Module 9 : Robot Dynamics & controls

Lecture 35 : Lyapunovs theorems (applications to robots and stability analysis)

Objectives

In this course you will learn the following

- Lyapunov theory
- Energy of system is Monotonically decreasing
- Lyapunov'sTheoerm on Stability
- Lyapunov's Theorem on Asymptotic Stability
- Lyapunov Stability: PD Control of Robot
- PD Control

LaSalle's Theorem

- Trajectory tracking problem

Lyapunov theory:

Some connection with 4th lecture Fundamentals of Lyapunov Theory :

Suppose **0** is equilibrium which is obtained by solving the differential equation $\dot{x} = 0$ for x.

Total energy (E) is zero at origin and positive otherwise.

Now suppose system is perturbed from origin, if we observe E, it will be either

- Nonincreasing.
- Monotonically decreasing.
- Increasing.

Recall that $\dot{V}(t,x) = \frac{\partial V}{\partial t}(t,x) + \nabla V(t,x) f(t,x)$

1. **Energy of system is nonincreasing :**

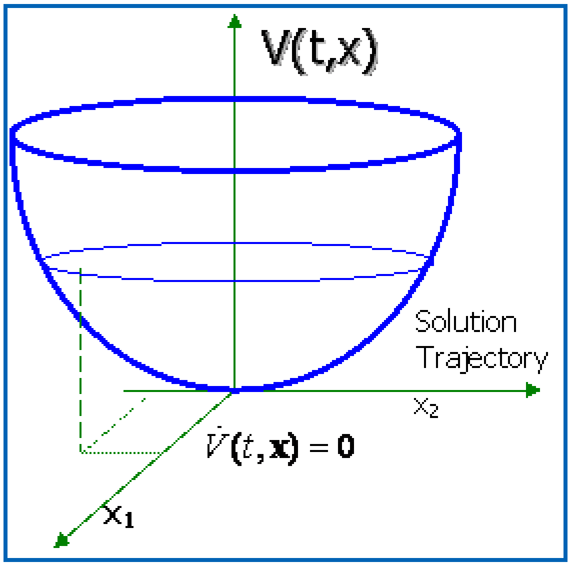

Figure 35.1(a)

Figure 35.1(b)

We are considering only 2nd order systems only. (i.e. state vecor consists of two variables.)In this case energy function is constant. Such kind of system is stable in sense of Lyapunov. e.g: If we disturb bob of Ideal simple pendulum, it will start performing oscillations of constant magnitude.The energy of bob at any instant of time will be constant. Hence the equilibrium of system is stable in this case.

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Lyapunov Stability: PD Control of Robot

Nonlinear dynamic equations of robotic manipulator for one joint

$$\sum_{j=1}^{n} d_{jk}(q)\ddot{q}_j + \sum_{i,j=1}^{n} C_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k \quad \textbf{(A)}$$

$$J_{m_k}\ddot{\theta}_{m_k} + \left(B_{m_k} + \frac{K_b K_m}{R}\right)\dot{\theta}_{m_k} = \frac{K_m}{R}v_k - r_k\tau_k$$ equation of motor dynamics.

but $\theta_m = \frac{1}{r_k}q_k$

where r_k is gear reduction ratio for link k. substituting that in above equation of motor dynamics

$$\frac{1}{r_k^2}J_m\ddot{q}_k + \frac{1}{r_k^2}B\dot{q}_k = \frac{K_m}{r_k R}v_k - \tau_k$$

i.e. $$\tau_k = \frac{K_m}{r_k R}v_k - \frac{1}{r_k^2}J_m\ddot{q}_k - \frac{1}{r_k^2}B\dot{q}_k$$

substituting this in **A** ,

$$\underbrace{\frac{1}{r_k^2} J_m}_{J}\ddot{q}_k + \sum_{j=1}^{n} d_{jk}\ddot{q}_j + \sum_{i,j=1}^{n} C_{ijk}\dot{q}_i\dot{q}_j + \frac{1}{r_k^2} B\dot{q}_k + g_k = \underbrace{\frac{K_m}{r_k R} v_k}_{u_k}$$

$$\textit{Where } B = B_{m_k} + \frac{K_b Lm}{R}$$

J = equivalent inertia of motor transformed to the joint variables.

In matrix form these equations can be written as

$$\left(D(q) + J\right)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = u \quad \textbf{(A)}$$

The above one is Robot & Actuator Dynamics Equation expressed in joint variables which will be used by us in next part of lectures for developing control for non-linear system.

$$J_{eff}\ddot{\theta}_{mk} + B_{eff}\dot{\theta}_{mk} = KV_k - d_k r_k$$

The above one is Robot & Actuator Dynamics Equation expressed in motor variables which was used by us in developing control laws for linear system with non-linear disturbance. But it is very important to note that both equations are representing same dynamics but the variables are different.

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2. **Energy of system is Monotonically decreasing.**

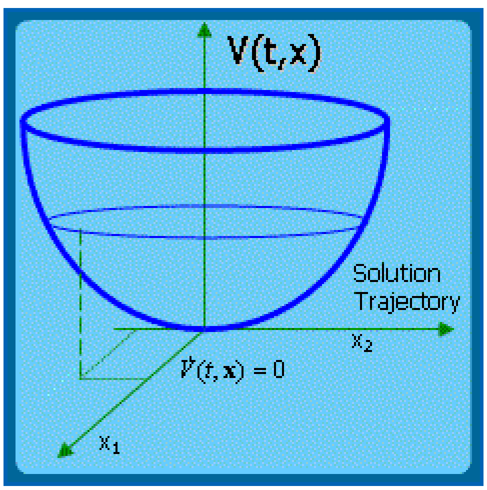

Figure 35.2(a)

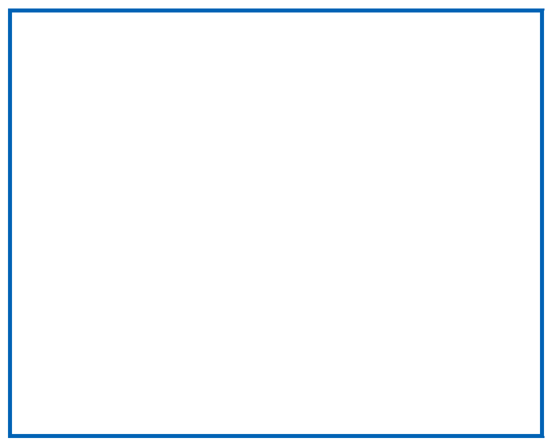

Figure 35.2(b)

In this case energy function is continuously decreasing.Such type of system will return to equiblirium position no matter where it is perturbed e.g Simple pendulum with damping. Energy of bob will go on decreasing. In this case the equilibrium of system is stable & attaractive in Lyapunov sense. We know that equilibrium is asymptotically stable if it is stable & attaractive.

Based on previous analysis we can conclude: –

- When derivative of energy type PDF function is zero the equilibrium of the system is stable.
- If derivative of energy type PDF function is strictly less than zero then the equilibrium of the system is stable and attractive i.e.asymptotically stable.
- There is definite relationship between stability and properties of V and $\dot{V}$.

Lyapunov generalized this relationship and came up with theorems on stability .

Lyapunov'sTheoerm on Stability :

Consider a nonlinear system $\dot{x} = f(t, x)$

The equilibrium **0** of the system is stable if there exist a C^1 lpdf $V: R_+ \times R^n \to R$.

(C^1-continuously differentiable on time.)

And a constant r>0 such that $\dot{V}(t,x) \leq 0, \forall t \geq t_0, \forall x \in B_r$

Where $\dot{V}$ is evaluated along the trajectories of the system

Lyapunov's Theorem on Asymptotic Stability:

The equilibrium 0 of the system is uniformly asymptotically stable if there exist a C^1 decrescent lpdf V such that $-\dot{V}$ is an lpdf .

Figure 35.3 **Lyapunov's Theorem on Asymptotic Stability**

In the above graph we have shown function V in between two class K function. If V satisfies following two conditions

$$\alpha(\|\mathbf{x}\|) \leq V(t, \mathbf{x}), \forall t \geq 0, \forall \mathbf{x} \in B_r$$

$$V(t,x) \leq \beta(\|x\|), \forall t \geq 0$$

Then the equilibrium **0** is stable & attaractive.

We will not derive Lyapunov's theorem of stability but we will apply these theorems to simple cases & to Robot manipulator in later part of lecture.

Now consider one simple example in which we will apply Lyapunov theorem of stability.

Figure .35.4

We want to move block to X^d We want point to point control or regulation problem.We know that equation of dynamics is

$$m\ddot{x} = F$$

For PD control:

Control strategy $F = K_p(x_d - x) + K_d(\dot{x}_d - \dot{x})$

$$\dot{e} = \dot{x}$$

$$\because \textit{In final position } \dot{X}_d = 0$$

$$F = -K_p e - K_d \dot{x}$$

$$m\ddot{x} = -K_p e - K_d \dot{x}$$

$$m\ddot{e} + K_d \dot{e} + K_p e = 0$$

Here we are having e & $\dot{e}$ as state spaces.

Now we will define Lyapunov function candidate which will contain e & $\dot{e}$ $V = \underbrace{\frac{1}{2} m\dot{e}^2}_{\mathrm{I}} + \underbrace{\frac{1}{2} k_p e^2}_{\mathrm{II}}$

Here we are using Lyapunov function candidate which is Energy based . We are calling the function as candidate because it has to be tested for being Lyapunov function.Note that *V* is a positive definite function. For most of robot manipulator we can use energy based Lyapunov function candidate.The first term I represents Kinetic part of energy & term II represents spring elastic energy.Taking derivative of V & to make it along the trajectory we will substitute for $\ddot{e}$ in it.

$$\begin{aligned} \dot{V} &= m\dot{e}\ddot{e} + K_p e\dot{e} \\ &= \dot{e}(-K_p e - K_d \dot{e}) + K_p e\dot{e} \\ &= -K_d \dot{e}^2 \quad \leq 0 \end{aligned}$$

Whatever may be the sign of error, its square will be positive & gain will be also positive so the term $\dot{V}$ will be less than or equal to zero. So the system is stable.Note that $\dot{V}$ does not contain e.(State vector consists of e & $\dot{e}$). So $\dot{V}$ is negative semi-definite function. We cannot prove its asymptotic stability with Lyapunov theorem only. We will use another theorem called Lasalle's theorem to prove asymptotic stability of given system. Now we will apply Lyapunov's theorem to Robot manipulator. In the above example we write the Lyapunov function in scalar form while in case of Robot manipulator we will write it in matrix form.

PD Control :

An independent joint PD-control scheme can be written in vector from as $u = K_p \tilde{q} - K_D \dot{q}$ Where $\tilde{q}$ is the error vecor $\tilde{q} = q^d - q$ .It is defined for entire Robot.

K_p and K_D are diagonal matrix of proportional and derivative gain.

To show that the above control law achieves zero steady state error consider an energy based Lyapunov function candidate in vecor form for Robot Maipulator

$$V = \underbrace{\frac{1}{2} \dot{q}^T \left(D(q) + J\right) \dot{q}}_{\mathrm{I}} + \underbrace{\frac{1}{2} \tilde{q}^T K_p \tilde{q}}_{\mathrm{II}}$$

The first term I is the kinetic energy of the robot and the second term II accounts for the proportional feedback (spring elastic energy) $K_p \tilde{q}$.

Time derivative of V is given by $\dot{V} = \frac{1}{2} \dot{q}^T \left(D(q) + J\right) \ddot{q} + \frac{1}{2} \dot{q}^T \dot{D}(q) \dot{q} - \dot{q}^T K_p \tilde{q}$

Solving(**A**) for $\left(D(q) + J\right) \ddot{q}$ taking g(q)=0 and substituting it in the above expression gives

$$\begin{aligned} \dot{V} &= \dot{q}^T \left(u - C(q,\dot{q}) + B\dot{q}\right) + \frac{1}{2} \dot{q}^T \dot{D}(q) \dot{q} - \dot{q}^T K_p \tilde{q} \\ &= \dot{q}^T \left(u - B\dot{q} - K_p \tilde{q}\right) + \frac{1}{2} \underbrace{\dot{q}^T \left(\dot{D}(q) - 2C(q,\dot{q})\right) \dot{q}}_{=0} \end{aligned}$$

Recall that $\dot{D} - 2C$ is a skew- symmetric matrix.

$$\because \dot{q}^T\left(\dot{D}(q)-2C(q,\dot{q})\right)\dot{q}=0$$

$$\dot{V}=\dot{q}^T\left(u-B\dot{q}-K_p\tilde{q}\right)$$

Recall $u=K_p\tilde{q}-K_D\dot{q}$ from PD control law

$\dot{V}=-\dot{q}^T\left(K_D+B\right)\dot{q}\le 0$ Notice that there is no term corresponding to q or $\tilde{q}$.

Above analysis shows that V is decreasing as long as $\dot{q}$ is not zero. This is sufficient to prove that manipulator can reach a position where $\dot{q}=0$ but it is not sufficient to prove that $\tilde{q}$ =0 or $q=q^d$ .So the system is not asymptotically stable as $\dot{V}$ is negative semi-definite.

To prove the asymptotic stability of above system we will use Lasalle's theorem. We will not prove it but we will use it.

LaSalle's Theorem :

Suppose the system is autonomous $\dot{x}=f(x)$

- Suppose there exist a function C^1 such that
- V is a pdf and a radially unbounded .
- $\dot{V}(t,0)\le 0, \forall t\ge 0, \forall x\in R^n$.
- Define $R=\left\{x\in R^n : \exists t\ge 0 \text{ such that } \dot{V}(t,x)=0\right\}$

R is a domain where $\dot{V}$ is zero & condition is that R does not contain any trajectories of the system other than the trivial trajectory x=0. Then the equilibrium 0 is globally uniform asymptotically stable.
This means that if we get $\dot{V}=0$ when $\dot{q}=0$ & $\ddot{q}=0$ .i.e. there is no other trajectory other than this. Then the equiblirium 0 is asymptotically stable.

PD Control :

We now apply this theorem to our PD control problem,

We have $\dot{V}=-\dot{q}^T\left(K_D+B\right)\dot{q}$

For $\dot{V}$ =0 we must have $\dot{q}$ =0 & $\ddot{q}$ =0 if we put it in equation of motion for PD control which is given as

$$\left(D+J\right)\ddot{q}+C\left(q,\dot{q}\right)\dot{q}=-K_p\tilde{q}-K_D\dot{q}$$

we must then have $0=-K_p\tilde{q}$ which implies that $\tilde{q}=0$.

LaSalle's theorem then implies that system is asymptotically stable. If in case gravitational term is present equation,then it must be modified to read $\dot{V}=\dot{q}^T\left(U-g(q)-B\dot{q}-K_p\tilde{q}\right)$. The presence of the gravitational term in above equation means that PD control alone cannot guarantee asymptotic stability for point-to-point movement.

Lasalle's theorem is applicable to autonomous system but if system is time dependent like the one used in case of trajectory tracking problem we can't apply Lasalle's theorem . But Lasalle's theorem can be applied to periodic systems.Now we will move to trajectory tracking problem.

Trajectory tracking problem :

In case of trajectory tracking problem we have to move end-effecor from initial position to final position along a predefined path. The desired joint angles as a function of time are carried out from the given trajectory. We assume that the end effector is needed to move along smooth (i.e.continuous & differentiable) trajectory.

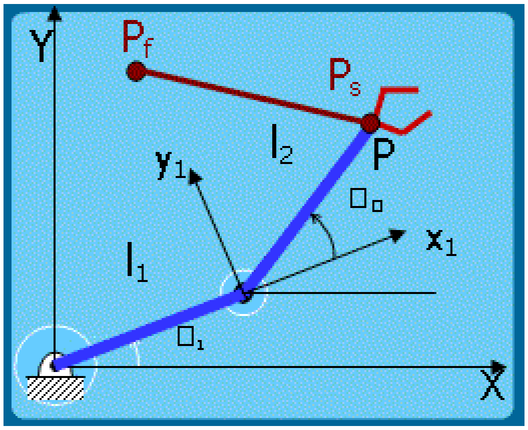

Figure 35.5 Trajectory tracking

Analysis of computed torque controllers can be carried out in this case. It usually leads to cases in which situation is similar to what we saw for PD control.We can't prove asymptotic stability of given system.($\dot{V}$ is semi-definite in all these cases).LaSalle's theorems could not be applied in this case and more advanced tools are necessary to establish asymptotic stability proofs. We will study one of them in next lectures which was developed by Li-Slotine .It is a high performance tracking controller which we will see in next lecture.

Recap

In this course you have learnt the following

- Lyapunov theory
- Energy of system is Monotonically decreasing
- Lyapunov'sTheoerm on Stability
- Lyapunov's Theorem on Asymptotic Stability
- Lyapunov Stability: PD Control of Robot
- PD Control
- **LaSalle's Theorem**
- Trajectory tracking problem

Congratulations, you have finished Lecture 35. To view the next lecture select it from the left hand side menu of the page