Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 24

Semi-Infinite Plates

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Lecture Overview

- Introduction
- Assumptions
- Different Cases to be Studied
- Solutions for Some Cases

Introduction

- Here, we will study how governing equations for composite plates as developed earlier, can be used to calculate the response of semi-infinite plates.

- We analyze semi-infinite plates because of following reasons:
  – Closed form solutions for such plates are easy to develop. The same may not be true for more "common" geometries and boundary conditions.
  – Analysis of semi-infinite plates involves significant idealization, thereby reducing the complexity of the problem. Such simplified analyses provide us with useful insights related to role of material properties, boundary conditions and external loads on the response of plates.
  – The closed form solutions developed here, may be used to check validity of other solution approaches, e.g. finite element method.

Assumptions about Semi-Infinite Plates

- Consider a plate which is infinitely long in one direction and simply supported along both of its short-edges as shown in Fig. 24.1. The plate is loaded in the z-direction with constant load intensity q.

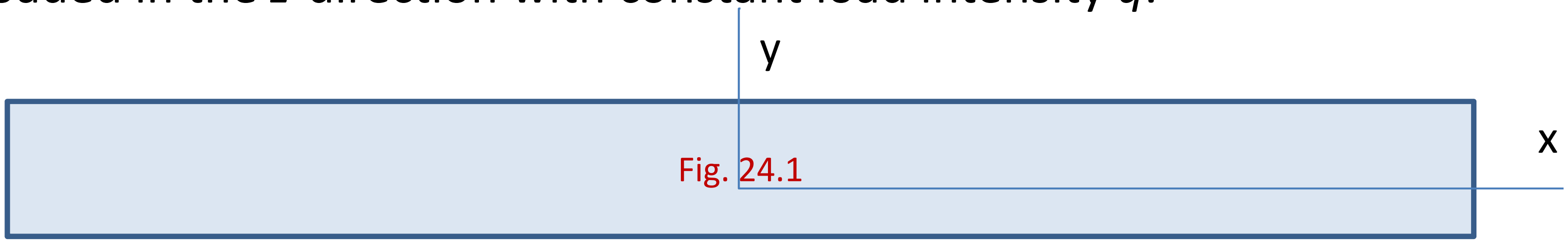

Fig. 24.1

- Avery long plate, with dimensions a, and b, such that $a >> b$, will approximately mimic the response of an "infinitely" long plate in an approximate sense. Here, a is plate's dimension in the x-direction.

- Given that the plate is very long in the x-direction, it can be assumed that partial derivatives of all entities in the y-direction for such a plate are zero. Thus, for such a plate partial derivative of an entity in the x-direction will equal total derivative in the x direction. This can be written as:

$$\frac{\partial}{\partial y} = 0, \quad \text{and} \quad \frac{\partial}{\partial x} = \frac{d}{dx}$$

Different Cases to be Studied

- Finally we define the boundary conditions and lamination sequence of the plates to be studied. Overall, four cases will be studied. The BCs and lamination sequences for these cases are shown in Fig. 24.2.

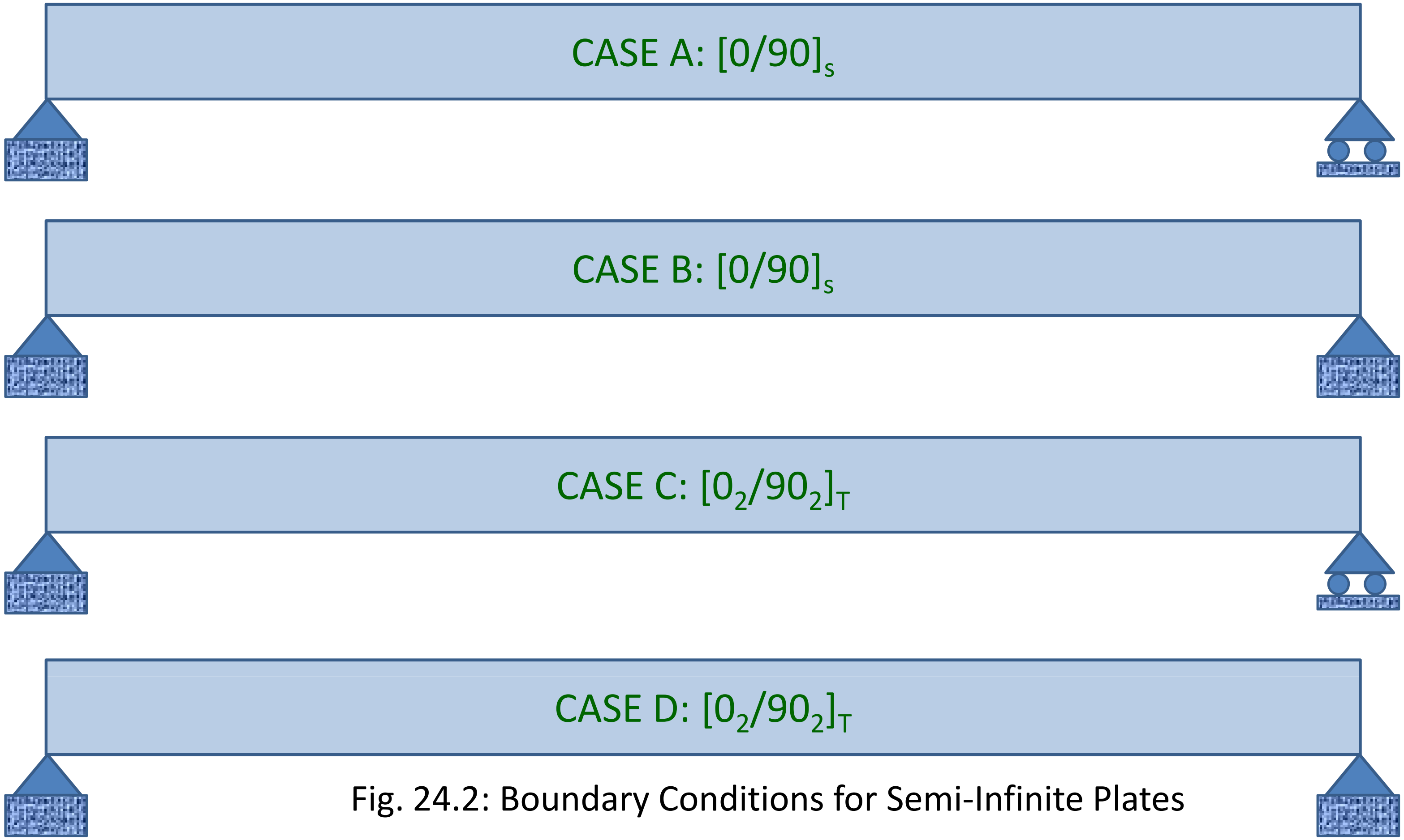

Fig. 24.2: Boundary Conditions for Semi-Infinite Plates

Solutions for Response of Infinitely Long Plates

- Kinematic Equations: Rewriting Eqs. 16.4, 16.5 and 16.6 we get:

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad \text{where,}$$

$$\begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u^o}{\partial x} \\ \dfrac{\partial v^o}{\partial y} \\ \dfrac{\partial u^o}{\partial y} + \dfrac{\partial v^o}{\partial x} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial^2 w^o}{\partial x^2} \\ \dfrac{\partial^2 w^o}{\partial y^2} \\ 2\dfrac{\partial^2 w^o}{\partial x \partial y} \end{Bmatrix}$$

- Given that for such plates partial derivatives in the *y*-direction are zero, we can write these relations as:

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad \text{where,}$$

(Eq. 24.1)

$$\begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{du^o}{\partial x} \\ 0 \\ \dfrac{dv^o}{\partial x} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{d^2 w^o}{\partial x^2} \\ 0 \\ 0 \end{Bmatrix}$$

Solutions for Response of Infinitely Long Plates

- Equilibrium Equations: Next, we simplify equilibrium equations, in the same way as we simplified kinematic equations, i.e. by using the fact that partial derivatives the *y*-direction for such a plate are zero.

- Thus, Eqs. 19.7, 19.8 and 19.9 may be simplified as:

$$\frac{\partial N_x}{\partial x} = \frac{dN_x}{dx} = 0$$

$$\frac{\partial N_{xy}}{\partial x} = \frac{dN_{xy}}{dx} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + q = \frac{d^2 M_y}{dx^2} + \mathrm{q} = 0$$

Integrating these equations yields:

$N_x = c_1$

$N_{xy} = c_2$ (Eq. 24.2)

$M_x = -qx^2/2 + c_3 x + c_4$

Solutions for Response of Infinitely Long Plates

- Stiffness Matrices: For cases A and B, as defined in Fig. 24.2, the lamination sequence is $[0/90]_s$. Thus, the laminate is symmetric, as well as cross-ply. For such a laminate:
  - $[B] = [0]$
  - And because all the plies in the laminate are oriented either at 0° or at 90°:
    - $A_{16} = A_{26} = 0$
    - $D_{16} = D_{26} = 0$ (Eq. 24.3)

- For cases C and D, as defined in Fig. 24.2, the lamination sequence is $[0_2/90_2]_T$. This is a non-symmetric but cross-ply laminate. For such a laminate:
  - $A_{16} = A_{26} = 0$
  - $B_{16} = B_{26} = 0$
  - $D_{16} = D_{26} = 0$ (Eq. 24.4)

Solution for Case A

- For Case A, we now combine Eqs. 24.1, 24.2 and 24.3 to get:

$$N_x = c_1 = A_{11} \frac{du^o}{dx} \qquad \text{(Eq. 24.5)}$$

Integrating this relation, we get:

$c_1 x = A_{11} u^0(x) - c_5$ or,

$$\boldsymbol{u^0(x) = \frac{c_1 x + c_5}{A_{11}}} \qquad \text{(Eq. 24.6)}$$

Similarly, in definition of N_y, only 1st term is non-zero, as all other terms are zero because A_{16} and A_{26} are zero, and also partial derivative in *y* direction is zero.

$$N_y = A_{11} \frac{du^o}{dx}$$

Thus, using Eq. 16.5, we get: (Eq. 24.7)

$$\boldsymbol{N_y = A_{12} \frac{c_1}{A_{11}}}$$

Finally, for equilibrium equation in *y* direction, we get:

$$\boldsymbol{v^0(x) = \frac{c_2 x + c_6}{A_{66}}} \qquad \text{(Eq. 24.8)}$$

Solution for Case A

- Also, considering the momentum equation, and assuming that external normal load per unit area q, is a constant, we get from Eq. 24.2:

  $M_x = -qx^2/2 + C_3x + C_4$ (Eq. 24.9)

- And expressing M_x in terms of strains and curvatures, we can write:

  $M_x = D_{11}(d^2w^o/dx^2)$ (Eq. 24.10)

  because terms D_{16}, D_{26}, and B_{ij} are zero.

- Combining these two equations, and integrating twice, we get:

  $w^o(x) = \{-qx^4/24 + C_3x^3/6 + C_4x^2/2 + C_7x + C_8\}/D_{11}$ (Eq. 24.11)

- Also, writing expression for M_y, and using Eq. 24.8, we get:

  $M_y = D_{12}(d^2w^o/dx^2) = (D_{12}/ D_{11})M_x$ (Eq. 24.12)

Solution for Case A

- It may be noted here, that Eqs. 24.5 through 24.12 are valid for Cases A and B, because the problem definition for either cases is same except for boundary conditions. As we have not yet applied boundary conditions, Equations 24.5-24.12 are applicable to Case B as well.

- Boundary Conditions for Case A
  - As shown in Figure 24.2, the BCs for this plate are:

BCs at x = -a/2	BCs at x = +a/2
$u^{o-} = 0$	$N_x^{+} = 0$
$v^{o-} = 0$	$N_{xy}^{+} = 0$
$M_x^{-} = 0$	$M_x^{+} = 0$
$w^{o-} = 0$	$w^{o+} = 0$

Solution for Case A

- Plugging 1st set of BCs, i.e. $u^{o-} = 0$ and $N_x^+ = 0$ in Eqs. 24.5 and 24.6, yields:

  $C_1 = C_5 = 0$.

  Thus,

  $N_x(x) = 0 \qquad u^o(x) = 0 \qquad N_y(x) = 0 \qquad \text{(Eq. 24.13)}$

- Similarly, second set of BCs, i.e. $v^{o-} = 0$ and $N_{xy}^+ = 0$ yields:

  $C_2 = C_6 = 0$.

  Thus,

  $N_{xy}(x) = 0 \qquad v^o(x) = 0 \qquad \text{(Eq. 24.14)}$

- Further, the 3rd set of BCs , i.e., $M^{0\pm} = 0$, at $x = \pm a/2$ implies (from Eq. 24.9):

  $0 = -qa^2/8 + C_3 a/2 + C_4 \qquad \text{(for } x = +a/2\text{)}$

  $0 = -qa^2/8 - C_3 a/2 + C_4 \qquad \text{(for } x = -a/2\text{)}$

  Thus,

  $C_3 = 0, \qquad C_4 = qa^2/8, \text{ and } \qquad M_x = q(a^2/4 - x^2)/2 \qquad \text{(Eq. 24.15)}$

Solution for Case A

- Finally, plugging 4th set of BCs, i.e. $w^{0\pm} = 0$, at $x = \pm a/2$ in Eqs. 24.11,and also using Eq. 24.15, yields following two simultaneous equations.

  $0 = \{-qa^4/384 + \cancel{C_3 a^3/48} + C_4 a^2/8 + C_7 a/2 + C_8\}/D_{11}$

  $0 = \{-qa^4/384 - \cancel{C_3 a^3/48} + C_4 a^2/8 - C_7 a/2 + C_8\}/D_{11}$

  Thus,

  $C_7 = 0$, $C_8 = -5qa^2/384$, and

  $w^o(x) = \{-16(x/a)^4/24 + 24(x/a)^2/16 - 5\}(qa^4)/(384D_{11})$ (Eq. 24.16)

  Also,

  $M_y = (D_{12}/ D_{11})M_x$ and $M_{xy} = 0$ (Eq. 24.17)

- Equations 24.13 through 24.17 constitute the solution for Case A.

Comments on Solution for Case A

- It is seen from Eqs. 24.13 through 24.17, that the solutions for mid-plane displacements, resultant forces, and resultant moments is symmetric with respect to *x* axis.

- However, as seen in Fig. 24.2, the boundary conditions for Case A are not symmetric about the *x*-axis. Further, even though the beam is free to move in *x* direction at $x = +a/2$, the solution for displacement at this end, u^o, is zero. There are two reasons for this.
  - Presence of symmetric laminate ensures that there is no coupling between $u(x)$ and M_x. Hence, the moment imposed by external load intensity *q*, cannot create inplane resultant force N_x, and thus, no inplane displacement in x-direction is getting generated.
  - Further, we have assumed linearity of strains in our strain-displacement formulation.

Comments on Solution for Case A

- Next, consider Fig. 24.3(a), which shows the un-deformed shape of a small portion of the plate being currently analyzed.

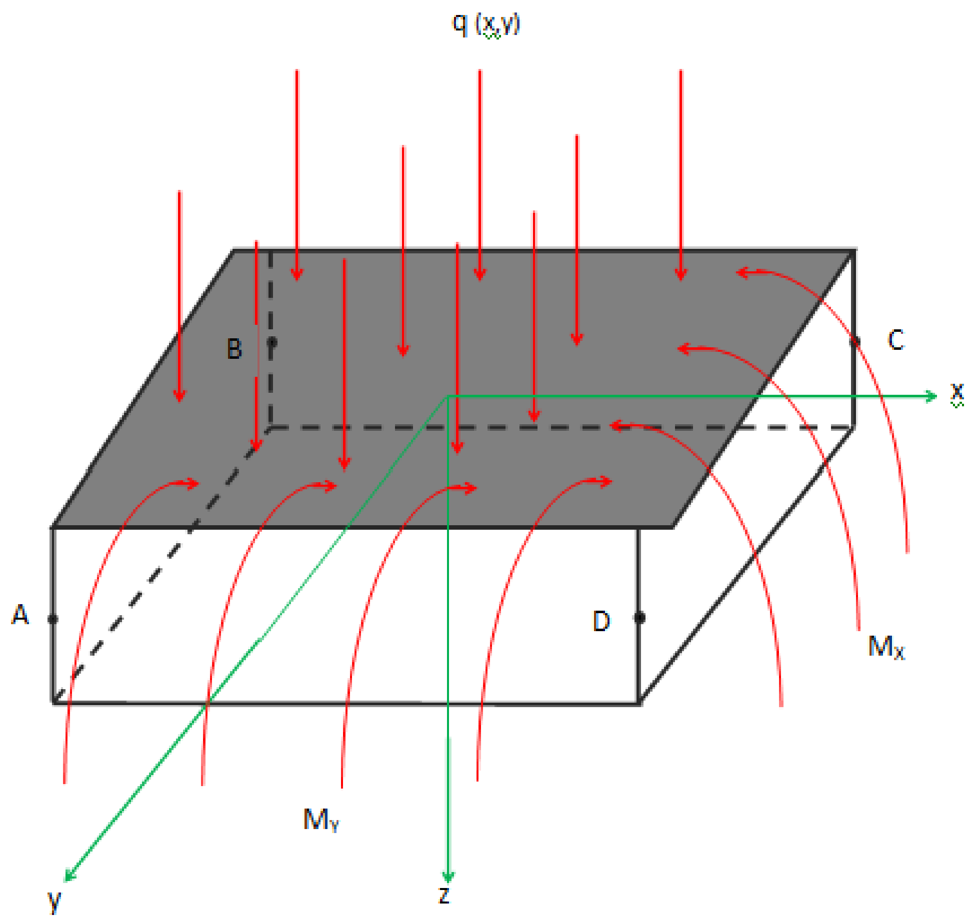

Fig. 24.3(a): Undeformed Shape of a Small Portion of Plate Subjected to Various Loads and Moments which may Change Its Curvature

Comments on Solution for Case A

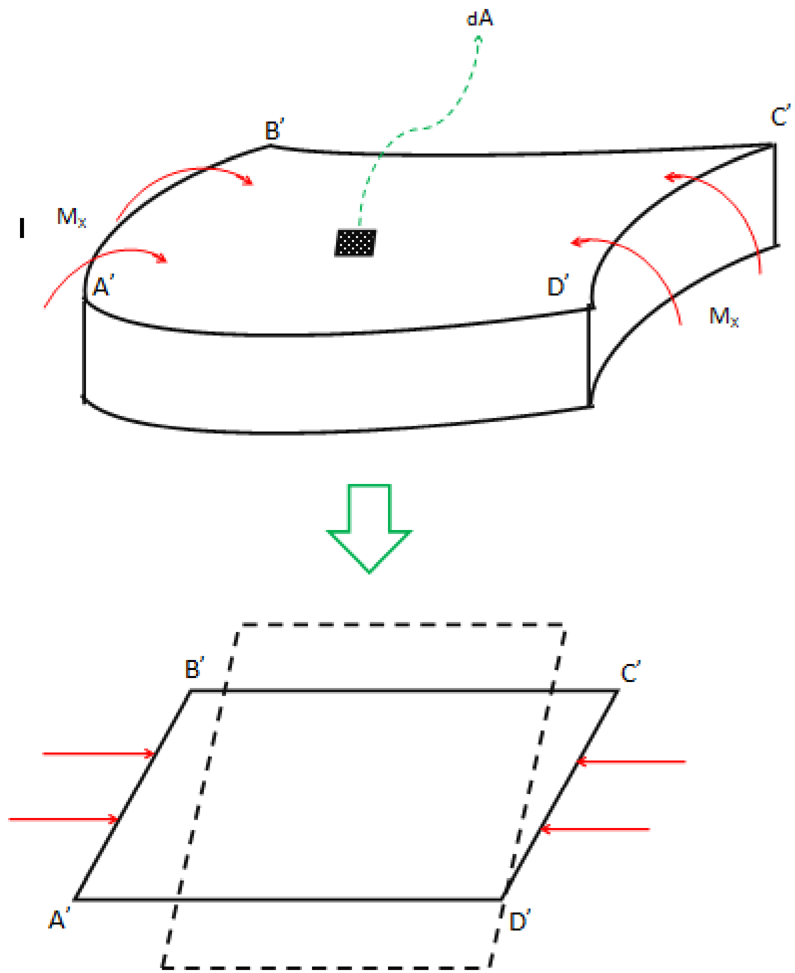

Fig. 24.3(b): Anticlastic curvature in plate under loads and moments

Comments on Solution for Case A

- This plate, has three forces/moments acting on the laminate. These are; q, M_x, and M_y. All other forces have been found to be zero as per our calculations.

- Surface ABCD is the mid-plane for the plate in Fig. 24.3a. Now, because in-plane displacements (u^o, and v^o) and curvature dw^o/dy for midplane ABCD are zero (see Eqs. 24.13 and 24.14), lines AB, and CD, which were straight prior to deformation, will remain straight after deformation as well. This is counter-intuitive, as external load q would tend to induce curvature on these line-segments.

- However, presence of non-zero M_y ensures that these line segments remain straight. This is explained as follows.

Comments on Solution for Case A

- Vertical force due to q is balanced by M_x, and M_y. However, given that M_y/M_x equals D_{12}/D_{11}, and typically, $D_{12} << D_{11}$, q is predominantly balanced by M_x.

- However, in absence of M_y plate would curve as shown in Fig. 24.3b, and a straight line AB, would curve upwards to position A'B' due to load.

- This curvature, which would have existed in absence of M_y, is known as *anticlastic curvature*.

- However, the presence of non-zero M_y causes segment A'B' to become once again straight.

- Also, note that M_y is non-zero because D_{12} is non-zero, and the latter term is directly proportional to Poisson's ratio ν_{12} Hence, presence of M_y, and straightening of AB is a demonstration of Poisson's effect.

- Finally, absence of M_y would have induced a non-zero curvature in the *y*-direction. This would have violated the condition that partial derivatives in the *y*-direction should be zero. Hence, we see that it is the material, which drives the condition that partial derivative in the *y*-direction should be zero.

Solution for Case B

- Case B is similar to Case A, except for some differences in in-plane boundary conditions. BCs for Case B are given below.

BCs at x = -a/2	BCs at x = +a/2
$u^{o-} = 0$	$u^{o+} = 0$
$v^{o-} = 0$	$V^{o+} = 0$
$M_x^- = 0$	$M_x^+ = 0$
$w^{o-} = 0$	$w^{o+} = 0$

- Plugging 1st set of BCs, i.e. $u^{o\pm} = 0$ at x = ±a/2 in Eqs. 24.5 and 24.6, yields:

  $C_1 = C_5 = 0$. Thus,

  $N_x(x) = 0 \qquad u^o(x) = 0 \qquad N_y(x) = 0$ (Eq. 24.18)

- Similarly, second set of BCs, i.e. $v^{o\pm} = 0$ at x = ±a/2 yields:

  $C_2 = C_6 = 0$. Thus,

  $N_{xy}(x) = 0 \qquad v^o(x) = 0$ (Eq. 24.19)

- Since the out-of-plane problem is identical to that of Case A, and since it is not coupled with the inplane problem (because [B] is zero), thus, the solution for out-of-plane problem is same as that for Case A.