Smart Materials, Adaptive Structures, and Intelligent Mechanical Systems

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Lecture 23

Equilibrium Equations for Plates

References for this Lecture

1. Analysis and Performance of Fiber Composites, Agarwal, B.D. and Broutman, L. J., John Wiley & Sons.

2. Mechanics of Composite Materials, Jones, R. M., Mc-Graw Hill

3. Structural Analysis of Laminated Composites, Whitney, J. M., Technomic

4. Nonlinear Analyis of Plates, Chia, C., McGraw-Hill International Book Company

Lecture Overview

- Introduction
- Equilibrium Equations
  - Assumptions
  - Equations for force
  - Equations for moments
- Boundary Conditions

Introduction

- Earlier, following equations, which govern stress-strain relations in a composite laminate, have been developed.

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix} \begin{Bmatrix} \varepsilon^{o} \\ \kappa \end{Bmatrix}$$

- These equations work well in terms of predicting stress-strain response of plate if {*N*} and {*M*} are known. However, if they are unknown a new set of relations have to be developed, which have to be used in addition to above equations.

- These relations can be developed using one of following two approaches.
  - Equilibrium approach: It is based on Newton's 1st Law of Motion.
  - Variational approach: It is based on energy equilibrium.

- Here, we start our discussion with the Newtonian approach.

Equilibrium Equations

- According to Newton's Laws of Motion, for a body to remain in equilibrium, the sum of external forces and moments on it, should be zero. This can be mathematically expressed as:
  - $\sum F_i = 0$
  - $\sum M_i = 0$,

  where, *i*, is an index which can assume values *x*, *y*, or *z*.

- Thus, there are a total of six equilibrium equations; three for force, and three for moment. Consider an infinitesimal portion of a plate, as shown in Fig. 23.1. Here, forces in only *x*-direction have been depicted.

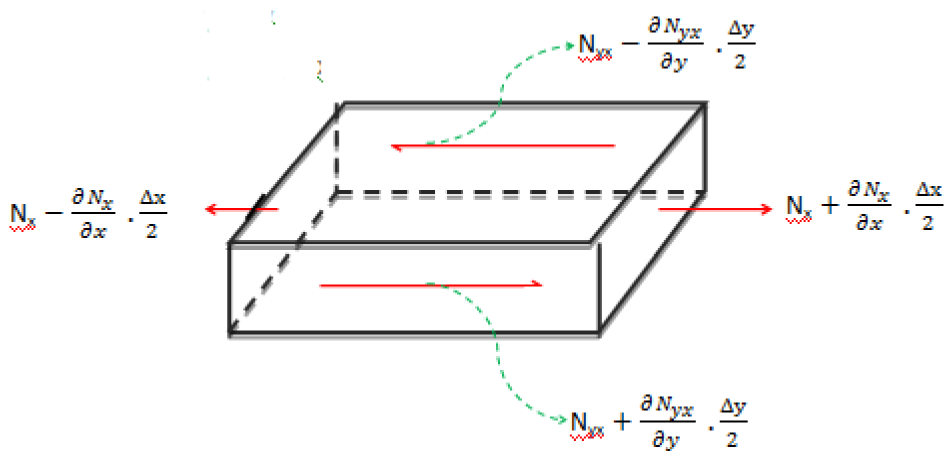

Fig. 23.1: *x*-direction forces acting on the mid-plane of a plate (origin assumed to be at geometric center of plate element)

Equilibrium Equations

- In Fig. 23.1, the geometric center of a very small portion of the plate coincides with the origin of the coordinate system. Also, the dimensions of this plate are Δx, Δy, Δz. Further, we assume that force and moment resultants at the origin are N_x, N_y, N_{xy}, M_x, M_y, and M_{xy}. *Figure 23.1 shows force resultants on different faces of the plate, acting in x-direction.*

- For equilibrium in *x* direction, the following equation can be written as shown below.

$$\Sigma F_x = 0 = \left\{\left(N_x + \frac{\partial N_x}{\partial x}\frac{\Delta x}{2}\right) - \left(N_x - \frac{\partial N_x}{\partial x}\frac{\Delta x}{2}\right)\right\}\Delta y + \left\{\left(N_{yx} + \frac{\partial N_{yx}}{\partial y}\frac{\Delta y}{2}\right) - \left(N_{yx} - \frac{\partial N_{yx}}{\partial y}\frac{\Delta y}{2}\right)\right\}\Delta x$$

or,

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0$$

(Eq. 23.1)

- Similarly, the equilibrium equation for *y* direction can be written as:

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

(Eq. 23.2)

Equilibrium Equations

- Here, it should be noted that force resultants N_z, N_{xz}, and N_{yz} are zero, and so are moment resultants M_{xz}, and M_{yz}. This is an outcome of Kirchhoff's assumptions for a plate due to which out-of-plane shear and normal strains are zero.

- Next, we develop the equilibrium equation for the *z*-direction. Consider an infinitesimal portion of a plate subjected to normal load per unit area, *q*. Figure 23.2 shows force resultants on different faces of the plate, acting in *z*-direction.

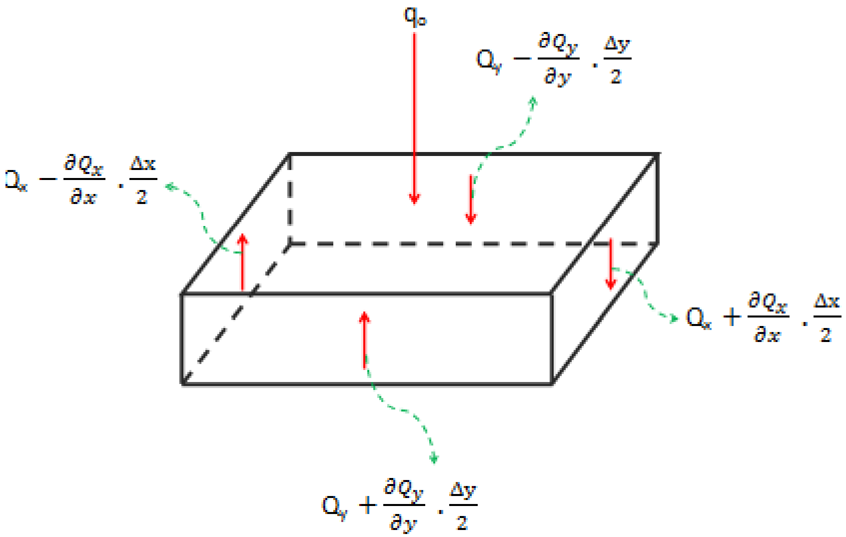

Fig. 23.2: Equilibrium schematic for *z*-direction on an infinitesimal portion of a composite plate

Equilibrium Equations

- As per Kirchhoff's assumptions, out of plane shear stresses (τ_{xz}, and τ_{yz}) should be zero. This in turn implies that the equilibrium equation for *z* direction should be:

  $q \cdot dx \cdot dy = 0$ or $q = 0$.

- Above relation implies that for a plate to be in equilibrium in *z* direction, it should not be subjected to any external load acting in out-of-plane direction.

- However, this is not consistent with reality since plates indeed capable of bearing significant out-of-plane loads. Thus we have to revisit Kirchhoff's assumptions concerning out-of-plane shears, and it can be inferred that these entities are not zero in plates.

- Thus, we apply a correction to Kirchhoff's plate as shown further.

Equilibrium Equations

- As per this correction, we do not assume out-of-plane shear stresses to be zero, and thus define two *non-zero* entities as follows.

$$Q_x = \int_{t/2}^{t/2} \tau_{xz} dz, \qquad \text{and} \qquad Q_y = \int_{t/2}^{t/2} \tau_{yz} dz$$

- Thus, the equation for force equilibrium in *z* direction is:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \mathrm{q} = 0 \qquad \text{(Eq. 23.3)}$$

- Next we look at moment equilibrium about origin. Consider Fig. 23.3, which depicts various moments acting about the *y*-axis.

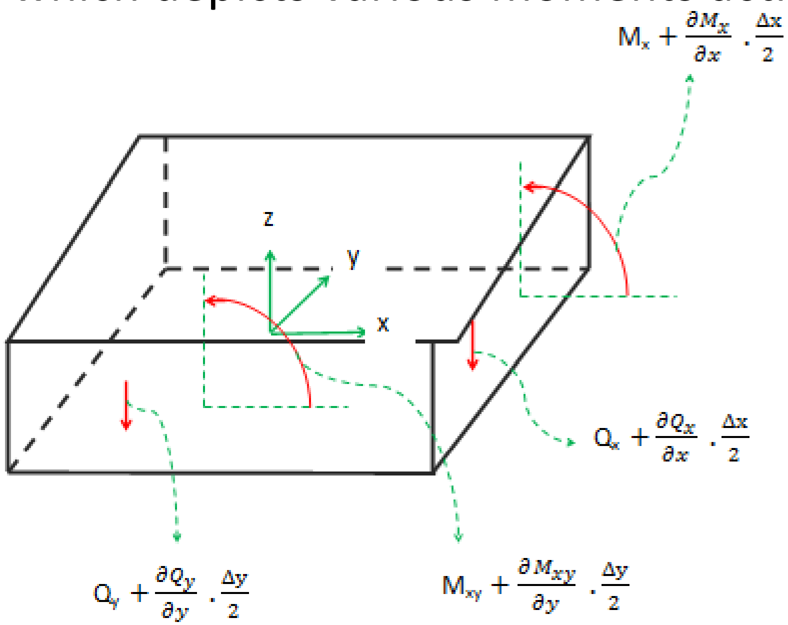

Fig. 23.2: Depiction of moments and forces needed to develop moment equilibrium equation about *y*-axis (moments and forces shown only on positive faces of the plate element).

Equilibrium Equations

- Considering the equilibrium of moments about *y*-axis, we get:

$$\left\{\left(M_x + \frac{\partial M_x}{\partial x}\frac{\Delta x}{2}\right) - \left(M_x - \frac{\partial M_x}{\partial x}\frac{\Delta x}{2}\right)\right\}\Delta y + \left\{\left(M_{xy} + \frac{\partial M_{xy}}{\partial y}\frac{\Delta y}{2}\right) - \left(M_{xy} - \frac{\partial M_{xy}}{\partial y}\frac{\Delta y}{2}\right)\right\}\Delta x$$
$$+\left\{\left(Q_x + \frac{\partial Q_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y\frac{\Delta x}{2} - \left(Q_x - \frac{\partial Q_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y\frac{(-\Delta x)}{2}\right\} = 0$$

- Neglecting higher order terms, we get moment equilibrium about *y* axis.

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \qquad \text{(Eq. 23.4)}$$

Similarly, the equation for moment equilibrium about y axis is:

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \qquad \text{(Eq. 23.5)}$$

Equilibrium Equations

- Finally, we consider moment equilibrium about the *z*-axis. Here, we note the following:
  - N_x, and N_y, do not contribute to moments about the *z*-axis, as they per definition, act on the mid-plane.
  - $q(x, y)$ does not contribute to moments about the *z*-axis, since its contribution is of a higher order.
  - Only N_{xy}, and N_{yx}, contribute to moments about the *z*-axis. However, their contributions add up to zero. Thus, the moment equilibrium equation for the *z*-direction, which is given below, is identically satisfied.

    $$N_{xy} - N_{yx} = 0 \qquad \text{(Eq. 23.6)}$$

- Equations 23.1, 23.2, and 23.3 represent force equilibrium in *x*, *y*, and *z*, directions, respectively.

- Equations 23.4, 23.5, and 23.6 represent moment equilibrium in *x*, *y*, and *z*, directions, respectively.

Equilibrium Equations

- Differentiating Eq. 23.4 and 23.5 with respect to *x*, and *y*, respectively, summing them up, and then substituting Eq. 23.3 in this "summed up" relation, we get:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0$$

- Thus, there are a total of four equilibrium equations. These are:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0 \qquad \text{Eq. (23.7)}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \qquad \text{Eq. (23.8)}$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q = 0 \qquad \text{Eq. (23.9)}$$

$$N_{xy} - N_{yx} = 0. \qquad \text{Eq. (23.10)}$$

- Embedded within Eq. 23.10 are three equilibrium conditions; force balance in *z* direction, and moment balance in *x* and *y* directions.

Boundary Conditions

- Equations 23.7 to 23.10 are valid for any composite plate of arbitrary shape or dimensions. These equations have to be solved to getting displacement-field for the plate. For a *rectangular* plate there are four sets of boundary conditions (BCs) corresponding to each edge.

- These four sets of BCs correspond to four equations of equilibrium.
  - First two sets of boundary conditions correspond to inplane equilibrium equations, i.e. Eqs. 23.7 and 23.8.
  - Second two sets boundary conditions correspond to out-of-plane equilibrium equation, i.e. Eq. 23.9.

- Each edge of a rectangular plate could have five different force and moment resultants. For instance, the $x = a/2$ edge could be subjected to external force and moments resultants as, N_x^+, N_{xy}^+, N_y^+, M_x^+, and M_{xy}^+.

- *However, there are only four BCs for each edge. Here, we directly write these BCs for each edge. A detailed proof for these BCs will be provided later.*

Boundary Conditions

- Boundary Conditions at $x = +a/2$.

(i) $(N_x - N_x^+)\delta u^o = 0,$

where δu^o implies variation in midplane displacement u_o.

Thus, either $N_x = N_x^+$ or u_o must be known.

(ii) $\left(N_{xy} - N_{xy}^+\right)\delta v^o = 0,$

where δv^o implies variation in midplane displacement v_o.

Thus, either $N_{xy} = N_{xy}^+$ or v_o must be known.

(iii) $\left\{\left(\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y}\right) - \left(Q_x^+ + \frac{\partial M_{xy}^+}{\partial y}\right)\right\}\delta w^o = 0$

where δw^o implies variation in midplane displacement w_o.

Thus, either $\left(\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y}\right) = \left(Q_x^+ + \frac{\partial M_{xy}^+}{\partial y}\right)$ or, or w_o must be known.

(iv) $(M_x - M_x^+)\delta(\frac{\partial w^o}{\partial x}) = 0$

Thus, either $M_x = M_x^+$ or $\frac{\partial w^o}{\partial x}$ must be known. (Eq. 23.11)

Boundary Conditions

- Boundary Conditions at $x = -a/2$.

(i) $(\boldsymbol{N_x - N_x^-})\boldsymbol{\delta u^o = 0},$

where δu^o implies variation in midplane displacement u_o.

Thus, either $N_x = N_x^-$, or u_o must be known.

(ii) $(\boldsymbol{N_{xy} - N_{xy}^-})\boldsymbol{\delta v^o = 0},$

where δv^o implies variation in midplane displacement v_o.

Thus, either $N_{xy} = N_{xy}^-$, or v_o must be known.

(iii) $\left\{\left(\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y}\right) - \left(Q_x^- + \frac{\partial M_{xy}^-}{\partial y}\right)\right\}\boldsymbol{\delta w^o = 0}$

where δw^o implies variation in midplane displacement w_o.

Thus, either $\left(\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y}\right) = \left(Q_x^- + \frac{\partial M_{xy}^-}{\partial y}\right)$ or, or w_o must be known.

(iv) $(\boldsymbol{M_x - M_x^-})\boldsymbol{\delta}\left(\frac{\partial w^o}{\partial x}\right) = \mathbf{0}$

Thus, either $M_x = M_x^-$ or $\frac{\partial w^o}{\partial x}$ must be known.

(Eq. 23.12)

Boundary Conditions

- Boundary Conditions at $x = -b/2$.

(i) $\left(N_y - N_y^-\right)\delta v^o = 0,$

where δv^o implies variation in midplane displacement v_o.

Thus, either $N_y = N_y^-$, or v_o must be known.

(ii) $\left(N_{xy} - N_{xy}^-\right)\delta u^o = 0,$

where δu^o implies variation in midplane displacement u_o.

Thus, either $N_{xy} = N_{xy}^-$, or u_o must be known.

(iii) $\left\{\left(\frac{\partial M_y}{\partial y} + 2\frac{\partial M_{xy}}{\partial x}\right) - \left(Q_y^- + \frac{\partial M_{xy}^-}{\partial x}\right)\right\}\delta w^o = 0$

where δw^o implies variation in midplane displacement w_o.

Thus, either $\left(\frac{\partial M_y}{\partial y} + 2\frac{\partial M_{xy}}{\partial x}\right) = \left(Q_y^- + \frac{\partial M_{xy}^-}{\partial x}\right)$ or, or w_o must be known.

(iv) $\left(M_y - M_y^-\right)\delta\left(\frac{\partial w^o}{\partial y}\right) = 0$

Thus, either $M_y = M_y^-$ or $\frac{\partial w^o}{\partial y}$ must be known.

(Eq. 23.13)

Boundary Conditions

- Boundary Conditions at $x = +b/2$.

(i) $\left(\boldsymbol{N_y - N_y^+}\right)\boldsymbol{\delta v^o = 0,}$

where δv^o implies variation in midplane displacement v_o.

Thus, either $N_y = N_y^+$, or v_o must be known.

(ii) $\left(\boldsymbol{N_{xy} - N_{xy}^+}\right)\boldsymbol{\delta u^o = 0,}$

where δu^o implies variation in midplane displacement u_o.

Thus, either $N_{xy} = N_{xy}^+$, or u_o must be known.

(iii) $\left\{\left(\frac{\partial M_y}{\partial y} + 2\frac{\partial M_{xy}}{\partial x}\right) - \left(Q_y^+ + \frac{\partial M_{xy}^+}{\partial x}\right)\right\}\boldsymbol{\delta w^o = 0}$

where δw^o implies variation in midplane displacement w_o.

Thus, either $\left(\frac{\partial M_y}{\partial y} + 2\frac{\partial M_{xy}}{\partial x}\right) = \left(Q_y^+ + \frac{\partial M_{xy}^+}{\partial x}\right)$ or, or w_o must be known.

(iv) $\left(\boldsymbol{M_y - M_y^+}\right)\delta\left(\frac{\partial w^o}{\partial y}\right) = \boldsymbol{0}$

Thus, either $M_y = M_y^+$ or $\frac{\partial w^o}{\partial y}$ must be known.

(Eq. 23.14)

Boundary Conditions

- In Equations 23.11 through 23.14, the BC on equality of out-of-plane shears requires careful consideration.

- As per this condition, either displacement in *z* direction should be known, or the sum of *Q* and partial derivative of M_{xy} should be known.

- This condition does not require knowledge of either *Q*, or partial derivative of M_{xy}, individually. Rather, what is needed to be known is their sum. It is for this reason, five variables on each face of plate, are linked by only four boundary conditions.

- Overall, there are three displacements, three strains, three stresses, and six force and moment resultants; a total of 15 unknown variable. These can be calculated by solving 15 simultaneous equations; three kinematic relations, six equations of equilibrium, and three relationships between force and momentum relations and strains using *A*, *B* and *D* matrices. The integration constants/functions may be determined from boundary conditions.

Boundary Conditions

- Finally, we note that for the edge, x =+a/2, one of the boundary condition is:

  $$\left(\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y}\right) = \left(Q_x^+ + \frac{\partial M_{xy}^+}{\partial y}\right)$$

  or, mid-plane displacement w^o, should be known.

- The term $\left(Q_x^+ + \frac{\partial M_{xy}^+}{\partial y}\right)$ represents, net out-of-plane shear force acting at edge, x=+a/2. It is sometimes also referred as $Q^+_{x\_eff}$.

- Similar boundary condition requirements also exist for other edges of the plate.

Classification of Boundary Conditions

- We have seen that there are four sets of boundary conditions for each edge.

- For each such set, the BC involves a force resultant (or moment resultant) and a displacement (or displacement gradient). Thus, we either have to know the displacement (or its gradient) or externally applied force resultant (or moment resultant).

- These BCs can be classified in two groups. These are:
  - Essential boundary conditions: All variables in formulation which are either displacements or their gradients are termed as *primary variables*. BCs associated with primary variables are termed *essential boundary conditions (EBC)*.

  - Natural boundary conditions: All variables in formulation which are directly related to forces or moments (or their gradients) are *secondary variables*. BCs associated with primary variables are termed *natural boundary conditions (NBC)*.

Classification of Boundary Conditions

- Essential and natural boundary conditions occur invariably in pairs. In a given situation, only one element of each pair can be specified. The following table categorizes various BCs for a rectangular plate as EBCs and NBCs.

Classification of Boundary Conditions for a Rectangular Composite Plate							
$x = a/2$		$x = -a/2$		$y = b/2$		$y = -b/2$	
EBC	NBC	EBC	NBC	EBC	NBC	EBC	NBC
u^o	N_x	u^o	N_x	u^o	N_{xy}	u^o	N_{xy}
v^o	N_{xy}	v^o	N_{xy}	v^o	N_y	v^o	N_y
w^o	Out-of-plane shear	w^o	Out-of-plane shear	w^o	Out-of-plane shear	w^o	Out-of-plane shear
$\partial w^o/\partial x$	M_x	$\partial w^o/\partial x$	M_x	$\partial w^o/\partial y$	M_y	$\partial w^o/\partial y$	M_y