Module 3: Internal Flows

Lecture 13: Fully Developed Pipe Flow

The Lecture Contains:

- Introduction
- Velocity profile in a fully devloped pipe flow

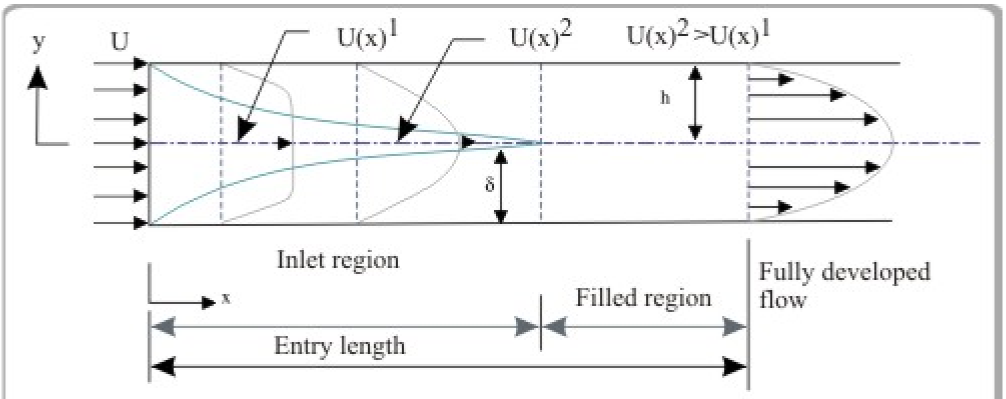

Figure 3.1 : Flow Development in Parallel Plate Channel

Introduction

The flow development in a parallel plate channel has been shown in **Figure 3.1**. Beyond the entrance region, effect of viscosity extends over the entire cross section and the flow is said to be fully developed.

$$Re_D = \frac{\rho u_m D}{\mu} = \frac{\rho D^2 u_m^2}{\mu D u_m} = \frac{\rho D^2 u_m (D/t)}{D^2 \times \mu \frac{u_m}{D}} = \frac{\rho D^3 (u_m/t)}{D^2 \times \mu \frac{u_m}{D}} = \frac{\text{Inertia Force}}{\text{Viscous Force}}$$

U_m is the mean velocity of the uid. The critical Reynolds number is usually considered as $Re_{D,C} = 2300$

Flow is **turbulent** for $Re_D > 4000$ (generally) and **laminar** for $Re_D \leq 2300$

In between is the transition zone. The estimation of the entrance length (x_e, h) can be done in the following way.

$$\left(\frac{x_{e,h}}{D}\right)_{lam} \approx 0.05 Re_D$$

$$10 \leq \left(\frac{x_{e,h}}{D}\right)_{tur} \leq 60$$

We shall assume turbulent fully developed flow for $(x_{e,h}/D) > 10.$

Mean Velocity :

Because velocity varies over the cross section, it is necessary to work with a mean velocity u_m.

$$\dot{m} = \rho u_m A_c = \rho u_m \pi D^2/4, \quad Re_D = \frac{4\dot{m}}{\pi D \mu} \qquad \textbf{(3.1)}$$

(3.2)

$$u_m = \frac{\int_{A_c} \rho u(r) dA_c}{\rho A_c} = \frac{2\pi\rho \int_0^{r_0} u(r) r dr}{\rho \pi r_0^2} = \frac{2}{r_0^2} \int_0^{r_0} u(r) r dr$$

Velocity profile in a fully developed pipe flow :

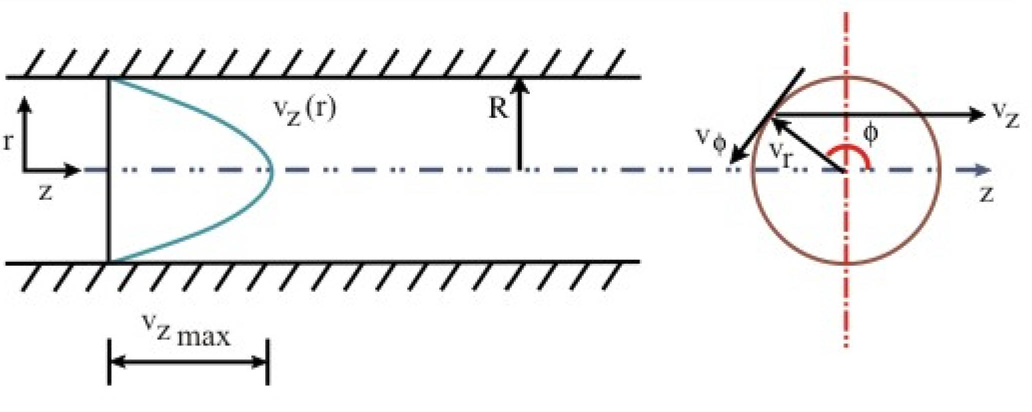

Figure 3.2: Hagen-Poiseullie Flow through a pipe

The fully developed pipe flow is often called **Hagen-Poiseullie flow (Figure 3.2)**. The general governing equations are**:**

- **Continuity**

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0 \qquad \textbf{(3.3)}$$

- **r Momentum**

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2}{r} + v_z\frac{\partial v_r}{\partial z}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \phi^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi}\right) \qquad \textbf{(3.4)}$$

- ϕ **Momentum**

$$\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + v_z\frac{\partial v_\phi}{\partial z} \qquad \textbf{(3.5)}$$

Previous Next

- **Z momentum**

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_z}{\partial \phi} + v_z \frac{\partial v_z}{\partial z}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \phi^2} + \frac{\partial^2 v_z}{\partial z^2}\right) \qquad \textbf{(3.6)}$$

For fully developed flow, vz is only non trivial component. $v_r = v_\phi = 0$ **From continuity**, $(\partial v_z/\partial z) = 0$ therefore, $v_z = v_z(r, \phi, t)$

Under steady state, $\partial(\text{anything})/\partial t = 0$ **flow is axisymmetric**, $(\partial\,[\text{any variable}]/\partial\phi) = 0$

Therefore,

$$v_z = v_z(r)$$

From **r** and ϕ **momentum equation,** we get $p \neq p(r, \phi);\ p = p(z)$ only

From **z momentum equation ,** we get

$$\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} = \frac{1}{\mu}\frac{dp}{dz}$$

or,

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dv_z}{dr}\right] = \frac{1}{\mu}\frac{dp}{dz} \qquad \textbf{(3.7)}$$

or,

$$r\frac{dv_z}{dr} = \frac{1}{\mu}\frac{dp}{dz}\frac{r^2}{2} + C_1$$

or,

$$\frac{dv_z}{dr} = \frac{1}{\mu}\frac{dp}{dz}\frac{r}{2} + \frac{C_1}{r}$$

or,

$$v_z = \frac{1}{4\mu}\frac{dp}{dz}r^2 + C_1 \ln r + C_2 \qquad \textbf{(3.8)}$$

The boundary conditions are**:**

at r = 0, v_z is finite = C_1 = 0

at $r = r_0, v_z = 0 \Rightarrow 0 = \frac{1}{4\mu}\frac{dp}{dz}r_0^2 + C_2$

$$C_2 = -\frac{1}{4\mu}\frac{dp}{dz}r_0^2$$

$$v_z = -\frac{1}{4\mu}\left(\frac{dp}{dz}\right)r_0^2\left[1 - \left(\frac{r}{r_0}\right)^2\right]$$

using, $v_z \leftrightarrow u$ and $z \leftrightarrow x$, .

$$u(r) = -\frac{1}{4\mu}\left(\frac{dp}{dx}\right)r_0^2\left[1 - \left(\frac{r}{r_0}\right)^2\right] \qquad \textbf{(3.9)}$$

Previous Next

$$u = u_{max} \text{ at } r = 0$$

$$u_{\max} = -\frac{1}{4\mu}\left(\frac{dp}{dx}\right) r_0^2 \tag{3.10}$$

$$u_m = \frac{2}{r_0^2}\int_0^{r_0} -\frac{r_0^2}{4\mu}\left(\frac{dp}{dz}\right)\left[1-\left(\frac{r}{r_0}\right)^2\right] r dr$$

or,

$$u_m = \frac{2}{r_0^2}\left(\frac{r_0^2}{4\mu}\frac{dp}{dx}\right)\left[\frac{r^2}{2}-\frac{r^4}{4r_0^2}\right]_0^{r_0} = -\frac{1}{8\mu}\left(\frac{dp}{dx}\right) r_0^2 = \frac{u_{max}}{2} \tag{3.11}$$

or,

$$u_m = \frac{1}{2}u_{max} \quad \text{and} \quad \frac{u(r)}{u_m} = 2\left[1-\left(\frac{r}{r_0}\right)^2\right] \tag{3.12}$$

Darcy friction factor,

$$h_f = \frac{\Delta p}{\rho g} = \frac{fLu_m^2}{2Dg} \tag{3.13}$$

$$f = \frac{(\Delta p/L)D}{\rho u_m^2/2} = \frac{(-dp/dx)D}{(\rho u_m^2/2)} \tag{3.14}$$

Fanning's friction factor

$$C_f = \frac{\tau_w}{(\rho u_m^2/2)} \tag{3.15}$$

$$C_f = \frac{(-dp/dx)\ (D/4)}{(\rho u_m^2/2)} \tag{3.16}$$

As such

$$\tau_w = -\mu \frac{\partial u}{\partial r}\,\Big|_{r=r_0}$$

$$\tau_w = -\mu\left(+\frac{1}{4\mu}\ \frac{dp}{dx}\right)\ 2r_0^2\ \frac{r_0}{r_0^2}\quad = -\frac{1}{2}\ \frac{dp}{dx} r_0 = \frac{1}{2}\ \left(-\frac{dp}{dx}\right)\ \frac{D}{2}$$

Therefore, $C_f = f/4$ For fully developed laminar flow

$$f = \frac{(-dp/dx)\ (D)}{(\rho u_m^2/2)}$$

Substitute

$$u_m = -\frac{1}{8\mu}\ \frac{dp}{dx}\ r_0^2,\ \ Re_D = \frac{\rho u_m D}{\mu}$$

Or,

$$f = \frac{8\mu u_m D/r_0^2}{\rho u_m^2/2} = \frac{64}{\frac{\rho u_m D}{\mu}} = \frac{64}{Re_D} \tag{3.17}$$

fully developed laminar flow $f = \frac{64}{Re_D}$

Module 3: Internal Flows

Lecture 13: Fully Developed Pipe Flow

For fully developed turbulent flow, the analysis is quite involved. At this stage, we may

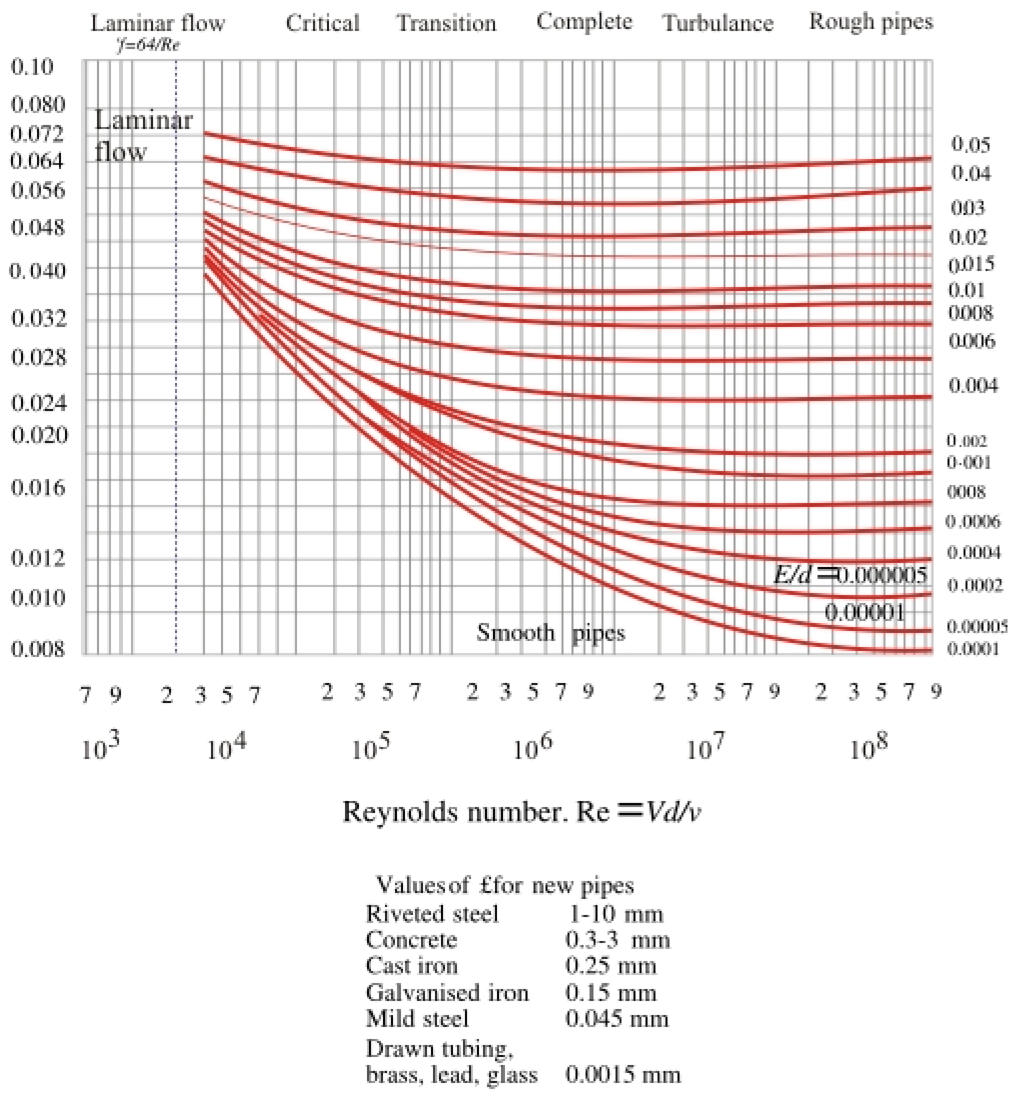

Figure 3.3: Friction factors for pipes rely

rely on the experimental results, Moody diagram **(Figure3.3)**. Friction factor is a functionof Reynolds number and surface roughness. Correlations that reasonably approximatethe smooth surface**:**

$$f = 0.316\ Re^{-1/4} .... Re_D \leq 2 \times 10^4 \qquad \textbf{(3.18)}$$

$$f = 0.184\ Re^{-1/5} .... Re_D \geq 2 \times 10^4 \qquad \textbf{(3.19)}$$

Note that, $C_f = f/4$ is **constant in fully developed region**

$$f = \frac{-(dp/dx)D}{\rho u_m^2/2}$$

$$\Delta p = -\int_{p_1}^{p_2} dp = \int_1^2 f\frac{\rho u_m^2}{2D}dx = \frac{f\rho u_m^2}{2D}(x_2 - x_1) = \frac{f\ \rho u_m^2}{2D}\ L$$

$f = \frac{64}{Re_D}$ in **laminar flow**, and $f = 0.184\ Re_D^{-1/5}$ for $Re_D > 2 \times 10^4$ **(for turbulent flow)**

However,from the above relation we get back and $f = 0.184\ Re_D^{-1/5}$ for $Re_D > 2 \times 10^4$ (for turbulent flow) $\Delta p/\rho g = \frac{fLu_m^2}{2Dg} = h_f$