

Module 4: Interferometry

Lecture 23: Introduction to tomography, CBP

The Lecture Contains:

- ☰ Computerized Tomography
- ☰ Convolution Backprojection
- ☰ Iterative Tehniques
- ☰ ART
 - Simple ART
 - Gordon ART
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COMPUTERIZED TOMOGRAPHY

The three-dimensional temperature field can be reconstructed from its interferometric projections using principles of tomography. Tomography is the process of recovery of a function from a set of its line integrals evaluated along some well-defined directions. In interferometry, the source of light (the laser) and the detector (CCD camera) lie on a straight line with the test cell in between. Further a parallel beam of light is used. This configuration is called *transmission* tomography and the ray configuration as the parallel beam geometry [28]. Tomographic algorithms used in interferometry reconstruct two-dimensional fields from their one-dimensional projections. Reconstruction is then applied sequentially from one plane to the next until the third dimension is filled.

Tomography can be classified into: (a) transform (b) series expansion, and (c) optimization methods. Transform methods generally require a large number of projections for a meaningful answer [92]. In practice, projections can be recorded either by rotating the experimental setup or the source-detector combination. In interferometry, the latter is particularly difficult and more so with the Mach-Zehnder configuration. With the first option, It is not possible to record a large number of projections, partly owing to inconvenience and partly due to time and cost. Hence, as a rule, a large number of projections cannot be acquired with interferometry and one must look for methods that converge with just a few projections. Limited-view tomography is best accomplished using the series expansion method [29]. As limited-view tomography does not have a unique solution, the algorithms are expected to be sensitive to the initial guess of the field the start the iterations. Optimization-based algorithms are known to be independent of initial guess, but the choice of the optimization functional plays an important role in the result obtained. Depending on the mathematical definition used, the entropy extremization route may yield good results, while the energy minimization principle may be suitable in other applications. For the algebraic techniques considered in the present study, an unbiased initial guess such as a constant profile was seen to be good enough to predict the correct temperature field. A complete random number guess can also be viewed as an unbiased initial guess. Tomography begin an inverse technique, was seen to preserve (and amplify under certain conditions) the noise in the initial data. The dominant trend in the field variable was seen to be however captured during tomography inversion.

Convolution Backprojection

The convolution backprojecton (CBP) algorithm for three-dimensional reconstruction classifies as a transform technique. It has been used for medical imaging of the human brain for the past several decades. Significant advantages of this method include (a) its noniterative character, (b) availability of analytical results on convergence of the solution with respect to the projection data, and (c) established error-estimates. A disadvantage to be noted is the large number of projections normally required for good accuracy. In engineering applications, this translates to costly experimentation, and nonviability of recording data in unsteady experiments the use of CBP continues to be seen in steady flow experiments, particularly when the region to mapped is physically small in size. The statement of the CBP algorithm is presented below.

Let the path integral equation be written as

$$P(s, \theta) = \int_C f(r, \phi) dz \quad (17)$$

where p is projection data recorded in the experiments and f is the function to be computed by inverting the above equation. In practice, the function f is a field variable such as density, void fraction, attenuation coefficient, refractive index, and temperature. The symbols s, θ, r and ϕ stand for the ray position, view angle, position within the object to be reconstructed, and the polar angle, respectively (fig. 4.50). The integration is performed with respect to the variable z along the chord C of the ray defined by s and θ . Following Herman [28], the *projection slice theorem* can be employed in the form

$$\bar{p}(R, \theta) = \bar{f}(R \cos(\theta), R \sin(\theta)) \quad (18)$$

Where the overbar indicates the Fourier transform and R is the spatial frequency. In words, this theorem states the equivalence of the one-dimensional Fourier transform of $p(s, \theta)$ with respect to s and the two-dimensional Fourier transform of $f(r, \phi)$ with respect to r and r and ϕ . A two-dimensional Fourier inversion of this theorem leads to the well-known Radon transform

$$f(r, \phi) = \int_0^\pi \int_{-\infty}^{\infty} \bar{p}(R, \theta) \exp(i2\pi R r \cos(\theta - \phi)) |R| dR d\theta$$

where

$$\bar{p}(R, \theta) = \bar{f}(R \cos(\theta), R \sin(\theta))$$

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The first integral in the form given above is divergent with respect to the spatial frequency R . A practical implementation of the formula replaces $|R|$ by $W(R)|R|$, where W is a window function that vanishes outside the interval $[-R_c, R_c]$. The cut-off frequency R_c can be shown to be inversely related to the ray-spacing for a consistent numerical calculation of the integral. When the filter is purely of the band-pass type, the Radon formula can be cast as a convolution integral:

$$f(r, \phi) = \iint_{0-\infty}^{\pi \infty} P(s, \theta) q(s' - s) ds d\theta \quad (19)$$

where

$$q(s) = \int_{-\infty}^{\infty} |R| W(R) \exp(i2\pi R s) dR$$

and

$$s' = r \cos(\theta - \phi)$$

The inner integral over s is one dimensional convolution and the outer integral, an averaging operation over θ is called *back projection*. This implementation of the convolution backprojection algorithm is commonly used in medical imaging. Applications of the CBP algorithm to flow and heat transfer problems have been reviewed by Munshi [94].

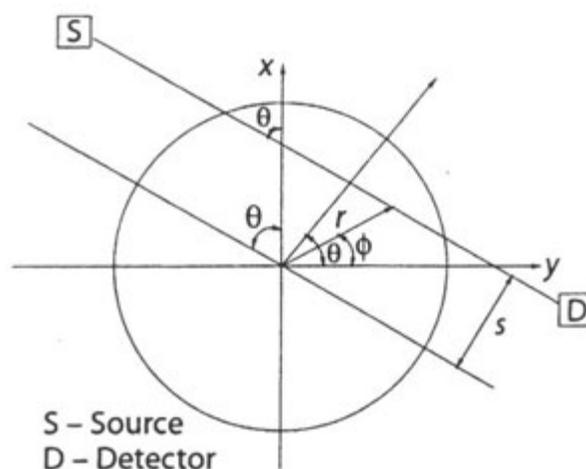


Figure 4.63: Nomenclature for the convolution backprojection algorithm

Iterative Techniques

Series expansion methods are perhaps the most appropriate tomographic technique for interferometry since they work limited projection data. These methods are iterative in nature and consist necessarily of four major steps, namely:

- initial assumption of the field to be reconstructed over a grid,
- calculation of the correction for each pixel,
- application of the correction, and
- test for convergence,

The central idea behind the calculation of the correction (step2) is the following. With the assumed field, one can explicitly compute projection values by numerical integration. The difference between the computed projection and experimentally recorded projection data is a measure of the error in the assumed solution this error can be redistributed to the pixels so that error is reduced to zero. Repetition of these steps is expected to converge to a meaningful solution. The series expansion techniques differ only in the manner in which the errors are redistributed over the grid.

The word *convergence* in step 4 is used in an engineering sense as a stopping criterion for the iterations, and not in the strict mathematical sense, where a formal proof is needed to show convergence of the numerical solution to the exact solution

The iterative methods require the discretization of the plane to be reconstructed by a rectangular grid (Figure 4.66). The length of the intercept of the i -th ray with the j -th cell in given projection is known as the weight function w_{ij} . It can be shown that

$$\phi_i = \sum_{j=1}^N w_{ij} f_j \quad i = 1, 2, \dots, M \quad (20)$$

where ϕ refers to the projection data. The number of unknowns N in most cases is much larger than the number of unknowns M . This discretization produces a matrix equation

$$[w_{ij}] \{f_j\} = \{\phi_i\} \quad (21)$$

The problem of reconstruction thus is a problem of inversion of a rectangular matrix. Iterative techniques that are used in the tomography can be viewed as developing a generalized inverse of the matrix $[w_{ij}]$. This matrix in a typical laser tomography problem has large dimensions for the differentially heated fluid layer, the greatest matrix size encountered was 560×14400 . This is a sparse matrix with many of its elements being zero. General purpose matrix libraries cannot be used to invert such matrices since they are highly ill-conditioned and rectangular in structure. The tomographic algorithms can be seen as a systematic route towards a meaningful inversion of the matrix equation.

Series expansion methods being discussed in the present section can be classified into: ART (Algebraic Reconstruction Technique) and MART (multiplicative Algebraic Reconstruction Technique). The optimization of the entropy and minimization of the energy functions.

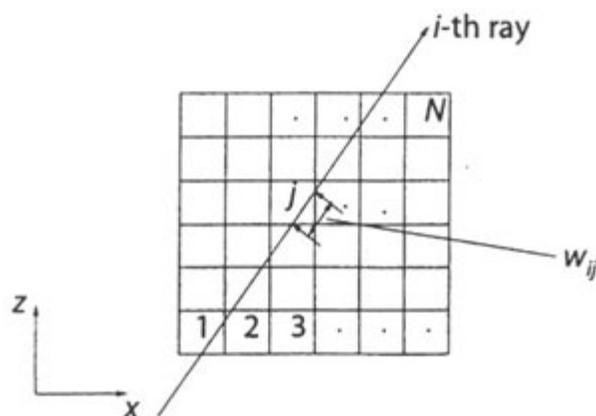


Figure 4.64: Discretization of a plane of fluid layer for art calculations

The ART and MART families of algorithms differ only in the method of updating the field parameters in each iteration. In ART, the correction is additive while for MART, the correction is multiplicative. In both cases, the numerical procedure is based on the comparison of the estimated projection from an initial guess with the measured projection data obtained through experiments. This gives a correction term for the field variables. The values of the variables are then updated. Once an iteration is over, the field value differs from the previous guess. The extent of the difference is then calculated. If the difference is within acceptable limits, the field value is taken to represent the physical field. Otherwise, the iterations continue until the convergence criterion is satisfied.

Since the original field in real experiments is unknown, an estimate of the number of iterations can be found by using test functions (*called phantoms*) that are similar in nature to the original field. The test functions are also perturbed with noise to gauge the sensitivity of the algorithms to issues such as initial guess and error in the projection data. This method can only be adopted where an exact estimate of noise in the projection data and a good knowledge of the original field is known beforehand. Variations in the noise level and nature of the noise in the projection data can alter the convergence rates.

Tomographic algorithms used in the present work are iterative in nature and intermediate steps may also involve iterations in the form of FOR loops. To identify the beginning and the ending of each iterative loop, start and close labels with statement number have been indicated in the description of each algorithm. These algorithms are briefly surveyed in the following sections.

ART

Various ART algorithms are available in the literature owing their origin to Kaczmarz [95] and Tanabe [96]. They differ from each other in the way the correction is applied. Those presented below have been tested successfully by the author and his coworkers in the context of interferometry.

Simple ART

This algorithm has been suggested by Mayinger [6]. The corrections are applied through a weight factor. Computed as an average correction along a ray. The difference between the calculated projections with the measured projection data gives the total correction to be applied for a particular ray. The average correction is then the contribution to each cell falling in the path of the ray. This is computed by dividing the total correction obtained with the length of the ray. The calculated projection are computed once for a particular angle. Though the field values are continuously updated the calculated projection values remain unchanged until the completion of all the rays for a given angle. This algorithm will be referred to as ART1 in future discussions.

Let $\phi_{i\theta}$ be the projection due to the i -th ray in the θ direction of projection and \tilde{f}_i be the initial guess of the field value. Numerically the projection $\tilde{\phi}_{i\theta}$ using the current field values is defined as:

$$\tilde{\phi}_{i\theta} = \sum_{j=1}^N w_{i\theta,j} f_i \quad i\theta = 1, 2, \dots, M_\theta \quad (22)$$

The individual steps in the algorithm are listed below.

Calculate the total value of weight function ($W_{i\theta}$) along each ray as:

Start:1 For each projection angle (θ):

Start:2 For each ray ($i\theta$):

Start:3 For each cell (j):

$$W_{i\theta} = \sum_{j=1}^N w_{i\theta,j}$$

Close:3

Close:2

Close:1

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Start:4 start iterations (k):

Start:5 For each projection angle (θ):

Start:6 For each ray ($i\theta$):

Compute the numerical projection (Equation 21)

Close:6

Start:7 For each ray ($i\theta$):

Calculate the correction as:

$$\Delta\phi_{i\theta} = \phi_{i\theta} - \tilde{\phi}_{i\theta}$$

Calculate the average of correction as:

$$\overline{\Delta\phi_{i\theta}} = \frac{\Delta\phi_{i\theta}}{W_{i\theta}}$$

Close:7

Start:8 For each ray ($i\theta$):

Start: 9 For each cell (j):

If $w_{i\theta,j}$ is non-zero then

$$f_j^{new} = f_j^{old} + \mu \frac{\Delta\phi_{i\theta} \times w_{i\theta,j}}{W_{i\theta}}$$

where μ is a relaxation factor.

Close:9

Close:8

Close:5

Check for convergence as:

If

$$abs \left[\frac{f^{k+1} - f^k}{f^{k+1}} \right] \times 100 \leq e$$

(where e is the prescribed convergence, say 0.01%)

STOP:

Else: continue

close:4 (k)

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Gordon ART

Algorithm contributed by Gordon et al. [97] is considered. Mayinger's ART is similar to this original version under the condition that no two ray simultaneously pass through a particular cell for a given projection. In this method corrections are applied to all the cells through which the i -th passes, using the weight factor which is exactly the proportion of w_{ij} to the total length of the ray. The projection data gets updated after calculations through each ray. This procedure will be referred to as ART2. The individual steps are:

Calculate the total value of weight function ($W_{i\theta}$) along each ray as:

Start:1 For each projection angle (θ):

Start:2 For each ray ($i\theta$):

Start:3 For each cell (j):

$$W_{i\theta} = \sum_{j=1}^N w_{j\theta,j} \times w_{i\theta,j}$$

Close:3

Close:2

Close:1

Start:4 start iterations (k)

Start:5 For each projection angle (θ):

Start:6 For each ray ($i\theta$):

Compute the numerical projection (Equation 22)

Calculate the correction as:

$$\Delta\phi_{i\theta} = \phi_{i\theta} - \tilde{\phi}_{i\theta}$$

Start:7 For each cell (j)

If is non-zero then:

$$f_{j^{new}} = f_{j^{old}} + \mu \frac{\Delta\phi_{i\theta} \times w_{i\theta,j}}{w_{i\theta}}$$

where μ is a relaxation factor:

Close:7

Close:6

Close:5

Check for convergence as:

If

$$\text{abs} \left[\frac{f^{k+1} - f^k}{f^{k+1}} \right] \times 100 \leq e$$

STOP:

Else: Continue

Close: 4 (*k*)

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Gilbert ART

Gilbert [98] has developed independently a form ART known as SIRT (Simultaneous Iterative Reconstruction Algorithm). In SIRT, the elements of the field function are modified after all the corrections corresponding to individual pixels have been calculated. This will be referred to as ART3. The numerically generated projection is computed once for all the angles and gets updated only after the completion of calculations through all the rays. For each ray from all angles, all the cells are examined to look for those rays which pass through a particular cell. For each cell, the rays which pass through it will contribute a correction that is decided by the weight factor w_{ij} . The algebraic average of all these corrections is implemented on the cell. This procedure will be called ART3. Its individual steps are: Calculate the total value of weight function ($W_{i\theta}$) along each ray as:

Start:1 For each projection angle (θ):

Start:2 For each ray ($i\theta$):

Start:3 For each cell (j):

$$W_{i\theta} = \sum_{j=1}^N w_{j\theta,j} \times w_{i\theta,j}$$

Close: 3

Close: 2

Close: 1

Start: 4 Start iterations (k):

Start: 5 For each projection angle (θ):

Start: 6 For each ray ($i\theta$):

Compute the numerical projection (Equation 21)

Calculate the correction as:

$$\Delta\phi_{i\theta} = \phi_{i\theta} - \phi_{i\theta}$$

Close: 6

Close: 5

Start:7 For each cell (j):

Identify all the rays passing through given cell (j) let be the total number of rays passing through the j -th cell) and corresponding $i\theta$. $w_{i\theta,j}$ and $w_{i\theta}$ and $\Delta\phi_{i\theta}$. and

Apply correction as:

$$f_{i\text{new}} = f_{i\text{old}} + \frac{1}{M_{cj}} \sum_{M_{cj}} \mu \frac{w_{i\theta,j} \Delta\phi_{i\theta}}{w_{i\theta}}$$

Where μ is relaxation factor.

Close: 7

Check for convergence as:

If

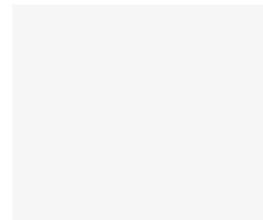
$$\text{abs} \left[\frac{f^{k+1} - f^k}{f^{k+1}} \right] \times 100 \leq e$$

STOP:

Else:

close: 4 (*k*)

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Anderson ART

Anderson and Kak [99] have proposed a variation to the ART algorithm. This algorithm is abbreviated as SART (Simultaneous Algebraic Reconstruction Technique). The method of implementing the correction is similar to ART1. The only difference this algorithm has from ART1 is in the calculation of correction for each cell. The weight factor used here is the exact intersection of a ray with the concerned cell. In contrast, ART1 uses the average correction for all the cells. This algorithm will be referred to as ART4. The individual steps are:

Calculate the total value of weight function ($W_{i\theta}$) along each ray as:

start:1 For each projection angle (θ):

start: 2 For each ray ($i\theta$):

start: 3 For each cell (j):

$$W_{i\theta} = \sum_{j=1}^N w_{i\theta,j} \times w_{i\theta,j}$$

close: 3

close: 2

close: 1

start: 4 start iterations (k):

start: 5 For each projection angle (θ):

start: 6 For each ray ($i\theta$):

Compute the numerical projection (Equation21)

close: 6

start: 7 For each ray ($i\theta$):

Calculate the correction as:

$$\Delta\phi_{i\theta} = \phi_{i\theta} - \tilde{\phi}_{i\theta}$$

start:8 For each cell (j):

If $w_{i\theta,j}$ non-zero then:

$$f_{j^{new}} = f_{j^{old}} + \mu \frac{\Delta\phi_{i\theta} \times W_{i\theta,j}}{W_{i\theta}}$$

where μ is a relaxation factor.

close: 8

close: 7

close: 5

Check for convergence as:

If

$$abs \left[\frac{f^{k+1} - f^k}{f^{k+1}} \right] \times 100 \leq e$$

STOP
Else: continue
close: 4 (*k*)

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