Module 3: Internal Flows

Lecture 15: Exact Solution for Constant Wall Heat Flux Condition

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Laminar Flow in a Circular Tube

The problem of laminar flow in a circular tube has been treated theoretically and the results has been derived to express convection coefficients. We wish to calculate heat transfer under hydrodynamically and thermally fully developed condition. At any point in the tube the boundary layer approximations may be applied.

Heat transfer through a circular tube for hydrodynamically developed and thermally developed flow with Uniform Wall Heat Flux (UHF) condition:

The governing equation is

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{3.41}$$

Hydrodynamically fully developed conditions: $v = 0,\ \frac{\partial u}{\partial x} = 0$ and $u(r) = 2u_m[1-(r/r_o)^2]$

$$2u_m\left[1-\left(\frac{r}{r_0}\right)^2\right]\frac{dT_m}{dx} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{3.42}$$

At any $x, \frac{dT}{dx} = \frac{dT_m}{dx};$ we are solving for T (at any x) with respect to r. At a given x (where dT_m/dx is known), T becomes a function of r only. We can write

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dT}{dr}\right] = \frac{2u_m}{\alpha}\frac{dT_m}{dx}\left[1-\left(\frac{r}{r_0}\right)^2\right]$$

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or,

$$r\frac{dT}{dr} = \frac{2u_m}{\alpha}\frac{dT_m}{dx}\left[\frac{r^2}{2} - \frac{r^4}{4r_0^2}\right] + C_1$$

or,

$$\frac{dT}{dr} = \frac{2u_m}{\alpha}\frac{dT_m}{dx}\left[\frac{r}{2} - \frac{r^3}{4r_0^2}\right] + \frac{C_1}{r}$$

or,

$$T(r) = \frac{2u_m}{\alpha}\frac{dT_m}{dx}\left[\frac{r^2}{4} - \frac{r^4}{16r_0^2}\right] + C_1 \ln r + C_2 \qquad \textbf{(3.43)}$$

$$@\ r = 0, T \text{ is finite; } C_1 = 0 \ @\ r = r_0,\ T = T_w$$

$$C_2 = T_w - \frac{2u_m}{\alpha}\left(\frac{dT_m}{dx}\right)\frac{3r_0^2}{16}$$

Now,

$$T_m = \frac{2}{u_m r_0^2} \int_0^{r_0} u(r)\ T(r)\ r dr$$

$$T_m = \int_0^{r_0} u(r)\ T(r)\ rdr / \int_0^{r_0} u(r)\ \ rdr$$

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$$T_m = \frac{\int_0^{r_0} 2u_m\left(1-\frac{r^2}{r_0^2}\right) r\left[T_w + \frac{q_w'' r_0}{k}\left\{\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{3}{4}\right\}\right] dr}{\int_0^{r_0} 2u_m\left[1-\left(\frac{r}{r_0}\right)^2\right] r dr}$$

$$T_m = \frac{2u_m\left[T_w\left(\frac{r_0^2}{2} - \frac{r_0^4}{4r_0^2}\right) + \frac{q_w'' r_0}{k}\int_0^{r_0}\left(r - \frac{r^3}{r_0^2}\right)\left(\frac{r^2}{r_0^2} - \frac{1}{4}\frac{r^4}{r_0^4} - \frac{3}{4}\right)\right] dr}{2u_m\left[\frac{r_0^2}{2} - \frac{r_0^4}{4r_0^2}\right]}$$

$$T_m - T_w = \frac{q_w'' r_0}{k}\left(-\frac{11}{24}\right) \qquad \textbf{(3.44)}$$

$$\frac{q_w''}{(T_w - T_m)}\frac{2r_0}{k_f} = Nu_D = \frac{2\times 24}{11} = 4.36 \qquad \textbf{(3.45)}$$

Where, $q_w'' = h(T_w - T_m) = k_f \frac{dT}{dr}\mid_{r=r_0}$

For, Uniform Wall Heat Flux (UHF) boundary condition, laminar fully developed flow in a tube, N_{uD}= 4.36 [independent of R_e and P_r] Finally, it canbe shown that

$T(r) = T_w + \frac{q_w'' r_0}{k}\left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{7}{24}\right]$ is the fully developed temperature profile in a tube.

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3.4.2 Slug Flow

$$u\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \qquad \textbf{(3.47)}$$

Slug flow means u is independent of r; u = U = constant

$$U\frac{\partial T}{\partial x} = \alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] \qquad \textbf{(3.48)}$$

or,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{U}{\alpha}\frac{\partial T}{\partial x} \qquad \textbf{(3.49)}$$

T is a function of x and r

Substituting $\frac{T-T_w}{T_\infty - T_w} = \theta$ and $\frac{r}{r_0} = Y$ and $x = \frac{U r_0^2}{\alpha} X$ we get

$$\frac{\partial^2 \theta}{\partial Y^2} + \frac{1}{Y}\frac{\partial \theta}{\partial Y} = \frac{\partial \theta}{\partial X} \qquad \textbf{(3.50)}$$

Assume $\theta = F(X)G(y)$ where $F = F(X)$ and $G = G(Y)$

$$F\frac{d^2 G}{dY^2} + \frac{F}{Y}\frac{dG}{dY} = G\frac{dF}{dX}$$

or,

$$\frac{1}{G}\left[\frac{d^2 G}{dY^2} + \frac{1}{Y}\frac{dG}{dY}\right] = \frac{1}{F}\frac{dF}{dX} = -\beta_n^2$$

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The temperature of the fluid must approach that of the wall with increasing **X** so it isnecessary to assume **F** as a decaying function

$$\frac{1}{F}\frac{dF}{dX} = -\beta_n^2$$

or,

$$\frac{dF}{F} = -\beta_n^2 dX$$

or,

$$\log F = -\beta_n^2 X + \log C \qquad \textbf{(3.51)}$$

$F = Ce^{-\beta_n^2 X}$ Where **C** is a **constant.**

Considering the left hand side of the equation, we get

$$\frac{1}{G}\left[\frac{d^2G}{dY^2} + \frac{1}{Y}\frac{dG}{dY}\right] = -\beta_n^2$$

$$\frac{d^2G}{dY^2} + \frac{1}{Y}\frac{dG}{dY} + G\beta_n^2 = 0$$

$$Y^2\frac{d^2G}{dY^2} + Y\frac{dG}{dY} + Y^2G\beta_n^2 = 0$$

Substituting $Y^2\beta_n^2 = v^2$, we get

$$\frac{v^2}{\beta_n^2}\frac{d^2G}{d\left(\frac{v}{\beta_n}\right)^2} + \frac{v}{\beta_n}\frac{dG}{d\left(\frac{v}{\beta_n}\right)} + Gv^2 = 0$$

(3.52)

$$v^2\frac{d^2G}{dv^2} + v\frac{dG}{dv} + G(v^2 - 0) = 0$$

Boundary conditions:

$@\, r = r_0, T = T_w$ for $x > 0$

$@Y = 1, \theta = 0$ (basically for $v = \beta_n$)

This is **standard Bessel equation** and its general solution is given by the equation

$$G = AJ_0(v) + BY_0(v) \tag{3.53}$$

where $J_0(v)$ and $Y_0(v)$ are zero order Bessel functions of the first kind and second kind. Now we can write

$$\theta = C\ .\ e^{-\beta_n^2 X}\left\{AJ_0(v) + BY_0(v)\right\}$$

$$\theta = \left\{A_1\ J_0(v) + B_1\ Y_0(v)\right\}\ e^{-\beta_n^2 X} \tag{3.54}$$

Boundary conditions:

$@\ r = r_0, T = T_w \text{ for } x > 0$

$@Y = 1, \theta = 0 \text{ (basically for } v = \beta_n)$

$@\ r < r_0 \text{ and } x = 0,\quad T = T_\infty, \text{ this leads to } @\ X = 0,\quad \theta = 1$

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Boundary conditions:

$@ r=r_0, T=T_w$ for $x>0$

$@Y=1, \theta = 0$ (basically for $v = \beta_n$)

$@\ r < r_0$ and $x=0, \quad T=T_\infty$, this leads to $@\ X=0, \quad \theta = 1$

$$\theta = \{A_1\ J_0\ (Y\ \beta_n)\ +\ B_1\ Y_0\ (Y\beta_n)\}\ e^{-\beta_n^2 X}$$

$@\ r=0, Y=0, \theta=$ finite. Since $Y_0(0)=-\infty, B_1=0$

$$\theta = A_1\ J_0\ (Y\ \beta_n)\ e^{-\beta_n^2 X} \quad \textbf{(3.55)}$$

$@\ r=r_0, T=T_w$; this leads to $@\ Y=1,\ \theta=0$

$$J_0(\beta_n)=0 \quad \textbf{(3.56)}$$

$\beta_n = 2.4048, 5.5201, 8.6537, 11.7915, 14.93309.........$

For each value of β_n, , the problem is satised thus far, and each must be considered

$$\theta = \sum_{n=1}^{\infty} A_n\ J_0(\beta_n Y)\ \exp\ (-\beta_n^2 X) \quad \textbf{(3.57)}$$

The remaining boundary condition $@x=0, T=T_\infty$ or $@X=0, \theta=1$ demands

$$\sum A_n J_0(\beta_n\ Y) = 1 \quad \textbf{(3.58)}$$

Which is a requirement to express in a Fourier series of Bessel functions over the range $0 \leq Y \leq 1$

$$\int_0^1 Y\ J_0\ (\beta_n Y)\ dY = \int_0^1 A_n\ Y J_0^2\ (\beta_n Y)\ dY$$

$$A_n = \frac{\int_0^1 Y J_0\ (\beta_n Y)\ dY}{\int_0^1 Y J_0^2\ (\beta_n Y)\ dY} = \frac{\int_0^{\beta_n} \frac{v}{\beta_n}\ J_0\ (v)\ \frac{dv}{\beta_n}}{\frac{1}{2}\ J_1^2\ (\beta_n)} \quad \textbf{(3.59)}$$

For the above expression, let us refer to the following recurrence relations of Bessel functions:

Relation I :

$$\frac{d}{dv}\left[v^n J_n(v)\right] = v^n J_{n-1}(v)$$

$$\frac{d}{dv}\left[vJ_1(v)\right] = vJ_0(v) \Rightarrow \int vJ_0(v)dv = vJ_1(v)$$

Also,

$$\frac{n}{x}\left[J_n(x)\right] - J_n'(x) = J_{n+1}(x)$$

Which reduces for n=0, to the form, $J_0'(x) = -J_1(x)$

Relation II :

$$\int_0^a xJ_n^2(\lambda x)\ dx = \frac{a^2}{2}\left[\left\{J_n'(\lambda a)\right\}^2 + \left(1 - \frac{n^2}{\lambda^2 a^2}\right) J_n^2(\lambda a)\right]$$

or,

$$\int_0^a xJ_0^2(\lambda x)\ dx = \frac{a^2}{2}\left[(-J_1(\lambda a))^2 + J_0^2(\lambda a)\right]$$

Now,

$$\int_0^1 YJ_0^2\left(\beta_n Y\right) dYy = \frac{1}{2}\left[(-J_1(\beta_n))^2 + J_0^2(\beta_n)\right]$$

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In our case $J_0(\beta_n) = 0$

Therefore

$$A_n = \frac{\frac{1}{\beta_n^2} \int_0^{\beta_n} v J_0(v)\, dv}{\frac{1}{2} J_1^2(\beta_n)} = \frac{\frac{1}{\beta_n^2} \left[v J_1(v) \right]_0^{\beta_n}}{\frac{1}{2} J_1^2(\beta_n)}$$

$$A_n = \frac{2}{\beta_n J_1(\beta_n)}$$

$$\theta = 2 \sum_{n=1}^{\infty} \frac{J_0(\beta_n Y)}{\beta_n J_1(\beta_n)} e^{-\beta_n^2 X} \qquad \textbf{(3.60)}$$

$$T_m = \frac{2}{u_m r_0^2} \int_0^{r_0} u(r) T(r) r dr$$

$$\theta_m = 2 \int_0^1 \theta Y dY = 4 \int_0^1 \frac{Y J_0(\beta_n Y)\, dY}{\beta_n J_1(\beta_n)} e^{-\beta_n^2 X}$$

$$\theta_m = 4 \sum_{n=1}^{\infty} \frac{e^{-\beta_n^2 X}}{\beta_n^2} \qquad \textbf{(3.61)}$$

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Again,

$$\frac{d}{dY}\{J_0(\beta_n Y)\} = -\beta_n J_1(\beta_n Y)$$

$$\left.\frac{d\theta}{dY}\right|_{Y=1} = -2\sum_{n=1}^{\infty} e^{(-\beta_n^2 X)} \qquad \textbf{(3.62)}$$

$$h(T_w - T_m) = k\frac{\partial T}{\partial r}\,|_{r=r_0} \qquad \textbf{(3.63)}$$

$$\frac{hD}{k} = \frac{2\times 2\sum_{n=1}^{\infty} e^{-\beta_n^2 X}}{\theta_m} = \frac{\sum_{n=1}^{\infty} e^{-\beta_n^2 X}}{\sum_{n=1}^{\infty}(e^{-\beta_n^2 X})/\beta_n^2} \qquad \textbf{(3.64)}$$

$$\text{For } \beta_1, \quad \frac{hD}{k} = \frac{e^{-\beta_1^2 X}\ \beta_1^2}{e^{-\beta_1^2 X}} = \beta_1^2$$

$$\text{or,} \quad Nu_D = 5.7831 \qquad \textbf{(3.65)}$$

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Parabolic Velocity profile and Thermally Developed Flow (For Uniform Wall Temperature Boundary Condition) :

The governing equation is**:**

$$\rho c_p u \frac{\partial T}{\partial x} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \quad \textbf{(3.66)}$$

$\theta(r) = \frac{T-T_w}{T_c-T_w}$, where Tc is the centerline temperature and T_w is the wall temperature;introducing $\eta = \frac{r}{r_o}, z = \frac{x}{r_0}, U(\eta) = \frac{u}{u_m}$

We get,

$$\frac{\rho c_p u_m U(\eta)}{r_0}\frac{\partial T}{\partial z} = \frac{k}{r_0^2}\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial T}{\partial \eta}\right)$$

$$\left(\frac{\rho \nu c_p}{k}\right)\frac{u_m r_0}{\nu}U\frac{\partial T}{\partial z} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial T}{\partial \eta}\right)$$

$$\frac{Pr Re_D}{2}U\frac{\partial T}{\partial z} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial T}{\partial \eta}\right)$$

Now, $T = T_w + (T_c - T_w)\theta(r);\ T_c \equiv f_n(z)$

$$\frac{\partial T}{\partial z} = \theta(r)\frac{\partial T_c}{\partial z}$$

$$\frac{\partial T}{\partial r} = (T_c - T_w)\frac{\partial \theta}{\partial r}$$

$$\frac{\partial T}{\partial \eta} = (T_c - T_w)\frac{\partial \theta}{\partial \eta}$$

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On substitution,

$$Pe\,\frac{U\theta}{2}\,\frac{\partial T_c}{\partial z}=\frac{1}{\eta}\,\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right)(T_c-T_w)$$

or,

$$\left[\frac{1}{T_c(z)-T_w}\right]Pe\,\frac{dT_c}{dz}=\frac{\frac{1}{\eta}\,\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right)}{\frac{U(\eta)}{2}\theta(\eta)}=-\lambda^2\quad(\text{constant}) \tag{3.67}$$

$$\frac{Pe\;dT_c/dz}{T_c-T_w}=-\lambda^2$$

$$T_c(z)=T_w+Ce^{-\lambda^2 z/Pe}$$

The second equation gives

$$\frac{1}{\eta}\,\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \theta}{\partial \eta}\right)+\frac{\lambda^2}{2}U\theta=0$$

Where, $\frac{u}{u_m}=2\left[1-\left(\frac{r}{r_0}\right)^2\right]\;=2[1-\eta^2]$

$$\underbrace{\theta''+\frac{1}{\eta}\theta'+\lambda^2(1-\eta^2)\theta=0}_{\text{Euler Cauchy Equation}} \tag{3.68}$$

Boundary conditions,

$$@\;r=0\Rightarrow @\;\eta=0\;\;\frac{\partial\theta}{\partial\eta}=0\;(\text{symmetric})$$

$$@\;r=r_0\Rightarrow @\;\eta=1\qquad\theta=0$$

The solution of above **equation is an infinite series.**

Let the general solution be

$$\theta = \sum_{n=0}^{\infty} C_n \eta^n$$

Now,

$$\theta = C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3 + ... + C_n\eta^n$$

$$\theta' = C_1 + 2\,C_2\eta + 3\,C_3\eta^2 + ... + (n+1)\,C_{n+1}\eta^n$$
$$\theta'' = 2$$

Plugging in to **the original equation**, one gets,

$$2\,C_2 + 3\cdot 2\,C_3\eta + ... + (n+2)(n+1)\,C_{n+2}\eta^n$$

$$+\quad \frac{C_1}{\eta} + 2\,C_2 + 3\,C_3\eta + ... + (n+1)\,C_{n+1}\eta^{n-1} + (n+2)\,C_{n+2}\eta^n$$

$$+\quad \lambda^2 C_0 + \lambda^2 C_1\eta + \lambda^2 C_2\eta^2 + ... + \lambda^2 C_n\eta^n$$

$$=\quad \lambda^2 C_0\eta^2 + \lambda^2 C_1\eta^3 + \lambda^2 C_2\eta^4 + ... + \lambda^2 C_n\eta^{n+2}$$

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Equating the **equal powers of** η

$$C_1 = 0$$

$$2.1C_2 + 2C_2 + \lambda^2 C_0 = 0$$

$$3.2C_3 + 3C_3 + \lambda^2 C_1 = 0$$

$$4.3C_4 + 4C_4 + \lambda^2 C_2 = \lambda^2 C_0$$

$$5.4C_5 + 5C_5 + \lambda^2 C_3 = \lambda^2 C_1$$
$$(n+2)(n+1)C_{n+2} + (n+2)C_{n+2} + \lambda^2 C_n = \lambda^2 C_{n-2}$$

As $C_1 = 0, C_3, C_5.....$ **all odd coefficients are zero.**

Putting $n = 2m$

$$(2m+2)(2m+1)C_{2m+2} + (2m+2)C_{2m+2} + \lambda^2 C_{2m} = \lambda^2 C_{2m-2}$$

$$(2m+2)^2 C_{2m+2} + \lambda^2 C_{2m} = \lambda^2 C_{2m-2}$$

$$C_{2m+2} = \frac{\lambda^2}{(2m+2)^2}(C_{2m-2} - C_{2m})$$

$$C_{2m} = \frac{\lambda^2}{(2m)^2}(C_{2m-4} - C_{2m-2})$$

Therefore, we can write

$$\theta = \sum_{n=0}^{\infty} C_{2n}\eta^{2n} \qquad \textbf{(3.69)}$$

Where
$$C_{2n} = \frac{\lambda^2}{(2n)^2}[C_{2n-4} - C_{2n-2}] \quad \text{for} \quad n \geq 2$$

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We can also write

$$\theta = \sum_{n=0}^{\infty} C_{2n}\eta^{2n} = C_0 + C_2\eta^2 + C_4\eta^4 + C_6\eta^6 + \ldots$$

Invoking the boundary conditions, $@\eta = 0, \theta = 1$

We get

$$C_0 = \theta_{\text{at } \eta=0} = 1$$

We also know, $@r = r_0$, i.e., $@\eta = 1, \theta = 0$

$$C_0 + C_2 + C_4 + C_6 + ... = 0$$

$$C_0 - \frac{\lambda^2}{4}C_0 + \frac{\lambda^2}{16}\left(1 + \frac{\lambda^2}{4}\right)C_0 - \frac{\lambda^2}{36}\left(\frac{5\lambda^2}{16} + \frac{\lambda^4}{64}\right)C_0 + ... = 0$$

$$C_0\left[1 - \frac{\lambda^2}{4} + \frac{\lambda^2}{16} + \frac{\lambda^4}{64} - \frac{5\lambda^4}{36 \times 16} - \frac{\lambda^6}{36 \times 64}\right] + ... = 0$$

Considering **upto 4th power of the series**

$$C_0\left[1 - \frac{\lambda^2}{4} + \frac{\lambda^2}{16} + \frac{\lambda^4}{64} - \frac{5\lambda^4}{576}\right] = 0$$

Since, $C_0 \neq 0$

$$1 - \frac{\lambda^2}{4} + \frac{\lambda^2}{16} + \frac{\lambda^4}{144} = 0$$

$$\lambda^4 - 27\lambda^2 + 144 = 0$$

Solving numerically by **Newton Rapshon method,**

$$\lambda = 2.704364 \qquad \textbf{(3.70)}$$

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Nusselt number for UWT

From the governing equation, we get

$$\int_0^{r_0} \rho c_p u \frac{\theta}{r_0}\frac{dT_c}{dz} r\,dr = k r_0 \frac{\partial T}{\partial r}\Big|_{r=r_0}$$

or,

$$\int_0^{r_0} k\frac{\nu}{\alpha}\,\frac{u_m 2r_0}{\nu}\,\frac{r}{2r_0^2}\,\frac{u}{u_m}\,(T-T_w)\,\frac{dT_c/dz}{(T_c-T_w)}\,dr = r_0 q_w''$$

$$\int_0^{r_0} \frac{Pr\,Re_D \frac{dT_c}{dz}}{(T_c-T_w)}\,k\,\frac{r}{2r_0^2}\,\frac{u}{u_m}\,(T-T_w)\frac{2\pi}{2\pi}dr = r_0 q_w''$$

or,

$$\int_0^{r_0} -\frac{\lambda^2 k}{2}\,\frac{2\pi u r (T-T_w)dr}{2\pi u_m r_0^2} = r_0 q_w''$$

or,

$$-\frac{\lambda^2 k}{4}\,\left\{\frac{\int_0^{r_0} uT2\pi r\,dr}{u_m \pi r_0^2} - \frac{T_w \int_0^{r_0} u2\pi r\,dr}{u_m \pi r_0^2}\right\} = r_0 q_w''$$

or,

$$-\frac{\lambda^2 k}{4}(T_m - T_w) = r_0 q_w''$$

or,

$$\frac{q_w''}{(T_w-T_m)}\,\frac{2r_0}{k} = \frac{hD}{k} = \frac{\lambda^2}{2}$$

Nusselt number for hydrodynamically and thermally fully developed flow **(subject to uniform wall temperature)**

$$Nu_D = \frac{\lambda^2}{2} = 3.656 \qquad \textbf{(3.71)}$$

For parallel plate channels, fully developed Nusselt number

$Nu = 7.5407$ (constant wall temperature)

$Nu = 8.2352$ (constant wall heat flux)

In the above two expressions, the Nusselt number has been calculated based on characteristics length 2b, where b is the channel height.