Module 1 : Conduction

Lecture 9 : Duhamel's theorem (contd.) Solids in motion

Objectives

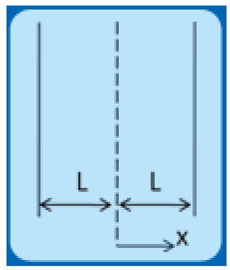
In this class:

- An example for the use of the Duhamel's theorem is discussed.
- Solution of the equations using separation of variables has been used in earlier classes. The use of the Laplace transform method is discussed for an example problem. Other methodologies exist but are not discussed further in this course.
- A situation where a solid is moving is discussed though not completely.

Duhamel's Theorem-21

- Let us consider an example. Slab with temperature varying linearly with time on both edges.
- Use of the Duhamel's theorem is required to obtain the solution

$$[T]_{x=-L} = T_\infty + kt$$



$$[T]_{x=L} = T_\infty + kt$$

Duhamel's Theorem-21

- The governing equation system for the constant temperature at x = $\pm$ L case which forms the basic building block for the use of the Duhamel's theorem are:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{9.1}$$

$$[T]_{x=L} = T_\infty; \tag{9.2}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0; \tag{9.3}$$

$$T(x,0) = T_i \tag{9.4}$$

Duhamel's Theorem-22

- Solution using earlier methodology gives:

$$\frac{T-T_\infty}{T_i-T_\infty} = \theta = 2\sum_1^\infty \frac{(-1)^n}{\lambda_n} e^{-\lambda_n^2 t*} \cos \lambda_n x* \qquad \lambda_n = (2n-1)\frac{\pi}{2} \tag{9.5}$$

- To use the Duhamel's theorem, need to construct the function Ψ. Recall that it is the response of the system to a unit step input. In addition, at t $\leq$ 0, disturbance = 0 and the output is zero.

Duhamel's Theorem-23

- Construct therefore the following function

$$1-\theta = \frac{T-T_i}{T_\infty - T_i}$$

- Notice that the above function is the ψ that we are looking for, provided $(T_\infty - T_i) = 1$. It is zero for $t \le 0$ and the disturbance is a unit step rise in the surrounding temperature over the initial value.

- Disturbance is zero for $t \le 0$ implies that the disturbance should be of the form $(T_\infty - T_i)$

Duhamel's Theorem-24

- The disturbance is given to be $[T]_{x=L} = T_\infty + kt$. The disturbance in the form that can be used in the Duhamel's theorem is:

$$D(t) = [T]_{x=L} - T_i = \underbrace{T_\infty - T_i}_{D(0)} + kt$$

- The Duhamel's theorem can now be directly used in the form:

$$\varphi(r,t_{final}) = D(0)\,\psi(r,t_{final}) + \int_0^{t_{final}} \psi(r,t_{final}-s)\frac{dD}{ds}ds$$

Duhamel's Theorem-25

- Note that the t_{final} in the above equation is the time at which the solution is desired. Note that there is no loss of generality here since t_{final} is an arbitrary time and can be any time value.
- Use of the Duhamel's theorem gives:

$$\varphi(r,t^*) = D(0)\psi(r,t^*_{final}) + k\left[\int_0^{t^*_{final}} 1 - 2\sum_1^\infty \frac{(-1)^n}{\lambda_n} e^{-\lambda_n^2(t^*_{final}-s)} \cos\lambda_n x^* ds\right]$$

Duhamel's Theorem-25

- Simplifying the expression further gives:

$$\begin{aligned}
\varphi(r,t) &= D(0)\,\psi(r,t^*_{final}) + k\left[t^*_{final} - 2\sum_1^\infty \frac{(-1)^n}{\lambda_n}\cos\lambda_n x^* \int_0^{t_{final}} e^{-\lambda_n^2(t^*_{final}-s)}ds\right] \\
&= D(0)\,\psi(r,t^*_{final}) + k\left[t^*_{final} - 2\sum_1^\infty \frac{(-1)^n}{\lambda_n}\cos\lambda_n x^* \left[\frac{e^{-\lambda_n^2(t^*_{final}-s)}}{\lambda_n^2}\right]_0^{t_{final}}\right] \\
&= D(0)\,\psi(r,t^*_{final}) + k\left[t^*_{final} - 2\sum_1^\infty \frac{(-1)^n}{\lambda_n}\cos\lambda_n x^* \left[\frac{1-e^{-\lambda_n^2(t^*_{final})}}{\lambda_n^2}\right]\right]
\end{aligned} \tag{9.6}$$

Laplace Transform Methods-1

- Another solution methodology for differential equations is the use of Laplace transforms. The Laplace transform of a function is obtained by multiplying the function by e^{-st} and integrating between 0 and ∞. The inverse transform of the Laplace transform is the function itself. Mathematically:

$$L(f(t)) = \int_0^\infty e^{-st} f(t)dt = F(s) \quad (9.7)$$

$$L^{-1}(F(s)) = f(t) \quad (9.8)$$

Laplace Transform Methods-2

- Some useful Laplace transforms, that are relevant to the solution procedure to be demonstrated are as follows:

$$L(k) = \int_0^\infty e^{-st} k dt = \frac{k}{s} \quad (9.9)$$

$$L(c_1 f(t) + c_2 g(t)) = c_1 F(s) + c_2 G(s) \quad (9.10)$$

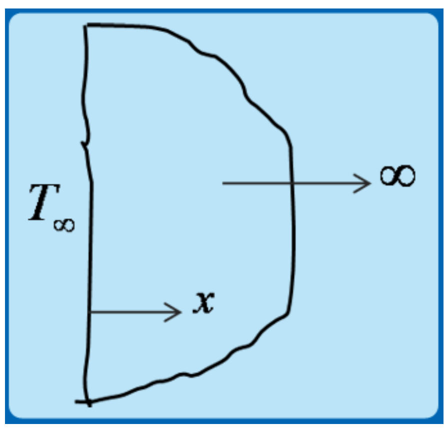
$$L\left(\frac{df(t)}{dt}\right) = sL(f(t)) - f(0^+) \quad (9.11)$$

$$L\left(\frac{\partial^n f}{\partial x^n}\right) = \frac{d^n}{dx^n} L(f(t)) \quad (9.12)$$

- All the above are easily obtainable from the basic definition of the Laplace transform.

Laplace Transform Methods -3
Semi-infinite solid

- Consider a solid which extends to a very large distance in the x direction. Initially the solid temperature is T_0 everywhere and suddenly the face at x = 0 is raised to a temperature T_∞



Laplace Transform Methods -4
Semi-infinite solid

- The governing equation and corresponding boundary conditions are:

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \quad (9.13)$$

$$\theta(x, 0) = \theta_0 \quad (9.14)$$

$$\theta(0, t) = 0 \quad (9.15)$$

$$\theta(\infty, t) = \theta_0 \quad (9.16)$$

$$\text{where } \theta = T - T_\infty ; \frac{k}{\rho c} = \alpha \quad (9.17)$$

Laplace Transform Methods -5
Semi-infinite solid

- Take Laplace transform on both sides and use equ^{n}s (9.11) and (9.12), the equ^n (9.13), (9.15), (9.16) become:

$$\frac{d^2 v}{dx^2} - \frac{sv}{\alpha} = \frac{-\theta_0}{\alpha} \quad (9.18)$$

$$v\big|_{0,s} = L(\theta)\big|_{0,s} = 0 \quad (9.19)$$

$$v\big|_{\infty,s} = L(\theta)\big|_{\infty,s} = \frac{\theta_0}{s} \quad (9.20)$$

$$where\ v = L(\theta) \quad (9.21)$$

Laplace Transform Methods -6
Semi-infinite solid

- Note that the above equation is an ordinary differential equation. Note that now the initial condition is part of the governing equation
- The solution to equ^n (9.18) is

$$v = Ae^{-qx} + Be^{qx} + \boxed{\frac{\theta_0}{s}} \leftarrow \text{Part integral}$$

$$where\ \frac{s}{\alpha} = q^2$$

Laplace Transform Methods -7
Semi-infinite solid

- Now look at the boundary conditions – equ^{n}s (9.19) and (9.20):

$$v\big|_{\infty,s} = \frac{\theta_0}{s} \Rightarrow B = 0$$

$$v\big|_{0,s} = 0 \Rightarrow A + B = -\frac{\theta_0}{s} \Rightarrow A = -\frac{\theta_0}{s}$$

- Therefore the solution becomes:

$$v = -\frac{\theta_0}{s} e^{-qx} + \frac{\theta_0}{s} \Rightarrow \frac{v}{\theta_0} = -\frac{e^{-qx}}{s} + \frac{1}{s} \quad (9.22)$$

Laplace Transform Methods -8
Semi-infinite solid

- Note that v = L(θ) and interest is in θ. The inverse Laplace transform, L^{-1} , is therefore required. The following inverse transforms obtained from a standard table are useful:

$$L^{-1}\left(\frac{1}{s}\right) = 1, \qquad L^{-1}\left(\frac{e^{-qx}}{s}\right) = erfc\,\frac{x}{2\sqrt{\alpha t}}$$

- Use in the solution equ^n (9.22) to get:

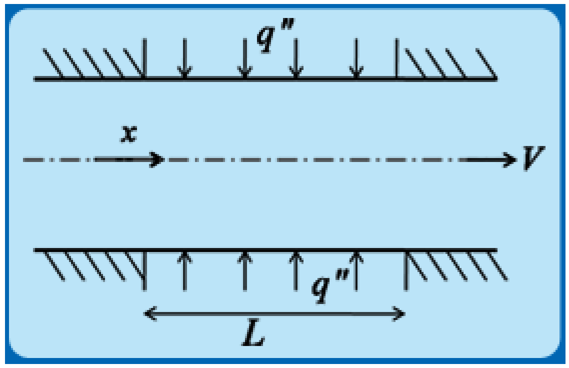
$$\theta = \frac{T - T_\infty}{T_0 - T_\infty} = 1 - erfc\,\frac{x}{2\sqrt{\alpha t}} \quad = erf\,\frac{x}{2\sqrt{\alpha t}} \quad (9.23)$$

Laplace Transform Methods -9
Semi-infinite solid

- The semi-infinite slab problem was solved using the Laplace transform technique.
- The problem can be solved by other methodologies also.
- Analytical solutions are available for a very small set of geometries and boundary conditions. For complex geometries numerical methods are required which we will not employ here

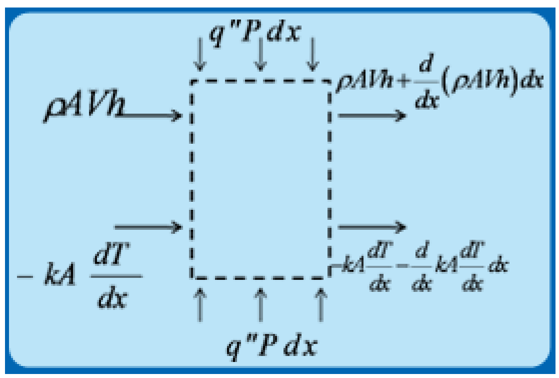
Solids in Motion-1

- All examples considered so far are for conduction in stationary media
- Consider the situation where a sheet moves with uniform velocity from left to right while experiencing a uniform flux q'' from x = 0 to L



Solids in Motion-2

- Consider a control volume and perform an energy balance:



A = area of cross-section
P = Perimeter
c = specific heat capacity
h = enthalpy = cT

Solids in Motion-3

- Energy balance yields:

$$\frac{d}{dx}\left[kA\frac{dT}{dx}\right]dx - \frac{\rho c V}{k}\frac{dT}{dx}dx + \frac{q''Pdx}{k} = 0$$

- Assuming the properties of the sheet to be constant, area of cross section of the sheet to be uniform and thickness of the sheet negligible, above equation becomes:

$$\frac{d^2T}{dx^2} - \frac{\rho c V}{kA}\frac{dT}{dx}dx + \frac{q''P}{kA} = 0 \qquad (9.24)$$

Solids in Motion-4

- Since only the central portion is heated and the rest is insulated the equations for each region are written separately. The three portions are assumed to have temperatures T_1 , T_2 and T_3. The equations now become:

$$\frac{d^2T_1}{dx^2} - \frac{\rho c V}{k}\frac{dT_1}{dx} = 0; -\infty < x \le 0 \qquad (9.25)$$

$$\frac{d^2T_2}{dx^2} - \frac{\rho c V}{k}\frac{dT_2}{dx} + \frac{q'''}{k} = 0; 0 \le x \le L \qquad (9.26)$$

$$\frac{d^2T_3}{dx^2} - \frac{\rho c V}{k}\frac{dT_3}{dx} = 0; L \le x < \infty \qquad (9.27)$$

Solids in Motion-5

- There are three second order differential equations and therefore need six conditions for obtaining a solution.
- Temperature at a large distance from the beginning is the initial temperature:

$$T_1\big|_{x=-\infty} = T_0 \qquad (9.28)$$

- There are no other obvious 'boundary conditions'

Solids in Motion-6

- Additional conditions can be used to determine the constants in the solution to the differential equation. The temperature must be continuous and heat transferred must be continuous at the location x = 0 and x = L

$$T_1(0) = T_2(0); \qquad (9.29)$$

$$\left.\frac{dT_1}{dx}\right|_{x=0} = \left.\frac{dT_2}{dx}\right|_{x=0} \qquad (9.30)$$

$$T_2(L) = T_3(L); \qquad (9.31)$$

$$\left.\frac{dT_2}{dx}\right|_{x=L} = \left.\frac{dT_3}{dx}\right|_{x=L} \qquad (9.32)$$

Solids in Motion-7

- So far five conditions have been obtained and one more is required. The last condition is not obvious. The following condition is useful:

$$T_1\big|_{x=\infty} = Finite \qquad (9.33)$$

- This condition may or may not work but that's the only option available here. In this case it works but the detailed solution for the differential equation and boundary conditions is not presented here and can be obtained without much difficulty.

Recap

In this class:

- An example for the use of the Duhamel's theorem is discussed
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