Module 3 : Radiation

Lecture 29 : Discrete Ordinates Methodology

Objectives
In this class:

- The solution for the RTE using the Discrete Ordinates Methodology is discussed.
- No analytical solutions will be discussed in this course for the RTE

Discrete Ordinates Methodology (DOM)-1

- The main problem with the RTE is the occurrence of the integral term and differential terms together.
- In the Discrete Ordinates Method the integral term in the RTE is replaced by a summation.

$$\int f(x)dx \cong \sum_{i=1}^{n} w_i f(x_i) \tag{29.1}$$

- Regular numerical schemes can then be used to solve for the intensities

DOM-2

- Depending on the number of directions chosen the methods are referred to as the S_2, S_4, S_6 ….. S_N methods.

- The weights depend on the directions chosen and are computed mathematically. Weights are different for 1D, 2D and 3D situations. Weights and directions (μ = cosθ) for 1D and are given in the table on next slide from ref[5].

DOM- 1 D weights & Directions

S_N	Weights	± μ
S_2	1	0.50000
S_4	1/2	0.211325
	1/2	0.788675
S_6	1/3	0.146446
	1/3	0.500000
	1/3	0.853554
S_8	1/4	0.102672
	1/4	0.406205
	1/4	0.593795
	1/4	0.897327

- Weights for 1D case. Multiply weights by 2п.
  (Taken from Ref.[5])

DOM: Summation terms

- Consider the radiative transport equation:

$$\frac{dI}{ds} = \varepsilon I_b - \alpha I - \rho I + \int_0^{4\pi} \rho\, \mathrm{I} \frac{\hat{\Phi}(\theta_i, \theta_e)}{4\pi} d\theta_i \tag{29.2}$$

- The integral term is represented by:

$$\int_0^{4\pi} \rho(s)\, \mathrm{I}(s,\theta_i) \frac{\hat{\Phi}(s,\theta_i,\theta_e)}{4\pi} d\theta_i$$

$$= \frac{\rho(s)}{4\pi} \sum_{j=1}^{n} w_j . \mathrm{I}(s,\theta_j) . \hat{\Phi}(s,\theta_j,\theta_e) \tag{29.3}$$

DOM: Directions-1

- The direction 's' is along that specified by the discrete ordinates method. We now look at the cartesian coordinate system. An arbitrary unit vector along a direction which makes angle θ with the z-axis and whose projection on the x-y plane makes an angle ψ with the x axis is given by:

$$\sin\theta \sin\psi\, \hat{i} + \sin\theta\cos\psi\, \hat{j} + \cos\theta\, \hat{k}$$

DOM: Directions-2

- Therefore change in intensity per unit distance is given as:

$$\frac{dI}{ds} = \frac{\partial I}{\partial x}\frac{dx}{ds} + \frac{\partial I}{\partial y}\frac{dy}{ds} + \frac{\partial I}{\partial z}\frac{dz}{ds} \tag{29.4}$$

- And:

$$\frac{dx}{ds} = \sin\theta\sin\psi, \frac{dy}{ds} = \sin\theta\cos\psi, \frac{dz}{ds} = \cos\theta \tag{29.5}$$

DOM: Directions-3

- Therefore:

$$\frac{dI}{ds} = \frac{\partial I}{\partial x}\sin\theta\sin\psi + \frac{\partial I}{\partial y}\sin\theta\cos\psi + \frac{\partial I}{\partial z}\cos\theta \tag{29.6}$$

- Therefore since the discrete ordinate directions are known one can take a cartesian grid and determine the gradients in the x, y and z directions and then compute the change in the intensity in a given direction.

DOM: Directions-4

- For the sake of simplicity define the following:

$$\begin{aligned} \sin\theta\sin\psi &\equiv l \\ \sin\theta\cos\psi &\equiv m \\ \cos\theta &\equiv n \end{aligned} \tag{29.6a}$$

- Therefore:

$$\frac{dI}{ds} = \frac{\partial I}{\partial x} l + \frac{\partial I}{\partial y} m + \frac{\partial I}{\partial z} n \tag{29.6b}$$

DOM: Formulation-1

- Integrate equ^n (29.2) over a typical control volume:

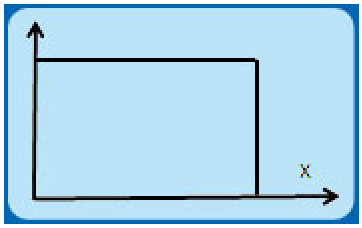
$$\iiint \frac{dI}{ds} dV = \iiint \varepsilon I_b dV - \iiint (\alpha + \rho) I dV$$
$$\iiint \left(\int_0^{4\pi} \rho\, \mathrm{I} \frac{\hat{\Phi}(\theta_i,\theta_e)}{4\pi} d\theta_i \right) dV \tag{29.7}$$

Using the Gauss divergence theorem the first term of the above equation is written as:

$$\iiint \frac{dI}{ds} dV = \iint I \; l dA_x + \iint I \; m dA_y + \iint I \; n dA_z \qquad (29.8)$$

DOM: Formulation-2

- Equ^n (29.8) is substituted in equ^n (29.7) for the first term.
- The boundary conditions are given and the temperature within the enclosure is to be determined. The entire volume is divided into a number of smaller volumes.



DOM: Formulation-3

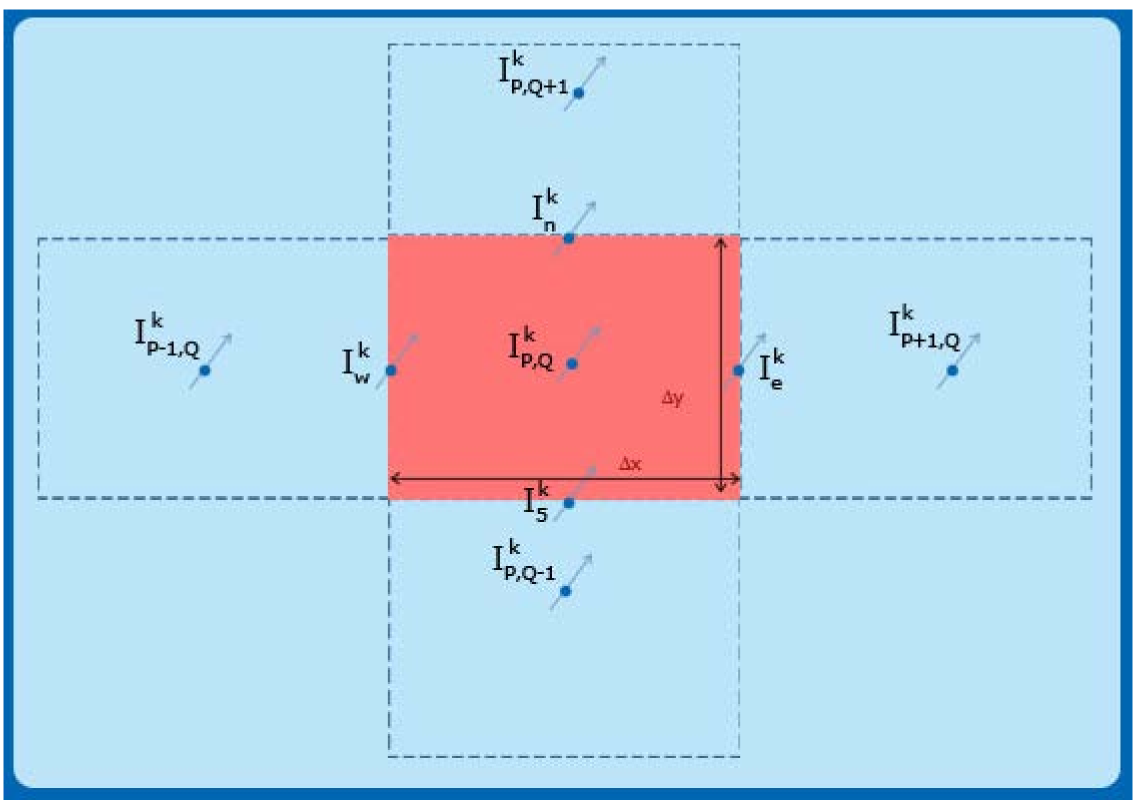
- The following nomenclature is used during the integration process:
  A_N, A_S, A_E, A_W: Areas of the North, South, East and West faces respectively.
  $I^k_{P,Q}$:the Intensity in the 'kth' direction at the P, Q location/node
  Direction cosines:

$$\frac{dx}{ds} = \sin\theta \sin\psi \equiv l; \frac{dy}{ds} = \sin\theta \cos\psi \equiv m; \frac{dz}{ds} = \cos\theta \equiv n$$

DOM: Formulation-4

- Intensity is different in different directions. Consider the direction k. Consider an arbitray point **P, Q** within the volume. The volume averaged intensity $I^k_{P,Q}$ for the control volume is considered to exist at this point. Intensity at neighbours are $I^k_{P+1,Q}, I^k_{P-1,Q}, I^k_{P,Q+1}, I^k_{P,Q-1}$ and the intensity at the control volume faces are given as $I^k_e, I^k_w, I^k_n, I^k_s$. All these are shown in the figure on the next page where the control volume is in red.

DOM: Formulation-5



DOM: Formulation-6

- For a 2D situation the variation of I^k in the **x** direction does not exist, therefore equ^n (29.8) is rewritten as:

$$\iiint \frac{dI^k}{ds} dV = \iint I^k m dA_y + \iint I^k n dA_z$$
$$= mI^k_e A_e - mI^k_w A_w + nI^k_n A_n - nI^k_s A_s \qquad (29.9)$$

- The intensities at the faces are not known and therefore some sort of an interpolation has to be used to relate the cell face and volume intensities.

DOM: Formulation-7

- A common interpolation is:

$$I^k_{P,Q} = \eta I^k_n + (1-\eta) I^k_s \;;\; I^k_{P,Q} = \eta I^k_e + (1-\eta) I^k_w$$
$$\Rightarrow I^k_n = \frac{I^k_{P,Q} - (1-\eta) I^k_s}{\eta};\; I^k_e = \frac{I^k_{P,Q} - (1-\eta) I^k_w}{\eta} \qquad (29.10)$$

- Substitute equ^n (29.10) in equ^n (29.9) to get:

$$\iiint \frac{dI^k}{ds} dV = m\frac{I^k_{P,Q} - (1-\eta) I^k_w}{\eta} A_e - mI^k_w A_w + n\frac{I^k_{P,Q} - (1-\eta) I^k_s}{\eta} A_n - nI^k_s A_s$$
$$= I^k_{P,Q}\left(\frac{nA_n}{\eta} + \frac{mA_e}{\eta}\right) - nI^k_s\left(A_s + A_n\frac{1-\eta}{\eta}\right) - I^k_w m\left(A_w + \frac{1-\eta}{\eta}A_e\right) \qquad (29.11)$$

DOM: Formulation-8

- Substitute equ^n (29.11) in equ^n (29.7) to get

$$I^k_{P,Q}\left(\frac{mA_e}{\eta} + \frac{nA_n}{\eta}\right) - nI^k_s\left(A_s + A_n\frac{1-\eta}{\eta}\right) - I^k_w m\left(A_w + \frac{1-\eta}{\eta}A_e\right)$$
$$= \varepsilon I_b - (\alpha + \rho) I^k_{P,Q} V_{P,Q} + \frac{\rho(s)}{4\pi}\sum_{j=1}^{n} w_j . \mathrm{I}(s,\theta_j) . \hat{\Phi}(s,\theta_j,\theta_e) V_{P,Q} \qquad (29.12)$$

- This equation is now simplified so that the intensity at the point **P, Q** is explicitly obtained.

DOM: Formulation-9

- Simplifying equ^n (29.12) gives the expression to use for solving for the intensity at the point **P, Q** :

$$I^k_{P,Q} = \frac{I^k_w m(\eta A_w + (1-\eta)A_e) + nI^k_s(\eta A_s + (1-\eta)A_n) + \eta\varepsilon I_b V_{P,Q} + \eta\frac{\rho(s)}{4\pi}\sum_{j=1}^{n} w_j . \mathrm{I}(s,\theta_j) . \hat{\Phi}(s,\theta_j,\theta_e) V_{P,Q}}{(mA_e + nA_n + \eta(\alpha + \rho) V_{P,Q})} \qquad (29.13)$$

- The term marked red is the numerator and that marked blue is the denominator

DOM: Formulation-10

- So far the boundary conditions have not been discussed. At the boundary the intensity will have contributions due to emission and reflection of incident radiation. Recall the equ^n (26.15) where the definition of intensity was given. This was for intensity of radiation from a surface losing energy. The same magnitude incoming intensity can be visualized as supplying the same energy to a very similar surface.

DOM: Formulation-11

- The incident energy on a surface is therefore:

$$\frac{E_{\text{total}}}{dA_{\text{i}}} = \int I \cos\theta d\omega \tag{29.14}$$

- Assuming the surface to be diffuse, the reflected intensity will be equal in all directions. The reflected intensity is therefore:

$$I = \frac{E_{\text{total}}}{\pi dA_{\text{i}}} = \frac{(1-\varepsilon)}{\pi} \int I \cos\theta d\omega \tag{29.15}$$

DOM: Formulation-12

- The boundary condition at the wall is therefore written as

$$I_{wall}^{k} = \varepsilon I_{bwall} + \frac{(1-\varepsilon)}{\pi} \sum_{j=1}^{n/2} \cos(\theta_j) w_j \mathrm{I}^j \tag{29.16}$$

- Note that the θ_j is the angle between the normal to the surface and the I^j direction.
- Also note that the summation is only till n/2 i.e. only directions towards the surface are considered in the calculations. Only what falls on a surface can get reflected

DOM: Formulation-13

- The steady state radiative energy equation assuming convection, conduction and internal heat generation are negligible is:
- Energy emitted = energy absorbed
- Energy emitted = $\int \varepsilon I_b d\omega = 4\pi\varepsilon \frac{\sigma T^4}{\pi}$ (29.17)
- Energy absorbed = $\alpha \int I d\omega$ (29.18)
- Therefore

$$\varepsilon \int I d\omega = 4\varepsilon\sigma T^4 \Rightarrow T^4 = \frac{1}{4\sigma} \int I d\omega \tag{29.19}$$

Recap
In this class:

- The solution for the RTE using the Discrete Ordinates Methodology is discussed.
- No analytical solutions will be discussed in this course for the RTE