Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 27

Simply Supported Plates with Normal Load

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Lecture Overview

- Analysis of a Simply Supported Plate on All Four Sides
  - Governing equations
  - Boundary conditions
  - Stiffness matrices
  - Series solution

- Assessing Convergence of Solution

Simply-Supported Plate on All Sides

- Consider a plate of dimensions *a* X *b* in *x* and *y* directions, respectively, and which is simply supported on all four sides.

- Further, we assume that this plate has a symmetric and specially orthotropic lamination sequence. Thus,
  - [B] = [0]   due to symmetry
  - $A_{16} = A_{26} = D_{16} = D_{26} = 0$

- Finally we assume that the plate is normally loaded as shown in Fig. 27.1 with a load intensity of $q(x,y)$.

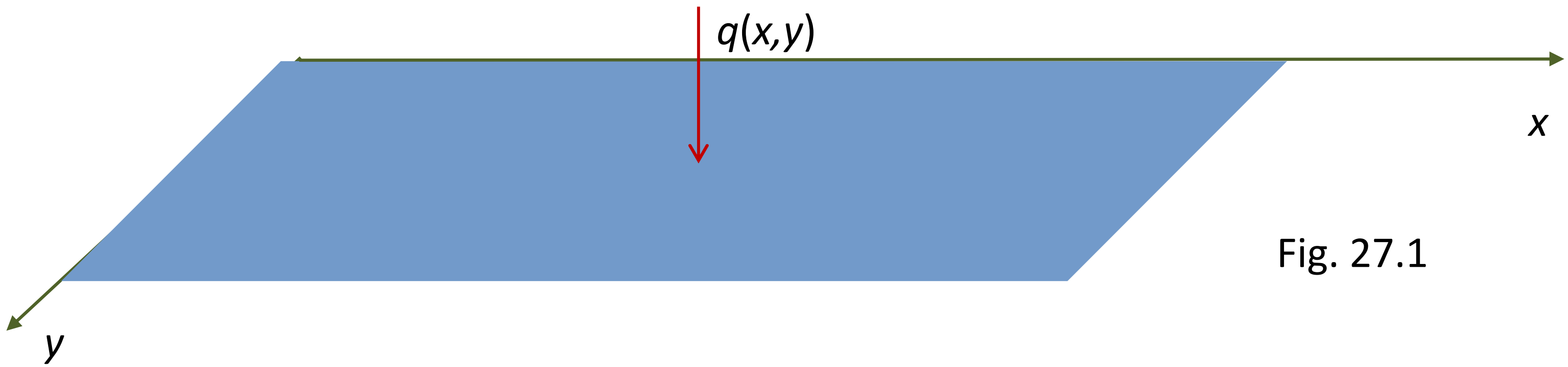

Fig. 27.1

Simply-Supported Plate on All Sides

- For such a plate, the out-of-plane boundary conditions are:
  - $w^{\pm} = 0$ on all four sides.
  - $M_x^{\pm} = 0$ at $x = 0, a$.
  - $M_y^{\pm} = 0$ at $y = 0, b$.

- As explained earlier, the governing equation for equilibrium for out-of-plane direction for such a plate is decoupled with in-plane equations, because the plate's lamination sequence is symmetric. This equation is reproduced below.

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

- Expressing this equation in terms of derivatives of w^o, we get:

$$q(x,y) = D_{11}\frac{\partial^4 w^o}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w^o}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w^o}{\partial y^4} \qquad \text{(Eq. 27.1)}$$

Simply-Supported Plate on All Sides

- Equation 27.1 has to be solved for w^o. At this stage, we assume a series function for w^o, which satisfies the boundary conditions listed earlier. Let this assumed function be:

$$w^o(x,y) = \sum\sum w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

  where, *m*, and *n* are integers which can vary between 1 and infinity.

- Such a series expression for w^o satisfies out-of-plane boundary conditions. If this expression also satisfies Eq. 27.1, then it is a valid solution. For this expression to satisfy Eq. 27.1, we put it into the equation, and get the following condition, which has to be satisfied for getting a valid solution for w^o.

$$q^o(x,y) = \sum\sum\left\{D_{11}\left(\frac{m\pi}{a}\right)^4 + 2(D_{12}+2D_{66})\left(\frac{m\pi}{a}\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4\right\} w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

(Eq. 27.2)

Simply-Supported Plate on All Sides

- Now, if $q(x,y)$ can also be expressed as a double Fourier series, as shown below:

  $$q(x,y) = \sum\sum q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

  then, for Eq. 22.2 to be true, the following condition must hold.

  $$w_{mn} = \frac{q_{mn}}{d_{mn}}$$

  where,

  $$d_{mn} = D_{11}\left(\frac{m\pi}{a}\right)^4 + 2(D_{12} + 2D_{66})\left(\frac{m\pi}{a}\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4$$

- Hence, if we know q_{mn}, we can find out-of-plane deflections for such a plate.

Simply-Supported Plate on All Sides

- The values for q_{mn} can be gotten by decomposing $q(x,y)$ into a double Fourier series as shown below.

$$\int_0^b \int_0^a q(x,y)\left[\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\right]dxdy = \int_0^b \int_0^a q_{mn}\left[\sin\frac{m\pi x}{a}\sin\frac{m\pi y}{b}\right]\left[\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\right]dxdy$$

- While solving for q_{mn}, following identities may be used.

$$\int_0^a \left[\sin\frac{m\pi x}{a}\sin\frac{i\pi x}{a}\right]dx = \frac{1}{2}\delta_{im}$$

$$\int_0^b \left[\sin\frac{n\pi y}{b}\sin\frac{j\pi y}{b}\right]dx = \frac{1}{2}\delta_{jn}$$

- If functions are more complex, then Gaussian integration may be used.

- Also, if there are point loads, then Dirac-Delta functions $\delta[(x-x_o), (y-y_o)]$ may be used while computing q_{mn} using above formula.

Convergence of Solution

- While computing the out-of-plane response for the plate, one has to estimate the number of terms to be used in serial expansion of $q(x,y)$. Theoretically, the larger the number of terms the more accurate (or converged) is the solution.

- But then, one question that requires careful consideration relates to the context in which accuracy/convergence is being assessed. Are we looking for convergence of displacement, or stress, or load, or … ?

- In this context, consider that $q(x,y)$ is constant over the plate, and has a value q_o. For such a load, the Fourier series expansion is:

$$q(x,y) = q_o = \sum\sum q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = \frac{16 q_o}{\pi^2}\sum\sum\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

where, m = 1, 3, 5, …. and n = 1, 3, 5, …

Convergence of Solution

- Here, for a two-term solution (i.e. m, and n can assume two distinct values):

  $q_{1,1} = (16q_o/\pi^2)$ $\qquad$ $q_{3,3} = (16q_o/\pi^2) \times (1/3^2)$

- Similarly, for a 20-term solution:

  $q_{1,1} = (16q_o/\pi^2)$ $\qquad$ $q_{39,39} = (16q_o/\pi^2) \times (1/39^2)$

- Hence, we see that load converges in a $1/m^2$ way.

- Next, we look at convergence of deflection. Earlier, for this plate, it has been shown that:

  $w_{mn} = q_{mn}/d_{mn}$

- But, we know that d_{mn} and q_{mn} are directly proportional to m^4 and $1/m^2$, respectively. Thus, deflection converges in a $1/m^6$ fashion.

Convergence of Solution

- Similarly, we can find the pace of convergence for other parameters. The following table lists the rate of convergence for different mechanical parameters as well.

Load	m^{-2}
Out-of-plane shear	m^{-3}
Curvature	m^{-4}
Resultant Moment	m^{-4}
Stress	m^{-4}
Displacement (w)	m^{-6}

- Hence, we find that among the unknowns, out-of-plane shear converges much more slowly than out-of-plane displacement.