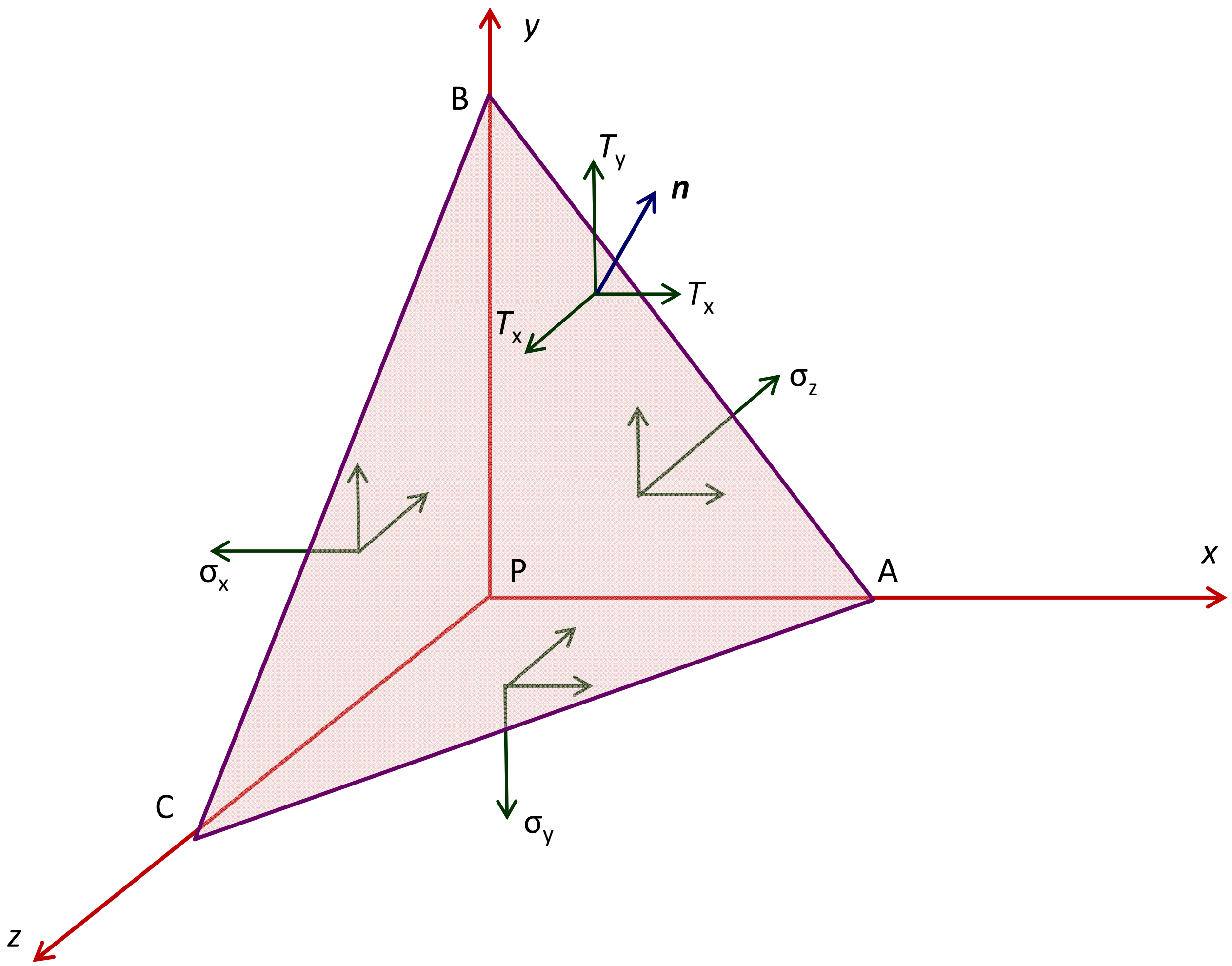

Fig. 19.1: Tetrahedron at Point P

Engineering Constants for a 2-D Orthotropic Lamina

- Further, we assume that the body is in equilibrium, and thus, other three faces experience normal and shear stresses.

- Given that the body is in equilibrium, we can write the following three equilibrium equations.

$$\sigma_{xx}An_x + \tau_{yx}An_y + \tau_{zx}An_z = T_xA$$
$$\tau_{xy}An_x + \sigma_{yy}An_y + \tau_{zy}An_z = T_yA$$
$$\tau_{xz}An_x + \tau_{yz}An_y + \sigma_{zz}An_z = T_zA,$$

- Eliminating A from above equations, we get Cauchy stress-formulae, as follows.

$$\sigma_{xx}n_x + \tau_{yx}n_y + \tau_{zx}n_z = Tn_x = T_x$$
$$\tau_{xy}n_x + \sigma_{yy}n_y + \tau_{zy}n_z = Tn_y = T_y \qquad \text{(Eq. 19.1)}$$
$$\tau_{xz}n_x + \tau_{yz}n_y + \sigma_{zz}n_z = Tn_z = T_z$$

- In Eq. 19.1, T_x, T_y, and T_z, are x, y, and z, components of stress-vector $\boldsymbol{T}$. Further, $\boldsymbol{T}$, can also be resolved in terms of its normal and tangential component, with respect to surface $\boldsymbol{A}$. The normal component can be expressed as:

$$\sigma_n = T_xn_x + T_yn_y + T_zn_z \qquad \text{(Eq. 19.2)}$$

Engineering Constants for a 2-D Orthotropic Lamina

- Combining Equations 19.1 and 19.2, we can write the relation for normal stress as:

$$\sigma_n = \sigma_{xx} n_x^2 + \sigma_{yy} n_y^2 + \sigma_{zz} n_z^2 + 2\tau_{xy} n_x n_y + 2\tau_{yz} n_y n_z + 2\tau_{zx} n_z n_x \qquad \text{(Eq. 19.3)}$$

- Eq. 19.3 can be used to transform normal stress from one set of axes to another set of axes.

- Now, let us consider Fig. 19.2. Here, we assume that the stress state at a point with respect to an arbitrary set of axes, *x*, *y*, and *z* is known. We would like to use this information to calculate the stress-state with respect to natural material axes of the system.

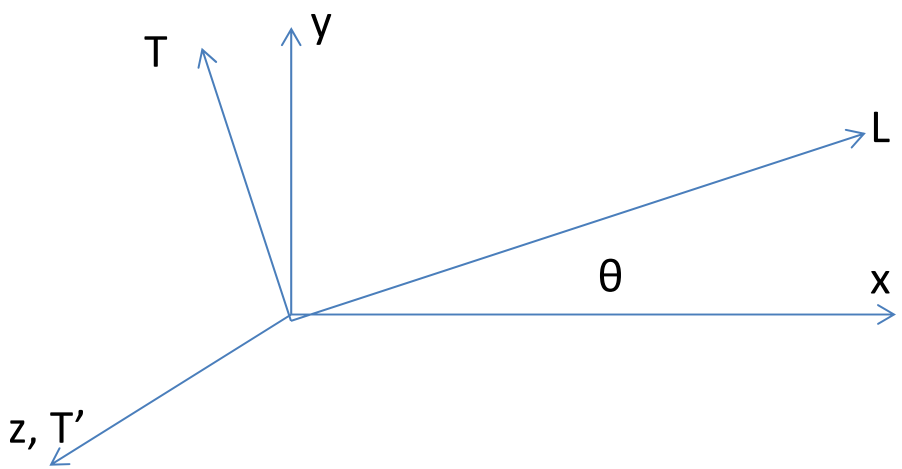

Fig. 19.2: Orientation of Natural Material Axes with reference to Arbitrary Axes

Engineering Constants for a 2-D Orthotropic Lamina

- From Fig. 19.2, it is seen that the material axis system is essentially a rotation of *x*-*y* axes, around *z* axis by an angle θ. Thus the direction cosines for the material axis system (*L*-*T*-*T'*), with respect to *x*-*y*-*z* system are, cos θ,sin θ and 1.

- Thus, normal stresses $\sigma_{xx,}$ and σ_{yy} can be written as:

  $\sigma_L = \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + 2\tau_{xy} \cos\theta \sin\theta$ (Eq. 19.4a)

  $\sigma_T = \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta + 2\tau_{xy} \cos\theta \sin\theta$ (Eq. 19.4b)

- Using similar approach, we can also write the equation for shear stress can be written as as:

  $\tau_{LT} = -\sigma_{xx} \cos\theta \sin\theta + \sigma_{yy} \cos\theta \sin\theta + \tau_{xy} \cos^2\theta \sin^2\theta$ (Eq. 19.4c)

- Eqs. 19.4a-c, can also be written in matrix form as:

$$\begin{Bmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$ Eq. (19.5)

Engineering Constants for a 2-D Orthotropic Lamina

- Similar equations can also be used to transform strains from one coordinate system to another one. The strain transformation equations are:

$$\begin{Bmatrix} \varepsilon_L \\ \varepsilon_T \\ \frac{1}{2}\gamma_{LT} \end{Bmatrix} = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{Bmatrix} \quad \text{Eq. (19.6)}$$

- In Equations 19.5 and 19.6, [M] is transformation matrix, and is defined as:

$$[T] = \begin{bmatrix} cos^2\theta & sin^2\theta & 2sin\theta\ cos\theta \\ sin^2\theta & cos^2\theta & -2sin\theta\ cos\theta \\ -sin\theta\ cos\theta & sin\theta\ cos\theta & cos^2\theta - sin^2\theta \end{bmatrix} \quad \text{Eq. (19.7)}$$

- It may be noted here, that unlike stress transformation equations, strain transformation equations have a factor of ½ within strain vectors. This is because such a transformation requires usage of tensor strains, and not engineering strains. While mathematical definitions of normal tensor strain and normal engineering strains are identical, tensor shear strain is one-half times that of engineering shear strain.

Transformation of Engineering Constants

- Now, that we have relations which can be used to transform strains from one system to other, we proceed to develop relations which will help us transform engineering constants. Pre-multiplying Eq. 19.5 by $[T]^{-1}$ on either sides, we get:

  $[T]^{-1}\{\sigma\}_{L\text{-}T} = [T]^{-1}\,[T]\,\{\sigma\}_{x\text{-}y}$ or $\{\sigma\}_{x\text{-}y} = [T]^{-1}\{\sigma\}_{L\text{-}T}$ (Eq. 19.8)

  where, $\{\sigma\}_{L\text{-}T}$ and $\{\sigma\}_{x\text{-}y}$ are stresses measured in *x-y*, and *L-T* reference frames, respectively.

- Further, as shown in earlier lecture, we can write stress-strain relation as:

  $\{\sigma\}_{L\text{-}T} = [Q]\,\{\varepsilon\}_{L\text{-}T}$ (Eq. 19.9)

- Putting RHS of Eq. 19.9 in RHS of Eq. 19.8, we get:

  $\{\sigma\}_{x\text{-}y} = [T]^{-1}\,[Q]\,\{\varepsilon\}_{L\text{-}T}$ (Eq. 19.10)

Transformation of Engineering Constants

- And finally expressing $\{\varepsilon\}_{L\text{-}T}$ in terms of $\{\sigma\}_{x\text{-}y}$, using appropriate transformations, in Eq. 19.10, we get:

$$\{\sigma\}_{x\text{-}y} = [T]^{-1}[Q]\,[T]\{\varepsilon\}_{x\text{-}y} \qquad \text{or,}$$

$$\{\sigma\}_{x\text{-}y} = [\underline{Q}]\{\varepsilon\}_{x\text{-}y} \qquad \text{(Eq. 19.11)}$$

- Equation 19.11 helps us compute stresses measured in *x-y* coordinate system in terms of strains measure in the same system. Here, $[\underline{Q}]$ is the transformed stiffness matrix, and its individual components are:

$$\underline{Q}_{11} = Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta$$

$$\underline{Q}_{22} = Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta$$

$$\underline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\cos^4\theta + \sin^4\theta)$$

$$\underline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\cos^4\theta + \sin^4\theta)$$

$$\underline{Q}_{16} = (Q_{11} - Q_{22} - 2Q_{66})\sin\theta\cos^3\theta - (Q_{22} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta$$

$$\underline{Q}_{26} = (Q_{11} - Q_{22} - 2Q_{66})\sin^3\theta\cos\theta - (Q_{22} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta$$

(Eq. 19.12)

Transformation of Engineering Constants

- Following observations can be made from Eq. 19.12.
  - Unlike, [Q] matrix, [$\underline{Q}$] matrix is fully populated.

  - Terms Q_{16}, and Q_{26} are identically zero. However, terms $\underline{Q}_{16}$, and $\underline{Q}_{26}$, are not necessarily zero, and their definition involves linear combinations of four elements of [Q] matrix.

  - For a *specially* orthotropic lamina, i.e. when its loading direction coincides with lamina's material axes, application of normal stresses produce only normal strains, and application of shear stresses produce pure shear strains.

  - For a generally orthotropic lamina, i.e. when loading direction and material axes are not coincidental, application of normal stresses produce normal as well as shear strains. This occurs because of non-zero values for terms $\underline{Q}_{16}$, and $\underline{Q}_{26}$, which couple normal and shear responses. These terms are also known as cross-coupling stiffness coefficients.

Transformation of Engineering Constants

- Using a transformation procedure similar to the one used to transform stiffness matrix [Q], we can also transform the compliance matrix [S] to an arbitrary coordinate system. The elements of transformed compliance matrix [$\underline{S}$] are defined below.

$$\underline{S}_{11} = S_{11} \cos^4\theta + S_{22} \sin^4\theta + (2S_{12} + S_{66}) \sin^2\theta \cos^2\theta$$

$$\underline{S}_{22} = S_{11} \sin^4\theta + S_{22} \cos^4\theta + (2S_{12} + S_{66}) \sin^2\theta \cos^2\theta$$

$$\underline{S}_{12} = (S_{11} + S_{22} - S_{66})\sin^2\theta \cos^2\theta + S_{12} (\cos^4\theta + \sin^4\theta)$$

$$\underline{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})\sin^2\theta \cos^2\theta + S_{66} (\cos^4\theta + \sin^4\theta)$$

$$\underline{S}_{16} = 2(2S_{11} - 2S_{22} - S_{66})\sin\theta \cos^3\theta - 2(2S_{22} - 2S_{12} - S_{66})\sin^3\theta \cos\theta$$

$$\underline{S}_{26} = 2(2S_{11} - 2S_{22} - S_{66})\sin^3\theta \cos\theta - 2(2S_{22} - 2S_{12} - S_{66})\sin\theta \cos^3\theta$$

(Eq. 19.13)

Strength of an Orthotropic Lamina

- In isotropic materials, failure prediction requires calculating principal stresses or strains and comparing them to their respective allowable stress and strain limits.

- In non-isotropic materials such an approach does not work.
  - The notion of principal stress makes no sense for these materials, as material strength changes with direction, and direction of principal stress may not in most of the cases coincide with direction of maximum strength.

- For an isotropic material, we can fully describe allowable stress field by knowing the material's tensile, compressive and shear strength.

Failure in Isotropic v/s Transversely Isotropic Materials

- Similarly, for 2-D orthotropic materials, we evaluate allowable stress field in context of five different strengths of material measured with respect to its principal material directions. These are:
  - Longitudinal tensile strength (σ_{LU})
  - Lateral or transverse tensile strength (σ_{TU})
  - Longitudinal compressive strength (σ'_{LU})
  - Lateral or transverse compressive strength (σ'_{LU})
  - In-plane shear strength (τ_{LTU})

- These material strength parameters for an orthotropic lamina are its fundamental material properties.

Failure in Orthotropic Materials

- Similar to isotropic materials, several theories have been developed to predict failure in orthotropic materials. Some of the more widely used theories are based on maximum stress, maximum strain, and maximum work.

- Maximum Stress Theory: As per this theory, failure will occur once stresses measured with respect to principal material axes, exceed their respective allowable limits. Thus, for failure at least one of the following conditions must be violated.

  For tensile loads:

  $\sigma_L < \sigma_{LU}$, $\sigma_T < \sigma_{TU}$, $\tau_{LT} < \tau_{LTU}$.

  For compressive loads:

  $\sigma_L < \sigma'_{LU}$, $\sigma_T < \sigma'_{TU}$. (Eq. 19.14)

- One limitation of this theory is that different modes of potential failure do not interact with each other.

Failure in Orthotropic Materials

- Maximum Strain Theory: As per this theory, failure will occur once strains measured with respect to principal material axes, exceed their respective allowable limits. Thus, for failure, at least one of the following five conditions must be violated.

  For normal tensile strains the conditions are:

  $\varepsilon_L < \varepsilon_{LU}$, $\varepsilon_T < \varepsilon_{TU}$, $\gamma_{LT} < \gamma_{LTU}$.

  And if normal strains are compressive, then failure criteria are:

  $\varepsilon_L < \varepsilon'_{LU}$, $\varepsilon_T < \varepsilon'_{TU}$. (Eq. 19.15)

- *If material is linearly elastic, then* Eq. 10.15 can be re-written as:

  $\varepsilon_L < \sigma_{LU}/E_L$, $\varepsilon_T < \sigma_{TU}/E_T$, $\gamma_{LT} < \tau_{LTU}/G_{LT}$.

  $\varepsilon_L < \sigma'_{LU}/E_L$, $\varepsilon_T < \sigma'_{TU}/E_T$. (Eq. 19.16)

- Predictions from maximum stress and maximum strain theories are very similar, with minor differences being attributable to role of Poisson's ratio. This is true for linear elastic materials. For non-linear elastic materials, Eq. 19.16 should not be used, and significant difference should be expected between results from these two theories.

Failure in Orthotropic Materials

- Tsai-Hill or Maximum Work Theory: As per this theory, failure occurs when the following inequality condition is violated.

$$(\sigma_L / \sigma_{LU})^2 - (\sigma_L / \sigma_{LU}) (\sigma_T / \sigma_{TU}) + (\sigma_T / \sigma_{TU})^2 + (\tau_{LT} / \tau_{LTU})^2 < 1 \qquad \text{(Eq. 19.16)}$$

- Here, if normal stresses are compressive then compressive strength should be used in the equation. Also, if the lamina is subjected to unidirectional normal stress, then above equation can be simplified as:

$$(\cos^2\theta / \sigma_{LU})^2 - (\cos\theta \sin\theta / \sigma_{LU})^2 + (\sin^2\theta / \sigma_{TU})^2 + (\cos\theta \sin\theta / \tau_{LTU})^2 < (1/ \sigma_x)^2 \qquad \text{(Eq. 19.17)}$$

- Unlike maximum stress and strain theories, Eq. 19.16 provides a single criterion for predicting failure. It also accounts for interaction between different strengths of the material. Predictions of strength from this theory are slightly lesser than those from maximum stress and maximum strain theories.

- All the theories discussed till so far work only for a lamina subjected to bi-axial stress state and not for tri-axial stress state.

What you learnt in this lecture?

- Transformation of stress and strains
- Stress-strain relations for a lamina with any orientation
- Strength of an orthotropic lamina