NPTEL Video Course

An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-Tori and Elliptic Curves

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Lecture 9: A First classification of Riemann surfaces

Riemann surfaces may be classified to some extent on the basis of their universal covering spaces and their fundamental groups. This can be achieved by using Covering Space Theory and studying properties of certain special subgroups of Moebius transformations, as will be seen in later lectures. In the following statements, $\cong$ denotes a holomorphic (or analytic- or conformal-) isomorphism of Riemann surfaces, sometimes also called a biholomorphic map of Riemann surfaces. It is simply a holomorphic map with a holomorphic inverse. Further X denotes a Riemann surface.

9.1 Theorem: Surfaces covered by the Riemann sphere

If the universal covering space of X is $\mathbb{P}^1_{\mathbb{C}}$ (the Riemann sphere), then $X \cong \mathbb{P}^1_{\mathbb{C}}$.

9.2 Theorem: Surfaces covered by the complex plane

If the universal covering space of X is $\mathbb{C}$, then the fundamental group $\Pi_1(X)$ has to be abelian and in fact has to be isomorphic to $(0), \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. The Riemann surfaces X corresponding to these cases are as follows:

- If the fundamental group $\Pi_1(X) = 0$, then $X \cong \mathbb{C}$.
- If $\Pi_1(X) = \mathbb{Z}$, then $X \cong \mathbb{C}^*$.
- If $\Pi_1(X) = \mathbb{Z} \oplus \mathbb{Z}$, then $X \cong$ a complex torus.

9.3 Theorem: Surfaces X covered by the upper half-plane or the unit disc

If the universal covering space of X is $\mathbb{U} = \{z \in \mathbb{C} : \text{Im}(z) > 0\} \cong \Delta \ = \{z \in \mathbb{C} : |z| < 1\}$, then X is given by the following cases:

- If $\Pi_1(X)$ is abelian and $\Pi_1(X) = 0$, then $X \cong \mathbb{U} \cong \Delta$.
- If $\Pi_1(X)$ is abelian and $\Pi_1(X) \cong \mathbb{Z}$, then either $X \cong \Delta^* = \Delta \setminus \{0\}$ or $X \cong \Delta_r = \{z \in \mathbb{C} : r < |z| < 1\}$ for a unique $r \in (0, 1)$. There are no other possibilities, and for different r's the Δ_r's are not biholomorphic.

- If $\Pi_1(X)$ is not abelian and X is compact, then X is isomorphic to a Riemann surface structure on a g-torus where $g > 1$.

In the following we re-state the above results relative to the fundamental group Π_1.

9.4 Theorem: Surfaces with trivial Π_1

If X has trivial fundamental group, then X is given by the following cases:

- If the universal covering space of X is the Riemann sphere, then $X \cong$ the Riemann sphere. This occurs iff X is compact.
- If the universal covering space of X is the complex plane, then $X \cong$ the complex plane or to the unit disc (equivalently the upper half-plane). These occur iff X is non-compact.

9.5 Theorem: Surfaces with $\Pi_1 \cong \mathbb{Z}$

- If X has universal covering the complex plane, then $X \cong$ the punctured plane.
- If X has universal covering the unit disc (equivalently the upper half-plane), then $X \cong$ the punctured unit disc or to an annulus of the form Δ_r for a unique $r \in (0,1) \subset \mathbb{R}$.

9.6 Theorem: Compact Surfaces with $\Pi_1 \cong \mathbb{Z} \oplus \mathbb{Z}$

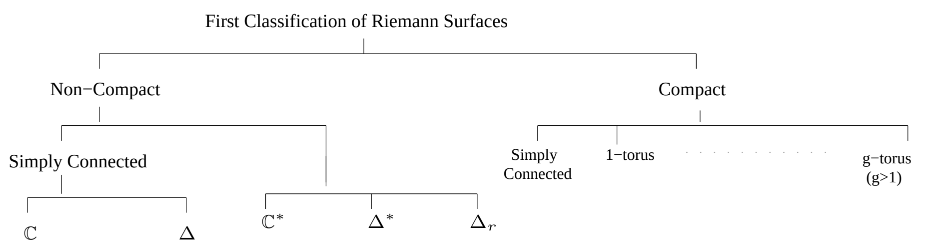
X has to be isomorphic to a complex torus.

9.7 Theorem: Compact Surfaces with Non-abelian Π_1

X has to be isomorphic to a Riemann surface structure on a real g-torus for $g > 1$.

9.8 Table of Riemann Surfaces based on Topological Properties

We define the moduli space for Riemann surfaces with given topological properties to be the set of isomorphism classes of Riemann surfaces satisfying those properties. The number of moduli is then defined as the dimension of the moduli space in an intuitive sense. The term moduli was coined by G. F. B. Riemann about 156 years ago who described moduli spaces and computed the number of moduli. Finite sets are given dimension zero. The resulting picture of the classification based on Fundamental Groups and topological properties like compactness, non-compactness and simply-connectedness or otherwise is given below.



No. of Moduli (Dimension of Moduli space)	0			1 real parameter		0	1 complex parameter	· · · · · · · · · · ·	3g−3 complex parameters
Moduli space	$\{[\mathbb{C}], [\mathbb{U}]\}$			$\{[\mathbb{C}^*], [\Delta^*]\} \cup \{[\Delta_r] : r \in (0,1)\}$		$\{[\mathbb{P}^1_{\mathbb{C}}]\}$	$\mathcal{M}_1 \cong \mathbb{C}$		$\mathcal{M}_g$
Universal covering	$\mathbb{C}$	Δ	$\mathbb{C}$	Δ	Δ	$\mathbb{P}^1_{\mathbb{C}}$	$\mathbb{C}$		$\mathbb{U}$
Fundamental group	0	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$		Not abelian