Module 2 : Convection

Lecture 25 : Integral Boundary Layer Equations

Objectives

In this class:

- Integral equations for the momentum and thermal boundary layers are obtained
- Solution of these equations for flow over a flat surface is demonstrated

Integral Boundary Layer Equations

- Similarity methods are differential methods. Give local information but mostly need complicated methodologies for solution of the resulting equations.
- Integral methods give overall information but are very useful since the results are simple and not too far from reality. Major disadvantage is the necessity of profiles for temperature and velocity.

Momentum Equation-Derivation-1

- Consider the 2D continuity equation and integrate over the height of the boundary layer

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\Rightarrow\int_0^\delta\frac{\partial u}{\partial x}dy+\int_0^\delta\frac{\partial v}{\partial y}dy=0 \qquad (25.1)$$

$$\int\frac{\partial u}{\partial x}dy+v\big|_\delta-v\big|_0=0\Rightarrow v\big|_\delta=-\int_0^\delta\frac{\partial u}{\partial x}dy \qquad (25.2)$$

- Note that the velocity at the wall is zero and at the edge of the boundary layer is the free stream value

Momentum Equation-Derivation-2

- Now, consider the momentum equation.

$$\frac{\partial u^2}{\partial x}+\frac{\partial}{\partial x}uv=\upsilon\frac{\partial^2 u}{\partial y^2}+\frac{1}{\rho}\frac{dp}{dx} \qquad (25.3)$$

- Integrate this over the thickness of the boundary layer

$$\int_0^\delta\frac{\partial u^2}{\partial x}dy+\int_0^\delta\frac{\partial}{\partial y}(uv)dy=\upsilon\int_0^\delta\frac{\partial^2 u}{\partial y^2}dy+\int_0^\delta\frac{1}{\rho}\frac{dp}{dx}dy \qquad (25.4)$$

Momentum Equation-Derivation-3

- Perform the integration for equ^n (25.4):

$$\int_0^\delta\frac{\partial u^2}{\partial x}dy+uv\big|_\delta-uv\big|_0=\upsilon\left[\frac{\partial u}{\partial y}\bigg|_\delta-\frac{\partial u}{\partial y}\bigg|_0\right]+\frac{1}{\rho}\frac{dp}{dx}\delta \qquad (25.5)$$

- Simplify equ^n (25.5) using equ^n (25.2), velocity is zero at wall and the velocity gradient is zero for free stream at y = δ and replace pressure gradient with free stream velocity gradient.

Momentum Equation-Derivation-4

- The equ^n (25.5) therefore becomes:

$$\int_0^\delta \frac{\partial u^2}{\partial x}dy - u_\infty \int_0^\delta \frac{\partial u}{\partial x}dy = -\upsilon \frac{\partial u}{\partial y}\bigg|_{y=0} - \frac{1}{\rho}\frac{dp}{dx}\delta$$

$$= -\frac{\tau_w}{\rho} + u_\infty \frac{du_\infty}{dx}\delta \qquad (25.6)$$

- Modify the second term on the LHS of equ^n(25.6) to get:

$$\int_0^\delta \frac{\partial}{\partial x}(u^2 - uu_\infty)dy + \int_0^\delta u\frac{du_\infty}{dx}dy = -\frac{\tau_w}{\rho} + u_\infty \frac{du_\infty}{dx}\delta \qquad (25.7)$$

Momentum Equation-Derivation-5

- The momentum equation becomes:

$$\int_0^\delta \frac{\partial}{\partial x}(u^2 - uu_\infty)dy + \underbrace{\int_0^\delta (u - u_\infty)dy}_{\text{Displacement thickness, } \delta^*}\frac{du_\infty}{dx} = -\frac{\tau_w}{\rho} \qquad (25.8)$$

- The Leibnitz rule, given below, is used to simplify the first term

$$\frac{d}{d\alpha}\int_{\phi_1(\alpha)}^{\phi_2(\alpha)} F(x,\alpha)dx = \int_{\phi_1(\alpha)}^{\phi_2(\alpha)} \frac{\partial}{\partial \alpha}F(x,\alpha)dx +$$

$$F(\phi_2(\alpha),\alpha)\frac{d\phi_2(\alpha)}{dx} - F(\phi_1(\alpha),\alpha)\frac{d\phi_1(\alpha)}{dx} \qquad (25.9)$$

Momentum Equation-Derivation-6

- Using the Leibnitz rule for the term marked in red in equ^n (25.8):

$$\int_0^\delta \frac{\partial}{\partial x}(u^2 - uu_\infty)dy = \frac{d}{dx}\int_0^\delta (u^2 - uu_\infty)dy - (u_\infty{}^2 - u_\infty{}^2)\frac{d\delta}{dx}$$

$$\Rightarrow \int_0^\delta \frac{\partial}{\partial x}(u^2 - uu_\infty)dy = -\frac{d}{dx}u_\infty{}^2\theta$$

- Substitute in equ^n (25.8) to get the Momentum Integral Equation

$$\boxed{\frac{d}{dx}u_\infty{}^2\theta + \delta^* u_\infty \frac{du_\infty}{dx} = \tau_w / \rho} \qquad (25.10)$$

Momentum Equation: Flat plate-1

- Equ^n (25.10) can be solved to obtain the quantities of interest. However, unless a profile for the velocity is known the solution cannot be obtained.
- Assume the following velocity profile for zero pressure gradient flat plate boundary layer flow:

$$\frac{u}{u_\infty} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2 + d\left(\frac{y}{\delta}\right)^3 \qquad (25.11)$$

Momentum Equation: Flat plate-2

- The four unknown constants need four conditions and the following are often used:

$$\frac{y}{\delta}=0, \quad \frac{u}{u_\infty}=0 \text{ - no slip}$$

$$\frac{y}{\delta}=0, \quad \frac{d^2(u/u_\infty)}{d(y/\delta)^2}=0 \text{ - no slip and equ}^{\text{n}} \text{ (25.3);}$$

$$\frac{y}{\delta}=1, \quad \frac{u}{u_\infty}=1 \text{ - velocity=free stream velocity at y= }\delta$$

$$\frac{d(u/u_\infty)}{d(y/\delta)}=0- \text{ velocity profile smooth at y= }\delta \tag{25.12}$$

Momentum Equation: Flat plate-3

- Equ^n (25.12) and equ^n (25.11) together give:

$$\begin{array}{ll} a=0 & 1=b+c+d \\ 0=b+2c+3d & 0=c \\ \Rightarrow d=-1/2, b=3/2 & \end{array}$$

- The profile and wall shear are calculated as

$$\frac{u}{u_\infty}=\frac{3}{2}(y/\delta)-\frac{1}{2}(y/\delta)^3 \tag{25.13}$$

$$\tau_w=\mu\frac{du}{dy}\bigg|_{y=0}=-\mu\left(\frac{3}{2\delta}\right)u_\infty \tag{25.14}$$

Momentum Equation: Flat plate-4

- Substitute the velocity profile in the momentum integral equ^n (25.10) and simplify to get: (Assume η = y/δ)

$$\frac{\mathrm{d}}{\mathrm{dx}}\int_0^\delta\left(\frac{3}{2}\eta-\frac{1}{2}\eta^3\right)\left[\frac{3}{2}\eta-\frac{1}{2}\eta^3-1\right]d\eta\,\delta=-\frac{3}{2}\frac{\mu}{\delta\rho u_\infty}$$

$$\frac{d}{dx}\left(-\frac{39}{280}\delta\right)=-\frac{3\mu}{2\delta\rho u_\infty}$$

$$\frac{d}{dx}\delta^2=\frac{280}{13}\frac{\upsilon}{u_\infty} \tag{25.15}$$

Momentum Equation: Flat plate-5

- Integrate equ^n (25.15) to get:

$$\delta^2=\frac{280}{13}\frac{\upsilon}{u_\infty}x+C \tag{25.16}$$

- Even though it is not correct assume that the boundary layer starts from x = 0 i.e. x = 0, δ = 0 we use this condition to determine C = 0. Therefore:

$$\delta=4.64\sqrt{\frac{\upsilon x}{u_\infty}} \qquad \Rightarrow \frac{\delta}{x}=\frac{4.64}{\sqrt{\mathrm{Re}_x}} \tag{25.17}$$

Momentum Equation: Flat plate-6

- The growth of the boundary layer predicted by this expression is remarkably close to the one obtained by the more accurate similarity solution.
- A typical velocity profile has been chosen in the analysis shown. One could use several other profiles and get reasonably good agreement with the numerically obtained similarity solutions.

Energy Equation Derivation-1

- Similar to the momentum integral equation an energy integral equation can be derived by integrating the energy equation:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \qquad (25.18)$$

- Add continuity equation on the LHS of equ^n (25.18) to obtain:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+T\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\partial}{\partial x}uT+\frac{\partial}{\partial y}vT=\alpha\frac{\partial^2 T}{\partial y^2} \qquad (25.19)$$

Energy Equation Derivation-2

- Integrate equ^n (25.19) over the thickness of the thermal boundary layer:

$$\int_0^{\delta_T}\frac{\partial}{\partial x}uT\,dy+\int_0^{\delta_T}\frac{\partial}{\partial y}(vT)\,dy=\int_0^{\delta_T}\alpha\frac{\partial^2 T}{\partial y^2}dy$$

$$\int_0^{\delta_T}\frac{\partial}{\partial x}uT\,dy+vT\big|_{\delta_T}-vT\big|_0=\alpha\left[\left.\frac{\partial T}{\partial y}\right|_{y=\delta_T}-\left.\frac{\partial T}{\partial y}\right|_{y=0}\right] \qquad (25.20)$$

- Equ^n (25.2) can be used to replace the second term in the equ^n (25.20).

Energy Equation Derivation-3

- In addition assume the temperature profile smoothly meshes with the free stream temperature making the gradient of temperature zero at the edge of the boundary layer. Equ^n (25.20) simplifies to:

$$\int_0^{\delta_T}\frac{\partial}{\partial x}uT\,dy+T_\infty\int_0^{\delta_T}-\frac{\partial u}{\partial x}dy=-\alpha\left.\frac{\partial T}{\partial y}\right|_{y=0}$$

$$\int_0^{\delta_T}\frac{\partial}{\partial x}u(T-T_\infty)\,dy=\frac{q_w}{\rho c_p} \qquad (25.21)$$

Energy Equation Derivation-4

- Use Leibnitz rule for equ^n (25.21) and noting that T = T_∞ at the edge of the thermal boundary layer obtain:

$$\frac{d}{dx}\int_0^{\delta_T}u(T-T_\infty)\,dy=\frac{q_w}{\rho c_p} \qquad (25.22)$$

- Equ^n (25.22) is the integral form of the energy equation. Here also a temperature profile is required and therefore assume:

$$T=a_1+b_1y+c_1y^2+d_1y^3 \qquad (25.23)$$

Energy Equation: Flat Plate-1

- Conditions required for evaluation of the constants are:

$$y=0,\quad \mathrm{T=T_w}\text{- Wall temperature specified}$$
$$y=0,\quad \frac{d^2T}{dy^2}=0 \text{ - no slip and equ}^{\mathrm{n}}\ (25.18)$$
$$y=\delta_{\mathrm{T}},\quad \mathrm{T=T_\infty}\text{ - tempeature=free stream tempeature at y=}\delta_{\mathrm{T}}$$
$$y=\delta_{\mathrm{T}},\quad \frac{dT}{dy}=0 \text{ - temperature profile smooth at y= }\delta_{\mathrm{T}} \tag{25.24}$$

Energy Equation: Flat Plate-2

- Using conditions in equ^n (25.24) in equ^n (25.23):

$$\begin{aligned}
&T_w=a_1\\
&T_\infty=T_w+b_1\delta_t+c_1\delta_t^2+d_1\delta_t^3\\
&0=b_1+2c_1\delta_t+3d_1\delta_t^2\\
&0=2c_1
\end{aligned} \tag{25.25}$$

- Solving the equ^{n}s (25.25) gives:

$$b=-3d_1\delta_t^2; d=-\frac{T_\infty-T_w}{2\delta_t^3} \tag{25.26}$$

Energy Equation: Flat Plate-3

- The temperature profile therefore becomes:

$$\begin{aligned}
&T=T_w-\frac{3}{2}\frac{(T_w-T_\infty)y}{\delta_t}+\frac{T_w-T_\infty}{2}\frac{y^3}{\delta_t^3}\\
&\frac{T-T_w}{T_\infty-T_w}=\frac{3}{2}(y/\delta_t)-\frac{1}{2}(y/\delta_t)^3\\
&\Rightarrow\frac{T-T_\infty}{T_w-T_\infty}=1-\frac{3}{2}(y/\delta_t)+\frac{1}{2}(y/\delta_t)^3
\end{aligned} \tag{25.27}$$

- Substitute in the energy integral equ^n (25.21)

$$\frac{d}{dx}\int_0^{\delta_t}u_\infty\left(\frac{3}{2}(y/\delta)-\frac{1}{2}(y/\delta)^3\right)(T_w-T_\infty)\left(1-\frac{3}{2}(y/\delta_t)+\frac{1}{2}(y/\delta_t)^3\right)=\frac{q_w''}{\rho c_p} \tag{25.28}$$

Energy Equation: Flat Plate-4

- Both the thermal and momentum boundary layers are together in the equ^n (25.28). Define:

$$\frac{y}{\delta_t}=n;\frac{\delta_t}{\delta}=\varphi \tag{25.29}$$

- Substitute in equ^n (25.28):

$$\begin{aligned}
&\frac{d}{dx}\int_0^1 u_\infty(T_w-T_\infty)\left(\frac{3}{2}\phi n-\frac{1}{2}\phi^3n^3\right)\left(1-\frac{3}{2}n+\frac{1}{2}n^3\right)\delta_t dn=\frac{-k}{\rho c_p}\left[-\frac{3}{2}\frac{(T_w-T_\infty)}{\delta_t}\right]\\
&\frac{d}{dx}u_\infty(T_w-T_\infty)\int_0^1\phi\left(\frac{3}{2}n-\frac{9}{4}n^2+\frac{3}{4}n^4\right)+\phi^3\left(-\frac{1}{2}n^3+\frac{3}{4}n^4-\frac{n^6}{4}\right)\delta_t dn=\alpha\left[\frac{3}{2}\frac{(T_w-T_\infty)}{\delta_t}\right]
\end{aligned} \tag{25.30}$$

Energy Equation: Flat Plate-5

- Equ^n (25.30) can be integrated assuming φ to be a constant since φ is not a function of y:

$$u_\infty (T_w - T_\infty) \frac{d}{dx} \delta_t \left(\frac{3}{20} \phi - \frac{3}{280} \phi^3 \right) = \frac{3}{2} \frac{\alpha}{\delta_t} (T_w - T_\infty) \tag{25.31}$$

- In this equation, in general φ can be less than or greater than unity. The solutions are likely to be different for these two cases.

- When $\phi = \frac{\delta_t}{\delta} < 1, \frac{3}{280} \phi^3 << \frac{3}{20} \phi$

$$u_\infty (T_w - T_\infty) \frac{d}{dx} \delta_t \frac{3}{20} \phi = \frac{3}{2} \frac{\alpha}{\delta_t} (T_w - T_\infty) \tag{25.32}$$

Energy Equation: Flat Plate-6

- Simplify equ^n (25.32):

$$\begin{aligned} &\frac{u_\infty}{10} \delta_t \frac{d}{dx} \delta \phi^2 = \alpha \Rightarrow \frac{u_\infty}{10} \delta_t \left[\phi^2 \frac{d\delta}{dx} + 2\delta\phi \frac{d\phi}{dx} \right] = \alpha \\ &\Rightarrow \frac{u_\infty}{10} \left[\delta\phi^3 \frac{d\delta}{dx} + 2\delta^2 \phi^2 \frac{d\phi}{dx} \right] = \alpha \end{aligned} \tag{25.33}$$

- Equ^n (25.15) for the growth of the momentum boundary layer gives:

$$\delta \frac{d\delta}{dx} = \frac{140}{13} \frac{\upsilon}{u_\infty}; \delta^2 = \frac{280}{13} \frac{\upsilon x}{u_\infty} \tag{25.34}$$

Energy Equation: Flat Plate-7

- Substitute equ^n (28.34) in equ^n (28.33) and simplify:

$$\begin{aligned} &\frac{u_\infty}{10} \left[\phi^3 \frac{140}{13} \frac{\upsilon}{u_\infty} + 2\phi^2 \frac{280}{13} \frac{\upsilon x}{u_\infty} \frac{d\phi}{dx} \right] = \alpha \\ &\frac{14}{13} \upsilon \phi^3 + \frac{56}{13} \upsilon x \phi^2 \frac{d\phi}{dx} = \alpha \\ &\Rightarrow \phi^3 + \frac{4}{3} x \frac{d\phi^3}{dx} = \frac{\alpha 13}{\upsilon 14} \end{aligned} \tag{25.35}$$

- Equ^n (28.35) is a first order differential equation in φ^3,is solved using the solution of the homogenous part and particular integral

Energy Equation: Flat Plate-8

- Solution for Equ^n (25.35) is:

$$\phi^3 = Cx^{-3/4} + \frac{1}{1.08\,\mathrm{Pr}}$$

- Assuming that the thermal boundary layer starts growing from $x = x_0$, the constant can be evaluated and the solution becomes:

$$\phi^3 = -\frac{x_0^{3/4}}{1.08\mathrm{Pr}} x^{-3/4} + \frac{1}{1.08\,\mathrm{Pr}} = \frac{1}{1.08\mathrm{Pr}} \left[1 - \left(\frac{x_0}{x} \right)^{3/4} \right]$$

- If $x_0 << x$ then:

$$\phi = \frac{1}{1.08}\mathrm{Pr}^{-1/3} \tag{25.36}$$

Energy Equation: Flat Plate-9

- When both the boundary layers start at nearly the same location the ratio is simply the Prandtl number which is a constant.
- Use the temperature profile equ^n (25.27) to evaluate the heat transfer coefficient:

$$hA(T_w - T_\infty) = -kA\left.\frac{dT}{dy}\right|_{y=0} = -\frac{k}{\delta_t}(-\frac{3}{2})(T_w - T_\infty)$$
$$\Rightarrow h = \frac{3}{2}\frac{k}{\delta_t} = \frac{3}{2}\frac{k}{\delta\phi} \tag{25.37}$$

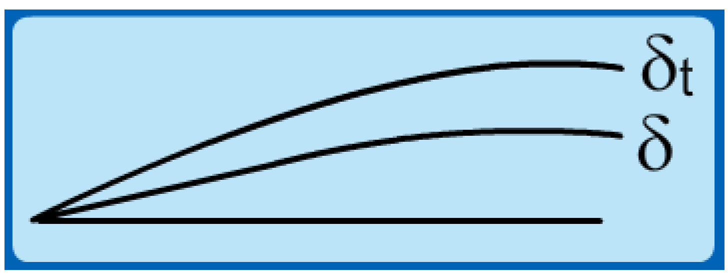
Energy Equation: Flat Plate-10

- Notice that in equ^n (25.37) the heat transfer coefficient is inversely proportional to the thermal boundary layer thickness.
- Use equ^{n}s (25.36) and (25.34) in equ^n (25.37) to obtain:

$$Nu = \frac{hx}{k} = \frac{3(1.03)}{2(4.64)}\mathrm{Re}^{1/2}\,\mathrm{Pr}^{1/3} = 0.332\,\mathrm{Re}^{1/2}\,\mathrm{Pr}^{1/3}$$

Energy Equation: Flat Plate-11

- Now consider the case where $\delta_t > \delta$



- For a given 'x' location, when y > δ u = U∞ The momentum boundary layer is fully developed within the thermal boundary layer.

Energy Equation: Flat Plate-12

- Start with the energy integral equ^n (25.28) and split the integral into two parts since the velocity is different in the two regions:

$$\int_0^{\delta_T} u_\infty\left(\frac{3}{2}(y/\delta) - \frac{1}{2}(y/\delta)^3\right)(T_w - T_\infty)\left[1 - \frac{3}{2}(y/\delta_t) + \frac{1}{2}(y/\delta_t)^3\right]dy$$
$$= \int_0^{\delta} u_\infty(T_w - T_\infty)\left[\frac{3}{2}(y/\delta) - \frac{1}{2}(y/\delta)^3\right]\left[1 - \frac{3}{2}(y/\delta_t) + \frac{1}{2}(y/\delta_t)^3\right]dy$$
$$+\int_{\delta}^{\delta_T} u_\infty(T_w - T_\infty)\left[1 - \frac{3}{2}(y/\delta_t) + \frac{1}{2}(y/\delta_t)^3\right]dy \tag{25.38}$$

Energy Equation: Flat Plate-13

- Define $\hat{n} = y/\delta; \phi = \frac{\delta_t}{\delta}; n = y/\delta_t$ and substitute in equ^n (25.38)

$$u_\infty(T_w-T_\infty)\left[\int_0^1\left[\frac{3}{2}\hat{n}-\frac{1}{2}\hat{n}^3\right]\left[1-\frac{3}{2}\frac{\hat{n}}{\phi}+\frac{1}{2}\frac{\hat{n}^3}{\phi^3}\right]d\hat{n}\delta+\int_{1/\phi}^1\left[1-\frac{3}{2}n+\frac{1}{2}n^3\right]dn\delta_t\right]$$

$$=u_\infty(T_w-T_\infty)\left[\int_0^1\left[\frac{3}{2}\hat{n}-\frac{1}{2}\hat{n}^3-\frac{9}{4}\frac{\hat{n}^2}{\phi}+\frac{3}{4}\frac{\hat{n}^4}{\phi}+\frac{3}{4}\frac{\hat{n}^4}{\phi^3}-\frac{1}{4}\frac{\hat{n}^6}{\phi^3}\right]\delta+\left[n-\frac{3}{4}n^2+\frac{n^4}{8}\right]_{1/\phi}^1\delta_t\right]$$

$$=u_\infty(T_w-T_\infty)\left[\left[\frac{5}{8}-\frac{72}{120\phi}+\frac{16}{140\phi^3}\right]\delta+\left[\frac{3}{8}\phi-1+\frac{3}{4\phi}-\frac{1}{8\phi^3}\right]\delta_t\right] \tag{25.39}$$

Energy Equation: Flat Plate-14

- Equ^n (25.39) is further simplified to:

$$u_\infty(T_w-T_\infty)\left[\frac{3}{8}\phi-\frac{3}{8}+\frac{1}{\phi}\left(\frac{3}{4}-\frac{72}{120}\right)+\frac{1}{\phi^3}\left(-\frac{1}{8}+\frac{16}{140}\right)\right]$$

$$=u_\infty(T_w-T_\infty)\left[\frac{3}{8}\phi-\frac{3}{8}+\frac{3}{20\phi}-\frac{3}{280\phi^3}\right] \tag{25.40}$$

- Equ^n (25.40) is the integral portion of LHS marked green of the energy integral equ^n (25.22). Including the RHS also it becomes:

$$u_\infty(T_w-T_\infty)\frac{d}{dx}\left[\frac{3}{8}\phi-\frac{3}{8}+\frac{3}{20\phi}-\frac{3}{280\phi^3}\right]\delta=\alpha\left[\frac{3}{2}\frac{(T_w-T_\infty)}{\delta_t}\right] \tag{25.41}$$

Energy Equation: Flat Plate-15

- Equ^n (25.41) simplifies to:

$$\frac{d}{dx}\delta\left(\frac{3}{8}\phi-\frac{3}{8}+\frac{3}{20\phi}-\frac{3}{280\phi^3}\right)=\frac{3}{2\,\mathrm{Pr}\,\delta_t}\frac{\mu}{\rho u_\infty} \tag{25.42}$$

- In addition, from the momentum boundary layer:

$$\frac{d\delta}{dx}\frac{39}{280}=\frac{3}{2\delta}\frac{\mu}{\rho u_\infty}$$

- When φ is large only the first term will survive and equation is very similar to equ^n (25.32) and the conclusions that were drawn earlier continue to hold i.e. $\phi\approx\frac{1}{\mathrm{Pr}^{1/3}}$

Energy Equation: Flat Plate-16

Therefore here too:

$$Nu_x = 0.332\ Re_x^{1/2}\ Pr^{1/3} \tag{25.43}$$

- Almost the same equation is obtained from the similarity analysis also
- Notice that the ratio of the boundary layer thicknesses is proportional to the Prandtl number to some power. Heat transfer coefficient is inversely proportional to the thickness of the thermal boundary layer.

Recap

In this class:

- Integral equations for the momentum and thermal boundary layers are obtained
- Solution of these equations for flow over a flat surface is demonstrated