Module 6 : Robot manipulators kinematics

Lecture 21 : Forward & inverse kinematics examples of 2R, 3R & 3P manipulators

Objectives

In this course you will learn the following

- Inverse position and orientation matrix for 6R serial chain robot

- Forward and inverse position problem for simple manipulators

Inverse Kinematics for 6R Manipulator using D-H Parameters

Here given the a_i, d_i for i=1……6 and the transformation matrix ${}^0T_6$ EE position & orientation; Find out the joint position θ_i for i=1…..6

Fig. 21.1 Concurrent wrist for PUMA robot

C is a point of concurrency is at distance d_6 from End Effector frame X_6, Y_6, Z_6 & at distance d_4 from ref frame X_3, Y_3, Z_3.

$$\begin{Bmatrix} x_{c0} \\ y_{c0} \\ z_{c0} \\ 1 \end{Bmatrix}_0 = \left[{}^0T_3\right]^3 \begin{Bmatrix} 0 \\ 0 \\ d_4 \\ 1 \end{Bmatrix}_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Where equation (A) contains $[{}^0T_3]$ with θ_1, θ_2, θ_3 as only unknowns.

$$x_{c0} = \cos\theta_1(d_4 \sin\theta_3 \cos\theta_2 + a_2 \cos\theta_2) + (d_2 + d_3)\sin\theta_1$$
$$y_{c0} = \sin\theta_1(d_4 \sin\theta_3 \cos\theta_2 + d_4 \cos\theta_3 \sin\theta_2) - (d_2 + d_3)\cos\theta_1$$
$$zc0 = d_4 \sin\theta_3 \sin\theta_2 - d_4 \cos\theta_3 \cos\theta_2 + a_2 \sin\theta_2 + d_1$$
multiplying (1) *with* $\sin\theta_1$ *and* (2) *with* $\cos\theta_1$ *and subtracting,*
$$x_{c0}\sin\theta_1 - y_{c0}\cos\theta_1 = d_2 + d_3$$
Let $x_{c0} = \cos\beta(\sqrt{(x_{c0}{}^2 + y_{c0}{}^2)})$; $yc0 = \sin\beta(\sqrt{(x_{c0}{}^2 + y_{c0}{}^2)})$

$\therefore$ *from above,*

$$x_{c0} = c\theta_1(d_4\, s\theta_3 c\theta_2 + d_4\, c\theta_3 s\theta_2 + a_2 c\theta_2) + (d_2 + d_3)s\theta_1$$
$$y_{c0} = s\theta_1(d_4\, s\theta_3 c\theta_2 + d_4 c\theta_3 s\theta_2 + a_2 c\theta_2) - (d_2 + d_3)c\theta_1$$
$$z_{c0} = d_4 s\theta_3 s\theta_2 - d_4 c\theta_3 c\theta_2 + a_2 s\theta_2 + d_1$$
multiplying (1) *with* $s\theta_1$ *and* (2) *with* $c\theta_1$ *and subtracting,*
$$x_{c0}s\theta_1 - y_{c0}c\theta_1 = d_2 + d_3$$
Let
$$c\beta = \frac{x_{c0}}{\sqrt{(x^2{}_{c0} + y^2{}_{c0})}};\ and\ s\beta = \frac{y_{c0}}{\sqrt{(x^2{}_{c0} + y^2{}_{c0})}}$$
$\therefore$ *above equation gives,*
$$\theta 1 = \sin^{-1}\left(\frac{d_2 + d_3}{\sqrt{x^2{}_{c0} + y^2{}_{c0}}}\right) + \tan^{-1}\left(\frac{y_{c0}}{x_{c0}}\right)$$

Using numerical value of θ_1, from (1) or (2)
$$d_4 \sin(\theta_2 + \theta_3) + a_2 \cos\theta_2 = \frac{x_{c0} - (d_2 + d_3)\sin\theta_1}{\cos\theta_1}$$
$$\text{or } = \frac{y_{c0} + (d_2 + d_3)\cos\theta_1}{\sin\theta_1}$$
$$= B_1 \text{ (say)}$$
from equation (3)
$$-d_4 \cos(\theta_2 + \theta_3) + a_2 \sin\theta_2 = z_{c0} - d_1 = B_2 \text{ (say)}$$
eliminating $\theta_2 + \theta_3$
$$d_4{}^2 = B_1{}^2 + B_2{}^2 + a_2{}^2 - 2a_2 B_1 \cos\theta_2 - 2a_2 B_2 \sin\theta_2$$

$$B_1 \cos\theta_2 + B_2 \sin\theta_2 = \frac{B_1{}^2 + B_2{}^2 + a_2{}^2 - d_4{}^2}{2a_2}$$
$$\theta_2 = \cos^{-1}\left(\frac{B_1{}^2 + B_2{}^2 + a_2{}^2 - d_4{}^2}{2a_2\sqrt{B_1{}^2 + B_2{}^2}}\right) + \operatorname{atan2}(B_2, B_1) \_\_\_\_\_\_\_\_ (8)$$

using θ_1, θ_2, determine $\cos(\theta_2 + \theta_3), \sin(\theta_2 + \theta_3)$
from (6) , (7); then
$$\theta_3 = \operatorname{atan2}\left(\frac{B_1 - a_2\cos\theta_2}{d_4}, \frac{-B_2 + a_2\sin\theta_2}{d_4}\right) - \theta_2$$

We observe that θ_1 does not exist when sin (arg) >1. This implies that end effecter is beyond the reach as shown Figure 21.2. That is when sin (arg) is grater than 1, numerator is grater than denominator in θ_1 expression. And hence two solutions exist for θ1.

Fig.21.2 Workspace for Puma

Figure 21.3 Work space for Puma Robot

(Hint for θ_2)Use cosine rule knowing 3 sides, find angles; 2 solution for θ_2 as $\tan^{-1}$ (arg) +/- $\cos^{-1}$ (arg) . Thus we have 2 solutions for θ_1 & θ_2 each and therefore total 4 solutions for kinematics of Puma robot (refer figure 21.3). Now θ_3 will have COROSPONDING 4 solutions.

Part II

$$\text{Determination of wrist angles: } \theta_4, \theta_5, \theta_6$$
$${}^0R_6 = {}^0R_3\,{}^3R_6 \rightarrow {}^3R_6 = \left[{}^0R_3\right]^T {}^0R_6$$

Here ${}^0\mathrm{R}_3$ is known in terms of θ_1, θ_2, θ_3, with ${}^0\mathrm{R}_6$ as given and thus RHS is known as

$$\begin{bmatrix} \cos\theta_4\cos\theta_5\cos\theta_6+\sin\theta_4\sin\theta_6 & -\cos\theta_4\cos\theta_5\sin\theta_6+\sin\theta_4\cos\theta_6 & \cos\theta_4\sin\theta_5 \\ \sin\theta_4\cos\theta_5\cos\theta_6-\cos\theta_4\sin\theta_6 & -\sin\theta_4\cos\theta_5\sin\theta_6-\cos\theta_4\cos\theta_6 & \sin\theta_4\sin\theta_5 \\ \sin\theta_5\cos\theta_6 & -\sin\theta_5\sin\theta_6 & -\cos\theta_5 \end{bmatrix}$$

$$\theta_5 = \cos^{-1}(-r_{33})$$

$$\text{if } (\sin\theta_5 \neq 0)$$

$$\theta_4 = \operatorname{atan2}\left(\frac{r_{23}}{\sin\theta_5}, \frac{r_{13}}{\sin\theta_5}\right)$$

$$\theta_6 = \operatorname{atan2}\left(-\frac{r_{32}}{\sin\theta_5}, \frac{r_{31}}{\sin\theta_5}\right)$$

$$\text{elseif } (\sin\theta_5 = 0, \text{ and } \cos\theta_5 = 1)$$

$$\theta_4 - \theta_6 = \operatorname{atan2}(r_{21}, r_{11})$$

$$\text{elseif } (\sin\theta_5 = 0, \text{ and } \cos\theta_5 = -1)$$

$$\theta_4 + \theta_6 = \operatorname{atan2}(-r_{21}, -r_{11})$$

$$\text{endif}$$

Thus all θ_i for i=1....6 is known and problem of inverse kinematics for 6R Puma robot is solved.

Example of SCARA Manipulator

Basically it has RRP pairs as arm and single DOF as wrist. It is as shown in figure. The homogeneous transformation matrices for i^{th} frame expressed in $(i-1)^{th}$ frame is as follows

i.e. The D-H representation of linkage parameters are as given in table here.

Joint	ai	αi	di	θi
1	a1	0	0	θ_1
2	a2	0	0	θ_2
3	0	180	d3	0
4	0	0	d4	θ_4

(Bold letters shows the joint variables)

The corresponding transformation matrices are

$$^0T_4 = {}^0T_1\ {}^1T_2\ {}^2T_3\ {}^3T_4$$

$$= \begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1C_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C4 & -S4 & 0 & 0 \\ S4 & C4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_{12}C_4 + S_{12}S_4 & -C_{12}S_4 + S_2C_4 & 0 & a_1C_1 + a_2C_{12} \\ S_{12}C_4 - C_{12}S_4 & -S_{12}S_4 - C_{12}C_4 & 0 & a_1S_1 + a_2S_{12} \\ 0 & 0 & -1 & d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The forward kinematics problem will be knowing ai, αi, di, θi i.e. joint variables, finding end effector position and orientation. This is achieved through above transformation matrices. The inverse position problem would be, knowing the $^0T_4$ elements in numerical values; finding the position and orientation of each link i.e. ai, αi, di, θi. This is achieved by equating numerical values of $^0T_4$ to above $^0T_4$ elements and solving for ai, αi, di, θi.

Fig. 21.2 SCARA Manipulator sketch of joint variables

Solve following

Exercise (A)

Do the Forward and inverse position calculation for concurrent wrist Manipulators

- **PUMA**
- **Spherical**
- **Cylindrical**
- **SCARA**

- Do the exercise 1 calculation for non-concurrent wrist

- Do the Exercise 1 for $\alpha_i \neq 0, \pm 90^0, \pm 180^0, a_i \neq 0, d_i \neq 0$

- **Exercise (B)**

- Do the forward and inverse position calculation for serial chain manipulator / robot with n>6

- [Hint: In inverse kinematics, we have only EE position & orientation as known. This has 6 independent variables as studied earlier. Therefore,

- Infinite solution are possible for such chains and are kinematically redundant]

- Do the exercise for n<6

- Generalised Stewart Platform is the one with fully parallel chain mechanism.

  [Hint: in forward kinematics, the local reference frame of all links are given & one has to find out the position & orientation of the Stewart Platform. In Inverse kinematics it is the reverse way of above]

  Hybrid links (combination of linear & rotary actuators) can be solved for above depending on no. of links in chains.

PUMA for non-concurrent wrist

We have C1(Xc1, Yc1, Zc1)0 and C2 (Xc2, Yc2, Zc2)0 as known points in terms of ref base ref frame and ref frame (X3, Y3, Z3) and (X6, Y6, Z6) respectively(refer figure 21.4). (Both position and orientation as follows).

$$\begin{Bmatrix} X_{c1} \\ Y_{c1} \\ Z_{c1} \\ 1 \end{Bmatrix}_0 = \left[{}^0T_3\right] * {}^3\begin{bmatrix} & \left[{}^3R_{c1}\right] & & X_{c13} \\ & & & Y_{c13} \\ & & & Z_{c13} \\ 0 & 0 & 0 & 1 \end{bmatrix}_3 = \left[{}^0T_3\right] * \begin{bmatrix} {}^3X_{c1x} & {}^3X_{c1y} & {}^3X_{c1z} & X_{c13} \\ {}^3Y_{c1x} & {}^3Y_{c1y} & {}^3Y_{c1z} & Y_{c13} \\ {}^3Z_{c1x} & {}^3Z_{c1y} & {}^3Z_{c1z} & Z_{c13} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here the only unknowns are θ_1, θ_2, θ_3 in $\left[{}^3R_{c1}\right]$

Similarly,

$$\begin{Bmatrix} X_{c2} \\ Y_{c2} \\ Z_{c2} \\ 1 \end{Bmatrix}_0 = \left[{}^0T_6\right] * {}^6\begin{bmatrix} & \left[{}^6R_{c2}\right] & & {}^6X_{c1} \\ & & & {}^6Y_{c1} \\ & & & {}^6Z_{c1} \\ 0 & 0 & 0 & 1 \end{bmatrix}_6 = \left[{}^0T_6\right] * \begin{bmatrix} {}^6X_{c2x} & {}^6X_{c2y} & {}^6X_{c2z} & {}^6X_{c2} \\ {}^6Y_{c2x} & {}^6Y_{c2y} & {}^6Y_{c2z} & {}^6Y_{c2} \\ {}^6Z_{c2x} & {}^6Z_{c2y} & {}^6Z_{c2z} & {}^6Z_{c2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where the $\left[{}^6R_{c2}\right]$contains θ_4, θ_5, θ_6 as unknowns which can be solved as we have done for concurrent wrist.

Fig.21.4 PUMA non-concurrent wrist sketch

Cylindrical robot with 3 dof wrist (non-concurrent C1 & C2)

Joint	ai	αi	di	θi
1	a1	α_1	d1	$\boldsymbol{\theta_1}$
2	a2	α_2	**d2**	θ_2
3	a3	α_3	**d3**	θ_3
4	a4	α_4	d4	$\boldsymbol{\theta_4}$
5	a5	α_5	d5	$\boldsymbol{\theta_5}$
6	a6	α_6	d6	$\boldsymbol{\theta_6}$

(Bold letters are the joint variables)

The procedure for obtaining θ_1, θ_2, θ_3, θ_4, θ_5, θ_6 (refer figure 21.5) remains the same as for PUMA non concurrent wrist with C1 & C2 as known.

Fig. 21.5 Sketch of cylindrical robot with joint variables
(All parameters listed in D-H table are not shown in figure)

Spherical robot with 3 dof wrist (non-concurrent C1 & C2)

Joint	ai	α_i	di	θ_i

1	a1	α_1	d1	$\theta_{\mathbf{1}}$
2	a2	α_2	d2	$\theta_{\mathbf{2}}$
3	a3	α_3	**d3**	θ_3
4	a4	α_4	d4	$\theta_{\mathbf{4}}$
5	a5	α_5	d5	$\theta_{\mathbf{5}}$
6	a6	α_6	d6	$\theta_{\mathbf{6}}$

(Bold letters are the joint variables)

Fig.21.6 Sketch of spherical robot with joint variables (All parameters listed in D-H table are not shown in figure)

The procedure for obtaining $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$ (refer figure 21.6) remains the same as for PUMA non concurrent wrist with C1 & C2 as known

Hybrid (3 DOF Stewart Platform) Manipulators

Forward Kinematics : Given the link parameters, find EE position & orientation.

The D-H table is given as

(Xp, Yp, Zp)

Joint	ai	αi	di	θi
1	b	0	0	θ**1**
2	0	90	**d2**	0
3	0	0	a	θ**3**

Fig. 21.7 Stewart Platform as 3 DOF hybrid manipulator

We have ${}^{i}T_{i\text{-}1}$ as given earlier, hence the forward kinematics is multiplication of these transformation matrices as ${}^{0}T_{3} = {}^{0}T_{1}\ {}^{1}T_{2}\ {}^{2}T_{3}$ which gives position and orientation point P.

To get position of top platform, sum up three top joints coordinates. And to get orientation, take cross product of orientation vectors from ${}^{0}T_{3}$.

Inverse kinematics: This will be as, given top platforms position and orientation, find out position and orientation of each link. i.e. given numerical values of 4x4 matrix, finding the joint variables $\theta 1$, d2, $\theta 3$ is a inverse kinematics problem. Equating and solving for joint variables is a solution.

Recap

In this course you will learn the following

- Inverse position problem for PUMA robot.
- Example of SCARA robot.
- PUMA for non-concurrent wrist as example for forward kinematics.
- Cylindrical robot with 3 dof wrist (non-concurrent C1 & C2).
- Example of Spherical robot with 3 DOF wrist (non-concurrent).
- Example of Stewart platform as 3 DOF hybrid manipulator

Congratulations, you have finished Lecture 21. To view the next lecture select it from the left hand side menu of the page.