Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 32

Energy Methods

References for this Lecture

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2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

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4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Introduction

- Till so far, we have deduced solutions for laminated composite plates by solving equilibrium equations.

- These equations are derived from Newton's Laws of Motion, as per which "for an equilibrium state there must be no net force acting on the body" (or system).

- There is an alternative approach to solve these problems without using Newton's method. This method was originally developed by John Bernoulli (1667-1745) and later pefected by Lagrange (1736-1783).

- Bernoulli conceived the notion of "virtual work" to solve the same problem. As per this method, "for all possible displacements (of a system), the sum of the products of force and initial displacement in the direction of the force (i.e. the virtual work) must balance for equilibrium.

Introduction

- Comments on Newton’s and Bernoulli approaches:
  - These are totally different concepts.
  - Newton’s method is force oriented, and free-body-diagrams have to be constructed to develop equilibrium equations.
  - Bernoulli’s method is displacement oriented (or configuration oriented).
  - Both these methods provide us final state of equilibrium.
  - These methods are mutually independent.

- The virtual work method is applicable only for static problems. However, its dynamic analogue is d’Alembert’s Principle, which is also known as Lagrange-d’Alembert’s Principle.
  - A special “version” of this method is known as “Hamilton’s Principle”.
  - Also, it can be mathematically shown that this principle maps to the Principle of Virtual Work in static systems.

Introduction

- Here, we will understand how these methods can be used to address problems of statics and dynamics.

- Specifically, we will use:
  - Principle of minimum potential energy (similar to virtual work principle) to address static problems.
  - Hamilton’s method to solve dynamic problems.

- There are several significant advantages of energy methods over Newton’s method. These are:
  - Problem formulation is relatively straightforward.
  - Such formulations can be expressed in “weak” and “strong” versions. Weak formulations, when automated in FEA lead to symmetric stiffness matrices for linear systems.
  - It is easy to discern boundary condition requirements using such methods.

- However, Newton’s approach is more intuitive vis-à-vis energy based methods.

Total Potential Energy Principle

- The total potential energy in a structural system can be expressed as:

$$\Pi = \int_V W dV + \int_V (B_x u_x + B_y u_y + B_z u_z) dV - \int_S (T_x u_x + T_y u_y + T_z u_z) dS$$

(Eq. 32.1)

- Here, *W*, is the strain-energy density function, B_i represents body force per unit volume in *i*-direction, and T_i represents traction force per unit area in *i*-direction. For purposes of brevity only, we will omit body forces going further.

- Now, we know that:

$W = [\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}]/2$

- For a laminated plate, due to assumption of normality, we can discard shear strains in y-z and z-x plane. Also, using Eq. 14.11 in above relation we get:

$W = [\underline{Q}_{11}\varepsilon^2_{xx} + 2\underline{Q}_{12}\varepsilon_{xx}\varepsilon_{yy} + 2\underline{Q}_{16}\varepsilon_{xx}\gamma_{xy} + 2\underline{Q}_{26}\gamma_{xy}\varepsilon_{yy} + \underline{Q}_{22}\varepsilon^2_{yy} + 4\underline{Q}_{66}\gamma^2_{xy}]/2$

(Eq. 32.2)

Total Potential Energy Principle

- Putting Eq. 32.2 in 32.1, we get:

$$\Pi = \frac{1}{2}\iint_A \int_{-t/2}^{t/2} \left[\underline{Q}_{11}\varepsilon_{xx}^2 + 2\underline{Q}_{12}\varepsilon_{xx}\varepsilon_{yy} + 2\underline{Q}_{16}\varepsilon_{xx}\gamma_{xy} + 2\underline{Q}_{26}\gamma_{xy}\varepsilon_{yy} + \underline{Q}_{22}\varepsilon_{yy}^2 + 4\underline{Q}_{66}\gamma_{xy}^2\right] dz\,dA - \int_S (T_x u_x + T_y u_y + T_z u_z)\,dS$$

(Eq. 32.3)

- Using strain definitions from Eq. 15.3 above, and integrating the strain energy term over thickness of plate, we get:

$$\Pi = \frac{1}{2}\iint_A H(u^\circ, v^\circ, w)\,dA - \int_S (T_x u_x + T_y u_y + T_z u_z)\,dS$$ (Eq. 32.4)

- In Eq. 32.4, $H(u^o, v^o, w^o)$ is defined in Eq. 32.5. Also, it should be noted here that the domain of integration for H is midplane area of the plate, while the domain of integration of tractions constitutes six external surfaces of a rectangular plate.

Total Potential Energy Principle

$$H(u^o,v^o,w)=$$
$$A_{11}\left(\frac{\partial u^o}{\partial x}\right)^2+2A_{12}\frac{\partial u^o}{\partial x}\frac{\partial v^o}{\partial y}+A_{22}\left(\frac{\partial v^o}{\partial x}\right)^2+2\left(A_{16}\frac{\partial u^o}{\partial x}+A_{26}\frac{\partial v^o}{\partial y}\right)\left(\frac{\partial v^o}{\partial x}+\frac{\partial u^o}{\partial y}\right)+$$
$$A_{66}\left(\frac{\partial v^o}{\partial x}+\frac{\partial u^o}{\partial y}\right)^2-2B_{11}\frac{\partial u^o}{\partial x}\frac{\partial^2 w^o}{\partial x^2}-2B_{12}\left(\frac{\partial v^o}{\partial y}\frac{\partial^2 w^o}{\partial x^2}+\frac{\partial u^o}{\partial x}\frac{\partial^2 w^o}{\partial y^2}\right)-2B_{22}\frac{\partial v^o}{\partial y}\frac{\partial^2 w^o}{\partial y^2}-$$
$$2B_{16}\left\{\frac{\partial^2 w^o}{\partial x^2}\left(\frac{\partial v^o}{\partial x}+\frac{\partial u^o}{\partial y}\right)+2\frac{\partial u^o}{\partial x}\frac{\partial^2 w^o}{\partial x\partial y}\right\}-2B_{26}\left\{\frac{\partial^2 w^o}{\partial y^2}\left(\frac{\partial v^o}{\partial x}+\frac{\partial u^o}{\partial y}\right)+2\frac{\partial v^o}{\partial y}\frac{\partial^2 w^o}{\partial x\partial y}\right\}-$$
$$4B_{66}\frac{\partial^2 w^o}{\partial x\partial y}\left(\frac{\partial v^o}{\partial x}+\frac{\partial u^o}{\partial y}\right)+D_{11}\left(\frac{\partial^2 w^o}{\partial x^2}\right)^2+2D_{12}\frac{\partial^2 w^o}{\partial x^2}\frac{\partial^2 w^o}{\partial y^2}+D_{22}\left(\frac{\partial^2 w^o}{\partial y^2}\right)^2+$$
$$4\left(D_{16}\frac{\partial^2 w^o}{\partial x^2}+D_{26}\frac{\partial^2 w^o}{\partial y^2}\right)\frac{\partial^2 w^o}{\partial x\partial y}+4D_{66}\left(\frac{\partial^2 w^o}{\partial x\partial y}\right)^2$$

(Eq. 32.5)

- According to Theorem of Minimum Potential Energy, “of all possible displacement fields which satisfy compatibility and prescribed boundary conditions, the displacement field which satisfies the equilibrium equation makes the total potential energy a minimum”. Mathematically, this implies that the condition for equilibrium is that the 1st variation of total potential energy should be zero. Thus,

  $\delta\Pi = 0.$ (Eq. 32.6)

Total Potential Energy Principle

- Thus, setting the 1st variation of Eqn. 29.4 to zero, we get:

$$\delta\Pi = \iint_A \Bigg[\left\{A_{11}\frac{\partial u^o}{\partial x} + A_{12}\frac{\partial v^o}{\partial y} + A_{16}\left(\frac{\partial v^o}{\partial x} + \frac{\partial u^o}{\partial y}\right) - B_{11}\frac{\partial^2 w^o}{\partial x^2} - B_{12}\frac{\partial^2 w^o}{\partial y^2} - 2B_{16}\frac{\partial^2 w^o}{\partial x \partial y}\right\}\frac{\partial \delta u^o}{\partial x} + \left\{A_{12}\frac{\partial u^o}{\partial x} + A_{22}\frac{\partial v^o}{\partial y} + A_{26}\left(\frac{\partial v^o}{\partial x} + \frac{\partial u^o}{\partial y}\right) - B_{12}\frac{\partial^2 w^o}{\partial x^2} - B_{22}\frac{\partial^2 w^o}{\partial y^2} - 2B_{26}\frac{\partial^2 w^o}{\partial x \partial y}\right\}\frac{\partial \delta v^o}{\partial y} + \left\{A_{16}\frac{\partial u^o}{\partial x} + A_{26}\frac{\partial v^o}{\partial y} + A_{66}\left(\frac{\partial v^o}{\partial x} + \frac{\partial u^o}{\partial y}\right) - B_{16}\frac{\partial^2 w^o}{\partial x^2} - B_{26}\frac{\partial^2 w^o}{\partial y^2} - 2B_{66}\frac{\partial^2 w^o}{\partial x \partial y}\right\}\left(\frac{\partial \delta v^o}{\partial x} + \frac{\partial \delta u^o}{\partial y}\right) + \left\{D_{11}\frac{\partial^2 w^o}{\partial x^2} + D_{12}\frac{\partial^2 w^o}{\partial y^2} + 2D_{16}\frac{\partial^2 w^o}{\partial x \partial y}\right\}\frac{\partial^2 \delta w^o}{\partial x^2} + \left\{D_{12}\frac{\partial^2 w^o}{\partial x^2} + D_{22}\frac{\partial^2 w^o}{\partial y^2} + 2D_{26}\frac{\partial^2 w^o}{\partial x \partial y}\right\}\frac{\partial^2 \delta w^o}{\partial y^2} + 2\left\{D_{16}\frac{\partial^2 w^o}{\partial x^2} + D_{26}\frac{\partial^2 w^o}{\partial y^2} + 2D_{66}\frac{\partial^2 w^o}{\partial x \partial y}\right\}\frac{\partial^2 \delta w^o}{\partial x \partial y}\Bigg] dA - \int_S (T_x(\delta u^o + z\frac{\partial \delta w^o}{\partial x}) + (T_y(\delta v^o + z\frac{\partial \delta w^o}{\partial y}) + T_z \delta w^o) dS = 0$$

- Plugging definitions of N_x, N_y, etc. in above equation, we get:

$$\delta\Pi = \iint_A \left[N_x\frac{\partial \delta u^o}{\partial x} + N_y\frac{\partial \delta v^o}{\partial x} + N_{xy}\left(\frac{\partial \delta v^o}{\partial x} + \frac{\partial \delta u^o}{\partial y}\right) + M_x\frac{\partial^2 \delta w^o}{\partial x^2} + M_y\frac{\partial^2 \delta w^o}{\partial y^2} + M_{xy}\frac{\partial^2 \delta w^o}{\partial x \partial y}\right] dA - \int_S (T_x(\delta u^o + z\frac{\partial \delta w^o}{\partial x}) + T_y(\delta v^o + z\frac{\partial \delta w^o}{\partial y}) + T_z \delta w^o) dS = 0$$

(Eq. 32.7)

Total Potential Energy Principle

- Till so far we have not addressed forces attributable to *known* tractions T_i. These forces may exist on various surfaces of the plate. For a *rectangular plate*, there are six such surfaces. The second integral (over surface S) must be evaluated over all of these six surfaces.

- We start this by assuming that the origin is located at geometric center of the plate, and the plate dimensions are a x b. Also, the plate thickness is assumed to be t.

- Now consider 1st surface (or edge) of the plate, $x = a/2$. At this edge, dS equals dydz. If we integrate the second integral on this surface, we get:

$$\int_S \left(T_x\left(\delta u^o + z\frac{\partial \delta w^o}{\partial x}\right) + T_y\left(\delta v^o + z\frac{\partial \delta w^o}{\partial y}\right) + T_z \delta w^o\right) dS$$

$$= \int_{-b/2}^{+b/2}\int_{-t/2}^{+t/2}\left\{T_x\left(+\frac{a}{2},y,z\right)\delta u^o\left(+\frac{a}{2},y\right) + T_x\left(+\frac{a}{2},y,z\right)z\frac{\partial \delta w^o}{\partial x}\left(+\frac{a}{2},y\right)\right.$$

$$+ T_y\left(+\frac{a}{2},y,z\right)\delta v^o\left(+\frac{a}{2},y\right)$$

$$\left.+ T_y\left(+\frac{a}{2},y,z\right)z\frac{\partial \delta w^o}{\partial y}\left(+\frac{a}{2},y\right) + T_z\left(+\frac{a}{2},y,z\right)\delta w^o\left(+\frac{a}{2},y\right)\right\} dz\,dy$$

Total Potential Energy Principle

- Integrating each component over the thickness, we get the following expression for second integral on surface x = a/2.

$$\int_S \left(T_x\left(\delta u^o + z\frac{\partial \delta w^o}{\partial x}\right) + T_y\left(\delta v^o + z\frac{\partial \delta w^o}{\partial y}\right) + T_z \delta w^o\right) dS$$
$$= \int_{-b/2}^{+b/2} \left\{ N_x^+(y)\delta u^o\left(+\frac{a}{2}, y\right) + M_x^+(y)\frac{\partial \delta w^o}{\partial x}\left(+\frac{a}{2}, y\right) + N_{xy}^+(y)\delta v^o\left(+\frac{a}{2}, y\right) \right.$$
$$\left. + M_{xy}^+(y)\frac{\partial \delta w^o}{\partial y}\left(+\frac{a}{2}, y\right) + Q_x^+ \delta w^o\left(+\frac{a}{2}, y\right) \right\} dy \qquad \text{(Eq. 32.8)}$$

where,

$$N_x^+(y) = \int_{-t/2}^{+t/2} \left\{T_x\left(+\frac{a}{2}, y, z\right)\right\} dz \qquad N_{xy}^+(y) = \int_{-t/2}^{+t/2} \left\{T_y\left(+\frac{a}{2}, y, z\right)\right\} dz$$

$$M_x^+(y) = \int_{-t/2}^{+t/2} z\left\{T_x\left(+\frac{a}{2}, y, z\right)\right\} dz \qquad M_{xy}^+(y) = \int_{-t/2}^{+t/2} \left\{T_y\left(+\frac{a}{2}, y, z\right)\right\} dz$$

$$Q_x^+(y) = \int_{-t/2}^{+t/2} \left\{T_z\left(+\frac{a}{2}, y, z\right)\right\} dz$$

(Eq. 32.9)

Total Potential Energy Principle

- It is seen in Eq. 32.8 and 32.9, that Kirchhoff assumption along with the surface integral for tractions shifts emphasis from known stresses to known stress-resultants.

- Hence, developing the solution of plate does not necessarily require us to know stresses on the boundary on a point-to-point basis. Rather, what is needed are overall integrals of those stresses over the edge of the plate.

- This in turn means, that several stress distributions can lead to same integrated value. Thus, the plate theory under discussion does not provide us with unique answers for a given stress distribution over a plate's boundary, as some other stress distribution may also lead to same stress resultants. This is a limitation of several *structural level* theories of laminated plates.

- However, these theories provide sufficient detail as we move away from the edge of the plate, as per St. Venant's principle.

Total Potential Energy Principle

- Using a similar approach as discussed earlier, we can evaluate surface integrals (2nd integral) for other edges of the plate as well, i.e. x = -a/2, y=-b/2, and y=+b/2, and get results similar to that in Eq. 32.8.

- Finally, we evaluate integrals for top and bottom surfaces of the plate, i.e. $z=\pm t/2$. On these surfaces, typically, T_x, and T_y, are zero, and dS is dxdy. Thus, for the top surface of the plate, this integral can be expressed as:

$$\int_S \left(T_x\left(\delta u^o + z\frac{\partial \delta w^o}{\partial x}\right) + T_y\left(\delta v^o + z\frac{\partial \delta w^o}{\partial y}\right) + T_z \delta w^o\right) dS$$
$$= \int_{-b/2}^{+b/2}\int_{-a/2}^{+a/2} T_z\left(x, y, +\frac{t}{2}\right) \delta w^o \, dx dy = q^+(x,y) \qquad \text{(Eq. 32.10)}$$

- Similarly, the integral for the bottom surface is $-q^-(x,y)$.

- We add them together such that:

  $q(x,y) = q^+ - q^-$

Total Potential Energy Principle

- Putting all these results for surface integral terms corresponding to six surfaces in Eq. 32.7, we get:

$$\delta\Pi = \iint_A \left[N_x \frac{\partial \delta u^o}{\partial x} + N_y \frac{\partial \delta v^o}{\partial x} + N_{xy}\left(\frac{\partial \delta v^o}{\partial x} + \frac{\partial \delta u^o}{\partial y}\right) + M_x \frac{\partial^2 \delta w^o}{\partial x^2} + M_y \frac{\partial^2 \delta w^o}{\partial y^2} + M_{xy} \frac{\partial^2 \delta w^o}{\partial x \partial y} - q(x,y)\delta w^o \right] dA - P1 + P2 - P3 + P4 = 0$$

$$P1 = \int_{-b/2}^{+b/2} \left\{ N_x^+(y)\delta u^o\left(+\frac{a}{2},y\right) + M_x^+(y)\frac{\partial \delta w^o}{\partial x}\left(+\frac{a}{2},y\right) + N_{xy}^+(y)\delta v^o\left(+\frac{a}{2},y\right) + M_{xy}^+(y)\frac{\partial \delta w^o}{\partial y}\left(+\frac{a}{2},y\right) + Q_x^+(y)\delta w^o\left(+\frac{a}{2},y\right) \right\} dy$$

(Eq. 32.11)

$$P2 = \int_{-\frac{b}{2}}^{+\frac{b}{2}} \left\{ N_x^-(y)\delta u^o\left(-\frac{a}{2},y\right) + M_x^-(y)\frac{\partial \delta w^o}{\partial x}\left(-\frac{a}{2},y\right) + N_{xy}^-(y)\delta v^o\left(-\frac{a}{2},y\right) + M_{xy}^-(y)\frac{\partial \delta w^o}{\partial y}\left(-\frac{a}{2},y\right) + Q_x^-(y)\delta w^o\left(-\frac{a}{2},y\right) \right\} dy$$

$$P3 = \int_{-a/2}^{+a/2} \left\{ N_{yx}^+(x)\delta u^o\left(x,+\frac{b}{2}\right) + M_{yx}^+(x)\frac{\partial \delta w^o}{\partial x}\left(x,+\frac{b}{2}\right) + N_y^+(x)\delta v^o\left(x,+\frac{b}{2}\right) + M_y^+(x)\frac{\partial \delta w^o}{\partial y}\left(x,+\frac{b}{2}\right) + Q_y^+(x)\delta w^o\left(x,+\frac{b}{2}\right) \right\} dx$$

$$P4 = \int_{-a/2}^{+a/2} \left\{ N_{yx}^-(x)\delta u^o\left(x,-\frac{b}{2}\right) + M_{yx}^-(x)\frac{\partial \delta w^o}{\partial x}\left(x,-\frac{b}{2}\right) + N_y^-(x)\delta v^o\left(x,-\frac{b}{2}\right) + M_y^-(x)\frac{\partial \delta w^o}{\partial y}\left(x,-\frac{b}{2}\right) + Q_y^-(x)\delta w^o\left(x,-\frac{b}{2}\right) \right\} dx$$

Total Potential Energy Principle

- Earlier, using total potential energy principle, we developed Eq. 32.11. This is shown below.

$$\delta\Pi = \iint_A \left[N_x \frac{\partial \delta u^o}{\partial x} + N_y \frac{\partial \delta v^o}{\partial x} + N_{xy}\left(\frac{\partial \delta v^o}{\partial x} + \frac{\partial \delta u^o}{\partial y}\right) + M_x \frac{\partial^2 \delta w^o}{\partial x^2} + M_y \frac{\partial^2 \delta w^o}{\partial y^2} + M_{xy}\frac{\partial^2 \delta w^o}{\partial x \partial y} - q(x,y)\delta w^o \right] dA - P1 + P2 - P3 + P4 = 0$$

$$P1 = \int_{-b/2}^{+b/2} \left\{ N_x^+(y)\delta u^o\left(+\frac{a}{2},y\right) + M_x^+(y)\frac{\partial \delta w^o}{\partial x}\left(+\frac{a}{2},y\right) + N_{xy}^+(y)\delta v^o\left(+\frac{a}{2},y\right) + M_{xy}^+(y)\frac{\partial \delta w^o}{\partial y}\left(+\frac{a}{2},y\right) + Q_x^+(y)\delta w^o\left(+\frac{a}{2},y\right)\right\} dy \qquad \text{(Eq. 32.11)}$$

$$P2 = \int_{-\frac{b}{2}}^{+\frac{b}{2}} \left\{ N_x^-(y)\delta u^o\left(-\frac{a}{2},y\right) + M_x^-(y)\frac{\partial \delta w^o}{\partial x}\left(-\frac{a}{2},y\right) + N_{xy}^-(y)\delta v^o\left(-\frac{a}{2},y\right) + M_{xy}^-(y)\frac{\partial \delta w^o}{\partial y}\left(-\frac{a}{2},y\right) + Q_x^-(y)\delta w^o\left(-\frac{a}{2},y\right)\right\} dy$$

$$P3 = \int_{-a/2}^{+a/2} \left\{ N_{yx}^+(x)\delta u^o\left(x,+\frac{b}{2}\right) + M_{yx}^+(x)\frac{\partial \delta w^o}{\partial x}\left(x,+\frac{b}{2}\right) + N_y^+(x)\delta v^o\left(x,+\frac{b}{2}\right) + M_y^+(x)\frac{\partial \delta w^o}{\partial y}\left(x,+\frac{b}{2}\right) + Q_y^+(x)\delta w^o\left(x,+\frac{b}{2}\right)\right\} dx$$

$$P4 = \int_{-a/2}^{+a/2} \left\{ N_{yx}^-(x)\delta u^o\left(x,-\frac{b}{2}\right) + M_{yx}^-(x)\frac{\partial \delta w^o}{\partial x}\left(x,-\frac{b}{2}\right) + N_y^-(x)\delta v^o\left(x,-\frac{b}{2}\right) + M_y^-(x)\frac{\partial \delta w^o}{\partial y}\left(x,-\frac{b}{2}\right) + Q_y^-(x)\delta w^o\left(x,-\frac{b}{2}\right)\right\} dx$$

Total Potential Energy Principle

- Equation 32.11 is an important form of the virtual work equilibrium equation. It has been derived using the total potential energy principle. Here total potential energy has been minimized by setting its 1st variation of to be zero.

- We see from Eq. 32.11, that the formulation by itself generates requisite boundary conditions in an appropriate form.

- Equation 32.11 is widely used since it serves as the basis of classic Rayleigh-Ritz formulations. These formulations are used for obtaining approximate solutions, and are very widely used in finite element formulations.

- Equation 32.11 is also known as “weak form”. Later, we will also develop a “strong form” for equilibrium. The rationale underlying such terminologies will be discussed later.

Generation of Equilibrium Equations

- Equation 32.11 can be further processed to generate equilibrium equations, which we have already developed through 1st principles earlier. We will see that such an approach not only generates requisite equilibrium equations, it also provides us with boundary conditions in appropriate form.

- First, we apply integration by parts to all the terms which are integrated over the area of the plate, in Eq. 32.11. There are a total of eight such terms. Consider the 1st part:

$$\iint_A N_x \frac{\partial \delta u^o}{\partial x} dA = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} N_x \frac{\partial \delta u^o}{\partial x} dx dy$$

- In above expression we have defined the domain of integration.
  - Thus, the area of integration is now a rectangle of size *a* X *b*.
  - Also, d*A*, has been substituted by d*x*d*y*.

Generation of Equilibrium Equations

- Next, consider following identities.

$$\frac{\partial(F_1 \delta u^o)}{\partial x} = F_1 \frac{\partial \delta u^o}{\partial x} + \delta u^o \frac{\partial F_1}{\partial x}$$

or,

$$F_1 \frac{\partial \delta u^o}{\partial x} = \frac{\partial(F_1 \delta u)}{\partial x} - \delta u^o \frac{\partial F_1}{\partial x}$$

- Applying this identity to the 1st term, we get:

$$\iint_A N_x \frac{\partial \delta u^o}{\partial x} dA = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} N_x \frac{\partial \delta u^o}{\partial x} dxdy = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left\{ \frac{\partial (N_x \delta u^o)}{\partial x} - \left(\delta u^o \frac{\partial N_x}{\partial x} \right) \right\} dxdy$$

(Eq.32.12)

Generation of Equilibrium Equations

- Next, we recall the component form of divergence (or gradient) theorem.

$$\iint_A \frac{\partial (F_1 \delta u^o)}{\partial x} dxdy = \oint_\Gamma F_1 \delta u^o n_x dx$$

and,

$$\iint_A \frac{\partial (F_1 \delta u^o)}{\partial y} dxdy = \oint_\Gamma F_1 \delta u^o n_y dy$$

- We use this theorem to compute the area integral of $\partial(N_x \delta u^o)/\partial x$ in Eq. 32.12, to get:

$$\iint_A N_x \frac{\partial \delta u^o}{\partial x} dA = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left(-\delta u^o \frac{\partial N_x}{\partial x} \right) dxdy + \oint_\Gamma N_x \delta u^o n_x dx \quad \text{(Eq. 32.13)}$$

- Here, n_x is the x-direction component of a unit vector normal to the boundary. Thus, its value is 1, 0, -1, and 0 along edges x=a/2, y=b/2, x=-a/2, and y=-b/2, respectively, of the composite plate being analyzed here.

Generation of Equilibrium Equations

- Using these values of n_x along the closed boundary of composite plate, we calculate the contour integral as defined in Eq. 32.13 as:

$$\oint_\Gamma N_x \delta u^o n_x dx = \int_{-\frac{b}{2}}^{\frac{b}{2}} N_x\left(\frac{+a}{2}, y\right)\delta u^o\left(\frac{+a}{2}, y\right)(1)dy + \int_{-\frac{a}{2}}^{\frac{a}{2}} N_x\left(x, \frac{+b}{2}\right)\delta u^o\left(x, \frac{+b}{2}\right)(0)dx$$

$$+ \int_{-\frac{b}{2}}^{\frac{b}{2}} N_x\left(\frac{-a}{2}, y\right)\delta u^o\left(\frac{-a}{2}, y\right)(-1)dy + \int_{-\frac{a}{2}}^{\frac{a}{2}} N_x\left(x, \frac{-b}{2}\right)\delta u^o\left(x, \frac{-b}{2}\right)(0)dx$$

$$= \int_{-\frac{b}{2}}^{\frac{b}{2}} N_x^+ \delta u^o\left(\frac{+a}{2}, y\right)dy - \int_{-\frac{b}{2}}^{\frac{b}{2}} N_x^- \left(\frac{-a}{2}, y\right)\delta u^o\left(\frac{-a}{2}, y\right)dy \quad \text{(Eq. 32.14)}$$

- Putting Eq. 32.14 in Eq. 32.13, the final form of 1st part of Eq. 32.11 is:

$$\iint_A N_x \frac{\partial \delta u^o}{\partial x} dA$$

$$= \int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\left(-\delta u^o \frac{\partial N_x}{\partial x}\right)dxdy + \int_{-\frac{b}{2}}^{\frac{b}{2}} N_x^+ \delta u^o\left(\frac{+a}{2}, y\right)dy$$

$$- \int_{-\frac{b}{2}}^{\frac{b}{2}} N_x^- \left(\frac{-a}{2}, y\right)\delta u^o\left(\frac{-a}{2}, y\right)dy \quad \text{(Eq. 32.15)}$$

Generation of Equilibrium Equations

- Similarly, we apply integration by parts to remaining seven terms in Eq. 32.11. Putting all these terms together, and reorganizing, them we get:

$$0 = -\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\left[\left(\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}\right)\delta u^o+\left(\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}\right)\delta v^o + \left(\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x \partial y}+q(x,y)\right)\delta w^o\right]dxdy + P5 + P6 + P7 + P8 + P9 \quad \text{(Eq. 32.16)}$$

- where, P5, P6, P7, P8, and P9 are defined as follows.

Generation of Equilibrium Equations

$$P5 = \int_{-\frac{b}{2}}^{+\frac{b}{2}} \left\{ \left(N_x\left(+\frac{a}{2},y\right) - N_x^{+}(y) \right) \delta u^{o}\left(+\frac{a}{2},y\right) + \left(M_x\left(+\frac{a}{2},y\right) - M_x^{+}(y) \right) \delta \beta_x^{0}\left(+\frac{a}{2},y\right) \right.$$
$$+ \left(N_{xy}\left(+\frac{a}{2},y\right) - N_{xy}^{+}(y) \right) \delta v^{o}\left(+\frac{a}{2},y\right)$$
$$\left. + \left(\frac{\partial M_x\left(+\frac{a}{2},y\right)}{\partial x} + 2\frac{\partial M_{xy}\left(+\frac{a}{2},y\right)}{\partial y} - \left[Q_x^{+}(y) + \frac{\partial M_{xy}^{+}(y)}{\partial x} \right] \right) \delta w^{o}\left(+\frac{a}{2},y\right) \right\} dy$$

$$P6 = -\int_{-\frac{b}{2}}^{+\frac{b}{2}} \left\{ \left(N_x\left(-\frac{a}{2},y\right) - N_x^{-}(y) \right) \delta u^{o}\left(-\frac{a}{2},y\right) + \left(M_x\left(-\frac{a}{2},y\right) - M_x^{+}(y) \right) \delta \beta_x^{0}\left(+\frac{a}{2},y\right) \right.$$
$$+ \left(N_{xy}\left(-\frac{a}{2},y\right) - N_{xy}^{-}(y) \right) \delta v^{o}\left(-\frac{a}{2},y\right)$$
$$\left. + \left(\frac{\partial M_x\left(-\frac{a}{2},y\right)}{\partial x} + 2\frac{\partial M_{xy}\left(-\frac{a}{2},y\right)}{\partial y} - \left[Q_x^{-}(y) + \frac{\partial M_{xy}^{-}(y)}{\partial x} \right] \right) \delta w^{o}\left(-\frac{a}{2},y\right) \right\} dy$$

Generation of Equilibrium Equations

$$P7 = \int_{-\frac{a}{2}}^{+\frac{a}{2}} \left\{ \left(N_{xy}\left(x, \frac{+b}{2}\right) - N_{yx}^{+}(x) \right) \delta u^{o}\left(x, \frac{+b}{2}\right) + \left(N_{y}\left(x, \frac{+b}{2}\right) - N_{y}^{+}(x) \right) \delta v^{o}\left(x, \frac{+b}{2}\right) \right.$$
$$+ \left(M_{y}\left(x, \frac{+b}{2}\right) - M_{y}^{+}(x) \right) \delta \beta_{y}^{0}\left(x, \frac{+b}{2}\right)$$
$$\left. + \left(\frac{\partial M_{y}\left(x, \frac{+b}{2}\right)}{\partial y} + 2 \frac{\partial M_{xy}\left(x, \frac{+b}{2}\right)}{\partial x} - \left[Q_{y}^{+}(x) + \frac{\partial M_{yx}^{+}(x)}{\partial y} \right] \right) \delta w^{o}\left(x, \frac{+b}{2}\right) \right\} dx$$

$$P8 = -\int_{-\frac{a}{2}}^{+\frac{a}{2}} \left\{ \left(N_{xy}\left(x, \frac{-b}{2}\right) - N_{yx}^{-}(x) \right) \delta u^{o}\left(x, \frac{-b}{2}\right) + \left(N_{y}\left(x, \frac{-b}{2}\right) - N_{y}^{-}(x) \right) \delta v^{o}\left(x, \frac{-b}{2}\right) \right.$$
$$+ \left(M_{y}\left(x, \frac{-b}{2}\right) - M_{y}^{-}(x) \right) \delta \beta_{y}^{0}\left(x, \frac{-b}{2}\right)$$
$$\left. + \left(\frac{\partial M_{y}\left(x, \frac{-b}{2}\right)}{\partial y} + 2 \frac{\partial M_{xy}\left(x, \frac{-b}{2}\right)}{\partial x} - \left[Q_{y}^{-}(x) + \frac{\partial M_{yx}^{-}(x)}{\partial y} \right] \right) \delta w^{o}\left(x, \frac{-b}{2}\right) \right\} dx$$

Generation of Equilibrium Equations

- and,

$$
\begin{aligned}
P9 = -&\left[M_{xy}\left(+\frac{a}{2},+\frac{b}{2}\right)-M_{xy}^{+}\left(+\frac{b}{2}\right)\right]\delta w^{o}\left(+\frac{a}{2},+\frac{b}{2}\right)\\
&+\left[M_{xy}\left(+\frac{a}{2},-\frac{b}{2}\right)-M_{xy}^{+}\left(-\frac{b}{2}\right)\right]\delta w^{o}\left(+\frac{a}{2},-\frac{b}{2}\right)\\
&+\left[M_{xy}\left(-\frac{a}{2},+\frac{b}{2}\right)-M_{xy}^{+}\left(+\frac{b}{2}\right)\right]\delta w^{o}\left(-\frac{a}{2},+\frac{b}{2}\right)\\
&-\left[M_{xy}\left(-\frac{a}{2},-\frac{b}{2}\right)-M_{xy}^{+}\left(-\frac{b}{2}\right)\right]\delta w^{o}\left(-\frac{a}{2},-\frac{b}{2}\right)\\
&-\left[M_{xy}\left(+\frac{a}{2},+\frac{b}{2}\right)-M_{xy}^{+}\left(+\frac{a}{2}\right)\right]\delta w^{o}\left(+\frac{a}{2},+\frac{b}{2}\right)\\
&+\left[M_{xy}\left(-\frac{a}{2},+\frac{b}{2}\right)-M_{xy}^{+}\left(-\frac{a}{2}\right)\right]\delta w^{o}\left(-\frac{a}{2},+\frac{b}{2}\right)\\
&+\left[M_{xy}\left(+\frac{a}{2},-\frac{b}{2}\right)-M_{xy}^{+}\left(+\frac{a}{2}\right)\right]\delta w^{o}\left(+\frac{a}{2},-\frac{b}{2}\right)\\
&-\left[M_{xy}\left(-\frac{a}{2},-\frac{b}{2}\right)-M_{xy}^{+}\left(-\frac{a}{2}\right)\right]\delta w^{o}\left(-\frac{a}{2},-\frac{b}{2}\right)
\end{aligned}
$$

Generation of Equilibrium Equations

- Till so far, we have used the process of integration of parts to transform Eq. 32.11 (weak form) into Eq. 32.16.

$$0 = -\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{b}{2}}^{\frac{b}{2}}\Bigg[\left(\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}\right)\delta u^{o}+\left(\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}\right)\delta v^{o} + \left(\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x \partial y}+q(x,y)\right)\delta w^{o}\Bigg]dxdy + P5 + P6 + P7 + P8 + P9 \quad \text{(Eq. 32.16)}$$

- where, P5, P6, P7, P8, and P9 have been defined earlier.

- Equation 32.16 is called the “strong form” of equilibrium condition. It is termed as “strong” because the differentiability conditions for an assumed displacement function required for such a form are stronger.

- For instance, here the equation involves second derivatives of moments, while in Eq. 32.11 did not involve any derivatives of moments.

Generation of Equilibrium Equations

- Given that moments are directly proportional to second derivatives of $w^o(x,y)$, it implies that:
  - A valid function for $w^o(x,y)$ has to have at least 4^{th} order differential continuity, if we use the strong formulation (i.e. Eq. 32.16) for solving the problem.
  - However, if we used Eq. 32.11 for solving the same problem, the formulation would require second order, i.e. C^2, continuity for $w^o(x,y)$.

- Thus, we see that Equation 32.16 requires higher order of continuity for assumed displacement functions, vis-à-vis Eq. 32.11. It is for this reason that Eq. 32.16 is also known as the "strong form", while Eq. 32.11 is known as the "weak form".

- Of course, in both cases, the assumed displacement function should satisfy essential boundary conditions for the problem.

- Moving further we now extract equilibrium equations from Eq. 32.16.

Generation of Equilibrium Equations

- Looking at Eq. 32.16, we realize that it is a sum of area integrals, line integrals (P5-P8), and point-specific values (P9). We further note, that there are an infinite number of mathematically valid variations in displacement field possible which may be used in this equation as long as they satisfy the following conditions.
  - Differentiability requirements
  - Essential boundary conditions

- Thus, Eq. 32.16 can be zero only if, area integrals, line integrals, and point-specific values (P9) are individually zero.

- Thus, from area integrals we get equilibrium equations, while terms P5-P9 give us a mathematically consistent set of boundary conditions.

Generation of Equilibrium Equations

- Thus, the equilibrium equations are:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \mathrm{q} = 0$$

- These equilibrium equations are identical to the ones developed earlier developed using the Newtonian approach.

- Next we look at boundary conditions. Terms P5, P6, P7, P8 correspond to boundaries x=a/2, x=-a/2, y=b/2, and y=-b/2, respectively.

Generation of Equilibrium Equations

- Looking at term P5, which corresponds to edge, x=a/2, we notice that there are four BCs along this edge. Further, for variational statement to be true, each of these BCs should be individually zero.

- Thus the boundary conditions along edge x=a/2 are:

1. $(N_x - N_x^{+})\delta u^o = 0$

   Implying, either $N_x = N_x^{+}$ or $u^o$is known.

2. $(N_{xy} - N_{xy}^{+})\delta v^o = 0$

   Implying, either $N_{xy} = N_{xy}^{+}$ or $v^o$is known.

3. $(M_x - M_x^{+})\delta \beta_x^0 = 0$

   Implying, either $M_x = M_x^{+}$ or $\beta_x^0 = \frac{\partial w^o}{\partial x}$ is known.

4. $\left(\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y} - Q_x^{+} - \frac{\partial M_{xy}^{+}}{\partial x}\right)\delta w^o = 0$

   Implying, either $\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y} = Q_x^{+} + \frac{\partial M_{xy}^{+}}{\partial x}$ or w^o is known.

Generation of Equilibrium Equations

- Similarly, BCs conditions for x=-a/2 are:

1. $(N_x - N_x^-)\delta u^o = 0$

   Implying, either $N_x = N_x^-$ or $u^o$is known.

2. $\left(N_{xy} - N_{xy}^-\right)\delta v^o = 0$

   Implying, either $N_{xy} = N_{xy}^-$ or $v^o$is known.

3. $(M_x - M_x^-)\delta\beta_x^0 = 0$

   Implying, either $M_x = M_x^-$ or $\beta_x^0 = \frac{\partial w^o}{\partial x}$ is known.

4. $\left(\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y} - Q_x^- - \frac{\partial M_{xy}^-}{\partial x}\right)\delta w^o = 0$

   Implying, either $\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y} = Q_x^- + \frac{\partial M_{xy}^-}{\partial x}$ or w^o is known.

- Likewise, BCs for other two edges of the plate, y=±b/2 may also be computed in a straight-forward way.

Generation of Equilibrium Equations

- Finally, we look at term in P9 as defined below.

$$
\begin{aligned}
P9 = -&\left[M_{xy}\left(+\frac{a}{2},+\frac{b}{2}\right)-M_{xy}^{+}\left(+\frac{b}{2}\right)\right]\delta w^{o}\left(+\frac{a}{2},+\frac{b}{2}\right)\\
&+\left[M_{xy}\left(+\frac{a}{2},-\frac{b}{2}\right)-M_{xy}^{+}\left(-\frac{b}{2}\right)\right]\delta w^{o}\left(+\frac{a}{2},-\frac{b}{2}\right)\\
&+\left[M_{xy}\left(-\frac{a}{2},+\frac{b}{2}\right)-M_{xy}^{+}\left(+\frac{b}{2}\right)\right]\delta w^{o}\left(-\frac{a}{2},+\frac{b}{2}\right)\\
&-\left[M_{xy}\left(-\frac{a}{2},-\frac{b}{2}\right)-M_{xy}^{+}\left(-\frac{b}{2}\right)\right]\delta w^{o}\left(-\frac{a}{2},-\frac{b}{2}\right)\\
&-\left[M_{xy}\left(+\frac{a}{2},+\frac{b}{2}\right)-M_{yx}^{+}\left(+\frac{a}{2}\right)\right]\delta w^{o}\left(+\frac{a}{2},+\frac{b}{2}\right)\\
&+\left[M_{xy}\left(-\frac{a}{2},+\frac{b}{2}\right)-M_{yx}^{+}\left(-\frac{a}{2}\right)\right]\delta w^{o}\left(-\frac{a}{2},+\frac{b}{2}\right)\\
&+\left[M_{xy}\left(+\frac{a}{2},-\frac{b}{2}\right)-M_{yx}^{+}\left(+\frac{a}{2}\right)\right]\delta w^{o}\left(+\frac{a}{2},-\frac{b}{2}\right)\\
&-\left[M_{xy}\left(-\frac{a}{2},-\frac{b}{2}\right)-M_{yx}^{+}\left(-\frac{a}{2}\right)\right]\delta w^{o}\left(-\frac{a}{2},-\frac{b}{2}\right)
\end{aligned}
$$

- These are 8 corner conditions.

Observations

- The equilibrium equations developed here and the associated BCs govern the plate behavior completely.
- Each edge of the plate is associated with four sets of BCs. Within each set, there is a pair of conditions of which, one must be satisfied.
- For each of these pairs, either a force or moment must be equated to externally applied force or moment, or a kinematic condition (displacement or slope) must be specified.
- Such a pairing of force/moment and kinematic conditions emerges automatically, when variational process is used to minimize the total potential energy of the system.

Observations

- The boundary conditions developed here are the only valid form of BCs which can be enforced on a plate's boundary, and also ensure problem's consistency with classical lamination theory.

- If BCs are specified in some other way then it quite likely that we will not be able to solve the problem.

- Specification of BCs in ways different than laid out earlier leads to an *ill-posed* problem.

- The strong form of variational formulation requires higher order continuity of displacement field. Such a formulation is used in Galerkin method. In contrast, the weak formulation is used in Rayleigh Ritz method.