Module 4: Solution of Navier-Stokes and Energy Equations for Incompressible Internal Flows

Lecture 20: Solution Technique

The Lecture Contains:

Numerical Stability Considerations :-

For accuracy, the mesh size must be chosen small enough to resolve the expected spatial variations in all dependant variables. Once a mesh has been chosen, the choice of the time increment is governed by two restrictions, namely, the **Courant-Fredrichs-Lewy (CFL)** condition and the restriction on the basis of grid-Fourier numbers. According to the CFL condition, material cannot move through more than one cell in one time step, because the difference equations assume fuxes only between the adjacent cells. Therefore the time increment must satisfy the inequality.

$$\delta t < \min\left\{\frac{\delta x}{|u|}, \frac{\delta y}{|v|}, \frac{\delta z}{|w|}\right\} \qquad \textbf{(4.19)}$$

where the minimum is with respect to every cell in the mesh. Typically, δt is chosen equal to one-fourth to one-third of the minimum cell transit time. When the viscous diffusion terms are more important, the condition necessary to ensure stability is dictated by the restriction on the grid Fourier numbers, which results in.

$$\nu \delta t < \frac{1}{2} \cdot \left(\frac{\delta x^2 \delta y^2 \delta z^2}{\delta x^2 + \delta y^2 + \delta z^2}\right) \qquad \textbf{(4.20)}$$

in dimensional form. After nondimensionilization, this leads to

$$\delta t < \frac{1}{2} \cdot \left(\frac{\delta x^2 \delta y^2 \delta z^2}{\delta x^2 + \delta y^2 + \delta z^2}\right) Re \qquad \textbf{(4.21)}$$

The final δt for each time increment is the minimum of the δt's obtained from **Equations (4.19) and (4.21)** The last quantity needed to ensure numerical stability is the upwind parameter . In general, α should be slightly larger than the maximum value of $|u\delta t/\delta x|$ or $|v\delta t/\delta y|$ occurring in the mesh, that is,

$$\max\left\{\left|\frac{u\delta t}{\delta x}\right|, \left|\frac{v\delta t}{\delta y}\right|, \left|\frac{w\delta t}{\delta z}\right|\right\} \le \alpha < 1 \qquad \textbf{(4.22)}$$

As a ready prescription, a value between **0.2 and 0.4** can be used for α. If is too large, an unnecessary amount of numerical diffusion **(articial viscosity)** may be introduced.

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Higher Order Upwind Differencing:

More accurate solutions are obtained if the convective terms are discretized by higher order schemes. **Davis and Moore (1982)** use the **MAC** method with a multidimensional

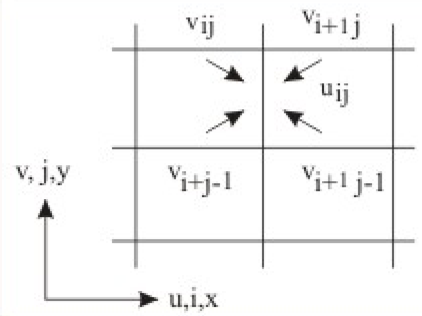

Figure 4.6:
Dependent variables $(u, v\ and\ \phi)$ on a rectangular mesh

third-order upwinding scheme. Needless to mention that their marching algorithm for the momentum equation is explicit and the stability restriction concerning the **CFL condition**

$[u\delta t/\delta x \leq 1 \text{ and } v\delta t/\delta y \leq 1]$ 1] is satisfied.The multidimensional third-order upwinding is,in principle similar to one-dimensional quadratic upstream interpolation scheme introduced **by Leonard (1979).** Consider **Figure 4.6**.Let be any property which can be convected and diffused. The convective term $\frac{\partial(u\phi)}{\partial x}$ may be represented as .

$$\frac{\partial(u\phi)}{\partial x} = \frac{(u\phi)_{i+\frac{1}{2},j} - (u\phi)_{i-\frac{1}{2},j}}{\delta x} \quad \textbf{(4.23)}$$

where the variables $\phi_{i+\frac{1}{2},j}$ and $\phi_{i-\frac{1}{2},j}$ j are defined as

$$\phi_{i+\frac{1}{2},j} = 0.5(\phi_{i,j} + \phi_{i+1,j}) - \frac{\xi}{3}(\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j}) \quad \text{for} \quad u_{i,j} \leq 0 \quad \textbf{(4.24)}$$

and

$$\phi_{i-\frac{1}{2},j} = 0.5(\phi_{i,j} + \phi_{i-1,j}) - \frac{\xi}{3}(\phi_{i-2,j} - 2\phi_{i-1,j} + \phi_{i,j}) \quad \text{for} \quad u_{i,j} > 0 \quad \textbf{(4.25)}$$

The parameter ξ can be chosen to increase the accuracy or to alter the diffusion-like characteristics. It may be pointed out $\xi = 3/8$ corresponds to the quick scheme of **Leonard(1979)** Let us consider two-dimensional momentum equation in weak conservative form which is given by.

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad \textbf{(4.26)}$$

In non-conservative form this may be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad \textbf{(4.27)}$$

Here we introduce a term transport-velocity .The transport velocities for the second and third terms on the left hand side are u and v respectively. While dealing with the equations in the conservative form, we shall keep this in mind. For example, during discretization of the term $\partial (uv)/\partial y$ of **Equation 4.26** we should remember that v is the transport-velocity associated with this term. It is customary to define the transportvelocity at the nodal point where the equation is being defined. In case of the term $\partial (uv)/\partial y$ we have to refer to **Figure 4.7 and write down the product term uv as.**

$$(uv)_{i,j} = 0.25 u_{i,j}(v_{i,j} + v_{i+1,j} + v_{i,j-1} + v_{i+1,j-1}) \quad \textbf{(4.28)}$$

Finally the discretization of the term $\partial (uv)/\partial y$for the **x-momentum equation** will be accomplished in the following way:

$$\begin{aligned} \frac{\partial uv}{\partial y} &= 0.25\frac{1}{8\delta y}[3u_{i,j+1}(v_{i,j+1} + v_{i,j} + v_{i+1,j+1} + v_{i+1,j}) \\ &+ 3u_{i,j}(v_{i,j} + v_{i,j-1} + v_{i+1,j} + v_{i+1,j-1}) \\ &- 7u_{i,j-1}(v_{i,j-1} + v_{i,j-2} + v_{i+1,j-1} + v_{i+1,j-2}) \\ &+ u_{i,j-2}(v_{i,j-2} + v_{i,j-3} + v_{i+1,j-2} + v_{i+1,j-3})] \quad \text{for} \quad V > 0 \end{aligned} \quad \textbf{(4.29)}$$

(4.30)

$$\begin{aligned}\frac{\partial(uv)}{\partial y} &= 0.25\frac{1}{8\delta y}[-u_{i,j+2}(v_{i,j+2}+v_{i,j+1}+v_{i+1,j+2}+v_{i+1,j+1})\\ &+ 7u_{i,j+1}(v_{i,j+1}+v_{i,j}+v_{i+1,j+1}+v_{i+1,j})\\ &- 3u_{i,j}(v_{i,j}+v_{i,j-1}+v_{i+1,j}+v_{i+1,j-1})\\ &- 3u_{i,j-1}(v_{i,j-1}+v_{i,j-2}+v_{i+1,j-1}+v_{i+1,j-2})] \quad \text{for} \;\; V<0\end{aligned}$$

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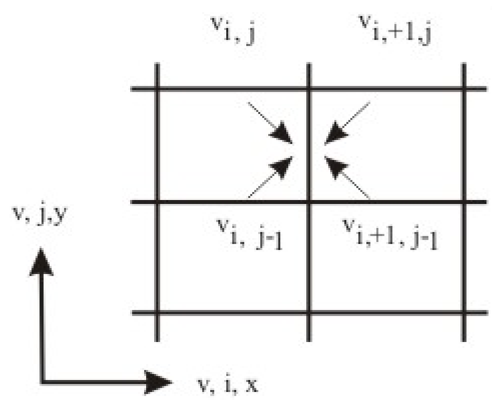

Figure 4.7:

Definition of the transport velocity at a point where the momentum equation is being discredited

where

$$V = v_{i,j} + v_{i+1,j} + v_{i,j-1} + v_{i+1,j-1}$$

Sample Results :-

For unsteady laminar flow past a rectangular obstacle in a channel, **Mukhopadhyay, Biswas and Sundararajan (1992)** use the MAC algorithm to explicitly march in time.Their results corroborated with the experimental observation of **Okajima (1982).** A typical example of numerical flow visualization depicting the development of **Von-Karmen vortex street is illustrated in Figure 4.8.** The cross-stream velocity vectors behind a delta-wing placed inside a channel are shown in Figure 4.9. These results were obtained by **Biswas and Chattopadhyay (1992)** who used MAC to solve for a three-dimensional flow field in a channel containing delta-wing as a vortex generator. The MAC algorithm has been extensively used by the researchers to solve flows in complex geometry. Braza, Chassaing and Ha-Minh (1986) investigated the dynamic characteristics of the pressure and velocity elds of the unsteady wake behind a circular cylinder using MAC algorithm. Recently, **Robichaux, Tafti and Vanka (1992) deployed MAC algorithm for Large Eddy Simulation (LES)** of turbulent channel flows. Of course, they performed the time integration of the discretized equations by using a fractional step method **(Kim and Moin, 1985)**. Another recent investigation by **Kim and Benson (1992)** suggests that the MAC method is signicantly accurate and at the same time the computational eort is reasonable

where,

$$V = v_{i,j} + v_{i+1,j} + v_{i,j-1} + v_{i+1,j-1}$$

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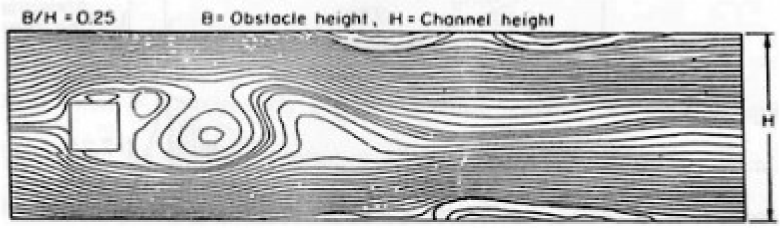

Figure 4.8:
Streamlines around a Blu-body in a duct: ReB = 162

Solution of Energy Equation :
The energy for incompressible flows, neglecting mechanical work and gas radiation, may be written as

$$\rho c_p \frac{DT}{Dt^*} = k\nabla^2 T + \mu\phi^* \qquad \textbf{(4.31)}$$

where is the viscous dissipation given as

$$\phi^* = 2\left[\left(\frac{\partial u^*}{\partial x^*}\right)^2 + \left(\frac{\partial v^*}{\partial y^*}\right)^2 + \left(\frac{\partial w^*}{\partial x^*}\right)^2\right] + \left\{\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right\}^2 + \left\{\frac{\partial w^*}{\partial y^*} + \frac{\partial v^*}{\partial z^*}\right\}^2 + \left\{\frac{\partial w^*}{\partial x^*} + \frac{\partial u^*}{\partial z^*}\right\}^2$$

Equation 4.31 may be nondimensionalized in the following way:

$$u = \frac{u^*}{U_\infty}, \quad v = \frac{v^*}{U_\infty}, \quad w = \frac{w^*}{U_\infty}, \quad \theta = \frac{T - T_\infty}{T_w - T_\infty}$$
$$x = \frac{x^*}{L}, \quad v = \frac{y^*}{L}, \quad z = \frac{z^*}{L}, \quad t = \frac{t^*}{L/U_\infty}$$

Substituting the above variables in **equation 4.31** we obtain

$$\frac{\rho c_p U_\infty (T_w - T_\infty)}{L}\left[\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z}\right] = \frac{(T_w - T_\infty)k}{L^2}\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right] + \frac{\mu U_\infty^2}{L^2}\phi$$

where is the nondimensional form of Finally, the normalized energy equation

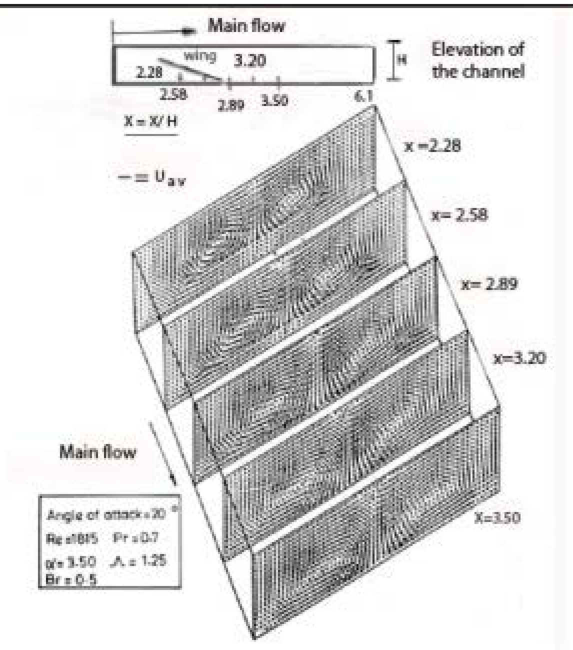

Figure 4.9 : Generation and deformation of longitudinal vortices in a channel

$$\frac{\partial\theta}{\partial t}+u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}+w\frac{\partial\theta}{\partial z}=\frac{1}{Pe}\left[\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}\right]+\frac{Ec}{Re}\phi$$

where Pe, the Peclet number is given as

$$\frac{1}{Pe}=\frac{(T_w-T_\infty)k}{L^2}\cdot\frac{L}{\rho c_p U_\infty(T_w-T_\infty)}$$

$$\frac{1}{Pe}=\frac{k}{L\rho c_p U_\infty}=\frac{k}{\mu c_p}\cdot\frac{\mu}{\rho L U_\infty}=\frac{1}{Pr}\cdot\frac{1}{Re}$$

Further, Ec, the Eckert number is

$$\frac{Ec}{Re}=\frac{\mu U_\infty^2}{L^2}\cdot\frac{L}{\rho c_p U_\infty(T_w-T_\infty)}=\frac{U_\infty^2}{c_p(T_w-T_\infty)}\cdot\frac{1}{\rho U_\infty L/\mu}$$

Retention of Dissipation

The dissipation term is frequently neglected while solving the energy equation for incompressible flows. As the Mach number $M\to 0, Ec\to 0.$ However, even at a low Mach number, can be important if (T_w-T_∞) is very small. Let us look at these aspects.
Since

$$Ec = \frac{U_\infty^2}{c_p(T_w - T_\infty)}, \; \frac{1}{Ec} = \frac{c_p T_\infty}{U_\infty^2}\left[\frac{T_w}{T_\infty} - 1\right]$$

and

$$\frac{c_p T_\infty}{U_\infty^2} = \frac{c_p \gamma R T_\infty}{\gamma R U_\infty^2}$$

where **R is the gas constant** $= c_p - c_v$, and $\gamma = c_p/c_v$. and = cp=cv.Let the local acoustic velocity $C = \sqrt{\gamma R T_\infty}$, and Mach number $M_\infty = U_\infty / C$

Then,

$$\frac{c_p T_\infty}{U_\infty^2} = \frac{c_p}{\gamma(c_p - c_v)}\left[\frac{1}{M_\infty^2}\right] = \frac{c_p}{\gamma c_p\left(1 - \frac{1}{\gamma}\right)}\left[\frac{1}{M_\infty^2}\right] = \frac{1}{(\gamma - 1)}\frac{1}{M_\infty^2}$$

hence

$$\frac{1}{Ec} = \frac{1}{(\gamma - 1)M_\infty^2}\left(\frac{T_w}{T_\infty} - 1\right)$$

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or

$$Ec = \frac{(\gamma - 1)M_\infty^2}{((T_w/T_\infty) - 1)}$$

In general for incompressible flows $M_\infty < 0.3 \text{ and } \gamma \geq 1$. Hence Ec is small. But for very small temperature difference, i.e if $(T_w - T_\infty)$ is slightly larger than 1, Ec might assume a large value and importance of including dissipation arises. However, for computing incompressible convective flows, the viscous dissipation is neglected in this chapter and we start with the steady state energy equation.

Solution Procedure

The steady state energy equation, neglecting the dissipation term, may be written in the following conservative form as.

$$\frac{\partial(u\theta)}{\partial x} + \frac{\partial(v\theta)}{\partial y} + \frac{\partial(w\theta)}{\partial z} = \frac{1}{Pe}\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right] \quad \textbf{(4.33)}$$

Equation 4.33 may be written as

$$\nabla^2\theta = Pe[CONVT]^m_{i,j,k} \quad \textbf{(4.34)}$$

where **[CONV T]m** $^{m}_{i,j,k}$ is the discretized convective terms on the left hand side of **Equation 4.33** and stands for the iterative counter. To start with, we can assume any guess value of θ throughout the flow field. Since **u; v; w** are known from the solution of momentum equation hence **Equation 4.33** is now a linear equation. However, from the guess value of θ and known correct values of u;v and w the left hand side of **Equation 4.33** is evaluated. A weighted average scheme or **QUICK** scheme may be adapted for discretization of the convective terms. After discretizing and evaluating right hand side of **Equation 4.34** we obtain a Poisson equation for temperature with a source term on the right hand side. Now, we shall follow SOR technique for solving **Equation 4.34.**

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Consider a discretized equation as.

$$\frac{\theta_{i+1,j,k} - 2\theta_{i,j,k} + \theta_{i-1,j,k}}{(\delta x)^2} + \frac{\theta_{i,j+1,k} - 2\theta_{i,j,k} + \theta_{i,j-1,k}}{(\delta y)^2} + \frac{\theta_{i,j,k+1} - 2\theta_{i,j,k} + \theta_{i,j,k-1}}{(\delta z)^2} = S^{*m}$$

Where

$$S^{*m} \equiv Pe[CONVT]^{m}_{i,j,k}$$

or

$$A^{*m} - \theta_{i,j,k}\left(\frac{2}{\delta x^2} + \frac{2}{\delta y^2} + \frac{2}{\delta z^2}\right) = S^{*m}$$

or

$$\theta_{i,j,k} = \frac{A^{*m} - S^{*m}}{\left(\frac{2}{\delta x^2} + \frac{2}{\delta y^2} + \frac{2}{\delta z^2}\right)}$$

Where

$$A^{*m} = \frac{\theta^m_{i+1,j,k} + \theta^m_{i-1,j,k}}{(\delta x)^2} + \frac{\theta^m_{i,j+1,k} + \theta^m_{i,j-1,k}}{(\delta y)^2} + \frac{\theta^m_{i,j,k+1} + \theta^m_{i,j,k-1}}{(\delta z)^2}$$

$\theta^m_{i,j,k}$ in **Equation 4.35** may be assumed to be the most recent value and it may be written as $\theta^m_{i,j,k}$. In order to accelerate the speed of computation we introduce an over relaxation factor ω. Thus

$$\theta^{m+1}_{i,j,k} = \theta^m_{i,j,k} + \omega\left[\theta^{m'}_{i,j,k} - \theta^m_{i,j,k}\right]$$

where $\theta^m_{i,j,k}$ is the previous value, $\theta^m_{i,j,k}$ the most recent value and $\theta^{m+1}_{i,j,k}$ the calculated better guess. The procedure will continue till the required convergence is achieved. This is equivalent to **Gauss-Siedel procedure** for solving a system of linear equations.

References

1. Biswas, G. and Chattopadhyay, H., Heat Transfer in a Channel with Built-in Wingtypw vortex Generators, Int. J. Heat Mass Transfer, Vol. 35, pp. 803-814, 1992. 24

2. Brandt, A., Debdy, J. E. and Ruppel, H., The Multigrid Method for Semi-Implicitly Hydrodynamic Codes, Journal of Comput. Phys., Vol. 34, pp. 348-370, 29-1980.

3. Braza, M. Chassaing, P. and Ha Minh, H., Numerical Study and Physical Analysis of the Pressure and Velocity Fields in the Near Wake of a Circular Cylinder, J. Fluid Mech., Vol. 165, pp. 79-130, 1986.

4. Chorin, A. J., A Numerical Method for Solving Incompressible Viscous Flow Problems,Journal of Comput. Phys., Vol. 2, pp. 12-26, 1967.

5. Davis, R. W., and Moore, E. F., A Numerical Study of Vortex Shedding from Rectangles, J. Fluid Mech., Vol. 116, pp. 475-50.

6. Harlow, F. H. and Welch, J. E., Numerical Calculation of Time-dependent Viscous Incompressible Flow of Fluid with Free Surfaces, The Phys. of Fluids, Vol. 8, pp. 2182-2188, 1965.

7. Hirt, C. W. and Cook, J. L., Calculating Three Dimensional Three Dimensional Flows around Structures and over Rough Terrain, Journal of Comput. Phys., Vol. 10, pp. 324-340,1972.

8. Hirt, C. W., Nichols, B. D. and Romero, N. C., SOLA - A Numerical Solution Algorithm for Transient Fluid Flows, LA - 5852, Los Alamos Scientic Laboratory Report, 1975.

9. Kim, J. and Moin, P., Application of a Fractional Step Method to Incompressible Navier-Stokes Equations, Journal of Comput. Phys., Vol. 59, pp. 308-323, 1985.

10. Kim, S. W. and Benson, T. J., Comparison of the SMAC, PISO and Iterative Time-Advancing Schemes for Unsteady Flows, Computers Fluids, Vol. 21, pp. 435-454, 1992. 25

11. Leonard, B. P., A Stable and Accurate Convective Modeling Procedure Based on Quadratic Upstream Interpolation, Comp. Methods Appl. Mech. Engr., Vol. 19,pp. 59-98, 1979.6, 1982.

12. Mukhopadhyay, A., Biswas, G. and Sundararajan, T., Numerical Investigation of Conned Wakes behind a Square Cylinder in a Channel, Int. J. Numer. Methods Fluids, Vol. 14, pp. 1473-1484, 1992.

13. Orlanski, I., A Simple Boundary Condition for Unbounded Flows, J. Comput. Phys., Vol. 21, pp. 251-269, 1976.

14. Okajima, A., Strouhal Numbers of Rectangular Cylinders, J. Fluids Mech., Vol. 123, pp. 379-398, 1982. Patankar, S. V. and Spalding, D. B., A Calculation Procedure for Heat Mass and Momentum Transfer in Three Dimensional Parabolic Flows, Int. J. Heat and Mass Transfer, Vol. 15, pp. 1787-1805, 1972.

15. Peyret, R. and Taylor, T. D., Computational Methods for Fluid Flow, Springer Verlag, New York, 1983.

16. Robichaux, J., Tafti, D. K. and Vanka, S. P., Large Eddy Simulations of Turbulence on CM-2, Numerical Heat Transfer, Part B, Vol. 21, pp. 267-388, 1992.

17. Vanka, S. P., Chen, B. C.-J., and Sha, W. T., A Semi-Implicit Calculation Procedure for Flow Described in Body-Fitted Coordinate Systems, Numerical Heat Transfer, Vol. 3, pp. 1-19, 1980.

18. Viecelli, J. A., A Computing Method for Incompressible Flows Bounded by Moving Walls, Journal of Comput. Phys., Vol. 8, pp. 119-143, 1971.

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