Module 2 : Convection

Lecture 16 : Fully Developed Pipe flow with Constant Wall temperature and Heat Flux

Objectives

In this class:

- The fully developed temperature profile for uniform circumferential heating with constant flux and the corresponding Nusselt number are derived
- For the constant temperature boundary condition, the methodology is established to the point where numerical calculations are required.

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-10

- The equ^n (15.18) therefore becomes:

$$u^*\frac{\partial T}{\partial z^*}=\frac{kZ}{r^2_0}\frac{1}{\rho c_p \bar{u} r^*}\frac{\partial}{\partial r^*}r^*\frac{\partial T}{\partial r^*}+\frac{k}{Z\rho c_p \bar{u}}\frac{\partial^2 T}{\partial z^{*2}} \qquad (16.1)$$

- The above equation suggests the choice for making the 'z' direction non-dimensional. The dimensional combination of Z can be obtained from either the first or the second term on the RHS of equ^n (16.1).

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-11

- Look at the first term to get

$$Z=\frac{\rho c_p \bar{u} r_0^2}{k}=\frac{(\bar{u}r_0)r_0}{\alpha}=\frac{\mathrm{Re}\,\mathrm{Pr}\,r_0}{2} \qquad (16.2)$$

$$\mathrm{Re}=\frac{(\bar{u}\,2r_0)}{\upsilon}$$

- Look at second term to get:

$$Z=\frac{k}{\rho c_p \bar{u}}=\frac{r_0^2 k}{r_0^2 \rho c_p \bar{u}}=\frac{2r_0}{\mathrm{Re}\,\mathrm{Pr}} \qquad (16.3)$$

- While 'Z' can be obtained from either (16.2) or (16.3) we will use 'Z' from equ^n (16.2)

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-12

- Now simplify equ^n (16.1) to get

$$u^*\frac{\partial T}{\partial z^*}=\frac{1}{r^*}\frac{\partial}{\partial r^*}r^*\frac{\partial T}{\partial r^*}+\frac{k}{\rho c_p \bar{u} r_0^2}k\frac{\partial^2 T}{\partial z^{*2}}\frac{1}{\rho c_p \bar{u}}$$

$$u^*\frac{\partial T}{\partial z^*}=\frac{1}{r^*}\frac{\partial}{\partial r^*}r^*\frac{\partial T}{\partial r^*}+\frac{4}{(\mathrm{Re}\,\mathrm{Pr})^2}\frac{\partial^2 T}{\partial z^{*2}} \qquad (16.4)$$

- Define RePr = Pe. Pe = Peclet number.
- High Pe means axial conduction is ignored wrt to convection(or radial conduction – since convection term arises from radial conduction from the wall).

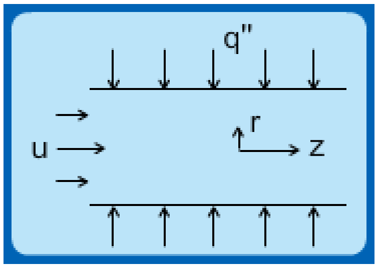
Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-13

- Peclet number high, implies ignoring the axial conduction term with respect to the radial conduction (or the convection term since radial conduction and convection terms are of similar magnitude)
- Now let us make an additional assumption that the Peclet number is high and so ignore the axial conduction term. The final equation is therefore:

$$\rho c_p u \frac{\partial T}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \quad (16.5)$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-1

- Solve equ^n (16.5) with appropriate boundary conditions
- Consider flow through a pipe with uniform wall flux at the wall. Assume flow is fully developed.



Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-2

- Use the nondimensional temperature profile and thermally fully developed flow:

$$\theta(r) = \frac{T(z,r) - T_w(z)}{T_b(z) - T_w(z)}$$

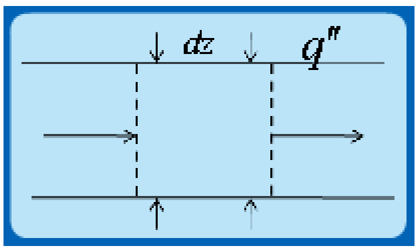
$$T = T_w + \theta\left(T_b - T_w\right)$$

- Equ^n (16.5) needs the temperature gradient in the z direction. Therefore:

$$\frac{\partial T}{\partial z} = \frac{dT_w}{dz} + \theta(r)\left(\frac{dT_b}{dz} - \frac{dT_w}{dz}\right) \quad (16.6)$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-3

- Consider a small slice of the pipe and use the First Law of Thermodynamics for an open system



$$q''(2\pi r_0 dz) = \dot{m} c p dT_b$$

$$\frac{dT_b}{dz} = \frac{2q''}{\rho \bar{u} r_0 c_p} \quad (16.7)$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-4

- For fully developed thermal profile heat transfer coefficient is constant. Wall heat flux is also constant. Therefore:

$$q_w = h\left(T_w - T_b\right)$$

$$\Rightarrow \left(T_w - T_b\right) = \text{constant} \Rightarrow \frac{dT_w}{dz} = \frac{dT_b}{dz} \quad (16.8)$$

- Substitute equ^n (16.8) in equ^n (16.6) gives:

$$\frac{\partial T}{\partial z} = \frac{dT_w}{dz} \quad (16.9)$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-5

- The governing equ^n (16.5) therefore becomes:

$$\rho c_p u \frac{dT_b}{dz} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{16.10}$$

- Substitute equ^n (16.7) in equ^n (16.10) to get:

$$\rho c_p u \frac{2q''}{\rho \bar{u} r_0 c_p} = \frac{k}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r}$$

$$\Rightarrow \frac{u}{\bar{u}}\frac{2q''}{r_0} = \frac{k}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \tag{16.11}$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-6

- Use the following non-dimensional variables:

$$r^* = \frac{r}{r_0}, u^* = \frac{u}{\bar{u}} = 2\left(1 - r^{*2}\right), \theta = \frac{T - T_w}{T_b - T_w} \tag{16.12}$$

- Non-dimensional energy equation with the boundary conditions is thus:

$$4\left(1 - r^{*2}\right)\frac{q''}{k r_0} = \frac{\left(T_b - T_w\right)}{r_0^{\,2} r^*}\frac{d}{dr^*} r^* \frac{d\theta}{dr^*} \tag{16.13}$$

$$r^* = 0; \frac{\partial T}{\partial r} = 0 \Rightarrow \left.\frac{\partial \theta}{\partial r^*}\right|_{r^*=0} = 0 \tag{16.14}$$

$$r = r_0, T = T_w \Rightarrow \left.\theta\right|_{r^*=1} = 0 \tag{16.15}$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-7

- Integrating equ^n (16.13) gives

$$\int 4q''\frac{r_0}{k}(1 - r^{*2}) r^* dr^* = (T_b - T_w)\int \frac{d}{dr^*} r^* \frac{d\theta}{dr^*}$$

$$q''\frac{r_0}{k}\left(\frac{r^{*2}}{2} - \frac{r^{*4}}{4}\right) + C_1 = (T_b - T_w) r^* \frac{d\theta}{dr^*} \tag{16.16}$$

- Use condition in equ^n (16.14) to get $C_1 = 0$. Now integrate equ^n (16.16) to get:

$$T - T_w = \int 4q''\frac{r_0}{k}\left(\frac{r^*}{2} - \frac{r^{*3}}{4}\right) dr^* \Rightarrow T - T_w = \frac{q'' r_0}{k}\left[r^{*2} - \frac{r^{*4}}{4}\right] + C_2$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-7

- Use condition in equ^n (16.15) to get:

$$T - T_w = \frac{q'' r_0}{k}\left[r^{*2} - \frac{r^{*4}}{4} - \frac{3}{4}\right] \tag{16.17}$$

- The temperature profile is known and thus the bulk temperature can be evaluated using:

$$T_b = \frac{\int \dot{m} c_p T}{\int \dot{m} c_p} = \frac{\int 2\pi r dr . uT}{\pi r_0^2 \bar{u}} \qquad (16.18)$$

$$\Rightarrow T_b - T_w = \frac{\int 2\pi r dr . uT}{\pi r_0^2 \bar{u}} - T_w \qquad (16.19)$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-8

- Equ^n (16.19) can be written as:

$$T_b - T_w = \frac{\int 2\pi r dr . u(T - T_w)}{\pi r_0^2 \bar{u}} \qquad (16.20)$$

- T_w is taken inside integral on the RHS of equ^n (16.19) since it is constant at any 'z' location
- Use equ^n (16.17) in equ^n (16.20) and integrate:

$$\therefore T_b - T_w = \int 4(1 - r^{*2}) r^* \frac{q'' r_0}{k} \left(r^{*2} - \frac{r^{*4}}{4} - \frac{3}{4} \right) dr^*$$

$$= 4 \frac{q'' r_0}{k} \left[\frac{1}{4} - \frac{1}{24} - \frac{3}{8} - \frac{1}{6} + \frac{1}{32} - \frac{3}{16} \right] = -\frac{11}{24} q'' \frac{r_0}{k} \qquad (16.21)$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-9

- Substitute (16.21) in (16.17) to get

$$\mathrm{T}\text{-}\mathrm{T}_b = \frac{q'' r_0}{k} \left[r^{*2} - \frac{r^{*4}}{4} - \frac{7}{24} \right] \qquad (16.22)$$

- Bulk temperature at any 'z' location can be written in terms of the inlet temperature using equ^n (16.7) and therefore the above equation can also be written as:

$$T = Ti + \frac{2q'' z}{\rho \bar{u} r_0 c_p} + \frac{q'' r_0}{k} \left[r^{*2} - \frac{r^{*4}}{4} - \frac{7}{24} \right] \qquad (16.23)$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-10

- Now evaluate the heat transfer coefficient using equ^n (16.21):

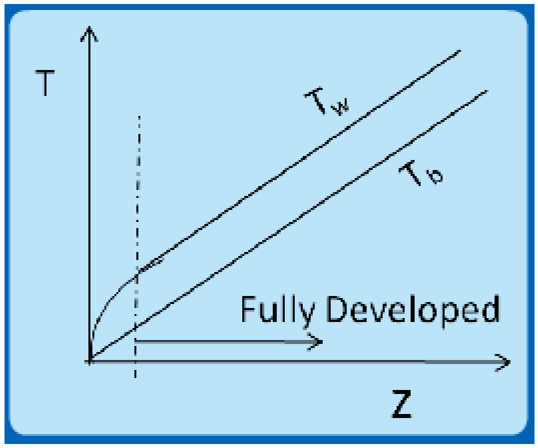
$$-q'' = h(T_b - T_w) = h\left(-\frac{11}{24} q'' \frac{r_0}{k}\right) = h \frac{11}{48} q'' \frac{2r_0}{k}$$

$$Nu = \frac{h(2r_0)}{k} = \frac{48}{11} = 4.36$$

- For constant wall flux case the Nusselt number is a constant.

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Flux-11

- Qualitative variation is shown as a function of axial location
- Note wall and bulk temperatures are parallel in the Fully Developed Zone (equ^n (16.8))



Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-1

- Slightly more difficult since $\frac{\partial T}{\partial z}$ cannot be replaced by a constant. Governing equation remains the same

$$\rho c_p u \frac{\partial T}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \tag{16.5}$$

- Try to see if the non-dimensional temperature gives some information

$$\frac{T - T_w}{T_b - T_w} = \theta(r) \quad \Rightarrow \frac{\partial T}{\partial z} = \theta(r)\frac{dT_b}{\partial z}$$
$$\Rightarrow \frac{\partial T}{\partial r} = \left(T_b - T_w\right)\frac{d\theta}{dr} \tag{16.24}$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-2

- Using the same non-dimensional variables get the equ^n (16.4) in which 'Pe' term neglected:

$$u*\frac{\partial T}{\partial z*} = \frac{1}{r*}\frac{\partial}{\partial r*} r*\frac{\partial T}{\partial r*}$$

- Use equ^n (15. 13), (16.24) in (16.5)

$$2(1-r*^2)\theta(r)\frac{dT_b}{dz*} = \frac{1}{r*}\frac{\partial}{\partial r*}\left(r*\frac{d\theta}{dr*}\right)\left(T_b - T_w\right) \tag{16.25}$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-3

- Collect like terms to write

$$\frac{dT_b/dz*}{(T_b - T_w)} = \frac{\frac{d}{dr*}r*\frac{\partial\theta}{\partial r*}}{2(1-r*^2)r*\theta} = -\lambda \tag{16.26}$$

- In the above equation first term is a function of 'z' only while the second term is a function of 'r only and therefore equal to a constant. The negative value is to keep the T_b bounded.

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-4

- Since wall temperature is constant, Rewrite the first term of equ^n (16.26) as:

$$\frac{dT_b}{dz^*(T_b - T_w)} \Rightarrow \frac{d(T_b - T_w)}{dz^*(T_b - T_w)} = -\lambda$$
$$\Rightarrow T_b - T_w = (T_i - T_w)e^{-\lambda z^*} \tag{16.26a}$$

- T_i is the temperature with which fluid enters the pipe at z = 0.

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-5

- The second term in the equ^n (16.26) in the 'r' direction gives:

$$\frac{d}{dr^*}\left(r^*\frac{d\theta}{dr^*}\right) + 2\lambda r^*(1 - r^{*2})\theta = 0 \tag{16.27}$$

- The boundary conditions are

$$T = T_w\Big|_{r^*=1} \Rightarrow \theta = 0 \tag{16.28}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r^*=0} \Rightarrow \theta'(0) = 0 \tag{16.29}$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-6

- Another condition is needed since λ is still unknown. This comes from the definition of the bulk temperature:

$$T_b = \frac{\int \rho c_p u T 2\pi r dr}{\pi r_0^2 \, \bar{u} \, \rho c_p} \tag{16.30}$$

$$= 2\int_0^{r_0} \frac{u}{\bar{u}}\left(\frac{r}{r_0}\right) T d\left(r/r_0\right)$$

$$(T_b - T_w) = 4\int_0^1 (1 - r^{*2}) r^* (T - T_w) dr^* \tag{16.31}$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-7

- Note that T_W is subtracted from both sides in equ^n (16.31). However, it is taken into the integral sign on the RHS since the term within the integral is unity for constant 'T' and T_W is constant. Term inside the integral being unity for constant 'T' is easier understood by viewing equ^n (16.30)
- Equ^n (16.31) is rewritten as:

$$\frac{1}{4} = \int_0^1 (1 - r^{*2})\theta \, r^* dr^* \tag{16.32}$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-8

- Equations (16.27), (16.28), (16.29), (16.32) are solved together for θ and λ. Analytical solution is difficult and therefore a numerical methodology is adopted.
- A methodology could be guess λ and solve equ^{n}s (16.27), (16.28), (16.29) for θ. However, the equation (16.32) needs to be satisfied also and since it does not have λ explicitly, using this to get a better guess for λ is not possible. One way is define a new variable F = λθ.

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-9

- The equations (16.27), (16.28), (16.29), (16.32) are therefore rewritten as:

$$\frac{d}{dr^*}\left(r^*\frac{dF}{dr^*}\right)+2\lambda r^*(1-r^{*2})F=0 \qquad (16.33)$$

$$F(0)=0; F'(1)=0 \qquad (16.34)$$

$$\frac{\lambda}{4}=\int_0^1(1-r^{*2})Fr^*dr^* \qquad (16.35)$$

- Guess a value for λ, solve for F using equ^{n}s (16.33), (16.34). Use equ^n (16.35) to get new λ, solve for F again and continue till convergence, i.e. change in λ is very small.

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-10

- Another methodology:
- Guess two values of λ and solve for θ using equ^{n}s (16.27), (16.28) and (16.29)
- Use equation (16.32) to compute the integral for two values of λ.
- Use secant method till integral becomes 0.25 to within a specified tolerance
- λ will converge to 3.66

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-11

- Sometimes bulk temperature for constant wall temperature case is calculated for the case when flow is fully developed.
- Energy balance gives:

$$\dot{m}C_p dT_b = hPdz(T_w - T_b) \qquad (16.36)$$

- Define: $\Theta = T_w - T_b$
- Equ^n (16.36) transforms to

$$-\frac{d\Theta}{dz}=\frac{hP\Theta}{\dot{m}C_p} \qquad (16.37)$$

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-12

- Solution for equ^n (16.37) is:

$$\Theta=\Theta_i e^{-\frac{hp_z}{\dot{m}C_p}} \qquad (16.38)$$

- The temperature profile is therefore exponential as observed earlier also.
- Bulk temperature variation for the constant wall flux and constant wall temperature can be evaluated by just doing an energy balance.

Recap

In this class:

- The fully developed temperature profile for uniform circumferential heating with constant flux and the corresponding Nusselt number are derived
- For the constant temperature boundary condition, the methodology is established to the point where numerical calculations are required.