Module 1 : Conduction

Lecture 6 : 2D conduction (contd.). 1D unsteady conduction

Objectives

In this class:

- The 2D conduction problem is completed.
- The zero dimension transient problem is presented and the significance of the Biot number explained.
- The solution for the 1D transient problem is started

2 D Conduction-10

- Only option remaining is + λ^2 and using it one gets:

$$X = C_1 \cos\lambda x + C_2 \sin\lambda x \quad (6.1)$$

$$Y = C_3 e^{-\lambda y} + C_4 e^{\lambda y} \quad (6.2)$$

- Recall B.C. for X equation: X = 0 for x = 0,L
- Apply to get:

$$X = 0 \text{ at } x = 0 \Rightarrow C_1 = 0; \quad (6.3)$$

$$X = 0 \text{ at } x = L \Rightarrow C_2 \sin\lambda L = 0 \quad (6.4)$$

2 D Conduction-11

- C_2 can be chosen as zero in equ^n (6.4) but again the solution becomes X = 0 which is an unacceptable solution. Therefore choose:

$$\sin\lambda L = 0$$

$$\Rightarrow \lambda L = n\pi, n = 1, 2, ... \quad (6.5)$$

$$\Rightarrow X = C_2 \sin\lambda x \quad (6.6)$$

- Now look at the y direction:

$$Y = 0 \text{ at } y = 0 \Rightarrow C_3 = -C_4 \quad (6.7)$$

2 D Conduction-12

- At the y = W end:

$$Y(W)X(x) = 1 \text{ at } y = W$$

$$\Rightarrow C_3(e^{-\lambda W} - e^{\lambda W})X(x) = 1 \quad (6.8)$$

- Equ^n (6.8) does not give any extra information about the solution and therefore revert back to Equ^n (6.2)

$$Y = C_3\left(e^{\frac{n\pi y}{L}} - e^{\frac{-n\pi y}{L}}\right)$$

$$= 2C_3 Sinh\left(\frac{n\pi y}{L}\right) \quad (6.9)$$

2 D Conduction-13

- Substitute equ^n (6.6) and equ^n (6.9) in equ^n (5.19) and again use the last boundary condition i.e. Y = 1 at y = W

$$\theta\left(x,W\right)=1=C\ \sin\frac{n\pi x}{L}\sinh\frac{n\pi W}{L} \tag{6.10}$$

- Note that $C = 2C_2C_3$ and is yet another constant. It is obvious that a single constant cannot satisfy the equ^n (6.10) since LHS is a constant while RHS has an explicit 'x' dependence.

2 D Conduction-14

- The governing equation is a linear differential equation and therefore a linear combination of the solutions is also a solution. Therefore even though a single constant cannot satisfy the boundary condition we attempt to satisfy the condition by taking multiple constants and using a summation of the terms:

$$\theta(x,y)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L} \tag{6.11}$$

2 D Conduction-15

- Note that for each value of 'n' there is a constant C_n that is assumed. The constants C_n are unknown and need to be determined
- From trigonometry the following result is borrowed:

$$\int_0^L \sin n\pi\frac{x}{L}\sin m\pi\frac{x}{L}dx=0\ \ for\ \ m\neq n$$

$$=\frac{L}{2}\ for\ m=n \tag{6.12}$$

2 D Conduction-16

- Now substitute the boundary condition:

$$\theta(x,W)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{L}\sinh\frac{n\pi W}{L} \tag{6.13}$$

- Multiply Right hand side of above equation with $\sin m\pi\frac{x}{L}$ and note that

$$\left[\sum C_n\sin n\pi\frac{x}{L}\sinh n\pi\frac{W}{L}\right]\sin\frac{m\pi x}{L}=$$

$$\sum\left[C_n\sin n\pi\frac{x}{L}\sinh n\pi\frac{W}{L}\sin\frac{m\pi x}{L}\right]$$

2 D Conduction-17

- Now integrate after multiplying both sides of equ^n (6.13) with $\sin m\pi\frac{x}{L}$:

$$\int_0^L \theta(x,W)\sin m\pi\frac{x}{L}\,dx = \int\sum C_n \sin n\pi\frac{x}{L}\sin\frac{m\pi x}{L}\sinh n\pi\frac{W}{L}dx$$
$$= \sum\int C_n \sin n\pi\frac{x}{L}\sin m\pi\frac{x}{L}\sinh n\pi\frac{W}{L}dx$$
$$= 0 \text{ for } n \neq m \quad \text{n is an integer}$$
$$= \sinh\left(n\pi\frac{W}{L}\right)\frac{L}{2} \text{ for } n = m \tag{6.14}$$

2 D Conduction-18

- Left hand side of equ^n (6.14) is easily integrated since $\theta(x,W)$ is known/specified (=1 here):

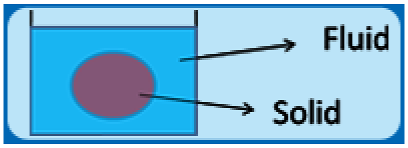
$$\frac{L}{n\pi}(1-\cos n\pi) = C_n\frac{L}{2}\sinh n\pi\frac{W}{L}$$
$$C_n = (1-\cos n\pi)\frac{2}{n\pi\sinh n\pi W/L}$$
$$\therefore \theta(x,y) = \frac{2}{\pi}\sum\frac{(1-\cos n\pi)}{n}\left(\sin n\pi\frac{x}{L}\right)\frac{\sinh n\pi\, y/L}{\sinh n\pi\, W/L} \tag{6.15}$$

2 D Conduction-19

- The algebra for analytically solving the governing equation for 2D with the associated boundary conditions is relatively involved even for the steady constant property case.
- One type of boundary condition was discussed here. Others are addressed in assignment.
- Separation of variables methodology can be used for other coordinate systems and conceptually similar to that seen here and therefore not discussed here.

Transient Conduction-1

- Simplest is the 'zero dimensional(in space)' model where temperature gradients within the solid are ignored (also called Lumped model).
- Consider a body at a certain temperature T_i suddenly put in a fluid of a different temperature, T_∞. Further assume that the fluid temperature remains constant.



Transient Conduction-2

- The body gains/looses heat by interacting with the fluid. Write the energy equation

$$\text{Heat gained by body} = m\,c\,\frac{dT}{dt}$$

$$\text{Heat lost by fluid} = hA\left(T_\infty - T\right)$$

$$\Rightarrow m\,c\,\frac{d\theta}{dt} = -hA\theta \quad \text{where } \theta = T - T_\infty$$

$$\frac{m\,c}{hA}\int_{\theta_i}^{\theta}\frac{d\theta}{\theta} = -\int_0^t dt$$

$$\ln\frac{\theta}{\theta_i} = -\frac{hA}{m\,c}t \Rightarrow \frac{\theta}{\theta_i} = e^{-\frac{hA}{m\,c}t} \qquad (6.16)$$

Transient Conduction-3

- Rewrite equ^n (6.15) in the following manner:

$$\frac{\theta}{\theta_i} = e^{-\frac{hA}{mc}t} = e^{-\frac{t}{\tau}} \text{ where } \tau = \frac{mc}{hA} \qquad (6.17)$$

- $\tau$is called the time constant. Time constant for a first order system with a step input is the time required for the system to reach 63.2% of the input step value. Often a thermocouple is modeled in this fashion to estimate its time response

Transient Conduction-4

- In equ^n (6.17):

$$\frac{\theta}{\theta_i} = e^{-1} = 0.367 \; for \; t = \tau$$

$$\theta_i = T_i - T_\infty$$

- The temperature of the body after time τ is:

$$\theta = T - T_\infty = .367\left(T_i - T_\infty\right)$$

$$\Rightarrow T - T_i = 0.633\left(T_\infty - T_i\right)$$

- Note that the temperature is 63.3 % of the step input above the initial value

Transient Conduction-5

- Rewrite the equ^n (6.15) as:

$$\frac{\theta}{\theta_i} = e^{-\left(\frac{hL}{k}\right)\left(\frac{k}{\rho c}\frac{t}{L^2}\right)} \qquad (6.18)$$

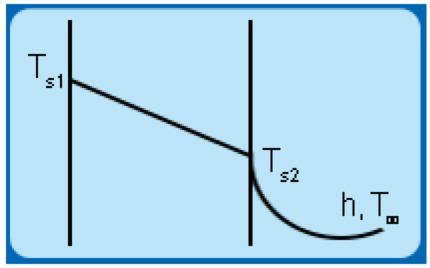
- V/A=L is some appropriate length scale. Thermal conductivity is artificially brought in here since it does not appear in the formulation of the zero dimension problem.

- Define : $\left(\frac{hL}{k}\right)$ = Biot number $\left(\frac{k}{\rho c}\frac{t}{L^2}\right)$ = Fourier number

Transient Conduction-6

- Consider a 1D example to understand the physical significance of the Biot number.
- A slab with specified temperatures on one wall and a convection boundary condition on the other

and at steady state.



Transient Conduction-7

- Heat transferred by conduction is equal to that carried by convection

$$kA\frac{(T_{S1}-T_{S2})}{L}=hA(T_{S2}-T_\infty)$$

- This rewritten as:

$$\frac{T_{S1}-T_{S2}}{T_{S2}-T_\infty}=\frac{L/kA}{1/hA}$$

- Using the concept of thermal resistances:

$$\frac{L/kA}{1/hA}=\frac{R_{cond}}{R_{conv}}$$

Transient Conduction-8

- The Biot number is a ratio of conduction resistance to convection resistance.

$$\frac{T_{S1}-T_{S2}}{T_{S2}-T_\infty}=\frac{L/kA}{1/hA}=\frac{R_{cond}}{R_{conv}}\equiv Bi(Biot\ No.) \tag{6.19}$$

- High Bi implies high conduction resistance and thus large temperature gradients in the solid. In the 'zero dim.' approximation temperature gradients within solid were ignored.

Transient Conduction-9

- Thus if Biot number is small, temperature gradients within solid can be ignored with respect to gradients within the surrounding fluid. Thus, the 'zero D' approximation within the solid can be used for small Bi – typically do so for Bi < 0.1 (some justification for this choice will be given later)

1D unsteady conduction - constant k
no heat generation-1

- Consider now another 2D situation. The two dimensions now are one space 'x' and one time 't'.
- The governing equation is the same as the one previously used, i.e. equ^n (2.1) but the storage term is also included. The equation is second order in space and first order in time. Two conditions needed for space dimension and one for time dimension.

1D unsteady conduction - constant k
no heat generation-2

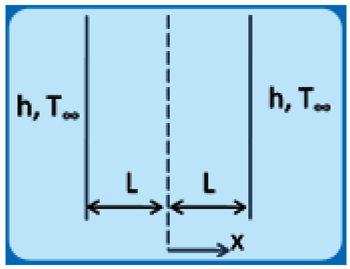
- The governing equation and the boundary and initial conditions are:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{6.20}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \tag{6.21}$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h(T - T_\infty)|_{x=L} \tag{6.22}$$

$$T(x,0) = T_i \tag{6.23}$$



1D unsteady conduction - constant k
no heat generation-3

- Non-dimensionalize the equation using:

$$\theta^* = \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} \qquad x^* = \frac{x}{L} \qquad t^* = \frac{t}{t_0}$$

- A suitable length for non-dimensionalizing 'x' is 'L' but a suitable time 't_0' for 't' is unknown. However, continue with the algebra and it is possible that the governing equation will suggest a suitable parameter

1D unsteady conduction - constant k
no heat generation-4

- The governing equation becomes:

$$\frac{\theta_i}{L^2}\frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{1}{\alpha}\frac{\theta_i}{t_0}\frac{\partial \theta^*}{\partial t^*} \Rightarrow \frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{L^2}{\alpha t_0}\frac{\partial \theta^*}{\partial t^*} \tag{6.24}$$

- Since the variables are nondimensional, the combination of constants on the right hand side must have no dimension. A suitable 't_0' can be obtained and t* is nothing but the Fourier number defined earlier:

$$t_0 = \frac{L^2}{\alpha}; t^* = \frac{\alpha t}{L^2} \equiv F_0$$

1D unsteady conduction - constant k
no heat generation-5

- Final equation for solving is therefore:

$$\frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial Fo} \tag{6.25}$$

- Assume again that the variables are separable and therefore attempt a solution of the form:

$$\theta^* = F(x^*)G(t^*) \tag{6.26}$$

- Substitute into the governing equation to get:

$$F''G = FG' \Rightarrow \frac{F''}{F} = \frac{G'}{G} = \pm\lambda^2 \quad (6.27)$$

1D unsteady conduction - constant k
no heat generation-6

- Where the following notation is used and λ^2 is an arbitrary constant

$$F'' \equiv \frac{d^2F}{dx^2}; G' \equiv \frac{dG}{dt}$$

- Assume $\lambda^2 = 0$ and solve equ^n (6.27) for G
- This implies G = constant which is not possible since this means no time dependence in the final solution. Thus $\lambda^2 = 0$ is not permissible.

1D unsteady conduction - constant k
no heat generation-7

- Assume + λ^2 to the choice for the constant. Again solve for 'G' in equ^n (6.27):

$$\frac{G'}{G} = +\lambda^2 \Rightarrow G = e^{\lambda^2 t}$$

- This implies that the solution is unbounded as time increases which is physically not possible. Therefore, this choice for the constant is also not permissible

1D unsteady conduction - constant k
no heat generation-8

- Using $-\lambda^2$ as the choice for the constant:

$$\frac{G'}{G} = -\lambda^2 \Rightarrow G = Ce^{-\lambda^2 t*}$$

- The function 'G' decays with time and this behaviour is therefore acceptable. Using $-\lambda^2$ as constant for the 'F' equation in equ^n (6.27) gives:

$$F'' + \lambda^2 F = \cos$$
$$\Rightarrow F = P\cos\lambda x^* + Q\sin\lambda x^*$$

1D unsteady conduction - constant k
no heat generation-9

- F and G are now determined. Therefore get the solution of the equ^n (6.25) by substituting these in equ^n (6.26) to get:

$$\begin{aligned}\theta^* = F(x^*)G(t^*) &= (P\cos\lambda x^* + Q\sin\lambda x^*)Ce^{-\lambda^2 t^*} \\ &= (A\cos\lambda x^* + B\sin\lambda x^*)e^{-\lambda^2 t^*}\end{aligned} \quad (6.28)$$

- The expression has three unknowns A, B, λ and three conditions are available to determine the constants.

1D unsteady conduction - constant k no heat generation-10

- Use condition equ^n (6.21)

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \Rightarrow F'G|_{x^*=0} = 0 \Rightarrow F'|_{x^*=0} = 0 \tag{6.29}$$

$$\Rightarrow (A\cos\lambda x^* + B\sin\lambda x^*)'|_{x^*=0} = 0$$

$$\Rightarrow (-A\lambda\sin\lambda x^* + B\lambda\cos\lambda x^*)|_{x^*=0} = 0$$

$$\Rightarrow B = 0 \tag{6.30}$$

1D unsteady conduction - constant k no heat generation-11

- Now use condition equ^n (6.21) i.e. the convective boundary condition at one end:

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h(T - T_\infty)|_{x=L}$$

$$\Rightarrow -\frac{k\theta_i}{L}\left.\frac{\partial\theta^*}{\partial x^*}\right|_{x^*=1} = h\theta_i\theta^*|_{x^*=1}$$

$$\Rightarrow -\frac{k}{L}GF'|_{x^*=1} = hFG|_{x^*=1}$$

$$\Rightarrow F'|_{x^*=1} = -BiF|_{x^*=1} \tag{6.31}$$

$$\Rightarrow -\frac{k}{L}\lambda A\sin\lambda = hA\cos\lambda \tag{6.32}$$

Recap

In this class:

- The 2D conduction problem is completed.
- The zero dimension transient problem is presented and the significance of the Biot number explained.
- The solution for the 1D transient problem is started