Module 2 : Convection

Lecture 12 : Derivation of conservation of energy

Objectives

In this class:

- Start the derivation of conservation of energy.
- Utilize earlier derived mass and momentum equations for simplification
- Show that the viscous dissipation term is always positive

Conservation of Energy-1

- Conservation of mass and momentum are complete and now the last conservation equation i.e. energy is derived.
- Again we start with the verbal form of the equation and then express this in mathematical terms
- Net rate of influx of energy into the Control Volume is equal to the rate of accumulation of energy within the Control Volume.

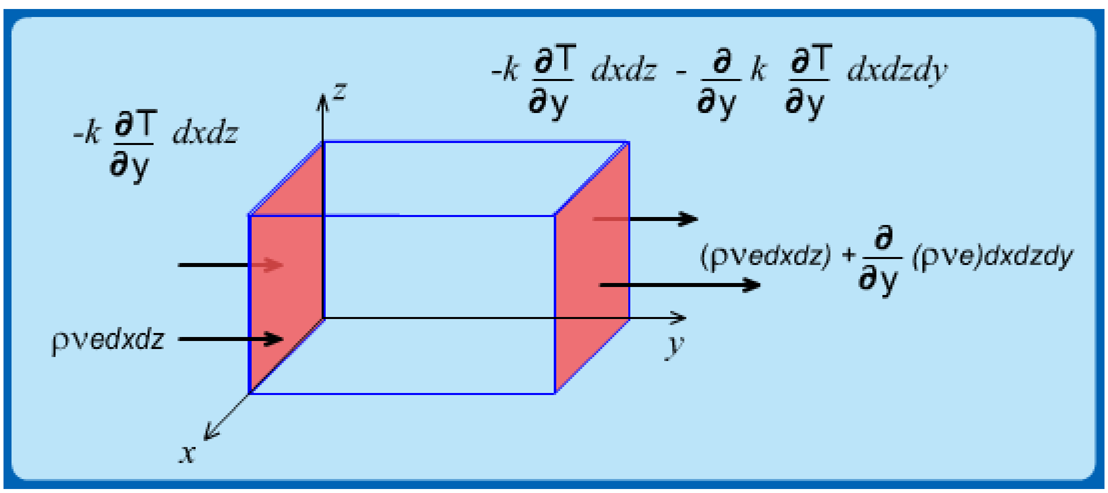
Conservation of Energy-2

- Energy can enter the Control Volume in the form of Conduction, Convection due to mass entering the Control Volume or Work done on the fluid in the Control Volume
- Energy (per unit mass) for fluid consists of kinetic, potential and thermal components:

$$\begin{aligned} energy &= e_{\text{thermal}} + e_{\text{kinetic}} + e_{\text{potential}} \\ &= \hat{u} + \frac{V^2}{2} + gh \end{aligned} \qquad (12.1)$$

Conservation of Energy-3

- Consider a Control Volume of size dx X dy X dz where energy transfer due to mass entering in the y direction is shown. Similar terms will exist for the x and z directions also but are not shown in the figure in the interest of clarity.



Conservation of Energy-4

- The conduction term has been seen already
- The rate of energy convected into the CV by virtue of mass entering in the ‘Y’ direction is shown in the figure on the y = 0 plane
- On the y = dy plane the regular Taylor series expansion has been used and only the leading term has been retained, as usual, for the conduction and convection terms

Conservation of Energy-5

- Net rate of heat conducted in:

(12.2)

$$\left[\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]dxdydz$$

- Net rate of heat convected in:

$$-\left[\frac{\partial(\rho ue)}{\partial x}+\frac{\partial(\rho ve)}{\partial y}+\frac{\partial(\rho we)}{\partial z}\right]dxdydz \tag{12.2a}$$

- Rate of storage of energy:

$$\frac{\partial}{\partial t}(\rho e)\,dxdydz \tag{12.3}$$

Conservation of Energy-6

- Subtract equ^n (12.2a) from equ^n (12.3) and group the appropriate terms to get:

$$\rho\left[\frac{\partial e}{\partial t}+u\frac{\partial e}{\partial x}+v\frac{\partial e}{\partial y}+w\frac{\partial e}{\partial z}\right]+e\underbrace{\left[\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}\rho u+\frac{\partial}{\partial y}\rho v+\frac{\partial}{\partial z}\rho w\right]}_{\substack{0\\ \text{continuity}}} \tag{12.4}$$

- Now look at energy transfer due to work. The rate of work done is computed as the dot product of the force and velocity:

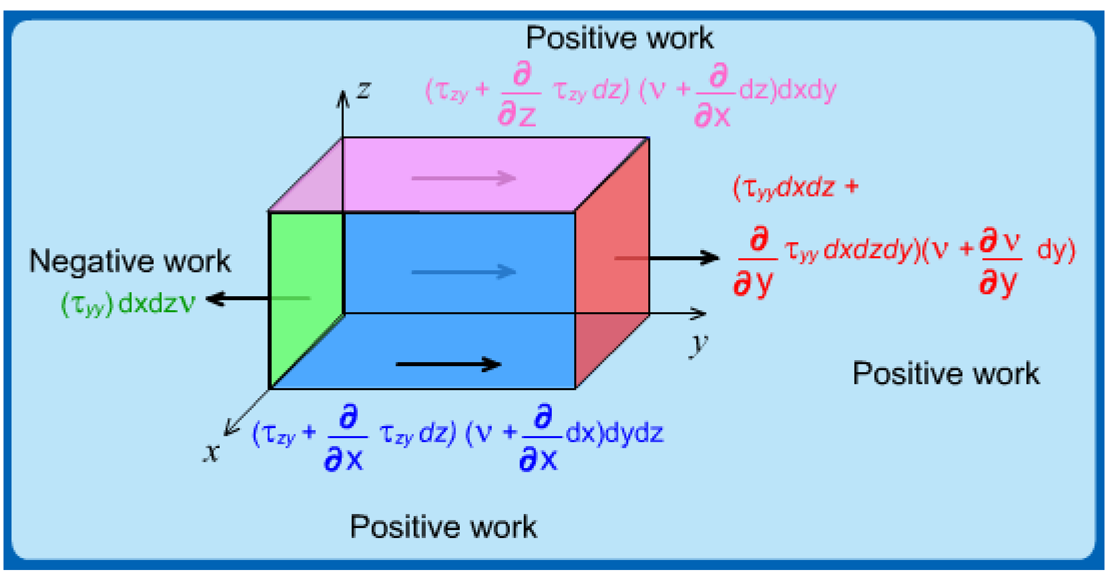
$$W=\overline{F}.\overline{v} \tag{12.5}$$

Conservation of Energy-7

- Rate of work done is assumed positive if the force and velocity vectors are in the same direction. Only surface forces are used for work calculations and not body forces since the gravitational potential energy has already been included in the energy per unit mass of fluid term. Of course, this argument is valid only for gravitational body force term. Need to consider the work done if other types of body forces exist.

Conservation of Energy-8

- Consider now the work terms on the Control Volume surfaces. Notice the signs of the work terms on the different faces of the CV



Conservation of Energy-8a

- Consider the top (pink), front (blue) and the right (red) face. The force due to the stress is in the direction from left to right. The positive 'v' direction is also from left to right. The dot product is

therefore positive as shown.
- The left face (green), however, has force due to stress from right to left and 'v' in the left to right direction, making the work negative. It is not possible to show the bottom and back face which also have negative values.

Conservation of Energy-9

- The net work rate **on the y = 0 and y = dy** face due to the force in the 'y' direction is therefore:

$$
\begin{aligned}
&(\tau_{yy}dxdz + \frac{\partial}{\partial y}\tau_{yy}dxdzdy)(v + \frac{\partial v}{\partial y}dy) - \left[\tau_{yy}dxdz\right]v \\
&= \left[\frac{\partial}{\partial y}\tau_{yy}dxdzdy\right]v + \left[\tau_{yy}dxdz\right]\left[\frac{\partial v}{\partial y}dy\right] + \left[\frac{\partial}{\partial y}\tau_{yy}dxdzdy\right]\left[\frac{\partial v}{\partial y}dy\right] \\
&= \frac{\partial}{\partial y}(v\tau_{yy}dxdzdy) + \left[\frac{\partial}{\partial y}\tau_{yy}dxdzdy\right]\left[\frac{\partial v}{\partial y}dy\right]
\end{aligned}
$$

- Work rate done per unit volume is therefore:

$$\frac{W}{dxdydz} = \frac{\partial}{\partial y}(v\tau_{yy}) + \left[\frac{\partial}{\partial y}\tau_{yy}\right]\left[\frac{\partial v}{\partial y}dy\right] \qquad (12.6)$$

Conservation of Energy-10

- Now let the volume 'dxdydz' be shrunk to zero
- Equ^{n} 12.6 can be modified as:

$$
\begin{aligned}
\frac{W}{dxdydz} &= \frac{\partial}{\partial y}(v\tau_{yy}) + \boxed{\cancelto{0}{\left[\frac{\partial}{\partial y}\tau_{yy}\right]\left[\frac{\partial v}{\partial y}dy\right]}} \\
&= \frac{\partial}{\partial y}(v\tau_{yy})
\end{aligned}
\qquad (12.7)
$$

- The last term is zero since it is explicitly multiplied by 'dy' which tends to zero.
- There are two other terms due to forces in the 'y' direction on x = 0, x = dx and z = 0, z = dz planes:

$$\frac{\partial}{\partial x}(v\tau_{xy}), \frac{\partial}{\partial z}(v\tau_{zy})$$

Conservation of Energy-11

- The total rate of work done due to forces in the y direction is therefore:

$$\frac{W}{dxdydz} = \frac{\partial}{\partial x}\tau_{xy}v + \frac{\partial}{\partial y}\tau_{yy}v + \frac{\partial}{\partial z}\tau_{zy}v \qquad (12.8)$$

- Work is a scalar and therefore there is an algebraic sum.
- Similarly there will be three terms each for the work due to forces in the 'x' and 'z' directions which will all be added together to the total work on the control volume.

Conservation of Energy-12

- Total rate of work is therefore:

$$\frac{W}{dxdydz} = \frac{\partial}{\partial x}(\tau_{xx}u) + \frac{\partial}{\partial y}(\tau_{yx}u) + \frac{\partial}{\partial z}(\tau_{zx}u)$$
$$+\frac{\partial}{\partial x}(\tau_{xy}v) + \frac{\partial}{\partial y}(\tau_{yy}v) + \frac{\partial}{\partial z}(\tau_{zy}v)$$
$$+\frac{\partial}{\partial x}(\tau_{xz}w) + \frac{\partial}{\partial y}(\tau_{yz}w) + \frac{\partial}{\partial z}(\tau_{zz}w) \qquad (12.9)$$

- Each of the product terms in equ^n (12.9) can be split into two terms and therefore a total of 18 terms exist on the right hand side.

Conservation of Energy-12a

- From the 18 terms in equ^n (12.9) there are 9 terms which also appear in the momentum equation (see equ^n (11.3) with $X_z = -g$). These 9 terms can therefore be simplified using the same approximations as used earlier to derive the equ^n (11.3). Note that equ^n (11.3) was simplified later using the continuity equation to obtain equ^n (11.10a)

Conservation of Energy-13

- The set of the 9 terms which can be simplified using the momentum equation is therefore: (with $X_z = -g$)

$$\begin{aligned}
u\left[\frac{\partial}{\partial x}\tau_{xx}+\frac{\partial}{\partial y}\tau_{yx}+\frac{\partial}{\partial z}\tau_{zx}\right] = \quad & \rho u\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right] \\
+v\left[\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial y}\tau_{yy}+\frac{\partial}{\partial z}\tau_{zy}\right] \quad & +\rho v\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right] \\
+w\left[\frac{\partial}{\partial x}\tau_{xz}+\frac{\partial}{\partial y}\tau_{yz}+\frac{\partial}{\partial z}\tau_{zz}\right] \quad & +\rho w\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]+\rho g w
\end{aligned} \qquad (12.10)$$

Conservation of Energy-14

- Simplify the RHS of equ^n (12.10). Note each coloured column adds to a single term below:

$$\begin{aligned}
\rho u\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right] \quad & \equiv \rho\left[\frac{1}{2}\frac{\partial u^2}{\partial t}+\frac{1}{2}u\frac{\partial u^2}{\partial x}+\frac{1}{2}v\frac{\partial u^2}{\partial y}+\frac{1}{2}w\frac{\partial u^2}{\partial z}\right] \\
+\rho v\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right] \quad & +\rho\left[\frac{1}{2}\frac{\partial v^2}{\partial t}+\frac{1}{2}u\frac{\partial v^2}{\partial x}+\frac{1}{2}v\frac{\partial v^2}{\partial y}+\frac{1}{2}w\frac{\partial v^2}{\partial z}\right] \\
+\rho w\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right] \quad & +\rho\left[\frac{1}{2}\frac{\partial w^2}{\partial t}+\frac{1}{2}u\frac{\partial w^2}{\partial x}+\frac{1}{2}v\frac{\partial w^2}{\partial y}+\frac{1}{2}w\frac{\partial w^2}{\partial z}\right] \\
+\rho g w \quad & +\rho g w \\
& \equiv \rho\left[\frac{1}{2}\frac{\partial V^2}{\partial t}+\frac{1}{2}u\frac{\partial V^2}{\partial x}+\frac{1}{2}v\frac{\partial V^2}{\partial y}+\frac{1}{2}w\frac{\partial V^2}{\partial z}\right] \\
\text{where } V^2 = u^2+v^2+w^2 \quad & +\rho g w
\end{aligned} \qquad (12.11)$$

Conservation of Energy-15

- 9 of the eighteen stress work terms therefore become:

(12.12)

$$\rho\left[\frac{1}{2}\frac{\partial V^2}{\partial t}+\frac{1}{2}u\frac{\partial V^2}{\partial x}+\frac{1}{2}v\frac{\partial V^2}{\partial y}+\frac{1}{2}w\frac{\partial V^2}{\partial z}\right]+\rho g w=\rho\frac{d\frac{V^2}{2}}{dt}+\rho g w$$

- The other 9 terms are:

$$\begin{aligned}&\tau_{xx}\frac{\partial u}{\partial x}+\tau_{yx}\frac{\partial u}{\partial y}+\tau_{zx}\frac{\partial u}{\partial z}\\&+\tau_{xy}\frac{\partial v}{\partial x}+\tau_{yy}\frac{\partial v}{\partial y}+\tau_{zy}\frac{\partial v}{\partial z}\\&+\tau_{xz}\frac{\partial w}{\partial x}+\tau_{yz}\frac{\partial w}{\partial y}+\tau_{zz}\frac{\partial w}{\partial z}\end{aligned}\tag{12.13}$$

Conservation of Energy-16

- Substitute the Stokes constitutive equ^n (11.5) in (12.13) gives:

$$\begin{aligned}&\left[(-P-\frac{2}{3}\mu\nabla.\vec{u})+2\mu\frac{\partial u}{\partial x}\right]\frac{\partial u}{\partial x}+\mu\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]\frac{\partial u}{\partial y}+\mu\left[\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right]\frac{\partial u}{\partial z}\\&+\left[(-P-\frac{2}{3}\mu\nabla.\vec{u})+2\mu\frac{\partial v}{\partial y}\right]\frac{\partial v}{\partial y}+\mu\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]\frac{\partial v}{\partial x}+\mu\left[\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right]\frac{\partial v}{\partial z}\\&+\left[(-P-\frac{2}{3}\mu\nabla.\vec{u})+2\mu\frac{\partial w}{\partial z}\right]\frac{\partial w}{\partial z}+\mu\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]\frac{\partial w}{\partial x}+\mu\left[\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right]\frac{\partial w}{\partial y}\end{aligned}\tag{12.14}$$

Conservation of Energy-16a

- The blue and yellow terms in the equation (12.14) are considered separately for convenience of algebraic manipulations. The terms coloured blue are first considered and simplified. The terms coloured yellow are then simplified and at the end the simplified equations are added together.

Conservation of Energy-17

- Consider the blue terms of (12.14) first. Terms coloured red, green and yellow on LHS are combined algebraically together to become the same colour terms on RHS as below:

$$\begin{aligned}&-P\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]\frac{\partial u}{\partial x}+2\mu\left(\frac{\partial u}{\partial x}\right)^2\equiv-P\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left[2\left(\frac{\partial u}{\partial x}\right)^2-\frac{\partial v}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right]\\&-P\frac{\partial v}{\partial y}-\frac{2}{3}\mu\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]\frac{\partial v}{\partial y}+2\mu\left(\frac{\partial v}{\partial y}\right)^2\quad-P\frac{\partial v}{\partial y}-\frac{2}{3}\mu\left[2\left(\frac{\partial v}{\partial y}\right)^2-\frac{\partial v}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\right]\\&-P\frac{\partial w}{\partial z}-\frac{2}{3}\mu\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]\frac{\partial w}{\partial z}+2\mu\left(\frac{\partial w}{\partial z}\right)^2\quad-P\frac{\partial w}{\partial z}-\frac{2}{3}\mu\left[2\left(\frac{\partial w}{\partial z}\right)^2-\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right]\end{aligned}\tag{12.15}$$

Conservation of Energy-18

- Consider equ^n (12.15). The terms with same colour on the RHS are grouped together to get the final compact form at the bottom:

$$-P\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]\frac{\partial u}{\partial x}+2\mu\left(\frac{\partial u}{\partial x}\right)^2 \equiv \frac{2}{3}\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2-\frac{\partial v}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right]-P\frac{\partial u}{\partial x}$$

$$-P\frac{\partial v}{\partial y}-\frac{2}{3}\mu\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]\frac{\partial v}{\partial y}+2\mu\left(\frac{\partial v}{\partial y}\right)^2 \quad \frac{2}{3}\mu\left[\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2-\frac{\partial v}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}\right]-P\frac{\partial v}{\partial y}$$

$$-P\frac{\partial w}{\partial z}-\frac{2}{3}\mu\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]\frac{\partial w}{\partial z}+2\mu\left(\frac{\partial w}{\partial z}\right)^2 \quad \frac{2}{3}\mu\left[\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2-\frac{\partial v}{\partial y}\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right]-P\frac{\partial w}{\partial z}$$

Add together to get $-P\nabla.\bar{u}+\frac{2}{3}\mu\left[\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}-\frac{\partial v}{\partial y}\right)^2\right]$

Conservation of Energy-19

- Now consider the terms marked yellow in equ^n (12.14). Terms marked with the same colour are combined together to obtain:

$$\begin{aligned}
&\mu\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]\frac{\partial u}{\partial y}+\mu\left[\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right]\frac{\partial u}{\partial z}\\
&+\mu\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]\frac{\partial v}{\partial x}+\mu\left[\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right]\frac{\partial v}{\partial z}\\
&+\mu\left[\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right]\frac{\partial w}{\partial x}+\mu\left[\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right]\frac{\partial w}{\partial y}\\
&=\mu\left[\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]^2+\left[\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right]^2+\left[\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right]^2\right]
\end{aligned}$$

Conservation of Energy-20

- The total contribution from the equ^n (12.14) is therefore

$$\frac{2}{3}\mu\left[\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}-\frac{\partial v}{\partial y}\right)^2\right]$$
$$+\mu\left[\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]^2+\left[\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right]^2+\left[\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right]^2\right] \qquad -P\nabla\cdot u$$

- The portion marked red in this term is always positive and is the viscous dissipation and we denote this as ‘Q’.

Conservation of Energy-21

- The energy equation therefore becomes:

$$\rho\frac{\mathrm{De}}{\mathrm{Dt}}=\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]+\rho\frac{D}{Dt}\frac{V^2}{2}+\rho g w-P\nabla.\bar{u}+Q$$

- Need to convert this into a more usable form i.e. variables that are easily measurable. Control volume manipulations are complete and now we need some thermodynamic manipulations to complete the derivation.

Recap

In this class:

- Start the derivation of conservation of energy.
- Utilize earlier derived mass and momentum equations for simplification
- Show that the viscous dissipation term is always positive