Module 1:Preliminiary Concepts and Basics Equations

Lecture 3: A Generalized Approach

The Lecture Contains:

Generalized approach for Energy Equation

Let us apply Reynolds transport theorem to an arbitrary control volume **(Figure 1.2)** for the conservation of energy. We can write

Rate of change of energy for the system,
(A) = Rate of change of energy in the CV,
(B) + Rate of efflux of energy through the control surface of the CV,
(C) Applying first law of thermodynamics,we can also write Rate of change of energy for the system
(A) = rate of energy production within the

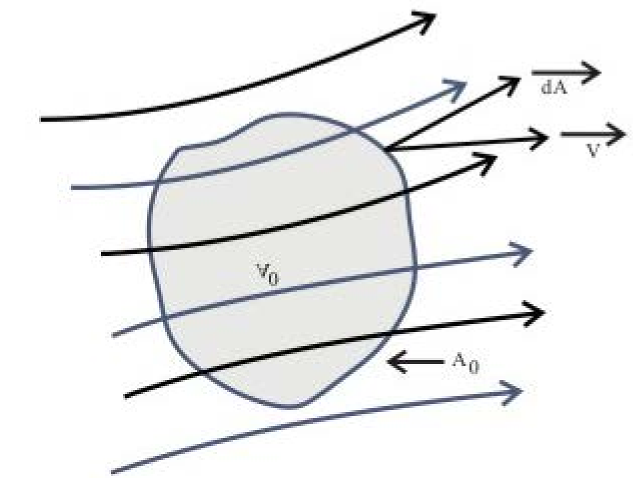

Figure 1.2:
Arbitrary Control Volume Fixed in Space

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system + rate of work done on the system.

We evaluate the quantities B & C in the following manner

$$(B) + (C) = \iiint \left[\frac{\partial}{\partial t}(\rho E) + \nabla \cdot (\rho \mathbf{V} E) - \nabla \cdot (k \nabla T)\right] d\forall$$

Here **E** is defined as $e + V^2/2.$ Now, (B) + (C) can be equated with the rate of energy production with in the system and the rate of work done on the system **:**

$$\iiint \left[\frac{\partial}{\partial t}(\rho E) + \nabla . \{\rho \mathbf{V} E\} - \nabla . (k \nabla T)\right] d\forall = \iiint \left[\rho\, \mathbf{f} . \mathbf{V} + \nabla . (\sigma . \mathbf{V}) + \dot{S}_{th}\right] d\forall \qquad \textbf{(1.51)}$$

For the homogeneous and isotropic medium :

$$\underbrace{\frac{\partial}{\partial t}(\rho E) + \nabla . \{\rho \mathbf{V} E\}}_{I} \underbrace{- k \nabla^2 T}_{II} = \underbrace{\rho \mathbf{f} . \mathbf{V}}_{III} + \underbrace{\nabla . (\sigma . \mathbf{V})}_{IV} + \underbrace{\dot{S}_{th}}_{V} \qquad \textbf{(1.52)}$$

We obtain the following from term I **:**

$$\begin{aligned} \frac{\partial}{\partial t}(\rho E) + \nabla \cdot \{\rho \mathbf{V} E\} &= E \frac{\partial \rho}{\partial t} + \rho \frac{\partial E}{\partial t} + \rho \mathbf{V} . \nabla E + E (\nabla \cdot \rho \mathbf{V}) \\ &= \rho \left[\frac{\partial E}{\partial t} + (\mathbf{V} \cdot \nabla) E\right] + E \left[\frac{\partial \rho}{\partial t} + \nabla . (\rho \mathbf{V})\right] \\ &= \rho \left[\frac{\partial E}{\partial t} + (\mathbf{V} \cdot \nabla) E\right] = \rho \frac{DE}{Dt} \end{aligned} \qquad \textbf{(1.53)}$$

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Term IV produces :

$$\begin{aligned}
\nabla \cdot (\sigma \cdot \mathbf{V}) &= \frac{\partial}{\partial x_i}(\sigma \cdot \mathbf{V})_i \\
&= \frac{\partial}{\partial x_i}(\sigma_{ij} \cdot \mathbf{V_j}) \\
&= \left(\frac{\partial}{\partial x_i}\sigma_{ij}\right)\mathbf{V_j} + \sigma_{ij}\frac{\partial \mathbf{V_j}}{\partial x_i} \\
&= (\nabla \cdot \sigma)_j \mathbf{V_j} + \sigma_{ij}(\nabla \mathbf{V})_{ij} \\
&= (\nabla \cdot \sigma) \cdot \mathbf{V} + \sigma : \nabla \mathbf{V}
\end{aligned} \tag{1.54}$$

Energy equation becomes :

$$\rho \frac{DE}{Dt} = k\nabla^2 T + \rho\, \mathbf{f} \cdot \mathbf{V} + (\nabla \cdot \sigma) \cdot \mathbf{V} + \sigma : \nabla \mathbf{V} + \dot{S}_{th} \tag{1.55}$$

N-S equation is given by :

$$\rho \frac{D\mathbf{V}}{Dt} = \rho\, \mathbf{f} + \nabla \cdot \sigma \tag{1.56}$$

or, taking dot product of both side with V :

$$\rho\left(\mathbf{V} \cdot \frac{\mathrm{D}\mathbf{V}}{\mathrm{Dt}}\right) = \rho\, \mathbf{f} \cdot \mathbf{V} + (\nabla \cdot \sigma) \cdot \mathbf{V} \tag{1.57}$$

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Or ,

$$\rho \frac{D}{Dt}\left(\frac{\mathbf{V^2}}{2}\right) = \rho\, \mathbf{f} \cdot \mathbf{V} + (\nabla \cdot \sigma) \cdot \mathbf{V} \tag{1.58}$$

Subtracting **Eqn. (1.58) from Eqn. (1.55)** one gets :

$$\rho \frac{De}{Dt} = k\nabla^2 T + \sigma : \nabla \mathbf{V} + \dot{S}_{th} \tag{1.59}$$

Here,

$$\sigma : \nabla \mathbf{V} = \left[- \left\{ p + \frac{2}{3}\mu \nabla \cdot \mathbf{V} \right\} I + \tau \right] : \nabla \mathbf{V} \tag{1.60}$$

Again,

$$\tau = 2\mu \ \mathrm{Def}\ \mathbf{V} \tag{1.61}$$

And

$$\mathrm{Def}\ \mathbf{V} = \frac{1}{2}\left[\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j}\right] \tag{1.62}$$

So

$$\sigma : \nabla \mathbf{V} = -pI : \nabla \mathbf{V} - \frac{2}{3}\mu(\nabla \cdot \mathbf{V})I : \nabla \mathbf{V} + \tau : \nabla \mathbf{V} \quad \textbf{(1.63)}$$

$$\begin{aligned} &= -p\,(\nabla \cdot \mathbf{V}) - \frac{2}{3}\mu(\nabla \cdot \mathbf{V})^2 + \tau : \nabla \mathbf{V} \\ &= -p(\nabla \cdot \mathbf{V}) + \mu\phi \end{aligned} \quad \textbf{(1.64)}$$

Where

$$\Phi = \left[\left\{\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right\}\left(\frac{\partial V_i}{\partial x_j}\right) - \frac{2}{3}\left(\frac{\partial V_i}{\partial x_i}\right)^2\right] \quad \textbf{(1.65)}$$

From continuity,

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla.\,(\rho \mathbf{V}) = 0 \\ \frac{\partial \rho}{\partial t} + (\mathbf{V}.\nabla)\,\rho + \rho\,(\nabla.\mathbf{V}) = 0 \end{aligned} \quad \textbf{(1.66)}$$

Or ,

$$\begin{aligned} -\nabla.\mathbf{V} &= \frac{1}{\rho}\left[\frac{\partial \rho}{\partial t} + (\mathbf{V}.\nabla)\,\rho\right] = \frac{1}{\rho}\frac{D\rho}{Dt} \\ -p\,(\nabla.\mathbf{V}) &= \frac{p}{\rho}\left[\frac{\partial \rho}{\partial t} + (\mathbf{V}.\nabla)\,\rho\right] = \frac{p}{\rho}\frac{D\rho}{Dt} \end{aligned} \quad \textbf{(1.67)}$$

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Energy **Equation (1.59)** can be written as :

$$\rho\frac{De}{Dt} = k\nabla^2 T + \frac{p}{\rho}\frac{D\rho}{Dt} + \mu\Phi + \dot{S}_{th} \tag{1.68}$$

$$i = e + \frac{p}{\rho} \tag{1.69}$$

$$\rho\frac{De}{Dt} = \rho\left[\frac{Di}{Dt} - \frac{D}{Dt}\left(\frac{p}{\rho}\right)\right] = \rho\left[\frac{Di}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}\right] \tag{1.70}$$

Now, the energy equation becomes :

$$\rho\frac{Di}{Dt} = -\frac{p}{\rho}\frac{D\rho}{Dt} + \frac{Dp}{Dt} + k\nabla^2 T + \frac{p}{\rho}\frac{D\rho}{Dt} + \mu\Phi + \dot{S}_{th} \tag{1.71}$$

Or ,

$$\rho\frac{Di}{Dt} = \frac{Dp}{Dt} + k\nabla^2 T + \mu\Phi + \dot{S}_{th} \quad \text{(General)} \tag{1.72}$$

For liquids, one can write **(from Eqn. (1.59) and Eqn. (1.64)),**

$$\rho\frac{De}{Dt} = k\nabla^2 T + \mu\Phi + \dot{S}_{th} \qquad \textbf{(1.73)}$$

where, the general viscous dissipation function is **:**

$$\Phi = \left[\left\{\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right\}\left(\frac{\partial V_i}{\partial x_j}\right) - \frac{2}{3}\left(\frac{\partial V_i}{\partial x_i}\right)^2\right] = \left[2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial y}\right.$$
$$+ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\frac{\partial u}{\partial z} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial x} + 2\frac{\partial v}{\partial y}\frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\frac{\partial v}{\partial z} \qquad \textbf{(1.74)}$$
$$\left. + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\frac{\partial w}{\partial x} + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\frac{\partial w}{\partial y} + 2\frac{\partial w}{\partial z}\frac{\partial w}{\partial z} - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2\right] \quad (1.74)$$

However, in absence of heat sources $(\dot{S}_{th} = 0)$ and negligible viscous dissipation $(\mu\Phi = 0,$ **which is often true for liquids)**, the energy equation for the liquids can be written as : **(from Eqn. (1.73)**

$$\rho c\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \qquad \textbf{(1.75)}$$

For gases, in general the energy equation can be written as **(from Equation (1.72))** ,

$$\rho c_p\frac{DT}{Dt} = \frac{Dp}{Dt} + k\nabla^2 T + \mu\Phi + \dot{S}_{th}$$

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Important Dimensionless Numbers

In order to nondimensionalize, we define :

$$u^* = \frac{u}{U_\infty}, \quad v^* = \frac{v}{U_\infty}, \quad w^* = \frac{w}{U_\infty}, \quad \theta = \frac{T - T_\infty}{T_w - T_\infty},$$
$$x^* = \frac{x}{L}, \quad y* = \frac{y}{L}, \quad z^* = \frac{z}{L}, \quad p^* = \frac{p}{\rho U_\infty^2}, \quad t^* = t\frac{U_\infty}{L}$$

Assumption of $\dot{S}_{th}=0$ produces,

$$\rho c_p \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = \left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right) + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \mu\Phi \qquad \textbf{(1.76)}$$

The time derivatives will vanish for the steady state situation. In any case,

$$\frac{\partial \theta}{\partial t^*} + u^*\frac{\partial \theta}{\partial x^*} + v^*\frac{\partial \theta}{\partial y^*} + w^*\frac{\partial \theta}{\partial z^*} = \frac{U_\infty^2}{c_p(T_w - T_\infty)}\left(\frac{\partial p^*}{\partial t^*} + u^*\frac{\partial p^*}{\partial x^*} + v^*\frac{\partial p^*}{\partial y^*} + w^*\frac{\partial p^*}{\partial z^*}\right) + \frac{k}{\rho c_p U_\infty L}\left(\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}} + \frac{\partial^2 \theta}{\partial z^{*2}}\right) + \frac{\mu U_\infty^2}{L}\frac{1}{\rho c_p(T_w - T_\infty)U_\infty}\bar{\Phi} \qquad (1 \qquad \textbf{(1.77)}$$

$$\frac{\partial \theta}{\partial t^*} + u^*\frac{\partial \theta}{\partial x^*} + v^*\frac{\partial \theta}{\partial y^*} + w^*\frac{\partial \theta}{\partial z^*} = Ec\left(\frac{\partial p^*}{\partial t^*} + u^*\frac{\partial p^*}{\partial x^*} + v^*\frac{\partial p^*}{\partial y^*} + w^*\frac{\partial p^*}{\partial z^*}\right) + \frac{1}{Re.Pr}\left(\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}} + \frac{\partial^2 \theta}{\partial z^{*2}}\right) + \frac{Ec}{Re}\bar{\Phi} \qquad \textbf{(1.78)}$$

$$Ec = \frac{U_\infty^2}{c_p(T_w - T_\infty)}, \quad Pr = \frac{\mu c_p}{k}, \quad Re = \rho\frac{U_\infty L}{\mu} \qquad \textbf{(1.79)}$$

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Let us look at the **term Ec (Eckert number) :**

$$Ec = \frac{U_\infty^2}{c_p(T_w - T_\infty)} = \frac{\text{Kinectic energy of the flow}}{\text{Boundary layer enthalpy difference}} \quad \textbf{(1.80)}$$

$$\begin{aligned} \frac{1}{Ec} &= \frac{c_p(T_w - T_\infty)}{U_\infty^2} = \frac{c_p T_\infty}{U_\infty^2}\left(\frac{T_w}{T_\infty} - 1\right) \\ &= \frac{c_p}{\gamma(c_p - c_v)}\frac{a^2}{U_\infty^2}\left(\frac{T_w}{T_\infty} - 1\right) = \frac{1}{(\gamma - 1)}\frac{1}{M^2}\left(\frac{T_w}{T_\infty} - 1\right) \end{aligned} \quad \textbf{(1.81)}$$

$$Ec = \frac{(\gamma - 1)M^2}{[(T_w/T_\infty) - 1)]} \quad \textbf{(1.82)}$$

where **M = (fluid velocity) / (local sound speed)** and a is the **local sound speed,** and M is known as **Mach Number.**

or low Mach number situation, energy equation in nondimensional form is :

$$\frac{\partial \theta}{\partial t^*} + u^*\frac{\partial \theta}{\partial x^*} + v^*\frac{\partial \theta}{\partial y^*} + w^*\frac{\partial \theta}{\partial z^*} = \frac{1}{Re.Pr}\left[\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}} + \frac{\partial^2 \theta}{\partial z^{*2}}\right] \quad \textbf{(1.83)}$$

Dimensional form will be :

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right] \qquad \textbf{(1.84)}$$

For forced convection, the variables can be presented as :

$$\theta=\theta(u,\quad v,\quad x,\quad y,\quad Re,\quad Pr,\quad Ec)$$

$$q_w=-k_f\frac{\partial T}{\partial n}=h(T_w-T_\infty) \qquad \textbf{(1.85)}$$

Where, n is the dimension in the direction normal to the surface.or,

$$h=\frac{-k_f\frac{\partial T}{\partial n}}{(T_w-T_\infty)}\quad=\quad\frac{-k_f\frac{(T_w-T_\infty)}{L}\frac{\partial\theta}{\partial\bar{n}}}{T_w-T_\infty} \qquad \textbf{(1.86)}$$

where $\bar{n}=\frac{n}{L}$

Or

$$\left(\frac{hL}{k}\right)\quad=Nu=-\left(\frac{\partial\theta}{\partial\bar{n}}\right) \qquad \textbf{(1.87)}$$

Therefore, **Nusselt number, Nu is the nondimensional temperature gradient at the surface** on which heat transfer takes place.

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