Module 2 : Convection

Lecture 23 : Momentum Boundary Layer Equations

Objectives

In this class:

- Basic definitions for boundary layer flows are introduced
- Differential form of the governing equation for two dimensional momentum boundary layer flow is derived
- A similarity solution is obtained for the equations

Boundary Layer Theory-1

- D'Alembert's paradox: Even for low viscosity fluids drag is experienced
- Prandtl introduced the concept of a boundary layer: there is a region very close to the wall where viscous effects are important
- No slip condition is valid at the wall.
- Even though viscosity of the fluid may be small, steep velocity gradients exist close to the wall since velocity goes from zero at the wall to its free stream value far away from the wall.

Boundary Layer Theory-2

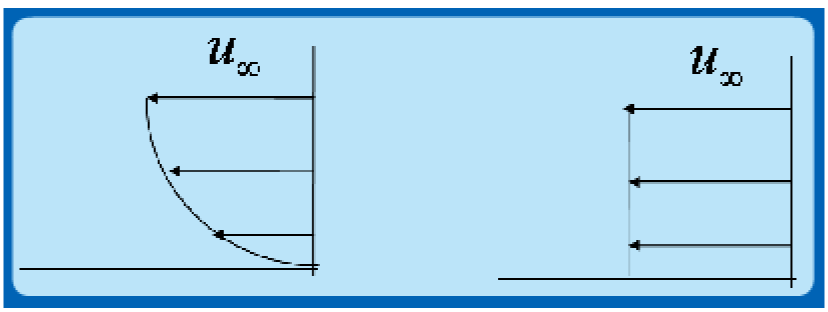
- The region close to the wall where the velocity gradients are important is the boundary layer
- Velocity goes from zero to free stream and the thickness of this boundary layer can be defined as that, at the outer edge of which the velocity reaches 99%, 99.5%, 99.9% ...... of the free stream velocity. The thickness is often denoted as δ_{99}, $\delta_{99.5}$, $\delta_{99.9}$ ......
- The physical thickness of the boundary layer can vary significantly therefore.

Integral Parameters-1

- A better definition would be one where the thickness does not vary so much.
- Integral parameters give a more 'constant' values.
- A comparison between the velocity profile between the cases where the fluid viscosity is present and absent forms the basis for another definition for the boundary layer thickness

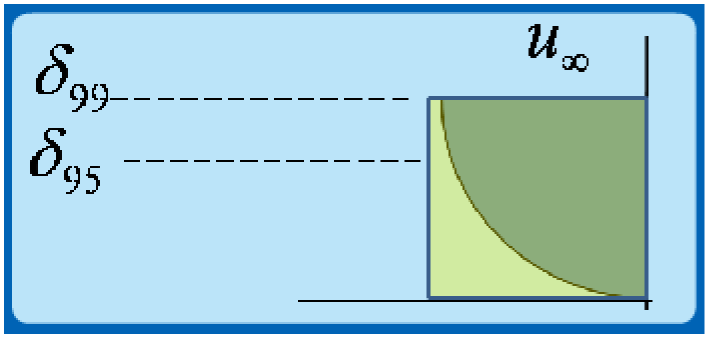
Integral Parameters-2

- The velocity profiles for a viscous and an inviscid fluid are as shown below with the length of the arrows representing the velocities:



Integral Parameters-3

- Now superpose the two profiles and the area marked yellow is the zone which is the mass flow deficit between the two cases.



- On the figure two boundary layer thicknesses are marked and it is easily seen that the mass deficit for these two is almost the same.

Integral Parameters-4

- Displacement thickness is defined as the thickness of an inviscid layer through which the mass deficit due to the presence of the boundary layer flows. This is mathematically represented as follows where δ* is the displacement thickness:

$$\rho u_\infty \delta^* = \int_0^\infty \rho (u_\infty - u) dy \tag{23.1}$$

$$\delta^* = \frac{1}{\rho u_\infty} \int_0^\infty \rho (u_\infty - u) dy = \frac{1}{\rho u_\infty} \int_0^\delta \rho (u_\infty - u) dy \tag{23.2}$$

Integral Parameters-5

- Displacement thickness is almost same whether you choose boundary layer thickness as being a value equal to δ_{99} , $\delta_{99.5}$, $\delta_{99.9}$, ...
- Similarly, momentum thickness is defined as the thickness of the inviscid layer through which the reduction in momentum due to the presence of the boundary layer passes through.

Integral Parameters-6

- The momentum thickness also is nearly independent of the boundary layer thickness chosen

$$\rho u_\infty{}^2 \theta = \int \rho u dy (u_\infty - u) \tag{23.3}$$

- Momentum defect can perhaps also be thought of as

$$\rho u_\infty{}^2 \theta = \int \rho dy (u_\infty^2 - u^2) \tag{23.4}$$

- However the definition in equ^n (23.3) appears naturally in the Momentum integral equation and is therefore the preferred choice

Differential Form of Equations-1

- The definitions for analyzing the boundary layer have now been established and now we try to derive the equations.
- The differential form of the equation will first be obtained and the integral form will be derived later by integrating the differential form
- The differential form is obtained by starting with the Navier Stokes equations derived earlier.

Differential Form of Equations-2

- The 2D Navier Stokes equations are scaled so that the order of the magnitude of the terms is unity – the concepts have already been discussed earlier during the discussion of the energy equation.
- The following variables are chosen for the non-dimensionalizing process:

$$u^* = u / u_\infty ; v^* = v / v_0 ; x^* = x / L ; y^* = y / \delta \tag{23.5}$$

Differential Form of Equations-3

- Consider the continuity equation first

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{23.6}$$

$$\frac{u_\infty}{L}\frac{\partial u^*}{\partial x^*}+\frac{v_0}{\delta}\frac{\partial v^*}{\partial y^*}=0 \tag{23.7}$$

- The non-dimensional parameters are of order unity. Thus the magnitude of the terms reside in their coefficients and since both terms must exist they must be of similar magnitude:

$$\frac{u_\infty}{L}\approx\frac{v_0}{\delta}\Rightarrow \mathbf{v}_0=u_\infty\delta/L \tag{23.8}$$

Differential Form of Equations-4

- Equ^n (23.8) suggests a value for the scaling parameter for the 'y direction' velocity
- Now look at the momentum equations. First look at the X-momentum equation:

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

- Non dimensionalize the equation using variables in equ^n (23.5)

Differential Form of Equations-5

- The following equation results:

$$\rho\frac{u_\infty^{\ 2}}{L}\frac{\partial u^*}{\partial x^*}u^*+\frac{\rho u_\infty v_0}{\delta}v^*\frac{\partial u^*}{\partial y^*}=-\frac{p_0}{L}\frac{\partial p^*}{\partial x^*}+\mu\left[\frac{u_\infty}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{u_\infty}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right]$$

- The last term in the above equation can be ignored since δ << L. The equation after a little manipulation becomes:

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{p_0}{\rho u_\infty^{\ 2}}\frac{\partial p^*}{\partial x^*}+\frac{\mu}{\rho u_\infty}\frac{L}{\delta^2}\frac{\partial^2 u^*}{\partial y^{*2}} \tag{23.8}$$

Differential Form of Equations-6

- Notice that the remaining viscous term in equ^n (23.8) cannot be ignored since then the equation would become the regular inviscid one and the viscous effects will disappear.
- Rearrange the variables in the magnitude of the second terms in the RHS of equ^n (23.8) to get:

$$\frac{\mu}{\rho u_\infty}\frac{L}{\delta^2}=\frac{\mu}{\rho u_\infty}\frac{L}{\delta^2}\frac{L}{L}=\frac{1}{\mathrm{Re}}(L/\delta)^2 \tag{23.9}$$

Differential Form of Equations-7

- The LHS of equ^n (23.8) has only non-dimensional variables of unit magnitude and is therefore of unit magnitude. Each of the terms on the RHS also are of unit magnitude since both of them continue to be present in the equation.

$$\frac{1}{\mathrm{Re}}(L/\delta)^2\approx 1;\qquad \frac{\delta}{L}\approx\frac{1}{\sqrt{\mathrm{Re}}} \tag{23.10}$$

Differential Form of Equations-8

- Boundary layer thickness δ << L. Equ^n (23.10) indicates that the boundary layer approximation is valid only for Re >> 1
- Now consider the Y-momentum equation and non-dimensionalize to get:

$$\rho\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho\left[\frac{u_\infty{}^2}{L^2}\delta\, u^*\frac{\partial v^*}{\partial x}+\frac{u_\infty{}^2}{L^2}\delta\frac{\partial v^*}{\partial y^*}\right]=-\frac{\partial p}{\partial y}+\frac{\mu}{L}\left[\frac{v_0}{L^2}\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{v_0}{\delta^2}\frac{\partial^2 v^*}{\partial y^{*2}}\right] \qquad (23.11)$$

Differential Form of Equations-9

- Consider the magnitude of the LHS of equ^{n}s (23.8) and (23.11). They are respectively:

$$\rho\frac{u_\infty{}^2}{L^2}\delta(Y \text{ mom}); \quad \rho\frac{u_\infty{}^2}{L}(X \text{ mom});$$

- The entire y-momentum equation is therefore one order of magnitude smaller than the x-momentum equation and can be completely neglected in comparison with the equation of x-momentum.

Differential Form of Equations-10

- Consider the pressure gradient term in equ^n (23.11). If it has to be retained it must be the same order of magnitude as the LHS. Therefore, equating the magnitudes:

$$\frac{\partial p}{\partial y}\approx\frac{\rho u_\infty^2\delta}{L^2}$$

- Integrating across the boundary layer:

$$\int_0^\delta\frac{\partial p}{\partial y}dy\approx\int_0^\delta\frac{\rho u_\infty^2\delta}{L^2}dy \;\Rightarrow p-p_\infty\approx\frac{\rho u_\infty^2\delta^2}{L^2} \qquad (23.12)$$

Differential Form of Equations-11

- Differentiate equ^n (23.12) with respect to ‘x’

$$\frac{\partial p}{\partial x}-\frac{\partial p_\infty}{\partial x}\approx\frac{\rho u_\infty^2 2\delta}{L^2}\frac{d\delta}{dx}\approx\frac{\rho u_\infty^2}{L}\frac{\delta^2}{L^2} \qquad (23.13)$$

- Accuracy of replacing pressure gradient within the boundary layer with free stream pressure gradient is $(\delta/L)^2$ times the order of the rest of the terms in the x-mom. equation. The free stream pressure gradient is known and therefore can be easily put in the equation.

Differential Form of Equations-12

- The streamwise pressure gradient term is caused by the change in the shape of the body and is calculated by the inviscid theory. The viscous effects will ‘change’ the shape of the body over which the ‘inviscid’ flow happens due to the presence of the thin boundary layer.

- Therefore:

$$\frac{\partial p}{\partial x}=\frac{\partial p_\infty}{\partial x}\left(\equiv\frac{dp_\infty}{dx}\right) \qquad (23.14)$$

Differential Form of Equations-13

- The boundary layer equations therefore become:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (23.15)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=u_\infty\frac{du_\infty}{dx}+\upsilon\frac{\partial^2 u}{\partial y^2} \qquad (23.16)$$

- The associated boundary conditions are

$$y = 0, u = v = 0; y = \delta, u = u_\infty \tag{23.16a}$$

Differential Form of Equations-14

- Note that since the free stream is inviscid, the Bernoulli equation has been used to convert the pressure gradient to a velocity gradient
- The boundary layer equations are a set of partial differential equations and solution is not very straight forward.
- Several strategies are used to convert the partial differential equations to a set of ordinary differential equations.

B.L. Equation Solution-1

- Solutions exist for a class of problems known as the 'wedge-flow' problems where the inviscid flow over a wedge is the free stream flow
- Blasius proposed an expression of the form

$$\frac{u}{u_\infty} = f(\eta); \eta = y/\delta \tag{23.17}$$

- The velocity varies only as 'η' which is a judicious combination of x and y.

B.L. Equation Solution-2

- Using the expression for δ obtained in equ^n (23.10):

$$\frac{\delta}{L} \approx \frac{1}{\sqrt{\mathrm{Re}}} \Rightarrow \frac{\delta}{x} \approx \frac{1}{\sqrt{u_\infty x / \upsilon}} \Rightarrow \delta \approx \sqrt{\frac{\upsilon x}{u_\infty}}$$

- Substitute in equ^n (23.17):

$$\frac{u}{u_\infty} = f\left(y\sqrt{\frac{u_\infty}{\upsilon x}} \right). \tag{23.18}$$

- The methodology requires lot of intuition

B.L. Equation Solution-3

- We now attempt a slightly more generalized formulation and assume

$$\eta = y.g(x) \tag{23.19}$$

- The hope is that the flow parameters will depend only on η which is a combination of the x and y coordinates. This will transform the partial differential equation into an ordinary differential equation

B.L. Equation Solution-4

- The problem is in 2D and therefore work with stream function which is a variable such that:

$$u = \frac{\partial \psi}{\partial y}; v = -\frac{\partial \psi}{\partial x} \tag{23.20}$$

- The continuity equation (23.15) is identically satisfied by virtue of the definition of the stream function
- The momentum equation needs to be converted into the stream function form

B.L. Equation Solution-5

- Obtain 'u' as a function of the stream function:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial \psi}{\partial \eta} g \qquad (23.21)$$

- Define a new function 'F' such that

$$\frac{u}{u_\infty} = f(\eta) = F'(\eta) \qquad (23.22)$$

- Note that the F' is the total derivative of F with respect to η. Substitute in (23.21) to get:

$$u_\infty F' = \frac{\partial \psi}{\partial \eta} g \qquad (23.23)$$

B.L. Equation Solution-6

- Integrate equ^n (23.22)

$$\psi = \frac{u_\infty F}{g} + h(x) \rightarrow \text{ set to zero arbitrarily}$$
$$\psi = \frac{u_\infty F}{g} \qquad (23.24)$$

- Now that ψ is available obtain the velocities and velocity gradients in the equ^n (23.16) in terms of the stream function

B.L. Equation Solution-7

- ' u' velocity is known from equ^n (23.23) and therefore now compute v velocity:

$$v = -\frac{\partial \psi}{\partial x} = -\left[\frac{u_\infty}{g} F' \frac{\partial \eta}{\partial x} + F \frac{\partial}{\partial x}\left(\frac{u_\infty}{g}\right)\right] \qquad (23.25)$$

- Now get the rest of the terms:

$$\frac{\partial u}{\partial y} = u_\infty F'' \frac{\partial \eta}{\partial y} = u_\infty F'' g; \qquad \frac{\partial^2 u}{\partial y^2} = u_\infty F''' g^2 \qquad (23.26)$$

$$\frac{\partial u}{\partial x} = u_\infty F'' \frac{\partial \eta}{\partial x} + F' \frac{du_\infty}{dx} \qquad (23.27)$$

B.L. Equation Solution-8

- Substitute equ^{n}s (23.22), (23.25), (23.26), (23.27) in the equ^n (23.16) to get

$$u_\infty F'\left[u_\infty F'' \frac{\partial \eta}{\partial x} + F' \frac{du_\infty}{dx}\right] - \left[\frac{u_\infty}{g} F' \frac{\partial \eta}{\partial x} + F \frac{\partial}{\partial x}\frac{u_\infty}{g}\right] u_\infty F'' g = u_\infty \frac{du_\infty}{dx} + \upsilon u_\infty g^2 F'''$$
$$\Rightarrow \frac{F'^2}{\upsilon g^2}\frac{du_\infty}{dx} - \frac{FF''}{\upsilon g}\frac{\partial}{\partial x}\left(\frac{u_\infty}{g}\right) = \frac{u_\infty}{\upsilon u_\infty g^2}\frac{du_\infty}{dx} + F''' \qquad (23.28)$$

- The above equation has F, derivatives of F, U_∞, x , g, v and η as the parameters.

B.L. Equation Solution-9

- If the equation (23.28) can be made into one where only 'F' and 'η' exist, then the equation becomes an ordinary differential equation. This is known as the similarity solution.
- Therefore the equation (23.28) suggests that the following should be true where C_1, C_2 are constants:

$$\frac{1}{\upsilon g^2}\frac{du_\infty}{dx} = C_1 \tag{23.29}$$

$$\frac{1}{\upsilon g}\frac{d}{dx}\left(\frac{u_\infty}{g}\right) = C_2 \tag{23.30}$$

B.L. Equation Solution-10

- Therefore the equ^n (23.28) can be written as:

$$C_1 F'^2 - C_2 FF'' = C_1 + F''' \tag{23.31}$$

- While it is true that the governing equation is now an ordinary differential equation, it has to be demonstrated that the U_∞ and 'g' can have functional forms that are indeed such that equations (23.29) and (23.30) are satisfied.

B.L. Equation Solution-11

- Simplify equ^n (23.30) to get:

$$\frac{1}{\upsilon g}\left[\frac{1}{g}\frac{du_\infty}{dx} - \frac{u_\infty}{g^2}\frac{dg}{dx}\right] = C_2$$

$$\Rightarrow \frac{u_\infty}{\upsilon g^3}\frac{dg}{dx} = C_1 - C_2 = C_3 \tag{23.32}$$

- C_3 is another constant. Divide equ^n (23.29) by equ^n (23.32) to get:

$$\frac{du_\infty}{u_\infty}\frac{g}{dg} = \frac{C_1}{C_3} = C_4 \tag{23.33}$$

B.L. Equation Solution-12

- Simplify equ^n (23.33) to get

$$\frac{du_\infty}{u_\infty} = C_4\frac{dg}{g} \Rightarrow \ln g = \frac{1}{C_4}\ln u_\infty \Rightarrow g = C_5 u_\infty^{1/C_4} \tag{23.24}$$

- Substitute 'g' in equ^n (23.34) in equ^n (23.29) to get:

$$\frac{du_\infty}{dx} = C_1 C_5^2 u_\infty^{2/C_4} \Rightarrow u_\infty = Ax^m \tag{23.35}$$

- The constants C_1 , C_2 etc. are replaced by two new constants – 'A' and 'm' for the sake of simplicity

B.L. Equation Solution-13

- Now substitute equ^n (23.35) in equation (23.29) to get:

$$\frac{1}{\upsilon g^2} A m x^{m-1} = C_1$$

- Choose C_1 = m and further simplify

$$\upsilon g^2 = Ax^{m-1} = \frac{Ax^m}{x} = \frac{u_\infty}{x} \Rightarrow g = \sqrt{\frac{u_\infty}{\upsilon x}} \tag{23.36}$$

- Use this 'g' in equation (23.19):

$$\eta = y.g(x) = y\sqrt{\frac{u_\infty}{\upsilon x}} \tag{23.37}$$

B.L. Equation Solution-14

- The similarity variable obtained in (23.37) has been obtained purely mathematically.
- Now substitute the ‘g’ obtained in equ^n (23.36) into equ^n (23.30) to get:

$$\frac{1}{\upsilon\sqrt{\frac{u_\infty}{\upsilon x}}}\frac{d}{dx}\sqrt{u_\infty \upsilon x} = C_2 \tag{23.38}$$

$$\frac{1}{\upsilon}\sqrt{\frac{\upsilon x}{u_\infty}}\sqrt{\upsilon}\,\frac{d}{dx}A^{1/2}x^{\frac{m+1}{2}} = C_2 \tag{23.39}$$

B.L. Equation Solution-15

- Simplifying equ^n (23.39):

$$\frac{1}{A^{1/2}x^{\frac{m-1}{2}}}A^{1/2}\frac{m+1}{2}x^{\frac{m+1}{2}-1} = C_2 \quad \Rightarrow C_2 = \frac{m+1}{2}$$

- Substitute C_1 and C_2 in equ^n (23.31) to get:

$$F'^2 m - \frac{m+1}{2}FF'' = m + F'''$$

$$or \quad F''' + \frac{m+1}{2}FF'' - (F'^2 - 1) = 0 \tag{23.40}$$

B.L. Equation Solution-16

- Notice that ‘m’ is not any arbitrary constant but one such that the freestream velocity U_∞ satisfies the equ^n (23.35). Whether such a distribution of free stream velocity exists or not has not yet been discussed.
- Equ^n (23.40) is the governing equation and needs appropriate boundary conditions to obtain a solution

B.L. Equation Solution-17

- The boundary conditions are obtained from (23.16a).
- Use equ^n (23.22) obtain:

$$F'(0) = 0,\ \text{using } u_{y=0} = 0 \tag{23.41}$$

$$F'(\infty) = 1,\ \text{using } u_{y=\infty} = u_\infty \tag{23.42}$$

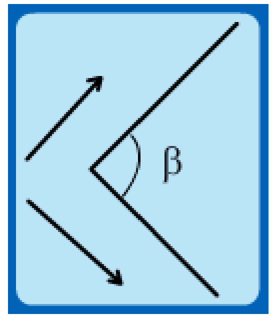
- Using equ^n (23. 25):

$$F(0) = 0 \text{ using } v_{y=0} = 0\ ;$$

B.L. Equation Solution-18

- The similarity solution that is governed by equ^{n}s (23.40)-(23.43) is valid only if the free stream velocity is of the form: $u_\infty = Ax^m$
- Flow over a wedge of angle ‘β’ does correspond to this profile

$$m = \frac{\beta / \pi}{2 - \beta / \pi}$$



B.L. Equation Solution-19

- Equations for the momentum equation for boundary layer over a surface have been established.
- The solution methodology when the velocity is expressed in terms of a single variable, thereby transforming the partial differential equation into an ordinary differential equation has been outlined.
- The energy equation will now be obtained.

Recap

In this class:

- Basic definitions for boundary layer flows are introduced
- Differential form of the governing equation for two dimensional momentum boundary layer flow is derived
- A similarity solution is obtained for the equations