The Lecture Contains:

Transient and Frequency Response

- Lumped Analysis
- Analysis with Spatial Variations

Transient and Frequency Response

There are three fundamental questions that need to be answered with respect to the temporal response of a probe and a measurement system which are subjected to a non-zero input. These are :

1. If the input is steady, how long will it take for the probe response to become steady?

2. If the input is steady, is the probe response oscillatory?

3. If the input is periodic, what is the critical frequency beyond which the output has a negligible amplitude?

Question 3 addresses the problem of attenuation of signals as they pass through the probe and the measurement system. Further attenuation of signals can take place in spatially distributed systems due to a non-uniform response of different parts of the probe. For example, in a pitot tube the fluid close to the wall is always at rest while the bulk of the fluid within it will move during a transient. In a hot-wire anemometer, the temperature may not be uniform along its length and in particular, the portion of the wire close to the prongs will be at the prong temperature. Hence it would take finite time to re-establish the temperature profile along the wire. The reciprocal of this time determines the cut-off frequency beyond which the signal amplitude is unacceptably small.

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Lumped Analysis

A lumped parameter analysis of probes is given below. The effect of spatial variability is discussed through specific examples later in this chapter. Let $x(t)$ be the flow input and $y(t)$ the probe output. The order of a probe, a transducer or a measurement system is determined by the order of the differential equation relating x and y with time t as the independent variable. Hence we have:

$$Zeroth\ \ order: \quad y = Kx$$

$$First\ \ order: \quad \frac{dy}{dt} + cy = Kcx$$

$$Second\ \ order: \quad \frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = a_0Kx$$

In the above equations K is the static sensitivity of the probe that can be determined once-and-for-all from a steady state experiment. Consider the response of these systems to a step input $x(t) = X$, a constant and a periodic input $x(t) = X\ \exp\ (i\omega t)$. Here ω is frequency and i the imaginary unit $\sqrt{-1}$. For a step input, we assume the initial conditions to be quiescent, i.e., $y(0) = dy(0)/dt = 0$. For a periodic input we assume that the system has reached a dynamic steady state and the output $y(t)$ oscillates with the same frequency as the forcing frequency ω. The second part of this assumption is strictly true only for linear systems, i.e. coefficients a_0, a_1, c_i and K are independent of x, y and t. In both laboratory and field experiments the fluctuations in the input will displace the measurement system only marginally with respect to the operating point and so its performance can be locally linearized. Hence the linear analysis presented here is not severely restrictive.

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For zeroth order system the output will match the input at every instant of time except for a scale factor K that is predetermined. There is no attenuation or phase lag for any value of X and ω. Hence it represents an *ideal* probe or an instrument.

The response of a first order system is:

$$Step\ \ input: \quad \frac{y}{KX} = 1 - \exp(-ct) \qquad \text{(Figure 2.20)}$$

$$Periodic\ \ input: \quad \frac{|y|}{KX} = \frac{1}{(1+\omega^2/c^2)^{1/2}} \qquad \text{(Figure 2.21)}$$

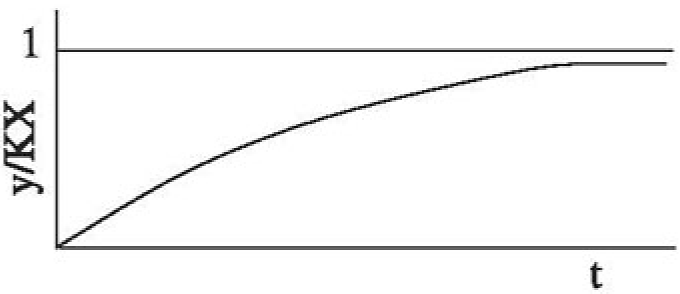

Figure 2.20: Step Response of a First Order System.

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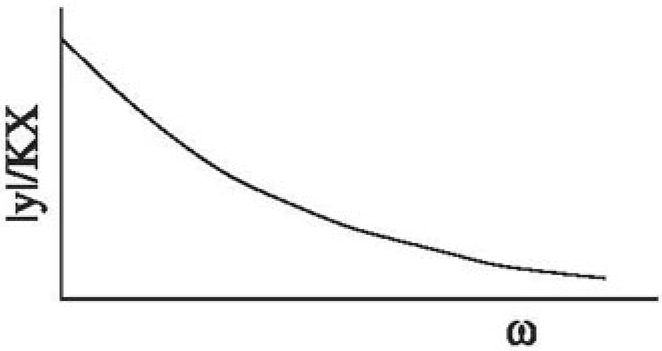

Figure 2.21: Periodic Response of a First Order System.

For a step input it takes a time $t = 3/c$ for y to reach within 95% of the steady state. Here, $1/c$ is called the time constant of the system. For a periodic input, the first order system shows attenuation for increasing values of ω. For a time constant of 0.2 second, the attenuation factor $|y|/KX$ is 0.04, when $\omega = 20$Hz, (=125.6 rad/s) and amplitude reduction by a factor of 25.

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The response of a second order system depends on the value of the damping ratio $a_1/2a_0(=S)$. For $S<1$ the system is underdamped and the response is oscillatory even for a steady input. For $S=1$ the system is critically damped and for $S>1$it is overdamped. If $S\geq 1$ the response is gradual and non-oscillatory with no overshoot (Figure 2.22). The system reaches steady state monotonically at the fastest rate if $S=1$. We have the following results for a second order system subject to a step input:

$$S<1, \frac{y}{KX}=1-\frac{\exp(-S\omega_n t)}{\sqrt{1-S^2}}\sin\left(\sqrt{1-S^2}\,\omega_n t+\ \phi\right)$$

$$S=1, \frac{y}{KX}=1-(1+\omega_n t)\exp\ (-\omega_n t)$$

$$S>1, \frac{y}{KX}=1-\frac{S+\sqrt{S^2-1}}{2\sqrt{S^2-1}}\exp\left(\left(-\mathrm{S}-\sqrt{\mathrm{S}^2-1}\right)\,\omega_\mathrm{n}\mathrm{t}\right)$$

$$+\frac{S-\sqrt{S^2-1}}{2\sqrt{S^2-1}}\exp\left(\left(-\mathrm{S}-\sqrt{\mathrm{S}^2-1}\right)\,\omega_\mathrm{n}\mathrm{t}\right)$$

Here the natural frequency $\omega_n=\sqrt{a_0}$ and the phase lag $\phi=\sin^{-1}(\sqrt{1-S^2})$ for $0<S<1$. For a periodic input:

$$\frac{|y|}{KX}=\left[1-\left(\frac{\omega^2}{\omega_n^2}\right)^2+\left(2S\,\frac{\omega}{\omega_2}\right)^2\right]^{-1/2}$$

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Module 2: Review of Probes and Transducers

Lecture 10: Temporal and frequency response

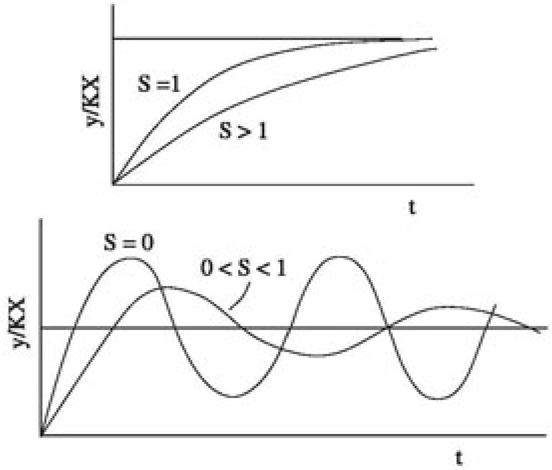

Figure 2.22: Step Response of a Second Order System.

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While the first order system unconditionally attenuates the input, second order can amplify it in the neighbourhood of the natural frequency ω_n for $S < 1$. A nearly uniform frequency response is obtained for *S* = 0.6 and $\omega < 3\omega_n$ (Figure 2.23) and this value of the damping factor is commonly used in the design of instruments.

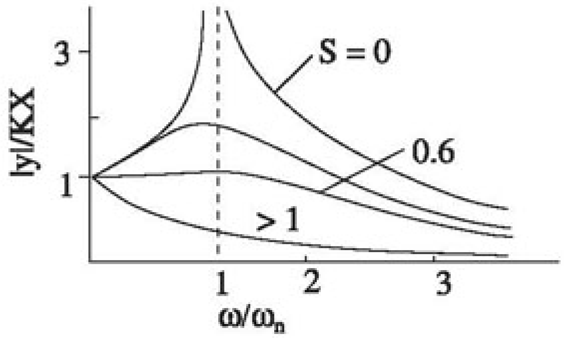

Figure 2.23: Periodic Response of a Second Order System.

As a rule mechanical systems are second order systems due to their inertia while fluid systems are of first order and electrical systems containing all the three elements R, L and c_i are of second order. RC systems are of first order and R- systems of zeroth order. A hot-wire anemometer working in the constant temperature mode with feedback will be shown later to be nearly a zeroth order system and hence close to an ideal probe.

The performance of certain probes that are commonly used in thermal sciences are described below by examining their transient behaviour.

Example: Consider a U-tube manometer carrying an isothermal incompressible liquid and subject to a pressure difference $p_a - p_b$ (Figure 2.24). The pressure difference is a constant; the initial liquid level is I and the final is F.

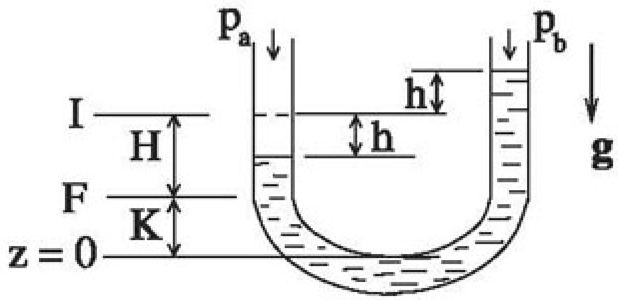

Figure 2.24: Dynamic Response of U-tube Manometer.

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The instantaneous liquid displacement is $h(t)$. It is of interest to determine the manner in which h increases from zero initially to a final value of H. Let A be the cross-sectional area of the tube and L be the length of the liquid column. The energy equation for the column is written as

$$\frac{d}{dt}\ [\,total\ \ energy\,] \quad = \quad Rate\ \ of\ \ external\ \ work\ \ done$$

Here the total energy of the manometer liquid consists of kinetic and potential energies and is written as

$$\int_0^L \int_0^R \frac{1}{2}\rho u^2 (2\pi r) dr dl \ + \ \rho g A\left\{\int_0^{K+H-h} \ zdz + \int_0^{K+H+h} \ zdz\right\}$$

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dl being a differential element in the liquid column and u is dh/dt. The external work done is due to the pressure difference $(p_a - p_b)$,a part of which is required to overcome viscous friction in the limbs of the manometer. Assuming a parabolic profile for velocity with a mean value dh/dt, the expression for work done can be written as

$$(p_a - p_b)\ A\ \frac{dh}{dt} - \int_0^L \int_0^R \left(\frac{\partial u}{\partial r}\right)^2 (2\pi r\mu\ dr)\ dl$$

The equation governing $h(t)$is then obtained after some algebraic manipulation as

$$h'' + \frac{6\mu}{R^2\rho}\ h' + \frac{3}{2}\frac{g}{L}h = \frac{3}{4}\left(\frac{p_a - p_b}{\rho\ L}\right)$$

where $h' = dh/dt$. Hence the U-tube manometer is a second order system with a natural frequency

$$\omega_n = \sqrt{\frac{3g}{2L}}$$

and a damping ratio

$$S = \frac{6\mu}{2R^2\rho}\sqrt{\frac{2L}{3g}}$$

For a manometer response without oscillations we require $S \geq 1$. As an example consider a column of length $L = 1$ m and water as the working fluid. Since μ= 80 $\times 10^{-5}$ Pa. S and $\rho = 1000$ kg/m^3 , the condition $S \geq 1$ results in $R \leq 0.39$ mm. Using tubes of larger radii will result in manometer oscillations.

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Example: Consider a pitot tube suddenly immersed in a flow for velocity measurements. At the mouth of the tube the pressure will increase to $p(t)$, the total pressure. The pitot tube response is governed by the time taken by the entire tube to attain a uniform pressure $p(t)$. If the fluid as taken as incompressible then the pressure within the tube will adjust to $p(t)$ instantaneously. Hence transients are related to the compressibility of the fluid. If $p_t - p_0$ is small, the fluid will behave as if it is nearly incompressible . However, the flexible tubing connecting the pitot tube to the manometer can be very long and of small diameter and hence have a large resistance. This can result in a finite time constant for the pitot tube and the associated tubing.

Let p_t be the instantaneous pressure in the tube and Rs be the resistance to the flow. Then the instantaneous flow is $(p_t - p)/Rs$. Assuming laminar Poisseulle flow behaviour in the tube Rs may be estimated as $8\mu L/R^4\pi$, where L is the tube length and R the radius. This is the minimum resistance offered by the tube. Effects of flow development will increase the resistance beyond this value. Let V be the volume of the tube $(=\pi R^2 L)$. The additional flow q will change the density of the fluid as

$$V d\rho = \rho \quad q \quad dt$$

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Let S be the compressibility of the fluid, i.e. $S = (1/\rho)d\rho/dp$. For an ideal gas at constant temperature $S = 1/p$. Hence

$$\rho VS \ \frac{dp}{dt} = \rho \ \ q = \rho(p_t - p)Rs$$

This is cast as a first order system

$$\frac{dp}{dt} + \frac{1}{VRsS}p = \frac{p_t}{VRsS} \quad and \quad p(0) = p_0$$

The time constant of the pitot tube is $VRsS$. For $R = 1$ mm, $L = 2$ m, at 300 K the time constant of a pitot tube exposed to flow can be calculated as 6 milliseconds. The cut-off frequency beyond which a fluctuating signal is severely attenuated can be estimated as $1/VRsS = 160$ Hz.

It is instructive to derive the pressure evolution equation in the pitot tube by suitable reduction of the Navier-Stokes equations. This discussion is left to the reader as an exercise.

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Example: First consider a hot-wire working in constant current mode (Figure 2.25). The wire whose resistance equals R is a part of a Wheatstone bridge with a control resistance R and a battery voltage V. The Joule heating of the wire is $V^2 R/(R+R_1)^2$ and the energy lost to the flow is $hA(T-T_f)$, where h is the heat transfer coefficient, A the surface area of the wire and T its temperature. T_f is the local fluid temperature. The difference between the two energy terms will change the wire temperature at a rate $mc_p dT/dt$and subsequently the wire resistance. We assume the wire resistance to change as

$$R_w = R_o \left(1 + \alpha \left(T_w - T_o\right)\right)$$

and so

$$\frac{dT}{dt} = \frac{1}{aR_0}\frac{dR}{dt}$$

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Hence the energy balance equation for the wire reads as

$$\frac{dR}{dt}+\left(\frac{hA}{mC_p}-\frac{\alpha R_0}{mC_p}\cdot\frac{V^2}{(R+R_1)^2}\right)\ R=\frac{hA}{mC_p}\ \ R_0$$

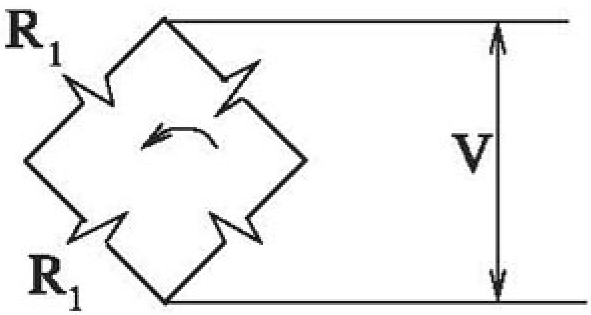

Figure 2.25: Transient Response of a Hot-wire Probe.

Here R_0 and T_0 are reference wire resistance and temperature respectively. The time constant of the wire is the reciprocal of the term multiplying R. It is desirable to reduce the time constant to a minimum value for improving the transient as well as the frequency response. This is accomplished under the following conditions:

$$small\ \ m\ \ C_p \qquad large\ \ h \qquad large\ \ R_1$$

When a hot wire is operated under constant temperature conditions through a feedback circuit $dR/dt=0$ and it becomes a zeroth order system. In principle the hot wire attains an infinite frequency response. In practice it is limited by the frequency response of the feedback circuit. This is in the range 10 to 100 kHz and is usually adequate for flow studies.

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Analysis with Spatial Variations

The effect of spatial distribution of the measured variable over the probe on the frequency response is considered through two examples.

Example: Spatial variability in the probe response can cause additional attenuation in the output signal at the high frequency end. Consider a pitot tube of radius R, length L and subjected to a pressure gradient

$$\frac{dp}{dz} = P \exp(i\omega t)$$

The resulting flow at dynamic steady state is not only a function of time but also the radial coordinate r since the velocity is zero at the tube walls and finite along its axis. Assuming a long tube and no edge effects, the Navier-Stokes equations can be reduced to

$$\rho \frac{\partial u}{\partial t} = \mu \left(u_{rr} + \frac{1}{r} u_r\right) - \frac{dp}{dz}$$

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Let $u(r,t) = \phi(r)\ \exp\ (i\omega t)$, where ϕ is the complex amplitude of u. Since $u = 0$ at $r = R$, $\phi(R) = 0$ and $\phi(0)$ must be finite. ϕ satisfies the equation

$$\phi_{rr} + \frac{1}{r}\ \phi_r - \frac{i\omega}{v}\ \phi = \frac{P}{\mu}$$

The solution of the equation is

$$\phi\ (r) = \frac{iP}{\rho\omega} + (c_r + ic_i)\left\{ber\left(\frac{\omega r^2}{v}\right)^{1/2} + i\ bei\left(\frac{\omega r^2}{v}\right)^{1/2}\right\}$$

where

$$c_i = \frac{(-P/\rho\omega)\ ber}{ber^2 + bei^2}$$

$$c_r = \frac{bei}{ber}\ c_i$$

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The Bessel functions ber and bei used in c_r and c_i have $(\omega R^2/\nu)^{1/2}$ as an argument. Further

$$ber\ \ x = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (x/2)^{4k}}{(2k!)^2}$$

$$bei\ \ x = \frac{x^2}{4} + \sum_{k=1}^{\infty} \frac{(-1)^k (x/2)^{4k+2}}{((2k+1)!)^2}$$

Define the complex mean velocity through the tube as

$$\bar{u} = \frac{1}{\pi R^2} \int_0^R 2\pi r\ \phi(r) dr$$

The equivalent resistance of the tube as $Rs = P/|\bar{u}|$ and the normalized resistance as $RS = Rs(\omega)/Rs(0)$. Values of RS as a function of $\omega R^2/\nu$ are given in Table 2. The reciprocal of the normalized resistance is a measure of the attenuation factor.

Table 2: Attenuation Factor for a Pitot Probe as a Function of the Dimensionless Frequency

$\omega R^2/\nu$	0.0	0.5	1	5	10	50	100	500
RS	1.0	1.0	1.014	1.315	1.968	7.63	14.4	66.5

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The effective time constant of a pitot tube can be determined by considering together the effect of lumping density changes and spatial variations in velocity. From lumped analysis the time constant T_{c1} is

$$T_{c1} = VRsS = \frac{8\mu L^2 S}{2} = 1.4161 \times 10^{-9} \ (LR)^2 \ seconds$$

for air flow. Thus the value of time constant T_{cl} for a pitot tube obtained from lumped analysis is a function of both the radius and the tube length.

Consider spatial variation next. The associated time constant is T_{c2}. The attenuation factor A can be expressed as the reciprocal of the equivalent resistance, namely

$$A = \frac{1}{\sqrt{1 + \omega^2 T_{c2}^2}}$$

Values of A for each frequency ω can then be obtained from Table 2 using this formula. The optimum value of A is determined using a least squares procedure through the reciprocal of the RS data in Table 2. This gives $\nu \ \ T_{c2}/R^2 = 0.145$ in dimensionless form and T_{c2} can be calculated for any value of R.

The effective total time constant τ can be taken as the sum of the individual time constant T_{c1} and T_{c2}. This value has been plotted in Figure 2.26 as a function of the radius for three different lengths. As the radius R increases the viscous resistance to flow reduces faster than the increase in the fluid volume. Hence T_{c1} decreases with R. In contrast to this it takes a longer time for the presence of the wall to be felt and T_{c2} increases with R. At some intermediate radius ($R: 0.5$ mm in Figure 2.25) the total time constant attains a minimum value. This radius as well as the magnitude of the total time constant depend on the length of the tubing. However Figure 2.26 shows that the effect of the length is not as pronounced as the radius. For a tube of radius greater than 2 mm the time constant is independent of the length and determined by R (and hence spatial variation) alone.

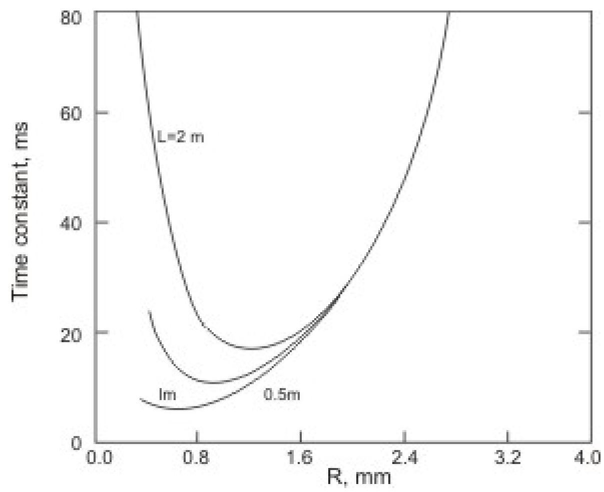

Figure 2.26: Time Constant of a Pitot Tube as a Function of Radius and Length.

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Example: When temperature in a hostile environment must be measured it is common to provide a metal shield to the thermocouple as shown in Figure 2.27.

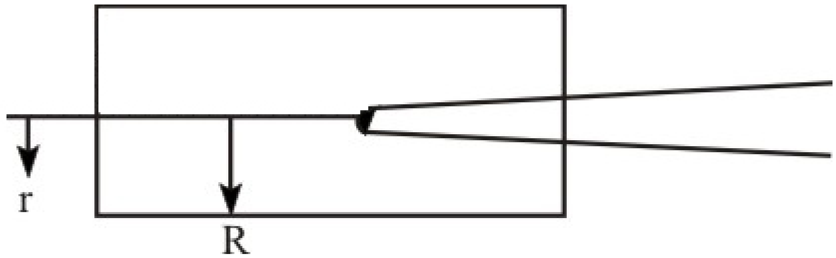

Figure 2.27: Shielded Thermocouple.

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To study the frequency response of the embedded thermocouple we consider the surface temperature to fluctuate as $\exp\ (i\omega t)$ with unit amplitude. The amplitude of temperature along the axis as a function of ω will determine the extent of attenuation of the surface signal before it reaches the thermocouple. At any location in the shield the diffusion equation is valid and expressed as

$$\frac{\partial T}{\partial t} = \alpha \left(T_{rr} + \frac{1}{r}\ T_r \right)$$

where α is thermal diffusivity of the metallic shield. Let $T = \phi(r)\exp(i\omega t)$. Hence ϕ satisfies

$$\phi_{rr} + \frac{1}{r}\ \phi_r - \frac{i\omega}{\alpha}\ \phi = 0 \quad with \quad \phi(R) = 1$$

where $\phi_r\ = d\phi/dr$ and the second boundary condition is

$$\phi(0) = \ finite$$

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The solution of this equation is

$$\phi(r) = (c_r + i c_i)\left\{ ber\left(\frac{\omega r^2}{\alpha}\right)^{1/2} + i \quad bei\left(\frac{\omega r^2}{\alpha}\right)^{1/2}\right\}$$

where

$$c_r = \frac{ber}{(ber^2 + bei^2)}$$

and

$$c_i = -\, c_r \frac{bei}{ber}$$

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In c_r and c_i of the above expressions, the argument for the ber and bei functions as defined in Example 14 is $(\omega R^2/\alpha)^{1/2}$.The amplitude of the centre-line temperature is

$$|\phi(0)| = (c_r^2 + c_i^2)^{1/2}$$

This relationship in numerical form is given in Table 3.

Table 3: Attenuation Factor for a Thermocouple as a Function of the Dimensionless Frequency

$\omega R^2/\alpha$	0	1	5	10	50	100
$\lvert\phi(0)\rvert$	1.0	0.984	0.744	0.47	0.044	0.033

For $\omega R^2/\alpha = 10$ the attenuation is nearly 50%. For a copper shield ($\alpha = 10^{-4}$m^2 /s) a diameter of 20 mm gives a corresponding frequency as 60 Hz. The temperature sensed by the thermocouple drops further at higher forcing frequencies applied at the surface of the shield.

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