Module 9 : Forced Vibration with Harmonic Excitation; Undamped Systems and resonance; Viscously Damped

Systems; Frequency Response Characteristics and Phase Lag; Systems with Base Excitation;

Transmissibility and Vibration Isolation; Whirling of Shafts and Critical Speed.

Lecture 21 : Harmonic vibration of single DOF system

Objectives

In this lecture you will learn the following

- Equation of motion for forced vibration
- Response to Harmonic Force
- Dynamic magnification factor
- Concept of resonance
- Role of stiffness, mass and damping in response

Mechanical systems are subject to various excitations and it is of interest to determine the response of the system. It is by no means a simple problem to determine the response of a complex real life engineering system to a general excitation. Consider for example, the response of an automobile when moving on a rough road. The excitation to the system comes from the road roughness which is random in nature and needs to be modeled using statistical techniques. Similarly the tire and the suspension of the vehicle itself could exhibit non-linear behavior and hence the mathematic model and its solution will turn out to be quite complex. As another example, consider the dynamic response of an aircraft (such as the indigenous Light Combat Aircraft) during take-off, flight maneuvers and landing. Accurate modeling and response estimation involves complex calculations often on a digital computer.

While we aim to perform complex vibration analysis on a computer on such practical systems, in order to get insight into the methods of analysis as well as the nature of the response, we begin our discussion (as we did for free vibration case) with a simple single degree of freedom spring – mass-damper system subjected to a sinusoidal excitation of single frequency. As mentioned earlier, it is not necessarily true that damping in every physical system is viscous in nature (i.e., damping force proportional to velocity) however we consider viscous damping as it is perhaps the easiest way of accounting for damping in the system as it leads to linear differential equations with constant coefficients.

Why Sinusoidal Excitation?

The reason for using sinusoidal excitation is simple - any general time varying forcing function can in principle be expressed as a summation of several sinusoids (Fourier series). Assuming linear system behavior, we will be able to superpose the response to individual sinusoids to get the total system response. Moreover, many mechanical systems involve a fundamental operating speed and the excitation source is often predominantly at this speed or its harmonics – for example, unbalance in a shaft causes an excitation at the running speed.

Why single d.o.f. spring-mass -damper system?

The simple spring-mass system exhibits several characteristics (such as resonance) similar to those of real life, complex engineering systems and hence we begin our discussion with this system.

Forced Vibration Analysis of an undamped single d.o.f system

Fig 9.1.1 Single degree of freedom system under harmonic excitation

Fig 9.1.2 Free body diagram of SDOF system

Consider a spring-mass system subjected to a sinusoidal force as shown in Fig. 9.1.1. A practical situation representing this is also shown in the figure. The figure shows a machine on its foundation, where the machine is operating at a rotational speed Ω . If the rotor has some unbalance and if 'k' denotes the stiffness of the foundation, we can represent the system as an equivalent single d.o.f system as shown in the figure. The sinusoidal excitation corresponds to the unbalance force as indicated in the figure. We will begin our discussion with the simple spring-mass system and later, return to this practical example to study it in greater detail.

The governing equation of motion can be readily obtained from the free body diagram shown in Fig. 9.1.2 as follows:

$$F_o \sin\Omega t - kx - c\dot{x} = m\ddot{x} \qquad (9.1.1)$$

The response of the system as a solution to the differential equation of motion has two parts viz., complementary solution and particular integral. For this system, the complementary solution represents the response of the system to given initial conditions and the particular integral is the response to the particular forcing function (sinusoid in this case). The complementary solution, as obtained in eqn. () is given by:

$$x_c(t) = e^{-\xi w_n t}\left(A\sin\omega_d t + B\cos\omega_d t\right)$$
$$\text{where} \quad \omega_d = \omega_n\sqrt{1-\xi^2} \quad \text{and} \quad \omega_n = \sqrt{k/m} \qquad (9.1.2)$$

The particular integral can be assumed to be of the form:

$$x_p(t) = X_0 \sin(\Omega t - \phi) \qquad (9.1.3)$$

Thus the total response of the system is given by:

$$x(t) = x_c(t) + x_p(t) \qquad (9.1.4)$$

The first part of the response, corresponding to the complementary solution representing the free vibration response, is at the damped natural frequency of the system and decays very fast exponentially !. In practice every system has some

form of dissipation or other and hence this will eventually die down (as we discussed in module 8.) leaving only the "steady state" response pertaining to the particular integral. The steady state response is a sinusoid at the frequency of excitation (called forcing frequency or driving frequency) and has some phase difference with the excitation as depicted in Fig. 9.1.3 of special interest is the amplitude of steady state which can be obtained as follows.

Fig 9.1.3 Phase relation of excitation and response

Substituting eq 9.1.3 in eq 9.1.1 we get

$$\begin{aligned} &-mX_0\Omega^2 \sin(\Omega t-\phi) \\ &+cX_0\Omega\cos(\Omega t-\phi) \\ &+kX_0 \sin(\Omega t-\phi)=F_o \sin\Omega t \end{aligned} \tag{9.1.6}$$

i,e.,

$$\begin{aligned} &(k-m\Omega^2)X_0[\sin\Omega t\cos\phi-\cos\Omega t\sin\phi] \\ &+cX_0\Omega[\cos\Omega t\cos\phi+\sin\Omega t\sin\phi]=F_o\sin\Omega t \end{aligned} \tag{9.1.7}$$

Comparing terms of $\sin\Omega t$ and $\cos\Omega t$ on either side we have

$$X_0(k-m\Omega^2)\cos\phi+cX_0\Omega\sin\phi=F_0 \tag{9.1.8}$$

$$-X_0(k-m\Omega^2)\sin\phi+cX_0\Omega\cos\phi=0 \tag{9.1.9}$$

which yield

$$X_0=\frac{F_0}{\sqrt{(k-m\Omega^2)^2+(c\Omega)^2}} \tag{9.1.10}$$

$$\tan\phi=\frac{c\Omega}{(k-m\Omega^2)} \tag{9.1.9}$$

Now we introduce the notation

undamped natural frequency $\omega_n=\sqrt{\dfrac{k}{m}}$

frequency ratio $\eta=\dfrac{\Omega}{\omega_n}$

critical damping $c_c=2m\omega_n$

damping factor $\xi=\dfrac{c}{c_c}$ i.e. $c=2m\omega_n\xi$

we can rewrite equations 9.1.9 and 9.1.10 as

$$X_0 = \frac{F_0 / k}{\sqrt{[(1-\eta^2)]^2 + [2\xi\eta]^2}} \tag{9.1.11}$$

and $$\tan\phi = \frac{2\xi\eta}{(1-\eta^2)} \tag{9.1.12}$$

If we consider that the force of magnitude F_0 were applied statically, the response would then be

$$X_{st} = \frac{F_0}{k} \tag{9.1.13}$$

Thus re-writing eq (9.1.11), we get,

$$\frac{X_0}{X_{st}} = \frac{1}{\sqrt{(1-\eta^2)^2 + (2\xi\eta)^2}} \tag{9.1.14}$$

Fig 9.1.4 Variation of dynamic magnification factor with frequency ratio

Since the R.H.S. indicates by how much the dynamic response is more than the static response, it is usually termed the Magnification Factor. If we now plot the variation of the magnification factor with frequency of excitation, we get a curve as shown in Fig. 9.1.4. There are primarily three regions of interest in such a graph.

When $\Omega \ll \omega_n$ ($\eta \ll 1$) [stiffness controlled regime]

In this case, the forcing frequency is much lower than the natural frequency of the system and the magnification factor is nearly equal to unity i.e., the dynamic response is almost same as static response. In other words, since the forcing frequency is so slow, the system responds almost as if the load were static. Thus the response in this regime is governed by the stiffness of the system.

For $\Omega = 0.33\omega_n$, the dynamic magnification factor equals 1.125 i.e., the dynamic response is just about 12.5% more than the static response. In such cases, one needs to critically assess whether it is indeed necessary to perform a full fledged dynamic analysis or it is sufficient to just perform a simple static analysis with the knowledge that the actual response could be slightly higher.

When $\Omega \gg \omega_n$ ($\eta \gg 1$) [inertia controlled regime]

In this case the excitation frequency is much higher than the natural frequency of the system and the magnification factor approaches zero i.e., the mass vibrates very little about the mean equilibrium position. In other words, the disturbance varies so fast that the inertia of the system cannot cope with it and kind of 'gives up'. To be sure the motion is still sinusoidal with the forcing frequency but the amplitude is extremely small. The dynamic response is just about 4% of the static response! Again, in this regime, one needs to critically examine whether it is necessary to perform a full fledged dynamic analysis. It is of course to be noted that high frequency vibration could result in eventual fatigue failure and secondly that even though vibratory displacements are small, the velocity need not be small since $\dot{X}_0 = \Omega X_0$ for a sinusoid.

When $\Omega \approx \omega_n$ ($\eta \approx 1$) [damping controlled]

In this region the excitation frequency is close to the natural frequency of the system and we see a huge build-up of dynamic response. It is this region that is critical and could cause potential damage to many structures and machine elements and needs to be avoided as far as possible. You now appreciate why it is important to accurately determine the natural frequencies of a system and make sure that the operating frequencies are not in the same range. When the excitation and the natural frequencies match, we call this phenomenon "resonance".

$$\frac{X_0}{X_{st}} \approx \frac{1}{2\xi} \qquad (9.1.15)$$

Practical Implication

A real-life complex engineering system may have several natural frequencies and may be experiencing excitation at various frequencies. When any of these excitation frequencies comes close to any of the natural frequencies of the system, we can expect large amplitudes of vibration and this condition should be avoided as far as possible.

Thus we see that the peak amplitude near resonanceis limited purely by the damping present in the system. The lower the damping in the system, the higher the peak amplitude and the sharper the peak. Thus one can guess the extent of damping in a system by merely looking at the dynamic response curves as depicted in Fig. 9.1.4. From the value of vibration amplitude as given by equation (9.1.15) one can think of a method of experimentally determining the damping coefficient viz., measure the response at this condition and use eqn (9.1.15) to estimate the damping factor. However, another easy method of determining damping is commonly employed, which is known as the half-power-point method. When the magnification factor is 0.707 times the peak magnification factor i.e., $\frac{1}{\sqrt{2}}$ times , the corresponding frequency values on either side of resonance are given by (see Fig.9.1.5)

$$\frac{1}{\sqrt{(1-\eta^2)^2 + (2\xi\eta)^2}} = \frac{1}{\sqrt{2}}\left(\frac{1}{2\xi}\right) \qquad (11.1.17)$$

$$\eta^4 + (4\xi^2 - 2)\eta^2 + (1-\xi^2) = 0 \tag{11.1.18}$$

$$\eta_{1,2}^2 = 1 - 2\xi^2 \pm 2\xi\sqrt{1+\xi^2} \tag{11.1.19}$$

Usually, $\xi << 1, \quad \xi^2 << 1 \quad \text{and so,} \quad \eta_{1,2}^2 \approx 1 \pm 2\xi$

$$\therefore \quad 4\xi \approx \frac{\Omega_2^2 - \Omega_1^2}{\omega_n^2} = \left(\frac{\Omega_2 + \Omega_1}{\omega_n}\right)\left(\frac{\Omega_2 - \Omega_1}{\omega_n}\right) \tag{11.1.20}$$

$$\frac{\Omega_2 + \Omega_1}{2} \approx \omega_n, \quad \xi = \frac{\Omega_2 - \Omega_1}{2\omega_n} = \cdot\frac{\eta_2 - \eta_1}{2} \tag{11.1.21}$$

This is an effective method of determining the damping coefficient in a system using a forced vibration test.Logarithmic decrement is another effective technoque and is based on free vibration test.

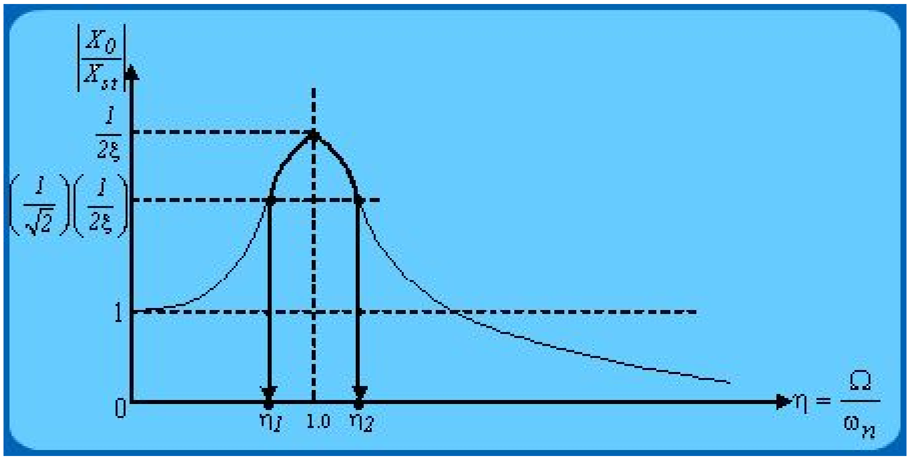

Figure 9.1.5

Recap

In this lecture you have learnt the following

- Solution to the equation for the forced oscillations
- Magnification factor ie ratio between the deflection due to force when applied as sinusoidal vs static
  $$X_0 = \frac{F_0/k}{\sqrt{[(1-\eta^2)]^2 + [2\xi\eta]^2}}$$
- Effect on vibrations due to change in ratio of the forced frequency to natural frequency of the system
  $\eta = \frac{\Omega}{\omega_n}$

Congratulations, you have finished Lecture 1. To view the next lecture select it from the left hand side menu of the page