

**An Introduction to Riemann Surfaces and Algebraic Curves:
Complex 1-Tori and Elliptic Curves**

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Lecture 3: Maximal Atlases and Holomorphic Maps of Riemann Surfaces

Recall what we have done so far. We started with a real surface X .

- We wanted to do complex analysis on the surface, for which we had to first define holomorphicity of a function at a point of the surface. To do this we use charts of the form (U, ϕ) , where $\phi : U \rightarrow V \subset \mathbb{C}$ is a homeomorphism of the open subset U (containing the point) onto an open subset V of \mathbb{C} .
- To ensure that the notion of holomorphicity is well-defined and intrinsic, we required that whenever two intersecting charts are used, they must be compatible.
- A collection of charts $\{(U_i, \phi_i) : i \in I\}$ that are pairwise compatible and that cover X is called an atlas, and an atlas gives a Riemann surface structure on X .
- We have seen examples of Riemann surface structures on the plane \mathbb{R}^2 and on the unit 2-sphere \mathbb{S}^2 .

In particular, recall that the complex plane \mathbb{C} is the Riemann surface structure on \mathbb{R}^2 given in the following way:

$$\text{Atlas} = \{(U, \phi)\}, U \subset \mathbb{R}^2, \phi : U \rightarrow \mathbb{C} : (x, y) \mapsto x + iy = z. \quad (*)$$

Consider also the Riemann surface structure on \mathbb{R}^2 given by the following atlas:

$$\{(U, \phi_U) \mid U \subset \mathbb{R}^2 \text{ is open and } \phi_U \text{ is } \phi \text{ in } (*) \text{ restricted to } U\}.$$

Since a function is holomorphic with respect to one of the atlases above if and only if it is holomorphic with respect to the other, it is obvious that both atlases will give the complex plane \mathbb{C} as the Riemann surface structure on \mathbb{R}^2 and therefore we do not want to distinguish between these two atlases. This is the motivation for making the definition of a Riemann surface a little more sophisticated as follows.

Definition 1 *Two atlases on X are said to be equivalent if every chart of one atlas is compatible with every chart of the other.*

We can easily check that “equivalence of atlases” is indeed an equivalence relation. Further for two equivalent atlases, their union is a new atlas which is equivalent to these two and the Riemann surface structure given by each of these three is the same.

An easy Zorn’s lemma argument can be used to prove the following theorem.

Theorem 1 *Given an atlas on X , there is a unique maximal atlas containing the given one.*

Because of the above theorem, we may make the following definition.

Definition 2 *A Riemann surface structure on a real surface X is specified by a maximal atlas.*

Recall the following theorems on the Riemann surface structures on \mathbb{R}^2 and \mathbb{S}^2 .

Theorem 2 Uniformisation for simply connected non-compact Riemann surfaces:

Any simply connected non-compact Riemann surface has to be holomorphically isomorphic to exactly one of the following:

- \mathbb{C} , the complex plane,
- Δ , the unit disc (or \mathbb{U} , the upper half plane).

Remarks :

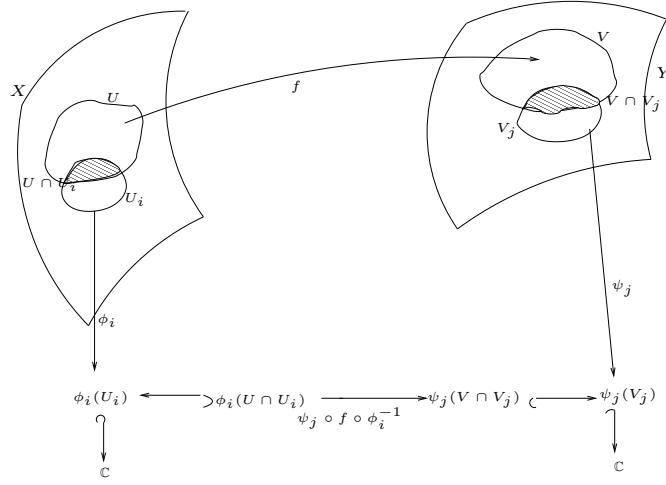
- We know from the Riemann mapping theorem that \mathbb{C} and Δ (or \mathbb{U}) are not holomorphically isomorphic.
- Any disc is holomorphically isomorphic to any half plane.

Theorem 3 Uniformisation for simply connected compact Riemann surfaces:

Any simply connected compact Riemann surface is holomorphically isomorphic to the Riemann sphere $\mathbb{P}_{\mathbb{C}}^1$.

We have as yet not been precise as to what a (holomorphic) isomorphism of Riemann surfaces means. In the above theorems, what does ”isomorphic” mean? More generally suppose we are given a function $f : U \rightarrow V$ where $U \subset X$ and $V \subset Y$ are open subsets of two Riemann surfaces X and Y . When do we say that f is holomorphic?

The answer to this is simple! We just have to decide holomorphicity locally using charts!! We say that f is holomorphic if it is continuous and for any charts (U_i, ϕ_i) of X and (V_j, ψ_j) of Y with $U \cap U_i \neq \emptyset \neq V \cap V_j$ and $f(U \cap U_i) \subset V \cap V_j$, the composition $\psi_j \circ f \circ \phi_i^{-1}$ is holomorphic. This makes sense as $\psi_j \circ f \circ \phi_i^{-1}$ is a function from an open subset of the complex plane to an open subset of the complex plane. If f is defined on all of X and



if for all possible charts (U_i, ϕ_i) and (V_j, ψ_j) the composition $\psi_j \circ f \circ \phi_i^{-1}$ is holomorphic, we then say that f is holomorphic. Further if f is one-to-one and if the inverse function f^{-1} is also holomorphic, we say that f is an isomorphism.

Remarks

- Note that as per our definition of a Riemann surface being specified by the maximal atlas containing a given atlas, the occurrence of any chart (U, ϕ) in the given atlas implies that for each open subset $U' \subset U$, the chart $(U', \phi|_{U'})$ also occurs in the maximal atlas. So our checking above would also involve checking for all sub-charts (U'_i, ϕ'_i) and (V'_j, ψ'_j) !
- Note that with the above definition of a holomorphic map between open subsets of Riemann surfaces, the homeomorphism $\phi: U \rightarrow \phi(U)$ of any chart (U, ϕ) in the (maximal) atlas defines a holomorphic isomorphism between U and $\phi(U)$! For here, $\psi_j \circ f \circ \phi_i^{-1}$ is $i \circ \phi \circ \phi_i^{-1}$ which is i the identity map which is holomorphic.

In conclusion, a Riemann surface structure on a real surface X can be thought of as being obtained by “glueing” together open subsets $\phi_i(U_i) \subset \mathbb{C}$ by the holomorphic transition functions $g_{ij} = \phi_i \circ \phi_j^{-1}$ for an open covering $\{U_i : i \in I\}$ of X .