Module 1: Preliminiary Concepts and Basics Equations

Lecture 4: Boundary Layers

The Lecture Contains:

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Velocity Boundary Layer :

Transition of zero velocity at the surface to the freestream velocity U_∞ takes place through a very thin layer δ With the increase in y from the surface, the x velocity component, u, must increase until it approaches U_∞.The quantity is termed as boundary layer thickness **(Figure 1.3)** and it is formally defined as the value of y for which u $= 0.99\, U_\infty$.The flow field has two regions **:**

(a) The region where $(\partial u/\partial y)$ and consequent shear stress is significant.
(b) The region where $(\partial u/\partial y)$ and shear stress is negligible .

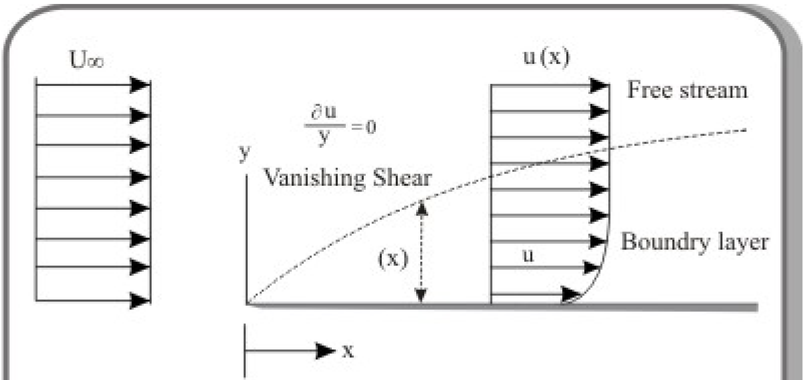

Figure 1.3: Velocity Boundary Layer on a Flat Plate

Here $(\partial u/\partial y)$ determines the local friction coefficient, and $\delta_x = 5.0x/\sqrt{Re_x}$

$$C_f = \tau_w/\frac{1}{2}\rho U_\infty^2 \quad \textbf{(1.88)}$$

$$\tau_w = \mu\frac{\partial u}{\partial y}|_{y=0} \quad \textbf{(1.89)}$$

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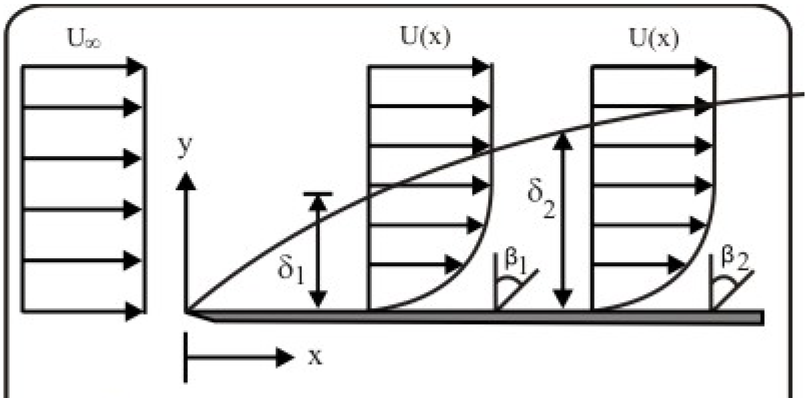

Figure 1.4: Change of Slope in the Downstream

Thermal Boundary Layer :

Just as the velocity boundary layer, a thermal boundary layer should develop if the temperatures at the fluid free stream and the solid-surface differ. Fluid particles that come into contact with the plate achieve thermal equilibrium of the plate's surface temperature and the temperature gradient develops in the fluid. The region of the fluid in which these temperature gradients exist, is thermal boundary layer **(Figure1.5)**. We can write $\delta_T \equiv$ value of y for which, $(T_w - T)/(T_w - T_\infty) = 0.99$

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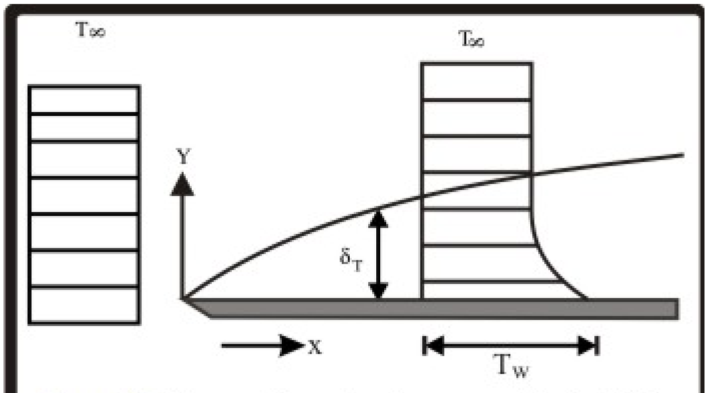

Figure 1.5: Thermal Boundary Layer on a Heated Plate

At any distance from leading edge, $\dot{q_x} = -k_f \frac{\partial T}{\partial y}|_{y=0}$, because at the surface, energy transfer is through conduction.The expression is exact because at the surface, there is no fluid motion. So,we can write :

$$h(T_w - T_\infty) = -k_f \frac{\partial T}{\partial y}|_{y=0} \quad \textbf{(1.90)}$$

$$h = \frac{-k_f \left(\frac{\partial T}{\partial y}\right)|_{y=0}}{(T_w - T_\infty)} \quad \textbf{(1.91)}$$

Hence, the conditions in the thermal boundary layer which strongly inuence the wall temperature gradient $\left(\frac{\partial T}{\partial y}|_{y=0}\right)$, determine the rate of heat transfer across the boundary layer. Since $(T_w - T_\infty)$ is a constant **(independent of x),** and δ_T increases with increasing **x**,temperature gradient in the boundary layer must decrease with increasing **x (Figure 1.6)**. This means $\frac{\partial T}{\partial y}|_{y=0}$ decreases with increase in **x,** whereby $\dot{q_x}$ and h also decrease with increase in **x.**

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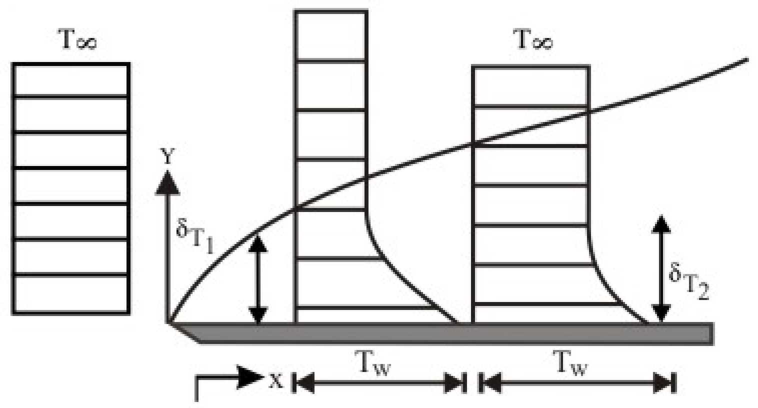

Figure 1.6: Change of Temperature Gradient within the Thermal Boundary Layer

More about Velocity Boundary Layer and Thermal Boundary Layer :

Navier-Stokes equation along with the equation of continuity:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad \textbf{(1.92)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \textbf{(1.93)}$$

In the region, very near to the wall, fluid motion is retarded until it adheres to the surface. The transition of main-stream velocity from zero at the surface to full magnitude takes place through the boundary layer. Its thickness is δ which is function of **x**.

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In order to find out velocity distribution in the field and velocity gradients at the wall, full **Navier-Stokes equations** should be solved.This is almost impossible analytically and can be solved only by numerical techniques.

Easier approach would be to divide the flow field in two regions, - inviscid and viscous **(boundary layer zone).** The inviscid flow is irrotational in this case. Here we solve different governing equations in the inviscid zone and the boundary layer zone.

To solve the governing equation in the boundary layer zone, boundary layer equation should be derived.

Flow over a flat plate (steady)

Let the dimensionless variables be defined as the following :
The equations (1.92) - (1.94) become

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{Re}\left[\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right] \quad \textbf{(1.95)}$$

$$u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}=-\frac{\partial p^*}{\partial y^*}+\frac{1}{Re}\left[\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right] \quad \textbf{(1.96)}$$

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0 \quad \textbf{(1.97)}$$

Following are the scales for the boundary layer variables:

Variable	Dimensional scale	Nondimensional scale
u	U_∞	1
x	L	1
y	δ	$\epsilon=(\delta/L);\epsilon<<1$

Consider, the continuity equation: $\partial u^*/\partial x^* \sim O(1) \Rightarrow \partial v^*/\partial y^*$ should be O(1). [We are not allowed to drop any term from the continuity equation, we do not allow accumulation or annihilation of mass]. Now, v^* has to be of order ϵ because y^* at its maximum is ϵ $(\delta/L<<1)$.

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Take a recourse to the **Navier-Stokes equations :**

x momentum :

$$\begin{aligned} u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} &= -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re}\left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right] \\ (1)\frac{(1)}{(1)} \quad (\epsilon)\frac{(1)}{(\epsilon)} &= \quad (1) \quad (\epsilon^2)\left[\frac{(1)}{(1)} \quad \frac{1}{(\epsilon^2)}\right] \end{aligned} \tag{1.98}$$

y momentum :

$$\begin{aligned} u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} &= -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re}\left[\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right] \\ (1)\frac{(\epsilon)}{(1)} \quad (\epsilon)\frac{(\epsilon)}{(\epsilon)} &= \quad (?) \quad (\epsilon^2)\left[\frac{(\epsilon)}{(1)} \quad \frac{\epsilon}{(\epsilon^2)}\right] \end{aligned} \tag{1.99}$$

Continuity :

$$\begin{aligned} \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} &= 0 \\ \frac{(1)}{(1)} \quad \frac{(\epsilon)}{(\epsilon)} & \end{aligned} \tag{1.100}$$

Within the boundary layer $I_f \sim V_f$ **(Inertia force = Viscous force)**

$$\rho\frac{U_\infty U_\infty}{L} \approx \mu\frac{U_\infty}{\delta^2} \tag{1.101}$$

or ,

$$\frac{\rho U_\infty L}{\mu} \approx \frac{L^2}{\delta^2} \qquad \Rightarrow \qquad Re \approx \frac{1}{\epsilon^2} \tag{1.102}$$

After performing order of magnitude analysis **x-momentum** equation can be re-written based on order of magnitude approximation **(retaining the terms of order 1)**

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re}\left[\frac{\partial^2 u^*}{\partial y^{*2}}\right] \tag{1.103}$$

and

(1.104)

$$\frac{\partial p^*}{\partial y^*} = O(\epsilon)$$

The meaning of **Equation (1.104)** is the following. There is no variation in pressure in **y** direction within the boundary layer. Pressure is impressed on the boundary layer by the outer flow. The pressure, **p** is only a function of **x** within the boundary layer.

At the outer edge of the boundary layer if we substitute **x-momentum** equation, we shall obtain :

$$U^* \frac{dU^*}{dx^*} = -\frac{dp^*}{dx^*} \quad \textbf{(1.105)}$$

In dimensional form :

$$U \frac{dU}{dx} = -\frac{1}{\rho} \frac{dp}{dx} \quad \textbf{(1.106)}$$

or,

$$p + \frac{1}{2}\rho U^2 = \text{constant} \quad \textbf{(1.107)}$$

Boundary conditions:

$$\begin{aligned} &\text{at} \quad y = 0,\ u = 0 = v \\ \text{or,}\ &\text{at} \quad y^* = 0,\ u^* = 0 = v^* \\ &\text{at} \quad y = \delta,\ u = U_\infty(x) \\ \text{or,}\ &\text{at} \quad y^* = \epsilon,\ u^* = 1 \end{aligned} \quad \textbf{(1.108)}$$

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Derivation of the equation for Thermal Boundary layer :

$$Pr = \frac{\mu c_p}{k} = \frac{\mu}{\rho}\,\frac{\rho c_p}{k} = \frac{\nu}{\alpha} = \frac{\text{kinematic viscosity}}{\text{thermal diffusivity}} \sim \left(\frac{\delta}{\delta_T}\right) \qquad \textbf{(1.1.09)}$$

The energy equation, for the incompressible flows can be written as :

$$\begin{array}{ccc} u^*\dfrac{\partial\theta}{\partial x^*} + v^*\dfrac{\partial\theta}{\partial y^*} & = & \dfrac{1}{Re\ Pr}\left[\dfrac{\partial^2\theta}{\partial x^{*2}} + \dfrac{\partial^2\theta}{\partial y^{*2}}\right] \\ (1)\dfrac{(1)}{(1)} \quad (\epsilon)\dfrac{(1)}{(\epsilon)} & = & \dfrac{(\epsilon^2)}{1}\left[\dfrac{(1)}{(1)} \quad \dfrac{1}{(\epsilon^2)}\right] \end{array} \qquad \textbf{(1.110)}$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}$$

$$T_{min} = T_\infty, \quad T_{max} = T_w \quad \text{and} \quad 0 < \theta < 1 \qquad \textbf{(1.111)}$$

The Prandtl number varies in a wide range from value of **order of 0.01** for liquid metals to a value of order of **1000** for viscous oils. Simplications are possible for very small or very large Prandtl numbers. We shall avoid such simplications in order to keep the boundary layer equations general. However, Re is always large in our consideration. We get finally..

$$u^*\frac{\partial\theta}{\partial x^*} + v^*\frac{\partial\theta}{\partial y^*} = \frac{1}{Re\ Pr}\left[\frac{\partial^2\theta}{\partial y^{*2}}\right] \quad \text{(nondimensional)} \qquad \textbf{(1.112)}$$

or

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} \quad \text{(dimensional)} \qquad \textbf{(1.113)}$$

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Boundry conditions:

$$at\ y = 0, T = T_\omega\ \ or\ at\ y^* = o, Q = 1$$

$$at\ y = \delta_T, T = T_\infty,\ or\ at\ y^* = \varepsilon, Q = 1$$

Physical significance is $\frac{\partial T}{\partial y} >> \frac{\partial T}{\partial x}$, axial conduction in fluids much less than that of the transport(difusion) rate in the normal direction. Please note that $\frac{\partial T}{\partial x}$ cannot be zero as long as heat transfer is taking place. However, $\frac{\partial T}{\partial y} >> \frac{\partial T}{\partial x}$.

$Nu = (hL/k)$= Nusselt number = Dimensionless temperature gradient at the surface.

$Nu = Nu(u^*, v^*, x^*, y^*, Re, Pr)$ for forced convection in the absence of dissipation and volumetric heat generation.

This can also be shown that the non-dimensional form of the thermal boundary layer equation becomes :

$$u^*\frac{\partial \theta}{\partial x^*} + v^*\frac{\partial \theta}{\partial y^*} = \frac{1}{RePr}\left[\frac{\partial^2 \theta}{\partial y^{*2}}\right] + \frac{Ec}{Re}\left[2\left(\frac{\partial u^*}{\partial y^*}\right)^2\right] \quad \textbf{(1.114)}$$

In the presence of viscous dissipation and the subsequent dimensional form can be written as :

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left[\frac{\partial^2 T}{\partial y^2}\right] + \mu\left[2\left(\frac{\partial u}{\partial y}\right)^2\right] \quad \textbf{(1.115)}$$

In the above analysis, has been considered as the leading representative term of the viscous dissipation function.

$$Nu = Nu(u^*, v^*, x^*, y^*, Re, Pr, Ec)$$

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