Module 1 : Conduction

Lecture 3 : Conduction with heat

Objectives

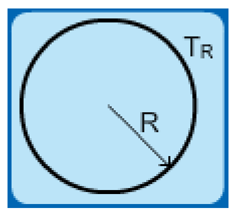
In this class:

- Conduction with heat generation is completed.
- The fin equation for extended surfaces is established and solutions obtained.
- The problem of fin size optimization for a given volume is discussed and solutions obtained.

Conduction with heat generation-1
Cylindrical geometry

- Assume uniform heat generation in a solid pipe. Outer wall temperature is fixed at T_R
- Governing equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{q'''}{k}=0 \tag{3.1}$$



- Integrate to get solution:

$$T=-q'''\frac{r^2}{4k}+C_1\ln r+C_2 \tag{3.2}$$

Conduction with heat generation-2
Cylindrical geometry

- Now apply boundary conditions and evaluate the constants:

$$T \text{ finite at } r=0 \Rightarrow C_1=0$$

$$T=T_R \Rightarrow C_2=T_R+q'''\frac{R^2}{4k}$$

- Now get the solution

$$T(r)=q'''\frac{r_0^2}{4k}\left(1-\frac{r^2}{R^2}\right)+T_R \tag{3.3}$$

Conduction with heat generation-3

Cylindrical geometry

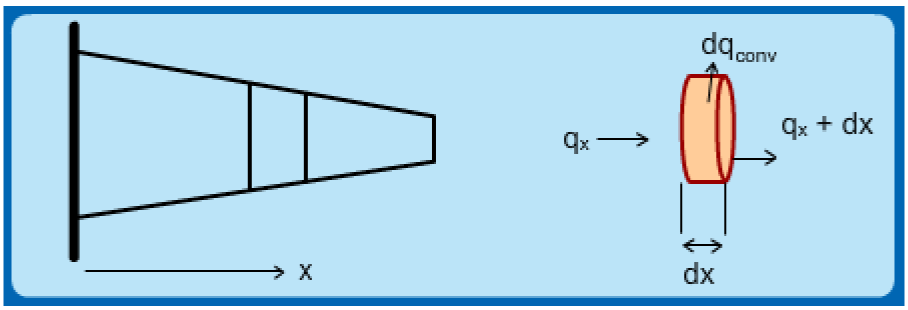
- Notice that here again the heat flux cannot be cast into the potential difference, current and resistance form.
- Therefore use the Ohm's law analogy only for steady, 1D constant thermal conductivity, no heat generation cases.

Extended surface heat transfer-1

- Extended surface heat transfer is an important engineering application. A new governing equation is often derived.
- For the derivation assume temperature varies only in the x direction. However, heat is lost in the y direction and therefore must be accounted for also. A two(or higher) dimension problem is converted to a one dimensional situation – this is why a new governing equation needs to be derived.

Extended surface heat transfer-2

- Consider a surface attached to a parent surface as shown:



- Take small slice at a given distance 'x' – need not be circular as shown

Extended surface heat transfer-3

- An energy balance gives:

$$q_x = q_{x+dx} + dq_{conv} \tag{3.4}$$

- Use the Fourier Law, Newton's law and ignoring higher order terms in the Taylor series expansion for q_{x+dx} :

$$\begin{aligned} q_x &= -kA_c \frac{dT}{dx}; q_{x+dx} = q_x + \frac{dq_x}{dx}dx \\ dq_{conv} &= hdA\left(T - T_\infty\right) \end{aligned} \tag{3.5}$$

Extended surface heat transfer-4

- Using these in the energy balance equation:

$$\frac{d}{dx}\left(-kA_c \frac{dT}{dx}\right)dx + h\left(T - T_\infty\right)dA_p = 0 \tag{3.5}$$

- Ac is the cross section area and A_p is the perimeter

- Assume k = constant to get

$$\frac{d^2T}{dx^2}+\left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx}-\left(\frac{1}{A_c}\frac{h}{k}\frac{dA_p}{dx}\right)\left(T-T_\infty\right)=0 \qquad (3.6)$$

Extended surface heat transfer-5

- Using surface area dA_p = Pdx gives:

$$\frac{d^2T}{dx^2}+\left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx}-\left(\frac{h}{k}\frac{P}{A_c}\right)\left(T-T_\infty\right)=0 \qquad (3.7)$$

- The above equation is the general equation for a fin attached to a surface. Would the general conduction equation derived earlier not be appropriate here ?

Extended surface heat transfer-6

- The general heat diffusion equation is applicable here too provided you consider the problem as a 2D/3D problem. Here the problem has been simplified to a 1D problem even though it is not a 1D problem. This is why a new governing equation had to be derived.

Fin with constant cross-section-1

- In equation (3.7) assume A_c = constant and get the equation for a constant area fin:

$$\frac{d^2T}{dx^2}-\frac{hP}{kA_c}\left(T-T_\infty\right)=0$$

- The above is a non homogenous 2^{nd}order differential equation. Use the following transformation to make it homogenous:

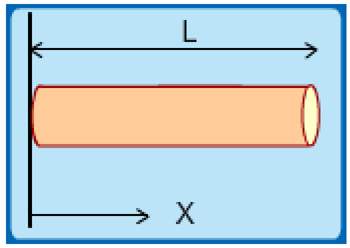
$$T\left(x\right)-T_\infty=\theta\left(x\right);m^2=\frac{hP}{kA_C}$$
$$\Rightarrow\frac{d^2\theta}{dx^2}-m^2\theta=0 \qquad (3.8)$$

Fin with constant cross-section-2

- Equation (3.8) is a linear 2^{nd}order homogenous differential equation whose solution can be written as:

$$\theta=C_1e^{mx}+C_2e^{-mx} \qquad (3.9)$$

- Now apply boundary conditions. One condition each is required at x = 0 and x = L



Fin with constant cross-section-3

- If base (x = 0) temperature is given and the tip (x = L) is assumed to have a convective boundary:

$$\left.\frac{-k\,d\theta}{dx}\right|_{x=L} = h\left(\theta\right)_L \tag{3.10}$$

$$\Rightarrow h\left(C_1 e^{mL} + C_2 e^{-mL}\right) = -km\left(C_1 e^{mL} - C_2 e^{-mL}\right);$$

$$\text{At base } \theta\left(0\right) = \theta_b \tag{3.11}$$

$$\Rightarrow \theta_b = C_1 + C_2;$$

- Now solve to get the constants C_1 and C_2

Fin with constant cross-section-4

- Assume tip is adiabatic i.e. h = 0 at tip. Convert the solution from exponential to hyperbolic functions:

$$\begin{aligned}\theta &= C_1 e^{mx} + C_2 e^{-mx} \\ &= C_3 \cosh mx + C_4 \sinh mx\end{aligned} \qquad \begin{aligned}e^{mx} + e^{-mx} &= 2\cosh mx \\ e^{mx} - e^{-mx} &= 2\sinh mx\end{aligned}$$

- Now apply boundary conditions to get:

$$x = 0, \theta = \theta_b \Rightarrow \theta_b = C_3$$

$$\frac{d\theta}{dx} = 0 = C_3 m \sinh mx + C_4 m \cosh mx$$

$$\Rightarrow C_4 = -\theta_b \tanh mL$$

Fin with constant cross-section-5

- The solution therefore becomes:

$$\theta = \theta_b \cosh mx - \theta_b \tanh mL \sinh mx$$

$$= \theta_b \left[\frac{\cosh mx \cosh mL - \sinh mL \sinh mx}{\cosh mL}\right] \tag{3.12}$$

- Heat flux from the base becomes:

$$q = -kA_c \left.\frac{d\theta}{dx}\right|_{x=0} = -kA_c \frac{\theta_b}{\cosh mL} m \sinh m(L-x)$$
$$= -kA_c m\theta_b(-\tanh mL)$$
$$= \sqrt{hPkA_c}\,\theta_b \tanh mL \tag{3.13}$$

Fin with constant cross-section-6

- Other boundary conditions may be specified and a solution for such cases can be obtained exactly in the manner followed for the adiabatic tip.
- First get the temperature profile and then the heat flux at the base
- The optimal size of the fin needs to be calculated and it will depend on certain constraints

Fin with constant cross-section-7

- Fin with specified temperatures

$$\theta = \left[\frac{\theta_L \sinh mx + \theta_b \sinh m(L-x)}{\sinh mL}\right]$$
$$q = \sqrt{hPkA_c}\,\theta_b \left[\frac{\cosh mL - \theta_L / \theta_b}{\sinh mL}\right]$$

- Infinitely long fin

$$\theta = \theta_b e^{-mx}$$
$$q = \sqrt{hPkA_c}\,\theta_b$$

Fin Optimization-1

- A common constraint is that the fin volume is fixed. Need to maximize the heat transfer from the base. The geometry of fin is:
  t = thickness, L= length, W = width
- Volume = LtW. In addition assume that the width is fixed and is constant.
- Assume the fin tip is adiabatic – this is only to keep the algebra simple

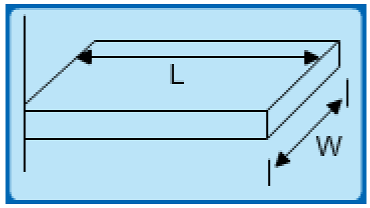
Fin Optimization-2

- Assume fin tip is adiabatic – for simplicity. Need to maximize:

$$Q = \sqrt{hPkA}\,\theta_b \tanh mL \tag{3.14}$$

- Evaluate the various parameters in the above expression assuming that W >> t

$$P = 2W + 2t \approx 2W\ ;\ A = Wt;\ L = V / Wt \tag{3.15}$$



Fin Optimization-3

- Use (3.15) in (3.14):

$$\sqrt{hPkA} = \sqrt{h(2W)kWt} = W\sqrt{2hk}\sqrt{t}$$

$$mL = \sqrt{\frac{hP}{kA}}L = \sqrt{\frac{h2W}{kWt}}\frac{V}{Wt} = \sqrt{\frac{2h}{k}}\frac{V}{W}t^{-\frac{3}{2}}$$

$$\therefore Q = \theta_b W\sqrt{2hk}\sqrt{t}\tanh\sqrt{\frac{2h}{k}}\frac{V}{W}t^{\frac{-3}{2}} \qquad (3.16)$$

- Fin thickness is the only variable in the above expression

Fin Optimization-4

- Equation is rewritten as (3.16):

$$Q = a\sqrt{t}\tanh bt^{-\frac{3}{2}} \qquad (3.17)$$

$$\textbf{where a} = \theta_b W\sqrt{2hk};\ \mathbf{b} = \sqrt{\frac{2h}{k}}\frac{V}{W}$$

- Notice that in the above equation ‘t’ is the only unknown. We need to maximize heat removed by the fin ‘Q’.

Fin Optimization-5

- Differentiate Q and equate to zero: $\frac{dQ}{dt} = 0$

$$a\tanh(bt^{-\frac{3}{2}})\frac{1}{2}t^{-\frac{1}{2}} + a\sqrt{t}\sec h^2 bt^{\frac{-3}{2}}\left(-\frac{3}{2}bt^{\frac{-5}{2}}\right) = 0$$

$$\frac{\tanh bt^{\frac{-3}{2}}}{2} + \left[1 - \tanh^2\left(bt^{\frac{-3}{2}}\right)\right]\frac{-3}{2}\left(bt^{\frac{-3}{2}}\right) = 0 \qquad (3.18)$$

$$\text{let } bt^{\frac{-3}{2}} = A$$

$$\tanh A + \left[1 - \tanh^2 A\right](-3A) = 0$$

$$\text{solve equation to get } A \approx 1.42$$

Fin Optimization-6

- This implies that

$$\sqrt{\frac{2h}{k}}\frac{V}{W}t^{\frac{-3}{2}} = 1.42 \Rightarrow t \approx \left(\frac{V}{W}\right)^{\frac{2}{3}}\left(\frac{h}{k}\right)^{\frac{1}{3}}; \qquad (3.19)$$

$$\text{since volume is fixed } L = \left(\frac{V}{W}\right)^{\frac{1}{3}}\left(\frac{h}{k}\right)^{\frac{-1}{3}} \qquad (3.20)$$

- When the width 'W' and the volume 'V' are fixed, the thickness and length are given above for maximum heat transfer from the fin.

Recap

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- Conduction with heat generation is completed.
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