Module 6: Free Convections

Lecture 25: Buoyancy Driven Flows

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Free Convection

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Introduction

Now we shall consider situations for which there is no forced velocity,
yet convection currents exist within the fluid.
They originate when a body force acts on a fluid in which there are density gradients. The net effect is a buoyancy force, which induces fluid velocity. The density gradient is due to a temperature gradient, and the body force is due to the gravitational field. The book by Gebhart, **Jaluria et al.(1988)** provides excellent insight in to natural convective flows and its importance.

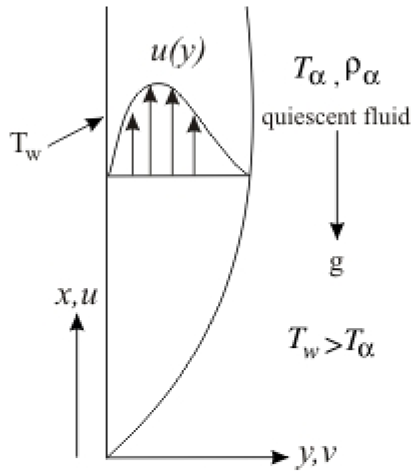

Figure 6.1: Boundary layer development on a heated vertical plate

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Free convection over a vertical flat plate

We focus on a classical example of boundary layer development on a heated vertical plate $T_w > T_\infty$. The fluid close to plate is less dense than fluid that is further removed.

Buoyancy forces therefore induce a free convection boundary layer in which the heated fluid rises vertically,entraining fluid from quiescent region. The resulting velocity distribution is unlike that associated with forced convection boundary layers.

In particular the velocity is zero as $y \to \infty$ as well as at

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}-\rho g+\mu\frac{\partial^2 u}{\partial y^2}$$

outside the boundary layer, we can write

$$\frac{\partial p}{\partial x}=-\rho_\infty g$$

substituting **(6.2) into (6.1),** we obtain the following equation.

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=(\rho_\infty-\rho)g+\mu\frac{\partial^2 u}{\partial y^2}$$

Introducing the volume coefficient of expansion $\beta=\frac{1}{\forall}\left(\frac{\partial\forall}{\partial T}\right)_p,\ \forall\equiv$ volume,we can write

$$\beta=\frac{1}{\forall_\infty}\left(\frac{\forall-\forall_\infty}{T-T_\infty}\right)=\frac{\rho_\infty}{1}\left(\frac{\rho_\infty-\rho}{\rho\rho_\infty}\right)\frac{1}{T-T_\infty}$$

$$\beta=\frac{(\rho_\infty-\rho)}{\rho(T-T_\infty)}=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p$$

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From **(6.3) and (6.4)** we can write

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\beta g\rho(T-T_\infty)+\mu\frac{\partial^2 u}{\partial y^2} \qquad \textbf{(6.5)}$$

Like forced flow boundary layer, y momentum equation is $\frac{\partial p}{\partial y}=0$

The set of governing equation is now

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \textbf{(6.6)}$$

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=g\beta(T-T_\infty)+\nu\frac{\partial^2 u}{\partial y^2} \qquad \textbf{(6.7)}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \qquad \textbf{(6.8)}$$

with $\beta=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p=\frac{1}{\rho}\left(\frac{p}{RT^2}\right)=\frac{1}{T}$

[this might be needed for calculating] **(for perfect gas)**

Boundary condition $@\ \ y=0: u=0, v=0\ ;\ @\ \ y=\delta: u=0, \frac{\partial u}{\partial y}=0$

$@\ \ y=0: T=T_w;\ @\ \ y=\delta: T=T_\infty \text{ and } \frac{\partial T}{\partial y}=0$

Integration (6.7) within the boundary layer yields

$$\int_o^{\delta}\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]dy=\int_o^{\delta}g\beta(T-T_\infty)dy+\int_o^{\delta}\nu\frac{\partial^2 u}{\partial y^2}dy$$

$$\int_o^{\delta}\left[\frac{\partial u^2}{\partial x}+\frac{\partial}{\partial y}(uv)\right]dy=g\beta\int_o^{\delta}(T-T_\infty)dy-\nu\left.\frac{\partial u}{\partial y}\right|_{y=0}$$

or

$$\frac{d}{dx}\int_o^{\delta}u^2dy+[uv]_o^{\delta}=g\beta\int_o^{\delta}(T-T_\infty)dy-\nu\left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$\frac{d}{dx}\int_o^{\delta}u^2dy=g\beta\int_o^{\delta}(T-T_\infty)dy-\nu\left.\frac{\partial u}{\partial y}\right|_{y=0} \qquad \textbf{(6.9)}$$

integration of **equations (6.8)** yields

$$\int_o^{\delta}\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)dy=\int_o^{\delta}\alpha\frac{\partial^2 T}{\partial y^2}dy$$

or ,

$$\int_o^{\delta}\left(u\frac{\partial T}{\partial x}\right)dy+\int_o^{\delta}T\frac{\partial u}{\partial x}dy-T_\infty\int_o^{\delta}\frac{\partial u}{\partial x}dy=-\alpha\left.\frac{\partial T}{\partial y}\right|_{y=0}$$

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(Invoking $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$ from equation (6.6))

$$\int_o^\delta \frac{\partial}{\partial x}\{u(T-T_\infty)\}\,dy = -\alpha \left.\frac{\partial T}{\partial y}\right|_{y=0} \qquad \textbf{(6.10)}$$

$$\frac{d}{dx}\int_o^\delta u(T-T_\infty)dy = -\alpha \left.\frac{\partial T}{\partial y}\right|_{y=0}$$

Defining $\theta = T - T_\infty$ and substituting this in **equation (6.10),** yields

$$\frac{d}{dx}\int_o^\delta u\theta dy = -\alpha \left.\frac{\partial \theta}{\partial y}\right|_{y=0} \qquad \textbf{(6.11)}$$

The temperature distribution may be assumed as

$$\frac{\theta}{\theta_m} = \frac{T-T_\infty}{T_w - T_\infty} = \left(1 - \frac{y}{\delta}\right)^2 \qquad \textbf{(6.12)}$$

This satisfies
$@\ y = 0, T = T_w$ i.e., $@\ y = 0, \theta = \theta_m$
$@\ y = \delta, T = T_\infty$ i.e., $@\ y = \delta, \theta = 0$
$@\ y = \delta, \frac{\partial T}{\partial y} = 0$ i.e., $@\ y = \delta,\ \frac{\partial}{\partial y}\left[(T_w - T_\infty)\left(1 - \frac{y}{\delta}\right)^2\right] = 0$

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The velocity in the boundry layer is given by

$$u = a + by + cy^2 + dy^3 \qquad \textbf{(6.13)}$$

where u_x is some arbitrary function of ***x***.

The boundary conditions are

$$u \text{ @ } y = 0, u = 0; \text{ @ } y = \delta, u = 0$$
$$\text{@} y = 0, \nu \frac{\partial^2 u}{\partial y^2} = -\beta g (T_w - T_\infty); \text{ @ } y = \delta, \frac{\partial u}{\partial y} = 0$$

These boundary conditions will finally produce

$$a = 0, \quad b = \frac{\delta \beta g}{4\nu} (T_w - T_\infty), \quad c = -\frac{\beta g}{2\nu} (T_w - T_\infty), \quad d = \frac{\beta g}{4\nu\delta} (T_w - T_\infty)$$

Substituting these values in the polynomial we get

$$u = \frac{g\beta\delta^2 \, (T_w - T_\infty)}{4\nu} \frac{y}{\delta} \left(1 - \frac{y}{\delta}\right)^2 \qquad \textbf{(6.14)}$$

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The terms $(T_w - T_\infty)$, δ^2, $\frac{g\beta}{4\nu}$ can be included in u_x so that we can write

$$\frac{u}{u_x} = \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 \qquad \textbf{(6.15)}$$

The maximum velocity and its position ***(distance in y direction)*** at any **x** can be obtained from **equation (6.15)**

$$\begin{aligned} &\frac{d}{dy}(u) = \frac{d}{dy}\left[u_x\left\{\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2\right\}\right] = 0 \\ &\text{or,}\quad \frac{d}{dy}\left[\frac{y}{\delta}\left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right)\right] = 0 \end{aligned} \qquad \textbf{(6.16)}$$

$$\begin{aligned} &\text{or,}\quad \frac{1}{\delta} - \frac{4y}{\delta^2} + \frac{3y^2}{\delta^3} = 0 \\ &\text{or,}\quad \delta^2 - 4\delta y + 3y^2 = 0 \\ &\text{or,}\quad \delta^2 - 3\delta y - \delta y + 3y^2 = 0 \\ &\text{or,}\quad (\delta - y)\,(\delta - 3y) = 0 \end{aligned} \qquad \textbf{(6.17)}$$

but $@\ y = \delta$ velocity is zero

Therefore $@\, y = \frac{\delta}{3}$, the velocity is maximum $\frac{U_{max}}{U_x} = \frac{1}{3}\left[1 - \frac{1}{3}\right]^2$ or, $U_{max} = \frac{4}{27}u_x$

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Energy equation (6.11) can be now written as

$$\frac{d}{dx}\int_o^\delta \theta_m \left(1-\frac{y}{\delta}\right)^2 u_x \frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2 dy = -\alpha\,\theta_m \frac{\partial}{\partial y}\left[\left(1-\frac{y}{\delta}\right)^2\right]\Bigg|_{y=0}$$

$$\frac{1}{30}\frac{d}{dx}(u_x\,\delta)\,\theta_m = \frac{2\alpha\,\theta_m}{\delta} \qquad \textbf{(6.18)}$$

The momentum **equation (6.9)** may be written as

$$\frac{d}{dx}\int_o^\delta \left[u_x \frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2\right]^2 dy = g\beta\int_o^\delta \theta_m\left(1-\frac{y}{\delta}\right)^2 dy - \nu u_x \frac{d}{dy}\left[\frac{y}{\delta}\left(1-\frac{y}{\delta}\right)^2\right]_{y=0}$$

or,

$$\frac{1}{105}\frac{d}{dx}(u_x^2\,\delta) = \frac{g\,\beta\,\theta_m\,\delta}{3} - \frac{\nu u_x}{\delta} \qquad \textbf{(6.19)}$$

In **equation (6.18) and (6.19),** and u_x are dependent variables and ***x*** is the independent variable.
To solve the above two equations ,u_x and δ are considered as function of ***x***

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Assume u_x and δ very according to the following functions **:-**

$$u_x = C_1 x^m \ \text{and} \ \ \delta = C_2 x^n$$

Substitution in **(6.18)** will produce

$$\frac{m+n}{30}\, C_1\, C_2\, (x)^{m+n-1} = \frac{2\alpha}{C_2}\, x^{-n} \qquad \textbf{(6.20)}$$

Substitution in **(6.19)** will produce

$$\frac{2m+n}{105}\, C_1^2\, C_2\, (x)^{2m+n-1} = \frac{1}{3}\, g\, \beta\, \theta_m\, C_2 x^n - \frac{\nu C_1 x^{m-n}}{C_2} \qquad \textbf{(6.21)}$$

The equations have to be treated in the following way.
First let us consider **equation (6.20)**. If a monomial nonlinear expression is dependent on a variable (say x) that is equivalent to an expression independent of ***x***, then for universal validity of the equation as a function of *x*,the power of the nonlinear expression must vanish.If we divide entire **equation (6.20)** by its right hand side, then for the universal validity of the expression,
we can say

$$m + 2n - 1 = 0$$

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Now, let us look at **equation (6.21)**. Any number of non-zero degree can combine to result into one term, provided each of the terms has same degree. The degree of the resulting term will also be equal to each of the terms. This will lead to

$$2m + n - 1 = n \quad \text{and} \quad 2m + n - 1 = m - n$$

$$2m - 1 = 0 \quad \text{and} \quad m + 2n - 1 = 0$$

The conclusion derived from **equation (6.20)** and one of the two-conclusions derived from **equation (6.21)** is same. Now by solving these, we get

$$m = 1/2 \qquad \text{and} \qquad n = 1/4$$

Substituting the values of m and n in **(6.20) and (6.21)**, we get

$$C_1 C_2^2 = 80\alpha \tag{6.22}$$

$$\frac{C_1^2 C_2}{84} = \frac{1}{3} g\,\beta\,\theta_m C_2 - \frac{C_1}{C_2}\,\nu \tag{6.23}$$

Solving the above two equations, we get

$$C_1 = 5.17\,\nu \left[\frac{\nu}{\alpha} + \frac{20}{21}\right]^{-1/2} \left(\frac{g\,\beta\,\theta_m}{\nu^2}\right)^{1/2} \tag{6.24}$$

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$$\text{and,}\quad C_2 = 3.93\left(\frac{\nu}{\alpha}\right)^{-1/2}\left[\frac{\nu}{\alpha}+\frac{20}{21}\right]^{1/4}\left(\frac{g\,\beta\,\theta_m}{\nu^2}\right)^{-1/4} \qquad \textbf{(6.25)}$$

Substituting C_1 and C_2 for evaluating u_x and δ and substituting for $\Pr = \nu/\alpha$

$$u_x = (5.17)\,\nu\left(Pr+\frac{20}{21}\right)^{-1/2}\left(\frac{g\,\beta\,\theta_m}{\nu^2}\right)^{1/2} x^{1/2}$$

$$\delta = 3.93\,(Pr)^{-1/2}\left(Pr+\frac{20}{21}\right)^{1/4}\left(\frac{g\,\beta\,\theta_m}{\nu^2}\right)^{-1/4} x^{1/4}$$

or,

$$\frac{\delta}{x} = 3.93\,(Pr)^{-1/2}\left(Pr+\frac{20}{21}\right)^{1/4}\left(\frac{g\,\beta\,\theta_m}{\nu^2}\right)^{-1/4} x^{-3/4}$$

or,

$$\frac{\delta}{x} = 3.93\,(Pr)^{-1/2}\left(Pr+\frac{20}{21}\right)^{1/4}\left(\frac{g\,\beta\,\theta_m x^3}{\nu^2}\right)^{-1/4}$$

Introducing $Gr_x = \text{Grashof number} \equiv \frac{g\,\beta\,\theta_m x^3}{\nu^2}$ Grashof number we get

$$\frac{\delta}{x} = 3.93(Pr)^{-1/2}\left(Pr+\frac{20}{21}\right)^{1/4} Gr_x^{-1/4}$$

$$\frac{\delta}{x} = 3.93\left[\frac{Pr+\frac{20}{21}}{Pr^2\,Gr_x}\right]^{1/4} \qquad \textbf{(6.26)}$$

Equation (6.26) gives the variation of boundary layer thickness along the height of the plate. Here thermal boundary layer thickness and hydrodynamic boundary layer thickness are same.

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