Module 2 : Convection

Lecture 14a : Illustrative examples

Objectives

In this class:

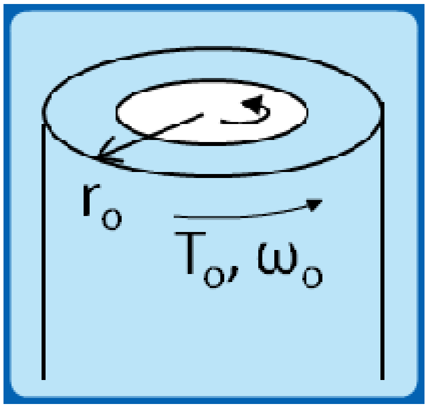
- Two examples will be taken where the conservation laws are used to obtain the temperature and velocity profiles. The solution methodology is discussed.

Example 1 (Problem statement-1)

- Consider steady incompressible constant property flow of Newtonian fluid between infinitely long concentric cylinders. Radius, angular velocity and temperature of inner cylinder is $r_i\omega_i$ and T_i respectively and the corresponding values for the outer cylinder are r_0 ω_0 and T_0 respectively. The flow is induced by the differential rotation of the cylinders and therefore no velocity in the axial direction is present. Assume viscous dissipation is important.

Example 1 (Problem statement-2)

- Assume that the body forces are negligible and that $\frac{\partial}{\partial\theta}$ of all variables is zero. Obtain the dimensionless velocity and temperature fields.



Example 1 (Solution-1)

- Start with the continuity equation:

$$\frac{1}{r}\frac{\partial}{\partial r}rv_r + \frac{1}{r}\frac{\partial}{\partial\theta}v_\theta + \frac{\partial}{\partial z}v_z = 0$$

- Now apply the following conditions as given in the problem statement:

$$\frac{\partial}{\partial\theta} = 0 \text{ and } v_z = 0$$

- Therefore:

$$\frac{\partial}{\partial r}rv_r = 0 \Rightarrow rv_r = k \Rightarrow v_r = 0$$

Example 1 (Solution-2)

- Since body force is ignored ‘z-momentum’ equation gives:

$$\frac{\partial p}{\partial z} = 0$$

- r-momentum equation is:

$$\cancel{\frac{\partial v_r}{\partial t}}+\cancel{(\bar{v}.\nabla)v_r}-\frac{v_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\cancel{g_r}+\nu\left(\cancel{\nabla^2 v_r}-\cancel{\frac{v_r}{r^2}}-\frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial\theta}}\right)=0$$

$\boxed{\text{0 since } v_r = 0}$ $\boxed{\text{0 since no body force}}$ $\boxed{\text{0 since } v_r = 0}$ $\boxed{\text{0 since no gradients in } \theta \text{ direction}}$

Example 1 (Solution-3)

- r-momentum equation becomes $\dfrac{v_\theta^2}{r}=\dfrac{1}{\rho}\dfrac{\partial p}{\partial r}$
- Now, look at the 'θ' momentum equation:

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{1}{r}v_\theta\frac{\partial v_\theta}{\partial\theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_z}{r}$$

$$=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+g_\theta+\nu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}\right)$$

$$\text{where, } \nabla^2 v_\theta=r\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial v_\theta}{\partial r}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}$$

Example 1 (Solution-4)

- Use the following conditions to simplify the momentum in the 'θ' direction:

  steady, $v_r=0$, $v_z=0$, body force $=0$ and $\dfrac{\partial}{\partial\theta}=0$
- The final equation is therefore:

$$\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r^2}=0$$

- Now non-dimensionalize the momentum equations

Example 1 (Solution-5)

- Choose the following variables:

$$r^*=\frac{r}{r_0};\ v_\theta^*=\frac{v_\theta}{v_0};\ p^*=\frac{p}{\rho {v_0}^2}$$

- The momentum equations therefore become:

$$\frac{\partial^2 v^*_\theta}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial v_\theta^*}{\partial r^*}-\frac{v_\theta^*}{r^{*2}}=0$$

$$\frac{\partial p^*}{\partial r^*}=1$$

Example 1 (Solution-6)

- Now look at the energy equation:

$$\rho c\left[\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{1}{r}v_\theta\frac{\partial T}{\partial\theta}+v_z\frac{\partial T}{\partial z}\right]=k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r}\frac{\partial^2 T}{\partial\theta^2}+\frac{\partial^2 T}{\partial z^2}\right]$$

$$+\mu\left[2\left(\varepsilon_{rr}^2+\varepsilon_{\theta\theta}^2+\varepsilon_{zz}^2\right)+\varepsilon_{\theta z}^2+\varepsilon_{rz}^2+\varepsilon_{r\theta}^2\right]$$

- Where:

$$\varepsilon_{rr} = \frac{\partial v_r}{\partial r} = 0;\ \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\mathrm{v_r}}{\mathrm{r}} = 0;\ \varepsilon_{zz} = \frac{\partial v_z}{\partial z} = 0$$

$$\varepsilon_{\theta z} = \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z};\ \varepsilon_{rz} = \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z};\ \varepsilon_{r\theta} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}$$

Example 1 (Solution-7)

- Now simplify using the same methodology as done for the momentum equation. The energy equation therefore becomes:

$$0 = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)\right] + \mu\left[\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right]^2$$

- Use the following non dimensional numbers for the above equation:

$$r^* = \frac{r}{r_0};\ v^* = \frac{v_\theta}{v_0};\ T^* = \frac{T - T_0}{T_1 - T_0}$$

Example 1 (Solution-8)

- The energy equation therefore becomes:

$$0 = k\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right)\right] + \underbrace{\frac{\mu v_0^2}{k\left(T_1 - T_0\right)}}_{\text{Brinkmann number}}\left[\frac{\partial v_\theta^*}{\partial r^*} - \frac{v_\theta^*}{r^*}\right]^2$$

- The energy and momentum equations are obtained and the temperature and velocity profiles are obtained by solving these.

Example 1 (Solution-9)

- Final Momentum equation:

$$\frac{\partial^2 v^*{}_\theta}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial v_\theta{}^*}{\partial r^*} - \frac{v_\theta{}^*}{r^{*2}} = 0$$

- Boundary conditions

$$\mathrm{v}_\theta^*\Big|_{r^*=1} = 1; \qquad \mathrm{v}_\theta^*\Big|_{r^*=\frac{r_i}{r_0}} = \frac{\omega_i r_i}{\omega_0 r_0}$$

- The above differential equation is easily solved using $v^*_\theta = Ar^{*m}$ in the equation and determining values of 'm'.

Example 1 (Solution-10)

- Substituting in the differential equation gives:

$$Am(m-1)r^{*m-2} + \frac{1}{r^*}Amr^{*m-1} - A\frac{r^{*m}}{r^{*2}} = 0$$

$$\Rightarrow m(m-1) + m - 1 = 0 \Rightarrow m = \pm 1$$

- The solution is therefore of the form:

$$v^*_\theta = Ar^* + Br^{*-1}$$

- Use Boundary conditions to get:

$$A + B = 1; \qquad \frac{v_1}{v_0} = A\frac{r_1}{r_0} + B\frac{r_0}{r_1}$$

Example 1 (Solution-11)

- The constants therefore are:

$$A = \frac{v_1/v_0 - r_0/r_1}{r_1/r_0 - r_0/r_1}; \quad B = 1 - A$$

- The solution becomes:

$$v^* = \frac{1}{\omega_0}\left(\frac{\omega_1 r_1^2 - \omega_0 r_0^2}{r_1^2 - r_0^2}\right)r^* + \left[\frac{r_1^2(\omega_0 - \omega_1)}{(r_1^2 - r_0^2)\omega_0}\right]\frac{1}{r^*}$$

- Now the solution for the temperature profile needs to be obtained.

Example 1 (Solution-12)

- Governing equation for temperature is:

$$\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right)\right] + Br\left[\frac{\partial v_\theta^*}{\partial r^*} - \frac{v_\theta^*}{r^*}\right]^2 = 0$$

- Use already obtained velocity profile to get:

$$\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) + Br\frac{4B^2}{r^{*3}} = 0$$

- Integrate to obtain:

$$T^* = -Br.B^2\frac{1}{r^{*2}} + C_1 \ln r^* + C_2$$

Example 1 (Solution-13)

- Now use the boundary conditions to obtain the constants:

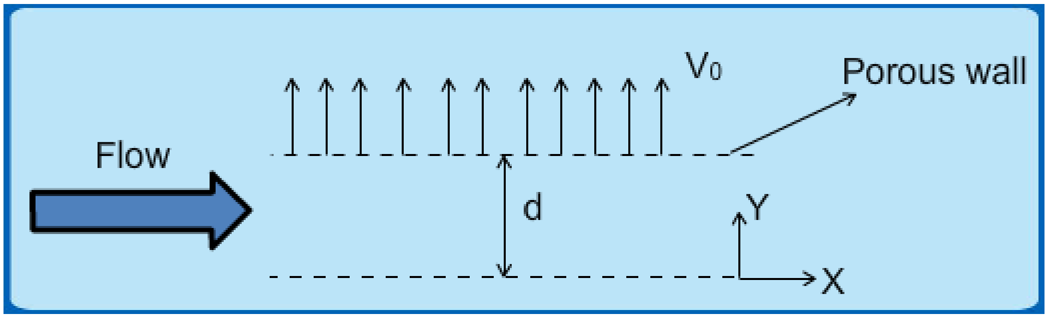
$$\left.T^*\right|_{r^*=1} = 0 \Rightarrow C_2 = Br.B^2$$

$$\left.T^*\right|_{r^*=\frac{r_1}{r_0}} = 1 \Rightarrow C_1 = \frac{1 + Br.B^2\frac{r_0^2}{r_1^2} - Br.B^2}{\ln\frac{r_1}{r_0}}$$

- The constant ‘B’ is known and therefore by using C_1 and C_2 above, temperature profile can be obtained.

Example 2 (Problem Statement-1)

- Consider the steady incompressible flow of a Newtonian fluid between 2 porous plates of infinite extent separated by a distance 'd'. Fluid exits the top plate at a constant velocity v_0 as shown. For the fully developed flow assume that $\frac{\partial u}{\partial x}, \frac{\partial T}{\partial x} = 0$



Example 2 (Problem Statement-2)

A constant pressure gradient dp/dx is imposed to sustain the flow. There is a heat generation q''' in the fluid. The top and bottom plates are maintained at temperatures 'T_1' and 'T_2' respectively. Assuming negligible viscous dissipation, no body forces determine v(x,y) and T(x,y). Note that you will need to use only the continuity and energy equations

Example 2 (solution-1)

- Use continuity equation and make similar arguments to those used in main text:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \Rightarrow v = k \Rightarrow v = v_0$$

- Now use the energy equation:

$$\rho C_p \left[\cancel{u \frac{\partial T}{\partial x}} + v \frac{\partial T}{\partial y} \right] = k \left(\cancel{\frac{\partial^2 T}{\partial x^2}} + \frac{\partial^2 T}{\partial y^2} \right) + \mu \left(\frac{\partial u}{\partial y} \right)^2$$

$$\boxed{= 0 \text{ fully developed assumption}}$$

Example 2 (solution-2)

- Energy equation therefore becomes:

$$\rho C_p v_0 \frac{dT}{dy} = k \frac{d^2 T}{dy^2} + q'''$$

$$T\big|_{y=0} = T_1; \quad T\big|_{y=d} = T_2$$

- This is an ordinary differential equation that can be solved by solving the homogenous part and adding the particular integral. For this case the particular integral, which is 'a' solution to the differential equation is: $\frac{q''' y}{\rho C_p v_0}$

Example 2 (solution-3)

- The homogeneous part is solved to obtain the solution for the temperature profile:

$$\frac{d}{dy}\frac{dT}{dy} - \frac{\rho C_p v_0}{k}\frac{dT}{dy} = 0 \Rightarrow \frac{dT}{dy} = c_0 e^{my}$$

$$T = \int c_0 e^{my} dy + c_1 = \frac{c_0}{m} e^{my} + c_1 + PI$$

$$= c_0 e^{\frac{\rho C_p v_0}{k} y} + c_1 + \frac{q''' y}{\rho C_p v_0}$$

Example 2 (solution-4)

- c_0 and c_1 are arbitrary constants that can be obtained by using the temperatures of the top and bottom plates.

$$c_1 = T_1; \qquad c_0 = \frac{T_2 - T_1 - \dfrac{q''' d}{\rho C_p v_0}}{e^{\frac{\rho C_p v_0}{k} d}}$$

- Temperature profile is therefore established by substituting the constants c_0 and c_1.

Recap

In this class:

- Two examples will be taken where the conservation laws are used to obtain the temperature and velocity profiles. The solution methodology is discussed.