Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

Bishakh Bhattacharya & Nachiketa Tiwari
Indian Institute of Technology Kanpur

Lecture 28

Virtual Work Approaches

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Introduction

- Till so far, we have solved problems using exact approaches. Two general approaches have been used in this regard:
  - Simplification of the problem by converting PDEs into ODEs, and then integrating them to get exact solutions.
  - Simplification of the problem, and then using series approach to satisfy PDEs , and through such a process extracting constants used in the series.

- However, such approaches, even though they provide exact solutions have a limited range of applicability. In a very large number of problems such approaches are difficult to implement because:
  - Either closed form solutions do not exist.
  - Or indentifying them is not very much obvious.

- Hence, a slew of other solution approaches have been developed to address relatively more complex problems.

Introduction

- Consider a rectangular plate of dimensions 4*a* X 4*b*, which is clamped on all four edges. This is shown below in Fig. 28.1.

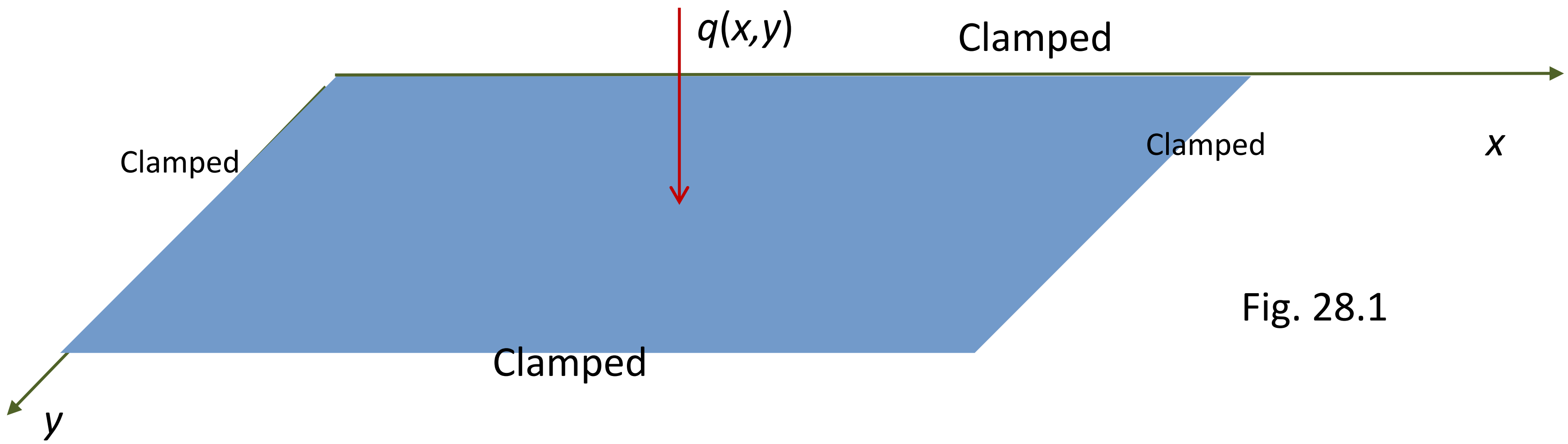

Fig. 28.1

- For such a problem the boundary conditions require *w* and its slope to be zero at all four edges of the plate.

Introduction

- Thus, we assume a 1-term solution, which satisfies these boundary conditions. This 1-term solution and its derivatives are given below.

$$w^o(x,y) = w_{11}\left(1 - cos\frac{\pi x}{2a}\right)\left(1 - cos\frac{\pi y}{2b}\right)$$

Thus,

$$\frac{\partial w^o}{\partial x} = w_{11}\left(\frac{\pi}{2a}sin\frac{\pi x}{2a}\right)\left(1 - cos\frac{\pi y}{2b}\right)$$

and,

$$\frac{\partial w^o}{\partial y} = w_{11}\left(1 - cos\frac{\pi x}{2a}\right)\left(\frac{\pi}{2b}sin\frac{\pi y}{2b}\right)$$

- Such a function satisfies all the boundary conditions, which are:

$$w^o(x,y) = 0 \ at \ x = 0, a; \ and \ y = 0, b$$

$\frac{\partial w^o}{\partial x} = at \ x = 0, a$ and $\frac{\partial w^o}{\partial y} = 0 \ at \ y = 0, b$

Introduction

- Even though the assumed function satisfies all the BCs, it does not satisfy the equilibrium equation for w^o. This is shown below.

- Assume that the plate is specially orthotropic, and also symmetric. Thus, the out-of-plane equilibrium equation for such a plate is:

$$D_{11}\frac{\partial^4 w^o}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w^o}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w^o}{\partial y^4} = q(x,y)$$

- Substituting the expression for w^o in this relation, we get:

$$w_{11}\left[D_{11}\left(\frac{\pi}{2a}\right)^4 cos\frac{\pi x}{2a}\left(1 - cos\frac{\pi y}{2b}\right) + 2(D_{12} + 2D_{66})\left(\frac{\pi^2}{4ab}\right)^2 cos\frac{\pi x}{2a} cos\frac{\pi y}{2b} + D_{22}\left(\frac{\pi}{2a}\right)^4\left(1 - cos\frac{\pi x}{2b}\right) cos\frac{\pi y}{2a}\right] - q(x,y) = E(x,y) \neq 0$$

- We see here, that the LHS of the equation does not become zero. Thus, there is an error [E(x,y)] in the equation, and hence, our assumed solution is not a valid one. The units of this error N/m^2.

Virtual Work Approaches

- It may noted here that the fact statement error [E(x,y)] is not zero means:
  - There is an imbalance between external and internal forces in z direction because in this specific case, the error E(x,y) relates equilibrium equation for z-direction.
  - The value of error changes from point to point over the surface of the plate. Hence, the equilibrium equation is not satisfied in a point to point-wise sense.

- However, if we chose to equate virtual work over the entire *domain* (in this case the area of the plate), then we do have a solution which may be good enough, though not the best possible.

- “Virtual work” is the dot product of forces and virtual displacements.

Virtual Work Approaches

- A virtual displacement is an assumed infinitesimal change of system coordinates occurring while time remains stationary. Such displacements are called “virtual” and not “real” since no actual displacement can occur while time is held stationary.

- Virtual displacements should be such that all kinematic conditions of the problem are satisfied. This in turn implies that:
  - Boundary conditions on displacements and slopes must be satisfied.
  - Continuity within boundaries must be maintained.

- Finally we note that imposition of virtual displacements does not lead to change of forces.

Galerkin Method

- Consider $w^o(x,y)$ as the *actual* displacement field for a plate.
  - From this reference condition the plate may be virtually perturbed by some virtual displacement field $\varepsilon w^o_1(x,y)$. Such a virtual displacement field should satisfy kinematic conditions.
  - Here, ε is small scale parameter. It gives "smallness" to virtual displacements. Thus,
  - Total displacement in z-direction = $w^o(x,y) + \varepsilon w^o_1(x,y)$.

- In Galerkin's method we set virtual work to zero. Thus, for out-of-plane equation we can write virtual work statement as:
  - Virtual work = $E[w^o(x,y)] \cdot \varepsilon w^o_1(x,y)\, dx dy = 0$,
  - where,
  - $E[w^o(x,y)] \cdot dx dy$ is residual force.

- However, above statement of virtual work cannot be valid, because none of three terms in it, $E[w^o(x,y)]$, ε and $w^o_1(x,y)$ are zero.

Galerkin Method

- However, our earlier statement of virtual work cannot be valid because none of the three terms in it, $E[w^o(x,y)]$, ε and $w^o_1(x,y)$ are zero.

- Hence, we satisfy virtual work statement in an integral sense.

- Mathematically, this may be expressed as:

$$\int_\Omega \mathbf{E}[\boldsymbol{w}^o(\boldsymbol{x},\boldsymbol{y})]\cdot \boldsymbol{\varepsilon w}^o{}_1(\boldsymbol{x},\boldsymbol{y})\, \mathbf{d}\boldsymbol{x}\mathbf{d}\boldsymbol{y} = \mathbf{0} \qquad \text{(Eq. 28.1)}$$

- Equation 28.1 is the *General Galerkin Equation* corresponding to out-of-plane equilibrium equation.

- Similar relations may be developed corresponding to other equilibrium equations as well.

Comments on Galerkin Method

- We are able to satisfy the equilibrium equation only in an integral sense, and not in a point-by-point sense.
- Since as the differential equation is satisfied only in an integral sense (i.e. over the entire domain), we are actually *forcing* the integral of error to become zero. Hence Galerkin solution is stiffer vis-à-vis exact method solution.
- In this method there is no restriction on the choice of virtual displacement field except that it has to satisfy kinematic conditions. Hence, it is known as **General Galerkin Method**.
- In **Special Galerkin Method**, we choose virtual displacement field such that it is of the same form as that of $w^o(x,y)$. In such a method, statement of virtual work corresponding to the 3rd equilibrium equation is:

$$\int_\Omega \mathbf{E}[w^o(x,y)] \cdot \varepsilon w^o(x,y)\, dx\,dy = 0 \qquad \text{(Eq. 28.2)}$$

Comments on Galerkin Method

- There are several interpretations of Galerkin method. One of the interpretations has already been discussed earlier, i.e. total virtual work done over the domain is zero.

- Another common interpretation relates to the notion of orthogonality. We know that work can be defined as:

  $dW = \mathbf{F} \cdot d\mathbf{U}$,

  i.e. work is a scalar, which is inner product of force and displacement vectors.

- Thus, if **F** and d**U** are mutually orthogonal, then their inner product, i.e. dW will be zero. Now in Eq. 28.1, we can state that in an *average* sense:

  (Error in force) · Virtual Displacement = 0 is the condition for equilibrium.

- This implies that in an average sense, error forces and virtual displacements are mutually orthogonal.