Module 2: External Flows

Lecture 7: Solution of Thermal Boundary Layer Equation

The Lecture Contains:

Thermal Boundary layer equation :-

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} \qquad \textbf{(2.30)}$$

$$u=U_\infty f'(\eta), v=-\frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left\{f(\eta)-\eta f'(\eta)\right\} \qquad \textbf{(2.31)}$$

$$\theta=\frac{T-T_\infty}{T_w-T_\infty} \qquad \textbf{(2.32)}$$

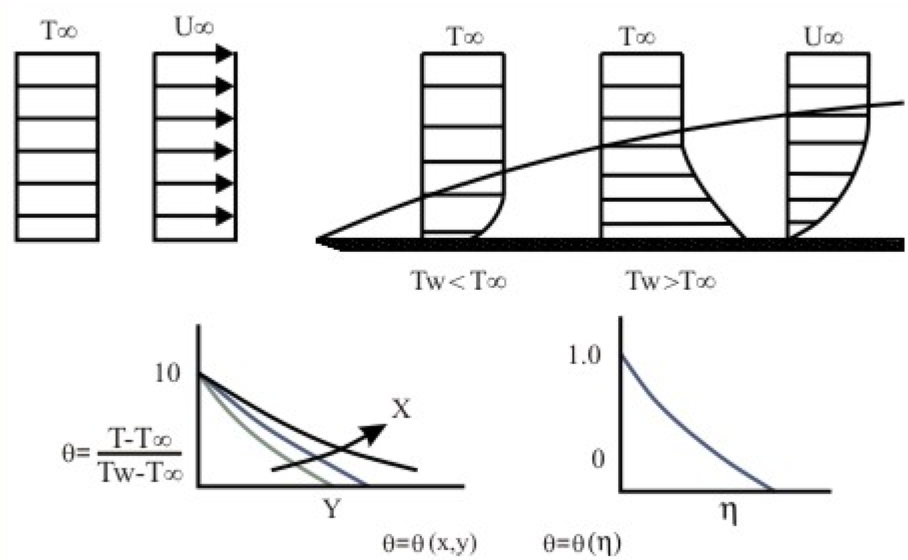

Figure 2.3: The growth of thermal boundary layer

describes the growth of thermal boundary layer on a heated at plate. The boundary condition for the situation described above is $\theta(0)=1,\ \theta(\infty)=0.$ Therefore , $T=T_\infty+\theta(T_w-T_\infty)$.

Let us assume $T_w=T_w(x)$ [a general case]

On substituting $\eta=(y/\sqrt{\frac{\nu x}{U_\infty}})$, we can write different derivatives as,

$$\frac{\partial T}{\partial x}=\theta\frac{dT_w}{dx}+(T_w-T_\infty)\left(-\frac{y}{2x}\right)\sqrt{\frac{U_\infty}{\nu x}}\frac{d\theta}{d\eta} \qquad \textbf{(2.33)}$$

$$\frac{\partial T}{\partial y}=\frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y}=(T_w-T_\infty)\frac{\partial \theta}{\partial \eta}\sqrt{\frac{U_\infty}{\nu x}} \qquad \textbf{(2.34)}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial \eta}\left[\frac{\partial T}{\partial y}\right]\frac{\partial \eta}{\partial y} = (T_w - T_\infty)\frac{\partial^2 \theta}{\partial \eta^2}\frac{U_\infty}{\nu x} \qquad \textbf{(2.35)}$$

On substituting the derivatives in thermal boundary layer **equation and multiplying by** $\frac{x}{U_\infty (T_w - T_\infty)}$ **on both sides of the equation,** we get :

$$\frac{f^{'}\theta x}{(T_w - T_\infty)}\frac{dT_w}{dx} - \frac{f^{'}y}{2}\sqrt{\frac{U_\infty}{\nu x}}\theta^{'} - \frac{(f - \eta f^{'})}{2}\theta^{'} = \frac{k}{\rho c_p}\frac{\theta^{''}}{\nu} \qquad \textbf{(2.36)}$$

$$f^{'}\theta x\frac{d}{dx}[\ln(T_w - T_\infty)] - \frac{f^{'}\eta\theta^{'}}{2} - \frac{(f - \eta f^{'})\theta^{'}}{2} = \frac{k}{\mu c_p}\theta^{''} \qquad \textbf{(2.37)}$$

Previous Next

Dividing by $f'\theta$

$$x\frac{d}{dx}[\ln(T_w - T_\infty)] - \frac{f\theta'}{2f'\theta} = \frac{k}{\mu C_p}\theta''\frac{1}{f'\theta} \qquad \textbf{(2.38)}$$

This equation could be put in the form **:**

$$\frac{x(dT_w/dx)}{T_w(x) - T_\infty} = \frac{k\theta''(\eta)}{\mu c_p f'(\eta)\theta(\eta)} + \frac{f(\eta)\theta'(\eta)}{2f'(\eta)\theta(\eta)} = \lambda \qquad \textbf{(2.39)}$$

This equation can be solved by using the method of separation of variables. This leads to :

$$\frac{x(dT_w/dx)}{T_w(x) - T_\infty} = \lambda \qquad \textbf{(2.40)}$$

Wall temperature has to follow this relation :

$$d\left\{\ln(T_w - T_\infty)\right\} = \frac{dx}{x}\lambda \qquad \textbf{(2.41)}$$

Integrating we get,

$$\ln\ (T_w - T_\infty) = \lambda \ln x + C \qquad \textbf{(2.42)}$$

So,

$$(T_w - T_\infty) = Cx^\lambda$$

When

$$\lambda = 0,\ T_w = T_\infty + C \qquad \text{(constant wall temperature)}$$

Again

$$\frac{k\theta''(\eta)}{\mu c_p f'(\eta)\theta(\eta)} + \frac{f(\eta)\theta'(\eta)}{2f'(\eta)\theta(\eta)} = \lambda \qquad \textbf{(2.43)}$$

$$\frac{\theta''(\eta)}{Pr} + \frac{1}{2}f\theta' = \lambda f'\theta \qquad \textbf{(2.44)}$$

$$\theta'' + \frac{Pr}{2}f\theta' = \lambda Pr f'\theta \qquad \textbf{(2.45)}$$

$$\theta'' + \frac{Pr}{2}f\theta' - \lambda Pr f'\theta = 0 \qquad \textbf{(2.46)}$$

In this equation **f** is known from **Blasius solution**. The boundary conditions are:

$$y = 0, T = T_w \rightarrow \eta = 0, \theta = 1$$
$$y = \infty, T = T_\infty \rightarrow \eta = \infty, \theta = 0$$
$$T = T_\infty + \theta(T_w - T_\infty)$$

where

$$(T_w - T_\infty) = Cx^\lambda$$

The final form of the boundary conditions are $\theta(0) = 1$ and $\theta(\infty) = 0$.
Let $Y_1 = \theta$ and $Y_2 = \theta'$ then

$$Y_2' = -\frac{Pr}{2}fY_2 + \lambda Pr f'Y_1 \qquad \textbf{(2.47)}$$

with $Y_1(0) = 1$ and $Y_1(\infty) = 0$

The numerical solution using Runge-Kutta method can be obtained via following steps :

1-Guess a value of $Y_2(0)$ or $\theta'(0)$

2- Solve for $Y_1,\ Y_2$

3- Check if $Y_1(\infty) = 0$, i.e., $\theta(\infty) = 0$?

4- If yes, stop. The calculated $\theta(\eta)$'s are correct solution

5- If no,correct and $\theta'(0)$ or $Y_2(0)$ repeat the calcualtion

Analytical solution, if $\lambda = 0$

$$\theta'' + \frac{Pr}{2} f\theta' = 0 \quad (2.48)$$

$$\frac{\theta''}{\theta'} = -\frac{Pr}{2} f \quad (2.49)$$

or,

$$\ln \theta' = -\frac{Pr}{2} \int f(\eta) d\eta + \ln C$$

or,

$$\theta' = Ce^{-\frac{Pr}{2} \int f(\eta) d\eta}$$

or,

$$\theta = C \int e^{-\frac{Pr}{2} \int f(\eta) d\eta} d\eta + D \quad (2.50)$$

or,

$$\theta(\eta) - \theta(0) = C \int_0^{\eta} e^{-\frac{Pr}{2} \int_0^{\eta} f(\eta) d\eta} d\eta$$

or,

$$\theta(\eta) = 1 + C \int_0^{\beta} e^{-\frac{Pr}{2} \int_0^{\beta} f(r) dr} d\beta$$

Now $\theta(\infty)=0$ **will result in :**

$$-1 = C\int_0^\infty e^{-\frac{Pr}{2}\int_0^\beta f(r)dr}d\beta \quad \textbf{(2.51)}$$

$$C = -\frac{1}{\int_0^\infty e^{-\frac{Pr}{2}\int_0^\beta f(r)dr}d\beta} \quad \textbf{(2.52)}$$

$$\theta(\eta) = 1 - \frac{\int_0^\eta e^{-\frac{Pr}{2}\int_0^\beta f(r)dr}d\beta}{\int_0^\infty e^{-\frac{Pr}{2}\int_0^\beta f(r)dr}d\beta} \quad \textbf{(2.53)}$$

$Nu_x = \text{Nusselt number} = \frac{hx}{k}$

$$h = \frac{-k(dT/dy)_{y=0}}{(T_w - T_\infty)} = \frac{-k\left[\frac{\partial T}{\partial \eta}\,\frac{\partial \eta}{\partial y}\right]_{\eta=0}}{(T_w - T_\infty)} \quad \textbf{(2.54)}$$

We know ,

$$T = T_\infty + \theta\,(T_w - T_\infty) \Rightarrow \frac{\partial T}{\partial \eta} = \frac{\partial \theta}{\partial \eta}\,(T_w - T_\infty)$$

$$h = -k\,\frac{\partial \theta}{\partial \eta}\,\frac{\partial \eta}{\partial y}\,|_{\eta=0} = -k\theta'(0)\sqrt{\frac{U_\infty}{\nu x}} \quad \textbf{(2.55)}$$

$$Nu_x = \frac{hx}{k} = -\theta'(0)\sqrt{\frac{U_\infty x}{\nu}} \quad \textbf{(2.56)}$$

$$\frac{Nu_x}{(Re_x)^{1/2}} = -\theta'(0) \simeq 0.332\,Pr^{1/3} \qquad (0.6 \le Pr \le 10) \quad \textbf{(2.57)}$$

The solution was obtained by Pohlhausen .

Previous Next