

Module 7 : Free Undamped Vibration of Single Degree of Freedom Systems; Determination of Natural

Frequency ; Equivalent Inertia and Stiffness; Energy Method; Phase Plane Representation.

Lecture 13 : Equivalent Stiffness and Inertia

Objectives

In this lecture you will learn the following

- Equivalent spring stiffness.
 - Serial combination
 - Parallel combination
- Equivalent mass / moment of inertia including geared shafts

While we have discussed so far the vibration behavior of a spring-mass system, in many practical situations we don't readily find such simple spring-mass systems. Many a time, we may find several springs and masses vibrating together and then we will have several second order differential equations to be solved simultaneously. In some special situations however, we will be able to simplify the system by considering equivalent stiffness and inertia. We may then still be able to model the system as a simple single d.o.f spring-mass case. We will now discuss some of those situations.

When multiple springs are used in an application, they are mainly found in two basic combinations.

- Series Combination
- Parallel Combination

Equivalent Spring Stiffness

Figure 7.3.1

Series Combination

A typical spring mass system having springs in series combination is shown above. The two springs can be replaced by a equivalent spring having equivalent stiffness equal to k as shown in the figure 7.3.2

Fig 7.3.2 Spring in series

When springs are in series, they experience the same force but under go different deflections.

For the two systems to be equivalent, the total static deflection of the original and the equivalent system must be the same.

$$\begin{aligned} \text{Deflection } \Delta &= \frac{mg}{k_{eq}} = \frac{mg}{k_1} + \frac{mg}{k_2} \\ \therefore \frac{1}{k_{eq}} &= \frac{1}{k_1} + \frac{1}{k_2} \end{aligned} \quad 7.3.1$$

Therefore if the springs are in series combination, the equivalent stiffness is equal to the reciprocal of sum of the reciprocal stiffnesses of individual springs. As an application example, consider the vertical bounce (up-down) motion of a passenger car on a road. Considering one wheel assembly we can develop what is known as a [quarter car model](#) as shown in fig 7.3.3a. For typical passenger cars, the tyre stiffness is of the order of 200,000N/m while the suspension stiffness is of the order of 20,000N/m. Also, the vehicle mass per wheel (sprung mass) can be taken to be of the order of 250kg while the un-sprung mass (i.e., mass of wheel, axle etc not supported by suspension springs) is less than 50kg. Reciprocal of tyre stiffness is negligible compared to reciprocal of suspension stiffness or in other words, the tyre is very rigid compared to the soft suspension. Hence an equivalent one d.o.f. model can be taken to be as shown in Fig. 7.3.3b

Fig 7.3.3a Quater Car model

Fig 7.3.3b Spring mass system

Fig 9.3.4 Spring in parallel

For the springs in parallel combination, the equivalent spring stiffness can be found out as:

Each of the individual spring supports part of the load attached to it but both the springs undergo same deflection.

Therefore the static deflection Δ of the mass is,

$$\text{Deflection } \Delta = \frac{mg}{k_{eq}} = \frac{f_1}{k_1} = \frac{f_2}{k_2}$$

$$\begin{aligned} \text{And} \quad mg &= f_1 + f_2 \\ \Rightarrow \quad k_{eq} \Delta &= k_1 \Delta + k_2 \Delta \\ \therefore k_{eq} &= k_1 + k_2 \end{aligned}$$

9.3.2

Therefore if the springs are in parallel combination, the equivalent spring stiffness is sum of individual stiffnesses of each spring.

Fig 7.3.6 Springs Connected by lever

Figure 7.3.6 shows a typical arrangement in which springs are connected with a lever.

We herein intend to replace the two springs by one equivalent spring. Let the displacement of the equivalent spring be, x_{eq} and the equivalent spring stiffness be k_{eq}

We select then the equivalent spring displacement as equal to either of the spring displacement. Lets say that its displacement is equal to x_1 . Then as the strain energy stored in the original system and the system with equivalent spring has to be the same.

7.3.3

$$\text{Strain energy } U = \frac{1}{2} \{k_1 x_1^2 + k_2 x_2^2\} = \frac{1}{2} k_{eq} x_{eq}^2$$

$$\text{Letting } x_{eq} = x_1 \text{ and since } x_2 = x_1 \frac{l_2}{l_1}$$

$$\text{we get } k_{eq} = k_1 + k_2 \left(\frac{l_2}{l_1} \right)^2 \quad 7.3.4$$

Equivalent Inertia

We have discussed finding the equivalent stiffness. This equivalence can be termed “statically equivalent system” since the spring controls the static response of the system. However for complete dynamic response to a time varying load to be equivalent, we need to develop a dynamically equivalent model. Ideally, this dynamic equivalent system has analogous equations of motion and produces the same output response for a given input. For example consider the connecting rod of an IC Engine. It has complex shape and undergoes general planar motion. Thus computing the kinetic energy exactly will be a tedious task. Instead we can try and represent the connecting rod with a much simpler system as shown in Fig 7.3.5. Let us now study how to find such equivalent mass/inertia systems.

Fig 7.3.5 Connecting rod

Rigidly Connected Masses

Sometimes the members connecting the various masses could be considered rigid compared to other springy elements in the system. For example consider two masses connected together by a stiff rod as shown in the fig 7.3.7

- Both the masses have identical velocities since they are connected by a rigid link.
- Therefore the equivalent mass is the sum of individual masses

$$v_1 = v_2 \quad 7.3.5$$

$$\therefore m_{eq} = m_1 + m_2 \quad 7.3.6$$

Fig 7.3.7 Rigid connected mass

Masses connected by Lever

For the configuration shown in fig7.3.8 with small amplitude of angular motion we take the velocity of the equivalent mass as equal to velocity of either mass and considering the lever ratio and assuming that the lever used is rigid but massless we get,

$$v_{eq} = v_1 \quad 7.3.7$$

$$v_2 = v_1 \left(\frac{L_2}{L_1} \right) \quad 7.3.8$$

We then equate the kinetic energy of the existing system to the equivalent system and find out the equivalent mass from the equation below

$$T = \frac{1}{2} (m_1 v_1^2 + m_2 v_2^2) = \frac{1}{2} m_{eq} v_{eq}^2 \quad 7.3.9$$

$$\therefore m_{eq} = \left(m_1 + m_2 \left(\frac{L_2}{L_1} \right)^2 \right) \quad 7.3.10$$

Fig 7.3.8 Masses connected by lever

Inertia on geared shafts (rotation)

To find the equivalent inertia on a geared system, consider two mass moments of inertias, J_1 and J_2 , each rotating on shafts connected by massless gears. Let the number of teeth on each gear be N_1 and N_2 , respectively.

Fig 7.3.9

We select the angular speed of the equivalent shaft as equal to the angular speed of either of the shaft, Say

$$\dot{\theta}_{eq} = \dot{\theta}_1 \quad 7.3.11$$

Then to determine the polar moment of inertia of the equivalent shaft we evoke the [equivalence of kinetic energy](#),

$$J_{eq} = \left(J_1 + J_2 \left(\frac{N_1}{N_2} \right)^2 \right) \quad 7.3.12$$

Equivalence of kinetic energy

The basis of conversion is that the kinetic energy and potential energy for the equivalent system should be same as that for the original system. Then if θ_1 and θ_2 are the angular displacement of rotors J_1 and J_2 respectively, then neglecting the inertia's of the gears, the energy equations are given by:

$$K.E. = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$$
$$P.E. = \frac{1}{2} k_{t1} \theta_1^2 + \frac{1}{2} k_{t2} \theta_2^2$$

Where K_{t1} , K_{t2} , are the torsional stiffnesses of the shafts

since $\theta_2 = n \theta_1$ (gear ratio 'n') the expressions become:

$$K.E. = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 (n \dot{\theta}_1)^2$$

$$P.E. = \frac{1}{2} k_{t1} \theta_1^2 + \frac{1}{2} k_{t2} (n \theta_1)^2$$

$$K.E. = \frac{1}{2} (J_1 + n^2 J_2) \dot{\theta}_1^2$$

$$P.E. = \frac{1}{2} (k_{t1} + n^2 k_{t2}) \theta_1^2$$

finally

$$J_{eq} = J_1 + n^2 J_2 ; \quad K_{eq} = K_{t1} + n^2 k_{t2}$$

where

$$n = \frac{N_1}{N_2}$$

wherein N_1 and N_2 are the number of teeth on the two meshing gears.

Recap

In this lecture you have learnt the following

- Types of combinations in springs(series or parallel)
- Concept of equivalent stiffness

- Equivalent stiffness for springs in series $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$

- Equivalent stiffness for springs in parallel $k_{eq} = k_1 + k_2$

- Quarter car model and its simplified model
- Concept of equivalent inertia
- Derivation of equivalent inertia of rigidly connected masses, masses connected by lever and geared shafts

Congratulations, you have finished Lecture 3. To view the next lecture select it from the left hand side menu of the page

