NPTEL Video Course

An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-Tori and Elliptic Curves

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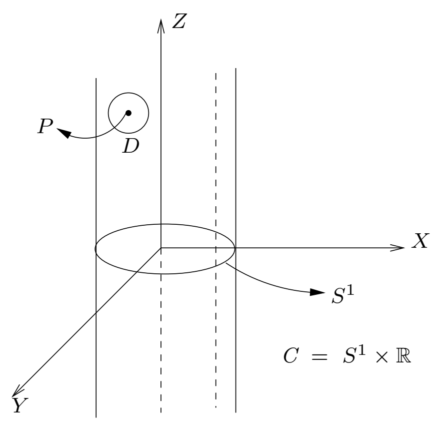
Lecture 4: A Riemann Surface Structure on a Cylinder

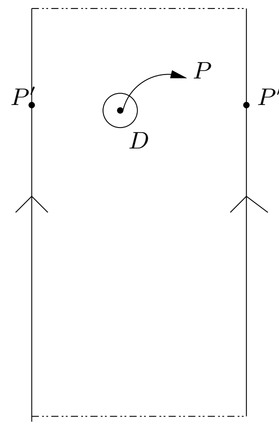
4.1 Riemann surface structure on a cylinder

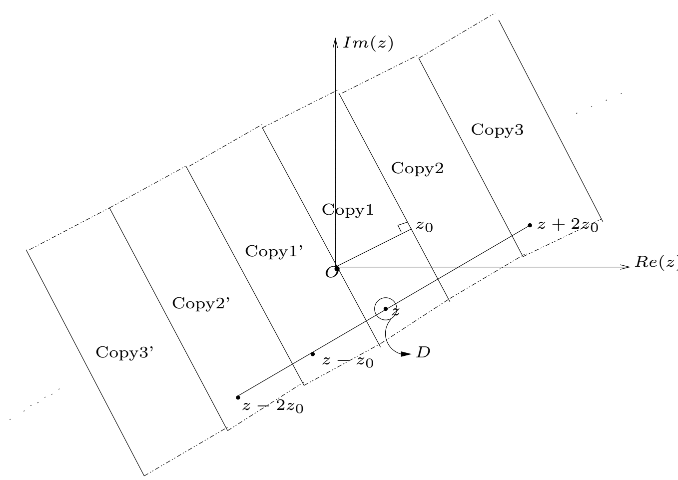
We want to make the cylinder C into a Riemann surface.



Cutting along the dotted line (not the negative Z-axis, but along a line on the cylinder parallel to its axis), we get a strip like the one shown on the next page. We can undo this operation by identifying the edges of the strip. If we put infinitely many copies of the strip side-by-side we get back the complex plane. We would also like to know how to undo that. For this, we choose a point on one of the edges of the strip, call it the origin and draw the axes. Refer to the image below. Translation by z_0 will move (map) copy 1 to copy 2; by $2z_0$ will move copy 1 to copy 3; by $-z_0$ will move copy 1 to copy 1' and so on. So translations by suitable integer multiples of z_0 are precisely the operations that identify all these strips together to get back the strip that we started with.

Formalising
For a fixed $z_0 \in \mathbb{C}$, $z_0 \neq 0$, we define an equivalence relation on $\mathbb{C}$: if $z,\ z' \in \mathbb{C}$, we define $z \sim z' \Leftrightarrow z = z' + nz_0$, where $n \in \mathbb{Z}$. Consider the set of all equivalence classes $\mathbb{C}/\sim$, and the natural map $\pi : \mathbb{C} \longrightarrow \mathbb{C}/\sim$, given by, $z \mapsto [z]$.





Here $[z]$ is the equivalence class of z i.e., the set of all translates of z by integer multiples of z_0. By means of π, we can identify $\mathbb{C}/\sim$ with the cylinder C (each point in the interior of a strip is the only point in the strip in its equivalence class, and two points occur on the strip in the equivalence class of a point on one of the edges).

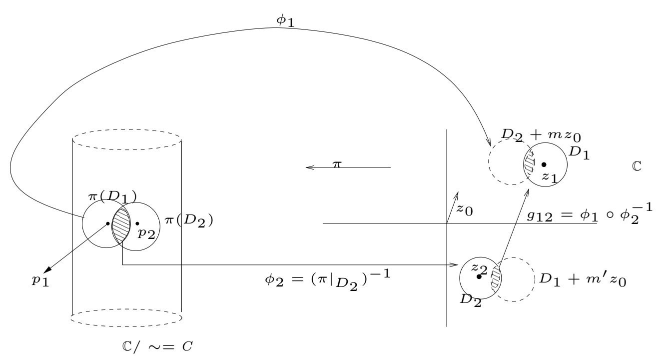
4.2 Remarks

- If $p \epsilon C$, then $\pi^{-1}(p)$ is just a copy of $\mathbb{Z}$. For if $z \epsilon \mathbb{C}$ such that $\pi(z) = p$, then z is one of the translates of p considered as a point in a fixed strip. So, $\pi^{-1}(p)$ = the set of points equivalent to z, i.e., $\pi^{-1}(p) = z + \mathbb{Z}z_0 = \{z + nz_0 \mid n \epsilon \mathbb{Z}\}$, which is bijective to $\mathbb{Z}$.
- If S is a subset of C and S' is a subset of $\mathbb{C}$ such that $\pi(S') = S$, and if

$\pi|_{S'}$ is bijective, then $\pi^{-1}(S) = S' + \mathbb{Z}z_0 = \{s' + nz_0 \mid s' \,\epsilon\, S',\ n \,\epsilon\, \mathbb{Z}\}$. If we take a sufficiently small disc about the point $z \,\epsilon\, \mathbb{C}$, then the image of this disc in C looks like a disc about the point p. The inverse image of this latter disc is just the union of translates of the original disc about z.

- The map $\pi : \mathbb{C} \longrightarrow \mathbb{C}/\sim$ is surjective. Therefore we can give $\mathbb{C}/\sim$ the quotient topology for which a subset S of $\mathbb{C}/\sim$ is open iff $\pi^{-1}(S)$ is open in $\mathbb{C}$. This definition automatically makes π continuous. With this the identification of C with $\mathbb{C}/\sim$ is not only as sets but as topological spaces as well. This is because if we take a point $z \,\epsilon\, \mathbb{C}$ and a small disc-like neighbourhood D around it, its image is a small disc-like neighbourhood $\pi(D)$ around $p = \pi(z) \,\epsilon\, C$. The set $\pi(D)$ is open because $\pi^{-1}(\pi(D))$ is simply the union of D and its translates by integer multiples of z_0, which is a union of open sets. Thus in fact π is an open map. Since π is also continuous by definition, π from D to $\pi(D)$ is a homeomorphism. All this shows that π is a surjective local homeomorphism.

So given a small disc D centred at z with $\pi(z) = p$ on the cylinder C, we see that $\pi^{-1}(\pi(D)) = D + \mathbb{Z}z_0 = \{z' + nz_0 \mid z' \,\epsilon\, D,\ n \,\epsilon\, \mathbb{Z}\}$ is open, being a union of open sets $D + \mathbb{Z}z_0 = \bigcup_{n\epsilon\mathbb{Z}} (D + nz_0)$, each of which is homeomorphic to D because it is just a translate of D. This will enable us to define a chart at p on C as follows. We take the pair $(\pi(D)\ ,\ (\pi|_D)^{-1})$ as the chart containing $p \,\epsilon\, C$. We can do this for every point in C. To show that this collection of charts is an atlas i.e., that it defines a Riemann surface structure on C, we only need to check the pairwise compatibility of this collection of charts that covers C.



Take two intersecting charts $\pi(D_1)$ and $\pi(D_2)$ about the points p_1 and p_2 where $p_1 = \pi(z_1)$, $p_2 = \pi(z_2)$ and $\phi_i = (\pi|_{D_i})^{-1}$ are homeomorphisms for $i = 1, 2$. We need $g_{12} = \phi_1 \circ \phi_2^{-1}$ to be holomorphic on $\phi_2(\pi(D_1) \cap \pi(D_2))$ (refer to the diagram above). As shown in the diagram the image of $\pi(D_1)$ under ϕ_2 is a translate of D_1 say $D_1 + m'z_0$ and that of $\pi(D_2)$ under ϕ_1 is also a translate $D_2 + mz_0$ of D_2; moreover these images respectively intersect D_2 and D_1. Thus g_{12} will be a translation by an integer multiple of the vector z_0. This forces $m = -m'$ and $g_{12} : w \mapsto w + mz_0$ for $w \,\epsilon\, D_1 \cap D_2$.

This is certainly holomorphic since it is just a translation. Thus the transition functions are holomorphic and we have a compatible collection of charts, giving a Riemann surface structure on the cylinder C. Since it was gotten beginning with an $z_0 \neq 0$ in $\mathbb{C}$, we sometimes denote this Riemann surface as C_{z_0}.

4.3 Group Theoretic Interpretation

In the above we fixed the vector z_0 and looked at translates by it. Recall that $Aut_{hol}(\mathbb{C}) = \{z \mapsto az + b \mid a \neq 0 \ ; \ a, b \ \epsilon \ \mathbb{C}\}$. Now instead of looking at z_0, we look at the translation T_{z_0} by z_0 as an element of $Aut_{hol}(\mathbb{C})$. Let $\mathbb{Z}.T_{z_0} := \{z \mapsto z + nz_0 \mid n \ \epsilon \ \mathbb{Z}\}$. This is an abelian subgroup of $Aut_{hol}(\mathbb{C})$ (under composition of mappings) and is isomorphic to the group $(\mathbb{Z}, +)$. This can be seen from the following diagram:

$$\begin{tikzcd} z \arrow[r, "T_{z_1}"] \arrow[dr, "T_{z_1+z_2}"'] & z + z_1 \arrow[d, "T_{z_2}"] \\ & z + z_1 + z_2 \end{tikzcd}$$

where $T_{z_1+z_2} = T_{z_1} \circ T_{z_2}$ and where the isomorphism is given by the map $\mathbb{Z}.T_{z_0} \longrightarrow (\mathbb{Z}, +)$ that sends T_{nz_0} to n. Now we have the group of translations $\mathbb{Z}.T_{z_0}$ acting on the set $\mathbb{C}$, and so we can talk about the orbits of this action. Recall that the orbit of a point is the collection of all points gotten by applying the elements of the group to the given point.

Now the orbit of z is the set of translates of z which is also the equivalence class of z. Thus, $\mathbb{C}/\sim$ is bijective to the set of orbits which is $\mathbb{C}/\mathbb{Z}.T_{z_0}$ and this latter set is bijective to $\mathbb{C}/\mathbb{Z}$. Since this last set is the quotient of the group of complex numbers under addition by the normal subgroup of integers, we may expect a group structure on C also and that is indeed the case.