Module 2: External Flows

Lecture 6: Exact Solution of Boundary Layer Equation

The Lecture Contains:

Blausius Soution

The simplest example of boundary layer flow is flow over a flat plate **(Figure 2.1)**

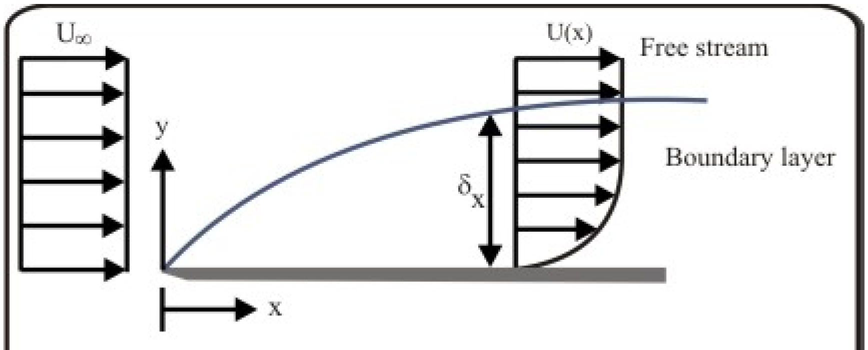

Figure 2.1: Flat Plate Boundary Layer

The governing equations are:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \qquad \textbf{(2.1)}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \textbf{(2.2)}$$

The boundary conditions are $\mathrm{(a)\ y=0,\ u=v=0,\ \ (b)\ y=\infty,\ u=U_\infty}$ Substitution of $-\frac{1}{\rho}\frac{dp}{dx}$ in the boundary layer momentum equation in terms of free stream velocity produces $U_\infty\ (dU_\infty/dx)$ which is equal to zero. Hence the governing equation does not contain any pressure gradient term. The characteristics parameters of this problem are u, U_∞, ν, x, y. Before we proceed further, let us discuss laws of similarity.

u component of velocity has the property that two velocity profiles of $u(x,y)$ at diffrerent x locations differ only by a scale factor.

The velocity profiles $u(x,y)$ at all values of x can be made same if they are plotted in coordinates which have been made dimensionless with reference to the scale factors.

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The local free stream velocity $U(x)$ at is an obvious scale factor for u, because the dimensionless **u (x)** varies with y between zero and unity at all sections. The scale factor for y, denoted by $g(x)$,,is proportional to local boundary layer thickness so that y itself varies between zero and unity . Finally

$$\frac{u[x_1,(y/g(x_1))]}{U(x_1)} = \frac{u[x_2,(y/g(x_2))]}{U(x_2)} \qquad \textbf{(2.3)}$$

Again, let us consider the statement of the problem **:**

$$u = u(U_\infty, \nu, x, y) \qquad \textbf{(2.4)}$$

Five variables involve two dimensions.Hence it is reducible for a dimensionless relation in terms of 3 quantities .For boundary layer equations a special similarity variable is available and only two such quantities are needed. When this is possible, the flow field is said to be self similar. For self similar flows the x-component of velocity has the property that two profiles of $u(x,y)/U_\infty$ at different x locations differ only by a scale factor that is at best a function of **x.**

$$\frac{u}{U_\infty} = F\left(\frac{y}{\delta}\right) = F\left(\frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}\right) \qquad \textbf{(2.5)}$$

For Blasuis problem the similarity law is **:**

$$\frac{u}{U_\infty} = F\left[\frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}\right] = F(\eta) \qquad \textbf{(2.6)}$$

Where

$$\eta = \frac{y}{\delta}, \quad \delta \sim \sqrt{\frac{\nu x}{U_\infty}} \qquad \textbf{(2.7)}$$

Or,

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}} \qquad \textbf{(2.8)}$$

Or,

$$\psi = \int u dy = \int U_\infty F(\eta)\sqrt{\frac{\nu x}{U_\infty}} d\eta = \sqrt{U_\infty \nu x}\int F(\eta) d\eta \qquad \textbf{(2.9)}$$

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$$\psi = \sqrt{U_\infty \nu x}\ f(\eta) + C(x) \qquad \textbf{(2.10)}$$

where $f(\eta) = \int F(\eta) d\eta,$ and $C(x) = 0$ if the stream function at the solid surface is set equal to 0.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = U_\infty f'(\eta) \qquad \textbf{(2.11)}$$

$$v = -\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial \eta}\ \frac{\partial \eta}{\partial x}\right) = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left[\eta f'(\eta) - f(\eta)\right] \qquad \textbf{(2.12)}$$

Similarly ,

$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2}\frac{\eta}{x}\ f''(\eta) \qquad \textbf{(2.13)}$$

$$\frac{\partial u}{\partial y} = U_\infty \sqrt{\frac{U_\infty}{\nu x}}\ f''(\eta) \qquad \textbf{(2.14)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{\nu x} f'''(\eta) \qquad \textbf{(2.15)}$$

Substituting these terms in **equation(2.1)** and simplifying we get :

$$2f'''(\eta) + f(\eta)f''(\eta) = 0 \qquad \textbf{(2.16)}$$

The **boundary conditions** are :

We know that at y = 0, u = 0 and v = 0. As a consequence, we can write

$$@\ \eta = 0, f^{'}(\eta) = 0 \ \ \text{and} \ \ f(\eta) = 0$$

Similarly at $y = \infty, u = U_\infty$ results in

$$@\ \eta = \infty, \ f^{'}(\eta) = 1$$

Equation (2.16) is a third order nonlinear differential equation. Blasius obtained this solution in the form of a series expanded around $\eta = 0$. Let us assume a power series expansion of the (for small values of η)

$$\begin{aligned} f(\eta) &= A_0 + A_1\eta + \frac{A_2}{2!}\eta^2 + \frac{A_3}{3!}\eta^3 + \frac{A_4}{4!}\eta^4 + \ldots\ldots\ldots \\ f^{'}(\eta) &= A_1 + A_2\eta + \frac{A_3}{2!}\eta^2 + \frac{A_4}{3!}\eta^3 + \ldots\ldots\ldots \\ f^{''}(\eta) &= A_2 + A_3\eta + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3 + \ldots\ldots\ldots \\ f^{'''}(\eta) &= A_3 + A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \ldots\ldots\ldots \end{aligned} \qquad \textbf{(2.17)}$$

Boundary conditions $@\ \eta = 0, \ f^{'}(\eta) = 0 \text{ and } @\ \eta = 0, \ f(\eta) = 0$ applied to above will produce $A_0 = 0; A_1 = 0$

We derive another boundary condition from the physics of the problem: $@\ y = 0, \ (\partial^2 u/\partial y^2) = 0$ which leads to $@\ \eta = 0 : f^{'''}(\eta) = 0$; invoking this into above we get $A_3 = 0$.. Finally **equation (2.17)** is substituted for $f, f^{''}, f^{'''}$ into the Blasius equation, we find

$$2\left[A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \ldots.\right] + \left[\frac{A_2}{2!}\eta^2 + \frac{A_4}{4!}\eta^4 + \frac{A_5}{5!}\eta^5 + \ldots\ldots\ldots\right]$$

$$\times\left[A_2 + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3\right] = 0$$

Collecting diffrenet powers of η and equating the corresponding coefficients equal to zero,we obtain

$$2A_4 = 0, \qquad 2\frac{A_5}{2!} + \frac{A_2^2}{2!} = 0, \qquad 2\frac{A_6}{3!} = 0$$

$$2\frac{A_7}{4!} + \frac{A_2 A_4}{4!} + \frac{A_2 A_4}{2!\ 2!} = 0$$

$$2\frac{A_8}{5!} + \frac{A_2 A_5}{5!} + \frac{A_2 A_5}{2!\ 3!} = 0$$

This will finally yield :

$$A_4 = A_6 = A_7 = 0$$

$$A_5 = -\frac{A_2^2}{2}, \qquad A_8 = \frac{11}{4}A_2^3, \qquad A_{11} = -\frac{375}{8}A_2^4$$

Now substituting the series for **f** (η)in terms of and η and A_2 :

$$f(\eta) = \frac{A_2}{2!}\eta^2 - \frac{1}{2}\frac{A_2^2}{5!}\eta^5 + \frac{11}{4}\frac{A_2^3}{8!}\eta^8 - \frac{1}{8}\frac{375}{11!}A_2^4\eta^{11}$$

(2.18)

$$f(\eta) = A_2^{1/3}\left[\frac{(A_2^{1/3}\eta)^2}{2!} - \frac{1}{2}\frac{(A_2^{1/3}\eta)^5}{5!} + \frac{11}{4}\frac{(A_2^{1/3}\eta)^8}{8!} - \frac{1}{8}\frac{375(A_2^{1/3}\eta)^{11}}{11!} + ...\right] \quad (2.18)$$

or

$$f(\eta) = A_2^{1/3}\ F(A_2^{1/3}\eta)$$

Equation (2.18) satisfies boundary conditions at $\eta = 0$. Applying boundary conditions at $\eta = \infty$,we have

$$\lim_{\eta\to\infty}\left[A_2^{2/3}F'(A_2^{1/3}\eta)\right] = f'(\infty) = 1$$

(21.9)

$$A_2 = \left[\frac{1}{\lim_{\eta\to\infty}F'(\eta)}\right]^{3/2}$$

or ,

The value of A_2 can be determined numerically to a good degree of accuracy.Howarth found $A_2 = 0.33206.$

Numerical Approach :-

Let us rewrite **Equation (2.16)**

$$2f'''(\eta) + f(\eta)\, f''(\eta) = 0$$

as three first order differential equations in the following way:

$$f' = G \qquad \textbf{(2.20)}$$

$$G' = H \qquad \textbf{(2.21)}$$

$$H' = -\frac{1}{2} f H \qquad \textbf{(2.22)}$$

The condition $f(0) = 0 \;[\text{or } \eta = 0, f(\eta) = 0]$ **remains valid and** $f'(0) = 0 \;[\text{or } \eta = 0, f'(\eta) = 0]$ means $G(0) = 0.$ $\text{Finally } \eta = \infty,\ f'(\eta) = 1 \text{ gives} \Rightarrow G(\infty) = 1$

Note that the **equations** for f and **G** have initial values. The value for **H(0)** is not known. This is not an usual initial value problem We handle the problem as an initialvalue problem by choosing values of **H(0)** and solving by numerical methods **f** (η);G (η) and **H** (η). In general **G**(∞)= 1 will not be satisfied for the function G arising from the numerical solution. We then choose other initial values of **H** so that we find an **H (0)** which results in **G** (∞) = 1. This method is called **SHOOTING TECHNIQUE.**

In Equations (2.20-2.22) the primes refer to dierentiation with respect to η. **The integration steps following a Runge-Kutta method are given below.**

$$f_{n+1} = f_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad \textbf{(2.23)}$$

$$G_{n+1} = G_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \qquad \textbf{(2.24)}$$

$$H_{n+1} = H_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \qquad \textbf{(2.25)}$$

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as one moves from η_n to $\eta_{n+1} = \eta_n + h$. The values of **k;l and m** are as follows :

$$\begin{aligned} \frac{df}{d\eta} &= f' = F_1(f, G, H, \eta) \\ \frac{dG}{d\eta} &= G' = F_2(f, G, H, \eta) \\ \frac{dH}{d\eta} &= H' = F_3(f, G, H, \eta) \end{aligned} \qquad \textbf{(2.26)}$$

$$\begin{aligned} k_1 &= h\ F_1(f_n, G_n, H_n, \eta_n) \\ l_1 &= h\ F_2(f_n, G_n, H_n, \eta_n) \\ m_1 &= h\ F_3(f_n, G_n, H_n, \eta_n) \end{aligned} \qquad \textbf{(2.27)}$$

$$\begin{aligned} k_2 &= h\ F_1\left(f_n + \frac{1}{2}k_1,\ G_n + \frac{1}{2}l_1,\ H_n + \frac{1}{2}m_1,\ \eta_n + \frac{h}{2}\right) \\ l_2 &= h\ F_2\left(f_n + \frac{1}{2}k_1,\ G_n + \frac{1}{2}l_1,\ H_n + \frac{1}{2}m_1,\ \eta_n + \frac{h}{2}\right) \\ m_2 &= h\ F_3\left(f_n + \frac{1}{2}k_1,\ G_n + \frac{1}{2}l_1,\ H_n + \frac{1}{2}m_1,\ \eta_n + \frac{h}{2}\right) \end{aligned} \qquad \textbf{(2.28)}$$

In a similar way k_3, l_3, m_3 and k_4, l_4, m_4 ; are calculated following standard formulae for Runge-Kutta integration.

The functions F_1, F_2 and F_3 are G, H and $-fH/2$

Then at a distance $\Delta\eta$ from the wall, we have :

$$\begin{aligned} f(\Delta\eta) &= f(0) + G(0)\Delta\eta \\ G(\Delta\eta) &= G(0) + H(0)\Delta\eta \\ H(\Delta\eta) &= H(0) + H'(0)\Delta\eta \\ H'(\Delta\eta) &= -\frac{1}{2}\ f(\Delta\eta)\ H(\Delta\eta) \end{aligned} \qquad \textbf{(2.29)}$$

As it has been mentioned $f''(0) = H(0)$ is unknown. **H(0)=S** must be determined such that the condition $f'(\infty) = G(\infty) = 1$ is satisfied. The condition at innity is usually approximated at a nite value of η (around $\eta = 10$). The value of H(0) now can be calculated by finding **H (0)** at which **G** (∞) crosses unity **(Figure 2.2)** Refer to figure **(Figure 2.2) (b)**

$$\frac{\tilde{H}(0) - H(0)_1}{1 - G(\infty)_1} = \frac{H(0)_2 - H(0)_1}{G(\infty)_2 - G(\infty)_1}$$

Repeat the process by using H (0) and better of two initial values H(0).Thus the correct initial value will be determined.

Table 2.1

η	f	$f' = G$	$f'' = H$
0	0	0	0.33206
0.2	0.00664	0.06641	0.33199
0.4	0.02656	0.13277	0.33147
0.8	0.10611	0.26471	0.32739
1.2	0.23795	0.39378	0.31659
1.6	0.42032	0.51676	0.29667
2.0	0.65003	0.62977	0.26675
2.4	0.92230	0.72899	0.22809
2.8	1.23099	0.81152	0.18401

0	0	0		0.33206
3.2	1.56911	0.87609		0.13913
3.6	1.92954	0.92333		0.09809
4.0	2.30576	0.95552		0.6424
4.4	2.69238	0.7587		0.03897
4.8	3.08534	0.98779		0.2187
5.0	3.28329	0.99155		0.01591
8.8	7.07923	1.00000		0.00000

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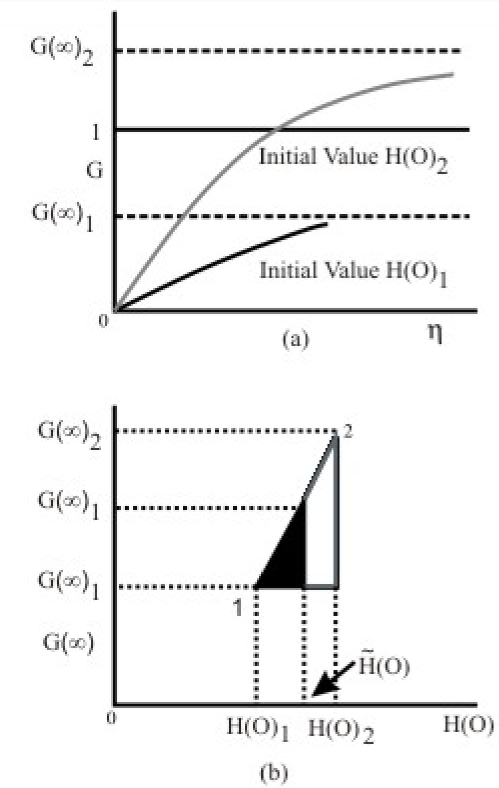

Figure 2.2 :-Corrections of the initial guess for H (0)

Shear Stress

$$\tau_{wall} = \mu \frac{\partial u}{\partial y}|_{y=0}$$

$$\tau_{wall} = \mu U_\infty \frac{\partial}{\partial \eta} f'(n) \frac{\partial \eta}{\partial y}|_{\eta=0}$$

$$\tau_{wall} = \mu U_\infty [0.33206] \frac{1}{\sqrt{\nu x / U_\infty}}$$

$$\tau_{wall} = \frac{0.332 \rho U_\infty^2}{\sqrt{Re_x}}$$

Each time examine $G(\eta)$ versus η for proper G (∞). Compare the values with Schlichting. The values are available in **Table 2.1**

Local skin friction coefficient :

$$C_{fx} = \frac{\tau_{wall}}{\frac{1}{2}\rho U_\infty^2}$$

$$C_{fx} = \frac{0.664}{\sqrt{Re_x}}$$

Total friction force per unit width

$$F = \int_0^L \tau_{wall}\, dx = \int_0^L \frac{0.332\, \rho\, U_\infty^2}{\sqrt{\frac{U_\infty}{\nu}}} \frac{dx}{\sqrt{x}} = \left[\frac{0.332\, \rho\, U_\infty^2 x^{1/2}}{\sqrt{\frac{U_\infty}{\nu}}\ \frac{1}{2}} \right]_0^L$$

$$F = 0.664\, \rho\, U_\infty^2 \sqrt{\frac{\nu L}{U_\infty}}$$

The average skin friction coefficient :

$$\overline{C}_f = \frac{F}{1/2\rho U_\infty^2 L} = \frac{1.328}{\sqrt{Re_L}}$$

From the table, it is seen that : $f' = G = 0.99$ for $\eta = 5$. So, u/U_∞ reaches 0.99 at $\eta = 5$.

It is possible to write

$$\frac{\delta}{\sqrt{\frac{\nu x}{U_\infty}}} = 5; \quad \text{or} \quad \frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}}$$