Module 1 : Conduction

Lecture 2 : Solution of Heat Diffusion Equation

Objectives

In this class:

- The derivation of the heat diffusion equation is continued. The boundary conditions and how they are to be applied correctly is discussed.
- Examples for cartesian and cylindrical geometries for steady constant property situations without heat generation are discussed and the electrical analogy obtained. Examples with heat generation are begun.

Heat Diffusion Equation-9

- In the equation (1.14) we use the Fourier law of heat conduction i.e. equation (1.1)
- The equation therefore transforms into one with temperature as variable

$$\because q'' = -k\nabla T$$

$$\boxed{\nabla.\left(k\nabla T\right) + q''' = \frac{\partial}{\partial t}\rho C\, T} \qquad (2.1)$$

Heat Diffusion Equation-10

- For a cartesian system the divergence term is

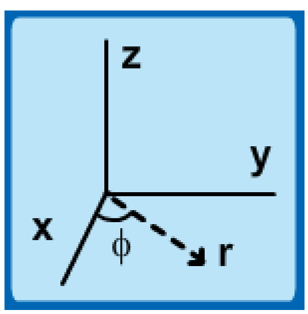
$$\nabla.(k\nabla T) = \frac{\partial}{\partial x} k \frac{\partial T}{\partial x} + \frac{\partial}{\partial y} k \frac{\partial T}{\partial y} + \frac{\partial}{\partial z} k \frac{\partial T}{\partial z} \qquad (2.2)$$

- In the equation (2.1) ρ, C and k can be functions of position. Appropriate simplifications are possible for constant property situations

Heat Diffusion Equation-11
Cylindrical Coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left[kr\frac{\partial T}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[k\frac{\partial T}{\partial \theta}\right] + \frac{\partial}{\partial z}\left[k\frac{\partial T}{\partial z}\right] + q''' = \rho C\frac{\partial T}{\partial t}$$

$$q_r'' = \left[-k\frac{\partial T}{\partial r}\right]; q_\theta'' = \left[-\frac{k}{r}\frac{\partial T}{\partial \theta}\right]; q_\phi'' = \left[-k\frac{\partial T}{\partial z}\right] \qquad (2.3a)$$



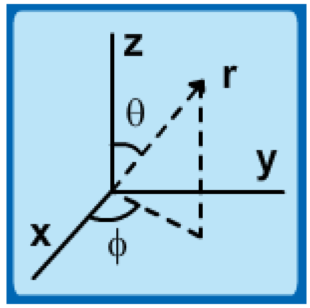
Heat Diffusion Equation-12
Spherical Coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[kr^2\frac{\partial T}{\partial r}\right] + \frac{1}{r^2 Sin\theta}\frac{\partial}{\partial \theta}\left[k\sin\theta\frac{\partial T}{\partial \theta}\right] + \frac{1}{r^2 Sin^2\theta}\frac{\partial}{\partial \varphi}\left[k\frac{\partial T}{\partial \varphi}\right]$$

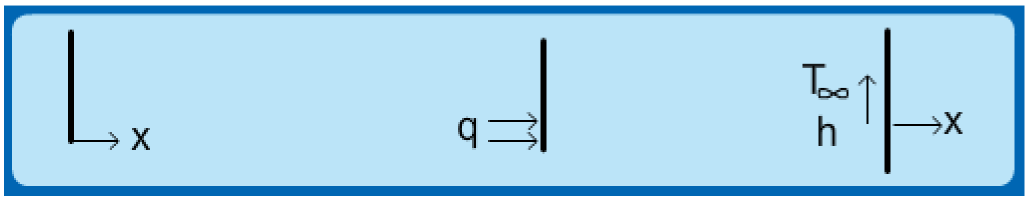
$$+ q''' = \rho C\frac{\partial T}{\partial t}$$

$$q_r'' = \left[-k\frac{\partial T}{\partial r}\right]; q_\theta'' = \left[-\frac{k}{r}\frac{\partial T}{\partial \theta}\right]; q_\phi'' = -\left[\frac{k}{rSin\theta}\frac{\partial T}{\partial \varphi}\right] \qquad (2.3b)$$

$$z = r\cos\theta; x = r\sin\theta\cos\phi$$



Boundary conditions-1



$T(x,0) = T_0$ $\qquad$ $q'' = -k\frac{\partial T}{\partial x}\bigg|_{X=0}$ $\qquad$ $-k\frac{\partial T}{\partial x}\bigg|_{X=0} = h(T_\infty - T_{wall})$

- Dirichlet bc
- Specified temp

const. flux

Neuman BC

q'' = 0 Ξ insulated

Mixed BC

Boundary conditions-2

- Mixed boundary condition is the most difficult to apply
- Notice that $q'' = -k\frac{\partial T}{\partial x}\bigg|_{x=0}$ implies transfer of heat in the direction of increasing 'x'. Similarly the equ^n $q'' = h(T_\infty - T_{wall})$ implies transfer of heat from '∞' towards the 'wall' condition and equating these as shown in the previous slide implies convection also in the positive 'x' direction. This consistency is required to get proper results.

Heat Conduction equ^n solns-1

- The basic heat diffusion equation has been derived. Several examples are now considered to illustrate the use of the equation and boundary conditions
- Consider first a slab infinite in two directions and of length 'L' in the 'x' direction. The equation is solved for steady state with constant thermal conductivity. The slab is maintained at temperatures equal to T_0 and T_L at x = 0 and x = L respectively.

Heat Conduction equ^n solns-2

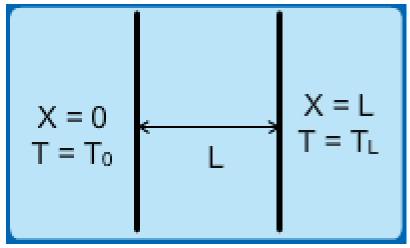
- 1 D plane wall steady state, no heat generation.

$$\nabla.k\nabla T + \cancel{q'''} = \frac{\partial}{\partial t}\left(\cancel{\rho c T}\right) \qquad (2.4)$$

No heat gen. $\qquad$ Steady

- The equation becomes

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \tag{2.5}$$



- Assume thermal conductivity k = constant

Heat Conduction equ^n solns-3

- Integration of equation (2.5) gives

  $T = C_1 x + C_2$

- Use boundary conditions to get the constants C_1 and C_2
- Steady conduction with no heat generation, constant conductivity and with specified temperatures gives a linear profile

Heat Conduction equ^n solns-4

- The details of the solution are given below:

$$\left\{\begin{array}{l} x = 0\ ; T = T_0 ; x = L ; T = T_L \\ C_2 = T_0 ; C_1 = \dfrac{T_L - T_0}{L} \\ T = \dfrac{T_L - T_0}{L} x + T_0 \end{array}\right\} \tag{2.6}$$

- Now consider a situation where the heat transfer coefficients and fluid temperatures are specified at x = 0 and x = L as ' h_1, h_2, $T_{\infty 1}$, $T_{\infty 2}$' respectively.

Heat Conduction equ^n solns-5

- When the boundary condition is convective, the two temperatures at x = 0 and x = L are still constant, and temperature profile within the solid continues to be linear. Use the solution obtained for the constant temperature case to get the solution for this case also.
- The boundary conditions are:

$$\left\{\begin{array}{l} x = 0 ; q'' = h_1\left(T_{\infty 1} - T_0\right) = h_1\left(T_{\infty 1} - C_2\right) \\ x = L ; q'' = h_2\left(T_L - T_{\infty 2}\right) = h_2\left(C_1 L + C_2 - T_{\infty 2}\right) \end{array}\right\}$$

Heat Conduction equ^n solns-6

- Notice that the two wall temperatures are assumed as T_0 and T_L. Therefore

$$\therefore\ T_0 - T_L = q'' \frac{L}{k} \tag{2.7}$$

- Now express the wall temperatures (which are unknown) in terms of the known convective boundary temperatures

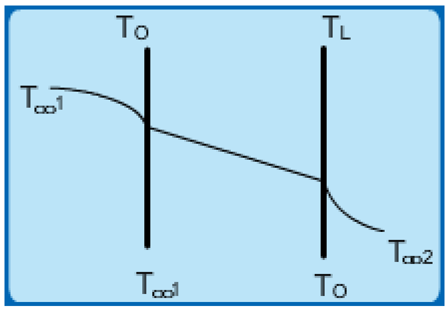
$$
\begin{aligned}
T_{\infty 1} - T_{\infty 2} &= q''\left[\frac{1}{h_1} + \frac{1}{h_2}\right] + \left[T_0 - T_L\right] \\
&= q''\left[\frac{1}{h_1} + \frac{1}{h_2} + \frac{L}{k}\right]
\end{aligned}
\tag{2.8}
$$

Heat Conduction equ^n solns-7

- In equation (2.7) and (2.8) the temperatures are like potentials and the heat flow is like current
- Compare with Ohm's law and obtain an electrical analogy. Defines a thermal resistance
- Easy to understand and use but has its limitations also

Electrical Analogy-1

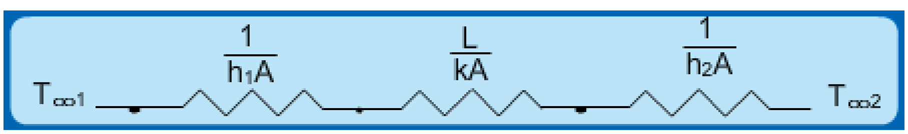
- The physical situation is as follows:



- Heat flows from $T_{\infty 1}$ to T_0 and then across the slab to the outer fluid

Electrical Analogy-2

- The electrical circuit analogy becomes:

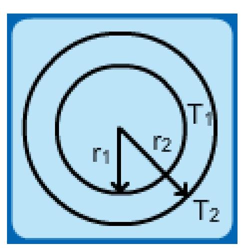


- Using resistances in series the heat flow is:

$$
Q = \frac{T_{\infty 1} - T_{\infty 2}}{\dfrac{1}{h_1 A} + \dfrac{L}{kA} + \dfrac{1}{h_2 A}}
\tag{2.10}
$$

Cylindrical System Examples-1

- Consider the cylindrical system shown below with uniform temperatures on inner and outer walls



- Heat transfer only in the radial direction and therefore one dimensional. True for spherical systems also

Cylindrical System Examples-2

- Solution for the one dimensional heat diffusion equation with constant wall temperatures is outlined below:

$$\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) = 0 \tag{2.11}$$

$$T(r) = C_1 \ln r + C_2 \tag{2.12}$$

$$T_1 = C_1 \ln r_1 + C_2 \tag{2.13}$$

$$T_2 = C_1 \ln r_2 + C_2 \tag{2.14}$$

$$\Rightarrow C_1 = \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}}; C_2 = T_2 - C_1 \ln r_2 \tag{2.15}$$

$$T = \frac{T_1 - T_2}{\ln \frac{r_1}{r_2}} \ln \frac{r}{r_2} + T_2 \tag{2.16}$$

Cylindrical System Examples-3

- Extract the heat flux expressions from the solution and put it in an appropriate form:

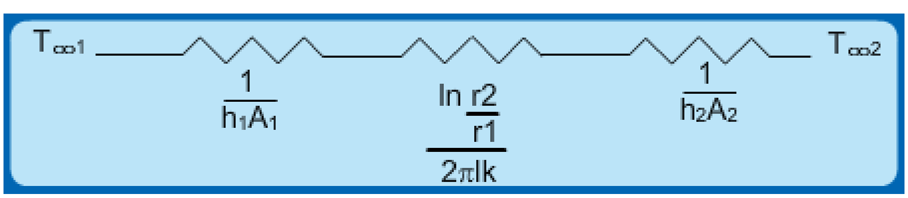
$$q'' = -k\frac{\partial T}{\partial r} = -k\frac{(T_1 - T_2)}{\ln \frac{r_1}{r_2}}\frac{1}{r} \tag{2.17}$$

$$Q = -\frac{k(T_1 - T_2)}{\ln \frac{r_1}{r_2}} 2\pi l \tag{2.18}$$

- Note that the heat flux is variable the total heat transferred is constant

Cylindrical System Electrical Analogy-1

- Now use the same methodology for the convective resistances as earlier cartesian case to get:



- Note that $A_1 = 2\pi r_1 L$ and $A_2 = 2\pi r_2 L$

Cylindrical System Electrical Analogy-2

- Now use the resistances in series to get the total heat transferred. Notice that this is not the flux since while the flux varies with radial distance, the total heat transferred remains constant.

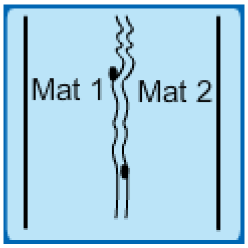
(2.19)

$$Q = \frac{T_{\infty 2} - T_{\infty 1}}{\dfrac{1}{h_1 2\pi l r_1} + \dfrac{\ln \dfrac{r_2}{r_1}}{2\pi l k} + \dfrac{1}{h_2 2\pi l r_2}}$$

- Note that we considered an annulus to obtain resistance and not a solid pipe
- Does it make sense to consider solid pipe and try to compute resistance ?
  The logarithmic term will give trouble. A specified temperature at the outer surface at steady state means temperature is same throughout – no need for any calculations !!
  A resistance for a solid cylinder is meaningless whereas for a solid block it is meaningful !!

Contact resistance

- Materials joined together to form a composite have resistance to transfer of heat across the interface.



$$R_{contact} = \frac{\Delta T}{(q''A)} \text{ or } \frac{\Delta T}{Q} \tag{2.20}$$

$$Q = \frac{\Delta T}{R} \tag{2.21}$$

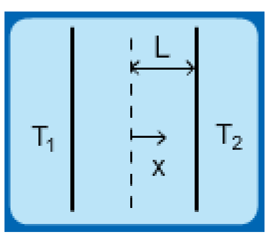
- Typical units are m^2K/W

Resistance Analogy-Comments

- Electrical resistance works only if the situation is one dimensional, steady and constant thermal conductivity
- Many situations exist where this approximation can be made and reasonable solutions can be obtained
- We next consider a situation where heat generation is present where the electrical resistance analogy will not work

Conduction with heat generation-1
Cartesian geometry

- Consider the plain wall (1D, steady state, constant k) with heat generation per unit volume q''' and specified wall temperatures. The governing equation, boundary conditions and the corresponding solution are:



$$\frac{d^2T}{dx^2} + \frac{q'''}{k} = 0 \tag{2.22}$$

$$T(-L) = T_1; T(L) = T_2 \tag{2.23}$$

$$T = -\frac{q'''}{k}\frac{x^2}{2} + C_1 x + C_2 \tag{2.24}$$

Conduction with heat generation-2
Cartesian geometry

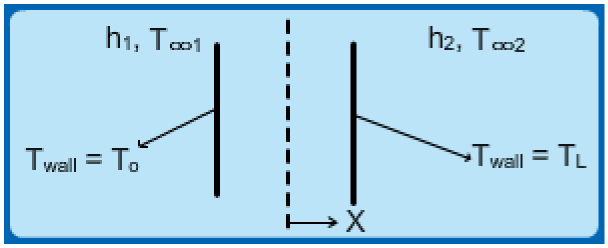
- Solving for the constants gives the following solution

$$T(x) = \frac{q'''L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_2 - T_1}{2}\frac{x}{L} + \frac{T_1 + T_2}{2} \tag{2.25}$$

- Calculating the flux from the above equation clearly indicates that it cannot be cast into a current, resistance and potential drop form. The electrical analogy does not work.

Conduction with heat generation-3
Cartesian geometry

- Consider the plain wall (1D, steady state, constant k) with heat generation per unit volume q''' and specified wall temperatures. The governing equation, boundary conditions and the corresponding solution are:



- Can easily get the solution although it is algebra intensive

Conduction with heat generation-4
Cartesian geometry

- Sometimes the physical problem will suggest easier ways of solution even though the standard methods work.
- Consider a case where $h_1 = h_2 = h$; $T_{\infty 1} = T_{\infty 2}$ then the problem is symmetric. For any of the two extremes the following expression is written using energy balance:

$$h\left(T_L - T_\infty\right) = q''' \frac{(2L)}{2}$$

Conduction with heat generation-5
Cartesian geometry

- Obtain T_L from the above expression which from symmetry is equal to T_0.
- Now substitute in the solution obtained for the specified temperature case and get the solution for the specified heat transfer coefficient case.

Recap

In this class:

- The derivation of the heat diffusion equation is continued. The boundary conditions and how they are to be applied correctly is discussed.
- Examples for cartesian and cylindrical geometries for steady constant property situations without heat generation are discussed and the electrical analogy obtained. Examples with heat generation are begun.