## The Lecture Contains:

## 2.5 Approximate Methods for Flat Plate Boundary Layer

Flow over a heated flat plate $\left[Pr \sim \frac{\delta}{\delta_T} \text{ from } \frac{\delta}{\delta_T} \sim (Pr)^{1/3} \text{ or } \frac{\delta}{\delta_T} \sim (Pr)^{1/2}\right]$ has been illustrated.

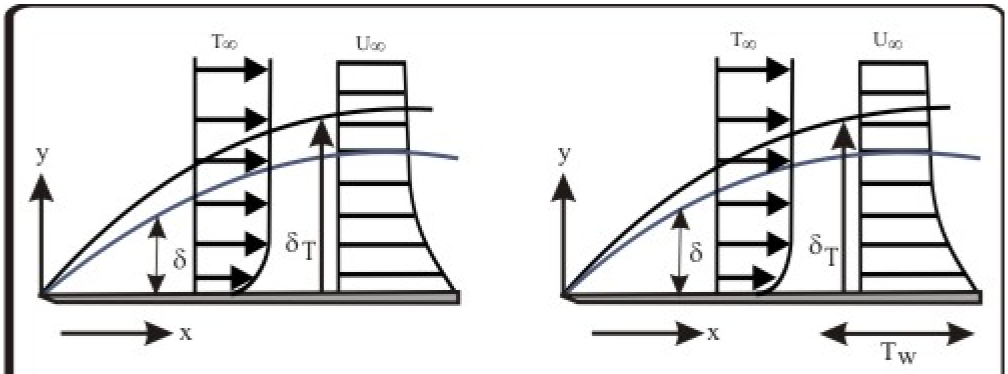


Figure 2.4: Velocity and Thermal Boundary Layers

**Governing Equations** :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.85}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{2.86}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} \tag{2.87}$$

The boundary conditions are :

$$@y = 0, u = 0 = v, T = T_w$$
$$@y = \delta,\ u = U_\infty,\ @y = \delta_T, T = T_\infty$$

Using Integral method due to **von Karman and Pohlhausen is written as eqn. (2.86)**

$$\underbrace{\int_0^\delta u\frac{\partial u}{\partial x}dy}_{\text{I}} + \underbrace{\int_0^\delta v\frac{\partial u}{\partial y}dy}_{\text{II}} = \underbrace{\int_0^\delta -\frac{1}{\rho}\frac{dp}{dx}dy}_{\text{III}} + \underbrace{\int_0^\delta \nu\frac{\partial^2 u}{\partial y^2}dy}_{\text{IV}}$$

**Term III** is zero for flow over flat plate **(because dp/dx = 0)**

**II term:**

$$[vu]_0^\delta - \int_0^\delta \frac{\partial v}{\partial y}udy = \left[vu|_\delta - vu|_0\right] + \int_0^\delta u\frac{\partial u}{\partial x}dy \quad (\text{from continuity}) \tag{2.88}$$

Again, from continuity :

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

or ,

$$\int_0^\delta \frac{\partial v}{\partial y} dy = -\int_0^\delta \frac{\partial u}{\partial x} dy$$

or ,

$$v|_{y=\delta} = -\int_0^\delta \frac{\partial u}{\partial x} dy$$

II term finally becomes :

$$-U_\infty \int_0^\delta \frac{\partial u}{\partial x} dy + \int_0^\delta u \frac{\partial u}{\partial x} dy$$

**Equation (2.88)** can be written as :

$$2\int_0^\delta u \frac{\partial u}{\partial x} dy - U_\infty \int_0^\delta \frac{\partial u}{\partial x} dy = -\nu \frac{\partial u}{\partial y}|_{y=0}$$

or ,

$$\frac{d}{dx}\int_0^\delta u^2 dy - U_\infty \frac{d}{dx}\int_0^\delta u dy = -\nu \frac{\partial u}{\partial y}|_{y=0} \qquad \textbf{(2.89)}$$

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**Equation (2.89)** is **Momentum Integral Equation.** Let us assume **:**

$$u = C_0 + C_1 y + C_2 y^2 + C_3 y^3$$
$$\frac{u}{U_\infty} = C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3, \quad \text{where } \eta = y/\delta$$

η is called similarity parameter; despite the growth of buoyancy layer in x direction $\frac{u}{U_\infty}$ remains similar for same $\frac{y}{\delta}$ at any **x** .

Boundary conditions:

$$@y = 0, u = 0, \quad \text{or,} \quad @\eta = 0, \frac{u}{U_\infty} = 0, \quad \Rightarrow C_0 = 0$$
$$@y = \delta, u = U_\infty, \quad \text{or,} \quad @\eta = 1, \frac{u}{U_\infty} = 1$$
$$\text{or,} \quad [C_1\eta + C_3\eta^3]_{\eta=1} = 1, \quad \text{or,} \quad C_1 + C_3 = 1$$
$$@y = 0, \frac{\partial^2 u}{\partial y^2} = 0 \quad \left(\text{comes from } \; u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}\right),$$
$$\text{or,} \quad @\eta = 0, \frac{\partial^2 (u/U_\infty) = 0}{\partial \eta^2} = 0,$$
$$\text{or,} \quad [2C_2 + 6C_3\eta]_{\text{at } \eta=0} = 0 \Rightarrow C_2 = 0$$
$$@y = \delta, \frac{\partial u}{\partial y} = 0 \quad \text{or,} \quad @\eta = 1, \frac{\partial}{\partial \eta}\left[\frac{u}{U_\infty}\right] = 0$$
$$\text{or,} \quad [C_1 + 3C_3\eta^2]_{\eta=1} = 0, \quad \Rightarrow C_1 + 3C_3 = 0$$

Finally we get,

$$C_1 = \frac{3}{2}, C_3 = -\frac{1}{2} \quad \text{So,} \quad \frac{u}{U_\infty} = \frac{3}{2}\eta - \frac{1}{2}\eta^3$$

Integral of the first term in **equation (2.89)**

$$\int_0^\delta u^2 dy = \frac{U_\infty^2}{4}\int_0^1 [9\eta^2 + \eta^6 - 6\eta^4]\;\delta d\eta$$
$$= \frac{\delta U_\infty^2}{4}\left[\frac{9}{3} + \frac{1}{7} - \frac{6}{5}\right] = \frac{68}{35}\frac{\delta U_\infty^2}{4}$$

Integral associated with the second term of **equation (2.89)**

$$\int_0^\delta u dy = \frac{U_\infty}{2}\int_0^1 [3\eta - \eta^3]\ \delta d\eta = \frac{5}{4}\frac{U_\infty \delta}{2}$$

The third term of **equation (2.89)** $-\nu \frac{\partial u}{\partial y}\Big|_{y=0}$

$$= -\frac{\nu}{\delta}\frac{\partial}{\partial \eta}\left[U_\infty\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\right]\Big|_{\eta=0}$$

$$= -\frac{3}{2}\frac{\nu U_\infty}{\delta}$$

Finally **equation (2.89)** becomes :

$$\frac{U_\infty^2}{4}\frac{d\delta}{dx}\left[\frac{68}{35} - \frac{5}{2}\right] = -\frac{3}{2}\frac{U_\infty \nu}{\delta}$$

which reduces to

$$\delta \frac{d\delta}{dx} = \frac{140}{13}\frac{\nu}{U_\infty}$$

On integration this gives

$$\frac{1}{2}\frac{13}{140}\delta^2 = \frac{\nu x}{U_\infty} + C \qquad \textbf{(2.90)}$$

Initial condition $@x = 0, \delta = 0 \text{ gives } C = 0$

**Finally :**

$$\delta = 4.64\sqrt{\frac{\nu x}{U_\infty}}$$

On substituting $Re_x = \frac{U_\infty x}{\nu}$ we get,

$$\delta = 4.64\frac{x}{\sqrt{Re_x}} \qquad \textbf{(2.91)}$$

Casting the integral form of thermal boundary layer equation :

$$\int_0^{\delta_T} u\frac{\partial T}{\partial x}dy + \int_0^{\delta_T} v\frac{\partial T}{\partial y}dy = \int_0^{\delta_T} \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}dy \qquad \textbf{(2.92)}$$

Integrating the second term by parts and applying continuity, we get :

$$\int_0^{\delta_T} u\frac{\partial T}{\partial x}dy + [vT]_0^{\delta_T} - \int_0^{\delta_T} T\frac{\partial v}{\partial y}dy = \left[\frac{k}{\rho c_p}\frac{\partial T}{\partial y}\right]_0^{\delta_T}$$

$$\text{or } \int_0^{\delta_T} u\frac{\partial T}{\partial x}dy + \int_0^{\delta_T} T\frac{\partial u}{\partial x}dy + T_\infty\, v|_{\delta_T} = -\frac{k}{\rho c_p}\frac{\partial T}{\partial y}|_{y=0}$$

where $v|_{\delta_T} = -\int_0^{\delta_T}\frac{\partial u}{\partial x}dy$

or

$$\int_0^{\delta_T}\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right)dy - T_\infty\int_o^{\delta_T}\frac{\partial u}{\partial x}dy = -\frac{k}{\rho c_p}\frac{\partial T}{\partial y}|_{y=0}$$

or,

$$\int_0^{\delta_T}\frac{\partial}{\partial x}[u(T-T_\infty)]dy = -\frac{k}{\rho c_p}\frac{\partial T}{\partial y}|_{y=0}$$

or ,

$$\frac{d}{dx}\int_0^{\delta_T}[u(T-T_\infty)]dy = -\frac{k}{\rho c_p}\frac{\partial T}{\partial y}|_{y=0} \qquad \textbf{(2.93)}$$

This **equation is called Energy Integral Equation**. In order to solve this, let us assume the temperature profile $\theta = \frac{T-T_w}{T_\infty - T_w} = C_0 + C_1\zeta + C_2\zeta^2 + C_3\zeta^3$. Where $\zeta = \frac{y}{\delta_T}$ is a similarity parameter.

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Applying thermal boundary conditions**:**

$$@\ y=0; T=T_w, \Rightarrow\ \ at\ \zeta=0,\ \ \theta=0: C_0=0$$
$$@\ y=0; \frac{\partial^2 T}{\partial y^2}=0, \Rightarrow\ \ at\ \zeta=0, \frac{d^2\theta}{d\eta^2}=0:\ \ C_2=0$$
$$@\ y=\delta_T; T=T_\infty, \Rightarrow\ \ at\ \zeta=1,\ \ \theta=1:\ \ C_1+C_3=1$$
$$@\ y=\delta_T; \frac{\partial T}{\partial y}=0, \Rightarrow\ \ at\ \zeta=1,\ \ \frac{d\theta}{d\eta}=0:\ \ C_1+3C_3=0$$

Solving for $C_1$ and $C_3$ $C_1=\frac{3}{2}$, and $C_3=-\frac{1}{2}$. Therefore

$$\frac{T-T_w}{T_\infty-T_w}=\frac{3}{2}\zeta-\frac{1}{2}\zeta^3 \qquad \textbf{(2.94)}$$

Now **,**

$$\theta-1\ =\frac{T-T_w-T_\infty+T_w}{T_\infty-T_w}\ =\frac{T-T_\infty}{T_\infty-T_w}$$

or ,

$$(T-T_\infty)\ =\ (\theta-1)\ (T_\infty-T_w)$$

First term of the energy integral **(2.93)** is:

$$(T_\infty-T_w)\frac{d}{dx}\int_0^{\delta_T} u(\theta-1)dy$$

$$U_\infty(T_\infty-T_w)\frac{d}{dx}\int_0^{\delta_T}\left[\frac{3}{2}\eta-\frac{1}{2}\eta^3\right]\left[\frac{3}{2}\zeta-\frac{1}{2}\zeta^3-1\right]dy$$

$$U_\infty(T_\infty-T_w)\frac{d}{dx}\int_0^{\delta_T}\left[\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[\frac{3}{2}\left(\frac{y}{\delta_T}\right)-\frac{1}{2}\left(\frac{y}{\delta_T}\right)^3-1\right]dy$$

Substituting $\xi=\frac{\delta_T}{\delta}$ and rearranging, we get :

$$U_\infty(T_\infty-T_w)\frac{d}{dx}\left[\delta\left\{\frac{3}{4}\xi^2-\frac{3}{20}\xi^4-\frac{3}{20}\xi^2+\frac{1}{28}\xi^4-\frac{3}{4}\xi^2+\frac{1}{8}\xi^4\right\}\right]$$
$$=U_\infty(T_\infty-T_w)\frac{d}{dx}\left[\delta\left\{-\frac{3}{20}\xi^2+\frac{3}{280}\xi^4\right\}\right]$$

The **RHS** of energy integral **equation (2.93)** is evaluated in the following way **:**

$$(T - T_w) = (T_\infty - T_w)\left[\frac{3}{2}\left(\frac{y}{\delta_T}\right) - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3\right]$$

$$\frac{\partial T}{\partial y} = (T_\infty - T_w)\left[\frac{3}{2}\frac{1}{\delta_T} - \frac{3}{2}\frac{y^2}{\delta_T^3}\right]$$

$$-\frac{k}{\rho c_p}\frac{\partial T}{\partial y}|_{y=0} = -\alpha\frac{3}{2}\frac{(T_\infty - T_w)}{\delta_T}$$

Now **equation (2.93)** becomes **:**

$$(T_\infty - T_w)U_\infty\frac{d}{dx}\left[\delta\left\{-\frac{3}{20}\xi^2 + \frac{3}{280}\xi^4\right\}\right] = -\frac{3}{2}\frac{\alpha(T_\infty - T_w)}{\delta_T} \qquad \textbf{(2.95)}$$

For $Pr > 1, \ \delta > \delta_T, \ \xi = \frac{\delta_T}{\delta}$ becomes small, and so $\xi^4 << \xi^2$

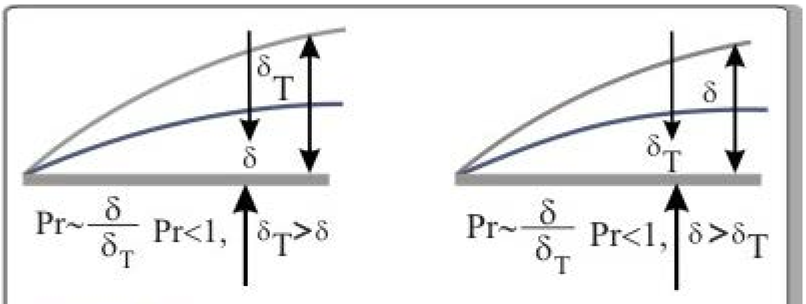


Figure 2.5: Different Situations with Pr < 1 and Pr > 1

For $Pr > 1$ **(Figure 2.5)**, **equation (2.95)** becomes **:**

$$\delta U_\infty\frac{d}{dx}\left[\delta\frac{\xi^2}{20}\right] = \frac{\alpha}{2\xi}$$

$$\frac{10\alpha}{U_\infty\xi} = \delta\frac{d\delta}{dx}\xi^2 + \delta^2\frac{d}{dx}\xi^2$$

$$\text{where} \quad \delta\frac{d\delta}{dx} = \frac{140}{13}\frac{\nu}{U_\infty}, \text{and} \quad \delta^2 = \frac{280}{13}\frac{\nu x}{U_\infty}$$

Finally, on substitution of the above two relations,**equation (2.95)** becomes :

$$\xi^3 + 2x\xi \frac{d}{dx}(\xi^2) = \frac{13}{14}\frac{1}{Pr} \qquad \textbf{(2.96)}$$

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Put $\chi = \xi^3;\ \frac{d\chi}{dx} = 3\xi^2\frac{d\xi}{dx}$ substituting in **equation (2.96)**

$$\chi + \frac{4x}{3}\frac{d\chi}{dx} = \frac{13}{14}\frac{1}{Pr}$$

$$\frac{d\chi}{dx} + \frac{3}{4x}\chi = \frac{13}{14}\frac{1}{Pr}\frac{3}{4x}$$

This linear differential equation for which the integrating factor can be seen to be $x^{3/4}$

$$x^{3/4}\frac{d\chi}{dx} + \frac{3}{4}\frac{\chi}{x}x^{3/4} = \frac{13\times 3}{14\times 4}\frac{1}{Pr}x^{-1/4} \qquad \textbf{(2.97)}$$

or

$$\frac{d}{dx}(x^{3/4}\chi) = \frac{39}{56}\frac{1}{Pr}x^{-1/4} \qquad \textbf{(2.98)}$$

$$\chi = \frac{13}{14}\frac{1}{Pr} + bx^{-3/4} \qquad \textbf{(2.99)}$$

**where b is the constant of integration :**

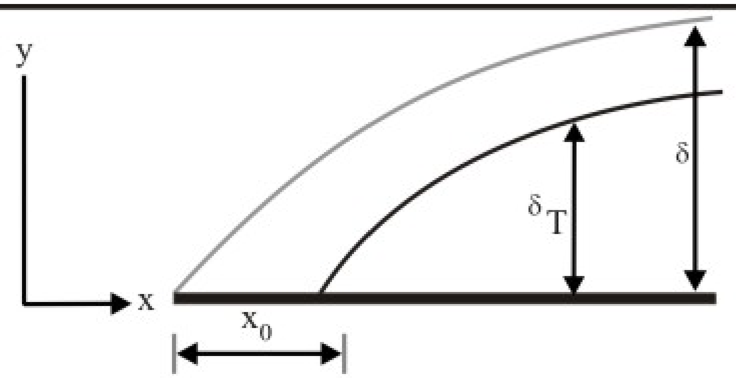


Figure 2.6:
The Flat Plate Situation with Unheated Initial Length

Applying the boundary conditions as described in Figure2.6,we get $@x = x_0,\ \delta_T = 0,\ \xi = 0$; and so, $@x = x_0,\ \chi = 0$ Substituting in **equation (2.99)**, we get **:**

$$b = -\frac{13}{14}\frac{1}{Pr}x_0^{3/4} \qquad \textbf{(2.100)}$$

After substituting in **equation (2.99) for b,** we get **:**

$$\xi^3 = \chi = \frac{13}{14}\frac{1}{Pr} - \frac{13}{14}\frac{1}{Pr}\left(\frac{x_0}{x}\right)^{3/4}$$

For $x_0 \longrightarrow 0,$ we finally have **:**

$$\delta_T = \frac{\delta}{1.026}(Pr)^{-1/3} \qquad \textbf{(2.101)}$$

Again we know that **:**

$$h(T_w - T_\infty) = -k\frac{\partial T}{\partial y}|_{y=0}$$

On substituting for $\frac{\partial T}{\partial y}|_{y=0}$ we get **:**

$$h = \frac{-k\frac{(T_\infty - T_w)}{\delta_T}\frac{3}{2}}{(T_w - T_\infty)}$$

or,

$$h = k\frac{3}{2}\frac{1.026}{4.64x}(Pr)^{1/3}(Re_x)^{1/2}$$

(after invoking **equations (2.91) and (2.101)**

The above expression leads to **:**

$$\frac{hx}{k} = 0.332(Re_x)^{1/2}(Pr)^{1/3}$$

The **local Nusselt number** can be expressed as**:**

$$Nu_x = 0.332(Re_x)^{1/2}(Pr)^{1/3} \qquad \textbf{(2.102)}$$

Now, the average heat transfer coefficient $\bar{h} = \frac{\int_0^L h_x dx}{\int_0^L dx}$ and so the **average Nusselt number**

$$\overline{Nu_L} = \frac{\bar{h}L}{k} = 0.664(Re_L)^{1/2}(Pr)^{1/3} \qquad \textbf{(2.103)}$$

Where , $Re_L = \frac{\rho U_\infty L}{\mu}$