Lecture 24: Two Equations Model

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Computational approach and Model of turbulence

The time averaged governing equations are

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \qquad (5.112)$$

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_i' u_j'}\right) \qquad (5.113)$$

$$\frac{\partial \overline{T}}{\partial t} + \overline{u}_i \frac{\partial \overline{T}}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\alpha \frac{\partial \overline{T}}{\partial x_i} - \overline{u_i' T'}\right] \qquad (5.114)$$

with

$$-\overline{u_i' u_j'} = \nu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - \frac{2}{3} k \delta_{ij} \qquad (5.115)$$

$$-\overline{u_i' T'} = \alpha_t \frac{\partial \overline{T}}{\partial x_i} \qquad (5.116)$$

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$$\alpha_t = \nu_t / \sigma_t \qquad \textbf{(5.117)}$$

where σ_t is turbulent Prandtl number. The entire system of equations can be closed if σ_t is known and ν_t is determined correctly.The turbulent viscosity, ν_t can be related to turbulent kinetic energy *k* and its dissipation rate, ε in the following way

$$\nu_t = C_\mu \frac{k^2}{\epsilon} \qquad \textbf{(5.118)}$$

The quantity C_μ is a model constant to be determined accurately for a specific flow. The kinetic energy **k** and its dissipation rate ϵ are evaluated at each point in the domain from their governing differential equations. The equation for k can be derived from the **Navier- Stokes equations** by subtracting the mean equations from the unaveraged equations to obtain an equation for the fluctuating velocity. Taking the scalar product of this equation with the fluctuating velocity and averaging yields the equation for the turbulent kinetic energy. An equation for the dissipation can be derived from the Navier Stokes equations by an extension of the method used to derive the **kssinetic energy equation.** The governing equation for ***k*** is given by **.**

$$\overbrace{\frac{\partial k}{\partial t}}^{(1)} + \overbrace{\overline{u_k}\frac{\partial k}{\partial x_k}}^{(2)} = \overbrace{-T_{ik}\frac{\partial \overline{u}_i}{\partial x_k}}^{(3)} - \overbrace{\epsilon}^{(4)}$$
$$\underbrace{-\frac{\partial}{\partial x_k}\left(\frac{1}{2}\overline{u_i' u_i' u_k'} + \frac{1}{\rho}\overline{p'\, u_k'}\right)}_{(5)} + \underbrace{\nu\left(\underline{\underline{nu}}\right)\nabla^2 k}_{(6)} \qquad \textbf{(5.119)}$$

In equation (5.119), term 1 is rate of change of kinetic energy k, term 2 is the convective transport of k, term 3 is the production by shear, term 4 is the dissipation, term 5 is the third-order turbulent diffusive transport of k and term 6 is the viscous diffusion of kinetic energy.

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The governing equation for ϵ is given by

$$\frac{\partial \epsilon}{\partial t} + \overline{u_i}\frac{\partial \epsilon}{\partial x_i} =$$

$$\left.\begin{array}{ll} -2\nu\overline{\frac{\partial u_j'}{\partial x_i}\frac{\partial u_j'}{\partial x_k}}\frac{\partial \overline{u_i}}{\partial x_k} & \text{production by mean velocity gradient} \\ -2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}} & \text{turbulent production} \\ -2\nu\frac{\partial^2 \overline{u_i}}{\partial x_k \partial x_j}\overline{u_k'\frac{\partial u_i'}{\partial x_j}} & \text{gradient production} \\ -2\nu\frac{\partial \overline{u_i}}{\partial x_k}\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_k'}{\partial x_j}} & \text{mixed production} \end{array}\right\} P_\epsilon, \quad \text{production of dissipation}$$

$$\left.\begin{array}{l} -\nu\frac{\partial}{\partial x_k}\left[\overline{u_k'\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right] \\ -2\nu\frac{\partial}{\partial x_k}\left[\overline{\frac{1}{\rho}\frac{\partial p'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}\right] \end{array}\right\} D_\epsilon, \quad \text{Turbulent diffusion of dissipation} \tag{5.120}$$

$$\left.-2\nu^2\left(\overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m}\frac{\partial^2 u_i'}{\partial x_k \partial x_m}}\right)\right\} \Phi_\epsilon, \quad \text{Turbulent destruction of dissipation}$$

$$+\nu\nabla^2\epsilon, \quad \text{Viscous diffusion of dissipation}$$

In equation (5.120), the first four terms on the right hand side (denoted by P_ϵ) represent production of dissipation, the next two terms(denoted by D_ϵ) represent the turbulent diffusion of dissipation and the seventh term (denoted by ϕ_ϵ) represents the turbulent destruction of dissipation. It can be seen that equation (5.120) for turbulent dissipation rate is analogous to equation (5.119) for the turbulent kinetic energy.

The turbulent viscosity concept,in analogy to kinetic theory of gases, can be expressed as

$$\nu_t \propto v_0 l_0$$

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where v_0 and l_0 are velocity and length scales. The above relation can be further reduced to

$$\nu_t \propto l_0^2/t_0 \qquad \textbf{(5.121)}$$

where t_0 the turbulent time scale

5.8.1 The $k-\epsilon$ Model

In the $k-\epsilon$ model, the length and the time scales are built up from the turbulent kinergy *k* and the turbulent dissipation rate, ϵ using the dimensional arguments.

$$l_0 \propto \frac{k\sqrt{k}}{\epsilon} \text{ and } t_0 \propto \frac{k}{\epsilon} \qquad \textbf{(5.122)}$$

Substitution of **Equation (5.122)** in **Equation (5.121)** yields

$$\nu_t = C_\mu \frac{k^2}{\epsilon} \qquad \textbf{(5.123)}$$

in which, C_μ is an empirical constant.

The turbulent kinetic energy, *k* and the dissipation rate, ϵ re obtained by solving the modeled transport equations. The transport equation for *k* **(Eqn.5.119)** is expressed as

$$\begin{aligned}\frac{\partial k}{\partial t} &+ \overline{u}_k \frac{\partial k}{\partial x_k} = \nu_t \left[\frac{\partial \overline{u}_i}{\partial x_k} + \frac{\partial \overline{u}_k}{\partial x_i}\right] \frac{\partial \overline{u}_i}{\partial x_k} \\ &+ \frac{\partial}{\partial x_k}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_k}\right) - \epsilon + \nu \nabla^2 k\end{aligned} \qquad \textbf{(5.124)}$$

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The transport equation for ϵ is a modeled version of the transport equation given earlier. **Equation (5.120)** is modeled as.

$$\frac{\partial \epsilon}{\partial t} + \overline{u}_i \frac{\partial \epsilon}{\partial x_i} = \nu \nabla^2 \epsilon + P_\epsilon + D_\epsilon - \phi_\epsilon \qquad \textbf{(5.125)}$$

As mentioned earlier, P_ϵ, D_ϵ and ϕ_ϵ represent the production of dissipation , the turbulent diffusion of dissipation and the turbulent destruction of dissipation respectively. The diffusion term, D_ϵ can be modeled using the gradient approximation as is done in the case of modeled ***k-equation***.

$$D_\epsilon = \frac{\partial}{\partial x_k}\left(\frac{\nu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_k}\right) \qquad \textbf{(5.126)}$$

in which σ_ϵ is an empirical constant. Analogous to modeling of ϵ in the ***k-equation*** for the **one-equation models**, the destruction of dissipation is assumed to depend only on the length and the time scales.This can be written in a functional form as.

$$\phi_\epsilon = f_\phi(l_0, t_0) \qquad \textbf{(5.127)}$$

or using **Equation (5.122),**

$$\phi_\epsilon = f_\phi(k, \epsilon) \qquad \textbf{(5.128)}$$

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The production of dissipation is assumed to be proportional to the production of the turbulent kinetic energy, the length scale and the time scale of turbulence. **A simple dimensional analysis for P_ϵ along with the above assumption leads to**

$$\begin{aligned}\frac{\partial \epsilon}{\partial t} &+ \overline{u}_i \frac{\partial \epsilon}{\partial x_i} = \nu \nabla^2 \epsilon + \frac{\partial}{\partial x_k}\left(\frac{\nu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_k}\right) \\ &+ C_{\epsilon 1}\frac{\epsilon}{k}\nu_t\left(\frac{\partial \overline{u}_i}{\partial x_k} + \frac{\partial \overline{u}_k}{\partial x_i}\right)\frac{\partial \overline{u}_i}{\partial x_k} - C_{\epsilon 2}\frac{\epsilon^2}{k}\end{aligned} \qquad \textbf{(5.129)}$$

Equation (5.129) is the modeled version of the transport equation for the turbulent dissipation rate.

The standard $k-\epsilon$ model has five empirical constants $C_\mu, \sigma_k, \sigma_\epsilon, C_{\epsilon 1}$ and $C_{\epsilon 2}$ in its formulation. $C_{\epsilon 2}$ is determined using the experiments on the decay of k behind a grid and a value of $C_{\epsilon 2} = 1.92$ is suggested. The experiments on the local-equilibrium-shear-layers suggested that $C_\mu \approx 0.09$. The constants $\sigma_k, \sigma_\epsilon$ and $C_{\epsilon 1}$ are fixed using the computer optimization and their suggested values are 1.0,1.3 and 1.44 respectively. It should be noted that the above values for the five constants are not universal and the $k-\epsilon$ model may require fine tuning in order to obtain correct results.

Rodi (1980) lists several cases where the $k-\epsilon$ model has been successfully applied using the above suggested values of the five empirical constants.

Reference

Barenblatt, G. I. And Chorin, A. J., 1998, Scaling of the Intermediate Region in Wall-bounded Turbulence: The Power Law, Phys. Fluids, Vol. 10, pp. 1043-1044.

Rodi, W., 1980, Turbulence Models and their Applications in Hydraulics, IAHR Mono- graph.

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