Module 1 : Conduction

Lecture 5 : 1D conduction example problems. 2D conduction

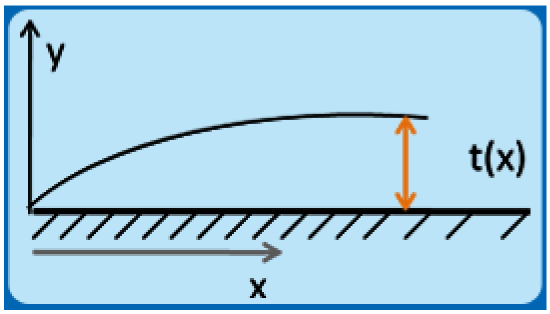
Objectives

In this class:

- An example of optimization for insulation thickness is solved. The 1D conduction is considered completed.
- A few example problems are solved for 1D conduction.
- 2D conduction in cartesian coordinate system using separation of variables is started.

Minimization of Insulation Thickness-1

- Need to insulate a plate such that the heat loss is minimum. Temperature of the plate varies with 'x' and the variation is known
- Volume of insulation is fixed.
- Determine the thickness of insulation



Minimization of Insulation Thickness-2

- Assume plate width (normal to x-y plane) is W and length is L. Assume heat transfer is one dimensional in the y direction.
- Assume outer insulation is at T_∞

$$\text{Heatloss} = Q = \int_0^L \frac{kW\left(T(x) - T_\infty\right)dx}{t(x)} \tag{5.1}$$

$$V = \int_0^L t(x)W dx \tag{5.2}$$

Minimization of Insulation Thickness-3

- t(x) has to be determined. Take help from variational calculus
- Need to find the maximum of the integral stated below

$$I = \int_a^b F\left(x, y, y'\right)dx \tag{5.3}$$

- If 'I' has an extremum for a 'y' the Euler equation below is satisfied:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{5.4}$$

Minimization of Insulation Thickness-4

- Suppose 'y' also satisfies a constraint given by:

$$C = \int G(x, y, y')\,dx \tag{5.5}$$

- C is a constant. The modified Euler equation given below is also satisfied:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \tag{5.6}$$

- Where $H = F + \lambda G$
- λ is the Lagrange multiplier and needs to be determined also (5.7)

Minimization of Insulation Thickness-5

- Now return to the problem at hand i.e. substitute eq^n 5.1 and eq^n 5.2 in eq^n (5.7):

$$H = \frac{kW(T(x) - T_\infty)}{t(x)} + \lambda W t(x) \tag{5.8}$$

$$\text{Here } \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0 \quad \text{since } H \neq f(y')$$

$$\therefore \frac{\partial H}{\partial t} = 0 \Rightarrow \frac{-kW(T(x) - T_\infty)}{t^2} + \lambda W$$

$$\text{i.e. } t_{opt}(x) = \sqrt{\frac{k}{\lambda}}\sqrt{T(x) - T_\infty} \tag{5.9}$$

Minimization of Insulation Thickness-6

- λ is unknown still but use the constraint equation to determine its value

$$V = \int_0^L \sqrt{\frac{k}{\lambda}}\sqrt{T(x) - T_\infty}\,W dx$$

$$\therefore \sqrt{\frac{k}{\lambda}} = \frac{V}{W\int (T(x) - T_\infty)dx} \tag{5.10}$$

- Now obtain t(x) using (5.9)

1D conduction: some comments-1

- We have seen some aspects of one dimensional conduction.
- The most important is the use of the resistance analogy for making calculations – many times this is very useful although a little crude
- Another important aspect is the conversion of a higher dimension problem into a lower dimension situation

1D conduction: some comments-2

- Conversion of a higher dimension situation to a lower dimension one necessitates the derivation of a new governing equation. Again this is done to get results in an easy fashion
- Some optimization situations were looked at and some methodologies were presented.
- One or two illustrative problems will be solved and some given as assignment before proceeding to the 2D situation.

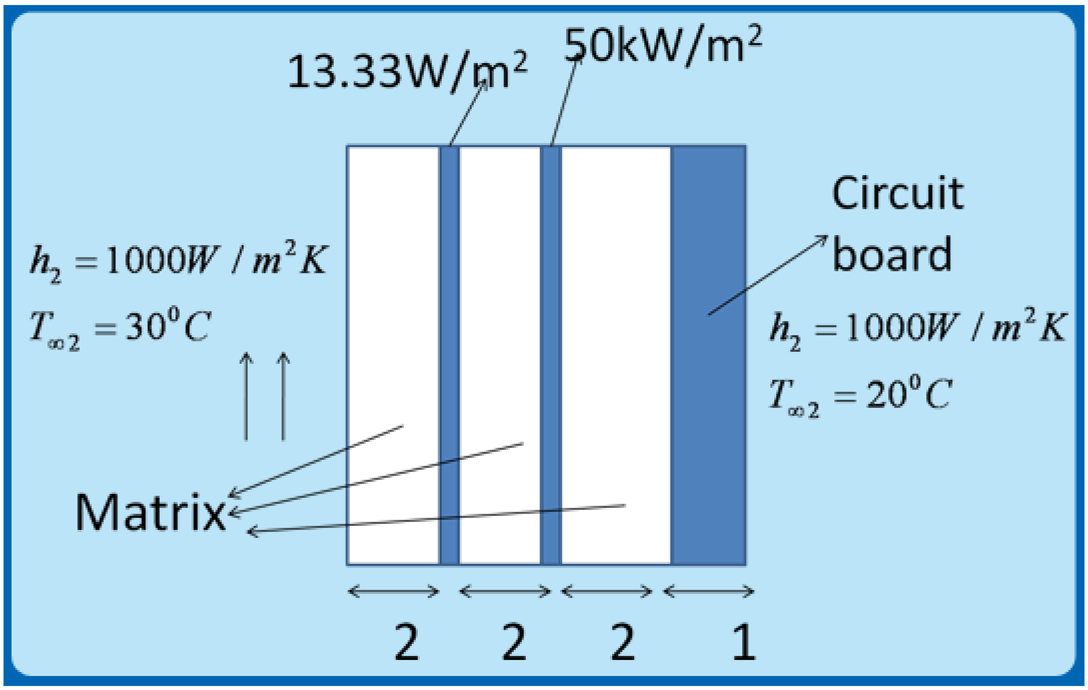
End of 1D conduction: Problem1-1

A special computer chip has 2 thin regions '1' and '2' embedded in a matrix which has a thermal conductivity of 4 W/mK. Heat is generated in regions 1 and 2 only and equal to 50 kW/m^2 and 13.33 kW/m^2 respectively. The chip is convectively cooled at its outer surface with a fluid stream h_1 = 1000 W/m^2 and $T_{\infty 1}$ = 30° C. The chip is joined to the circuit board at its inner surface. The thermal contact resistance between the chip and the board surface can be ignored.

End of 1D conduction: Problem1-2

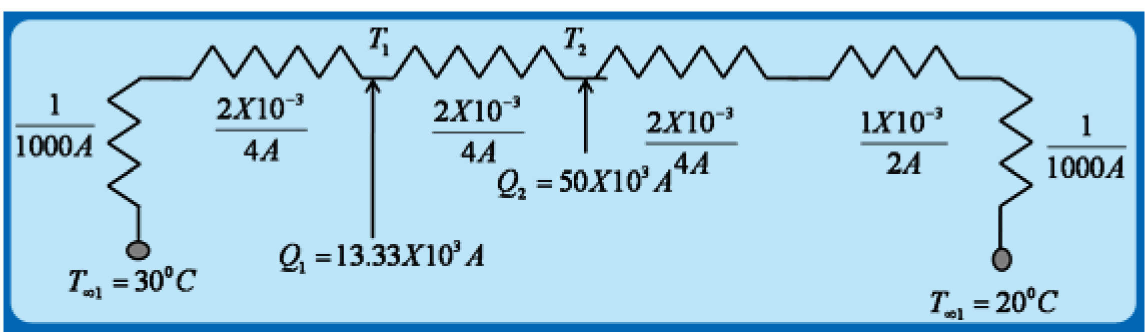
- The board thickness and thermal conductivity are L = 1 mm and k = 2 W/mK. The outer surface of the board is exposed to cooling air with temperature 20° C and heat transfer coefficient 1000 W/m^2 K. Assume heat generation regions to have negligible thickness, all surfaces not exposed to cooling air are perfectly insulated. Assume steady conditions and determine the temperature of the region '1'.

End of 1D conduction: Problem1-3



End of 1D conduction: Problem1-4

- Conditions are steady and properties are constant. The heat generation is present in areas of negligible thickness. Therefore the electrical analogy is used. The heat generation is input at the nodes and resistances are indicated on the figure.



End of 1D conduction: Problem1-5

- Now obtain the solution using the standard Ohm's law expressions. First write the equation for heat flow between nodes at T_2 and 30° C ambient:

$$\frac{30-T_1}{\frac{1}{1000A}+\frac{2}{1000A(4)}}+13.33(1000)A=\frac{T_1-T_2}{\frac{2}{1000(4)}}$$

$$\frac{30-T_1}{1.5}+13.33=\frac{T_1-T_2}{0.5}\Rightarrow 30-T_1+20=3(T_1-T_2)$$

$$\Rightarrow 50=4T_1-3T_2$$

End of 1D conduction: Problem1-6

- Now obtain the heat flow between the T_2 and 20° C ambient:

$$\frac{T_1-T_2}{\frac{2}{1000A(4)}}+50(1000)A=\frac{T_2-20}{\frac{1}{1000A}+\frac{2}{1000A(4)}+\frac{2}{1000A(4)}}$$

$$\Rightarrow\frac{T_1-T_2}{0.5}+50=\frac{T_2-20}{2}\Rightarrow 4(T_1-T_2)+100=T_2-20$$

$$\Rightarrow 4T_1+120=5T_2$$

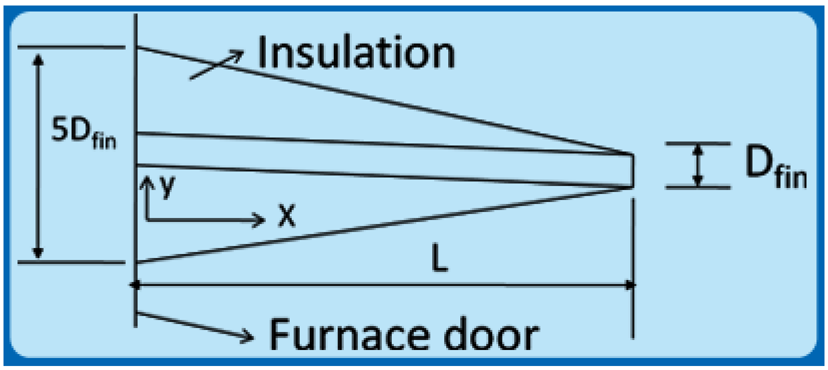
- Solve to get T_1 = 85° C and T_2 = 76.25° C

End of 1D conduction: Problem2-1

- A rod of length L which acts as a handle protrudes from a furnace door at temperature Tw . Since there is an air draft present, you have decided to insulate it. A linearly varying thickness of insulation as shown below with the diameter of the insulation at the base being 5 times the diameter of the fin is provided. Assume that transverse (y direction) temperature gradients can be ignored in the rod but not for the insulation.

End of 1D conduction: Problem2-2

- Set up the equations required to get the temp. profile within the rod. Assume that the surrounding air is at T_∞ , the heat transfer coefficient for air over the insulation is very large, thermal conductivity of insulation in x,y directions is k_x and zero respectively.



End of 1D conduction: Problem2-3

- Transverse gradients are present in insulation and absent in rod. The surrounding heat transfer coefficient is very large and therefore the outer insulation temperature is equal to T_∞
- The diameter of insulation at any distance D_x is given by:

$$\frac{1.5D}{L}=\frac{D_x}{L-x}\Rightarrow D_x=\frac{1.5D(L-x)}{L}$$

End of 1D conduction: Problem2-4

- Consider an annular disk of insulation of length dx at location x. Heat loss due to conduction in the y direction is :

$$Q = \frac{T_x - T_\infty}{\dfrac{\ln(D_x/D_{fin})}{2\pi dx k_x}}$$

- Use the same formulation for the rod as that done for the fin. Here the loss is due to conduction whereas there it was due to convection and this is the only difference.

End of 1D conduction: Problem2-5

- The governing equation therefore becomes:

$$-k_{insu} A \frac{d^2T}{dx^2} dx + \frac{(T - T_\infty) 2\pi dx k_{in}}{\ln(D_x/D)} = 0$$

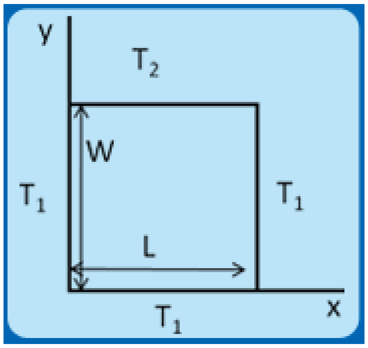
- The boundary conditions are:

$$T\big|_{x=0} = T_w ; T\big|_{x=L} = T_\infty$$

- The variation of Dx has already obtained and therefore the solution of the above equation will give the temperature profile.

2 D Conduction-1

- Analytical solution available for simple geometries.
- Consider steady situation without heat generation, constant thermal conductivity



2 D Conduction-2

- The governing equation is:

$$\nabla . k \nabla T + q''' = \rho C_p \frac{\partial T}{\partial t} \tag{5.11}$$

- Steady, constant properties, no heat generation, cartesian coordinates gives:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{5.12}$$

- Transform the variable 'T' using:

$$\theta = \frac{T - T_1}{T_2 - T_1} \tag{5.13}$$

2 D Conduction-3

- Governing equation and boundary conditions become:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \quad (5.14)$$

$$x=0, y=y \ \ \theta=0 \quad (5.15)$$

$$x=L, y=y \ \ \theta=0 \quad (5.16)$$

$$y=0, x=x \ \ \theta=0 \quad (5.17)$$

$$y=W, x=x \ \ \theta=1 \quad (5.18)$$

2 D Conduction-4

- Attempt a separation of variables methodology for obtaining solution:

$$\theta\left(x,y\right)=X(x).Y(y) \quad (5.19)$$

- Variable which is a function of ‘x’ and ‘y’ is assumed to be a product of two variables – one purely a function of ‘x’ and the other purely a function of ‘y’

2 D Conduction-5

- Substitute in the governing equation to get:

$$Y\frac{\partial^2 X}{\partial x^2}+X\frac{\partial^2 Y}{\partial y^2}=0 \quad (5.20)$$

$$-\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{Y}\frac{d^2Y}{dy^2}=\lambda^2 \quad (5.21)$$

- The original governing partial differential equation is transformed into two ordinary differential equations. λ^2 is an arbitrary constant which can be zero, positive or negative.

2 D Conduction-6

- Consider the two equations separately. Consider first the ‘x’ direction along with the associated boundary conditions:

$$\frac{d^2X}{dx^2}+\lambda^2 X=0 \quad (5.22)$$

$$\theta(x=0)=0\Rightarrow X(0)Y(y)=0\Rightarrow X(0)=0 \quad (5.23)$$

$$\theta(x=L)=0\Rightarrow X(L)Y(y)=0\Rightarrow X(L)=0 \quad (5.24)$$

2 D Conduction-7

- Consider now the ‘y’ direction along with the associated boundary conditions:

$$\frac{d^2Y}{dy^2}-\lambda^2 Y=0 \quad (5.25)$$

$$\theta(y=0)=0\Rightarrow Y(0)X(x)=0\Rightarrow Y(0)=0 \quad (5.26)$$

$$\theta(y=W)=1\Rightarrow Y(W)X(x)=1 \quad (5.27)$$

- The constant λ^2 needs to be determined

2 D Conduction-8

- λ^2 is assumed = 0.

- Solution for (5.22):

$$
\begin{aligned}
&X = C_1 + C_2 x \\
&C_1 = 0 \because X(0) = 0 \\
&\therefore X = C_2 x \Rightarrow C_2 = 0 \\
&\Rightarrow X = 0
\end{aligned}
\tag{5.28}
$$

- X = 0 is a solution which cannot be the case since we know x dependence exists. $\lambda^2 = 0$ is therefore is not a valid option

2 D Conduction-9

- λ^2 is assumed = – ve. Solution for equ^n (5.22):

$$
\begin{aligned}
&X = C_5 e^{-\lambda x} + C_6 e^{\lambda x}; \\
&X(0) = 0 \Rightarrow C_5 + C_6 = 0 \Rightarrow C_5 = -C_6 \\
&X(L) = 0 \Rightarrow C_5 e^{-\lambda L} + C_6 e^{\lambda L} = 0 \\
&C_5 (e^{-\lambda L} - e^{\lambda L}) = 0 \Rightarrow C_5 = 0 \\
&\Rightarrow X = 0
\end{aligned}
\tag{5.29}
$$

- This is not acceptable since this again implies no dependence on X

Recap

In this class:

- An example of optimization for insulation thickness is solved. The 1D conduction is considered completed.
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- 2D conduction in cartesian coordinate system using separation of variables is started.