Module 1 : Conduction

Lecture 7 : 1D unsteady conduction (contd.)

Objectives

In this class:

- The solution to the 1D transient problem is completed. A general procedure obtaining the orthogonal functions required for completion of the solution is discussed.
- A 'fuzzy' argument is developed for justifying the the Bi < 0.1 'rule of the thumb' for using the lumped model.

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- Simplification of the equ^n (6.32) therefore yields:

$$\lambda \tan \lambda = Bi \Rightarrow \cot \lambda = \frac{\lambda}{Bi} \tag{7.1}$$

- Need to find out the roots of this equation
- Graphically plot the following curves:

$$y = \cot \lambda ; y = \frac{\lambda}{Bi}$$

- Intersection of the curves gives the roots of the equation (7.1)

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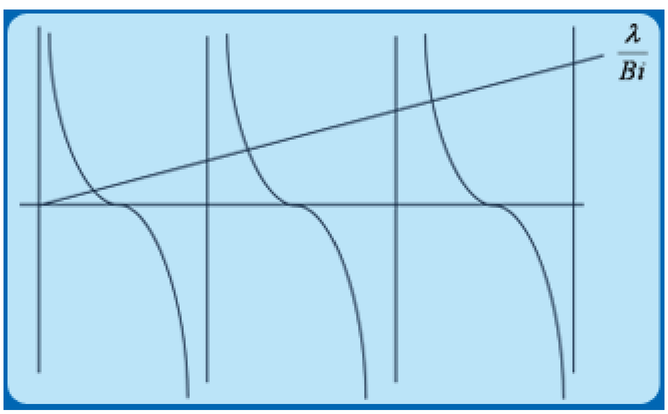
- The graphical procedure explained on the previous slide is shown below:



e.g. $Bi = 0.1$

$\lambda_1 = 0.31$

$\lambda_2 = 3.17$

$\lambda_3 = 6.3$

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- Can directly use a root finding routine to solve equ^n (7.1) without the above graphical method.

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- The λs have been obtained but still one constant 'A' is be determined and a condition(equ^n (6.23)) available. Using this gives:

$$1 = Ae^{-\lambda^2 t^*} \cos \lambda x^* \Big|_{t^*=0} \tag{7.2}$$

- Just like in the previous 2D steady conduction problem a single λ and A will not satisfy the condition and therefore attempt a series solution.

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- The solution is therefore assumed as:

$$\theta^* = \sum A_n e^{-\lambda_n^2 t^*} \cos \lambda_n x^* \tag{7.3}$$

- Use the initial condition to get:

$$1 = \sum_{1}^{\infty} A_n \cos \lambda_n x^* \tag{7.4}$$

- Recall that in the previous case a trignometric identity given by equ^n (6.12) was used to obtain the solution. If λ_ns were integer constants the identity would be useful here.

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- However λ_ns are fractions (Bi = 0.1, λ_1 = 0.31, λ_2 = 3.17 etc.). Therefore try to see if the methodology used earlier will work for this situation also.
- The idea is to see if the series solution can be manipulated so that the constants can be easily evaluated i.e. many terms become zero and only a single term remains that can be used to evaluate the constant. We now look at a generalized procedure for this.

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- Look again at the original differential equ^n and boundary conditions (6.27), (6.29), (6.31):

$$F'' + \lambda^2 F = 0; \tag{7.5}$$

$$F'\Big|_{x^*=1} = -BiF\Big|_{x^*=1};\ F'\Big|_{x^*=0} = 0. \tag{7.6}$$

- Let F_m and F_n be solutions corresponding to the solutions λ_m , λ_n which will satisfy the above equations. Therefore:

$$F_m'' + \lambda^2{}_m F_m = 0 \tag{7.7}$$

$$F_n'' + \lambda_n^2 F_n = 0 \tag{7.8}$$

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- Multiply equ^n(7.5) with F_n and equ^n(7.6) with F_m :

$$F_m'' + \lambda^2{}_m F_m = 0 \quad \text{x} \quad F_n \tag{7.9}$$

$$F_n'' + \lambda_n^2 F_n = 0 \quad \text{x} \quad F_m \tag{7.10}$$

- Subtract equ^n (7.10) from (7.9) to get:

$$F_m'' F_n - F_n'' F_m + (\lambda^2{}_m - \lambda^2{}_n) F_m F_n = 0$$

- Rewrite the above equ^n to get:

$$\frac{d}{dx}\left(F_m' F_n - F_n' F_m\right) + \left(\lambda^2{}_m - \lambda^2{}_n\right) F_m F_n = 0 \tag{7.11}$$

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- Now integrate the equ^n (7.9) between 0 and 1 to get:

$$F_m{}'F_n - F_n{}'F_m\Big|_0^1 + \left(\lambda^2{}_m - \lambda^2{}_n\right)\int_0^1 F_m F_n dx^* = 0$$

$$\left(F_m{}'F_n - F_n{}'F_m\right)\Big|_1 - \left(F_m{}'F_n - F_n{}'F_m\right)\Big|_0 + \left(\lambda^2{}_m - \lambda^2{}_n\right)\int_0^1 F_m F_n dx^* = 0 \tag{7.12}$$

- Since F_m and F_n are solutions to the equations (7.5) and (7.6):

$$\begin{aligned} F_m{}'\Big|_{x^*=1} &= -BiF_m\Big|_{x^*=1} ; F_m{}'\Big|_{x^*=0} = 0 \\ F_n{}'\Big|_{x^*=1} &= -BiF_n\Big|_{x^*=1} ; F_n{}'\Big|_{x^*=0} = 0 \end{aligned} \tag{7.13}$$

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- Using equ^n (7.13) in equ^n (7.12) gives:

$$\begin{aligned} &\left(-BiF_mF_n + BiF_nF_m\right)_1 - (0-0) = (\lambda^2{}_n - \lambda^2{}_m)\int_0^1 F_m F_n dx^* \\ &\Rightarrow 0 = (\lambda^2{}_n - \lambda^2{}_m)\int_0^1 F_m F_n dx^* \end{aligned} \tag{7.14}$$

- For $\lambda_m = \lambda_n$ the integral in the equ^n (7.14) is of the 0/0 form and approaches a finite value whereas for $\lambda_m \neq \lambda_n$ the integral in equ^n (7.14) is zero i.e.: $\int_0^1 F_m F_n dx^* = 0 \quad \text{for } \lambda_m \neq \lambda_n$

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- Recall that for the present situation F_m and F_n are $Cos(\lambda_m x)$ and $Cos(\lambda_n x)$ respectively.

- Now return to the solution of the unsteady problem: (equ^n (7.4))

$$1 = \sum_1^\infty A_n \cos \lambda_n x^*$$

- Use the property of the functions just derived to proceed to evaluate the unknown constants A_n.

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- Multiply both sides with $Cos(\lambda_n x)$ and integrate:

$$\int_0^1 1.\cos\lambda_m x^* dx^* = \sum_1^\infty \int_0^1 A_n \cos\lambda_n x^* \cos\lambda_m x^* dx^*$$

$$\Rightarrow \frac{\sin\lambda_m}{\lambda_m} = A_m \int_0^1 \cos^2\lambda_m x^* = \frac{A_m}{2}\int_0^1 \left[1 + Cos\,2\lambda_m x^*\right] dx^*$$

$$= \frac{A_m}{2}\left[x^* + \frac{\sin 2\lambda_m x^*}{2\lambda_m}\right]_0^1$$

$$\Rightarrow A_m = \frac{2\sin\lambda_m}{\lambda_m + \sin\lambda_m \cos\lambda_m} \tag{7.15}$$

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- Final solution is (equ^{n} (7.3) earlier obtained) :

$$\theta^* = \sum_{1}^{\infty} A_m e^{-\lambda^2 t^*} \cos \lambda x^* \qquad (7.3)$$

- Where:

$$\theta^* = \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty}; x^* = \frac{x}{L}; t^* = \frac{t}{t_0}$$

$$A_m = \frac{2 \sin \lambda_m}{\lambda_m + \sin \lambda_m \cos \lambda_m}$$

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- Note that once again the summation solution was obtained just like in the 2D case with two space dimensions. Here we had to look at the differential equation to obtain a methodology for determining the constants of the infinite series since the solution was not obtainable from trigonometry. This is a generalized methodology and would have worked even for the earlier case.

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- For t* > 0.2 the full series can be approximated by the first term with little error. Therefore:

$$\theta^* = A_1 e^{-\lambda^2 t^*} \cos \lambda x^*$$

$$\theta^* = \theta_0^* \cos \lambda x^*$$

- Θ_0^* is the mid plane temperature and is a function of time only.

- At x*=1, $\Theta^* = \Theta_0^* \cos \lambda$ which is the temperature variation at the boundary of the slab.

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- Let the temperature of the end of the slab be always 95 % of the centerline value. Then:

$$\cos \lambda = 0.95 \Rightarrow \lambda = 0.32$$

- Using equ^{n} (7.1):

$$\cot \lambda = \lambda / Bi \Rightarrow Bi = 0.1$$

- Thus a Biot number of 0.1 results in a temperature difference of 5 % between the center and one end of the slab.

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- If this 5 % variation is ignored then the temperature between the center and the boundary can be assumed to be the same which is the lumped approach (zero dimension). Therefore if the Bi < 0.1 the lumped approach can be used and this approximation yields quick and reasonably accurate results. We have not proved anything only demonstrated the ‘thumb rule’.

Recap

In this class:

- The solution to the 1D transient problem is completed. A general procedure obtaining the orthogonal functions required for completion of the solution is discussed.
- A ‘fuzzy’ argument is developed for justifying the the Bi < 0.1 ‘rule of the thumb’ for using the lumped model.