Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 20

Analysis of a Laminated Composite

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Introduction

- One very significant advantage offered by composites is that their properties can be tailor-made; layer-by-layer, to meet specific functional requirements.

- Further, each layer can be itself engineered by altering selection of fiber materials, having a mix of fibers, changing their orientation, using matrix material with appropriate properties, and controlling fiber volume fraction.

- Analytical models developed thus far help us calculate fairly accurately mechanical properties of each lamina. These models allow variability of properties of fibers and matrices, volume fractions, and fiber orientation.

- The next step in this journey is to develop a theoretical construct which will help us predict the mechanical response of a laminate, i.e. a collection of laminae, stacked up and bonded together. Each lamina in this stack-up may have different properties. The 1st step for predicting the response of a laminate involves developing stress-strain relations for a composite plate.

Strain-Field in a Laminate

- Before developing an understanding about variation of strains in a laminate, we will make certain assumptions about it. These are:
  - Laminates are manufactured so that they act as *single-layer* materials. In typical applications, such a response from the laminate is required so that its overall strength and stiffness can be maximized.
  - The requirement of “single-layer materials” necessitates that the adhesive bond between two adjacent layers is perfect in the sense it has:
    - Almost zero thickness
    - No shear deformation - Thus, adjacent lamina cannot slip over each other.
  - The assumption of “single-layer material” also implies that displacements are continuous across the bond between two adjacent layers.
  - Laminates are thin in the sense their overall thickness is significantly smaller other dimensions of the laminate.

Strain-Field in a Laminate

- Consider Fig. 20.1 The figure shows how a section of laminate, taken in *x-z* direction, appears after deformation due to application of forces. Here, *z*, is the thickness direction on reference coordinate system.

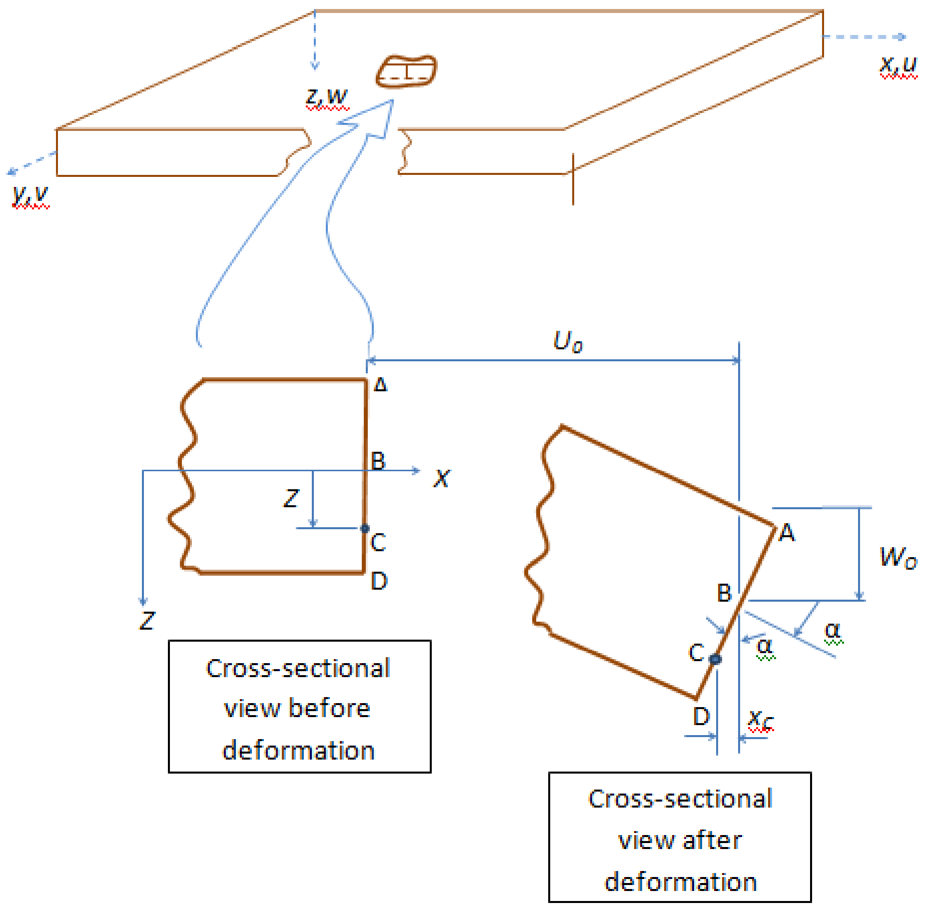

Fig. 20.1: Deformation of a Laminate as Viewed in *x-y* Plane

Strain-Field in a Laminate

- The lower left-side portion of Fig. 20.1 is a view of un-deformed laminate. The lower right-side portion of Fig. 20.1 shows the deformed state of laminate's section.

- In the un-deformed section, line ABCD, is perfectly straight and normal to mid-plane of the laminate. This line is assumed to remain straight and normal to mid-plane even after getting deformed. This implies that:
  - Out-of-plane shear strains γ_{xz}, and γ_{yz}, are zero.
  - There is no inter-laminar shear or slipping.

- Further, it is assumed that the length of line ABCD remains same after deformation. This in turn implies that strain in *z* direction, ε_{zz}, is zero.

Strain-Field in a Laminate

- Assumptions of normality, in-extensibility and straightness for line ABCD are together known as Kirchhoff-Love assumptions in shell theory, and Kirchhoff's assumptions in plate theory.

- Further, due to deformation of plate, point B undergoes translation by amount u^o, v^o, and w^o, in x, y, and z directions, respectively. Also, the line ABCD rotates about B by an angle α in the z plane. Figure 20. does not show v_o displacement explicitly because the figure is a side view of the laminate undergoing deformation.

- Thus, displacement of point C, which is z distance away from mid-plane is:

$$u(x,y,z) = u^o(x,y) - z\cdot\alpha(x,y) = \; = u^o - z\frac{\partial w_o}{\partial x} \qquad \text{(Eq. 20.1)}$$

- In Eq. 20.1, we use the fact that α is partial differential of w^o in x-direction.

Strain-Field in a Laminate

- Similarly, we can also write the relation for displacement $v(z)$, as:

$$v(x,y,z) = v^o(x,y) - z\frac{\partial w^o(x,y)}{\partial y}$$ (Eq. 20.2)

- Also, for small displacements, following relations hold for strains.

$$\epsilon_{xx} = \frac{\partial u}{\partial x}$$
$$\epsilon_{yy} = \frac{\partial v}{\partial y}$$ (Eq. 20.2a)
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

- Now, using definitions for *u*, and *v*, in above strain definitions, we get:

$$\varepsilon_{xx}(x,y,z) = \frac{\partial u^o(x,y)}{\partial x} - z\frac{\partial^2 w^o(x,y)}{\partial x^2}$$

$$\varepsilon_{yy}(x,y,z) = \frac{\partial v^o(x,y)}{\partial y} - z\frac{\partial^2 w^o(x,y)}{\partial y^2}$$ (Eq. 20.3)

$$\gamma_{xy}(x,y,z) = \frac{\partial u^o(x,y)}{\partial y} + \frac{\partial v^o(x,y)}{\partial x} - 2z\frac{\partial^2 w^o(x,y)}{\partial x \partial y}$$

- Also, as mentioned earlier, ε_{zz}, γ_{xz}, and γ_{yz} are zero.

Strain-Field in a Laminate

- As per Eq. 20.3, normal and shear strains at a point in a laminate can be decomposed into their mid-plane, and curvature components. Thus, Eq. 20.3 may be re-written as:

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{xx}^{0} \\ \epsilon_{yy}^{0} \\ \gamma_{xy}^{0} \end{Bmatrix} + z \begin{Bmatrix} k_{x} \\ k_{y} \\ k_{xy} \end{Bmatrix} \qquad \text{(Eq. 20.4)}$$

where the mid-plane strains are defined as:

$$\begin{Bmatrix} \varepsilon_{xx}^{o} \\ \varepsilon_{yy}^{o} \\ \gamma_{xy}^{o} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u^{o}(x,y)}{\partial x} \\ \dfrac{\partial v^{o}(x,y)}{\partial y} \\ \dfrac{\partial u^{o}(x,y)}{\partial y} + \dfrac{\partial v^{o}(x,y)}{\partial x} \end{Bmatrix} \qquad \text{(Eq. 20.5)}$$

Strain-Field in a Laminate

- And, mid-surface curvatures are defined as:

$$\begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial^2 w^o}{\partial x^2} \\ \dfrac{\partial^2 w^o}{\partial y^2} \\ 2\dfrac{\partial^2 w^o}{\partial x \partial y} \end{Bmatrix} \qquad \text{(Eq. 20.6)}$$

- In Eq. 20.6, the last term represents twist curvature of mid-surface of composite laminate.

- Equations 20.3-20.6 are valid only for plates and not for shells. This is because our strain definitions, as per Eq. 20.2a are valid only for plates as they do not account for a shell's curvature.

- Equation 20.4 shows that strains vary linearly over the thickness of a composite plate, with the average strain computed over plate's thickness equaling mid-plane strain.

Stresses in a Laminate

- If one were able to compute mid-plane strains and curvature of the plate, then predicting stresses over the laminate's thickness is simply a matter of multiplying these strains with stiffness constants using strain-stiffness relations on a layer-by-layer basis.

- Thus, stresses in k^{th} layer of the laminate may be calculated using following relations.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \underline{Q}_{11} & \underline{Q}_{12} & \underline{Q}_{16} \\ \underline{Q}_{12} & \underline{Q}_{22} & \underline{Q}_{26} \\ \underline{Q}_{16} & \underline{Q}_{26} & \underline{Q}_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \epsilon_{xx}^0 \\ \epsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \right\} \quad \text{(Eq. 20.7)}$$

- Since [Q] matrix varies discontinuously between two adjacent layers, variation of stresses between two layers need not be linear, or even continuous. Thus stresses are discontinuous between two adjacent layers, even though strain varies linearly across entire laminate thickness. However, over the thickness of a single lamina, stress variation is linearly continuous.