Module 1 : Conduction

Lecture 8 : Superposition in Conduction

Objectives

In this class:

- The superposition principle is discussed in detail by using an example.
- The use of superposition to convert a multi-dimension problem into a product of lower dimension problems is discussed
- The Duhamel's superposition theorem is discussed. The concept of the Duhamel's theorem is discussed using the zero dimension situation as the vehicle.

Superposition

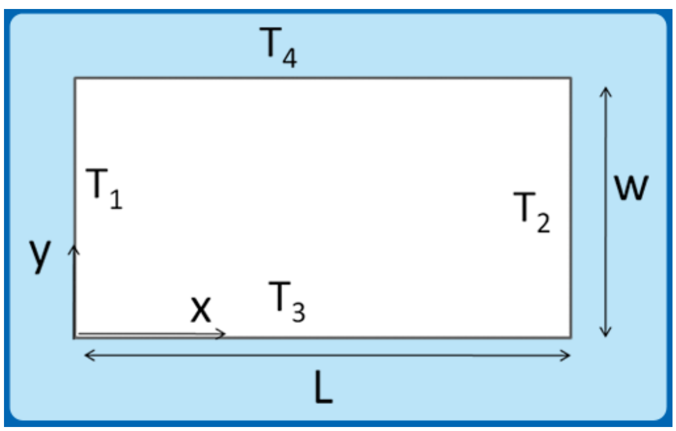
- The previous examples were those where solutions were obtained for relatively easy situations.
- The solutions to governing equations for more complex situations can be obtained as those which are a combination of solutions to simpler problems.
- This is possible since the governing equation is linear. However, care must be seen to satisfy boundary conditions also.

Example 1/1

- Consider the solution of the temperature in a 2D rectangular geometry with constant thermal conductivity. The 4 walls are at temperatures T_1, T_2, T_3, T_4. It is possible to reduce this problem to sub problems in such a way that the solution obtained for the problem in class can be used without needing to solve additional differential equations.

Example 1/2

- Explain your steps carefully and don't try to solve the subproblems.



Example 1/3

- In the previous presentation, we have seen the solution for a two dimensional situation where all walls were at 'zero' and only one wall was at temperature 'T_1'. Here we use superposition to 'split' the given problem into subproblems for which the solution is already known so that the solution for the given problem can be can be relatively easily obtained.

Example 1/4

- The governing equation along with the boundary conditions is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

$$T|_{x=0} = T_1;\, T|_{x=L} = T_2$$

$$T|_{y=0} = T_3;\, T|_{x=w} = T_4$$

- Let us assume a solution of the form:

$$T = A_1 + A_2 + A_3 + A_4$$

Example 1/5

- The governing equation is:

$$\frac{\partial^2 (A_1 + A_2 + A_3 + A_4)}{\partial x^2} + \frac{\partial^2 (A_1 + A_2 + A_3 + A_4)}{\partial y^2} = 0.$$

- It can be argued that the above governing equation being satisfied is equivalent to the following four equations being satisfied:

$$\begin{aligned} &\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} = 0; \frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} = 0; \\ &\frac{\partial^2 A_3}{\partial x^2} + \frac{\partial^2 A_3}{\partial y^2} = 0; \frac{\partial^2 A_4}{\partial x^2} + \frac{\partial^2 A_4}{\partial y^2} = 0. \end{aligned} \qquad \text{(A)}$$

Example 1/6

- The boundary conditions become:

$$(A_1 + A_2 + A_3 + A_4)\big|_{x=0} = T_1 \qquad \text{(a)}$$

$$(A_1 + A_2 + A_3 + A_4)\big|_{x=L} = T_2 \qquad \text{(b)}$$

$$(A_1 + A_2 + A_3 + A_4)\big|_{y=0} = T_3 \qquad \text{(c)}$$

$$(A_1 + A_2 + A_3 + A_4)\big|_{y=w} = T_4 \qquad \text{(d)}$$

- It is required to modify these conditions in an appropriate fashion

Example 1/7

- Consider the boundary condition (a):

$$(A_1 + A_2 + A_3 + A_4)\big|_{x=0} = T_1$$

- It can be argued that the above condition is satisfied if the following is satisfied:

$$A_1\big|_{x=0} = T_1; \;\; (A_2 + A_3 + A_4)\big|_{x=0} = 0 \qquad \text{(a1)}$$

- Note that the above is a choice for the variables which perhaps may simplify procedures

Example 1/8

- Now Consider the boundary condition (b):

$$(A_1 + A_2 + A_3 + A_4)\big|_{x=L} = T_2$$

- It can be argued that the above condition is satisfied if the following is satisfied:

$$A_2\big|_{x=L} = T_2; \;\; (A_1 + A_3 + A_4)\big|_{x=L} = 0 \qquad \text{(b1)}$$

- Similarly get from boundary conditions (c), (d):

$$A_3\big|_{y=0} = T_2; \;\; (A_1 + A_2 + A_4)\big|_{y=0} = 0 \qquad \text{(c1)}$$

$$A_4\big|_{y=w} = T_2; \;\; (A_1 + A_2 + A_3)\big|_{y=w} = 0 \qquad \text{(d1)}$$

Example 1/9

- Equations (a1), (b1), (c1), (d1) are all satisfied together if the following is true:

$$\begin{array}{llll} A_1\big|_{x=0} = T_1 & A_2\big|_{x=0} = 0 & A_3\big|_{x=0} = 0 & A_4\big|_{x=0} = 0 \\ A_1\big|_{x=L} = 0 & A_2\big|_{x=L} = T_2 & A_3\big|_{x=L} = 0 & A_4\big|_{x=L} = 0 \\ A_1\big|_{y=0} = 0 & A_2\big|_{y=0} = 0 & A_3\big|_{y=0} = T_3 & A_4\big|_{y=0} = 0 \\ A_1\big|_{y=w} = 0 & A_2\big|_{y=w} = 0 & A_3\big|_{y=w} = 0 & A_4\big|_{y=w} = T_3 \end{array} \qquad \text{(B)}$$

Example 1/10

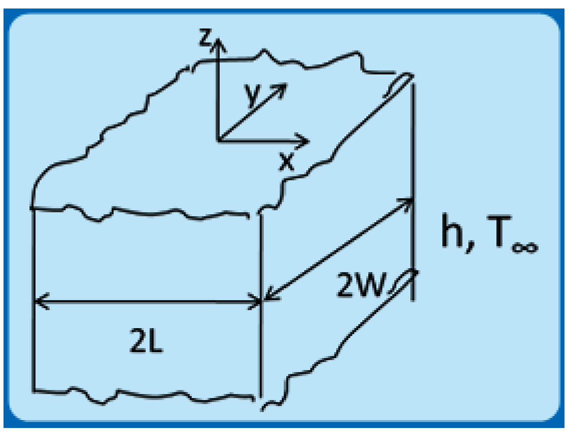
- Equation set (A) represents four governing equations. Equation set (B) represents four sets of boundary conditions.
- The original governing equation and boundary conditions are now converted to a set of four differential equations and four sets of boundary conditions
- These therefore represent four 'sub-problems' for which solution has already been obtained.

Example 1/11

- Final solution is therefore $A_1 + A_2 + A_3 + A_4$
- A_1, A_2, A_3, A_4 are the solutions to each of the sub problems with known temperature on one wall and zero temperature on all other walls
- Here a summation solution worked for the problem. Further we will see that a multiplication solution will be suitable for splitting the problem

Multidimensional Problems-1

- Consider 2D unsteady conduction. Normally analytical solutions are difficult. Sometimes analytical solutions can be obtained.
- Assume same heat transfer coeff. and free stream temperature boundary condition on all sides for a slab.



Multidimensional Problems-2

- The governing equation and boundary conditions are:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{8.1}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0}=0=\left.\frac{\partial T}{\partial y}\right|_{y=0} \tag{8.2}$$

$$\left[-k\frac{\partial T}{\partial x}\right]_{x=L}=h(T-T_\infty)=\left[-k\frac{\partial T}{\partial y}\right]_{y=W} \tag{8.3}$$

$$T_{t=0}=T_i \tag{8.4}$$

Multidimensional Problems-3

- Equ^n (8.1), (8.2), (8.3) are transformed to a more suitable form as shown:

$$\theta=\frac{T-T_\infty}{T_i-T_\infty} \tag{8.5}$$

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=\frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{8.6}$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0; \quad \left.\frac{\partial\theta}{\partial y}\right|_{y=0}=0 \tag{8.7}$$

$$\left[-k\frac{\partial\theta}{\partial x}\right]_{x=L}=h\theta\big|_{x=L}; \quad \left[-k\frac{\partial\theta}{\partial y}\right]_{y=W}=h\theta\big|_{y=W} \tag{8.8}$$

Multidimensional Problems-4

- Note that 'T' and 'θ' are both functions of x, y, t
- Attempt a solution by separation of variables:

$$\theta(x,y,t)=\theta_1(x,t)\theta_2(y,t)$$

$$\theta_2\frac{\partial^2\theta_1}{\partial x^2}+\theta_1\frac{\partial^2\theta_2}{\partial y^2}=\frac{1}{\alpha}\left[\theta_2\frac{\partial\theta_1}{\partial t}+\theta_1\frac{\partial\theta_2}{\partial t}\right] \tag{8.9}$$

$$\left.\frac{\partial\theta_1}{\partial x}\right|_{x=0}=0, \left.\frac{\partial\theta_2}{\partial y}\right|_{y=0}=0; \quad \theta_2\big|_{t=0}=1; \theta_1\big|_{t=0}=1 \tag{8.10}$$

$$\left.\frac{\partial\theta_1}{\partial x}\right|_{x=L}=\frac{h\theta_1}{k}; \quad \left.\frac{\partial\theta_2}{\partial y}\right|_{y=W}=\frac{h\theta_2}{k}; \tag{8.11}$$

Multidimensional Problems-5

- The equ^n 8.9 can be rewritten in the following form:

$$\frac{1}{\theta_1}\left(\frac{\partial^2\theta_1}{\partial x^2}-\frac{1}{\alpha}\frac{\partial\theta_1}{\partial t}\right)=\frac{1}{\theta_2}\left(\frac{1}{\alpha}\frac{\partial\theta_2}{\partial t}-\frac{\partial^2\theta_2}{\partial y^2}\right)$$

- The above equation is satisfied if the terms in the brackets are equated to zero. In addition if the boundary conditions are also satisfied the problem solution can be considered to have been obtained.

Multidimensional Problems-6

- Equ^n (8,9), (8.10), (8.11) can be rewritten as:

$$\frac{\partial^2\theta_1}{\partial x^2}=\frac{1}{\alpha}\frac{\partial\theta_1}{\partial t} \tag{8.12}$$

$$\left.\frac{\partial\theta_1}{\partial x}\right|_{x=0}=0; \tag{8.13}$$

$$\left.\frac{\partial\theta_1}{\partial x}\right|_{x=L}=\frac{h\theta_1}{k}; \tag{8.14}$$

$$\theta_1\big|_{t=0}=1 \tag{8.15}$$

$$\frac{\partial^2\theta_2}{\partial y^2}=\frac{1}{\alpha}\frac{\partial\theta_2}{\partial t} \tag{8.16}$$

$$\left.\frac{\partial\theta_2}{\partial y}\right|_{y=0}=0; \tag{8.17}$$

$$\left.\frac{\partial\theta_2}{\partial y}\right|_{y=w}=\frac{h\theta_2}{k}; \tag{8.18}$$

$$\theta_2\big|_{t=0}=1 \tag{8.19}$$

Multidimensional Problems-7

- Equ^n 7.3 and equ^n 7.15 can be used to obtain θ_1 , θ_2 which are the required solutions for the equ^n (8.12)-(8.15) and equ^n (8.16)-(8.18) respectively.
- θ_1 ,θ_2 also are solution equ^n (8.9)-(8.11). The 2D slab situation therefore has been split into two 1D slab situations. Note that this may not always be possible.

Multidimensional Problems-8

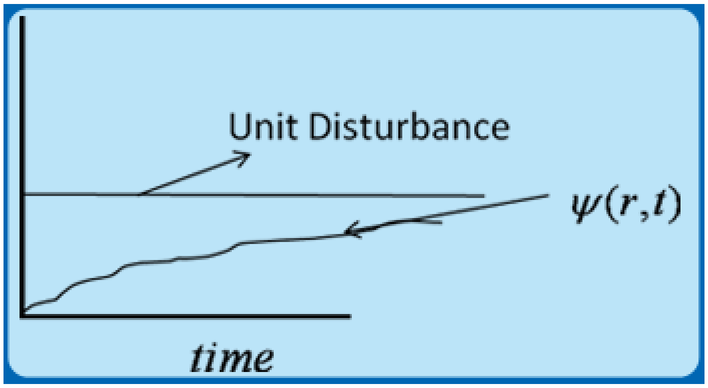
- Some multidimensional situations can be simplified to lower dimensional situations such that solutions for the lower dimension situations can be used to obtain the solution for higher dimension situations. However, we need to write the equations and ensure that the split is possible.
- Governing equation and boundary equations must both be amenable for the split

Duhamel's Theorem-1

- All the situations discussed above were those where a step change was imposed at the boundary.
- Analytical solutions are difficult for situations where the boundary condition is not a step change
- It is possible to use the step solution already obtained to get solutions for continuously varying boundary conditions.

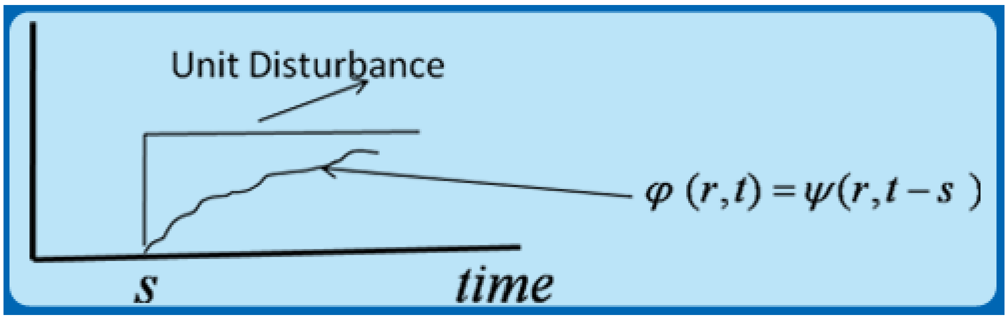
Duhamel's Theorem-2

- Consider a unit step disturbance. Let the solution be represented as$\Psi(r,t)$ as shown below where 'r' is the spatial and 't' the time coordinate. Note that for $t<0$, $\Psi(r,t) = 0$. The disturbance is also zero for $t<0$



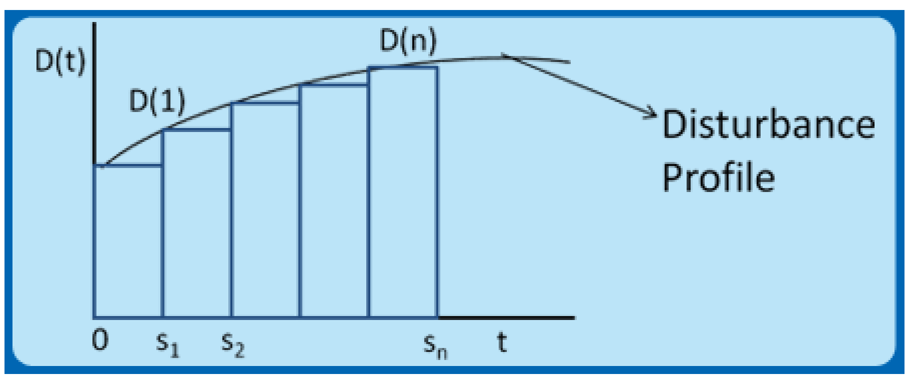
Duhamel's Theorem-3

- Consider another unit step disturbance. However, now the disturbance starts at a time 's' after zero. Note that the disturbance is zero for t < s. The solution φ(r,t) is represented as φ(r,t) = ψ(r,t-s) shown:



Duhamel's Theorem-4

- Now look at typical disturbance. The disturbance changes with time as shown. The disturbance can be considered to be a series of steps as shown; i.e. at time t = 0 disturbance is D(0), at time t = s_1 the disturbance is D(1) etc.



Duhamel's Theorem-5

- When the disturbance is not unity, the solution for a step D(n) at time 't_n' and zero disturbance until 't_n ' can be written as

$$\varphi_1(r,t) = \begin{cases} 0 & t < s \\ D(n)\psi(r,t-s) & t > s \end{cases} \tag{8.20}$$

- Final solution, the Duhamel's theorem, is written as:

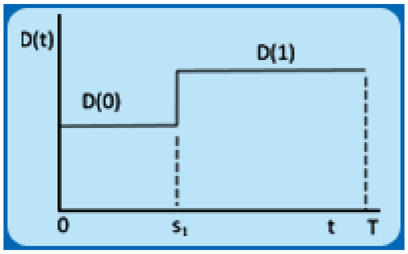
$$\begin{aligned} \varphi(r,t) = D(0)\,\psi(r,t) + (D(s_1)\text{-}D(0))\,\psi(r,t\text{-}s_1) \\ +[D(s_2) - D(s_1)]\,\psi(r,t\text{-}s_2) + \ldots.. \\ \ldots\ldots\ldots[D(s_n)\text{-}D(s_{n\text{-}1})]\,\psi(r,t\text{-}s_n) \end{aligned} \tag{8.21}$$

Duhamel's Theorem-6

- Equation (8.21) can be further explained as the following:
- The final solution is a summation of several individual solutions
- A given disturbance step begins at $t = s_n$ and continues till the final time until which the solution is required. The disturbance step at this time 'location' is $D(s_n) - D(s_{n-1})$

Duhamel's Theorem-7

- Assume the disturbance is as shown below. At time t = 0, disturbance D = D(0) until time $t = s_1$. At time $t = s_1$ the disturbance is suddenly changed to D(1) and this value stays till the end of the period for which the solution is desired i.e. time = T.



Duhamel's Theorem-8

- The solution is as follows:

$$\varphi(r,t) = D(0)\,\psi(r,T) + (D(s_1)-D(0))\,\psi(r,T-s_1)$$

- Note that the solution for the first disturbance is taken from the initial time t = 0 to the final time t = T. At time $t = s_1$, the disturbance is an incremental one over what already existed at the initial step. The solution for the incremental disturbance continues till the final time.
- Extension to more steps is now not difficult.

Duhamel's Theorem-9

- The general form of the equation can thus be written as:

$$\varphi(r,t) = D(0)\,\psi(r,t) + \sum_{1}^{\infty}([D(s_n)-D(s_{n-1})]\,\psi(r,t-s_n)) \qquad (8.22)$$

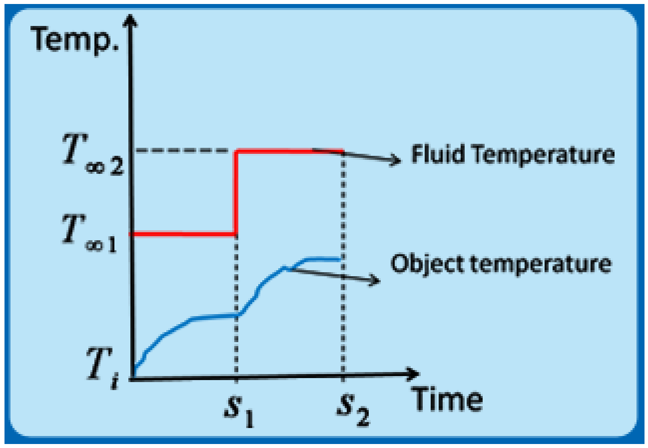
- The formulation is given for a finite number of steps. When the number of steps is very large, the summation can be replaced by an integral.

$$\phi(r,t) = D(0)\psi(r,t) + \int_0^t \psi(r,t-s)\frac{dD}{ds}ds \qquad (8.23)$$

Duhamel's Theorem-10

- We first take an example and demonstrate the working of the theorem
- A cold object with initial temperature T_i is inserted into a hot fluid. The fluid temperature is the disturbance. The fluid is at temperature equal to $T_{\infty 1}$ at the beginning and at time $t = s_1$ the temperature is suddenly changed to $T_{\infty 2}$. The temperature change of the object due to this is required. A qualitative variation is indicated in the figure.

Duhamel's Theorem-11



Duhamel's Theorem-12

- Assume the lumped approach for calculating the transient body temperature.
- Initial temperature is T_i . Initial transient is governed by the equation already discussed up to time $t = s_1$.

$$\frac{mc}{hA}\frac{dT}{dt} + T = T_{\infty 1} \text{ and } T\big|_{t=0} = T_i$$

- The equation is not in the form useful for use in the Duhamels theorem as discussed earlier.

Duhamel's Theorem-13

- Need suitable modifications to put it in the required form. The solution must start from zero as indicated in equ^n (8.20). The disturbance also must be zero for $t < 0$. Therefore transform the equations:

$$\text{Let } (T - T_i) = q_0, \ \ \tau = \frac{mc}{hA}; \ q_{i1} = T_{\infty 1} - T_i$$

$$\tau\frac{dq_0}{dt} + q_0 = q_{i1} \tag{8.24}$$

$$q_0\big|_{t=0} = 0 \tag{8.25}$$

Duhamel's Theorem-14

- Note that the solution q_0 starts from zero. Set $q_{i1} = 1$ to get initial disturbance as unity to get:

$$\tau\frac{d\bar{q}_0}{dt} + \bar{q}_0 = 1; \qquad \bar{q}_0\big|_{t=0} = 0 \tag{8.26}$$

- Solution for equn (8.26) ' $\bar{q}_0$ ' is the $\Psi(r,t)$ for use in equ^n (8.20).

- The solution for equ^n (8.24) is $q_0 = q_{i1}\,\bar{q}_0$. Substitute this in equ^n (8.26) to verify that this is indeed the solution for disturbance = q_{i1}

Duhamel's Theorem-15

- After $t = s_1$ the temperature of the fluid is suddenly raised to $T_{\infty 2}$. The governing equation remains same. The initial condition is 'q_0' which is the solution to equ^n (8.24) and (8.25) at $t = s_1$:

$$\tau \frac{dq_{01}}{dt} + q_{01} = q_{i2} \tag{8.27}$$

$$\text{where } q_{01} = T_{01} - T_i ; q_{i_2} = T_{\infty_2} - T_i$$

$$q_{01}\big|_{t=s_1} = q_0\big|_{t=s_1} \tag{8.28}$$

- Solution to equ^n (8.24), (8,25), (8.27), (8.28) is the required transient which is attempted next.

Duhamel's Theorem-16

- Let $q_{01} = q_{0s2} + q_{0s1}$
- Substitute in (8.24) to get:

$$\tau \frac{d}{dt}\left(q_{0s2} + q_{0s1}\right) + q_{0s2} + q_{0s1} = q_{i_1} + q_{i_2} - q_{i_1}$$

$$\left(q_{0s2} + q_{0s1}\right)\Big|_{t=s_1} = q_0\big|_{t=s_1}$$

- The above equation can be expanded and simplified. The differential equation and boundary condition can be split into two problems.

Duhamel's Theorem-17

- The two differential equations along with the associated conditions are:

$$\tau \frac{d}{dt}\left(q_{0s1}\right) + q_{0s1} = q_{i1}; \qquad \tau \frac{d}{dt}\left(q_{0s2}\right) + q_{0s2} = q_{i2} - q_{i1}$$

$$q_{0s1}\big|_{t=s_1} = q_0\big|_{t=s_1}; \qquad q_{0s2}\big|_{t=s_1} = 0$$

- Note now that both the above equations are valid from t = s_1 up to the time for which the solution is desired

Duhamel's Theorem-17

- Initial transient is governed by the equation for q_0 i.e. from time = 0 to t = s_1 .
- The next portion of the transient from t = s_1 to t = t_{final} is governed by q_{01} i.e. q_{os1} and q_{os2}
- Now combine the transient equations for q_o and q_{os1}. q_o is valid upto s_1 and after that the transient q_{os1} continues with initial condition identical to the value of qo at t = s_1.

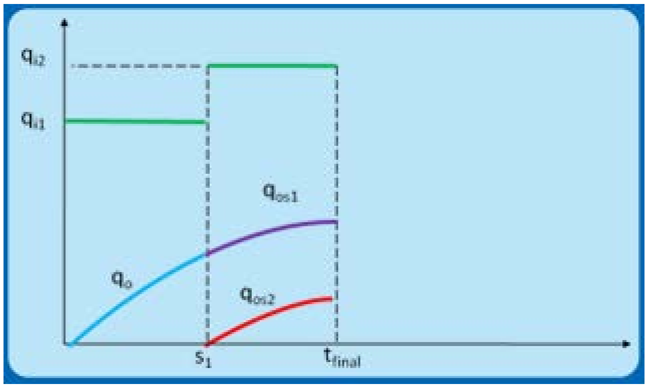
Duhamel's Theorem-19

- The combined transient for q_o and q_{os1} is equivalent to q_{i1} present from t = 0 to t = t_{final}.
- q_{os2} is another transient which has disturbance equal to $q_{i2} - q_{i1}$ and starts with zero initial condition and continues.
- Notice that the above statement is nothing but the verbal representation of Equ^n (8.22), i.e., Duhamel's theorem, which gives

$$\varphi(r,t) = q_{i1}\tilde{q}(t_{final}) + (q_{i2} - q_{i1})\tilde{q}(t_{final} - s_1)$$

Duhamel's Theorem-20

- Overall transient therefore looks like:



Green: Inputs q_{i1} and q_{i2}
Blue: transient for q_{i1} input from $t = 0$ to $t = s_1$
Purple: transient for q_{i1} input from $t = s_1$ to $t = t_{final}$
Red: transient for $(q_{i1} - q_{i2})$ input from $t = s_1$ to $t = t_{final}$

Duhamel's Theorem-21

- We have only demonstrated the working of the Duhamel's theorem and not proved it.
- The demonstration used a problem of lumped body as the vehicle but even if any other situation is used, the same methodology can be used to see that the Duhamel's theorem indeed gives the solution.

Recap

In this class:

- The superposition principle is discussed in detail by using an example.
- The use of superposition to convert a multi-dimension problem into a product of lower dimension problems is discussed
- The Duhamel's superposition theorem is discussed. The concept of the Duhamel's theorem is discussed using the zero dimension situation as the vehicle.