Module 11 : Free Vibration of Elastic Bodies; Longitudinal Vibration of Bars; Transverse Vibration of Beams;
Torsional Vibration of Shaft; Approximate Methods – Rayleigh's Method and Rayleigh-Ritz Method.

Lecture 34 : Torsional Vibration of Shafts

Objectives

In this lecture you will learn the following

- Significance of torsional vibrations in shafts

- Derivation of governing partial differential equation

- Natural frequencies and mode shapes for torsional vibrations

Shafts transmit power and in the process, are subjected to time-varying torques. For example an IC Engine crankshaft is subjected to pulsating torques as we have discussed in an earlier module. Such fluctuating torques set-up vibratory motion. However the shaft is already undergoing rotation. Thus the torsional (elastic twisting and untwisting) vibration is superposed on this rigid body rotation, making it somewhat complex to visualize. When subjected to excessive vibrations for a sufficiently long time, the shafts may fail and thus they need to be analyzed carefully for torsional vibrations. While most of the real-life shafts may be quite complex in shape, we begin our discussion with a simple, uniform cross-section circular shaft as shown in Figure 11.3.1.

Consider a small elemental length " Δx ". Under the action of the torque, let the left end rotate by ψ and the right end by $\psi + \Delta \psi$. For small deformations, the shear strain is given by:

$$\gamma = \lim_{\Delta x \to 0} \frac{CC'}{CD} = \lim_{\Delta x \to 0} \frac{r\Delta\psi}{\Delta x} = r\frac{d\psi}{dx} \qquad (11.3.1)$$

The shear stress is given by:

$$\tau = G\gamma = Gr\frac{d\psi}{dx} \qquad (11.3.2)$$

wherein $d\psi/dx$ represents rate of twist or angle of twist per unit length.

The shear stress acting on an elemental area "dA" at a radial distance "r" causes an elemental torque "dT" as shown in Fig. 11.3.2:

$$dT = r (\tau dA) \qquad (11.3.3)$$

Integrating over the whole area, the total twisting moment is obtained as:

$$T = \iint r\tau \, dA = \iint G\frac{d\psi}{dx}r^2 dA = G\frac{d\psi}{dx}\iint r^2 dA = GI_p\frac{d\psi}{dx} \qquad (11.3.4)$$

Considering a slice of length "dx" as shown in <u>Fig. 11.3.3</u>, from equilibrium considerations (Newton 's second law), we can write:

$$\left(T + \frac{\partial T}{\partial x} dx\right) - T = \left(\rho I_p \, dx\right)\frac{\partial^2 \psi}{\partial t^2} \qquad (11.3.5)$$

Substituting from eq. (11.3.4), we get:

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{G}{\rho}\frac{\partial^2 \psi}{\partial x^2} = c^2 \frac{\partial^2 \psi}{\partial x^2} \qquad (11.3.6)$$
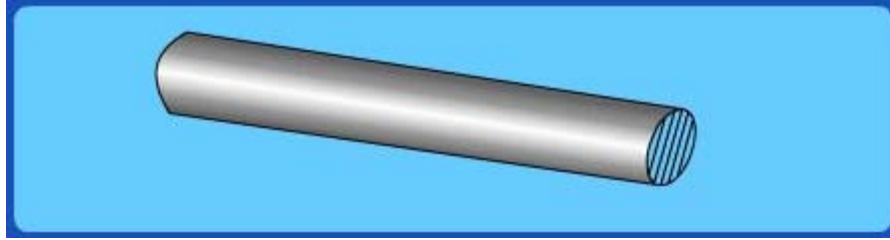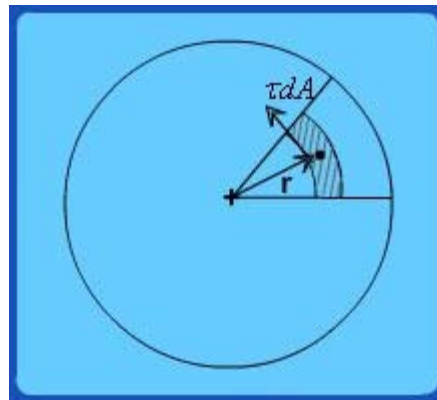

Figure 11.3.1

Figure 11.3.2

This is essentially similar to eq. (11.2.7), the wave equation for axial vibrations of rods. In both the cases, it is important to note that the governing equation does not contain any terms pertaining to area of cross-section or polar moment of area of cross-section etc. As long as the cross-section is uniform (and circular for torsion case), these equations can be used. The solution procedure also proceeds in a very similar fashion.

A typical multi-cylinder IC engine crankshaft is shown in Fig. 11.3.3 and it is readily observed that it deviates significantly from the idea uniform cross-section shaft we assumed so far. An equivalent uniform diameter shaft can be drawn-up as shown in Fig. 11.3.4 and the above equations can be used for its torsional dynamics. Such an analysis will help us determine the natural frequencies approximately. A more detailed analysis will require a full three dimensional finite element model as shown in Fig. 11.3.5. Finite Element Method, P.Seshu, Prentice Hall of India,2006.
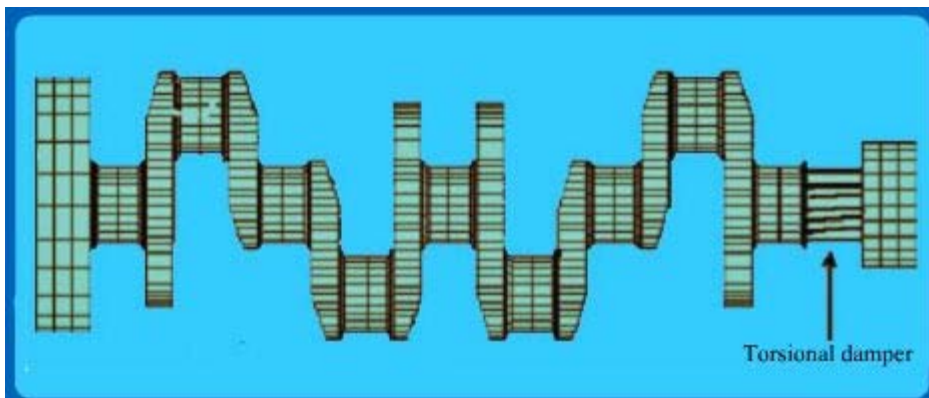

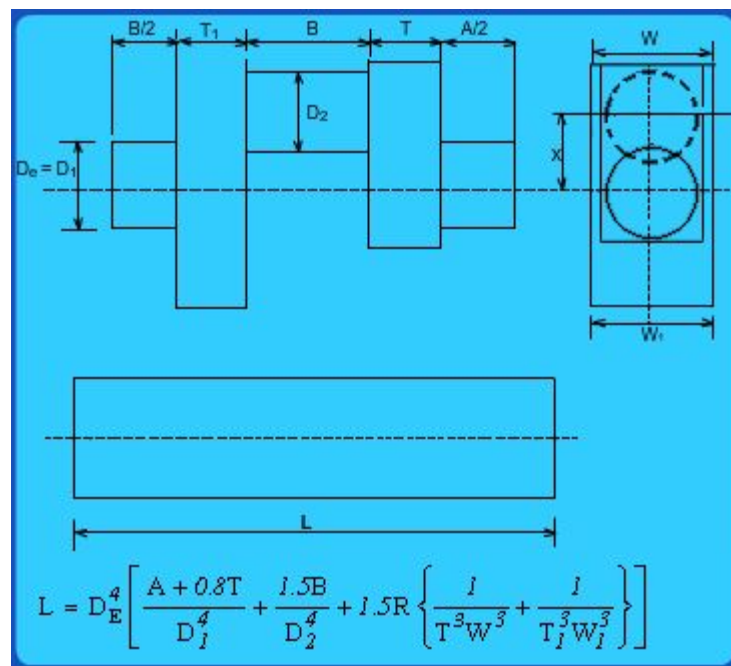**Fig. 11.3.3 A typical multi-cylinder engine crankshaft**

$$L = D_E^4 \left[\frac{A + 0.8T}{D_1^4} + \frac{1.5B}{D_2^4} + 1.5R \left\{ \frac{1}{T^3 W^3} + \frac{1}{T_1^3 W_1^3} \right\} \right]$$

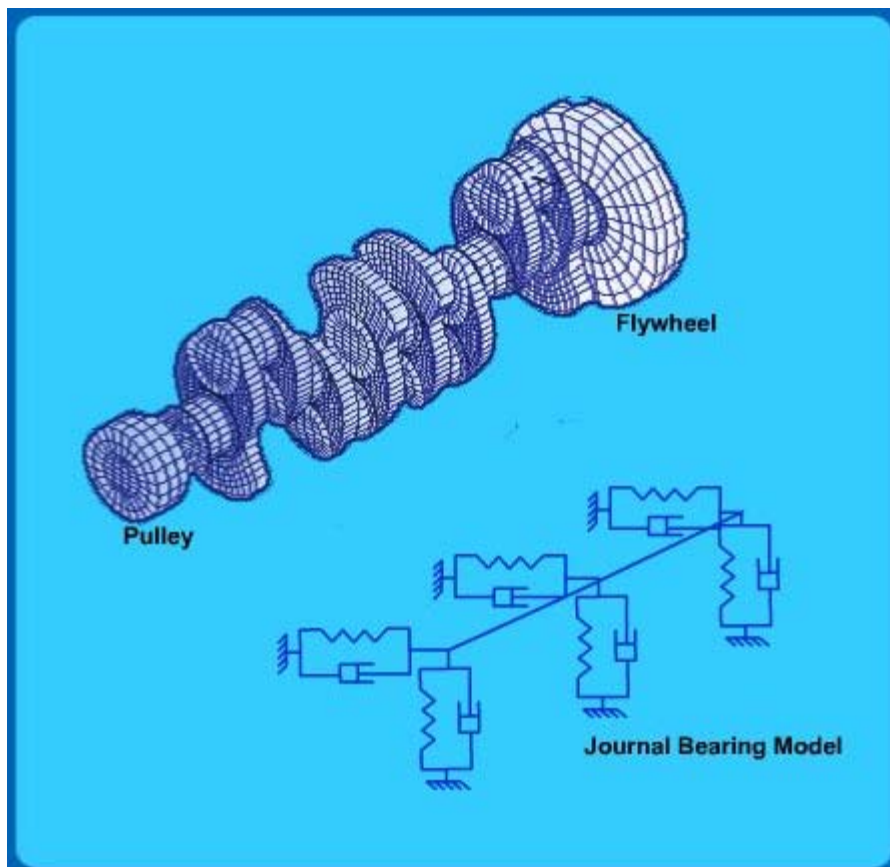**Fig. 11.3.4 Equivalent Shaft for Crankt**

Fig. 11.3.5 Representative 3-D finite element model of crank shaft

Recap

In this lecture you have learnt the following.

- Torsional vibrations of uniform circular cross-section shafting

- Derivation of the governing equation

- Simplifications required to model real-life crankshafts