Module 1 : Conduction

Lecture 4 : Fin Optimization (contd.). Underground Electric cable problem

Objectives

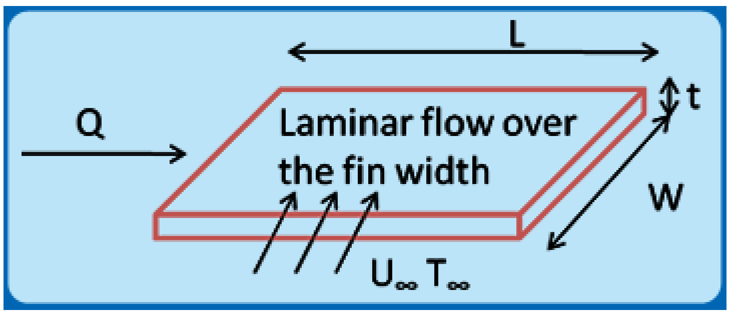
In this class:

- Fin size optimization is continued.
- Principle of superposition is introduced and the 1D underground electric cable problem solved.
- The solution for the problem of multiple cables is a straightforward extension and is left as an exercise.

Fin Optimization-7

(known fin thickness)

- Now consider another situation where the fin thickness and the volume are specified. Assume flow over fin is laminar



- When the width 'w' is a variable the average heat transfer coefficient will also be a variable.

Fin Optimization-8

(known fin thickness)

- Assume flow over the fin is modeled as flow over flat surface. Use the standard relationship for calculating average the heat transfer coefficients over flat surface:

$$Nu = 0.664 \Pr^{\frac{1}{3}} \mathrm{Re}^{\frac{1}{2}} \qquad (\Pr \geq 0.5) \tag{4.1}$$

$$h = \frac{k_f}{W} 0.664 \Pr^{\frac{1}{3}} \left(\frac{u_\infty W}{\nu} \right)^{0.5} \tag{4.2}$$

Fin Optimization-9

(known fin thickness)

- Get the heat transferred from the base from the previous example (equation 3.16) and then substitute the appropriate variables

$$Q = W \sqrt{2hkt}\theta_b \tanh \sqrt{\frac{2h}{k} \frac{V}{W}} t^{\frac{-3}{2}} \tag{4.3}$$

$$Q = W \sqrt{2 \frac{k_f}{W} 0.664 \Pr^{\frac{1}{3}} \left(\frac{u_\infty W}{\nu} \right)^{\frac{1}{2}} kt\theta_b} \tanh \sqrt{\frac{2}{k} \frac{k_f}{W} 0.664 \Pr^{\frac{1}{3}} \left(\frac{u_\infty W}{\nu} \right)^{\frac{1}{2}}} \frac{V}{W} t^{\frac{-3}{2}} \tag{4.4}$$

- Note k_f and k are the thermal conductivity of the fluid and fin respectively

Fin Optimization-10
(known fin thickness)
Rewrite the equation in a compact form as

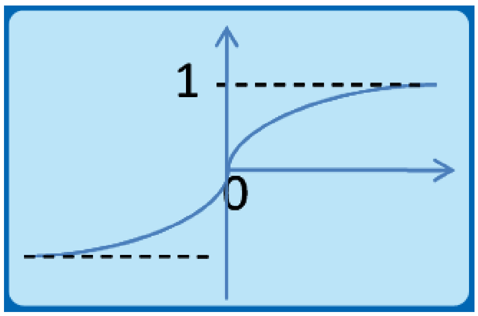
$$Q = \theta_b K (W)^{\frac{3}{4}} \tanh K_1 W^{\frac{-5}{4}} \tag{4.5}$$

- where

$$K = \sqrt{2k_f\, 0.664 \Pr^{\frac{1}{3}} \left(\frac{u_\infty}{\nu}\right)^{\frac{1}{2}} kt} \quad , \quad K_1 = \sqrt{2\frac{k_f}{k}\, 0.664 \Pr^{\frac{1}{3}} \left(\frac{u_\infty}{\nu}\right)^{\frac{1}{2}} Vt^{\frac{-3}{4}}} \tag{4.6}$$

Fin Optimization-11
(known fin thickness)

- The functional variation of the tanh(x) is:



- In equation (4.5) Q → 0 when W → 0 since tanh(x) tends to a constant value for large 'x'- Note that 'W' is in the denominator in the tanh term . Q also decreases as W increases since the tanh function approaches zero.

Fin Optimization-12
(known fin thickness)

- Q → 0 when W → 0 - this appears reasonable physically
- Q → 0 when W → ∞ – appears strange.
  However, since the volume is constant, L → 0 when W → ∞ since t = fixed, therefore again making Q small.
- The optimum 'W' lies in between these two extrema, which can be numerically obtained.

Fin Optimization-13
(known fin thickness)

- Need to solve the equation (4.4). Rewrite the equation in the following manner so that the result can be easily used

$$Q = \theta_b k t m^{\frac{3}{4}} C^{\frac{1}{2}} \left(\frac{W}{m\,t}\right)^{\frac{3}{4}} \tanh\left[\left(\frac{W}{m\,t}\right)^{\frac{-5}{4}}\right] \tag{4.7}$$

$$\text{Where} \quad m = \left[\frac{V}{t^3} C^{\frac{1}{2}}\right]^{\frac{4}{5}} \quad , \quad C = 1.328 \quad \Pr^{\frac{1}{3}} \frac{k_f}{k}\left(\frac{u_\infty t}{\nu}\right)$$

$$\frac{Q}{\theta_b k t m^{\frac{3}{4}} C^{\frac{1}{2}}} = \left(\frac{W}{m\,t}\right)^{\frac{3}{4}} \tanh\left(\frac{W}{m\,t}\right)^{\frac{-5}{4}} \tag{4.8}$$

Fin Optimization-14

(known fin thickness)

- The equation (4.8) has only one variable and can be solved numerically easily to give:

$$\frac{W}{mt} \approx 1.07$$

- Equation (4.5) can be directly solved also to get 'W' but the above non dimensional form is convenient to use since the equation need not be repeatedly solved for the optimal 'W'.

Superposition

- Governing equation is linear differential equation. A linear combination of the solutions is also a solution to the governing equation. Therefore if solutions to some situations are known, solutions to others can be obtained by using superposition.
- However, boundary conditions pose a problem and need to be carefully assessed to evaluate if superposition can be really used.

Underground electric cable problem-1

- Underground electric cable carrying current generates heat. Need to ensure that the insulation is below a certain temperature to prevent damage to the insulation. Need to understand single cable and the influence of neighbors. Superposition will be attempted.
- Assume cable is located at a distance 'L' from the ground surface and has heat generation of q' per unit length

Underground electric cable problem-2

- Governing Equation:

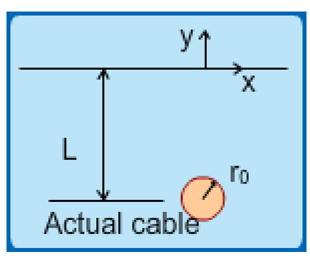
$$\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) = 0 \qquad (4.9)$$

- Boundary Conditions

$T = T_0$ at $r = r_0$

$T = T_{ground}$ at $y = L$ (4.10)

- Appears to be mixture of cartesian and radial. Assume $L >> r_0$. k in (4.9) is for ground.



Underground electric cable problem-3

- Solution temperature is obtained by integrating equation (4.9):

$$\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right) = 0 \Rightarrow \left(rk\frac{dT}{dr}\right) = C$$

- Heat generated within the cable is dissipated to the ground. We need not solve the temperature in the cable material.

Underground electric cable problem-4

- Therefore:

$$\left(2\pi L r k \frac{dT}{dr}\right) = 2\pi L C = q$$

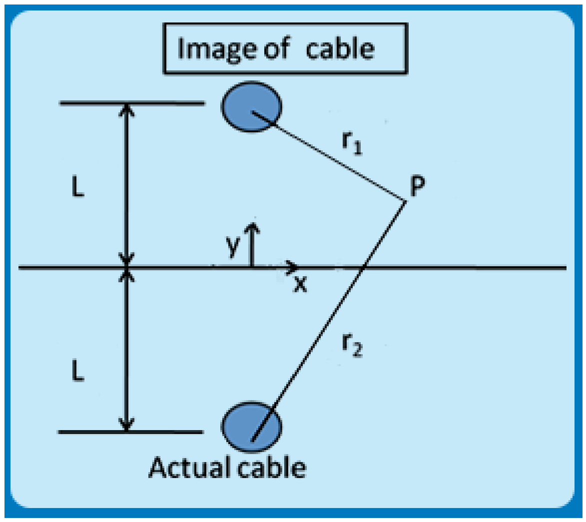
$$\Rightarrow C = \frac{q}{2\pi L} = \frac{q'}{2\pi}$$

- Integrating the above equation gives the required temperature profile:

$$T = \frac{q'}{2\pi k}\ln(r) + D \quad (4.11)$$

Underground electric cable problem-5

- Assume an image for the cable exists and superposition works for any given point in the domain. Consider a point ‘P’ at distance r_1 from image and r_2 from object



Underground electric cable problem-6

- Let solutions for temperature for image and actual cable be T_1 and T_2 respectively. Let q'_{image} be the heat generation in the cable image and a consistent value for this needs to be chosen

- T_1+T_2 is a solution of the governing equation:

$$T = T_1 + T_2 = \frac{q'_{image}}{2\pi k}\ln(r_1) + \frac{q'_{cable}}{2\pi k}\ln(r_2) + E \quad (4.12)$$

- The constant can be obtained using the boundary conditions.

Underground electric cable problem-7

- At y = L, i.e. $r_1 = r_2$, $T = T_{ground}$ Therefore:

$$T_{ground} = T_1 + T_2 = \frac{q'_{image}}{2\pi k}\ln(r_1) + \frac{q'_{cable}}{2\pi k}\ln(r_2) + E$$

- This condition can be satisfied only if the first two terms in the above equation are equal and opposite.
- Choose therefore

$$q'_{image} = -q'_{cable}$$

Underground electric cable problem-8

- The solution therefore becomes:

$$T = \frac{q'_{cable}}{2\pi k} \ln\left(\frac{r_2}{r_1}\right) + T_{ground}$$

- The next boundary condition is (from equation 4.10) $r_2 = r_{cable}$ $T = T_0$
- If L is assumed much larger than the radius of the cable r_{cable} then $r_1 = 2L$ when $r_2 = r_{cable}$

Underground electric cable problem-9

- Therefore,

$$T_0 = \frac{q'_{cable}}{2\pi k} \ln\left(\frac{r_2}{2L}\right) + T_{ground} \tag{4.13}$$

- Above expression has a probleml T_0 is not positive for positive r_2 as it is in reality.
- Get around this problem by assuming q'_{cable} to be negative. This means that q'_{image} is positive. Mathematically this is fine.

Underground electric cable problem-10

- This implies actual problem is replaced by a sink under the ground and source above !!
- Now the temperatures are consistent and the final solution is:

$$T_0 = \frac{q'_{actual}}{2\pi k} \ln\left(\frac{2L}{r_2}\right) + T_{ground}$$

$$T = \frac{q'_{actual}}{2\pi k} \ln\left(\frac{r_1}{r_2}\right) + T_{ground} \tag{4.14}$$

Underground electric cable problem-11

- Multiple cables generating heat:

$$T - T_{ground} = \sum_{i=1}^{N}\left(\frac{q'_{actual}}{2\pi k} \ln\left(\frac{r_{i1}}{r_{i2}}\right)\right) \tag{4.15}$$

- Can use this equation to obtain solution for temperature distribution in ground due to multiple cables.

Recap

In this class:

- Fin size optimization is continued.
- Principle of superposition is introduced and the 1D underground electric cable problem solved.
- The solution for the problem of multiple cables is a straightforward extension and is left as an exercise.