**Module 2 : Convection**

**Lecture 11 : Derivation of conservation of momentum (contd.)**

**Objectives**

**In this class:**

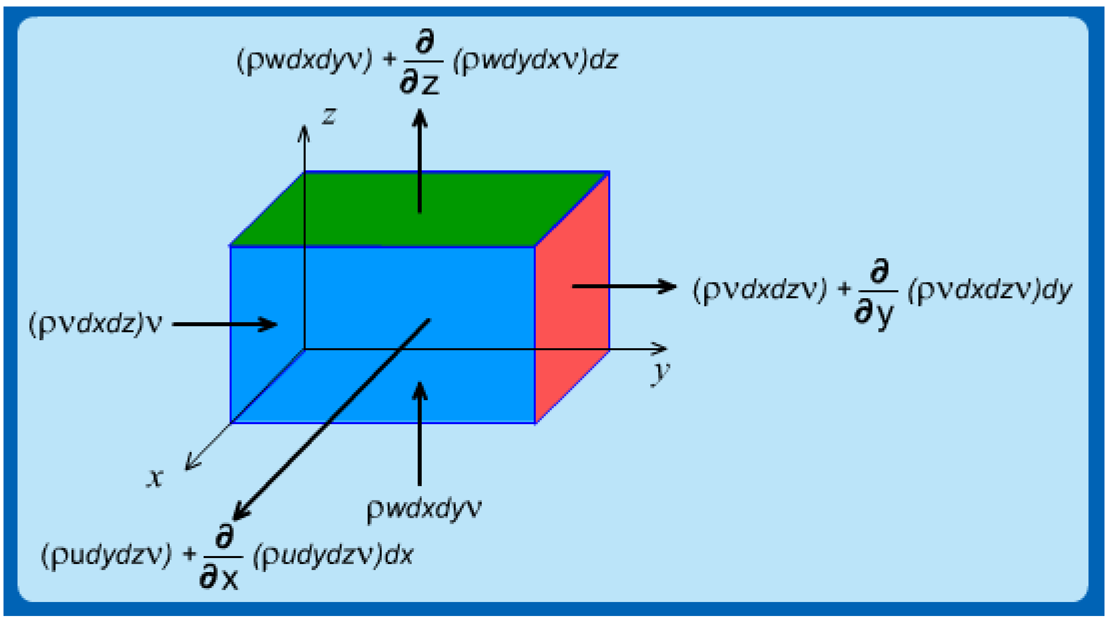
- Derivation of conservation of momentum equation is completed.

**Conservation of Momentum**
**Derivation-8**

- Now consider the influx of momentum due to mass entering the control volume. Let velocity be u, v, w in the x, y and z directions.
- Momentum entering the control volume in the 'y' direction due to mass entering the 'y = 0', 'z = 0' and 'x = 0' faces is ($\rho vdxdz)v$, $(\rho wdydxv)$ and $(\rho udydzv)$
- Momentum leaving due to mass leaving the control volume at y = dy, z = dz and x = dx is obtained from the Taylor series expansion with only the leading term retained

**Conservation of Momentum**
**Derivation-9**

- All the momentum terms in 'y' direction due to mass entering or leaving the control volume are given on the figure below; term on x = 0 face omitted for clarity




**Conservation of Momentum**
**Derivation-10**

- Net influx of momentum in 'y' direction due to mass influx

$$
\begin{aligned}
&\rho vvdxdz + \rho wvdxdy + \rho uvdydz \\
&-\left(\rho vv + \frac{\partial}{\partial y}\rho vv + \rho wv + \frac{\partial}{\partial z}\rho wv + \rho uv + \frac{\partial}{\partial x}\rho uv\right)dxdydz \\
&= -\left(\frac{\partial}{\partial y}\rho vv + \frac{\partial}{\partial z}\rho wv + \frac{\partial}{\partial x}\rho uv\right)dxdydz \qquad (11.1)
\end{aligned}
$$

- In addition to surface forces due to the stresses, assume body forces are present.

**Conservation of Momentum**
**Derivation-11**

- Assume body forces are present. Body force vector (per unit mass) is denoted by:

$$\left(X_x\hat{i}+X_y\hat{j}+X_z\hat{k}\right) \qquad (11.2)$$

- Net influx of momentum into control volume is due to:
  - mass entering (equ$^n$ 11.1)
  - force on the control volume faces (equ$^n$ 10.8)
  - Body force (equ$^n$ 11.2)
- Net accumulation rate is $\frac{\partial}{\partial t}\rho v dxdydz$

**Conservation of Momentum**
**Derivation-12**

- The overall momentum balance equation therefore becomes

$$-\left(\frac{\partial}{\partial y}\rho v^2+\frac{\partial}{\partial z}\rho wv+\frac{\partial}{\partial x}\rho uv\right)dxdydz+\left[\frac{\partial}{\partial y}\tau_{yy}+\frac{\partial}{\partial z}\tau_{zy}+\frac{\partial}{\partial x}\tau_{xy}\right]dxdydz$$
$$-\frac{\partial}{\partial t}\rho v dxdydz+\rho X_y dxdydz=0 \qquad (11.3)$$

- Stresses are hard to measure therefore convert to a more useful form using a constitutive relationship. We restrict ourselves to Newtonian fluids here.

**Conservation of Momentum**
**Derivation-13**

- Newton examined results of a large number of experiments and proposed the following relationship for shear stress: $\tau=\mu\frac{du}{dy}$ for 1D.
- This shear stress can be generalized using the nomenclature adopted earlier to get:

$$\tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\ \text{for i} \neq \text{j} \qquad (11.4)$$

- A relationship between velocities and stress is established using the above equation.

**Conservation of Momentum**
**Derivation-14**

- The following relationship, called the Stokes constitutive relationship, will be used here without deriving it.

$$\tau_{ij}=\left(-P-\frac{2}{3}\mu\ \nabla.\bar{u}\right)\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \qquad (11.5)$$
$$\delta_{ij}=0\ for\ i\neq j$$
$$=1\ for\ i=j$$
$$\bar{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$$

**Conservation of Momentum**
**Derivation-15**

- Now, consider the stress terms in the momentum equation and substitute the Stokes relationship to get:

$$\frac{\partial}{\partial y}\tau_{yy}+\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial z}\tau_{zy}=$$ ---------------- From momentum equation

$$
-\frac{\partial P}{\partial y}-\frac{\partial}{\partial y}\left(\frac{2}{3}\mu\ \nabla.\bar{u}\right)+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}\right)
+\frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+\frac{\partial}{\partial z}\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)
$$

-----After substituting Stokes relationship (11.6)

**Conservation of Momentum**
**Derivation-16**

- In addition if μ is assumed constant the equation becomes:

$$
\frac{\partial}{\partial y}\tau_{yy}+\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial z}\tau_{zy}=-\frac{\partial P}{\partial y}-\frac{2}{3}\mu\frac{\partial}{\partial y}(\nabla.\bar{u})
+2\mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \tag{11.7}
$$

- For an incompressible fluid it has been shown earlier that (refer equ$^{n}$ (10.7a))

$$
\nabla.\bar{u}=0 \tag{10.7 a}
$$

**Conservation of Momentum**
**Derivation-17**

- Since velocity is a continuous function, cross differentiation is permissible :

$$
\frac{\partial}{\partial y}\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\frac{\partial u}{\partial y} \tag{11.8}
$$

- Use equ$^{n}$ (10.7a) and equ$^{n}$ (11.8) in equ$^{n}$(11.6):

$$
\frac{\partial}{\partial y}\tau_{yy}+\frac{\partial}{\partial x}\tau_{xy}+\frac{\partial}{\partial z}\tau_{zy}\equiv-\frac{\partial P}{\partial y}-\frac{2}{3}\underbrace{\cancel{\frac{\partial}{\partial y}(\mu\nabla.\bar{u})}}_{=0}
+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\underbrace{\cancel{\frac{\partial v}{\partial y}}}_{=0}+\frac{\partial w}{\partial z}\right) \tag{11.9}
$$

**Conservation of Momentum**
**Derivation-18**

- Substituting Equ$^{n}$ 11.9 in equ$^{n}$ 11.3:

$$
\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}\rho uv+\frac{\partial}{\partial y}\rho v^2+\frac{\partial}{\partial z}\rho wv=
\rho X_y-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{11.10}
$$

- Above equation is called the conservative form of the momentum equation since it is the ‘original’ form obtained from the conservation equations and no simplifications are as yet applied.

**Conservation of momentum**
**Derivation-19**

- Expand LHS of equ$^{n}$ (11.10) to get:

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}\rho uv+\frac{\partial}{\partial y}\rho v^2+\frac{\partial}{\partial z}\rho wv$$
$$=\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]+v\underbrace{\left[\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}\rho u+\frac{\partial}{\partial y}\rho v+\frac{\partial}{\partial z}\rho w\right]}_{=0}$$

- Second term is zero from continuity (equ$^n$ 10.6)

**Conservation of momentum-19**
**Derivation-20**

- The 'y' component of the momentum equation therefore becomes (Note that $\upsilon=\dfrac{\mu}{\rho}$ ):

$$\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=$$
$$X_y+\upsilon\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]-\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{11.11}$$

**Conservation of Momentum**
**Derivation-21**

- The above Y-momentum equation is written in a compact form in the following fashion

$$\frac{dv}{dt}=X_y+\upsilon\nabla^2 v-\frac{1}{\rho}\frac{\partial p}{\partial y}$$
$$\nabla^2 v\equiv\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}$$
$$\frac{dv}{dt}\equiv\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}$$

- X and Z momentum can be similarly derived

**Conservation of momentum**
**Derivation-22**

- The final set of momentum equations are:

$$\frac{dv}{dt}=X_y+\upsilon\nabla^2 v-\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{11.12}$$
$$\frac{du}{dt}=X_x+\upsilon\nabla^2 u-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{11.13}$$
$$\frac{dw}{dt}=X_z+\upsilon\nabla^2 w-\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{11.14}$$

- The above equations are derived for laminar, incompressible, constant viscosity, Newtonian fluids

**Recap**

**In this class:**

- Derivation of conservation of momentum equation is completed.