Module 11 : Free Vibration of Elastic Bodies; Longitudinal Vibration of Bars; Transverse Vibration of Beams; Torsional Vibration of Shaft; Approximate Methods – Rayleigh's Method and Rayleigh-Ritz Method.

Lecture 35 : Transverse Vibration of Beams

Objectives

In this lecture you will learn the following

- Significance of transverse vibrations of beams.
- Derivation of governing partial differential.
- Natural frequencies and mode shapes.

So far we studied the axial and torsional vibrations of long, prismatic members. We will now study their transverse vibrations. We will assume that we could use the conventional Euler-Bernoulli beam model. This would imply that we invoke several assumptions. We assume that the beam is originally straight and uniform (in terms of both cross-section and material properties) along its length. We assume that the beam cross-section is symmetrical about the plane of loading as shown in Fig. 11.4.1 and the deformation is restricted to this plane of symmetry. We ignore any contribution of the shear force in the cross-section to the transverse deformation. The geometry of deformation is assumed to be such that plane cross-sections which are originally straight and normal to the “neutral axis” remain so even after bending. We assume that the longitudinal fibers are free to expand or contract in the lateral direction.

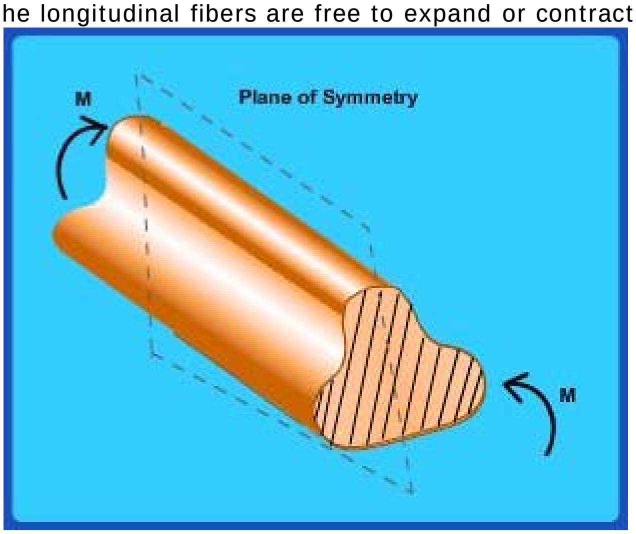

Fig. 11.4.1

When such a beam undergoes time dependent transverse deformations, we can draw a free body diagram of an elemental length as shown in Fig. 11.4.2 from which we can write:

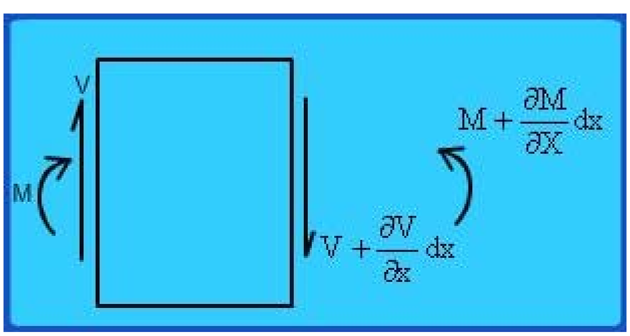

Fig. 11.4.2

$$-\left(V+\frac{\partial V}{\partial x}dx\right)+V=\left(\rho A dx\right)\frac{\partial^2 w}{\partial t^2} \qquad (11.4.1)$$

Where V is the shear force, ρ is the density, A is the cross-sectional area and w is the transverse displacement. However the shear force is related to the deformations by:

$$V = EI\,\frac{\partial^3 w}{\partial x^3} \qquad (11.4.2)$$

Thus we have:

$$EI\,\frac{\partial^4 w}{\partial x^4}+\rho A\,\frac{\partial^2 w}{\partial t^2}=0 \qquad (11.4.3)$$

Let $c^2 = {}^{EI}\!/_{\rho A}$. Then we can write:

$$\frac{\partial^2 w}{\partial t^2}=-c^2\,\frac{\partial^4 w}{\partial x^4} \qquad (11.4.4)$$

Contd....

This is the governing equation for free transverse vibration of a beam. The solution proceeds on lines similar to the case of axial vibrations of rods. Using the variables separable method, we can write:

$$w(x,t) = X(x)\,T(t) \qquad (11.4.5)$$

Substituting in (11.4.4), we get:

$$\frac{1}{T}\frac{d^2T}{dt^2}=-c^2\,\frac{1}{X}\frac{d^4X}{dx^4} \qquad (11.4.6)$$

Since the LHS is only a function of time and the RHS is only a function of spatial coordinate, then for (11.4.6) to hold good, each side should be equal to a constant, say $-\omega^2$. We get:

$$\begin{aligned}\frac{d^2T}{dt^2}&=-\omega^2 T\\ \frac{d^4X}{dx^4}&=\beta^4\,X\end{aligned} \qquad (11.4.7)$$

where $\beta^4 = {}^{\omega^2}\!/_{c^2} = \rho A\,{}^{\omega^2}\!/_{EI}$. Thus:

$$\begin{aligned}T(t)&=\alpha_1\cos\omega t+\alpha_2\sin\omega t\\ X(x)&=\alpha_3\cos\beta x+\alpha_4\sin\beta x+\alpha_5\cosh\beta x+\alpha_6\sinh\beta x\end{aligned} \qquad (11.4.8)$$

The coefficients in the above equation depend on the boundary/initial conditions. Let us consider a simple boundary condition viz., a beam simply supported at both ends. Thus we have:

$$
\begin{aligned}
&w(0,t) = w(l,t) = 0 \\
&\frac{\partial^2 w(0,t)}{\partial x^2} = \frac{\partial^2 w(l,t)}{\partial x^2} = 0
\end{aligned}
\qquad (11.4.9)
$$

We observe that since this is a fourth order differential equation in x, we have four boundary conditions. From these boundary conditions, we can write:

$$
\begin{aligned}
\alpha_3 + \alpha_5 &= 0 \\
-\alpha_3 + \alpha_5 &= 0
\end{aligned}
\qquad (11.4.10)
$$

From which we can write that $\alpha_3 = \alpha_5 = 0$. Similarly we can write:

$$
\begin{aligned}
\alpha_4 \, Sin \, \beta l + \alpha_6 \, Sinh \, \beta l &= 0 \\
-\alpha_4 \, Sin \, \beta l + \alpha_6 \, Sinh \, \beta l &= 0
\end{aligned}
\qquad (11.4.11)
$$

From (11.4.11), we can write ($\alpha_6 = 0$ and since α_4 can not also be zero):

Sin (βl) = 0 (11.4.12)

i.e., $\beta l = n\pi$, *n = 1,2,3...*

Contd....

Therefore, the natural frequencies of free vibration viz., ω are given by:

$$
\omega = n^2 \pi^2 \sqrt{\frac{EI}{\rho A l^4}} \qquad (11.4.11)
$$

Thus, for a given beam, the successive natural frequencies are in the ratio 1, 4, 9, 16, ….

Corresponding mode shapes of vibration are given by:

$$
X(x) = \alpha_4 \, Sin \frac{i \pi x}{l} \qquad (11.4.14)
$$

Typical mode shapes are shown in Fig. 11.4.3.

Fig. 11.4.3 Typical mode shapes of vibration of a simply supported beam

It is observed that the constant α_4 remains undetermined and X(x) remains the shape of vibration rather than any particular amplitude.

Beams with different boundary conditions will have different natural frequencies and mode shapes.

Contd....

Recap

In this lecture you have learnt the following.

- Transverse vibrations of beams
- Derivation of governing partial differential equation
- Solution for Natural frequencies and mode shapes for simple beams