Lecture 23: Universal Law of the Wall

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The Law of the Wall (Extended considering pressure gradient)

Let us consider a fully developed channel flow

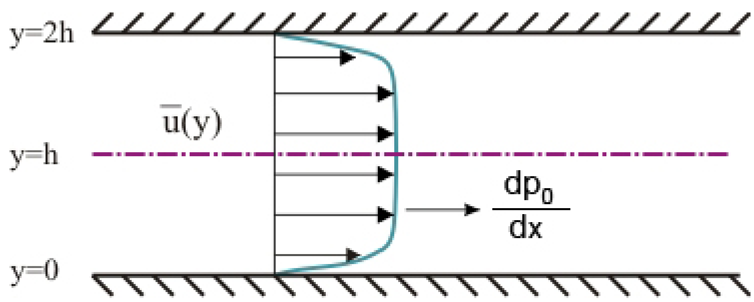

Figure 5.8: Channel flow with pressure gradient

Assumption: All derivatives with respect to x are zero except $\frac{dp}{dx}$

mean velocity field $\bar{u} = [\bar{u}(y), 0, 0]$

The averaged Ns equations give

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial}{\partial y}\left(\overline{u'v'}\right) + \nu\left[\frac{\partial^2 \overline{u}}{\partial y^2}\right] \qquad \textbf{(5.61)}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\partial}{\partial y}\left(\overline{v'^2}\right) \qquad \textbf{(5.62)}$$

Integrating **(5.62)** between **0 and y**,
with no-slip boundary condition,
we get

$$\frac{p}{\rho} + \overline{v'^2} = \frac{p_0}{\rho} \qquad \textbf{(5.63)}$$

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Thus

$$\frac{\partial p}{\partial x} = \frac{\partial p_0}{\partial x} \qquad \textbf{(5.64)}$$

also, to be noted $p_0 = p_0(x)$

From **(5.61) and (5.64)**

$$0 = -\frac{1}{\rho}\frac{dp_o}{dx} - \frac{d(u'v')}{dy} + v\left[\frac{d^2\bar{u}}{dy^2}\right] \qquad \textbf{(5.65)}$$

Integrating **(5.65)** between 0 and y, we get

$$0 = -\frac{y}{\rho}\frac{dp_0}{dx} - \overline{u'v'} + \nu\frac{d\bar{u}}{dy} - \nu\left.\frac{d\bar{u}}{dy}\right|_{y=0} \qquad \textbf{(5.66)}$$

set

$$u_\tau^2 = \nu\left.\frac{d\bar{u}}{dy}\right|_{y=o}$$

Where, u_z is the friction velocity. Te **equation (5.66)** becomes

$$0 = -\frac{y}{\rho}\frac{dp_0}{dx} - \overline{u'v'} + \nu\frac{d\bar{u}}{dy} - u_\tau^2 \qquad \textbf{(5.67)}$$

At $y = h$ at the center of the channel ,
due to symmetry.

$$-\overline{u'v'}|_{y=h} = \nu\left.\frac{d\bar{u}}{dy}\right|_{y=h} = 0$$

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Equation (5.67) becomes

$$-\overline{u'v'} + \nu\frac{d\bar{u}}{dy} = u_\tau^2\left(1 - \frac{y}{h}\right) \qquad \textbf{(5.68)}$$

put

$$u^+ = \frac{\bar{u}}{u_\tau}, \qquad y^+ = \frac{y u_\tau}{\nu}$$

Now **equation (5.68)** reads as

$$-\frac{\overline{u'v'}}{u_\tau^2} + \frac{du^+}{dy^+} = 1 - \frac{y^+}{R_\tau} \qquad \textbf{(5.69)}$$

Where $R_\tau = \frac{u_\tau h}{\nu}$

Now, let *y+* be finite, say of order one, in the limit of $R_\tau \longrightarrow \infty$ .The **equation (5.69)** becomes

$$-\frac{\overline{u'v'}}{u_\tau^2} + \frac{du^+}{dy^+} = 1 \qquad \textbf{(5.70)}$$

Assuming that the wall is smooth and no additional parameters appear in the **BC**,we expect the solution of **(5.70)** to be

$$u^+ = \frac{\bar{u}}{u_\tau} = f(y^+)$$
$$-\frac{\overline{u'v'}}{u_\tau^2} = g(y^+) \qquad \textbf{(5.71)}$$

Here, $f(y^+)$ and $g(y^+)$ are the laws of the wall.

(a) Assume $R_\tau >> 1, \overline{u'v'}$ to be negligible at $y^+ << R_\tau$. The **equation (5.69)** becomes.

$$\frac{du^+}{dy^+} = 1 \qquad \textbf{(5.72)}$$

Integrating **(5.72)** between ***0** and **y+*** and applying no-slip boundary condition, we obtain

$$u^+ = y^+ \qquad \textbf{(5.73)}$$

This law of the wall is valid for: $0 \leq y^+ < 5$. Such a region is called the viscous sublayer.

(b) Assuming at y/h << 1, there is a region where the viscous forces are negligible,then from **(5.69)** , we have (for $R_\tau \rightarrow \infty$)

$$-\overline{u'v'} = u_\tau^2 \qquad \textbf{(5.74)}$$

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Consider Prandtl's mixing length hypothesis

$$-\overline{u'v'} = l_m^2 \left(\frac{d\bar{u}}{dy}\right)^2 \quad \textbf{(5.75)}$$

Thus from **(5.67)**,we get

$$u_\tau^2 = -\frac{h}{\rho}\frac{dp_0}{dx}$$

Where, the mixing length

$$l_m = \chi y \quad \textbf{(5.76)}$$

parameter is called von-Karman constant. Here,
Karman assumed that l_m should be a function of the distance from the wall in a wall bounded turbulent flow, rather than a constant as taken by Prandtl in the case of free turbulent flows. Substituting **(5.75)** and **(5.76)** into **(5.74),** we obtain

$$\chi^2 y^2 \left(\frac{d\bar{u}}{dy}\right)^2 = u_\tau^2 \quad \textbf{(5.77)}$$

$$\chi y \frac{d\bar{u}}{dy} = u_\tau \quad \textbf{(5.78)}$$

$$\chi \frac{du^+}{dy^+} = \frac{1}{y^+} \qquad (5.79)$$

Integrating **(5.79)** between $0 \text{ and } y^+$, we get

$$u^+ = \frac{1}{\chi} \ln y^+ + C \qquad (5.80)$$

where $\chi \approx 0.41 \text{ and } C \approx 4.99.$

Equation (5.80) is called the **Logarithmic Law of the wall ,** which is valid for $30 \leq y^+ \leq 140$, the so called **inertial sublayer.**

Barenblatt and Chorin (1998) introduced a Law of the wall which is given by

$$u^+ = \beta (y^+)^\alpha \qquad (5.81)$$

where

$$\alpha = \frac{3}{2 \ln R_\tau}, \ \beta = \frac{\sqrt{3} + 5\alpha}{\alpha}$$

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Turbulent Heat Transfer in a Pipe (Simplified Analysis)

Heat flux due to conductivity of the fluid **(Figure 5.9)**

$$\dot{q}_l = -k\frac{\partial \overline{T}}{\partial y} \qquad \textbf{(5.82)}$$

The mean rate of turbulent heat transfer is given by

$$\dot{q}_t = \rho c_p \overline{v'T'} \qquad \left[c_p\overline{u'T'} < c_p\overline{v'T'}\right] \qquad \textbf{(5.83)}$$

The total heat flux is given by

$$\dot{q} = \dot{q}_l + \dot{q}_t = -k\frac{\partial \overline{T}}{\partial y} + \rho c_p \overline{v'T'} \qquad \textbf{(5.84)}$$

or

$$\dot{q} = -\rho c_p\left[\alpha\frac{\partial \overline{T}}{\partial y} - \overline{v'T'}\right] \qquad \textbf{(5.85)}$$

Turbulent heat exchange also considered equivalent to an increase in thermal diffusivity of the fluid similar to turbulent momentum exchange

$$\overline{v'T'} = -\alpha_t \frac{\partial \overline{T}}{\partial y} \qquad \textbf{(5.86)}$$

$$\dot{q} = -\rho c_p[\alpha + \alpha_t] \frac{\partial \overline{T}}{\partial y} \qquad \textbf{(5.87)}$$

where α_t is **eddy diffusivity**

Heat transport in a pipe or tube could be represented by

$$\frac{q}{\rho c_p A} = -(\alpha + \alpha_t) \frac{\partial \overline{T}}{\partial y} \qquad \textbf{(5.88)}$$

In a similar fashion,
shear stress in a turbulent flow could be written as

$$\frac{\tau}{\rho} = (\nu + \nu_t) \frac{\partial \overline{u}}{\partial y} \qquad \textbf{(5.89)}$$

We take a recourse to **Chilton-Colburn analogy** at this stage.
The analogy says that the heat and momentum are transported as the same rate; that is $\nu = \alpha$ **[Prandtl number,** $\nu/\alpha = 1$

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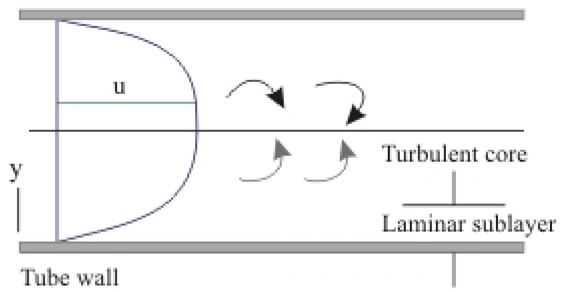

Figure 5.9 Viscous sublayer in a pipe flow

$and\ v_t = \alpha_t \left[Turbulent\ Prandt1\ number\ \frac{v_t}{\alpha_t} = 1 \right]$ Dividing **equation (5.88) by (5.89)** we get

$$\frac{q}{c_p A \tau} d\bar{u} = -d\overline{T} \tag{5.90}$$

$$(q/c_p\ A\ \tau)\ \ d\bar{u} = -d\ \overline{T}$$

An additional assumption is done here; ratio of heat transfer per unit area to shear stress is constant across the flow field. Thus,

$$\frac{q}{A\tau} = \text{constant} = \frac{q_w}{A_w \tau_w} \tag{5.91}$$

Now equating **equation (5.90)** between wall conditions and bulk conditions gives

$$\frac{q_w}{c_p\ A_w\ \tau_w}\int_{u=0}^{u=u_m} d\overline{u} = \int_{T_w}^{T_m} -d\overline{T} \qquad \textbf{(5.92)}$$

or

$$\frac{q_w\ u_m}{A_w\ \tau_w\ c_p} = T_w - T_m \qquad \textbf{(5.93)}$$

we know

$$q_w = hA_w\ \ (T_w - T_m) \qquad \textbf{(5.94)}$$

From **Figure 5.10**, one can obtain

$$\tau_w = \frac{\Delta p(\pi D^2)}{4\pi DL} = \frac{\Delta p}{4}\ \ \frac{D}{L} \qquad \textbf{(5.95)}$$

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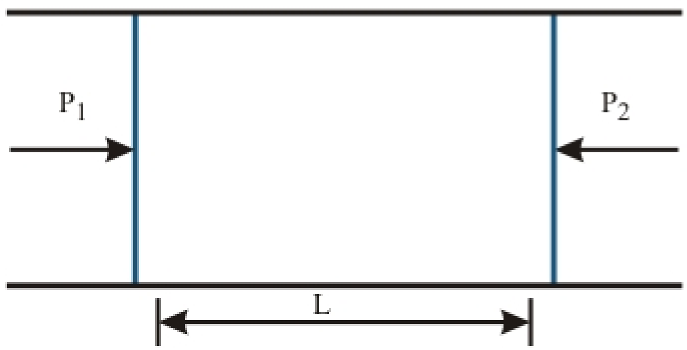

Figure 5.10: Pressure differential in a channel

Again

$$\frac{\Delta p}{\rho} = \frac{fLu_m^2}{2D}$$

so

$$\tau_w = \frac{\rho f L u_m^2}{4(2D)} \frac{D}{L} = \frac{f}{8} \ \rho u_m^2 \qquad \textbf{(5.96)}$$

Invoking **(5.94)** and **(5.96)** in **(5.93)** we get

$$\frac{h \ A_w \ (T_w - T_m) \ u_m 8}{A_w \ c_p \ \rho f u_m^2} = (T_w - T_m) \qquad \textbf{(5.97)}$$

$$\frac{h}{\rho c_p u_m} = \frac{f}{8} \qquad \textbf{(5.98)}$$

$$\frac{hD}{k} \cdot \frac{k}{\rho c_p} \cdot \frac{1}{\frac{u_m D}{\nu}} \cdot \frac{1}{\nu} = \frac{f}{8}$$

$$St = \frac{Nu_D}{Re_D\ Pr} = \frac{f}{8} \qquad \textbf{(5.99)}$$

Turbulent **friction factor** $f = 0.184\ \ Re_D^{-1/5}$ for $Re \geq 2 \times 10^4$

$$\frac{Nu_D}{Re_D\ Pr} = \frac{0.184\ Re_D^{-1/5}}{8}$$

or

$$Nu_D = 0.023\ Re^{4/5} \qquad \textbf{(5.100)}$$

This relation of turbulent heat transfer is highly restrictive because of the **Pr = 1.0** assumption.
The heat transfer-fluid friction analogy for flat plate problem (known as Reynolds analogy) indicates a Prandtl number dependence. Let us take a recourse to the Reynolds analogy for flow over a flat plate:

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} \qquad \textbf{(5.101)}$$

We know,

$$\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad \textbf{(5.102)}$$

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and ,

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right) \bigg|_{y=0} \qquad \textbf{(5.103)}$$

Invoking **Equation (5.102) and Equation (5.103) in Equation (5.101)**, we get

$$\frac{C_{fx}}{2} = 0.323 \, (Re_x)^{-1/2} \qquad \textbf{(5.104)}$$

We have already seen that for flow over a flat plate

$$Nu_x = 0.332 \; Re_x^{1/2} \; Pr^{1/3}$$

$$\frac{Nu_x}{Re_x \; Pr} = 0.332 \; Re_x^{-1/2} \; Pr^{-2/3} \qquad \textbf{(5.105)}$$

or

$$\frac{h}{\rho c_p U_\infty} = 0.332 \; Re_x^{-1/2} \; Pr^{-2/3}$$

$$St \; Pr^{2/3} = 0.332 \; Re^{-1/2} \qquad \textbf{(5.106)}$$

Compare **(5.104) and (5.106)** discrepancy is 3%

$$St \; Pr^{2/3} = \frac{C_{fx}}{2}$$

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Again we know that $C_f = f/4$

$$St\ Pr^{2/3} = \frac{f}{8} \qquad \textbf{(5.108)}$$

Now this dependence on Prandtl number works fairly well for turbulent pipe flow. **Accordingly (5.99)** can be modified as

$St\ Pr^{2/3} = \frac{f}{8}$ with friction factor $f = 0.184\ Re_D^{-1/5}$

or**,**

$$\frac{Nu_D}{Re_D\ Pr}\ Pr^{2/3} = \frac{0.184\ Re_D^{-1/5}}{8}$$

or,

$$Nu_D = 0.023\ [Re_D]^{4/5}\ [Pr]^{1/3} \qquad \textbf{(5.109)}$$

Dittus-Boelter (1930) established experimental correlation which is quite close to the above equation. **The correlation is:**

$$Nu_D = 0.023\ [Re_D]^{4/5}\ [Pr]^{n} \qquad \textbf{(5.110)}$$

n = 0.4 for heating, i.e., $T_w > T_m$ and

n = 0.3 for cooling, i.e., $T_w < T_m$. The result has been confirmed experimentally for the

range of conditions: $0.7 \le Pr \le 160$, $Re_D \ge 10000$ and $L/D \ge 10$.

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All fluid properties are evaluated at arithmetic mean of bulk temperature;

$$[\text{i.e., } \overline{T}_m = (T_{m,o} + T_{m,i})\ /2.]$$

For flows characterized by large property variations, the correlation due to **Sieder and Tate (1936)** is recommended

$$Nu_D = 0.027\ Re_D^{4/5} Pr^{1/3} \left[\frac{\mu}{\mu_w}\right]^{0.14} \qquad \textbf{(5.111)}$$

The range of applicability is $0.7 \leq Pr \leq 16700, Re_D \geq 10000, \text{ and } \ L/D \geq 10$
Where all the properties except μ_w are evaluated at $\overline{T}_m$. Note that, to a good degree of approximation , the correlations may be applied for both the constant surface temperature and constant wall heat flux conditions.
The μ_w is evaluated at the wall temperature.
The foregoing correlations are restricted to smooth surface conditions. Heat transfer will be enhanced by surface roughness. In fact,
heat transfer enhancement is often promoted by using internal ribs, fins or corrugations. However, the correlations do not apply to heat transfer in liquid metals $[3 \times 10^{-3} \leq \ Pr \leq 5 \times 10^{-2}]$ For fully developed turbulent flow $(L/D \geq 10)$, in smooth circular tubes with constant surface heat flux, the recommended correlation for liquid metals **[due to Skupinski, Tortel and Vautrey (1965)]** is

$$Nu_D = 4.82 + 0.0185\ (Pe_D)^{0.827} \quad (q_s'' = \text{constant})$$

Similarly for constant wall temperature, the correlation **[due to Seban and Shimazaki (1951)]** is

$$Nu_D = 5.0 + 0.025\ (Pe_D)^{0.8} \quad [T_w = \text{constant}]$$

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