Module 2 : Convection

Lecture 21 : Introductory Turbulent Flows

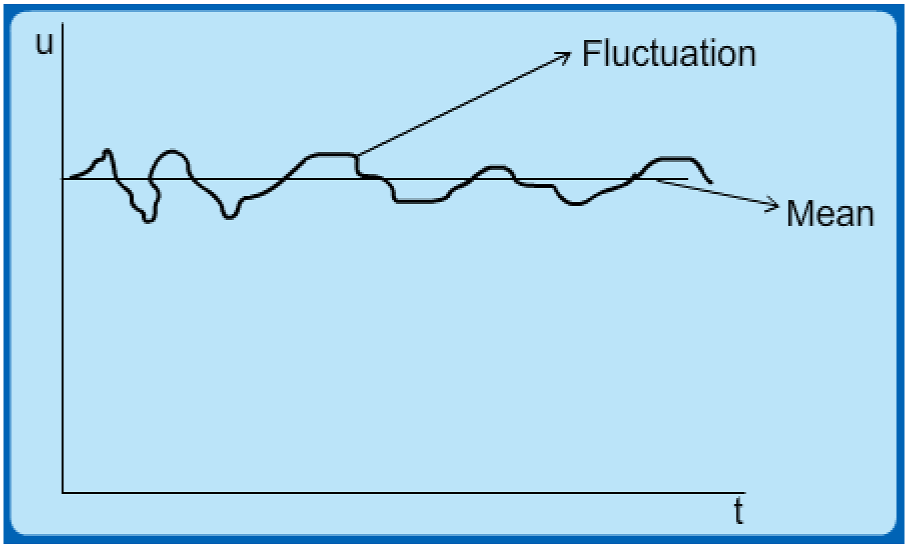
Objectives

In this class:

- Turbulent flows will be introduced
- Equations for the stresses will be developed
- Mixing length model will be discussed

Turbulent Flow Equations-1

- A turbulent flow is one where there is a mean variable and a fluctuating component. The variable could be velocity or temperature. The mean of the fluctuating component is zero.



Turbulent Flow Equations-2

- Consider velocity with a mean and fluctuating component being represented as:

$$u = \bar{u} + u' \qquad v = \bar{v} + v' \qquad w = \bar{w} + w' \tag{21.1}$$

- The mean is defined in the following manner:

$$\bar{u} = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{t}^{t+\Delta t} u\,dt \tag{21.2}$$

- Bar on top of the variable indicates averaging. Even though the Δt is infinite, practically it needs to be only of the order of a few seconds

Turbulent Flow Equations-3

- Using the definitions in equ^n (21.1) in equ^n (21.2), the mean becomes:

$$\bar{u} = \frac{1}{\Delta t} \int_{t}^{t+\Delta t} (\bar{u} + u')dt$$

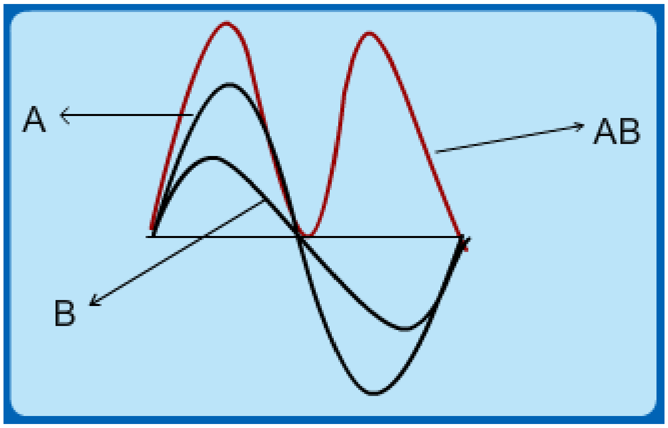
$$= \frac{1}{\Delta t} \int_{t}^{t+\Delta t} \bar{u}dt + \frac{1}{\Delta t} \int_{t}^{t+\Delta t} u'dt$$

$$= \quad \bar{u} \quad + \quad 0(=\overline{u'})$$

- Note that $\bar{u}' = 0$ but $\overline{u'u'} \neq 0$ necessarily

Turbulent Flow Equations-4

- Note that $\overline{u'} = 0$ but $\overline{u'u'} \neq 0$ necessarily
- In the figure below the averages of a variable 'A' and 'B' are individually zero. However, the average of 'AB' is obviously not zero.



Turbulent Flow Equations-5

- Variables are uncorrelated if $\overline{u'u'} = 0$
- Some correlation exists if $\overline{u'u'} \neq 0$
- A measure of the correlation is the correlation coefficient $C_{ij} = \dfrac{\overline{u_i' u_j'}}{\sqrt{\overline{u_i'^2}\;\overline{u_j'^2}}}$

Turbulent Flow Equations-6

- Turbulence intensity is defined as:

$$J = \frac{1}{\overline{V}}\sqrt{\frac{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}{3}}$$

- Turbulence is isotropic when

$$\overline{u'^2} = \overline{v'^2} = \overline{w'^2}$$

- We now derive the conservation equations again.

Turbulent Flow Equations-7

- Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (21.3)$$

- Substitute the definitions in equ^n (21.1) and equ^n (21.2) in equ^n (21.3):

$$\frac{\partial}{\partial x}(\bar{u} + u') + \frac{\partial}{\partial y}(\bar{v} + v') + \frac{\partial}{\partial z}(\bar{w} + w') = 0 \qquad (21.4)$$

- Averaging equ^n (21.4) gives:

$$\frac{\partial}{\partial x}(\bar{u}+u')+\frac{\partial}{\partial y}(\bar{v}+v')+\frac{\partial}{\partial z}(\bar{w}+w')=0 \tag{21.5}$$

Turbulent Flow Equations-8

- The first term of equ^n (21.5) can be simplified in the following manner:

$$\overline{\frac{\partial}{\partial x}(\bar{u}+u')}=\frac{\overline{\partial \bar{u}}}{\partial x}+\frac{\overline{\partial u'}}{\partial x}=\frac{\partial \bar{\bar{u}}}{\partial x}+\frac{\partial \overline{u'}}{\partial x}=\frac{\partial}{\partial x}(\bar{\bar{u}}+\overline{u'})$$

- Using this methodology for all terms and noting that $\overline{u'}=0$ the equ^n (21.4) is written as:

$$\frac{\partial}{\partial x}(\bar{\bar{u}}+\overline{u'})+\frac{\partial}{\partial y}(\bar{\bar{v}}+\overline{v'})+\frac{\partial}{\partial z}(\bar{\bar{w}}+\overline{w'})=0$$

$$\frac{\partial}{\partial x}(\bar{u})+\frac{\partial}{\partial y}(\bar{v})+\frac{\partial}{\partial z}(\bar{w})=0 \tag{21.6}$$

Turbulent Flow Equations-9

- Substituting equ^n (21.6) in equ^n (21.4)

$$\frac{\partial}{\partial x}(u')+\frac{\partial}{\partial y}(v')+\frac{\partial}{\partial z}(w')=0 \tag{21.7}$$

- The continuity equation is therefore satisfied by both the mean fluctuating components individually.

Turbulent Flow Equations-10

- Consider now the x-component of the momentum equation:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{\partial}{\partial x}\tau_{xx}+\frac{\partial}{\partial y}\tau_{yx}+\frac{\partial}{\partial z}\tau_{zx} \tag{21.8}$$

- Consider the first term and substitute the definitions to get:

$$u\frac{\partial u}{\partial x}=(\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u')=\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{u}\frac{\partial u'}{\partial x}+u'\frac{\partial \bar{u}}{\partial x}+u'\frac{\partial u'}{\partial x} \tag{21.9}$$

Turbulent Flow Equations-11

- Take average of the equ^n (21.9)

$$\overline{u\frac{\partial u}{\partial x}}=\bar{u}\frac{\partial \bar{u}}{\partial x}+\overline{\bar{u}\frac{\partial u'}{\partial x}}+\overline{u'\frac{\partial \bar{u}}{\partial x}}+\overline{u'\frac{\partial u'}{\partial x}}\equiv\bar{u}\frac{\partial \bar{u}}{\partial x}+\overline{u'\frac{\partial u'}{\partial x}} \tag{21.10}$$

- Expressions like those in equ^n (21.10) can be obtained for all the terms in equ^n (21.8). Now take the average of all the terms in equ^n (21.8) to obtain:

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\overline{u'\frac{\partial u'}{\partial x}}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\overline{v'\frac{\partial u'}{\partial y}}+\bar{w}\frac{\partial \bar{u}}{\partial z}+\overline{w'\frac{\partial u'}{\partial z}}=\overline{\frac{\partial}{\partial x}\tau_{xx}+\frac{\partial}{\partial y}\tau_{yx}+\frac{\partial}{\partial z}\tau_{zx}} \tag{21.11}$$

Turbulent Flow Equations-12

- Need to simplify the stress terms. The stress terms are the same as those in equ^n (11.5):

$$\tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\left(\frac{2}{3}\mu\nabla.\bar{u}+p\right)\delta_{ij}$$

The averaging process is illustrated below:

$$\tau_{xx} = -p + 2\mu\frac{\partial u}{\partial x}; \Rightarrow \overline{\tau_{xx}} = -\overline{p} + 2\mu\frac{\partial \overline{u}}{\partial x}$$

$$\overline{\tau_{xy}} = \mu\left[\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right] \tag{21.12}$$

Turbulent Flow Equations-13

- The momentum equation therefore becomes

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} = -\frac{\partial \overline{p}}{\partial x} + \mu\nabla^2\overline{u} - \left[\frac{\partial}{\partial x}\rho\overline{u'^2} + \frac{\partial}{\partial x}\rho\overline{u'v'} + \frac{\partial}{\partial x}\rho\overline{u'w'}\right] \tag{21.13}$$

- The second term on the RHS is the same as the RHS of equ^n (11.9) for the 'x' direction.
- Extra terms on the RHS shown in blue are the extra terms which look like the stress terms and are the Reynolds stress terms that need to be modeled appropriately.

Turbulent Flow Equations-14

- Now consider the energy equation. We ignore the heat generation terms here:

$$\rho c_p\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right] + Q$$

- In addition to velocity fluctuations, introduce temperature fluctuation T' such that T= $\overline{T}$ +T', in equn and average to get:

$$\rho c_p\left[\overline{(\overline{u}+u')\frac{\partial}{\partial x}(\overline{T}+T')} + \overline{(\overline{v}+v')\frac{\partial}{\partial y}(\overline{T}+T')}\right] = k\left[\frac{\partial^2}{\partial x^2}\overline{(\overline{T}+T')} + \frac{\partial^2}{\partial y^2}\overline{(\overline{T}+T')}\right] \tag{21.14}$$

Turbulent Flow Equations-15

- Look at the first term in equ^n (21.14):

$$\rho c_p\left\{\overline{(\overline{u}+u')\frac{\partial}{\partial x}(\overline{T}+T')}\right\}$$

$$= \rho c_p\left\{\overline{\overline{u}\frac{\partial \overline{T}}{\partial x}} + \overline{u'\frac{\partial \overline{T}}{\partial y}} + \overline{u'\frac{\partial T'}{\partial x}} + \overline{\overline{u}\frac{\partial T'}{\partial x}}\right\} = \rho c_p\left[\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{u'\frac{\partial T'}{\partial x}}\right]$$

- Other terms can be written similarly and the equ^n (21.14) becomes:

$$\rho c_p\left[\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial y}\right] = k\left[\frac{\partial^2 \overline{T}}{\partial x^2} + \frac{\partial^2 \overline{T}}{\partial y^2}\right] - \rho c_p\left[u'\frac{\partial T'}{\partial x} + v'\frac{\partial T'}{\partial y}\right] \tag{21.15}$$

Turbulent Flow Equations-16

- Use the continuity equ^n (21.7) and perform the following manipulations :

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0 \Rightarrow T'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right)=0$$

$$\frac{\partial T'u'}{\partial x}+\frac{\partial T'v'}{\partial y}-u'\frac{\partial T'}{\partial x}-v'\frac{\partial T'}{\partial y}=0$$

$$\frac{\partial \overline{T'u'}}{\partial x}+\frac{\partial \overline{T'v'}}{\partial y}-\overline{u'\frac{\partial T'}{\partial x}}-\overline{v'\frac{\partial T'}{\partial y}}=0$$

$$\overline{u'\frac{\partial T'}{\partial x}}+\overline{v'\frac{\partial T'}{\partial y}}=\frac{\partial \overline{T'u'}}{\partial x}+\frac{\partial \overline{T'v'}}{\partial y} \quad (21.16)$$

Turbulent Flow Equations-17

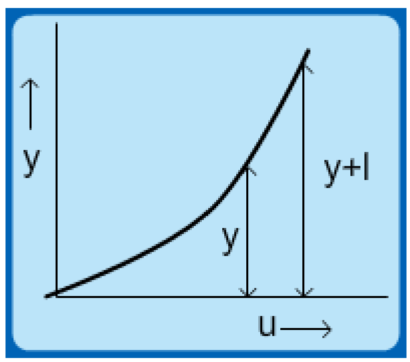
- Use equ^n (21.16) in equ^n (21.14) to get:

$$\rho c_p\left[\overline{u}\frac{\partial \overline{T}}{\partial x}+\overline{v}\frac{\partial \overline{T}}{\partial y}\right]=k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]-\rho c_p\left[\frac{\partial}{\partial x}\overline{u'T'}+\frac{\partial}{\partial y}\overline{v'T'}\right] \quad (21.17)$$

- Extra terms in the energy equation also exist that need to be modeled.
- Earliest attempt to model is the Prandtl mixing length theory

Mixing length model-1

- Consider a velocity profile as shown below. Fluid particles are exchanged between the fluid layers moving with average velocities



- Velocity fluctuations are due to mean velocity variations due to fluid particle motion

Mixing length model-2

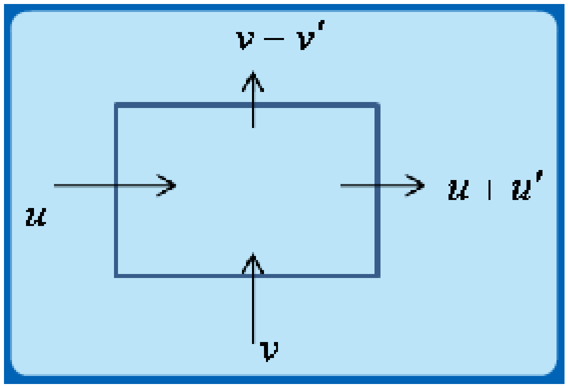
- Consider a fluid particle that moves from ‘y’ to ‘y + l’. Its change in velocity is

$$u' = u(y+l) - u(y) = l\frac{d\overline{u}}{dy} \quad (21.18)$$

- This change in velocity is the velocity fluctuation. Fluctuation is known if ‘l’ is known. ‘l’ is the Prandtl mixing length.
- The change in the ‘u’ and the ‘v’ components are of similar orders of magnitude

Mixing length model-3

- Consider a control volume. If there is a fluctuation in one direction there has to be a fluctuation of a similar order of magnitude on the other direction also



- Positive fluctuation in ‘u’ implies a negative fluctuation in ‘v’ from continuity considerations i.e. $v' = -u'$

Mixing length model-4

- Now look at the stress term that needs to be modeled in equ^n (21.13):

$$-\overline{u'v'} = |\overline{u'}||\overline{v'}| = l^2\left(\frac{d\bar{u}}{dy}\right)^2 = \frac{\tau}{\rho} \tag{21.19}$$

- Equ^n indicates that the shear stress is always positive but since it changes sign with sign of the velocity gradient it is re-written as:

$$\tau = \rho l^2\left(\frac{d\bar{u}}{dy}\right)^2 = \rho l^2\left(\frac{d\bar{u}}{dy}\right)\left|\left(\frac{d\bar{u}}{dy}\right)\right| \tag{21.20}$$

Mixing length model-5

- An expression for the turbulent viscosity is therefore obtained as:

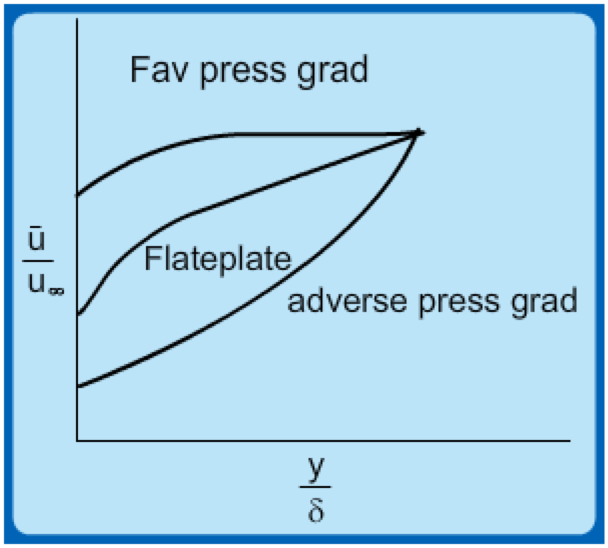
$$\tau_t = \mu_t\left(\frac{d\bar{u}}{dy}\right) \Rightarrow \mu_t = \rho l^2\left|\frac{d\bar{u}}{dy}\right| \tag{21.21}$$

- Overall stress is therefore a combination due to a molecular and a virtual kinematic viscosity:

$$\tau = \rho(\nu + \varepsilon_M)\left(\frac{d\bar{u}}{dy}\right) \text{ where } \varepsilon_M = l^2\left|\frac{d\bar{u}}{dy}\right| \tag{21.22}$$

Mixing length model-6

- Velocity measurements close to the wall typically indicate the following nature.



- Very close to the wall the velocity must go close to zero.

Mixing length model-7

- The following model was therefore proposed by Prandtl and Von Karman
  - The region very close to the wall dominated by viscous stresses is called the laminar sub layer
  - The region far away from the wall is dominated by turbulent stresses is called the overlap layer
  - The region in between is the buffer layer where both are important
  - Outside the overlap layer is the outer layer

Mixing length model-8

- Very close to the wall, only viscous effects are present. Prandtl assumed that the shear in this region is constant and equal to the value at the wall.
- Non dimensionalize the length and velocity in the expression for stress using:

$$u^+ = \frac{u}{u_\tau}; y^+ = \frac{y}{Y} \qquad (21.23)$$

Mixing length model-9

- The non-dimensional form of the stress is therefore:

$$\tau_w = \mu\frac{du}{dy} = \mu\left[\frac{u_\tau}{Y}\right]\frac{du^+}{dy^+} = \mu\left[\frac{u_\tau}{Y}\right]\frac{du^+}{dy^+}\frac{\rho u_\tau}{\rho u_\tau} \qquad (21.23)$$

- The equation suggests the following

$$Y = \nu / u_\tau; u_\tau = \sqrt{\frac{\tau_w}{\rho}} \Rightarrow \frac{du^+}{dy^+} = 1 \Rightarrow u^+ = y^+ \qquad (21.24)$$

- This region is seen to exist till $y^+ = 5$

Or,

Mixing length model-10

- A fully turbulent layer is postulated for $y^+ > 30$. Here laminar stresses are much smaller than the turbulent stresses

$$\tau_t = \rho\overline{u'v'} = \rho\varepsilon_M \frac{du}{dy} = \rho l^2 \left(\frac{du}{dy}\right)^2$$

$$\sqrt{\frac{\tau_t}{\rho}} = l\frac{du}{dy} \qquad (21.25)$$

- It is to be noted that 'l' is still unknown and needs to be modeled in some manner

Mixing length model-11

- One assumption that seems to work is

$$\begin{aligned} l &= ky \text{ for } y/\delta_{99} < 0.4 \\ l &= 0.085\delta_{99} \text{ for } y/\delta_{99} > 0.4 \end{aligned} \qquad (21.26)$$

- Substituting equ^n (21.26) in (21.25)

$$1 = ky^+ \frac{du^+}{dy^+} \Rightarrow u^+ = \frac{1}{k}\ln y^+ + B \qquad (21.27)$$

$$k = 0.41; B = 5.0 \qquad (21.28)$$

- This is called the logarithmic law of the wall and the constants are for boundary layer flow

Mixing length model-12

- We have so far discussed the inner layer – consisting of the viscous sub layer, log-law of the wall and a buffer layer that connects these two layers.
- There also exists an outer layer (usually $y^+ > 100$) where the pressure gradient effects are significant - we shall not discuss here
- All the above analysis was largely a result of boundary layer type flows

Mixing length model-13

- Interest here is in pipe flow situations
- Note that the shear has to go to zero at the pipe centerline and therefore the log law of the wall is really not valid in this region. A slightly modified equation matches the experimental data quite well (often called the Nikuradse equation):

$$u^+ = \frac{1}{0.4}\ln y^+ + 5.5 \qquad (21.29)$$

$$y = r_0 - r$$

Mixing length model-14

- The small difference in the constants from those in equ^n (21.28) is attributable to the favourable pressure gradient in the fully developed flow region
- Other empirical relations that avoid the problem of non-zero stress at the central tube region have been proposed. We shall use the Nikuradse equation.

Recap

In this class:

- Turbulent flows will be introduced
- Equations for the stresses will be developed
- Mixing length model will be discussed