NPTEL Video Course

An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-Tori and Elliptic Curves

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Lecture 5: A Riemann Surface Structure on a Torus

5.1 Recall: Riemann surface structure on a cylinder

In this lecture, we continue discussing Riemann surface structures on a cylinder. As in the previous lecture we fix a complex number $z_0 \neq 0$ and let T_{z_0} denote translation by z_0 i.e. $T_{z_0}(z) = z + z_0$. We let $nT_{z_0} = T_{nz_0}$ and let $G = \mathbb{Z}.T_{z_0}$, the subgroup of Moebius transformations generated by T_{z_0} under composition $\circ$ of mappings. As we noted in the previous lecture $(G, \circ) \cong (\mathbb{Z}, +)$ the isomorphism given by $nT_{z_0} \mapsto n$. Moreover $\mathbb{C}/G$ can be topologically identified with the cylinder C, because by going modulo G we identify two points iff they are translates by integer multiples of z_0. We let the map $\pi : \mathbb{C} \longrightarrow \mathbb{C}/G$ to be the natural one which sends every point to its equivalence class under G i.e. π sends every point to its orbit under G. We saw that if we give $\mathbb{C}/G$ the quotient topology, then π becomes continuous, open and is locally bijective. In other words π is a local homeomorphism. We used this property to define an atlas of compatible charts giving a Riemann surface structure on $\mathbb{C}/G$ and called it C_{z_0}. Then the natural map $\pi : \mathbb{C} \longrightarrow C_{z_0}$ is holomorphic. These assertions essentially follow from the simple fact that translation maps are holomorphic. How does the Riemann surface structure C_{z_0} depend on z_0? That it does not is what the following result says:

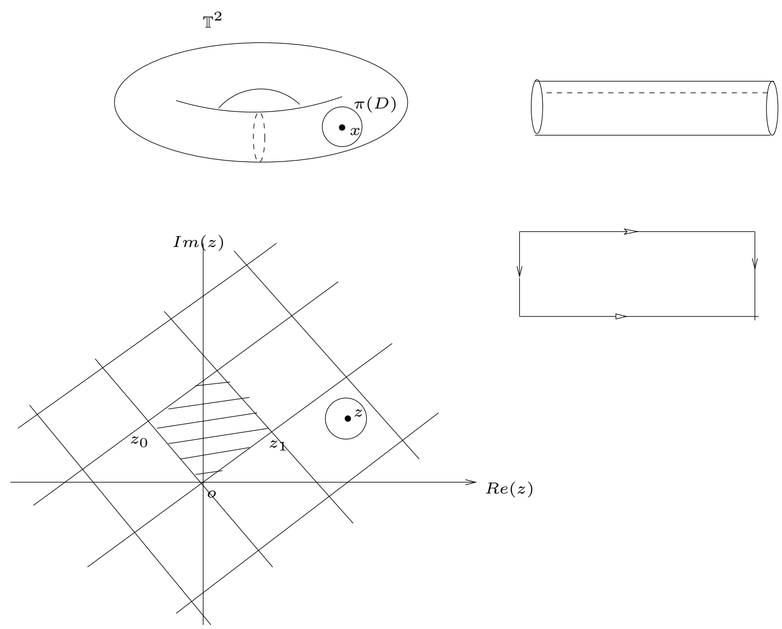
Theorem 1 *The set, of holomorphic isomorphism classes of Riemann surface structures $\mathcal{C}$ on the cylinder* C *which admit a non-constant holomorphic map $\mathbb{C} \longrightarrow \mathcal{C}$, is a singleton and it is represented by the natural Riemann surface structure on the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.*

The proof of this theorem follows from covering space theory which will be introduced in the subsequent lectures. For now we can see that C is isomorphic to the punctured plane at least at the level of groups. This is because $C = \mathbb{C}/G \simeq \mathbb{C}/\mathbb{Z}$ and the latter is isomorphic to $\mathbb{C}^*$, which can be seen as follows. Consider the sequence of groups and group homomorphisms:

$$\{0\} \longrightarrow \mathbb{Z} \xhookrightarrow{\phi_1} \mathbb{C} \xrightarrow{\phi_2} \mathbb{C}^* \longrightarrow \{1\}$$

Here ϕ_1 is the natural inclusion map and ϕ_2 is given by $z \mapsto e^{2\pi i z}$. ϕ_1 is injective. For a group with an incoming homomorphism and an outgoing homomorphism, we say that exactness holds at that group if the image of the incoming homomorphism is equal to the kernel of the outgoing homomorphism. So ϕ_1 is injective means that the sequence is exact at $\mathbb{Z}$. ϕ_2 is surjective since every non-zero complex number admits a logarithm; thus the sequence is exact at $\mathbb{C}^*$. Further Kernel $(\phi_2) = \mathbb{Z} =$ Image (ϕ_1). Hence the sequence is exact at $\mathbb{C}$. Therefore the above sequence is called a "short exact sequence". By the first isomorphism theorem in Group Theory, we have the isomorphism $\mathbb{C}^* \cong \mathbb{C}/\mathbb{Z}$. Thus $C = \mathbb{C}/G$ is isomorphic to $\mathbb{C}^*$ as groups. The theorem above says that there is also an isomorphism as Riemann surfaces; in fact this very group isomorphism would also be an isomorphism of Riemann surfaces as we shall see later!

5.2 Riemann surface structures on a Torus



Fix two complex numbers $z_0, z_1 \in \mathbb{C} \setminus \{0\}$ with $z_1/z_0 \notin \mathbb{R}$. Let

$$G = \{nT_{z_0} + mT_{z_1} |\ n, m \ \epsilon\ \mathbb{Z}\}.$$

Then we have an isomorphism of groups $G \cong \mathbb{Z} \times \mathbb{Z}$ defined by the map

$$(nT_{z_0} + mT_{z_1}) \mapsto (n, m).$$

Further $\mathbb{C}/G$ may be identified set-theoretically with the torus $\mathbb{T}^2 = S^1 \times S^1$. For the set of G-orbits may be gotten by taking the closed parallelogram with the vectors z_i as co-terminous edges (meeting at the origin) and identifying parallel sides leading to a torus. Consider the natural map $\pi : \mathbb{C} \longrightarrow \mathbb{C}/G$

given by $z \mapsto [z]$ where $[z]$ is the equivalence class of z (or the G-orbit of z). Giving $\mathbb{C}/G$ the quotient topology makes π continuous and an open map. For given $z \in \mathbb{C}$ and $x \in \mathbb{C}/G$ such that $\pi(z) = x$, if D is a sufficiently small disc centred at z then $\pi(D)$ will be an open set centred at x. We now define charts of the form $(\pi(D), (\pi|_D)^{-1})$ covering $\mathbb{C}/G$. These charts are compatible because the transition functions (as in the case of the cylinder) are translations which are holomorphic. This collection of charts thus gives us an atlas, i.e., a Riemann surface structure on the torus that we may denote by $\mathbb{T}^2_{z_0,z_1}$ to signify the dependence on z_0, z_1. It is natural to ask how this Riemann surface structure depends on z_0, z_1. We will answer this question in later lectures, but for now we state the following related result.

Theorem 2 *The set $\mathcal{M}_1$ of holomorphic isomorphism classes of Riemann surface structures on a real torus $\mathbb{T}^2$ is bijective to $\mathbb{C}$. In fact, $\mathcal{M}_1$ also acquires naturally the structure of a Riemann surface with respect to which this bijection with $\mathbb{C}$ becomes a holomorphic isomorphism (of Riemann surfaces)!*