Module 2 : Convection

Lecture 17 : Developed velocity and Developing temperature in Pipe flow with Constant Wall temperature and Heat Flux

Objectives

In this class:

- Constant wall temperature Nusselt number is obtained for the fully developed flow situation is completed.
- The developing temperature profile for fully developed velocity profile and uniform circumferential heating with constant flux is developed. Some numerical calculations are required, results for which are borrowed from available information.

Pipe Flow- Fully Developed Temp. and Velocity Profile Constant Wall Temp-13

- Interest is in the Nusselt number eventually therefore try to get it now.
- Recall the following equation already developed earlier:

$$h = -k \left.\frac{d\theta}{dr}\right|_{r=r_0} \tag{15.4}$$

$$Nu = \frac{h(2r_0)}{k} = -2 \left.\frac{d\theta}{dr^*}\right|_{r^*=1} \tag{17.1}$$

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- Take the governing equ^n (16.27) and integrate

$$\frac{d}{dr^*}\left(r^*\frac{d\theta}{dr^*}\right) + 2\lambda r^*\theta(1-r^{*2}) = 0 \tag{16.27}$$

$$\left. r^*\frac{d\theta}{dr^*}\right|_0^1 + 2\lambda\int_0^1 r^*\theta(1-r^{*2})dr^* = 0$$

$$\left.\frac{d\theta}{dr^*}\right|_1 = -2\lambda\left(\frac{1}{4}\right) = -\frac{\lambda}{2}$$

This is 1/4 form equ^n (16.32)

$$Nu = -2\left(-\frac{\lambda}{2}\right) = \lambda = 3.657 \tag{17.2}$$

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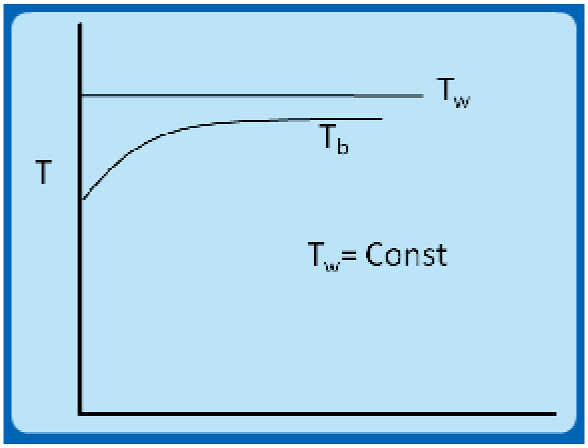
- Qualitative variation of bulk temperature as a function of axial distance 'z' is shown below. T_b rises exponentially(for fully developed case) as shown earlier



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- We have so far looked at fully developed temperature profiles.
- Now attempt solutions for the fully developed hydrodynamic profile and developing thermal profile.
- Assume the solution to the developing flow region is the sum of two temperature profiles. Maybe a developed temperature profile and something else superposed on it

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- The governing equation is still equ^n (16.5) which is:

$$\rho c_p u \frac{\partial T}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{16.5}$$

- We have already obtained the solution for the fully developed case. Therefore we try to attempt a solution which is the sum of the fully developed solution and some additional terms. Therefore assume:

$$T(z,r) = T^*(z,r) + T_1(z,r) \tag{17.3}$$

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- Substitute equ^n (17.3) in equ^n (16.5):

$$\rho c_p u \frac{\partial(T^*+T_1)}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}(T^*+T_1) \tag{17.4}$$

- The above equation can be rewritten as:

$$\rho c_p u \frac{\partial T^*}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r} r \frac{\partial T^*}{\partial r}; \quad \rho c_p u \frac{\partial T_1}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r} r \frac{\partial T_1}{\partial r} \tag{17.5}$$

- If T_1 and T^* satisfy the above equations then their sum will also satisfy the original differential equation. At the moment it is not clear what the variables, T_1 and T^* are.

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- Now look at the boundary conditions. Profile is symmetric about the center:

$$r = 0; \frac{\partial(T^*+T_1)}{\partial r} = 0 \tag{17.6}$$

- For this condition assume the following is true:

$$\frac{\partial T^*}{\partial r} = 0, \ \frac{\partial T_1}{\partial r} = 0 \tag{17.7}$$

- If equ^n (17.7) is satisfied for some values T^* and T_1 then equ^n (17.6) is also satisfied for these values.

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- Now look at the wall boundary condition:

$$r = r_0, \frac{\partial(T^*+T_1)}{\partial r} = \frac{q''}{k} \tag{17.8}$$

- Make the following choice:

$$\left.\frac{\partial T_1}{\partial r}\right|_{r=r0} = \frac{q''}{k} \text{ and } \left.\frac{\partial T^*}{\partial r}\right|_{r=r0} = 0 \tag{17.9}$$

- Now look at the initial condition

$$T|_{z=0} = T^*|_{z=0} + T_1|_{z=0} \tag{17.10}$$

- Rearrange to get:

$$T^*|_{z=0} = T|_{z=0} - T_1|_{z=0} \tag{17.11}$$

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- Equ^{n}s (17.5), (17.7), (17.9), (17.11) are rearranged in the following manner:

$$\begin{aligned} &\rho c_p u \frac{\partial T^*}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^*}{\partial r}\right) \\ &\left[\frac{\partial T^*}{\partial r}\right]_{r=0} = 0, \\ &\left[\frac{\partial T^*}{\partial r}\right]_{r=r_0} = 0 \\ &T^*_{z=0} = \left[T_i - T_1\right]_{z=0} \end{aligned} \tag{17.12}$$

$$\begin{aligned} &\rho c_p u \frac{\partial T_1}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_1}{\partial r}\right) \\ &\left[\frac{\partial T_1}{\partial r}\right]_{r=0} = 0 \\ &\left[\frac{\partial T_1}{\partial r}\right]_{r=r_0} = \frac{q''}{k} \end{aligned} \tag{17.13}$$

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- Equ^{n}s (17.13) for T_1 are the fully developed flow equations for constant wall flux case.
- Since the LHS of the T_1 equation is a constant there is no need for the initial condition i.e. initial condition cannot be specified and therefore the initial condition has been absorbed in T*.
- The fully developed solution is assumed to start at z = 0 which is not really true, physically.

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- Since $T_i = T_{fd} + T^*$ and T_{fd} is known the initial condition is also known. Using equ^n (16.23) for T_{1i} the final system equ^n (17.12) becomes:

$$\rho c_p u \frac{\partial T^*}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^*}{\partial r}\right) \tag{17.14}$$

$$\left.\frac{\partial T^*}{\partial r}\right|_{r=r_0} = 0; \quad \left.\frac{\partial T^*}{\partial r}\right|_{r=0} = 0; \tag{17.15}$$

$$T^*(0,r) = -q''\frac{r_0}{k}\left(r^{*2} - \frac{r^{*4}}{4} - \frac{7}{24}\right) \tag{17.16}$$

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- Use earlier defined non-dimensionalization

$$z^* = \frac{2z/r_0}{\rho c_p \bar{u} 2r_0/k}; \quad r^* = \frac{r}{r_0} \tag{17.17}$$

- Substitute to get final form that is to be solved:

$$\frac{\rho c_p u}{\frac{\rho c_p \bar{u} r_0^2}{k}}\frac{\partial T^*}{\partial z^*} = \frac{k}{r_0^2 r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) \tag{17.18}$$

$$2(1-r^{*2})\frac{\partial T^*}{\partial z^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) \tag{17.19}$$

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- Use separation of variables for equ^n (17.19):

$$T^* = Z(z^*)R(r^*) \tag{17.20}$$

- The equation therefore becomes:

$$2(1-r^{*2})[R(r^*)Z'(z^*)] = \frac{1}{r^*}Z\frac{d}{dr^*}r^*\frac{dR}{dr^*}$$
$$\Rightarrow \frac{Z'}{Z} = \frac{1}{r^*}\frac{\frac{d}{dr^*}\left(r^*\frac{dR}{dr^*}\right)}{2(1-r^{*2})R} = -w \tag{17.21}$$

- The Z portion of equ^n (17.21) gives:

$$Z' = -wZ \Rightarrow Z = ke^{-wz^*} \tag{17.22}$$

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- The 'R' portion of equ^n (17.21) becomes:

$$\frac{d}{dr^*}\left(r^*\frac{dR}{dr^*}\right) + 2r^*(1-r^{*2})Rw = 0 \tag{17.23}$$

- The required boundary conditions for solution of this equation are:

$$\left.\frac{\partial T^*}{\partial r}\right|_{r=r_0} = 0; \Rightarrow \left.\frac{dR}{dr^*}\right|_{r^*=1} = 0$$
$$\left.\frac{\partial T^*}{\partial r}\right|_{r=0} = 0; \quad \Rightarrow \left.\frac{dR}{dr^*}\right|_{r^*=0} = 0 \tag{17.24}$$

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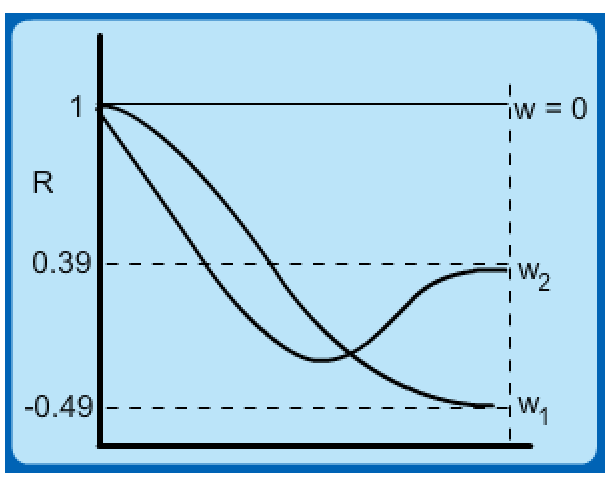
- Notice that the gradient is specified at both ends. A unique solution is therefore not possible – multiple solutions.
- The most obvious solution is R = constant provided w = 0 and keep increasing w for different solutions



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- Solution to the problem is therefore

$$T^* = Ce^{-wz^*}R(r^*)$$

- Note that the initial condition which has not yet been used shall be used to obtain the constant 'C'. As done previously use the summation approximation since a single 'C' cannot satisfy the equation:

$$T^* = \sum C\ e_m^{-w_m z^*} R\ (r_m^*) \tag{17.25}$$

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- Like we did earlier, we need to find out if the ‘R_m’ and ‘R_n’ can be multiplied and integrated to evaluate the constants.
- The equ^{n}s (17.23), (17.24) are a special case of the following general system known as the Sturm Lioville system of equations.

$$[r(\eta)R'(\eta)]' + [q(\eta) + wP(\eta)]R = 0$$
$$k_1 R(a) + k_2 R'(a) = 0 \qquad a \le \eta \le b$$
$$l_1 R(b) + l_2 R'(b) = 0 \tag{17.26}$$

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- For the equ^n (17.26) the following is true:

$$\int_a^b P(\eta) R_m R_n = 0 \text{ for } m \ne n \tag{17.27}$$

- Comparing the equ^n (17.23) and (17.24) with equ^n (17.26) the following can be obtained:

$$r = r^*; q = 0; P = 2r^*(1 - r^{*2})$$
$$\Rightarrow \int_a^b 2r^*(1 - r^{*2}) R_m R_n = 0 \text{ for } m \ne n \tag{17.28}$$

- P is the ‘weighting function’ that makes R_m and R_n ‘orthogonal’ i.e. amenable to use as suggested by equ^n (17.28)

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- Note that equations like (17.28) were used to simplify the algebra in the earlier cases where the separation of variables technique was used to obtain solutions. Now apply the condition given by equ^n (17.25)

$$T^*(0, r^*) = \sum_0^\infty C_m R_m(r^*)$$
$$\int T^*(0, r^*) 2r^*(1 - r^{*2}) R_n dr^* = \int 2r^*(1 - r^{*2}) R_n \sum_0^\infty C_m R_m(r^*) dr^* \tag{17.29}$$

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- Substitute equ^n (17.16) in equ^n (17.29)

$$\frac{q r_0}{k} \int_0^1 \left(-r^{*2} + \frac{r^{*4}}{4} + \frac{7}{24} \right) 2r^*(1 - r^{*2}) R_n dr^*$$
$$= \int_0^1 \sum_0^\infty C_m\ 2r^*(1 - r^{*2}) R_m R_n dr^*$$
$$= 2C_n \int_0^1 r^*(1 - r^{*2}) R_n^2 dr^* \tag{17.30}$$

- Equ^n (17.30) can be integrated to get different values of C_n

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- The constants can be now evaluated as

$$C_n = \frac{\dfrac{qr_0}{k}\displaystyle\int_0^1 \left(-r^{*2} + \frac{r^{*4}}{4} + \frac{7}{24}\right) r^*(1-r^{*2}) R_n \, dr^*}{\displaystyle\int_0^1 r^*(1-r^{*2}) R_n^2 \, dr^*} \qquad (17.31)$$

- Recall that we chose an arbitrary R_0 (= 1) corresponding to w_0 (= 0). It therefore appears that C_0 will be a function of the chosen R_0 when equ^n (17.31) is used for solution.

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- However, this is not the case

$$C_0 = \frac{qr_0}{k} \frac{\displaystyle\int_0^1 \left(-r^{*2} + \frac{r^{*4}}{4} + \frac{7}{24}\right) r^*(1-r^{*2}) \, dr^*}{R_0 \displaystyle\int_0^1 r^*(1-r^{*2}) \, dr^*} \qquad (17.32)$$

- The numerator in the above equation integrates to 0 and therefore $C_0 = 0$ always irrespective of the chosen value for R_0.

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- Also for any given R_m, kR_m is also a solution since it satisfies the governing equation and the boundary conditions.
- Any R_0 can be thought of an being a multiple of the $R_0 = 1$ case. The kR_m can be thought of as being the solution for the kR_0 initial value.
- Using equ^n (17.31) it is easy to see that:

$$C_{n,kR_m} = C_{n,R_m} / k \qquad (17.33)$$

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- The constants therefore will change with the value of R_0 chosen. However, the solution for temperature again remains unchanged since 'k' appears twice and cancels off as shown below:

$$T^* = \sum \frac{C_n}{k} e^{-w_n z^*} k R_n(r^*) \qquad (17.34)$$

- Typically therefore choose $R_0 = 1$ and obtain the solution for the profile. Any constant for R_0 will give the same solution.

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- Final solution for the problem is therefore:

$$\begin{aligned} T(z,r) &= T^*(z,r) + T_1(z,r) \\ &= \overbrace{\sum \frac{C_n}{k} e^{-w_n z^*} k R_n(r^*)}^{\text{Equ}^n\ (17.34)} + \overbrace{T_b + \frac{q'' r_0}{k}\left[r^{*2} - \frac{r^{*4}}{4} - \frac{7}{24}\right]}^{\text{Equ}^n\ (16.22)} \\ &= T_b + \frac{q'' r_0}{k}\left[\sum \frac{\hat{C}_n}{k} e^{-w_n z^*} k R_n(r^*) + \left[r^{*2} - \frac{r^{*4}}{4} - \frac{7}{24}\right]\right] \end{aligned} \qquad (17.35)$$

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- Wall temperature is obtained using $r = r_0$ in the temperature profile equ^n (17.35)

$$T_w = T_b + \frac{qr_0}{k}(1 - 1/4 - 7/24) + \frac{qr_0}{k}\left(\sum \hat{C}_n R_n e^{-w_n z^*}\right)$$

- Interest is again in Nusselt number. Therefore again using $q = h\,(T_w - T_b)$

$$T_w - T_b = \frac{qr_0}{k}\left[\frac{11}{24} + \sum_1^{\infty} \hat{C}_n R_n\big|_{r^*=1} e^{-wz^*}\right]$$

$$Nu = \frac{q\,2r_0}{k(T_w - T_b)} = \frac{2}{11/24 + \sum_1^{\infty} \hat{C}_n R_n\big|_{r^*=1} e^{-w_n z^*}} \qquad (17.36)$$

$$w_1 = 5, w_2 \approx 30, w_3 \approx 60 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

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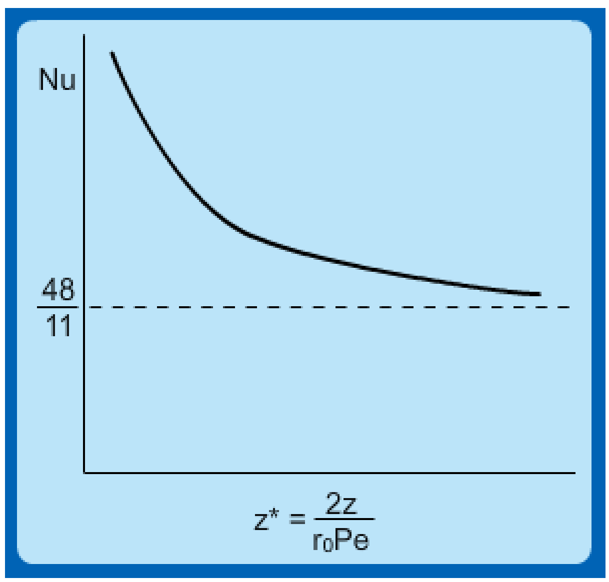
- The equ^n (17.36) for Nusselt number variation is as shown below. Note the dependence of Nu on the nondimensional z coordinate also called the Graetz number



Recap

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- Constant wall temperature Nusselt number is obtained for the fully developed flow situation is completed.
- The developing temperature profile for fully developed velocity profile and uniform circumferential heating with constant flux is developed. Some numerical calculations are required, results for which are borrowed from available information.