The Lecture Contains:

Previous Next

MAC Formulation

The region in which computations are to be performed is divided into a set of small cells having edge lengths δx,δy and δz **(Figure 4.3).** With respect to this set of computational cells, velocity components are located at the center of the cell faces to which they are normal and pressure and temperature are defined at the center of the cells. Cells are labeled with an index **(i,j,k)** which denotes the cell number as counted from the origin in the **x , y** and z directions respectively. Also $p_{i,j,k}$ is the pressure at the center of the cell **(i,j,k)**, while $u_{i,j,k}$ is the **x-direction** velocity at the center of the face between cells **(i,j,k)** and **(i+1, j, k)** and so on **(Figure 4.4).** Because of the staggered grid arrangements,the velocities are not defined at the nodal points, but whenever required, they are to to be found by interpolation. For example, with uniform grids, we can write $u_{i-1/2,j,k} = \frac{1}{2}[u_{i-1,j,k} + u_{i,j,k}]$.Where a product or square of such a quantity appears, it is to be averaged first and then the product to be formed. Convective terms are discretized using a weighted averaged of second upwind and space centered scheme **(Hirt et al,1975)**. Diffusive terms are discretized by a central diffrencing scheme. Let us consider the discretized terms of the x-momentum equation **(Figure 4.4):**

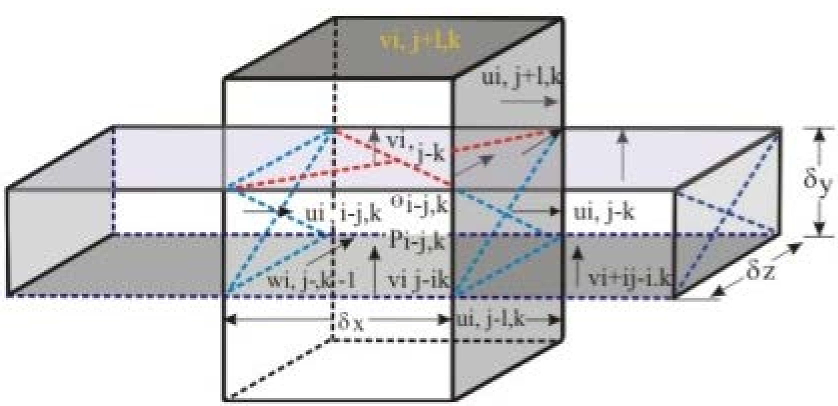

Figure 4.4:
Three-dimensional staggered grid showing the locations of the discretized variables

$$\begin{aligned}
\frac{\partial(u^2)}{\partial x} &= \frac{1}{4\delta x}[(u_{i,j,k}+u_{i+1,j,k})(u_{i,j,k}+u_{i+1,j,k}) \\
&+ \alpha|(u_{i,j,k}+u_{i+1,j,k})|(u_{i,j,k}-u_{i+1,j,k}) \\
&- (u_{i-1,j,k}+u_{i,j,k})(u_{i-1,j,k}+u_{i,j,k}) \\
&- \alpha|(u_{i-1,j,k}+u_{i,j,k})|(u_{i-1,j,k}-u_{i,j,k})] \\
&\equiv DUUDX
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(uv)}{\partial y} &= \frac{1}{4\delta y}[(v_{i,j,k}+v_{i+1,j,k})(u_{i,j,k}+u_{i,j+1,k}) \\
&+ \alpha|(v_{i,j,k}+v_{i+1,j,k})|(u_{i,j,k}-u_{i,j+1,k}) \\
&- (v_{i,j-1,k}+v_{i+1,j-1,k})(u_{i,j-1,k}+u_{i,j,k}) \\
&- \alpha|(v_{i,j-1,k}+v_{i+1,j-1,k})|(u_{i,j-1,k}-u_{i,j,k})] \\
&\equiv DUVDY
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(uw)}{\partial z} &= \frac{1}{4\delta z}[(w_{i,j,k}+w_{i+1,j,k})(u_{i,j,k}+u_{i,j,k+1}) \\
&+ \alpha|(w_{i,j,k}+w_{i+1,j,k})|(u_{i,j,k}-u_{i,j,k+1}) \\
&- (w_{i,j,k-1}+w_{i+1,j,k-1})(u_{i,j,k-1}+u_{i,j,k}) \\
&- \alpha|(w_{i,j,k-1}+w_{i+1,j,k-1})|(u_{i,j,k-1}-u_{i,j,k})] \\
&\equiv DUWDZ
\end{aligned}$$

Previous Next

$$
\begin{aligned}
\frac{\partial p}{\partial x} &= \frac{p_{i+1,j,k} - p_{i,j,k}}{\delta x} \equiv DPDX \\
\frac{\partial^2 u}{\partial x^2} &= \frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{(\delta x)^2} \equiv D2UDX2 \\
\frac{\partial^2 u}{\partial y^2} &= \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{(\delta y)^2} \equiv D2UDY2 \\
\frac{\partial^2 u}{\partial z^2} &= \frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{(\delta z)^2} \equiv D2UDZ2
\end{aligned}
$$

with

$$
\begin{array}{lll}
\alpha \longrightarrow 1 & \text{Scheme} \longrightarrow & \text{Second Upwind} \\
\alpha \longrightarrow 0 & \text{Scheme} \longrightarrow & \text{Space centered}
\end{array}
$$

Factor $\propto$ is chosen in such a way that the differencing scheme retains **"something"** of second order accuracy and the required upwinding is done for the sake of stability**.A typical value of is between 0.2 and 0.3.** As mentioned earlier,the quantity $\tilde{u}^{n+1}_{i,j,k}$ is now evaluated explicitly from the discretized form of **Equation (4.2)** as.

$$
\tilde{u}^{n+1}_{i,j,k} = u^n_{i,j,k} + \delta t \, [CONDIFU - DPDX]^n_{i,j,k}
$$

where

$$
\begin{aligned}
[CONDIFU - DPDX]^n_{i,j,k} &= [(-DUUDX - DUVDY - DUWDZ) \\
&- DPDX + (1/Re)(D2UDX2 \\
&+ D2UDY2 + D2UDZ2)]
\end{aligned}
$$

Similarly we evaluate.

$$
\begin{aligned}
\tilde{v}^{n+1}_{i,j,k} &= v^n_{i,j,k} + \delta t [CONDIFV - DPDY]^n_{i,j,k} \\
\tilde{w}^{n+1}_{i,j,k} &= w^n_{i,j,k} + \delta t [CONDIFW - DPDZ]^n_{i,j,k}
\end{aligned}
$$

As discussed earlier, the explicitly advanced tilde velocities may not necessarily lead to a flow field with zero mass divergence in each cell. This implies that, at this stage the pressure distribution is not correct. Pressure in each cell will be corrected in such a way that there is no net mass ow in or out of the cell. **In the original MAC method, the corrected pressures were obtained from the solution of a Poisson equation for pressure.**

A related technique developed by **Chorin (1967)** involved a simultaneous iteration on pressure and velocity components. **Vieceli (1971)** showed that the two methods as applied to MAC are equivalent. **We shall make use of the iterative correction procedure of Chorin (1967) in order to obtain a divergence-free velocity field**. The mathematical methodology of this iterative pressure-velocity procedure will be discussed herein. The relationship between the explicitly advanced velocity component and velocity at the previous time step may be written as.

$$\tilde{u}_{i,j,k}^{n+1} = u_{i,j,k}^{n} + \delta t\left[\frac{p_{i,j,k}^{n} - p_{i+1,j,k}^{n}}{\delta x}\right] + [CONDIFU]_{i,j,k}^{n}\delta t \qquad \textbf{(4.8)}$$

where, **[CONDIFU]** is only the contribution from convection and diffusion terms. On the other hand, the corrected velocity component (unknown) will be related to the corrected pressure (also unknown) in the following way:

$$u_{i,j,k}^{n+1} = u_{i,j,k}^{n} + \delta t\left[\frac{p_{i,j,k}^{n+1} - p_{i+1,j,k}^{n+1}}{\delta x}\right] + [CONDIFU]_{i,j,k}^{n}\delta t \qquad \textbf{(4.9)}$$

From equation **(4.8) and (4.9)**

$$u_{i,j,k}^{n+1} - \tilde{u}_{i,j,k}^{n+1} = \delta t\left[\frac{p'_{i,j,k} - p'_{i+1,j,k}}{\delta x}\right]$$

where the pressure correction may be defined as

$$p'_{i,j,k} = p_{i,j,k}^{n+1} - p_{i,j,k}^{n}$$

Neither the pressure correction nor $u_{i,j,k}^{n+1}$ are known explicitly at this stage. Hence,one cannot be calculated without the help of the other. Calculations are done in an iterative cycle and we write

Corrected Velocity Estimated Velocity $\pm$ Correction

$$u_{i,j,k}^{n+1} \longrightarrow \qquad \tilde{u}_{i,j,k}^{n+1} \qquad + \delta t\left[\frac{p'_{i,j,k} - p'_{i+1,j,k}}{\delta x}\right]$$

Previous Next

In a similar way, we can formulate the following array:

$$u_{i,j,k}^{n+1} \longrightarrow \tilde{u}_{i,j,k}^{n+1} + \delta t \left[\frac{p'_{i,j,k} - p'_{i+1,j,k}}{\delta x}\right] \quad \textbf{(4.10)}$$

$$u_{i-1,j,k}^{n+1} \longrightarrow \tilde{u}_{i-1,j,k}^{n+1} - \delta t \left[\frac{p'_{i,j,k} - p'_{i-1,j,k}}{\delta x}\right] \quad \textbf{(4.11)}$$

$$v_{i,j,k}^{n+1} \longrightarrow \tilde{v}_{i,j,k}^{n+1} + \delta t \left[\frac{p'_{i,j,k} - p'_{i,j+1,k}}{\delta y}\right] \quad \textbf{(4.12)}$$

$$v_{i,j-1,k}^{n+1} \longrightarrow \tilde{v}_{i,j-1,k}^{n+1} - \delta t \left[\frac{p'_{i,j,k} - p'_{i,j-1,k}}{\delta y}\right] \quad \textbf{(4.13)}$$

$$w_{i,j,k}^{n+1} \longrightarrow \tilde{w}_{i,j,k}^{n+1} + \delta t \left[\frac{p'_{i,j,k} - p'_{i,j,k+1}}{\delta z}\right] \quad \textbf{(4.14)}$$

$$w_{i,j,k-1}^{n+1} \longrightarrow \tilde{w}_{i,j,k-1}^{n+1} - \delta t \left[\frac{p'_{i,j,k} - p'_{i,j,k-1}}{\delta z}\right] \quad \textbf{(4.15)}$$

Previous Next

The correction is done through the continuity equation. Plugging-in the above relationship into the continuity **Equation (4.1) yields**

$$\left[\frac{u_{i,j,k}^{n+1}-u_{i-1,j,k}^{n+1}}{\delta x}+\frac{v_{i,j,k}^{n+1}-v_{i,j-1,k}^{n+1}}{\delta y}+\frac{w_{i,j,k}^{n+1}-w_{i,j,k-1}^{n+1}}{\delta z}\right]$$
$$=\left[\frac{\tilde{u}_{i,j,k}^{n+1}-\tilde{u}_{i-1,j,k}^{n+1}}{\delta x}+\frac{\tilde{v}_{i,j,k}^{n+1}-\tilde{v}_{i,j-1,k}^{n+1}}{\delta y}+\frac{\tilde{w}_{i,j,k}^{n+1}-\tilde{w}_{i,j,k-1}^{n+1}}{\delta z}\right]$$
$$-\delta t\left[\frac{p'_{i+1,j,k}-2p'_{i,j,k}+p'_{i-1,j,k}}{(\delta x)^2}+\frac{p'_{i,j+1,k}-2p'_{i,j,k}+p'_{i,j-1,k}}{(\delta y)^2}\right.$$
$$\left.+\frac{p'_{i,j,k+1}-2p'_{i,j,k}+p'_{i,j,k-1}}{(\delta z)^2}\right]$$

or

$$\left[\frac{u_{i,j,k}^{n+1}-u_{i-1,j,k}^{n+1}}{\delta x}+\frac{v_{i,j,k}^{n+1}-v_{i,j-1,k}^{n+1}}{\delta y}+\frac{w_{i,j,k}^{n+1}-w_{i,j,k-1}^{n+1}}{\delta z}\right]$$
$$=\left[\frac{\tilde{u}_{i,j,k}^{n+1}-\tilde{u}_{i-1,j,k}^{n+1}}{\delta x}+\frac{\tilde{v}_{i,j,k}^{n+1}-\tilde{v}_{i,j-1,k}^{n+1}}{\delta y}+\frac{\tilde{w}_{i,j,k}^{n+1}-\tilde{w}_{i,j,k-1}^{n+1}}{\delta z}\right]$$
$$+\frac{2\delta t(p'_{i,j,k})}{\delta x^2}+\frac{2\delta t(p'_{i,j,k})}{\delta y^2}+\frac{2\delta t(p'_{i,j,k})}{\delta z^2}$$

In deriving the above expression, **it is assumed that the pressure corrections in the neighboring cells are zero.** Back to the calculations,we can write .

$$0=(Div)_{i,j,k}+p'_{i,j,k}\left[2\delta t\left(\frac{1}{\delta x^2}+\frac{1}{\delta y^2}+\frac{1}{\delta z^2}\right)\right]$$

$$p'_{i,j,k}=\frac{-(Div)_{i,j,k}}{\left[2\delta t\left(\frac{1}{\delta x^2}+\frac{1}{\delta y^2}+\frac{1}{\delta z^2}\right)\right]} \qquad \textbf{(4.16)}$$

Previous Next

the pressure correction equation is modified as.

$$p'_{i,j,k} = \frac{-\omega_0 (Div)_{i,j,k}}{\left[2\delta t\left(\frac{1}{\delta x^2}+\frac{1}{\delta y^2}+\frac{1}{\delta z^2}\right)\right]} \qquad \textbf{(4.17)}$$

where ω_0 is the overrelaxation factor. A value of ω_0 = **1.7** is commonly used. The value of ω_0 giving most rapid convergence, should be determined by numerical experimentation. After calculating $p_{i,j,k}$, the pressure in the cell (i,j,k) is adjusted as.

$$p^{n+1}_{i,j,k} \longrightarrow p^{n}_{i,j,k} - p'_{i,j,k} \qquad \textbf{(4.18)}$$

Now the pressure and velocity components for each cell are corrected through an iterative procedure in such a way that for the final pressure field, the velocity divergence in each cell vanishes. The process is continued till a divergence-free velocity is reached with a prescribed upper bound; here a value of 0.0001 is recommended. Finally we discuss another important observation. If the velocity boundary conditions are correct and a divergence-free converged velocity field has been obtained, eventually correct pressure will be determined in all the cells at the boundary. Thus, this method avoids the application of pressure boundary conditions. **This typical feature of modified MAC method has been discussed in more detail by Peyret and Taylor (1983).** However, it was also shown by Brandt, **Dendy and Ruppel (1980)** that the aforesaid pressure-velocity iteration procedure of correcting pressure is equivalent to the solution of Poisson equation for pressure.

Boundary Conditions :-
So far we have not discussed the boundary conditions. However, they are imposed by setting appropriate velocities in the ctitious cells surrounding the physical domain (Figure 4.5).Consider, **for example, the bottom boundary of the computing (physical) mesh**. If this boundary is to be a rigid no-slip wall, the normal velocity on the wall must be zero and the tangential velocity components should also be zero. Here we consider a stationary wall. **With reference to the Figure 4.5,** we have

$$\left.\begin{aligned} v_{i,1,k} &= 0 \\ u_{i,1,k} &= -u_{i,2,k} \\ w_{i,1,k} &= -w_{i,2,k} \end{aligned}\right\} \begin{array}{l}\text{for } i=2 \ \text{ to ire} \\ \text{and } j=2 \ \text{ to kre}\end{array}$$

Previous Next

If the right side of **the wall is a free-slip (vanishing shear) boundary,** the normal velocity must be zero and the tangential velocities should have no normal gradient.

$$\left.\begin{array}{rcl} w_{i,j,1} & = & 0 \\ u_{i,j,1} & = & u_{i,j,2} \\ v_{i,j,1} & = & v_{i,j,2} \end{array}\right\} \begin{array}{l} \text{for } i=2 \ \text{ to ire} \\ \text{and} j=2 \ \text{ to jre} \end{array}$$

If the front plane is provided with in flow boundary conditions, it should be specified properly. Any desired functional relationship may be recommended. Generally, normal velocity components are set to zero and a uniform or parabolic axial velocity may be deployed. **Hence with reference to Figure 4.5**, we can write

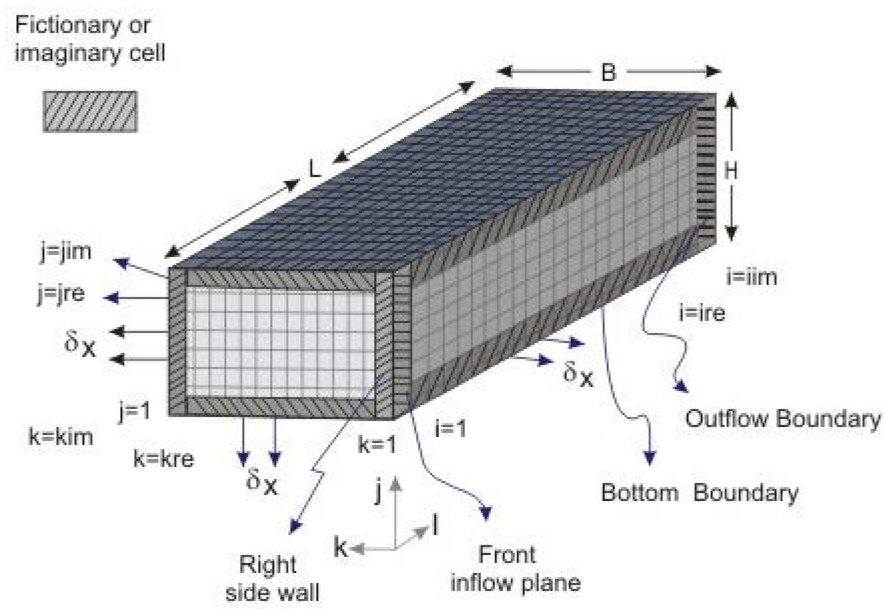

Figure 4.5:
Boundary conditions and ctitious boundary cells

where j_m is the horizontal midplane. Continuative or out flow boundaries always pose a problem for low-speed calculations, because whatever prescription is chosen it can affect the entire flow upstream. What is needed is a prescription that permits fluid to flow out of the mesh with a minimum of upstream influence. Commonly used conditions for such a boundary is $\nabla \mathbf{V} \cdot \mathbf{n} = 0,$ where n is the unit normal vector.

The boundary condition that has more generality at the out flow is described by **Orlanski (1976)**. This condition allows changes inside the flow field to be transmitted outward, but not vice-versa.

$$\frac{\partial \Psi}{\partial t} + U_{av}\frac{\partial \Psi}{\partial x} = 0$$

where U_{av} is the average velocity at the outflow plane and represents **u, v, w** or any dependant variable.

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