Module 2 : Convection

Lecture 13 : Derivation of conservation of energy (contd.)

Objectives

In this class:

- Derivation of conservation of energy equation is completed.
- The final form in terms of temperature is presented

Caution: Only in this class do we use 'v' for specific volume since this is standard terminology in thermodynamics. In all other places it is velocity.

Conservation of Energy-22

- The energy equation is:

$$\rho\frac{\mathrm{De}}{\mathrm{Dt}}=\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]+\rho\frac{D}{Dt}\frac{V^2}{2}+\rho gw-P\nabla.\bar{u}+Q$$

Using the definition $e=\hat{u}+\frac{V^2}{2}+gz$ and simplifying gives the above equation gives:

$$\rho\frac{\mathrm{D}}{\mathrm{Dt}}(\hat{u})=\nabla.k\nabla T-P\nabla.\bar{u}+Q \tag{13.1}$$

Conservation of Energy-23

- The equ^n (13.1) is not a very convenient to use since temperature is the desired variable. Use thermodynamics to modify this.
- Enthalpy ‘h’ is a function of pressure and temperature. Differentitate keeping one variable constant at a time to get:

$$h=h(T,P)$$

$$dh=\left.\frac{\partial h}{\partial T}\right)_P dT+\left.\frac{\partial h}{\partial P}\right)_T dP$$

$$=C_p dT+\left.\frac{\partial h}{\partial P}\right)_T dP \tag{13.2}$$

Conservation of Energy-24

- The second term in equ^n (13.2) is not easily usable and needs manipulation
- Entropy ‘s’ is also a function of T and P.

$$s=s(T,P)$$

$$ds=\left.\frac{\partial s}{\partial T}\right)_P dT+\left.\frac{\partial s}{\partial P}\right)_T dP$$

- For an isothermal process

$$dh=\left.\frac{\partial h}{\partial p}\right)_T dP \qquad ds=\left.\frac{\partial s}{\partial p}\right)_T dP \tag{13.3}$$

Conservation of Energy-25

- Again from thermodynamics we borrow the following relationship:

$$dh=Tds+vdp \tag{13.4}$$

- For an isothermal process substitute equ^n (13.3) in equ^n (13.4) :

$$\left.\frac{\partial h}{\partial P}\right)_T dP = T\left.\frac{\partial s}{\partial P}\right)_T dP + vdP \tag{13.5}$$

- The second term is not easily usable and needs modification

Conservation of Energy-26

- Again from thermodynamics borrow the equation for the Gibbs Free Energy

$$dg = vdP - sdT \tag{13.6}$$

- g is also a function P and T. Therefore

$$\begin{aligned} g &= g(T,P) \\ dg &= \left.\frac{\partial g}{\partial P}\right)_T dP + \left.\frac{\partial g}{\partial T}\right)_P dT \end{aligned} \tag{13.7}$$

- Comparing equ^n (13.6) and equ^n (13.7):

$$\left.\frac{\partial g}{\partial P}\right)_T = \frac{1}{\rho}; \tag{13.8}$$

$$\left.\frac{\partial g}{\partial T}\right)_P = -s \tag{13.9}$$

Conservation of Energy-27

- Differentiate equ^n (13.8) with T

$$\frac{\partial^2 g}{\partial T \partial P} = \frac{\partial}{\partial T}\left(\frac{1}{\rho}\right)_P \tag{13.10}$$

- Note that the derivative of 'g' with respect to 'P' can be a function of 'T' even though 'T' is maintained constant in the process
- Differentiate equ^n (13.9) with P

$$\frac{\partial^2 g}{\partial P \partial T} = -\left.\frac{\partial s}{\partial P}\right)_T \tag{13.11}$$

Conservation of Energy-28

- Equate equ^n (13.10) with equ^n (13.11):

$$\left.\frac{\partial s}{\partial P}\right)_T = -\frac{\partial}{\partial T}\left(\frac{1}{\rho}\right)_P \tag{13.12}$$

- Substitute equ^n (13.12) in equ^n (13.5)

$$\begin{aligned} \left.\frac{\partial h}{\partial P}\right)_T &= T\left.\frac{\partial s}{\partial P}\right)_T + \upsilon \\ &= -T\frac{\partial}{\partial T}\left(\frac{1}{\rho}\right) + \frac{1}{\rho} \\ &= T\frac{1}{\rho^2}\left.\frac{\partial \rho}{\partial T}\right)_P + \frac{1}{\rho} \end{aligned} \tag{13.13}$$

Conservation of Energy-29

- Using the definition of coefficient of thermal expansion

$$\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}\bigg)_P \tag{13.14}$$

- Use equ^n (13.14) in equ^n (13.13) to get:

$$\frac{\partial h}{\partial P}\bigg)_T = -\frac{\beta T}{\rho} + \frac{1}{\rho} = \frac{1}{\rho}(1-\beta T)$$

- Use in equ^n (13.2)

$$dh = C_p dT + \frac{1}{\rho}(1-\beta T)dP \tag{13.15}$$

Conservation of Energy-30

- Note that for an ideal gas β = 1/T and therefore

$$dh = C_p dT \tag{13.16}$$

- Note that for an incompressible fluid also

$$dh = C_p dT \tag{13.17}$$

- For an incompressible fluid h = h(T) i.e. putting β = 0 in equ^n (13.15) is meaningless
- From thermodynamics definition of enthalpy:

$$h = \hat{u} + P\nu \qquad \nu \equiv \frac{1}{\rho} \tag{13.18}$$

Conservation of Energy-31

- Differentiate equ^n (13.18)

$$\frac{\mathrm{d}\hat{u}}{\mathrm{dt}} = \frac{dh}{dt} - P\frac{d\nu}{dt} - \nu\frac{dP}{dt}$$

$$\Rightarrow \rho\frac{\mathrm{d}\hat{u}}{\mathrm{dt}} = \rho\frac{dh}{dt} + P\frac{1}{\rho}\frac{d\rho}{dt} - \frac{dP}{dt} \tag{13.19}$$

- Use continuity equ^n (10.7) in equ^n (13.19):

$$\rho\frac{\mathrm{d}\hat{u}}{\mathrm{dt}} = \rho\frac{dh}{dt} - P\nabla.\bar{u} - \frac{dP}{dt}$$

$$= \rho C_p\frac{dT}{dt} + \frac{\rho}{\rho}(1-\beta T)\frac{dP}{dt} - P\nabla.\bar{u} - \frac{dP}{dt}$$

Conservation of Energy-32

- The final relationship between internal energy and temperature is therefore

$$\rho\frac{d\hat{u}}{dt} = \rho C_p\frac{dT}{dt} - \beta T\frac{dP}{dt} - P\nabla.\bar{u} \tag{13.20}$$

- Now substitute in the energy equation

$$\rho C_p \frac{dT}{dt} - \beta T \frac{dP}{dt} - P\nabla.\bar{u} = \nabla.(k\nabla T) - P\nabla.\bar{u} + Q$$

$$\text{or,} \rho C_p \frac{dT}{dt} = \nabla.k\nabla T + \beta T \frac{dP}{dt} + Q \tag{13.21}$$

- The above is the final form of the energy equation we use in future.

Recap

In this class:

- Derivation of conservation of energy equation is completed.
- The final form in terms of temperature is presented