Module 10 : Vibration of Two and Multidegree of freedom systems; Concept of Normal Mode; Free Vibration Problems and Determination of Ntural Frequencies; Forced Vibration Analysis; Vibration Absorbers; Approximate Methods - Dunkerley's Method and Holzer Method

Lecture 27 : Free Vibration of Two d.o.f. systems

Objectives

In this lecture you will learn the following

- Equation of motion for free vibration of two degree of freedom systems
- Solution of the equations of motion; Natural frequencies
- Concept of Normal Modes

Free Vibration of Two d.o.f. Systems

In the previous modules we discussed the free and forced vibration behavior of single d.o.f. systems. In this lecture, we will extend the discussion to two d.o.f systems. We will begin with free vibrations. We will be interested in determining how the system vibrates given some initial disturbance and left free to vibrate on its own. We will thus determine the natural frequencies of vibration etc. We had observed earlier that damping has marginal effect on natural frequency of single d.o.f system but primarily controls the resonance amplitudes. Thus we will consider undamped two d.o.f. system for our discussion.

Fig 10.1.1 A two d.o.f. spring mass sysytem

Consider a two d.o.f. spring-mass system as shown in Fig. 10.1.1 with two masses and three springs. To specify the configuration of the system, we need to specify the position of both the masses and hence we refer to this as a two d.o.f system.

Fig 10.1.2 Free body diagrams for a two d.o.f. system

The free body diagrams are shown in Fig. 10.1.2. Using Newton 's second Law, we write the equations of motion of each mass as follows:

$$\begin{aligned} m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) &= 0 \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3 x_2 &= 0 \end{aligned} \qquad 10.1.1$$

Using matrix notation, we can re-write as follows:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad 10.1.2$$

Thus we have two coupled, second order ordinary differential equations of motion with constant coefficients.

Observing the derivation of the equation of motion for the single d.o.f and two d.o.f system we notice that the procedure remains essentially the same viz.

- Step 1 : Consider the system in a displaced Configuration
- Step 2 : Draw Free Body diagrams
- Step 3 : Use Newton 's second Law to write the equation of motion.

We use the same procedure to develop the equations of motion even for multi-d.o.f systems.

Solution of equations of motion

Considering harmonic vibrations, let us assume

$$\begin{aligned} x_1(t) &= x_1 \sin \omega t \\ x_2(t) &= x_2 \sin \omega t \end{aligned} \qquad 10.1.3$$

Substituting in Eq (10.1.2), we get,

$$-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad 10.1.4$$

Rewriting this, we get,

$$\begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_1\omega^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad 10.1.5$$

For a non-trivial solution, we can write,

$$\text{determinant} \begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2\omega^2 \end{bmatrix} = 0 \qquad 10.1.6$$

Therefore we have the frequency equation / Characteristic equation:

$$(k_1 + k_2 - m_1\omega^2)(k_2 + k_3 - m_2\omega^2) - (-k_2)(-k_2) = 0 \qquad 10.1.7$$

$$i.e.\ m_1 m_2 \left(\omega^2\right)^2 - [m_1(k_2 + k_3) + m_2(k_1 + k_2)]\,\omega^2 + [k_1k_2 + k_1k_3 + k_2k_3] = 0 \qquad 10.1.8$$

The “natural frequencies” of the system are obtained as the solutions of the characteristic frequency equation as follows:

$$\left(\omega_n^2\right)_{1,2} = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \qquad 10.1.9$$

where $\alpha = m_1 m_2$

$$\beta = m_1(k_2 + k_3) + m_2(k_1 + k_2) \qquad 10.1.10$$

$$\gamma = k_1 k_2 + k_1 k_3 + k_2 k_3$$

when $m_1 = m_2 = m$
$k_1 = k_2 = k_3 = k$

we get $\omega_{n1} = \sqrt{\frac{k}{m}}$
$\omega_{n2} = \sqrt{\frac{3k}{m}}$

Concept of Normal Modes

Consider $\omega_{n_1} = \sqrt{\frac{k}{m}}$ and $\omega_{n_2} = \sqrt{\frac{3k}{m}}$ 10.1.11

If we substitute ω_n = ω_{n1} in eq. (10.1.5), we get,

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad 10.1.10$$

Thus the ratio of the amplitudes of the two masses is:

$$\frac{X_1}{X_2} = 1 \qquad 10.1.13$$

Similarly, if we substitute ω_n = ω_{n2} in eq. (10.1.5), we get,

$$\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad 10.1.14$$

Thus the ratio of the amplitudes of the two masses for this case is:

$$\frac{X_1}{X_2} = -1 \qquad 10.1.15$$

Such synchronous motion of the masses with constant ratio of amplitudes is known as Normal Mode Vibration.

Physical Meaning of Normal Modes

We can attribute the following physical meaning to the mathematical solution obtained above – if we give initial conditions such that the two amplitudes are in the ratio given above (either eq. (10.1.13) or (10.1.15)) and leave the system free to vibrate on its own, it will continue to vibrate forever maintaining this ratio of amplitudes all the time. This free vibratory motion will be at the frequency ω_{n1} or ω_{n2} respectively. These are known as the natural modes of the system – two for a two d.o.f system and in general “n” for an “n” d.o.f. system. For the present case, these are depicted in Fig. 10.1.3

Fig 10.1.3 Depiction of Mode Shapes (Normal modes)

It must be appreciated that these are only ratios of amplitudes of the two masses and not absolute magnitudes of vibratory displacement. Thus they indicate a certain shape of vibrating system rather than any particular amplitudes and hence they are also known as Mode Shapes.

Physical Meaning of Normal Modes (contd….)

In the first mode of vibration, both the masses have the same amplitude and hence the middle spring is undeformed. Hence we can ignore the presence of this spring and each spring-mass system is operating independently. Hence the natural frequency is same as a simple spring-mass SDOF system i.e.

$$\omega_{n_1} = \sqrt{\frac{k}{m}} \qquad 10.1.16$$

In the second mode of vibration, the two masses are exactly out of phase i.e. the two ends of the intermediate spring move by the same amount in opposite directions. Thus the mid-point of the intermediate spring will be at rest and hence we can consider the intermediate spring to be cut into two and arrested at the middle as shown in Fig. 10.1.4 below. We know that when a spring is cut into two, its stiffness is doubled and when twosprings are in parallel, their stiffnesses add up. Thus the equivalent system is as shown in the figure. Thus the natural frequency is given by:

Fig 10.1.4 Interpretation of Second Normel Mode

$$\omega_{n_2} = \sqrt{\frac{k+2k}{m}} = \sqrt{\frac{3k}{m}} \qquad 10.1.17$$

Which is same as what we got by solving the two d.o.f system equations.

Recap

In this lecture you have learnt the following.

- Procedure for developing the equations of motion is same for single or two d.o.f systems.

- A two d.o.f system is represented by two second order ordinary differential equations of motion.

- The two d.o.f system has two natural modes of vibration – these are the shapes of vibration with constant amplitude ratios for the two masses under synchronous motion at the system natural frequency. If the system is set to vibrate initially in either of these two modes, it will continue to vibrate forever in that mode at that natural frequency.

Congratulations, you have finished Lecture 1. To view the next lecture select it from the left hand side menu of the page.