Module 2: External Flows

Lecture 11: Wedge Flows

## The Lecture Contains:


- More about similarity solution
- More of simlarly situation about energy equation
- Effect of Pressure Gradient on External Flows

## More About Similarly Situation

A systematic study of flow past a wedge **(Figure 2.9)** suggests.

$$\eta = Ayx^a \quad \text{and} \quad \frac{u}{U} = f'(\eta) \qquad \textbf{(2.119)}$$

$\psi$ is obtained by integration as $\psi = \frac{1}{A}x^{-a}[Uf(\eta)]$.The momentum equation for a flat plate boundary layer reduces to :

$$\frac{\nu A^2}{U}x^{2a}f''' - \frac{a}{x}ff'' = 0 \qquad \textbf{(2.120)}$$

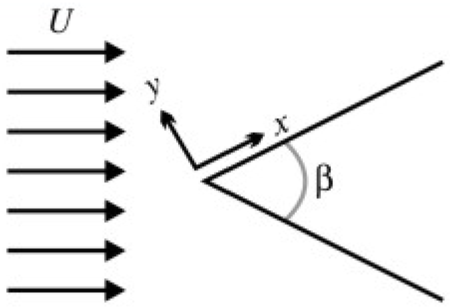


Figure 2.9: Flow past a wedge

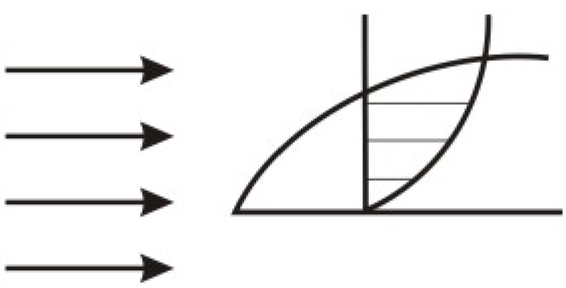

Figure 2.10: Flow past a flat plate

Choosing $a = -1/2$ cancels $x$ throughout the equation and choosing $A = \sqrt{U/\nu}$ keeps the equation free of the flow and fluid parameters Similarity solution can be obtained for a wider class of problems where $U = Cx^m$. This form of U represents flow past wedge shaped surface as shown above **(Figure 2.9)**

The relationship between m and $\beta$ is $m = \frac{\beta/\pi}{2-\beta/\pi}$ for $\beta = 0$ and $m = 0$ and the flat plate problem is recovered **(Figure 2.10)** for $\beta = \pi, m = 1$ and this is stagnation point flow **(Figure 2.11)**.The pressure field is. $(x) = \beta - \frac{1}{2}\rho U^2(x)$ hence the pressure gradient.

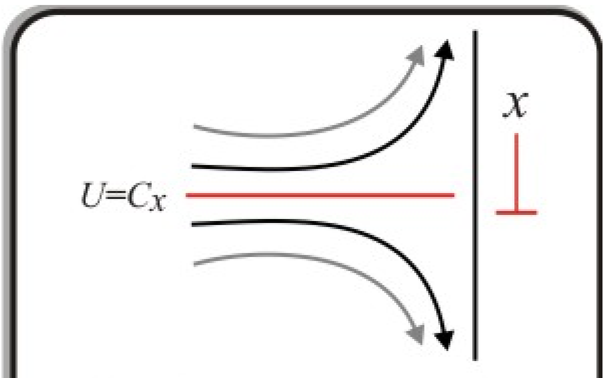


Figure 2.11: Stagnation point flow

$$\frac{dp}{dx} = -\rho U\frac{dU}{dx} = -\rho UCmx^{m-1} = -\frac{\rho U^2 m}{x} \qquad \textbf{(2.121)}$$

And the x momentum equation becomes :

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{mU^2}{x} + \nu\frac{\partial^2 u}{\partial y^2},\ U = Cx^m$$

Note that **x** and **y** do not refer to Cartesian coordinates any more. Instead they are boundary layer coordinates, parallel and perpendicular to the solid surface. The similarly variable $\eta = y\sqrt{U/\nu x}$ and the stream function $\psi = \sqrt{\nu x U}f(\eta)$ reduce the momentum equation to the two point boundary value problem .

$$f''' + \frac{(m+1)}{2}ff'' + m[1 - f'^2] = 0 \qquad \textbf{(2.122)}$$

With $f(0) = f'(0) = 0,\ f'(\infty) = 1$

The equation for **f** with m as a wedge parameter is called as the **Falkner-Skan equation**. Similarity solution can be obtained for a wide class of problem where $U = Cx^m$. This form of U represents flow past wedge shaped surfaces. The relationship between β and m is given by

$$m = \frac{\beta}{2\pi - \beta} = \text{wedge parameter}$$

$$m(2\pi - \beta) = \beta$$

**m < 0** signifies adverse pressure gradient. **Angle β** is negative and **U** is not a constant

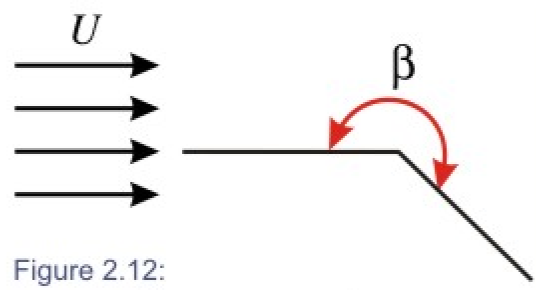


Figure 2.12:
Flow over a wedge

**for wedge flow (Figure 2.12 )**

we have $U = C\,x^m,\ \frac{dU}{dx} = C\,m\,x^{m-1}$

Therefore $\frac{x}{U}\frac{dU}{dx} = \frac{x}{U}\,C\,m\,x^{m-1} = \frac{x}{C\,x^m}C\,m\,x^{m-1} = m$

Also we have

$$\eta = yh(x), \quad \psi(x,\eta) = g(x)f(\eta)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad \textbf{(2.123)}$$

Here $-\frac{1}{\rho}\ \frac{\partial p}{\partial x}$ becomes $U\frac{dU}{dx}$ for wedge flows. We know that .

$$u = \left.\frac{\partial \psi}{\partial y}\right|_x \qquad v = -\left.\frac{\partial \psi}{\partial x}\right|_y$$

Again, if **(x, y) = fn (χ, η)**, the transformation is :

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \chi}\cdot\frac{\partial \chi}{\partial x} + \frac{\partial}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \chi}\cdot\frac{\partial \chi}{\partial y} + \frac{\partial}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

If **(x, y) = fn (x, η)**, the von Misses transformation is :

$$\left.\frac{\partial}{\partial x}\right|_y = \left.\frac{\partial}{\partial x}\right|_\eta \left(\frac{\partial x}{\partial x}\right)_y + \left.\frac{\partial}{\partial \eta}\right|_x \left(\frac{\partial \eta}{\partial x}\right)_y = \left.\frac{\partial}{\partial x}\right|_\eta + \left.\frac{\partial}{\partial \eta}\right|_x \left(\frac{\partial \eta}{\partial x}\right)_y \qquad \textbf{(2.124)}$$

and

$$\left.\frac{\partial}{\partial y}\right|_x = \left.\frac{\partial}{\partial \eta}\right|_x \left(\frac{\partial \eta}{\partial y}\right)_x + \left.\frac{\partial}{\partial x}\right|_\eta \left.\frac{\partial x}{\partial y}\right|_x = \left.\frac{\partial}{\partial \eta}\right|_x \left.\frac{\partial \eta}{\partial y}\right|_x \qquad \textbf{(2.125)}$$

Subsitutting we get :

$$u = \left.\frac{\partial \psi}{\partial y}\right|_x = \left.\frac{\partial}{\partial \eta}\left[g(x)\ f(\eta)\right]\right|_x \frac{\partial [y\ h(x)]}{\partial y} = f'\,g\,h$$

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$$v = -\left.\frac{\partial\psi}{\partial x}\right|_y = -\left[\frac{\partial}{\partial\eta}\left[g(x)\ f(\eta)\right]\Big|_x \left.\frac{\partial\eta}{\partial x}\right|_y + \left.\frac{\partial(g\ f)}{\partial x}\right|_\eta\right] = -[g\ f^{'} y\ h^{'} + f\ g^{'}]$$

Choose g (x)h (x) = U(x)

$$\frac{\partial u}{\partial x} = \left.\frac{\partial u}{\partial x}\right|_\eta + \left.\frac{\partial u}{\partial \eta}\right|_x \left.\frac{\partial \eta}{\partial x}\right|_y$$

$$= \frac{\partial}{\partial x}[f^{'}gh]\Big|_\eta + \frac{\partial}{\partial\eta}[f^{'}gh]\Big|_x\ yh^{'}$$

$$= f^{'}(gh^{'} + hg^{'}) + f^{''}ghyh^{'}$$

$$\frac{\partial u}{\partial y} = \left.\frac{\partial}{\partial\eta}\right|_x \left.\frac{\partial\eta}{\partial y}\right|_x = \left.\frac{\partial u}{\partial\eta}\right|_x \left.\frac{\partial\eta}{\partial y}\right|_x$$

$$= \frac{\partial}{\partial\eta}(f^{'}gh)\Big|_x\ h = f^{''}gh^2$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial\eta}\left[f^{''}gh^2\right]\Big|_x \left.\frac{\partial\eta}{\partial y}\right|_x = h\ f^{'''}gh^2 = gf^{'''}h^3$$

Consider the **x-momentum equation** :

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$f^{'}gh(f^{''}ghh^{'}y + f^{'}gh^{'} + hf^{'}g^{'}) - [fg^{'} + gf^{'}yh^{'}](f^{''}gh^2) = gh(g^{'}h + gh^{'}) + \nu gf^{'''}h^3 \quad \textbf{(2.126)}$$

After simplification, we get

$$f^{'''} + \frac{[g^{'}h + gh^{'}]}{\nu h^2}(1 - f^{'2}) + \frac{g^{'}}{\nu h}ff^{''} = 0 \quad \textbf{(2.127)}$$

As **f** is a function of η alone, both coefficients should be constants:

$$\frac{gh^{'} + hg^{'}}{\nu h^2} = C_1 \text{ and } \frac{g^{'}}{\nu h} = C_2$$

$$\frac{1}{\nu h^2}\frac{dU}{dx} = C_1$$

or,

$$\frac{U}{x}\frac{1}{\nu h^2}\left(\frac{x}{U}\frac{dU}{dx}\right) = C_1$$

or ,

$$\frac{U}{x\nu h^2}m = C_1$$

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Now choosing **C1 = m** ,the expression for small h reduces to that of flat plate, that is :

$$h = \sqrt{\frac{U}{\nu x}}$$

Again,

$$\frac{g}{\nu h} = C_2$$

or ,

$$\frac{dg}{dx} = C_2\ \nu\sqrt{\frac{U}{\nu x}} = C_2\sqrt{\frac{\nu\ U}{x}} = C_2\sqrt{\nu\ C\ x^{m-1}}$$

or ,

$$\frac{dg}{dx} = \sqrt{C_2^2\ C}\ \sqrt{\nu}\ (x)^{\frac{m-1}{2}}$$

After integrating, we get

$$g = \frac{C_2}{(m+1)/2}\sqrt{\nu U x}$$

Choosing $C_2 = \frac{m+1}{2}$ we obtain $g = \sqrt{\nu U x}$

The **result is compatible**, because $\psi = g(x) f(\eta) = \sqrt{\nu U x} f(\eta)$

The **final resulting** equation becomes

$$f''' + C_1(1 - f'^2) + C_2 f f'' = 0$$

$$f''' + m(1 - f'^2) + \frac{m+1}{2} f f'' = 0 \qquad \textbf{(2.128)}$$

This equation is called Falkner-Skan Equation. This equation can be solved as three initial value problems for which the boundary conditions are as follows:

$$\begin{aligned} u(0) = 0 \quad &\Rightarrow \quad f'(0) = 0 \\ u(\infty) = U \quad &\Rightarrow \quad f'(\infty) = 1 \\ v(0) = 0 \quad &\Rightarrow \quad f(0) = 0 \end{aligned} \qquad \textbf{(2.129)}$$

◆ The **equation (2.128)** can be solved using **SHOOTING TECHNIQUE**. The solutions for diffrent values of m has been shown in **Figure 2.13**.

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**More on Similarity Solution of Energy equation :**

The two-dimensional energy equation is given as

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2 \qquad \textbf{(2.130)}$$

For similarity solution of this equation put

$$\theta=\frac{T-T_\infty}{T_w-T_\infty}=\theta(\eta)$$

It is known that $\eta=y\sqrt{\frac{U}{\nu x}}; \quad \psi=\sqrt{\nu U x}f(\eta); \quad \frac{u}{U}=f'(\eta), \text{ or, } u=f'gh$

Also

$$v=-\left.\frac{\partial\psi}{\partial x}\right|_y=-\left[\left.\frac{\partial\psi}{\partial x}\right|_\eta+\frac{\partial\psi}{\partial\eta}\,\frac{\partial\eta}{\partial x}\right]=-\left[\sqrt{\nu x}\,\frac{1}{2}\,U^{-1/2}\frac{dU}{dx}f+\sqrt{\nu U}\,\frac{1}{2}\,x^{-1/2}f+\sqrt{\nu U x}\,f'\,\frac{\partial\eta}{\partial x}\right]$$

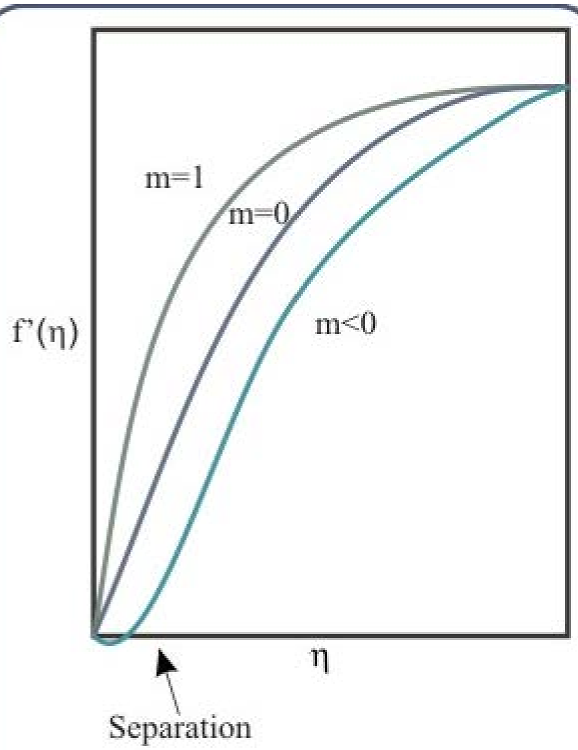


Figure 2.13: Variation of velocity profile for different m

$$= -\left[\sqrt{\frac{\nu U}{x}}\,\frac{f}{2} \;+\; \frac{f}{2}\,\sqrt{\frac{\nu x}{U}}\,\frac{dU}{dx} + \;\sqrt{\nu U x}\;f^{'}\;\frac{d\eta}{dx}\right]$$
$$= -\frac{1}{2}\,\sqrt{\frac{\nu U}{x}}\,(f - \eta f^{'})\; - \frac{f}{2}\,\sqrt{\frac{\nu x}{U}}\,\frac{dU}{dx}$$

Finally, the energy equation becomes

$$U f^{'}\;\left[\theta\,\frac{\partial T_w}{\partial x}\; + (T_w - T_\infty)\,\frac{\partial \theta}{\partial \eta}\,\left(-\frac{y}{2x}\right)\,\sqrt{\frac{U}{\nu x}}\right] + \left\{-\frac{1}{2}\,\sqrt{\frac{\nu U}{x}}\,(f - \eta f^{'})\; - \frac{f}{2}\,\sqrt{\frac{\nu x}{U}}\,\frac{dU}{dx}\right\}$$
$$\left\{(T_w - T_\infty)\,\frac{\partial \theta}{\partial \eta}\,\sqrt{\frac{U}{\nu x}}\right\} = \frac{k}{\rho c_p}\,(T_w - T_\infty)\,\frac{U}{\nu x}\,\theta^{''} + \frac{\mu}{\rho c_p}\,\frac{U^3}{\nu x}\,(f^{''})^2$$

or,

$$\frac{k}{\rho c_p}\,(T_w - T_\infty)\,\frac{U}{\nu x}\,\theta^{''} + \theta^{'}\,\frac{f}{2}\,(T_w - T_\infty)\;\left[\frac{U}{x} + \frac{dU}{dx}\right] - U f^{'}\;\frac{dT_w}{dx}\,\theta + \;\frac{U^3}{c_p x}\,(f^{''})^2 = 0$$

Finally it reduces to the form

$$\frac{\theta''}{Pr} + \frac{f}{2}\,(1+m)\,\theta' \; - x f'\,\theta\,\frac{(dT_w/dx)}{(T_w - T_\infty)} + Ec\,(f'')^2 = 0 \qquad \textbf{(2.131)}$$

It can be solved by the **Method of Separation of Variables**, and so

$$\frac{1}{f'\theta}\left\{\frac{\theta''}{Pr} + \frac{f}{2}\,(1+m)\,\theta' + Ec(f'')^2\right\} = \frac{x(dT_w/dx)}{(T_w - T_\infty)} = \lambda \qquad \textbf{(2.132)}$$

For m=0,we have at plate problem; $\lambda = 0$ leads to constant wall temperature case and $Ec = 0$ leads to the case without viscous dissipation. For this case, the simplified version of equation becomes.

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$$\frac{\theta''}{Pr} + \frac{1}{2} f\theta' = 0 \qquad \textbf{(2.133)}$$

with $\theta(0) = 1 \text{ and } \theta(\infty) = 0$ For this special case, $\lambda = 0 \Rightarrow T_w =$ constant Then the simplified equation becomes.

$$\theta'' + \frac{Pr}{2}\,f\theta' = 0 \qquad \textbf{(2.134)}$$

For Pr = 1 the system becomes $\theta'' + \frac{1}{2}f\theta' = 0 \text{ and } f''' + \frac{1}{2}ff'' = 0$

The boundary conditions are $f'(0) = 0,\; f'(\infty) = 1 \text{ and } f(0) = 0$

We can solve the energy equation using $f''' + \frac{1}{2}ff'' = 0$

let us say $\tilde{B} = f'$, then $\tilde{B}'' + \frac{1}{2}f\tilde{B}'$ with the boundary conditions $\tilde{B}(0) = 0, \tilde{B}(\infty) =$**1**

Now the energy equation is **:**

$$\theta'' + \frac{1}{2}f\theta' = 0 \quad \text{with} \quad \theta(0) = 1, \theta(\infty) = 0$$

Put $B = (1-\theta)$ the equations becomes

$$-B'' - \frac{1}{2}fB' = 0 \quad \text{with} \quad B(0) = 0, B(\infty) = 1$$

Which gives $\tilde{B} = B$ or, $\theta = (1 - B)$ The solution is plotted in **figure 2.14**

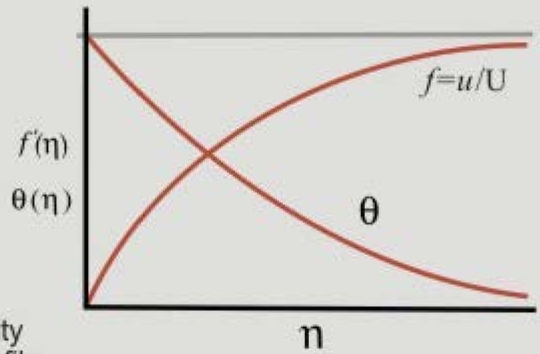


Figure 2.14:
Variation of velocity and temperature profile

## Effect of Pressure Gradient on External Flows :-

For the case of a boundary layer on a flat plate, the pressure gradient of the external stream is zero. Let us consider a body with curved surface **(Figure 2.15).** Upstream of the highest point the streamlines of the outer flow converge, resulting in an increase of the free stream velocity **U(x)** and a consequent fall of pressure with x. Downstream of the highest point the streamlines diverge, resulting in a decrease of U(x) and a rise in pressure. In this section we shall investigate the effect of such a pressure gradient on the shape of the boundary layer profile **u(x, y)**. The boundary layer equation is:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

where the pressure gradient is found from the external velocity field as $dp/dx = -\rho U(dU/dx)$, with **x** taken along the surface of the body. At the wall, the boundary layer equation becomes .

$$\mu\left(\frac{\partial^2 u}{\partial y^2}\right)_w = \frac{\partial p}{\partial x} \qquad \textbf{(2.135)}$$

In an accelerating stream $dp/dx < 0$ **(see Figure 2.15)** and therefore

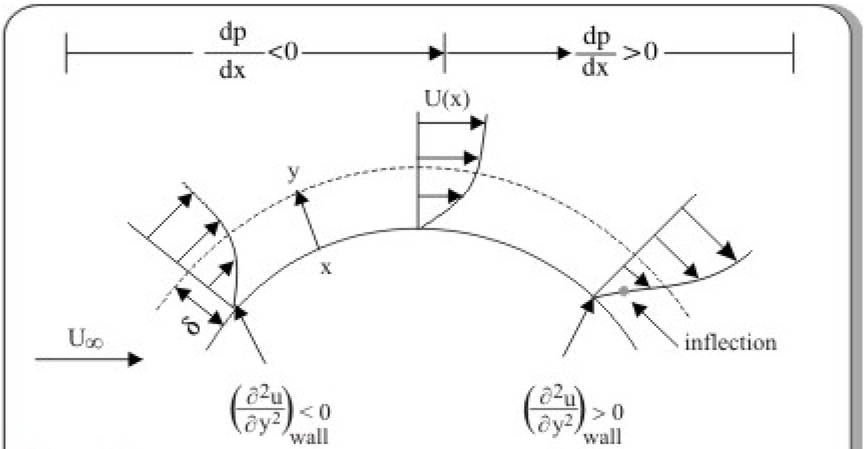


Figure 2.15:

Velocity profile associated with separation in a cross flow over a circular cylinder

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{wall} < 0 \qquad \textbf{(accelerating)} \qquad \textbf{(2.136)}$$

As the velocity profile has to merge smoothly with the external profile,the slope $\partial u/\partial y$ slightly below the edge of the boundary layer decreases with y from a positive value to zero; therefore, $\partial^2 u/\partial y^2$ slightly below the boundary layer edge is negative.**Equation (2.136)** then shows that $\partial^2 u/\partial y^2$ has the same sign at both the wall and the boundary layer edge, and presumably throughout the boundary layer **(Figure 2.16)**.In contrast,for a decelerating external stream, the curvature of the velocity profile at the wall is

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{wall} < 0 \qquad \textbf{(decelerating)} \qquad \textbf{(2.137)}$$

so that the curvature changes sign somewhere within the boundary layer.In other words,the boundary layer profile in a decelerating flow (dp /dx > 0) has a point of inflection where $\partial^2 u/\partial y^2 = 0$ **(Figure 2.17).**The shape of the velocity profiles in the figures suggests that an adverse pressure gradient tends to increase the thickness of the boundary layer. This can also be seen from the continuity equation.

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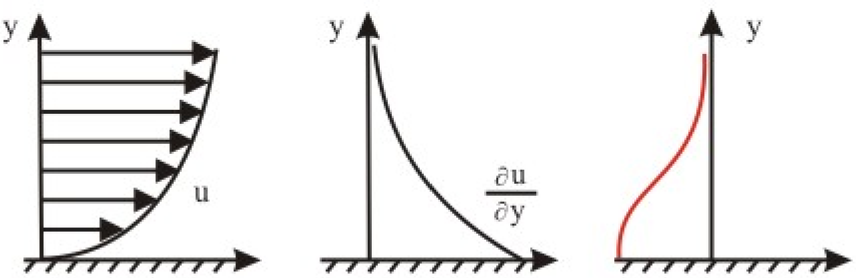


Figure 2.16: Profiles dp /dx < 0

$$v(y) = -\int_0^y \frac{\partial u}{\partial x}\, dy$$

Compared to a flat plate, a decelerating external stream causes a larger $-\partial u/\partial x$ within the boundary layer because the deceleration of the outer flow adds to the viscous deceleration within the boundary layer. From the above equation we observe that the v-field, directed away from the surface, is larger for a decelerating flow.The boundary layer therefore thickens not only by viscous diffusion but also by advection away from the surface, resulting in a rapid increase in the boundary layer thickness with **x**. If **p** falls along the direction of flow, dp / dx <0 and we say that the pressure gradient is ”favorable”. If, on the other hand, the pressure rises along the directions of flow,dp/dx >0 and we say that the pressure gradient is **”adverse”** or **”uphill”**.The rapid growth of the boundary layer thickness in a decelerating stream, and the associated large v-field, causes the important phenomena of separation, in which the external stream ceases to flow nearly parallel to the boundary surface.

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