Lecture 22: Turbulence Closure

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Prandtl's Mixing Length

In case of turbulent flow,
velocity, pressure and temperature at a fixed point in space do not remain constant with time but perform very irregular fluctuations of high frequency.
The velocity of the fluid in turbulent flow can be represented by two components.

(i) mean motion

(ii) fluctuating component (or eddying motion)

The velocity of fluid is given by

$$u = \overline{u} + u', \quad v = \overline{v} + v', \quad w = \overline{w} + w'$$

Similarly

$$p = \overline{p} + p', \quad T = \overline{T} + T'$$

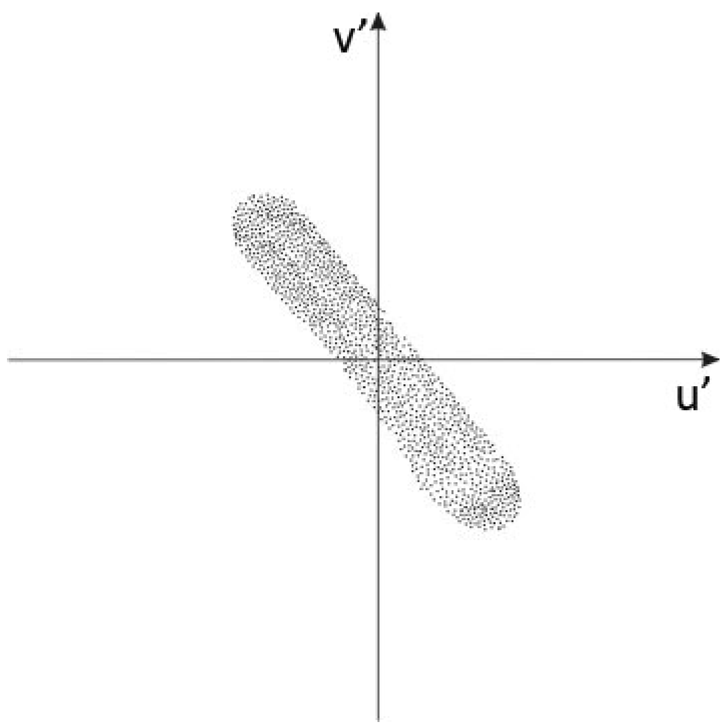

Figure 5.5: Each dot represents a *u'v'* pair at a given time

However,the fluctuating components do not bring about the bulk displacement of a fluid element.
The instantaneous displacement is $u'dt$ and if that is indeed not responsible for the bulk motion, we can conclude that

$$\int_{-t}^{t} u' dt = 0 \quad \text{for} \quad t \rightarrow \infty$$

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Due to the interaction of fluctuating components, macroscopic momentum transport takes place. Therefore, interaction effect between two fluctuating components over a long period is non-zero and this yields.

$$\int_{-t}^{t} u^{'} v^{'} dt \neq 0$$

We take time average of these two integrals and write

$$\overline{u^{'}} = \frac{1}{2t} \int_{-t}^{t} u^{'} dt = 0$$

$$\overline{u^{'} v^{'}} = \frac{1}{2t} \int_{-t}^{t} u^{'} \, v^{'} dt \neq 0$$

Similarly, we can write

$$\overline{v^{'}} = \overline{w^{'}} = \overline{p^{'}} = \overline{T^{'}} = 0$$

Let us explain the non-zero quadratic terms once again.

Consider continuity equation in two dimension $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$; If we substitute $u = \overline{u} + u^{'}$ and $v = \overline{v} + v^{'}$ and then perform time averaging, we shall obtain

$$\overline{\frac{\partial(\overline{u} + u^{'})}{\partial x}} + \overline{\frac{\partial(\overline{v} + v^{'})}{\partial y}} = 0$$

or

$$\left[\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y}\right] + \left[\frac{\partial \overline{u^{'}}}{\partial x} + \frac{\partial \overline{v^{'}}}{\partial y}\right] = 0 \qquad \textbf{(5.32)}$$

But we have seen that

$$\overline{u'} = \overline{v'} = \frac{\partial \overline{u'}}{\partial x} = \frac{\partial \overline{v'}}{\partial y} = 0$$

Invoking

$$\frac{\partial \overline{u'}}{\partial x} = \frac{\partial \overline{v'}}{\partial y} = 0$$

in **(5.32)** we get $\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$ whch yields

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

we can also write

$$\frac{\partial u'}{\partial x} = -\frac{\partial v'}{\partial y}$$

If we consider momentum exchange between two adjacent layers, then on the basis of above equation,It is postulated that if at any instant there is an increase in v' in the y direction it will be followed by an increase in u' in the x in the negative y direction. In otherwords $\overline{u'v'}$ is nonzero and negative.

This is discerned in **Figure 5.5** which shows a cloud of data points **(sometimes called a scattar plot)**.The dots represent the instantaneous values of ***u'v'*** pair at different times.

For unit area of the plane PP **(Figure 5.6)** the instantaneous turbulent mass transport rate across the plane is $\rho v'$. Associated with this mass transport is a change in the x component of velocity u' The net momentum flux per unit area,in the **x** direction,

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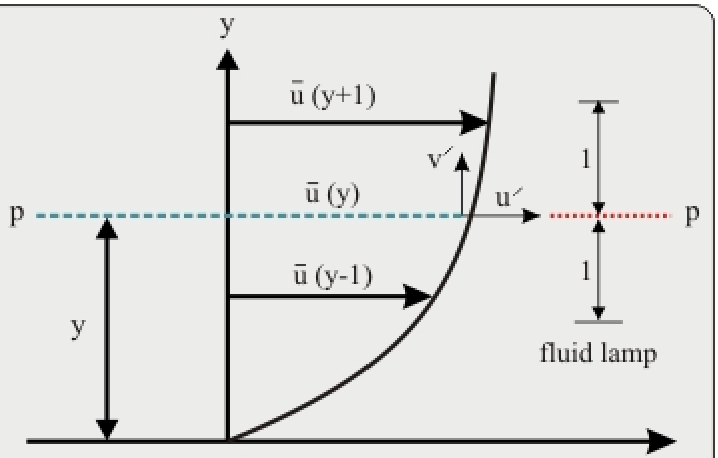

Figure 5.6: Mixing length hypothesis

represents the turbulent shear stress at the plane **PP** **(Figure 5.6)** which is $\rho u' v'$ When a turbulent lump movement is in the upward direction $(v' > 0)$ it enters a region of higher **u** and therefore likely to effect a slowing down fluctuation in u' , that is $u' < 0$. A similar argument can be made for $u' < 0$, so that the average turbulent shear stress will be given as

$$\tau_t = -\rho \overline{u' v'} \qquad \textbf{(5.34)}$$

Again,
let us imagine a turbulent fluid lump which is located a distance ***l*** above or below the plane PP. These lumps of fluid move back and forth across the plane and give rise to the eddy or turbulent-shear-stress effect.
At $(y + l)$, the velocity would be

$$\bar{u}(y + l) = \overline{u}\ (y) + l\ \frac{\partial \overline{u}}{\partial y} \qquad \textbf{(5.35)}$$

While at $(y - l)$ we can wirte

$$\bar{u}(y - l) = \overline{u}\ (y) - l\ \frac{\partial \overline{u}}{\partial y} \qquad \textbf{(5.36)}$$

Prandtl postulated that the turbulent fluctuation is proportional to the mean of the above two quantities

$$u' \approx l\frac{\partial \overline{u}}{\partial y} = C_1\ l\frac{\partial \overline{u}}{\partial y} \qquad \textbf{(5.37)}$$

Here *l* is Prandtl's mixing length. This is analogous to mean free path (average distance a particle travels between collisions) in molecular transport problems. Prandtl also postulated that is of the same order of magnitude as ***u*** so that

$$v' = C_2\ l\frac{\partial \overline{u}}{\partial y} \qquad \textbf{(5.38)}$$

Now, the turbulent shear stress could be written as

$$\tau_t = -\rho\overline{u'v'} = \rho l^2\ \left(\frac{\partial \overline{u}}{\partial y}\right)^2 = \rho\nu_t\frac{\partial \overline{u}}{\partial y} \qquad \textbf{(5.39)}$$

The constants **C1 and C2** can be included in still unknown mixing length, *l* The eddy viscosity thus becomes,

$$\nu_t = l^2\ \left|\frac{\partial \overline{u}}{\partial y}\right| \qquad \textbf{(5.40)}$$

Consider the boundary layer on a flat plate. The shear stress due to laminar flow is given by $\tau_l = \mu\ \frac{\partial \overline{u}}{\partial y}$. The total **shear stress** is

$$\tau = \tau_l + \tau_t = \rho\ \nu\ \frac{\partial \overline{u}}{\partial y} + \rho\nu_t\ \frac{\partial \overline{u}}{\partial y} \qquad \textbf{(5.41)}$$

or

$$\tau = \rho(\nu + \nu_t)\ \frac{\partial \overline{u}}{\partial y} \qquad \textbf{(5.42)}$$

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The turbulent viscosity,in **Equation (5.42)** can be determined from **Equation (5.40).**
However, our problem is still not resolved.

How to determine the value of *l*, the mixing length? Several correlations , using experimental results for νt have been proposed to determine l .In the regime of isotropic turbulence,the most widely used value of mixing length is.

$$l = \chi y$$

where **y** is the distance from the wall and χ is known as **von Karman** constant.

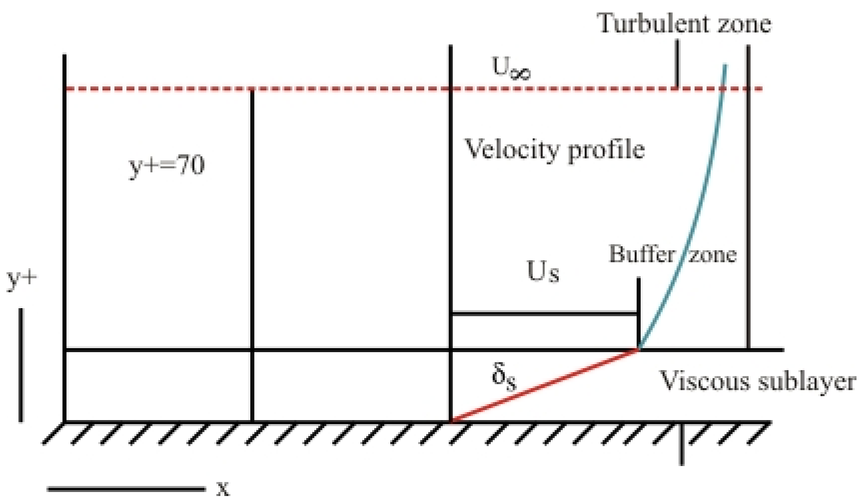

Figure 5.7: Different zones of a turbulent flow past a wall

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Universal Velocity Profile (On a Flat Plate)

A very thin layer next to the wall behaves like a near wall region of laminar flow.
The layer is known as laminar sub-layer and its velocities are such that the viscous forces dominate over the inertial forces **(Figure 5.7)**. No turbulence exists in it. As it has been stated that in the near wall region,
inertial effects are insignificant and we can write

$$\nu \frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial\, \overline{u'v'}}{\partial y} = 0 \qquad \textbf{(5.44)}$$

Which can be integrated as

$$\nu \frac{\partial \overline{u}}{\partial y} - \overline{u'v'} \;=\; \text{constant} \qquad \textbf{(5.45)}$$

Again, as we know that the fluctuating components vanish near the wall, the shear stress on the wall is purely viscous and it follows:

$$\nu \frac{\partial \overline{u}}{\partial y} = \frac{\tau_w}{\rho} \qquad \textbf{(5.46)}$$

or

$$\frac{\overline{u}-0}{y-0} = \frac{\tau_w}{\rho\,\nu} = \frac{u_\tau^2}{\nu} \quad \text{where} \quad u_\tau = \sqrt{\frac{\tau_w}{\rho}} \qquad \textbf{(5.47)}$$

The quantity **u** is known as friction velocity.
From the aforesaid expression,
It is possible to write

$$y\frac{u_\tau}{\nu} = \frac{\overline{u}}{u_\tau} \qquad \textbf{(5.48)}$$

Hence a nondimensional coordinate may be defined as $y^+ = yu_\tau/\nu$ and we write down the variation of the nondimensional velocity within the sublayer as

$$\overline{u}^+ = y^+ \qquad \textbf{(5.49)}$$

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In the turbulent zone, the turbulent shear stress from Prandtl's mixing length model can be written as

$$\tau_t = \rho\ l^2 \left[\frac{\partial \overline{u}}{\partial y}\right]^2 \qquad \textbf{(5.50)}$$

l is the **mixing length** χ_y and χ and is a von Karman constant we write

$$\tau_t = \rho\ l^2 \left[\frac{\partial \overline{u}}{\partial y}\right]^2 \qquad \textbf{(5.51)}$$

or

$$\left[\frac{\partial \overline{u}}{dy}\right]^2 = \left[\frac{u_\tau}{\chi y}\right]^2 \qquad [\text{since} \quad \tau_t = \tau_w] \qquad \textbf{(5.52)}$$

or

$$\overline{u} = \frac{u_\tau}{\chi} \quad \ln \quad y + c \qquad \textbf{(5.53)}$$

or

$$\frac{\overline{u}}{u_\tau} = \frac{1}{\chi}\ [\ln\ y\ -\ \ln\ y_0] \qquad \left[y_0 \ \text{ being very small} = \frac{\beta \nu}{u_\tau}\right] \qquad \textbf{(5.54)}$$

$$\overline{u}^+ = A_1 \ \ln y^+ \ + D_1 \qquad \left[\text{put } A_1 = \frac{1}{\chi}\right] \qquad (5.56)$$

These constants were determined from experiments.
For smooth ducts,
it has been observed $A_1 = 2.5 \text{ and } D_1 = 5.5$

$$\overline{u}^+ = 2.5 \ln y^+ + 5.5 \qquad (5.57)$$

or $\overline{u}^+ = 2.5 \ \ln \left[E y^+\right] \text{ where E} = 9.0$ for smooth walls

Finally the log-law is defined as.

$$\overline{u}^+ = \frac{1}{\chi} \ \ln \ \left[E y^+\right] \qquad (5.58)$$

The location where log profile and linear profile meet

$$\underbrace{\overline{u}^+ = \frac{1}{\chi} \ \ln \ \left[E y^+\right]}_{\text{log-law}} \qquad (5.59)$$

$$y^+ = \frac{1}{\chi} \ \ln \left[E y^+\right] \Rightarrow y^+ = 11.63 \qquad (5.60)$$

In terms of ***y+*** the boundary layer **thickness is around 1500.**

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