Module 3: Internal Flows

Lecture 16: Approximate Solution

The Lecture Contains:

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Graetz Problem

Hydrodynamically developed, thermally developing flow subject to uniform wall heat flux is shown in Figure 3.11. **The problem is also known as Graetz Problem.**

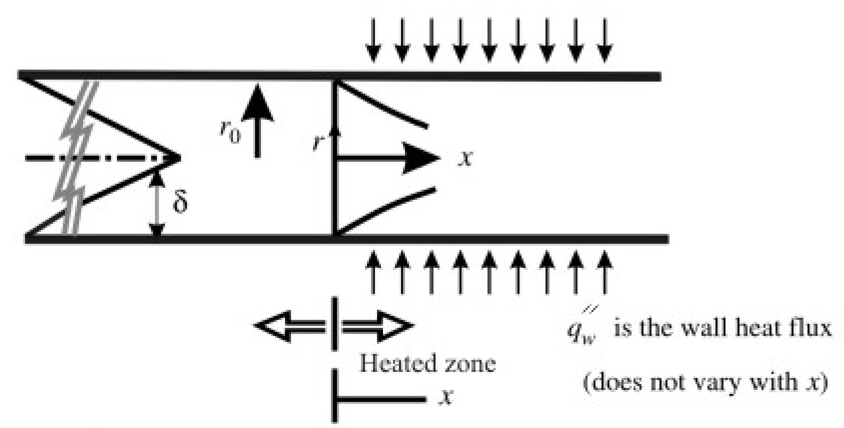

Figure 3.11:
Graetz problem with Constant Wall Heat Flux boundary condition

For $x < 0$: Both the **fluid and the wall** have uniform temperature, say T_0

For $x > 0$: Uniform all **heat flux** is applied

Preamble : We start with a situation which prevails at a very large x

Please refer to hydrodynamically and thermally fully developed flow with uniform wall heat flux.we got

$$T(r) = T_w + \frac{q''_w r_0}{k}\left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{3}{4}\right]$$

After substitution of $\theta_d(\tilde{x}, \tilde{r}) = \sum_{n=1}^{\infty} b_n \, R_n e^{-\beta_n^2 \tilde{x}}$

$$T(r) = T_w + \frac{q''_w r_0}{k}\left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{3}{4}\right] \qquad \textbf{(3.72)}$$

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from **(3.72) and (3.73)**

$$T(r) = T_m + \frac{q_w'' r_0}{k}\left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{7}{24}\right] \tag{3.74}$$

Also we have found from energy balance

$$T_m(x) = T_{m,i} + \frac{q_w'' P}{\dot{m} c_p} x$$

$$P = 2\pi r_0;\ \dot{m} = \rho u_m \pi r_0^2 \tag{3.75}$$

$$T_m(x) = T_{m,i} + \frac{2 q_w'' x}{\rho c_p u_m r_0}$$

Substitute **(3.75) in (3.74)** we get

$$T = T_{m,i} + \frac{2 q_w'' x}{\rho c_p u_m r_0} + \frac{q_w'' r_0}{k}\left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{7}{24}\right]$$

$$\frac{T - T_{m,i}}{(q_w'' r_0/k)} = \frac{4(x/r_0)}{Re\ Pr} + \left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{7}{24}\right] \tag{3.76}$$

For the present problem, the **equation (3.76)** can be written as

$$\frac{T_{fd} - T_0}{(q_w'' r_0/k)} = \frac{4(x/r_0)}{Re\ Pr} + \left[\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{7}{24}\right] \tag{3.77}$$

In the developing region, let us define Td in such a manner

$$T_d = T - T_{fd} \tag{3.78}$$

Here the suffix **"d"** means devloping and the suffix **"fd"** fully devloped .

The approximate form of energy equation
(boundary layer approximation has been invoked; also axial conduction is insignicant).

$$u\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{3.79}$$

$$u\frac{\partial T_d}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_d}{\partial r}\right) \qquad \textbf{(3.80)}$$

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Shape of the fully developed temperature prole is same as the temperature profile of thermally developing flow.

Therefore,

$$\left.\frac{\partial T_d}{\partial r}\right|_{r=r_0} = 0 = \left.\frac{\partial T_d}{\partial r}\right|_{r=0}$$

Also, to remember $@x \to \infty, T \to T_{fd} \, \text{and} \, T_d \to 0$

We know that,

$$\frac{u}{u_m} = 2\left[1 - \left(\frac{r}{r_0}\right)^2\right] \quad \textbf{(3.81)}$$

From **(3.80) and (3.81),** we get

$$2u_m\left[1 - \left(\frac{r}{r_0}\right)^2\right]\frac{\partial T_d}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T_d}{\partial r}\right] \quad \textbf{(3.82)}$$

Non-dimensional parameters $\tilde{r} = r/r_0$, $\tilde{x} = \frac{x/r_0}{Re \;\, Pr} = \frac{x\alpha}{2u_m r_0^2}$

$$\theta_d = \frac{T_d}{q''_w r_0/k} = \frac{T - T_{fd}}{q''_w r_0/k}$$

Therefore,

$$\frac{\partial T_d}{\partial x} = \frac{\alpha}{2u_m r_0^2} \cdot \frac{q''_w r_0}{k} \cdot \frac{\partial \theta_d}{\partial \tilde{x}}$$

Non-dimensional parameters

$$\tilde{r} = r/r_0,\ \tilde{x} = \frac{x/r_0}{Re\ Pr} = \frac{x\alpha}{2u_m r_0^2}$$

$$\theta_d = \frac{T_d}{q_w'' r_0/k} = \frac{T - T_{fd}}{q_w'' r_0/k}$$

$$\frac{\partial T_d}{\partial r} = \frac{q_w'' r_0}{k r_0}\frac{\partial \theta_d}{\partial \tilde{r}} = \frac{q_w''}{k}\frac{\partial \theta_d}{\partial \tilde{r}}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial T_d}{\partial r}\right) = \frac{q_w'' r_0}{k r_0}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\frac{\partial \theta_d}{\partial \tilde{r}}\right) = \frac{q_w''}{k}\frac{\partial}{\partial \tilde{r}}\left(\tilde{r}\ \frac{\partial \theta_d}{\partial \tilde{r}}\right)$$

Substituting these in **(3.82)** and simplifying, we finally get,

$$(1-\tilde{r}^2)\frac{\partial \theta_d}{\partial \tilde{x}} = \frac{\partial^2 \theta_d}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \theta_d}{\partial \tilde{r}} \qquad \textbf{(3.83)}$$

Boundary Conditions are

$$\theta_d(0,\tilde{r}) = \frac{T_0 - T_{fd}(0,\tilde{r})}{q_w'' r_0/k}$$

$$\left.\frac{\partial \theta_d}{\partial \tilde{r}}\right|_{\tilde{r}=0} = \left.\frac{\partial \theta_d}{\partial \tilde{r}}\right|_{\tilde{r}=1} = 0$$

We shall try the method of Separation of Variables
Now let,

$$\theta_d\ (\tilde{x},\ \tilde{r}) = X(\tilde{x})\ R\ (\tilde{r})$$

Substituting in **equation (3.83)** and rearranging, we get

$$\frac{1}{X}\frac{dX}{d\tilde{x}} = \frac{1}{R(1-\tilde{r}^2)}\left[\frac{d^2R}{d\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{dR}{d\tilde{r}}\right] \qquad \textbf{(3.84)}$$

Now that the variable are separated, each separated group can be equated to a constant 2 (say)

$$\frac{1}{X}\frac{dX}{d\tilde{x}} = -\beta^2 \qquad \textbf{(3.85)}$$

$$X = A_n e^{-\beta_n^2 \tilde{x}}$$

$$\frac{d^2R}{d\tilde{r}^2} + \frac{1}{\tilde{r}}\frac{dR}{d\tilde{r}} + R\beta_n^2(1-\tilde{r}^2) = 0 \qquad \textbf{(3.86)}$$

$$\tilde{r}\frac{d^2R}{d\tilde{r}^2} + \frac{dR}{d\tilde{r}} + R\tilde{r}\beta_n^2(1-\tilde{r}^2) = 0 \qquad \textbf{(3.87)}$$

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The solution of **equation (3.86 and 3.87**) is of the form

$$R = \sum_{n=1}^{\infty} C_n \; R_n$$

and obtainbale for some specic values of **n (Eigen values)** The complete solution is given by

$$\theta_d(\tilde{x}, \tilde{r}) = \sum_{n=1}^{\infty} b_n \; R_n e^{-\beta_n^2 \tilde{x}} \qquad \textbf{(3.88)}$$

where **Rn = Eigen functions and n = Eigen values of (3.86 or 3.87) Equation (3.86 and 3.87)** are called Strum Liouville equation. Symbolically stated as

$$\frac{d}{dx}[r(x) \; y'] + [q(x) + \lambda p(x)] \; y = 0$$

Boundary condition are y $(a) = h_1 \; y(a) \qquad 0 < a < \infty$

$$y'(b) = h_2 \; y \; (b) \qquad 0 < b < \infty$$

Finite upper and **lower bound h1 and h2** are real constants The solution of any function defined in the above domain is given by

$$f(x) = \sum_{n=1}^{\infty} a_n X_n(x)$$

Compare this with the form of solution we proposed for equation (3.86 and 3.87). In above the constants a_n are given by

$$a_n = \frac{\int_a^b f(x) \; X_n(x) dx}{\int_a^b X_n^2(x) dx}$$

Now the task is to compare our equation (**3.86 and 3.87)** with the strum Liouville and evaluate the constants Applying the boundary condition for equation **(3.86 and 3.87)** we get

$$\theta_d = (0, \tilde{r}) = \sum_{n=1}^{\alpha} b_n R_n$$

Substituting **x = 0 in equation (3.77)**

$$\theta_d = (0, \tilde{r}) = \frac{T_0 - T_{fd}}{q''_w r_0/k} = -\left[\tilde{r}^2 - \frac{1}{4}\tilde{r}^4 - \frac{7}{24}\right]$$

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Applying the boundary condition for **equation (3.86 and 3.87)** we get

$$\theta_d = (0,\tilde{r}) = \sum_{n=1}^{\alpha} b_n R_n$$

Substituting **x = 0 in equation (3.77)**

$$\theta_d = (0,\tilde{r}) = \frac{T_0 - T_{fd}}{q_w'' r_0 / k} = -\left[\tilde{r}^2 - \frac{1}{4}\tilde{r}^4 - \frac{7}{24}\right]$$

Comparing these two we get

$$f(r) = \sum_{n=1}^{\alpha} b_n R_n = -\left[\tilde{r}^2 - \frac{1}{4}\tilde{r}^4 - \frac{7}{24}\right] \qquad \textbf{(3.89)}$$

Now we also obtain

$$b_n = \frac{-\int_0^1 \left[(\tilde{r} - \tilde{r}^3)\right] R_n \left[\tilde{r}^2 - \frac{1}{4}\tilde{r}^4 - \frac{7}{24}\right] d\tilde{r}}{\int_0^1 \left[(\tilde{r} - \tilde{r}^3)\right] R_n^2 d\tilde{r}} \qquad \textbf{(3.90)}$$

where $\lambda = \beta_n^2$ are the **Eigen values**

Now by integrating equation **(3.86 and 3.87)** we get

$$\int_0^1 \frac{d}{d\tilde{r}}\left(\tilde{r}\,\frac{dR_n}{d\tilde{r}}\right) d\tilde{r} + \int^1 R_n(\tilde{r} - \tilde{r}^3)\,\lambda\, d\tilde{r} = 0 \quad \text{for} \quad R = R_n \qquad \textbf{(3.91)}$$

So,

$$\int_0^1 R_n(\tilde{r} - \tilde{r}^3)\lambda d\tilde{r} = 0$$

We can get λ from (3.91) and b_n can be valuated as

$$b_n = -\frac{\int_0^1 (\tilde{r} - \tilde{r}^3) R_n (\tilde{r}^2 - \frac{1}{4}\tilde{r}^4 - \frac{7}{24}) d\tilde{r}}{\int_0^1 (\tilde{r} - \tilde{r}^3) R_n^2 d\tilde{r}} \qquad \textbf{(3.92)}$$

Going back to the solution of **equation (3.88)**

$$\theta_d = \frac{T - T_{fd}}{q''_w r_0/k} = \sum_{n=1}^{\alpha} b_n e^{-\beta_n^2 \frac{x/R}{Re\ Pr}} R_n(\tilde{r})$$

Also, we know Tfd from **equations (3.77)** as Combining these two we get

$$\frac{T_{fd} - T_0}{q''_w r_0/k} = \frac{4(x/r_0)}{Re\ Pr} + \left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{7}{24}$$

Combining these two we get

$$T(x,r) = T_0 + \frac{q''_w r_0}{k}\left[\frac{4(x/r_0)}{Re\ Pr} + \left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4 - \frac{7}{24} + \sum_{n=1}^{\alpha} b_n\ R_n\ exp\left(\frac{-\beta_n^2 x/r_0}{Re\ Pr}\right)\right] \quad \textbf{(3.93)}$$

n	β_n^2	$R_n(1)$	b_n
1	25.6796	-0.492597	0.403483
2	83.8618	0.395508	-0.175111
3	174.167	-0.345872	0.105594
4	296.536	0.314047	-0.732804
5	450.947	-0.291252	0.0550357
6	637.387	0.273808	-0.043483
7	855.850	-0.259852	0.035597

Table 3.1
Eigen values and Eigen functions.

Applying the boundary condition for **equation (3.86 and 3.87)** we get

$$\theta_d = (0, \tilde{r}) = \sum_{n=1}^{\alpha} b_n R_n$$

Substituting **x = 0 in equation (3.77)** Comparing these two we get

$$\theta_d = (0, \tilde{r}) = \frac{T_0 - T_{fd}}{q''_w r_0 / k} = -\left[\tilde{r}^2 - \frac{1}{4}\tilde{r}^4 - \frac{7}{24}\right]$$

where $\lambda = \beta_n^2$ are the **Eigen values Now by integrating equation (3.86 and 3.87)** we get

$$\int_0^1 \frac{d}{d\tilde{r}}\left(\tilde{r}\,\frac{dR_n}{d\tilde{r}}\right) d\tilde{r} + \int_0^1 R_n(\tilde{r} - \tilde{r}^3)\,\lambda\, d\tilde{r} = 0 \qquad \text{for} \quad R = R_n$$

so,

$$\int_0^1 R_n(\tilde{r} - \tilde{r}^3)\lambda d\tilde{r} = 0 \qquad \textbf{(3.91)}$$

We can get λ from **(3.91)** and bn can be valuated as

$$b_n = -\frac{\int_0^1 (\tilde{r} - \tilde{r}^3) R_n (\tilde{r}^2 - \frac{1}{4}\tilde{r}^4 - \frac{7}{24}) d\tilde{r}}{\int_0^1 (\tilde{r} - \tilde{r}^3) R_n^2 d\tilde{r}} \qquad \textbf{(3.92)}$$

Going back to the **solution of equation (3.88)**

$$\theta_d = (0, \tilde{r}) = \frac{T_0 - T_{fd}}{q''_w r_0 / k} = -\left[\tilde{r}^2 - \frac{1}{4}\tilde{r}^4 - \frac{7}{24}\right]$$

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This is the complete solution. The terms bn and b_n and λ are to be determined from **equations(3.91 and 3.92).**

Now we proceed to find the **Nusselt number**

$$\theta_w = \frac{T_w}{q_w'' r_0/k}\ , \quad \theta_{fd} = \frac{T_{fd}}{q_w'' r_0/k}\ , \quad \theta_0 = \frac{T_0}{q_w'' r_0/k}$$

From **equation (3.88)**

$$\theta_d = \frac{T - T_{fd}}{q_w'' r_0/k} = \sum_{n=1}^{\alpha} b_n e^{-\beta_n^2 \frac{x/R}{Re\ Pr}} R_n(\tilde{r}) \qquad \textbf{(3.94)}$$

From **equation (3.77)** we can write

$$(\theta_{fd} - \theta_0) = 4\tilde{x} + \tilde{r}^2 - \frac{1}{4}\tilde{r}^4 - \frac{7}{24} \qquad \textbf{(3.95)}$$

Adding **equations (3.94) and (3.95)** at r=1

$$\theta_w - \theta_0 = 4\tilde{x} + \left(1 - \frac{1}{4} - \frac{7}{24}\right) + \sum_{n=1}^{\alpha} b_n R_n(1) e^{-\beta_n^2 \tilde{x}}$$

(3.96)

$$(\theta_w - \theta_0) = 4\tilde{x} + \frac{11}{24} + \sum_{n=1}^{\alpha} b_n R_n(1) e^{-\beta_n^2 \tilde{x}}$$

From energy balance:

$$T_m = T_{mi} + \frac{2q_w'' x}{\rho c_p u_m r_0}$$

Taking $T_{mi} = T_0$

(3.97)

$$(T_m - T_0) = \frac{4q_w^{''} r_0}{k}\left(\frac{x/r_0}{Re\ Pr}\right)$$

$$(\theta_m - \theta_0) = 4\tilde{x}$$

$$@\ \tilde{x} \to \infty, \quad Nu_D \to Nu_D|_{fd}$$

$$\text{Here } @\ x \to \infty$$

From **equations (3.96) and (3.97)** we get

$$(\theta_w - \theta_m) = \frac{11}{24} + \sum_{n=1}^{\alpha} b_n R_n(1) e^{-\beta_n^2 \tilde{x}}$$

By definition,

$$Nu_D = \frac{h 2r_0}{k} = \frac{q_w^{''}}{(T_w - T_m)}\frac{2r_0}{k} = \frac{2}{\theta_w - \theta_m}$$

Limiting case:

$$Nu_D = \frac{2}{\frac{11}{24} + \sum_{n=1}^{\infty}\ b_n R_n(1) e^{-\beta_n^2 \tilde{x}}}$$

$$@\ \tilde{x} \to \infty, \quad Nu_D \to Nu_D|_{fd}$$

$$\text{Here } @\ x \to \infty$$

$$Nu_D|_{fd} = \frac{48}{11} = 4.36$$

$$Nu_D|_{fd} = \frac{48}{11} = 4.36 \qquad \textbf{(3.98)}$$

The result matches with the case of **hydrodynamically and thermally fully developed flow.**