Module 2 : Convection

Lecture 10 : Derivation of conservation of mass and momentum equations

Objectives

In this class:

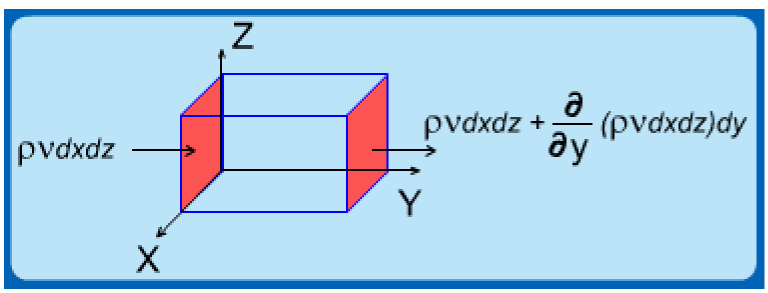
- Convection is introduced
- Conservation of mass equation for a control volume is derived
- Derivation of conservation of momentum equation is started. A standard convention for stresses is discussed which makes derivation of momentum equation easier.

Convection

- So far we have seen only conduction. We have seen only cartesian coordinate system for 2D but extension to other systems is not very difficult – however, the algebra can tend to be tedious
- We next look at convection. The continuity, momentum and energy are derived and then the applications are considered.

Conservation of Mass-1

- The conservation of mass for a control volume is derived first.
- The statement of the conservation of mass can be expressed in the form: Net mass flux into the control volume = Rate of accumulation of mass within the control volume
- Consider a small control volume of linear dimensions dx, dy and dz
- Consider the Y-direction as shown below:



- Net influx of mass in y direction:

$$= \rho v dxdz - \left(\rho v dxdz + \frac{\partial}{\partial y}(\rho v dxdz) dy \right)$$

Conservation of Mass-2

- Net mass influx in the y direction is therefore:

$$-\frac{\partial}{\partial y}(\rho v dxdz) dy \tag{10.1}$$

- Notice that since dx and dz are constants, the above equation can be written as:

$$-\frac{\partial(\rho v)}{\partial y}(dxdzdy) \tag{10.2}$$

Conservation of Mass-3

- Similarly get the net influx in the other directions, i.e. x and z, and obtain the net influx for the entire control volume to be

$$-\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right) dydzdx \tag{10.3}$$

- Rate of accumulation of mass within the control volume is expressed as:

$$\frac{\partial}{\partial t}(\rho dydxdz) \equiv \frac{\partial \rho}{\partial t}(dydxdz) \quad (10.4)$$

Conservation of Mass-4

- Conservation of mass requires the rate of accumulation of mass within the control volume to be equal to the net influx of mass:

$$-\left(\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right)dydzdx = \frac{\partial \rho}{\partial t}(dydxdz)$$

- Since volume is finite the conservation of mass is written as:

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0 \quad (10.5)$$

Conservation of Mass-5

- The equ^n 10.5 can written as the following in the vector notation:

$$\frac{\partial \rho}{\partial t}+\nabla.\rho\bar{u}=0 \quad (10.6)$$

- Using standard vector identities in equ^n (10.6):

$$\frac{\partial \rho}{\partial t}+\rho\nabla.\bar{u}+\bar{u}.\nabla\rho=0$$

$$\frac{d\rho}{dt}+\rho\nabla.\bar{u}=0 \quad (10.7)$$

- When density is constant:

$$\nabla.\bar{u}=0 \quad (10.7a)$$

Conservation of momentum-1

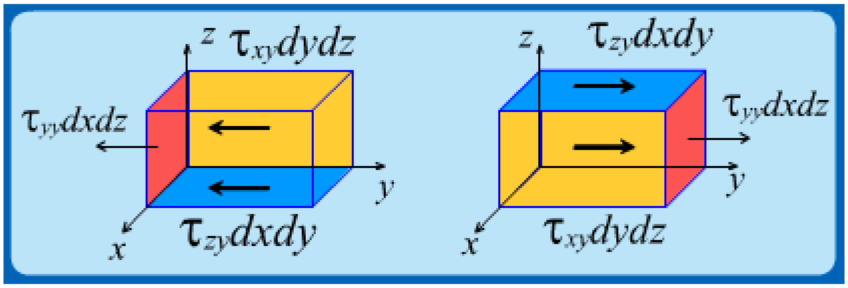
- Conservation of momentum is concerned with a force balance
- Force is a vector quantity and therefore has a direction
- Stress is force per unit area and since both force and area are vectors, stress is a tensor. Force due to a stress has a direction and this needs a sign convention. A nomenclature for stress is also required.
- We discuss these conventions first

Conservation of momentum-2
Nomenclature

- In general stress, τ_{ij} is assumed to act on ith plane in jth direction
- Force due to stress in the jth direction on the positive ith plane is in the positive jth direction
- Force due to stress in the jth direction on the negative ith plane is in the negative jth direction
- $\tau_{ij}dA_i$ is in the +ve 'j' direction for +ve and in the –ve 'j' direction for –ve dA_i

Conservation of momentum-3
Nomenclature

- In the control volume shown with sides dx, dy, dz, notice that the positive direction for $\tau_{zy}dxdy$ for z = 0 plane is towards left and for z = dz plane is towards right as shown. This is consistent with the nomenclature adopted.

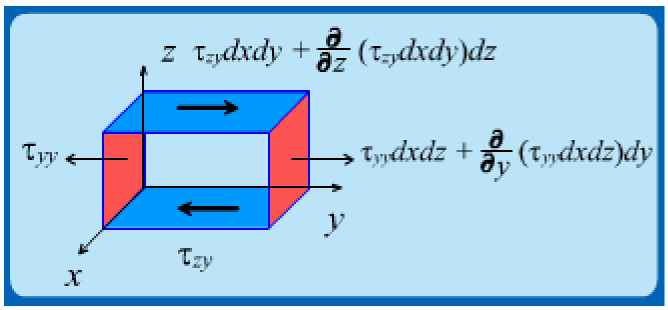


Conservation of momentum-4
Nomenclature

- The nomenclature is now complete.
- We will now derive the balance of momentum equation for a small control volume in the cartesian coordinates.
- The momentum principle in words is:
  Rate of accumulation of momentum = Net rate of influx of momentum due to velocity +
  Net rate of influx due to forces.

Conservation of momentum-5
Derivation

- Consider the 'y' direction. The forces due to the stresses are written on the y = 0 and z = 0 faces. The Taylor series expansion with the leading term are taken for the y = dy and z = dz faces. Terms on the x = 0 and x = dx face exist but are omitted for the sake of clarity.



Conservation of momentum-6
Derivation

- Force 'F_y' in the 'y' direction is therefore:

$$\left[\tau_{yy} + \frac{\partial}{\partial y}\tau_{yy}dy - \tau_{yy}\right]dxdz + \left[\tau_{zy} + \frac{\partial}{\partial z}\tau_{zy}dz - \tau_{zy}\right]dxdy + \left[\tau_{xy} + \frac{\partial}{\partial x}\tau_{xy}dx - \tau_{xy}\right]dydz$$

- Simplifying the above equation gives:

$$F_y = \frac{\partial}{\partial y}\tau_{yy}dxdydz + \frac{\partial}{\partial z}\tau_{zy}dxdydz + \frac{\partial}{\partial x}\tau_{xy}dxdydz \qquad (10.8)$$

- Force balance in the 'x' and 'z' directions can now be performed, in an identical manner, to get the net force in these directions

Conservation of momentum-7

Derivation

- The force due to stresses in the 'x' and 'z' directions are:

$$F_z = \frac{\partial}{\partial y}\tau_{yz}dxdydz + \frac{\partial}{\partial z}\tau_{zz}dxdydz + \frac{\partial}{\partial x}\tau_{xz}dxdydz \qquad (10.9)$$

$$F_x = \frac{\partial}{\partial y}\tau_{yx}dxdydz + \frac{\partial}{\partial z}\tau_{zx}dxdydz + \frac{\partial}{\partial x}\tau_{xz}dxdydz \qquad (10.10)$$

- The forces due to the stresses on the faces of the control volume have been established. Next step is to evaluate change in momentum due to mass entering and leaving the control volume.

Recap

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