Module 2 : Convection

Lecture 14 : Use of conservation equations

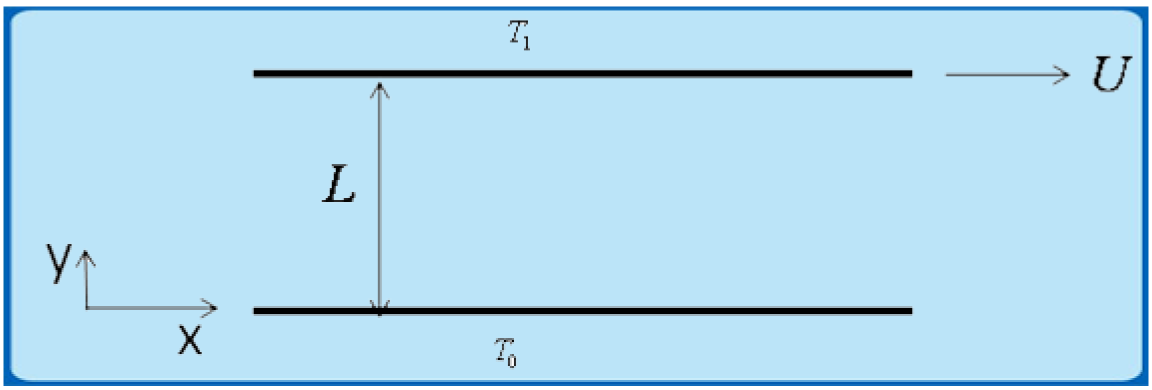
Objectives

In this class:

- Couette flow situation is used to understand non-dimensional numbers occurring in convection situations
- Fully developed approximation for mass momentum and energy equations is used to obtain velocity and temperature profiles
- Concept of adiabatic wall temperature for heat transfer coefficient definition is discussed.

2 D Couette Flow-1

- Two parallel plates of very long dimensions in the X and Z directions. Top plate moves at a velocity 'U' and is at temperature 'T_1'. Bottom plate stationary i.e. velocity = 0. Distance between plates = 'L'



2 D Couette Flow-2

- Steady, Laminar, Newtonian, Incompressible, constant viscosity
- In addition: Fully developed (Velocity and temperature profiles do not change in the flow direction), zero pressure gradient, no body forces, :

$$\underbrace{\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial T}{\partial x} = 0}_{\text{Fully Developed}} \qquad \underbrace{\frac{\partial P}{\partial x} = 0}_{\text{No Pressure Gradient}} \qquad \underbrace{\bar{X} = 0}_{\text{No Body Force}} \tag{14.1}$$

2 D Couette Flow-3

$$\text{Continuity: } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0; \tag{14.2}$$

$$v = \text{constant}$$

- Since v = constant and v = 0 at wall, v = 0 everywhere
- Now look at x-momentum

$$\rho\left[u\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{0} + \underbrace{\cancel{v}}_{0}\frac{\partial u}{\partial y}\right] = -\underbrace{\cancel{\frac{\partial p}{\partial x}}}_{0} + \mu\left[\underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{0} + \frac{\partial^2 u}{\partial y^2}\right] \tag{14.3}$$

2 D Couette Flow-4

- In equ^n (13.3) several terms are zero due to either the fully developed flow assumption or the v = 0 condition that was obtained from the continuity equation. The momentum equation therefore simplifies to:

$$\mu\frac{\partial^2 u}{\partial y^2}=0 \tag{14.4}$$

2 D Couette Flow-5

- Now look at the y-momentum equation. Since the v-velocity is itself zero all its gradients are also zero and drop out of the equation

$$\begin{aligned}&\rho\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial p}{\partial y}-\rho g+\mu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]\\ &\therefore \frac{\partial p}{\partial y}=0\end{aligned} \tag{14.5}$$

- Hydrostatic pressure gradient would exist if the body forces were present

2 D Couette Flow-6

- Now look at the energy equation. Simply get the 2D form from the general 3D form. The continuity equation and the fully developed flow assumption is again used to simplify the equation.

$$\begin{aligned}\rho c_p\left[u\cancelto{0}{\frac{\partial T}{\partial x}}+\cancelto{0}{v}\frac{\partial T}{\partial y}\right]&=\frac{\partial}{\partial x}\left(k\cancelto{0}{\frac{\partial T}{\partial x}}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\beta T\left[u\cancelto{0}{\frac{\partial p}{\partial x}}+\cancelto{0}{v}\frac{\partial p}{\partial y}\right]\\ &+\frac{2}{3}\mu\left(\cancelto{0}{\frac{\partial u}{\partial x}}-\cancelto{0}{\frac{\partial v}{\partial y}}\right)^2+\mu\left[\frac{\partial u}{\partial y}+\cancelto{0}{\frac{\partial v}{\partial x}}\right]^2\end{aligned} \tag{14.6}$$

2 D Couette Flow-7

- Energy equation therefore becomes:

$$k\frac{d^2T}{dy^2}+\mu\left(\frac{du}{dy}\right)^2=0 \tag{14.7}$$

- Non-dimensionalize the momentum and energy equation using the following combinations:

$$\theta=\frac{T-T_0}{T_1-T_0};u*=u/U;y*=y/L \tag{14.8}$$

2 D Couette Flow-8

- Non-dimensional Momentum equ^n becomes:

$$\frac{\mu U}{L^2}\left(\frac{d^2u*}{dy*^2}\right)=0 \tag{14.9}$$

- The boundary conditions are:

$$u*=1 \quad at\ y*=1 \tag{14.10}$$

$$u*=0 \quad at\ y*=0 \tag{14.11}$$

$$\theta=1 \quad at\ y*=1 \tag{14.12}$$

$$\theta=0 \quad at\ y*=0 \tag{14.13}$$

2 D Couette Flow-9

- Non-dimensional energy equation becomes:

$$\frac{k(T_1-T_0)}{L^2}\frac{d^2\theta}{dy^{*2}}+\mu\left(\frac{U}{L}\right)^2\left(\frac{du^*}{dy^*}\right)^2=0$$

$$\frac{d^2\theta}{dy^{*2}}+\frac{\mu U^2}{k(T_1-T_0)}\left[\frac{du^*}{dy^*}\right]^2=0$$

$$\frac{\mu U^2}{k(T_1-T_\infty)}\equiv\frac{\mu c_p}{k}\frac{U^2}{c_p(T_1-T_0)} \qquad (14.14)$$

Brinkman Number | Prandtl Number | Eckert Number

2 D Couette Flow-10

- The Brinkman number is the non-dimensional number representative of the viscous dissipation term. It is a combination of the Prandtl number and the Eckert number.
- Now we solve the equations to get a quantitative picture

2 D Couette Flow-11

- Solve momentum equation (equ^n (14.9))

$$\frac{d^2u^*}{dy^{*2}}=0\Rightarrow\frac{du^*}{dy^*}=K_1$$

$$u^*=K_1y^*+K_2$$

- Now use the Boundary conditions:

$$u^*=0 \quad y^*=0;\ u^*=1 \quad y^*=1$$

- Finally obtain

$$u^*=y^* \qquad (14.15)$$

2 D Couette Flow-12

- Now solve the energy equation

$$\frac{d^2\theta}{dy^{*2}}=-Ec\,\mathrm{Pr}\left[\frac{du^*}{dy^*}\right]^2=-Ec\,\mathrm{Pr}$$

$$\theta=-Ec\,\mathrm{Pr}\frac{y^{*2}}{2}+C_1y^*+C_2 \qquad (14.16)$$

- Using the boundary conditions, the constants are evaluated as:

$$y^*=0 \quad \theta=0\Rightarrow C_2=0$$

$$y^*=1 \quad \theta=1\Rightarrow C_1=1+\frac{E_c\,\mathrm{Pr}}{2}$$

2 D Couette Flow-13

- Solution of the equation is:

$$\theta = -E_c \Pr \frac{y^{*2}}{2} + \left(1 + \frac{E_c \Pr}{2}\right) y^*$$
$$= y^* + \frac{E_c \Pr}{2} y^*(1 - y^*) \qquad (14.17)$$

- When $E_c Pr << 1$ $\theta = y^*$. The nondimensional temperature profile is linear
- Pr is a fluid property. When Pr is of the order of unity, small Ec implies dissipation effects can be ignored.

2 D Couette Flow-14

- Now look at the wall flux:

$$q_w = -k \left.\frac{\partial T}{\partial y}\right)_{y=0} = \frac{-k(T_1 - T_0)}{L} \left.\frac{d\theta}{dy^*}\right|_{y^*=0}$$
$$= \frac{-k(T_1 - T_0)}{L}\left[1 + \frac{E_c \Pr}{2}\right] \qquad (14.18)$$
$$= +\frac{k}{L}\left[T_0 - \underbrace{\left(T_1 + \frac{\Pr U^2}{2c_p}\right)}_{T_{ref}}\right]$$

2 D Couette Flow-15

- If a heat transfer coefficient had to be defined it would perhaps be ‘k/L’ with the appropriate temperature difference being the difference between the wall temperature and some ‘reference’ temperature T_{ref} as in equ^n (14.18). The wall flux, q_w, heat transfer coefficient, h, and wall temperature T_w, are related as:

$$q_w = h(T_w - T_{ref}) \qquad (14.19)$$

- Notice that when $q_w = 0$, $T_w = T_{ref}$

2 D Couette Flow-16

- An insulated wall or an adiabatic wall temperature, T_{aw}, is therefore the most suitable reference temperature. Therefore:

$$q_w = h(T_w - T_{aw}) \qquad (14.20)$$

- Assume bottom wall insulated instead of bottom wall temperature specified in the Couette flow situation discussed earlier. The governing equation remains same and the equ^n (14.16) is still the solution.

2 D Couette Flow-17

- Use bottom wall insulated and top wall temperature specified conditions to get:

$$\left.\frac{d\theta}{dy^*}\right|_{y^*=0} = 0 \Rightarrow C_1 = 0; \qquad \theta\big|_{y^*=1} = 1 \Rightarrow C_2 = 1 + \frac{E_c \Pr}{2}$$

- The solution for the temperature field is therefore:

$$\theta = \frac{Ec \Pr}{2}(1 - y^{*2}) + 1 \qquad (14.21)$$

- The bottom wall temperature is therefore:

$$\theta\Big|_{y^*=0} = 1+\frac{E_C \Pr}{2} \qquad (14.22)$$

2 D Couette Flow-18

- Convert the non-dimensional form of equ^n (14.21) into a dimensional form:

$$\frac{T-T_0}{T_1-T_0} = 1+\frac{E_C \Pr}{2} \Rightarrow \frac{T-T_1}{T_1-T_0} = \frac{E_c \Pr}{2}$$

$$\Rightarrow T\Big|_{y^*=0} = T_1 + \frac{\Pr U^2}{2c_p} \qquad (14.23)$$

- Now compare this expression with the earlier expression for the T_{ref} in equ^n (14.18) and notice that the two are identical. This is indeed the adiabatic wall temperature

2 D Couette Flow-19

- We have shown by actually solving the equation that the most suitable temperature difference for heat transfer is the difference between the wall temperature and the temperature of the same wall if the wall is made adiabatic. It has already been stated earlier that this is indeed the case for a general case also.

2 D Couette Flow-20

- Now look at the heat flux on the top plate:

$$q_w = -k\frac{\partial T}{\partial y}\bigg)_{y^*=1} = -\frac{k}{L}(T_1-T_0)\frac{d\theta}{dy^*}\bigg|_{y^*=1}$$

- Using the solution in equ^n (14.17) for θ:

$$q_w\Big|_{y^*=1} = -\frac{k}{L}(T_1-T_0)\left(-E_C \Pr+1+\frac{E_C \Pr}{2}\right)$$

$$= -\frac{k}{L}(T_1-T_0)\left(1-\frac{E_C \Pr}{2}\right) \qquad (14.24)$$

2 D Couette Flow-21

- Notice that in equ^n (14.24) for $\frac{E_C \Pr}{2} < 1$ q_w is negative i.e. heat is transferred from the wall at higher temperature to the fluid. This is expected.
- However, when $\frac{E_C \Pr}{2} > 1$ heat is transferred from the fluid to the hot wall. Appears 'unphysical' but is not so since viscous heat generation exists and heat has to be removed from both walls.

2 D Couette Flow-22

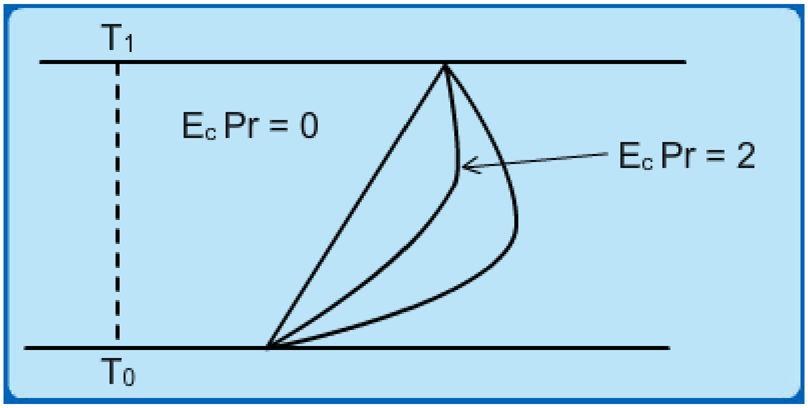
- Rewrite the wall flux in the following manner:

$$q_w = -\frac{k}{L}\left(T_1 - T_0 - \frac{U^2 \Pr}{2c_p}\right) = -\frac{k}{L}\left(T_1 - \left(T_0 + \frac{U^2 \Pr}{2c_p}\right)\right)$$

- Notice again the potential for heat transfer is not simply difference of two temperatures
- You can do the same exercise as we did for the low temperature (i.e. bottom) plate and show that the term in the brackets represents the adiabatic wall temperature for the top plate.

2 D Couette Flow-23

- Temperature profiles can be plotted for the situation considered.



- Note the change of slope at the wall for the top plate. Initially heat is input to the fluid for $E_cPr < 2$ and after $E_cPr > 2$ heat is input to the plate. Bottom plate always receives heat.

2 D Couette Flow-24

- When $E_cPr < 2$ heat generation within the fluid is small and therefore heat still comes in from the hotter top wall. However, when $E_cPr > 2$ the heat generation is so large that heat has to be removed from both the top wall and the bottom wall.

Recap

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- Couette flow situation is used to understand non-dimensional numbers occurring in convection situations
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