Module 11 : Free Vibration of Elastic Bodies; Longitudinal Vibration of Bars; Transverse Vibration of Beams;
Torsional Vibration of Shaft; Approximate Methods – Rayleigh's Method and Rayleigh-Ritz Method.

Lecture 33 : Longitudinal vibration of bars

Objectives

In this lecture you will learn the following

- Derivation of the governing partial differential equation for longitudinal vibration of bars
- Solution of the governing equations in terms of the natural frequencies and mode shapes

Consider a long, slender bar as shown in Fig. 11.2.1. We aim to study its vibration behavior in the longitudinal (i.e., axial) direction. Recall that when it undergoes axial deformation, we assume that the whole cross-section moves together by the same displacement. Thus the axial deformation "u" could vary from point to point along the length of the bar (i.e., u is a function of x) but all points in the cross-section at a given axial location (i.e., x) have the same displacement. Of course, the axial displacement at any given point varies with time as the system vibrates. Thus we write u(x,t).

When a rod undergoes deformation u, the strain at any point is given by:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad (11.2.1)$$

Assuming linear elastic homogeneous material obeying Hooke's Law, we have:

$$\sigma_x = E\,\varepsilon_x = E\frac{\partial u}{\partial x} \qquad (11.2.2)$$

When a cross-section of area A is subjected to this stress, the axial force is given by:

$$F = A\sigma_x = AE\frac{\partial u}{\partial x} \qquad (11.2.3)$$

Since axial displacement "u" is a function of x, all these quantities (the strain, the stress and the internal force in the cross-section) are all dependent on x and vary from point to point along the length of the bar.

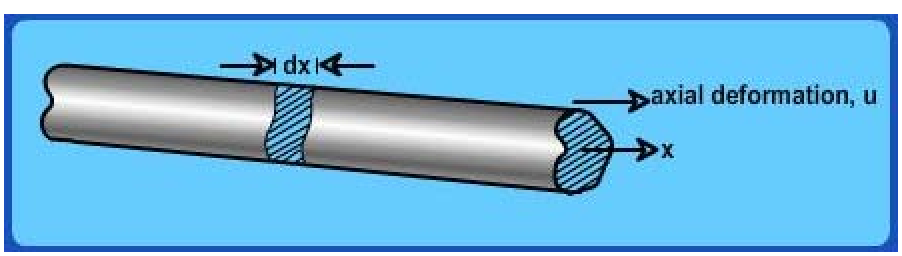

Figure 11.2.1

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Recall that the first order approximation to the Taylor 's series expansion of a function f(x) in the neighborhood of x is given by:

$$f(x+h) \approx f(x) + h\frac{df}{dx} \tag{11.2.4}$$

With this brief background, consider the free body diagram of a differential element of length (dx) shown in Fig. 11.2.2. From Newton 's second law, the algebraic sum of all the forces in the axial direction must equal mass time acceleration. Thus we can write:

$$F_x + \left(\frac{\partial F_x}{\partial x}\right)(dx) - F_x = \left(\frac{\partial F_x}{\partial x}\right)(dx) = \rho A(dx)\frac{\partial^2 u}{\partial t^2} \tag{11.2.5}$$

Substituting from Eq. (11.2.3) in Eq. (11.2.5), and assuming that the area of cross-section and Young's modulus are constant, we get:

$$AE\frac{\partial^2 u}{\partial x^2} = \rho A\frac{\partial^2 u}{\partial t^2} \tag{11.2.6}$$

Eq. (11.2.6) can be re-written as:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{11.2.7}$$

where $c = \sqrt{E/\rho}$, the wave speed i.e. the speed of sound wave (acoustic wave) propagation in that medium.

Particular solutions can be obtained for this wave equation when the boundary conditions are specified, for example the left end of the rod may be fixed (i.e. u(0,t)=0) etc. We will illustrate this on one set of boundary conditions here.

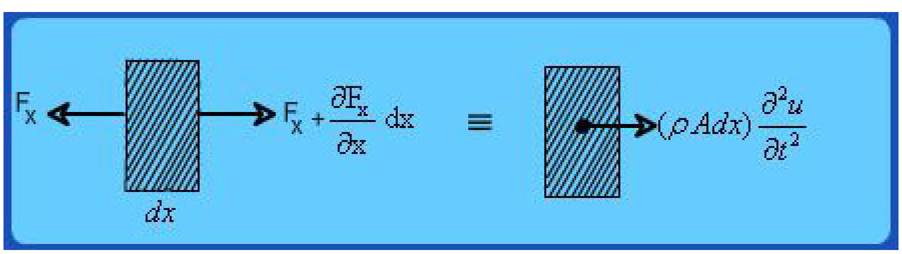

Figure 11.2.2

Example:

Let the left end of the rod be fixed and the right end be free i.e., no force on right end. Thus we get:

$$u(0,t) = 0; \quad \frac{\partial u}{\partial x}(l,t) = 0 \tag{11.2.8}$$

We can use the method of separation of variables i.e.,

$$u(x,t) = U(x)\,T(t) \tag{11.2.9}$$

Substituting in eq. (11.2.7), and re-arranging the terms, we get:

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = c^2\frac{1}{U(x)}\frac{d^2U}{dx^2} \quad (11.2.10)$$

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Since the left hand side is only a function of time and right hand side is only a function of spatial coordinate "x", each of them must be equal to a constant. Let this constant be $-\omega^2$. Thus we can write:

$$\frac{1}{T(t)}\frac{d^2T}{dt^2} = -\omega^2 \quad (11.2.11)$$

$$c^2\frac{1}{U(x)}\frac{d^2U}{dx^2} = -\omega^2 \quad (11.2.12)$$

Thus, we get,

$$T(t) = A\ Sin\ (\omega t) + B\ Cos\ (\omega t) \quad (11.2.11)$$

$$U(x) = C\ Sin\ (\omega x/c) + D\ Cos\ (\omega x/c) \quad (11.2.14)$$

Boundary condition that U(0) = 0 at all times requires that D = 0. The second boundary condition requires that:

$$Cos\ (\omega L/c) = 0 \quad (11.2.15)$$

i.e., $\omega L/c = \frac{n\pi}{2}$, for n = 1, 3, 5, ...

Thus harmonic vibration takes place at discrete frequencies called the natural frequencies of the system. The natural frequencies of the clamped-free bar under axial vibration are:

$$\boxed{\omega^2 = n^2\ \frac{\pi^2 E}{4\rho L^2}}\ n = 1,3,5,..... \quad (11.2.16)$$

The corresponding deformation shapes are given by:

$$u(x,t) = C\ Sin\ (\omega x/c)\ [A\ Sin\ (\omega t) + B\ Cos\ (\omega t)] \quad (11.2.17)$$

The constants A and B are determined using the prescribed initial conditions. The "shape" of vibratory displacement varies sinusoidally along the length of the bar. These are called the mode shapes.

Recap

In this lecture you have learnt the following.

- Developing governing partial differential equations for longitudinal vibration of rods
- Obtaining the solutions to the free vibration problem
- Natural frequencies and mode shapes of axially vibrating rods