## Module 2 : Convection

### Lecture 24 : Thermal Boundary Layer Equations

**Objectives**

**In this class:**

- Solution for thermal boundary layer equation is obtained

### Thermal Boundary Layer Solution-1

- The 2D boundary layer energy equation assumptions is easily obtained . Assume wedge flow with constant wall temperature and $\beta T \approx 1$

$$\rho c_p\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right]=k\frac{\partial^2 T}{\partial y^2}+\beta Tu\frac{\partial p}{\partial x}+\mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad (24.1)$$

- Boundary layer flow over a wedge possesses a similarity solution and therefore:

$$\eta = y\sqrt{\frac{u_\infty}{\upsilon x}}\ ; \frac{u}{u_\infty}=F';\ u_\infty = Ax^m$$

### Thermal Boundary Layer Solution-2

- In addition, we define a non-dimensional temperature as:

$$\theta=\frac{T-T_\infty}{T_w-T_\infty} \qquad (24.2)$$

- The attempt is once again to see if the energy equation can be transformed into one where the governing equation becomes an ordinary differential equation.

### Thermal Boundary Layer Solution-3

- Consider each term of equ$^n$ (24.1) separately.
  Consider the RHS first:

$$k\frac{\partial^2 T}{\partial y^2}=k(T_w-T_\infty)\frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial\eta}\frac{\partial\eta}{\partial y}\right)=k(T_w-T_\infty)\frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial\eta}\sqrt{\frac{u_\infty}{\upsilon x}}\right)$$

$$=k(T_w-T_\infty)\sqrt{\frac{u_\infty}{\upsilon x}}\frac{\partial}{\partial\eta}\left(\frac{\partial\theta}{\partial\eta}\right)\frac{\partial\eta}{\partial y}=\frac{ku_\infty}{\upsilon x}\frac{\partial^2\theta}{\partial\eta^2} \qquad (24.3)$$

### Thermal Boundary Layer Solution-4

- The next term on RHS is the viscous dissipation term:

$$\mu\left(\frac{du}{dy}\right)^2=\mu\left(\frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial y}\right)^2=\mu\frac{u_\infty}{\upsilon x}(u_\infty F'')^2 \qquad (24.4)$$

- The next term is the compressibility term. Use the Bernoulli equation to convert pressure into velocity and simplify:

$$u\frac{\partial p}{\partial x}=uu_\infty\frac{du_\infty}{dx}=u_\infty F'u_\infty m\frac{Ax^m}{x}=\frac{mF'u_\infty^{\ 3}}{x} \qquad (24.5)$$

### Thermal Boundary Layer Solution-5

- Now look at the LHS of equ$^n$ (24.1). The wall temperature is not a function of ‘y’. In addition assume that the $T_\infty$ is also not a function of ‘y’ (varying the free stream temperature in the ‘y’ direction is physically very difficult). The first term is:

$$u\frac{\partial T}{\partial x} = u_\infty F'\left[(T_w - T_\infty)\frac{\partial \theta}{\partial x} + \theta\frac{d}{dx}(T_w - T_\infty) + \frac{dT_\infty}{dx}\right] \quad (24.6)$$

**Thermal Boundary Layer Solution-6**

- It is hoped that a similarity solution will exist even when the energy equation is added. Derivatives must therefore be in terms of 'η'. Simplify equ$^n$ (24.6) using:

$$\eta = y\sqrt{\frac{u_\infty}{\upsilon x}} = y(A/\upsilon)^{1/2} x^{\frac{m-1}{2}}$$

- Therefore:

$$\frac{\partial \eta}{\partial x} = y\left(\frac{A}{\upsilon}\right)^{1/2}\frac{m-1}{2}x^{\frac{m-1}{2}-1}\frac{x}{x} = y\left(\frac{A}{\upsilon}\right)^{1/2}\frac{m-1}{2}x^{\frac{m+1}{2}}\frac{1}{x^2} \quad (24.7)$$

**Thermal Boundary Layer Solution-7**

- Simplify equ$^n$ (24.7) further to get:

$$\frac{\partial \theta}{\partial x} = \theta' y\left(\frac{Ax^m x}{\upsilon}\right)^{1/2}\frac{m-1}{2x^2} = \theta' y\left(\frac{u_\infty x}{\upsilon x^2}\right)^{1/2}\frac{m-1}{2}\frac{1}{x}$$

$$\frac{\partial \theta}{\partial x} = \theta'\eta\frac{m-1}{2x} \quad (24.8)$$

- Use this in equ$^n$ (24.6)

$$u\frac{\partial T}{\partial x} = u_\infty F'\left[(T_w - T_\infty)\theta'\eta\frac{m-1}{2x} + \theta\frac{d}{dx}(T_w - T_\infty) + \frac{dT_\infty}{dx}\right] \quad (24.9)$$

**Thermal Boundary Layer Solution-8**

- The next term is evaluated in parts. The first part is:

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\frac{\partial \theta}{\partial y} + \theta\frac{\partial}{\partial y}(T_w - T_\infty) + \frac{\partial}{\partial y}T_\infty$$

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\frac{\partial \theta}{\partial y} = (T_w - T_\infty)\theta'\sqrt{\frac{u_\infty}{\upsilon x}} \quad (24.10)$$

- The second part is the vertical component of the velocity 'v'

**Thermal Boundary Layer Solution-9**

- The 'v' component is evaluated as follows:

$$-v = \frac{\partial \psi}{\partial x} = \left[F\frac{\partial}{\partial x}\frac{u_\infty}{g} + \frac{u_\infty}{g}F'\frac{\partial \eta}{\partial x}\right] \qquad g = \sqrt{\frac{u_\infty}{\upsilon x}}$$

$$= F\frac{\partial}{\partial x}\sqrt{u_\infty \upsilon x} + \frac{u_\infty}{g}F'\frac{\partial}{\partial x}y\sqrt{\frac{u_\infty}{\upsilon x}}$$

$$= F\frac{\partial}{\partial x}\sqrt{Ax^{m+1}\upsilon} + \frac{u_\infty}{g}F'\frac{\partial}{\partial x}y\sqrt{\frac{Ax^{m-1}}{\upsilon}}$$

$$= F\sqrt{A\upsilon}\frac{m+1}{2}x^{\frac{m-1}{2}} + \frac{u_\infty F'}{g}\frac{y\sqrt{x}}{\sqrt{\upsilon}}\frac{m-1}{2}x^{\frac{m-3}{2}} \quad (24.11)$$

**Thermal Boundary Layer Solution-10**

- Simplify the expression (24.11) further:

$$-v = \frac{m+1}{2}\frac{Ax^{m}}{\sqrt{A}}\left[\sqrt{\upsilon}Fx^{-\frac{m-1}{2}} + \eta F'\frac{m-1}{m+1}\frac{u_\infty}{g^2}x^{-\frac{m-3}{2}}\right]$$

$$= \frac{m+1}{2}u_\infty\frac{1}{\sqrt{A}}\left[\sqrt{\upsilon}Fx^{-\frac{m-1}{2}} + \eta F'\frac{m-1}{m+1}\frac{u_\infty}{u_\infty}\frac{\upsilon x}{\sqrt{\upsilon}}x^{-\frac{m-3}{2}}\right]$$

$$= \frac{m+1}{2}u_\infty\frac{1}{\sqrt{A}}\left[\sqrt{\upsilon}Fx^{-\frac{m-1}{2}} + \eta F'\frac{m-1}{m+1}\sqrt{\upsilon}x^{-\frac{m+1}{2}}\right]$$

$$= \frac{m+1}{2}u_\infty\sqrt{\frac{\upsilon}{Ax^{m}x}}\left[F + \frac{m-1}{m+1}\eta F'\right] \qquad (24.12)$$

**Thermal Boundary Layer Solution-11**

- Combine equ$^n$ (24.9) and (24.11) to get:

$$v\frac{\partial T}{\partial y} = -\left[\frac{m+1}{2}u_\infty\sqrt{\frac{\upsilon}{u_\infty x}}\left(F + \frac{m-1}{m+1}\eta F'\right)\right]\theta'(T_w - T_\infty)\sqrt{\frac{u_\infty}{\upsilon x}}$$

$$= -\frac{m+1}{2}\frac{u_\infty}{x}\left(F + \frac{m-1}{m+1}\eta F'\right)\theta'(T_w - T_\infty) \qquad (24.13)$$

- All differential terms of the governing equation are now converted in terms of the variable 'η'.

**Thermal Boundary Layer Solution-12**

- Now substitute equ$^n$s (24.9) and (24.13) in in the LHS of governing equation (24.1)

$$u_\infty F'\left[(T_w - T_\infty)\theta'\eta\frac{m-1}{2x} + \theta\frac{d}{dx}(T_w - T_\infty) + \frac{dT_\infty}{dx}\right]$$
$$-\frac{m+1}{2}\frac{u_\infty}{x}\left(F + \frac{m-1}{m+1}\eta F'\right)\theta'(T_w - T_\infty)$$

- The terms marked yellow in the above expression cancel off

**Thermal Boundary Layer Solution-13**

- The LHS of governing equation (24.1) is thus:

$$\rho c_p\left(u_\infty F'\left[\theta\frac{d}{dx}(T_w - T_\infty) + \frac{dT_\infty}{dx}\right] - \frac{m+1}{2}u_\infty F\theta'(T_w - T_\infty)\right) \qquad (24.14)$$

- Substituting equ$^n$s (24.3)-(24.5), (24.14) in (24.1) gives:

$$\rho c_p\left(u_\infty F'\left(\theta\frac{d}{dx}(T_w - T_\infty) + \frac{dT_\infty}{dx}\right) - \frac{m+1}{2}\frac{u_\infty F\theta'}{x}(T_w - T_\infty)\right)$$
$$= k(T_w - T_\infty)\frac{u_\infty\theta''}{\upsilon x} + \frac{\mu u_\infty}{\upsilon x}(u_\infty F'')^2 - \frac{mF'u_\infty^{3}}{x} \qquad (24.15)$$

**Thermal Boundary Layer Solution-14**

- Multiply the equ$^n$ (24.15) throughout by the term $\dfrac{x}{u_\infty(T_w - T_\infty)\rho c_p}$ to get:

$$F'\left[\frac{\theta x}{T_w - T_\infty}\frac{d}{dx}(T_w - T_\infty) + \frac{x}{(T_w - T_\infty)}\frac{dT_\infty}{dx}\right] - \frac{m+1}{2}F\theta' = \frac{k\theta''}{\upsilon\rho c_p} + \frac{u_\infty^2 F''^2}{c_p(T_w - T_\infty)} - \frac{mF'u_\infty^2}{c_p(T_w - T_\infty)} \quad (24.16)$$

**Thermal Boundary Layer Solution-15**

- Minor adjustment of the equ$^n$ (24.16) results in the final form of the energy equation:

$$\frac{\theta''}{\Pr} + \frac{m+1}{2}F\theta' = F'\left[\frac{\theta x}{T_w - T_\infty}\frac{d}{dx}(T_w - T_\infty) + \frac{x}{(T_w - T_\infty)}\frac{dT_\infty}{dx}\right] - \frac{u_\infty^2}{c_p(T_w - T_\infty)}(F''^2 - mF') \quad (24.17)$$

- The term marked yellow will be assumed to be zero since varying $T_\infty$ in the 'x' direction is usually not possible.

**Thermal Boundary Layer Solution-16**

- For the equation (24.17) to be amenable to a similarity solution the following two conditions must be satisfied:

$$\frac{x}{T_w - T_\infty}\frac{d}{dx}(T_w - T_\infty) = \lambda = Const \quad (24.18)$$

$$\frac{u_\infty^2}{c_p(T_w - T_\infty)} = Ec = const \quad (24.19)$$

- λ and Ec are arbitrary constants. Ec is the Eckert number that has been seen earlier.

**Thermal Boundary Layer Solution-17**

- Equ$^n$ (24.18) can be integrated to give:

$$T_w - T_\infty = Dx^\lambda \quad (24.20)$$

- Substitute (24.20) in (24.19) to get:

$$\frac{u_\infty^2}{c_p(T_w - T_\infty)} = \frac{A^2 x^{2m}}{c_p Dx^\lambda} = \text{Eckert no.} = \text{constant} \quad (24.21)$$

$$\Rightarrow \lambda = 2m$$

- Even though λ was an 'arbitrary constant' to start with, it can now be seen that there are 'constraints' which it has to satisfy.

**Thermal Boundary Layer Solution-18**

- Now consider the special case of a flat plate. For this case m = 0 since $U_\infty = Ax^m = \text{constant}$
- The above condition gives $\lambda = 2m = 0$. This implies $T_w - T_\infty = Dx^\lambda = 0$ i.e. $T_w$ = constant.
- The similarity solution therefore will exist only for the constant wall temperature case. While this is a very important case one would also like to get the solution for the equally important constant wall flux case.

**Thermal Boundary Layer Solution-19**

- For any other value of 'm', corresponding to other wedge flow situations, the similarity solution will exist only for a very restrictive variation of wall temperature.
- One option is to neglect the presence of the viscous dissipation and compressibility term. The last term of equ$^n$ (24.17) therefore drops out and the condition given by equ$^n$ (24.19) no longer

is required to be satisfied.

**Thermal Boundary Layer Solution-20**

- Thus the λ = 2 condition is no longer required. Now the only restriction is $T_w = T_\infty + Dx^\lambda$ and solutions can be obtained for more cases.
- The governing equation and BCs become:

$$\frac{\theta''}{\Pr} + \frac{m+1}{2} F\theta' - \lambda F'\theta = 0 \qquad \theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{24.22}$$

$$\theta(0) = 1; \theta(\infty) = 0 \quad \lambda \text{ satisfies } T_w - T_\infty = Dx^\lambda$$

**Thermal Boundary Layer Solution-21**

- Now for $T_W$ = constant λ = 0 but the surface need not be flat plate anymore since λ and 'm' are no longer related. The governing equation becomes:

$$\frac{\theta''}{\Pr} + \frac{m+1}{2} F\theta' = 0 \tag{24.23}$$

- The equation can be integrated to obtain the solution for the temperature distribution after incorporating the boundary conditions.

**Thermal Boundary Layer Solution-22**

- Equ$^n$ (24.23) is rewritten in the following manner:

$$(\theta')' = -\frac{m+1}{2} \Pr F\theta' \tag{24.24}$$

- This equation is now integrated to obtain:

$$\ln(\theta') = -\frac{m+1}{2} \Pr \int F\eta \, d\eta \;\; \Rightarrow \theta = C_1 \int_0^\eta \left[ e^{-\frac{m+1}{2} \Pr \int_0^{\bar{\eta}} F d\bar{\bar{\eta}}} \right] d\bar{\eta} + C_2 \tag{24.25}$$

**Thermal Boundary Layer Solution-23**

- Now apply the boundary conditions and get solution

$$\theta_{\eta=0} = 1 \;\; \Rightarrow C_2 = 1$$

$$\theta_{\eta=\infty} = 0 \Rightarrow C_1 = \frac{-1}{\int_0^\infty \left[ e^{-\frac{m+1}{2} \Pr \int_0^{\bar{\eta}} F d\bar{\bar{\eta}}} \right] d\bar{\eta}}$$

$$\therefore \; \theta = 1 - \frac{\int_0^{\bar{\eta}} \left[ e^{-\frac{m+1}{2} \Pr \int_0^{\bar{\bar{\eta}}} F d\bar{\bar{\eta}}} \right] d\bar{\eta}}{\int_0^\infty \left[ e^{-\frac{m+1}{2} \Pr \int_0^{\bar{\bar{\eta}}} F d\bar{\bar{\eta}}} \right] d\bar{\eta}} \tag{24.26}$$

**Thermal Boundary Layer Solution-24**

- 'F' is known from the solution of the momentum equation and 'Pr' is a fluid property and therefore temperature profile can be obtained from equ$^n$ (24.26). Solutions would typically be numerical.
- Interest is typically more in the heat flux from the wall to the fluid which is evaluated as:

$$q = -k\frac{\partial T}{\partial y}\bigg|_{y=0} = -k(T_w - T_\infty)\sqrt{\frac{u_\infty}{\upsilon x}}\frac{d\theta}{d\eta}\bigg|_{\eta=0} \tag{24.27}$$

**Thermal Boundary Layer Solution-25**

- The similarity solution dictates that the free stream velocity and the wall temperature must follow the relations given by:

$$(T_w - T_\infty) = Dx^\lambda; \quad U_\infty = Ax^m$$

- Equ$^n$ (24.27) is therefore further simplified as:

$$q = -k(T_w - T_\infty)\left[\frac{Ax^{m-1}}{\upsilon}\right]^{1/2}\theta'(0)$$
$$\therefore \quad q \propto x^{\frac{2\lambda+m-1}{2}} \tag{24.28}$$

**Thermal Boundary Layer Solution-26**

- Consider the constant wall heat flux case. The equ$^n$ (24.28) suggests:

$$\frac{2\lambda + m - 1}{2} = 0$$

- For a flat plate m = 0 and therefore λ = 1/2
  The wall temperature variation for a constant wall flux case for boundary layer is given by:

$$T_w - T_\infty = Dx^{1/2}$$

**Thermal Boundary Layer Solution-27**

- Often the heat transfer coefficient is of interest which is

$$h = \frac{q}{T_w - T_\infty} = k\left(\frac{Ax^{m-1}}{\upsilon}\right)^{1/2}(-\theta'(0))$$

- $\theta'(0)$ can be numerically obtained

**Reynolds Analogy-1**

- The equations for the 2D thermal and momentum boundary layer are:

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial y^2}\right] \tag{24.29}$$

$$\rho c_p\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = k\frac{\partial^2 T}{\partial y^2} \tag{24.30}$$

- For α = ν and zero pressure gradient flows the equations are identical. The velocity and temperature profiles are therefore identical. Assume that the momentum equation solution is known

**Reynolds Analogy-2**

- The following quantities of interest are known:

$$\tau = \mu\left[\frac{\partial u}{\partial y}\right] \qquad \tau_w = \mu\left[\frac{\partial u}{\partial y}\right]_w \tag{24.31}$$

- The thermal quantities of interest are :

$$q = -k\left[\frac{\partial T}{\partial y}\right] \qquad q_w = -k\left[\frac{\partial T}{\partial y}\right]_w \tag{24.32}$$

- Equ$^n$s (24.31) and (24.32) are combined to give:

$$\left[\frac{\tau}{\tau_w}\right] = \left[\frac{q}{q_w}\right] \tag{24.33}$$

**Reynolds Analogy-3**

- Since the profiles are identical, momentum information is used to get the thermal information

$$u_1 = \int_0^y \frac{\tau}{\nu} dy \tag{24.34}$$

$$T - T_1 = \int_0^y \frac{q}{\rho C \alpha} dy = \int_0^y \frac{\tau q_w}{\tau_w \rho C \alpha} dy = \frac{u_1 q_w}{\tau_w \rho C} \tag{24.35}$$

- Use equn (24.34) to get 'u' as a function of 'y' and equn (24.35) to get 'T' as a function of 'y' by using $\alpha = \nu$

**Reynolds Analogy-4**

- Use equ$^n$ (24.35) to get:

$$T_w - T_\infty = \frac{u_\infty q_w}{\tau_w \rho C} \Rightarrow \frac{\tau_w}{\rho u_\infty^2} = \frac{q_w}{(T_w - T_\infty)\rho C u_\infty} \tag{24.35}$$

- The above equation is one form of the Reynolds analogy
- By definition:

$$\frac{\tau_w}{\rho u_\infty^2 / 2} \equiv C_f = \text{Friction coefficient}$$

**Reynolds Analogy-5**

- Therefore, the Reynolds analogy as derived in equ$^n$ (24.25) is often expressed as:

$$\frac{C_f}{2} = St$$

- The result is significant in the sense that one can get heat transfer data from shear stress data.
- The usefulness of the analogy is often limited due to the restrictions on pressure gradient and Prandtl number imposed

**Recap**

**In this class:**

- Solution for thermal boundary layer equation is obtained