Module 2 : Convection

Lecture 15 : Pipe flow- Simplification of energy equation

Objectives

In this class:

- The concept of fully developed temperature profile for uniform circumferential heating is discussed.
- Concept of bulk temperature is discussed
- Mass and momentum conservation equations are solved for a pipe geometry
- Solution of the energy equation is started

Pipe Flow- Hydrodynamically Fully Developed Flow-1

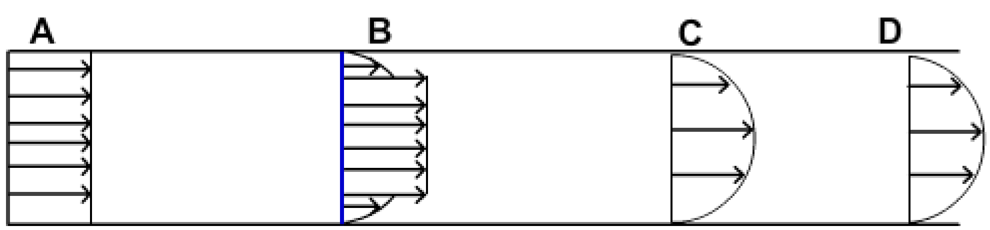
- Consider flow of incompressible fluid through a pipe.
- Fluid enters the pipe at some initial constant velocity. As the fluid flows through the pipe the influence of the wall which is at zero velocity will progress towards the center of the pipe.
- Process continues till the wall effect reaches the centerline.

Pipe Flow- Hydrodynamically Fully Developed Flow-2

- At the time the wall effect reaches the pipe centerline the velocity at the centerline reaches a maximum
- Beyond this point the velocity does not change with axial location and the flow is said to be fully developed. The velocity profile is a function of the radial coordinate only.
- More elegant explanations are possible using the concept of the boundary layer.

Pipe Flow- Hydrodynamically Fully Developed Flow-3

- At station A, Fluid enters pipe with uniform velocity. At station 'B', the wall 'zero velocity' propagates towards the center of the pipe. At station 'C' wall velocity has propagated to the center of pipe. At station 'D' profile is identical to that at 'C' and this continues axially.



Pipe Flow Thermally Fully Developed Flow-1

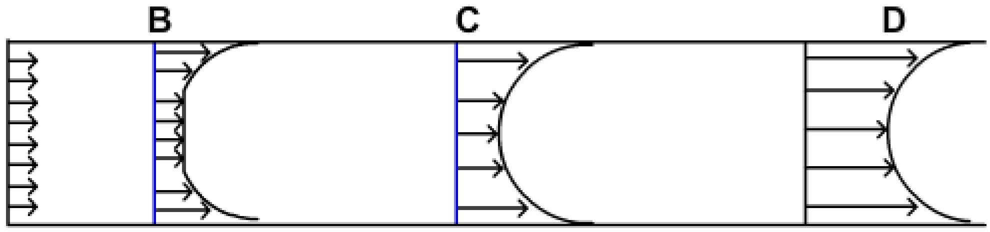
- Consider a pipe that is heated on the wall by a constant heat flux.
- Fluid enters the pipe at some initial temperature. Wall temperature is higher than the fluid temperature due to the input flux.
- Temperature in the fluid slowly increases till the wall temperature spreads to the center of the pipe. Beyond this distance, temperature continues to rise till the end of the pipe is reached.

Pipe Flow Thermally Fully Developed Flow-2

- Temperature is never independent of axial position unlike the velocity profile situation.
- Fully developed flow for parallel plate case with the top and bottom temperatures different was a case where the temperature profile was independent of axial distance.
- Even parallel plate case when both plates have identical temperature will not have axial temperature independence an fully developed flow definition needs modification.

Pipe Flow Thermally Fully Developed Flow-3

- At station A, Fluid enters pipe with uniform temperature. At station 'B', the temperature from the wall propagates towards the center of the pipe. At station 'C' wall temperature propagates to the center of pipe. At station 'D' profile is similar to that at 'C' but not identical.



Fully Developed Flow

- For a hydrodynamically fully developed flow, one can assume no change in velocity profile with change in axial location.
- For thermally fully developed flow, the temperature profile continues to change continuously even after the wall effects have reached the center of the pipe. Temperature profiles appear to be 'similar'.

Pipe Flow- Fully Developed Temp.

- Variation of temperature with axial distance cannot be used, therefore see if a non-dimensional temperature invariance is feasible. Define the following:

$$\theta = \frac{T(z,r) - T_w(z)}{T_b(z) - T_w(z)} \qquad (15.1)$$

- θ is a function of axial coordinate 'z' and radial coordinate 'r' in general. Hypothesize that it is a function of 'r' only for a fully developed flow

Pipe Flow- Fully Developed Temperature Profile-1

- For fully developed temperature profile this means that $\theta = \theta(r)$ instead of $\theta = \theta(z,r)$
- T_b is the bulk temperature of the fluid. Bulk fluid temperature is the mass weighted average temperature at a given axial location

$$T_b = \frac{\int \dot{m} c_p T}{\int \dot{m} c_p} \qquad (15.2)$$

- Bulk fluid temperature is a function of 'z' only

Pipe Flow- Fully Developed Temperature Profile-2

- Physically, the bulk fluid temperature is the temperature of the fluid which one would get if one took a slice of fluid at a location 'z' and then mixed it together adiabatically. The temperature that this fluid would attain is the bulk fluid temperature. The temperature therefore depends only on the axial location.

Pipe Flow- Fully Developed Temperature Profile-3

- Therefore, under the FD flow assumption rewrite equ^n (15.1) as:

$$T = (T_b - T_w)\theta(r) + T_w$$
$$\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = (T_b - T_w)\left.\frac{d\theta}{dr}\right|_{r=r_0} \qquad (15.2)$$

- Now use the concept of the heat transfer coefficient

$$q = -k\left.\frac{\partial T}{dr}\right|_{r=r_0} = h(T_b - T_w)$$
$$h = \frac{q}{(T_b - T_w)} = -k\frac{(T_b - T_w)}{(T_b - T_w)}\left.\frac{d\theta}{dr}\right|_{r=r_0} \qquad (15.3)$$

Pipe Flow- Fully Developed Temperature Profile-4

- Therefore,

$$h = -k\left.\frac{d\theta}{dr}\right|_{r=r_0} \tag{15.4}$$

- Since θ is a function of ‘r’ only, the gradient evaluated at a given ‘r’ location is a constant
- h is therefore constant for a fully developed flow – likewise a fully developed thermal profile results in a constant heat transfer coefficient.
- Notice that the sign of the gradient of θ in equ^n (15.4) will ensure that ‘h’ is positive.

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-1

- Now consider steady, incompressible fully developed flow in a pipe- cylindrical coordinates: r, Θ,z. Also assume Θ symmetry, no body forces and constant properties.
- General equation of continuity is

$$\underset{\substack{0\\ \text{Steady}}}{\cancel{\frac{\partial\rho}{\partial t}}} + \frac{1}{r}\frac{\partial}{\partial r}r\rho u_r + \underset{\substack{0\\ \text{No }\Theta\\ \text{variation}}}{\cancel{\frac{1}{r}\frac{\partial}{\partial\Theta}}}\rho u_\Theta + \underset{\substack{0\\ \text{No z variation(Fully Dev.)}}}{\cancel{\frac{\partial}{\partial z}(\rho u\ )}} = 0 \tag{15.5}$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}ru_r = 0 \tag{15.6}$$

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-2

- Equ^n (15.6) implies that:

$$r\rho u_r = k \Rightarrow u_r = \frac{k}{\rho r} \Rightarrow u_r = 0$$

- Notice that this is very similar to the parallel plate case where the continuity equation resulted in v = 0.
- Now consider the momentum equation. The Θ momentum equation will not exist since no variation in this direction has been assumed.

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-3

- Consider the ‘z’ momentum equation.

$$\rho\left[\underset{\substack{0\\ \text{continuity}}}{\cancel{u_r\frac{\partial u}{\partial r}}} + \underset{\substack{0\\ \Theta\text{ symmetry}}}{\cancel{\frac{u_\Theta}{r}}}\frac{\partial u}{\partial\Theta} + u\underset{\substack{0\text{ Fully Dev.}}}{\cancel{\frac{\partial u}{\partial z}}}\right] = -\frac{\partial p}{\partial z} + \mu\left[\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u}{\partial r}\right] + \underset{\substack{0\\ \Theta\text{ symmetry}}}{\cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial\Theta^2}}} + \underset{\substack{0\\ \text{Fully Dev.}}}{\cancel{\frac{\partial^2 u}{\partial z^2}}}\right] \tag{15.7}$$

- Using the simplifications of Fully Developed flow and $u_r = 0$

$$\frac{\partial p}{\partial z} = \frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) \tag{15.8}$$

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-4

- Now look at the ‘r’ momentum equation

$$\rho\left[u_r\frac{\partial u_r}{\partial r}+\frac{u_\Theta}{r}\frac{\partial u_r}{\partial\Theta}+u\frac{\partial u_r}{\partial z}\right]=-\frac{\partial p}{\partial r}+\mu\left[\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial u_r}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\Theta^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\Theta}\right] \qquad (15.9)$$

- Using $u_r = 0$ the equation becomes

$$\frac{\partial p}{\partial r}=0 \qquad \text{or } p=f(z) \text{ only}$$

- The ‘z’ momentum equ^n (15.8) is solved to give:

$$u=\frac{r_0^{\,2}}{4\mu}\left(-\frac{dp}{dz}\right)\left(1-\left(\frac{r}{r_0}\right)^2\right) \qquad (15.10)$$

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-5

- Average velocity can be computed as:

$$\bar{u}=\frac{\int_0^{r_0}u\,2\pi r\,dr}{\int_0^{r_0}2\pi r\,dr}\Rightarrow \pi r_0^{\,2}\bar{u}=\int_0^{r_0}\frac{r_0^{\,2}}{4\mu}\left(-\frac{dp}{dz}\right)\left(1-\left(\frac{r}{r_0}\right)^2\right)2\pi r\,dr \qquad (15.11)$$

$$=\frac{2\pi}{4\mu}\left(-\frac{dp}{dz}\right)\left[\frac{r_0^{\,2}r^2}{2}-\frac{r^4}{4}\right]_0^{r_0}=-\frac{\pi r_0^{\,4}dp}{8\mu dz}$$

$$\therefore\ \bar{u}=-\frac{r_0^2\,dp}{8\mu dz} \qquad (15.12)$$

- Substituting average velocity in equ^n (15.10)

$$u=2\bar{u}\left(1-\left(\frac{r}{r_0}\right)^2\right) \qquad (15.13)$$

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-6

- Now look at the energy equation

$$\rho c_p\left[u_r\frac{\partial T}{\partial r}+\frac{u_\Theta}{r}\frac{\partial T}{\partial\Theta}+u\frac{\partial T}{\partial z}\right]=k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial\Theta^2}+\frac{\partial^2 T}{\partial z^2}\right]$$
$$+Q+\beta T\left[\left(u_r\frac{\partial p}{\partial r}\right)+\frac{u_\Theta}{r}\frac{\partial p}{\partial\Theta}+u\frac{\partial p}{\partial z}\right] \qquad (15.14)$$

- The dissipation term ‘Q’ is given as:

$$Q=\mu\left[2(\varepsilon_{rr}^2+\varepsilon^2_{\Theta\Theta}+\varepsilon^2_{zz})+\varepsilon^2_{\Theta z}+\varepsilon^2_{rz}+\varepsilon^2_{r\Theta}\right] \qquad (15.15)$$
$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r};\ \varepsilon_{\Theta\Theta}=\frac{1}{r}\frac{\partial u_\Theta}{\partial\Theta}+\frac{u_r}{r}:\ \varepsilon_{zz}=\frac{\partial u}{\partial z}$$
$$\varepsilon_{\Theta z}=\frac{1}{r}\frac{\partial u}{\partial\Theta}+\frac{\partial u_\Theta}{\partial z};\ \varepsilon_{rz}=\frac{\partial u_r}{\partial z}+\frac{\partial u}{\partial r};\ \varepsilon_{r\Theta}=\frac{1}{r}\frac{\partial u_r}{\partial\Theta}+\frac{\partial u_\Theta}{\partial r}-\frac{u_\Theta}{r}$$

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-7

- Now using the simplifications to the flow the dissipation terms become:

$$Q = \mu\left(\frac{\partial u}{\partial r}\right)^2 \quad (15.16)$$

- The energy equation therefore becomes:

$$\rho c_p u \frac{\partial T}{\partial z} = k\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial T}{\partial r} + k\frac{\partial^2 T}{\partial z^2} + \mu\left(\frac{\partial u}{\partial r}\right)^2 \quad (15.17)$$

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-8

- Now let us non-dimensionalize the equation (15.17) using the following variables:

$$u^* = \frac{u}{\bar{u}}; r^* = \frac{r}{r_0}; z^* = \frac{z}{Z}; \theta = \frac{T - T_w}{T_b - T_w}$$

- There is no obvious parameter for 'z'. Also note that the temperature for terms marked yellow in equ^n (15.17) are not put in the non-dimensional form since wall boundary condition is not specified yet.

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-9

- The temperature in the terms marked red in the equ^n (15.17) is easily non dimensionalized since 'T_w' is not a function of the radial coordinate 'r'. The equ^n (15.17) therefore becomes:

$$\frac{\rho c_p \bar{u}}{Z} u^* \frac{\partial T}{\partial z^*} = \frac{k}{r_0^2}\frac{(T_b - T_w)}{r^*}\frac{\partial}{\partial r^*} r^* \frac{\partial \theta}{\partial r^*} + \frac{k}{Z^2}\frac{\partial^2 T}{\partial z^{*2}} + \frac{\mu \bar{u}^2}{r_0^2}\left(\frac{\partial u^*}{\partial r^*}\right)^2 \quad (15.18)$$

- The magnitude of the terms marked red reside in their coefficients since the nondimensional variables are of order unity.

Steady, Incomp., Θ symmetric, pipe Flow- Fully Dev. Temp., Vel. Profile-10

- Compare the coefficients of the radial conduction and dissipation terms(i.e. take the ratio)

$$\frac{\mu \bar{\mathrm{u}}^2}{r_0^2 \frac{k(T_b - T_w)}{r_0^2}} = \frac{\mu \bar{u}^2}{k(T_b - T_w)} = Br = Ec\,\mathrm{Pr}$$

- Notice the Br appears here also and like in the earlier case if Br << 1 the dissipation term can be ignored with respect to the radial conduction term.

Recap

In this class:

- The concept of fully developed temperature profile for uniform circumferential heating is discussed.
- Concept of bulk temperature is discussed
- Mass and momentum conservation equations are solved for a pipe geometry
- Solution of the energy equation is started