Lecture 2: Thermal Energy Equation

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Energy Equation using Specific Coordinate Sys-tem

Let us consider a differential control volume **(Figure 1.1)** in a flow field and account for the energy crossing the control volume (CV) boundary.

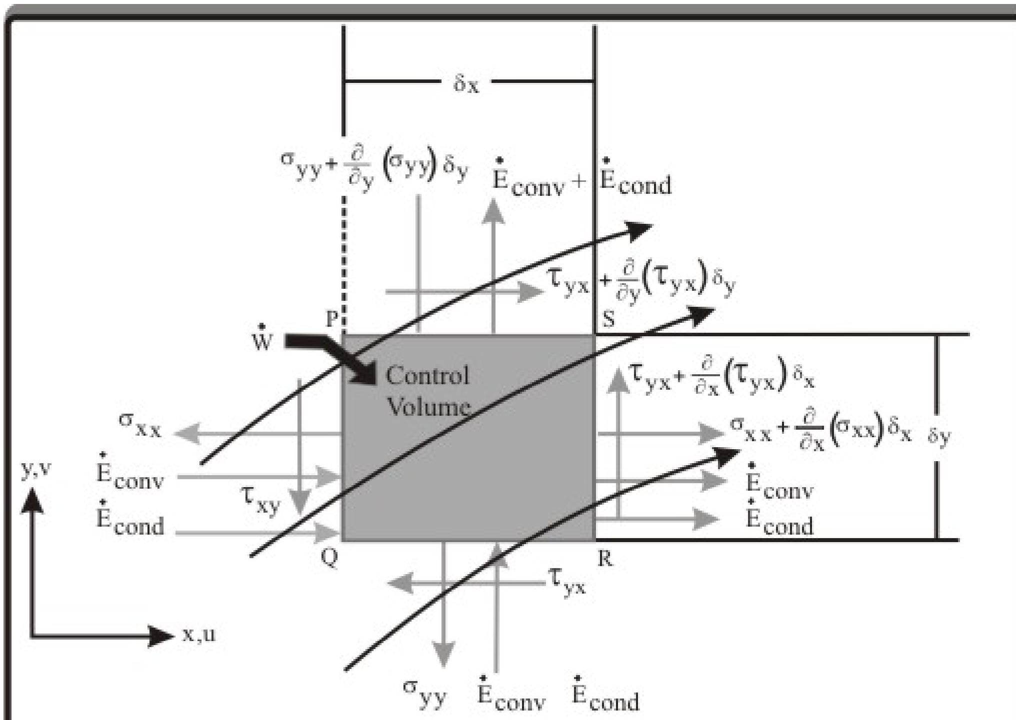

Figure 1.1: Differential Control Volume for Energy Conservation

Thermal energy per unit mass = e

Kinectic energy per unit mass = $V^2/2$ where $V^2 = u^2 + v^2$

Rate of change of total energy with in the CV

$$\frac{\partial}{\partial t}\left\{\left[\rho\left(e+\frac{V^2}{2}\right)\right]\delta x\delta y\right\} \qquad \textbf{(1.16)}$$

Rate of mass crossing $PQ = \rho u(\delta y.1)$

Rate of mass crossing $RS = \rho u(\delta y.1) + \frac{\partial}{\partial x}[\rho u(\delta y.1)]\delta x$

Rate of energy crossing $PQ = \rho u\ \delta y\left(e + \frac{V^2}{2}\right)$

Rate of energy crossing RS

$$\left\{\left(e+\frac{V^2}{2}\right)\rho u + \frac{\partial}{\partial x}\left[\left(e+\frac{V^2}{2}\right)\rho u\right]\delta x\right\}(\delta y.1) \qquad \textbf{(1.17)}$$

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Net convective efflux of energy across this pair of faces **(PQ and RS)**

$$\frac{\partial}{\partial x}\left[\left(e+\frac{V^2}{2}\right)\rho u\right]\delta x\delta y \qquad \textbf{(1.18)}$$

Similarly, we can find the efflux across the faces **PS and QR.**

efflux (both in x and y direction) of energy convected:

$$\left\{\frac{\partial}{\partial x}\left[\left(e+\frac{V^2}{2}\right)\rho u\right]+\frac{\partial}{\partial y}\left[\left(e+\frac{V^2}{2}\right)\rho v\right]\right\}\delta x\delta y \qquad \textbf{(1.19)}$$

Next, we shall consider diffusive transport of energy. Conduction (diffusive) flux across PQ into the control volume is given by

$$-k\frac{\partial T}{\partial x}\left(\delta y.1\right) \qquad \textbf{(1.20)}$$

Conduction **flux across RS**

$$-\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]\delta x\delta y \qquad \textbf{(1.21)}$$

Net conduction flux in **x** direction :

$$-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\delta x\delta y \qquad \textbf{(1.22)}$$

Similarly, net conduction flux across the whole control volume :

$$-\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]\delta x\delta y \qquad \textbf{(1.23)}$$

$$\left[\frac{\partial}{\partial x}\left\{\left(e+\frac{V^2}{2}\right)\rho u\right\}+\frac{\partial}{\partial y}\left\{\left(e+\frac{V^2}{2}\right)\rho v\right\}\right]\delta x\delta y - \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]\delta x\delta y \qquad \textbf{(1.24)}$$

In vector notation we can write that **efflux of energy through the Control Surface (CS):**

$$\left[\nabla.\left\{\rho\vec{V}\left(e+\frac{V^2}{2}\right)\right\}-\nabla.\left(k\nabla T\right)\right]\delta x\delta y \qquad \textbf{(1.25)}$$

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The Reynolds transport theorem can be stated as the Rate of change of energy for the system is summation of the rate of change of energy in the Control Volume (CV) and rate of efflux of energy through the Control Surface (CS) of the CV .

According to the First law of Thermodynamics :

Rate of energy production within the CV + Rate of work done on the CV = Rate of change of energy for the system.

Let us evaluate the quantity, rate of work done on the CV. The rate at which body force f does work is given by force time velocity

$$\rho\left[f_x u + f_y v\right]\delta x \delta y = \rho \vec{f}.\vec{V}\,\delta x \delta y \tag{1.26}$$

Work done (rate) by the surface forces on the **PQ plane of CV** :

$$-\sigma_{xx} u \delta y - \tau_{xy} v \delta y \tag{1.27}$$

Work done (rate) by the surface forces on the **RS plane is** :

$$\left[(\sigma_{xx}u) + \frac{\partial}{\partial x}(\sigma_{xx}u)\,\delta x + \tau_{xy}v + \frac{\partial}{\partial x}(\tau_{xy}v)\,\delta x\right]\delta y \tag{1.28}$$

Net work done (rate) by the surface forces in **x direction** :

$$\left[\frac{\partial}{\partial x}(\sigma_{xx}u) + \frac{\partial}{\partial x}(\tau_{xy}v)\right]\delta x \delta y \tag{1.29}$$

Similarly, we can determine the net work done (rate) by the surface forces in y direction,and adding these two we get work done (rate) on the CV:

$$\left[\frac{\partial}{\partial x}(\sigma_{xx}u) + \frac{\partial}{\partial y}(\sigma_{yy}v) + \frac{\partial}{\partial x}(\tau_{xy}v) + \frac{\partial}{\partial y}(\tau_{yx}u)\right]\delta x \delta y \tag{1.30}$$

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We assume the sources of heat such that the strength per unit volume is $\dot{S}_{th}$, the energy production within the CV $= S_{th}\delta x \delta y$

The conservation of energy requires **(invoking Eqns (1.16), (1.24), (1.26) and (1.30)**

$$\begin{aligned} &\frac{\partial}{\partial t}\left\{\rho\left(e+\frac{V^2}{2}\right)\right\}+\frac{\partial}{\partial x}\left\{\rho u\left(e+\frac{V^2}{2}\right)\right\} \\ &+\frac{\partial}{\partial y}\left\{\rho v\left(e+\frac{V^2}{2}\right)\right\}-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)-\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \\ &=\rho\left\{f_x u+f_y v\right\}+\left[\frac{\partial}{\partial x}\left\{\sigma_{xx}u+\tau_{xy}v\right\}+\frac{\partial}{\partial y}\left\{\sigma_{yy}v+\tau_{yx}u\right\}\right]+\dot{S}_{th} \end{aligned} \qquad \textbf{(1.31)}$$

In order to simplify these terms, let us take a recourse to the mechanical energy part through the NS equation. Multiplying the x momentum equation by u and y momentum equation by v and adding, we get

$$\begin{aligned} &\rho\left[\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right)+u^2\frac{\partial u}{\partial x}+uv\frac{\partial u}{\partial y}+uv\frac{\partial v}{\partial x}+v^2\frac{\partial v}{\partial y}\right] \\ &=\rho\left\{f_x u+f_y v\right]+u\left\{\frac{\partial}{\partial x}\sigma_{xx}+\frac{\partial}{\partial y}\tau_{xy}\right\}+v\left\{\frac{\partial}{\partial x}\tau_{yx}+\frac{\partial}{\partial y}\sigma_{yy}\right\} \end{aligned} \qquad \textbf{(1.32)}$$

First two convective terms of the **LHS of Eqn. (1.31)** can be rearranged as :

$$\begin{aligned} &\frac{\partial}{\partial x}\left\{\rho u\left(e+\frac{V^2}{2}\right)\right\}+\frac{\partial}{\partial y}\left\{\rho v\left(e+\frac{V^2}{2}\right)\right\}=\rho u\frac{\partial e}{\partial x}+e\frac{\partial}{\partial x}(\rho u)+\frac{V^2}{2}\frac{\partial}{\partial x}(\rho u) \\ &+\rho u\frac{\partial}{\partial x}\left(\frac{V^2}{2}\right)+\rho v\frac{\partial e}{\partial y}+e\frac{\partial}{\partial y}(\rho v)+\frac{V^2}{2}\frac{\partial}{\partial y}(\rho v)+\rho v\frac{\partial}{\partial y}\left(\frac{V^2}{2}\right) \end{aligned} \qquad \textbf{(1.33)}$$

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The temporal term of **(1.31)**

$$\frac{\partial}{\partial t}\left\{\rho\left(e+\frac{V^2}{2}\right)\right\}=\rho\frac{\partial e}{\partial t}+e\frac{\partial \rho}{\partial t}+\rho\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right)+\frac{V^2}{2}\left(\frac{\partial \rho}{\partial t}\right) \tag{1.34}$$

First three terms of **Eqn. (1.31)** will yield

$$\begin{aligned}
&\rho\left(u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}+\frac{\partial e}{\partial t}\right)+e\left(\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)\right)+\frac{V^2}{2}\left(\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)\right)\\
&\quad+\rho\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right)+\left[\rho\left\{u\frac{\left(2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}\right)}{2}\right\}+\rho\left\{v\frac{\left(2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}\right)}{2}\right\}\right]\\
&=\rho\left[\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right)+u^2\frac{\partial u}{\partial x}+uv\frac{\partial u}{\partial y}+uv\frac{\partial v}{\partial x}+v^2\frac{\partial v}{\partial y}\right]+\rho\left[\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}\right]
\end{aligned} \tag{1.35}$$

Invoking this and considering homogeneous isotropic medium, the energy **equation (1.31)** can be written as

$$\begin{aligned}
&\rho\left[\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}\right]+\rho\left[\frac{\partial}{\partial t}\left(\frac{V^2}{2}\right)+u^2\frac{\partial u}{\partial x}+uv\frac{\partial u}{\partial y}+uv\frac{\partial v}{\partial x}+v^2\frac{\partial v}{\partial y}\right]\\
&\quad+\left(-k\frac{\partial^2 T}{\partial x^2}-k\frac{\partial^2 T}{\partial y^2}\right)=\rho\left\{f_x u+f_y v\right]+\sigma_{xx}\frac{\partial u}{\partial x}+u\frac{\partial}{\partial x}\sigma_{xx}+\tau_{xy}\frac{\partial v}{\partial x}\\
&\quad+v\frac{\partial}{\partial x}\tau_{xy}+\sigma_{yy}\frac{\partial v}{\partial y}+v\frac{\partial}{\partial y}\sigma_{yy}+\tau_{yx}\frac{\partial u}{\partial y}+u\frac{\partial}{\partial y}\tau_{yx}+\dot{S}_{th}
\end{aligned} \tag{1.36}$$

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Subtracting **Eqn. (1.32) from Eqn. (1.36)** and considering the stress tensor symmetric we get

$$\rho\left[\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}\right]-k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right] \tag{1.37}$$
$$=\left\{\sigma_{xx}\frac{\partial u}{\partial x}+\tau_{xy}\frac{\partial v}{\partial x}+\sigma_{yy}\frac{\partial v}{\partial y}+\tau_{yx}\frac{\partial u}{\partial y}\right\}+\dot{S}_{th}$$

Let us look at terms within braces { }

$$\frac{\partial u}{\partial x}\sigma_{xx}=\left(-p+2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right\}\right)\frac{\partial u}{\partial x} \tag{1.38}$$
$$=-p\frac{\partial u}{\partial x}+2\mu\left(\frac{\partial u}{\partial x}\right)^2-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}\right)^2-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial v}{\partial y}\sigma_{yy}=\left(-p+2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right\}\right)\frac{\partial v}{\partial y} \tag{1.39}$$
$$=-p\frac{\partial v}{\partial y}+2\mu\left(\frac{\partial v}{\partial y}\right)^2-\frac{2}{3}\mu\left(\frac{\partial v}{\partial y}\right)^2-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right)$$

$$\tau_{xy}\frac{\partial v}{\partial x}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\right)=\mu\left(\frac{\partial v}{\partial x}\right)^2+\mu\left(\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}\right) \tag{1.40}$$

$$\tau_{yx}\frac{\partial u}{\partial y}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)=\mu\left(\frac{\partial u}{\partial y}\right)^2+\mu\left(\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}\right) \tag{1.41}$$

Equation (1.37) may now be written as

$$\rho\left[\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}\right]=k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]-p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{1.42}$$
$$\underbrace{+\mu\left[2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2\right]}_{\text{Viscous dissipation}\ \equiv\ \mu\Phi}+\dot{S}_{th}$$

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Equation (1.42) is the general form of energy equation, valid for both the compressible and incompressible flows.

Now, let us consider enthalpy per unit mass, $i = e + \frac{p}{\rho}$

Substituting ***i*** in equation**(1.42),** we get:

$$\rho\left[\frac{\partial i}{\partial t}+u\frac{\partial i}{\partial x}+v\frac{\partial i}{\partial y}\right]-\rho\left[\frac{\partial}{\partial t}\left(\frac{p}{\rho}\right)+u\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)+v\frac{\partial}{\partial y}\left(\frac{p}{\rho}\right)\right] = k\nabla^2 T - p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\mu\Phi+\dot{S}_{th} \quad \textbf{(1.43)}$$

or

$$\rho\left[\frac{\partial i}{\partial t}+u\frac{\partial i}{\partial x}+v\frac{\partial i}{\partial y}\right]-\left[\frac{\partial p}{\partial t}-\frac{p}{\rho}\frac{\partial \rho}{\partial t}+u\frac{\partial p}{\partial x}-\frac{up}{\rho}\frac{\partial \rho}{\partial x}+v\frac{\partial p}{\partial y}-\frac{vp}{\rho}\frac{\partial \rho}{\partial y}\right] = k\nabla^2 T - p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\mu\Phi+\dot{S}_{th} \quad \textbf{(1.44)}$$

or

$$\rho\left[\frac{\partial i}{\partial t}+u\frac{\partial i}{\partial x}+v\frac{\partial i}{\partial y}\right] = k\nabla^2 T + \underbrace{\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)}_{\frac{\partial p}{\partial t}+\vec{V}.\nabla(p)} -\frac{p}{\rho}\left[\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}\right]+\mu\Phi+\dot{S}_{th} \quad \textbf{(1.45)}$$

or

$$\rho\left[\frac{\partial i}{\partial t}+u\frac{\partial i}{\partial x}+v\frac{\partial i}{\partial y}\right] = k\nabla^2 T + \frac{Dp}{Dt} -\frac{p}{\rho}\left[\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)\right]+\mu\Phi+\dot{S}_{th} \quad \textbf{(1.46)}$$

or

$$\rho\left[\frac{\partial i}{\partial t}+\vec{V}.\nabla(i)\right]=k\nabla^2T+\frac{Dp}{Dt}+\mu\Phi+\dot{S}_{th} \quad \textbf{(1.47)}$$

or

$$\rho\frac{Di}{Dt}=k\nabla^2T+\frac{Dp}{Dt}+\mu\Phi+\dot{S}_{th} \quad \textbf{(1.48)}$$

Substituting , $i=c_pT$

$$\rho c_p\frac{DT}{Dt}=\frac{Dp}{Dt}+k\nabla^2T+\mu\Phi+\dot{S}_{th} \quad \textbf{(1.49)}$$

This is the general form of energy equation. Special Case: **For liquids,we can directly write from (1.42)**

$$\rho c\frac{DT}{Dt}=k\left[\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}\right]+\mu\Phi+\dot{S}_{th} \quad \textbf{(1.50)}$$

For liquids, $c_p=c_v=c \text{ and } p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0.$ In the case of liquids,

the viscous dissipation term will have to be modied by invoking the incompressibility condition.

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