

Module 8 : Free Vibration with Viscous Damping; Critical Damping and Aperiodic Motion; Logarithmic Decrement; Systems with Coulomb Damping.

Lecture 18 : Critical Damping and Aperiodic Motion

Objectives

In this lecture you will learn the following

- Solution of equation of motion for critically damped and overdamped systems
- Motion characteristics when $\zeta = 1$ and $\zeta > 1$

CRITICAL DAMPING

Critical damping is when a system has to undergo motion with the damping factor $\zeta = 1$. The two roots of the characteristic equation are given by the general equation:

$$s_{1,2} = \omega_n \left\{ -\zeta \pm \sqrt{\zeta^2 - 1} \right\} \quad 8.2.1$$

For $\zeta = 1$ the roots become:

$$s_1 = s_2 = -\omega_n \quad 8.2.2$$

The resulting motion of the system is given by

$$x = (A + Bt)e^{-\omega_n t} \quad 8.2.3$$

The above equation is solution to the system having critical damping, where A and B are to be obtained from initial conditions.

For finding the values of constants A and B we use the initial conditions: , let

$$\begin{aligned} x &= X_0 \quad \text{at } t=0 \\ \dot{x} &= 0 \quad \text{at } t=0 \end{aligned}$$

Differentiating the displacement equation (8.2.3), we get

$$\dot{x} = B e^{-\omega_n t} - (A + Bt) \omega_n e^{-\omega_n t} \quad 8.2.2$$

Substituting the initial conditions, we have

$$\begin{aligned} X_0 &= A \\ B - \omega_n A &= 0 \\ \text{which gives } A &= X_0 \\ \text{and } B &= \omega_n X_0 \end{aligned}$$

Substituting the values of A and B in the displacement equation

$$x = X_0 (1 + \omega_n t) e^{-\omega_n t} \quad 8.2.3$$

The value x in the above equation can be shown to decrease as t increases and ultimately tend to zero as t

tends to infinity see fig. 8.1.3.

APERIODIC MOTION

Aperiodic motion or over-damped system has the value of damping $\xi \geq 1$. The system has a very large value of damping. In this case roots of the differential equation are:

$$s_{1,2} = \omega_n \left\{ -\xi \pm \sqrt{\xi^2 - 1} \right\}$$

Both the roots are real and negative numbers. Therefore the general solution to the equation is

$$x = Ae^{s_1 t} + Be^{s_2 t}$$

which represents a decaying response.

We will now illustrate the determination of A and B from given initial conditions.

Consider the following initial conditions :

$$x = X_0 \quad \text{at } t=0$$

$$\dot{x} = 0 \quad \text{at } t=0$$

Differentiating the equation we get:

$$\dot{x} = AS_1 e^{s_1 t} + BS_2 e^{s_2 t} \quad 8.2.5$$

Substituting the initial conditions, we get

$$\begin{aligned} X_0 &= A + B \\ AS_1 + BS_2 &= 0 \end{aligned}$$

giving

$$\begin{aligned} A &= \left[\frac{\xi + \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}} \right] X_0 \\ B &= \left[\frac{-\xi + \sqrt{\xi^2 - 1}}{2\sqrt{\xi^2 - 1}} \right] X_0 \end{aligned}$$

Substituting the above values in the displacement equation :

$$x = \frac{X_0}{2\sqrt{\xi^2 - 1}} \left[\left[\xi + \sqrt{\xi^2 - 1} \right] e^{\left[-\xi + \sqrt{\xi^2 - 1} \right] \omega_n t} + \left[-\xi + \sqrt{\xi^2 - 1} \right] e^{\left[-\xi - \sqrt{\xi^2 - 1} \right] \omega_n t} \right] \quad 8.2.6$$

Since the exponent is negative in both the terms in the above equation, they both decrease exponentially with t. Higher the damping, more sluggish is the response of the system. Theoretically the system will take infinite time to come back to its equilibrium position. This type of motion is called aperiodic motion see fig. 8.1.4.

Underdamped system

Underdamped systems have the damping factor $\xi < 1$

The general equation of motion is as follows

$$x = Ae^{s_1 t} + Be^{s_2 t}$$

where $s_{1,2} = \omega_n \left\{ -\xi \pm \sqrt{\xi^2 - 1} \right\}$

With damping factor $\xi < 1$, the values of s_1 and s_2 will be as follows :

$$s_1 = \omega_n \left\{ -\xi + i\sqrt{\xi^2 - 1} \right\}$$

$$s_2 = \omega_n \left\{ -\xi - i\sqrt{\xi^2 - 1} \right\}$$

The general solution to the equation of motion is as follows:

$$x = e^{-\xi \omega_n t} \left[A \sin(\sqrt{1 - \xi^2} \omega_n t) + B \cos(\sqrt{1 - \xi^2} \omega_n t) \right] \quad 8.2.8$$

subjected to initial conditions as follows

$$x = X_0 \quad \text{at } t=0$$

$$\dot{x} = 0 \quad \text{at } t=0$$

solving for A and B we get

$$x = X_0 e^{-\xi \omega_n t} \left[\frac{\xi}{\sqrt{1 - \xi^2}} \sin(\sqrt{1 - \xi^2} \omega_n t) + \cos(\sqrt{1 - \xi^2} \omega_n t) \right] \quad 8.2.9$$

where $\omega_n \sqrt{1 - \xi^2}$ is called as damped natural frequency.

Recap

In this lecture you have learnt the following

- Concept of critical damping
- Solution of equation of motion for underdamped, critically damped and over damped systems.

Congratulations, you have finished Lecture 2. To view the next lecture select it from the left hand side menu of the page.

