Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 29

Application of Galerkin Method

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Application of Galerkin Method: Example 1

- Here, we consider a beam which is simply supported at both ends, as shown in Fig. 29.1. The overall length of the beam is *L*.

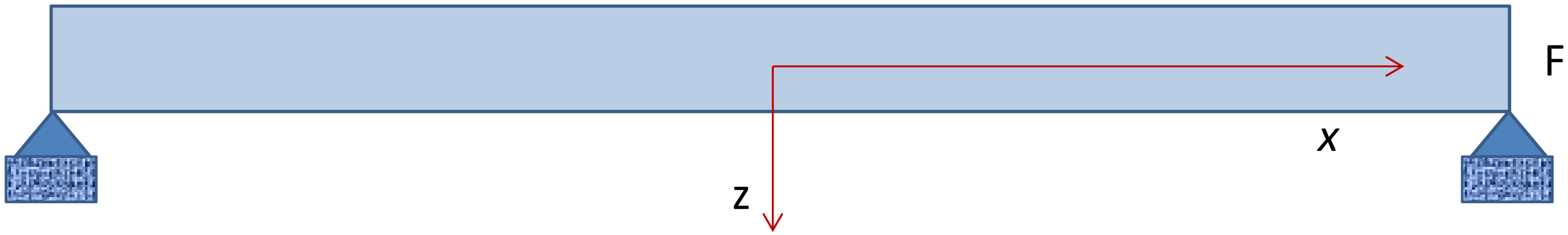

Fig. 29.1

- Here, we assume:
  - Material is isotropic.
  - Normal uniform load of intensity q_0 is applied over the length of the beam.
  - Strain-displacements relations are linear.

- For such a system, the governing equation for normal deflection is:

$$EI(d^4w/dx^4) - q_0 = 0. \qquad \text{(Eq. 29.1)}$$

Application of Galerkin Method: Example 1

- In this example, we will compare the accuracy of Galerkin solution for Eq. 29.1 with exact solution.

- In such a case, the boundary conditions are:
  - Displacement at ends of the beam is zero, i.e. $w = 0$ at $x = \pm L/2$.
  - Moment at both beam ends is zero.

- At this stage, we select a displacement function, which satisfies the kinematic boundary conditions. Thus, we assume:

  $w^o(x) = A \cos (\pi x/L)$

- Such a displacement function satisfies the displacement BC at both ends of beam. Substituting this function in governing equation gives us error in the force. The relation for this error is:

  $$E[w(x)] = AEI \cdot (\pi/L)^4 \cdot \cos (\pi x/L) - q_0 \qquad \text{(Eq. 29.2)}$$

Application of Galerkin Method: Example 1

- The virtual work done by this error force as defined in Eq. 29.2, when integrated over the length of beam should be zero. Thus, we get:

  $\int_\Omega E[w(x)] \cdot \varepsilon w_1(x)\, dx = 0,$ where integration limits are $-L/2$ and $L/2$.

  or,

  $\int_\Omega [AEI \cdot (\pi/L)^4 \cdot \cos(\pi x/L) - q_0] \cdot \varepsilon w_1(x)\, dx = 0.$ (Eq. 29.3)

- At this stage, we chose w_1 as defined below:

  $w_1(x) = A_1 \cos(\pi x/L)$

- Thus, Eq. 29.3 can be rewritten as:

  $\int_\Omega [AEI \cdot (\pi/L)^4 \cdot \cos(\pi x/L) - q_0] \cdot \varepsilon\, A_1 \cos(\pi x/L)\, dx = 0.$ (Eq. 27.4)

- From Eq. 29.4, we get:

  $A = 4q_0L^4/(\pi^5 EI)$ (Eq. 29.5)

Application of Galerkin Method: Example 1

- Thus, the approximate solution as per Galerkin method is:
  $w(x) = [4q_0L^4/(\pi^5 EI)] \cos(\pi x/L)$

- At x = 0, the beam deflection is:
  $w_{Galerkin}(0) = 0.01309[q_0L^4/(EI)]$

- Also, the exact solution for beam deflection is:
  $w_{Exact}(0) = 0.1302\ [q_0L^4/(EI)]$

- Comparing exact and approximate values (as per Galerkin method), we see the two answers are fairly close to each other.

Application of Galerkin Method: Example 2

- Now through Example 1, that the Galerkin solution for displacement is slightly less than as predicted by exact solution. This is because Galerkin method predicts higher stiffness for a system.

- To improve the accuracy of the solution we may use more than one term in the assumed solution form. Thus, we assume:

  $w^o(x) = A_1 \cos(\pi x/L) + A_3 \cos(3\pi x/L)$

- Such an assumed expression for $w(x)$ satisfied the kinematic boundary conditions at both ends of the simply supported beam. At this stage, we also assume an expression for virtual displacement. Thus,

  $w_1(x) = B_1 \cos(\pi x/L) + B_3 \cos(3\pi x/L)$

- Like the expression for $w(x)$, the expression for virtual displacement, $w_1(x)$, also satisfies kinematic boundary conditions.

Application of Galerkin Method: Example 2

- At this stage, we develop an expression for virtual work over the entire domain, and equate it to zero. Thus:

$$\int_{-L/2}^{L/2} E[w\{x)]w_1(x)dx = 0$$

or,

$$\int_{-L/2}^{L/2} \epsilon\left[EI\left\{\left(\frac{\pi}{L}\right)^4 A_1 cos\frac{\pi x}{L} + \left(\frac{3\pi}{L}\right)^4 A_3 cos\frac{3\pi x}{L}\right\} - q_o\right]\left(B_1 cos\frac{\pi x}{L} + B_3 cos\frac{3\pi x}{L}\right)dx = 0$$

or,

$$B_1 \int_{-L/2}^{L/2} \epsilon\left[EI\left\{\left(\frac{\pi}{L}\right)^4 A_1 cos\frac{\pi x}{L} + \left(\frac{3\pi}{L}\right)^4 A_3 cos\frac{3\pi x}{L}\right\} - q_o\right] cos\frac{\pi x}{L} dx$$
$$+ B_3 \int_{-L/2}^{L/2} \epsilon\left[EI\left\{\left(\frac{\pi}{L}\right)^4 A_1 cos\frac{\pi x}{L} + \left(\frac{3\pi}{L}\right)^4 A_3 cos\frac{3\pi x}{L}\right\} - q_o\right] cos\frac{3\pi x}{L} dx = 0$$

- But we know that B_1, and B_2 can have arbitrary magnitudes. Hence, integral of virtual work over domain can only be zero, if both terms in [ ] are individually zero.

Application of Galerkin Method: Example 2

- Thus we get two parallel equations in A_1, and A_3. Solving for these equations gives us:
  - $A_1 = 4q_0L^4/(\pi^5 EI)$
  - $A_3 = -A_1/35$

- Thus,

  $w(x) = [4q_0L^4/(\pi^5 EI)]\ [\cos(\pi x/L) - (1/35)\cos(3\pi x/L)]$.

- At the center, i.e. when $x = 0$, the value of displacement of the beam is:

  $w_{Galerkin}(0) = 0.01309[q_0L^4/(EI)]$ using 1-term solution

  $w_{Galerkin}(0) = 0.01302[q_0L^4/(EI)]$ using 2-term solution

  $w_{Exact}(0) = 0.1302\ [q_0L^4/(EI)]$ exact solution.

- Thus, we see that in this case, a 2-term solution brings us remarkably close to the exact solution.

Application of Galerkin Method: Example 2

- Based on this analysis, we make following observations.
  - As we increase the number of terms in an assumed form of Galerkin solution, it approaches the exact value monotonically.
  - A solution with lesser number of terms represents a stiffer system vis-à-vis a system which has more terms.
  - As shown later, this asymptotic convergence on stiffness (and displacements) occurs because Galerkin method is similar to a "least squares approach". As number of terms increase, so does the flexibility of the system, thereby reducing its overall energy. Hence, accuracy of solution increases with number of terms.
  - Galerkin method tends to help itself.
  - This method can be easily extended for plates.