NPTEL Video Course

An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-Tori and Elliptic Curves

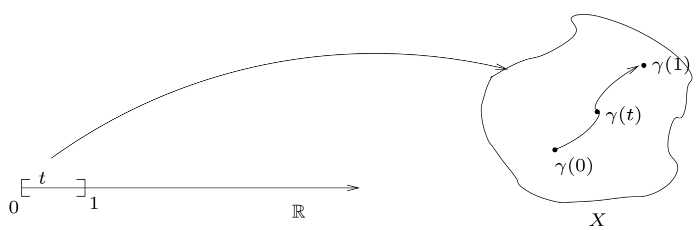
Notes of Lectures given by Dr. T. E. Venkata Balaji
with the assistance of Poorna Pushkala Narayanan
Department of Mathematics, IIT-Madras

Lecture 8: Homotopy and the First Fundamental Group

Let X be a Hausdorff topological space and $x \in X$. The first fundamental group of X based at x is denoted by $\Pi_1(X, x)$.

Definition 1 *Continuous maps* $f, g : X \longrightarrow Y$ *are said to be homotopic, written* $f \sim g$, *if we can find a continuous map* $F : X \times I \longrightarrow Y$, *where* $I = [0, 1] \subset \mathbb{R}$, $(x, t) \mapsto F(x, t)$ *such that* $F(x, 0) = f(x) \, \forall \, x \in X$ *and* $F(x, 1) = g(x) \, \forall \, x \in X$. F *is called a homotopy from* f *to* g. F *is essentially the collection* $\{F_t : X \longrightarrow Y : t \in [0, 1]\}$ *where* $F_t(x) = F(x, t)$ *so that* $F_0 = f$ *and* $F_1 = g$. *It is easy to see that* $\sim$ *is an equivalence relation.*

Definition 2 *A path or an arc in* X *is a continuous map* $\gamma : I = [0, 1] \longrightarrow X$.



Definition 3 *Two paths* γ_1 *and* γ_2 *are fixed-end-point (FEP-) homotopic if they are homotopic by a homotopy* $F : I \times I \longrightarrow X$, $(x, t) \mapsto F(x, t)$, $F_0 = \gamma_1, F_1 = \gamma_2$, *such that moreover* $F(0, t) = \gamma_1(0) = \gamma_2(0) = x_0$ *and* $F(1, t) = \gamma_1(1) = \gamma_2(1) = x_1$.

Definition 4 *If* $\alpha : I \longrightarrow X$ *and* $\beta : I \longrightarrow X$ *are paths such that* $\alpha(1) = \beta(0)$, *we define the concatenation of* α *and* β *(in that order) as the path* $\alpha\beta : I \longrightarrow X$ *which goes from* $x_0 = \alpha(0)$ *to* $x_1 = \alpha(1)$ *at "double the speed" or in "half the time" via* α *and then from* x_1 *to* $x_2 = \beta(1)$ *via* β *(again in "half the time"):*

$$\alpha\beta(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

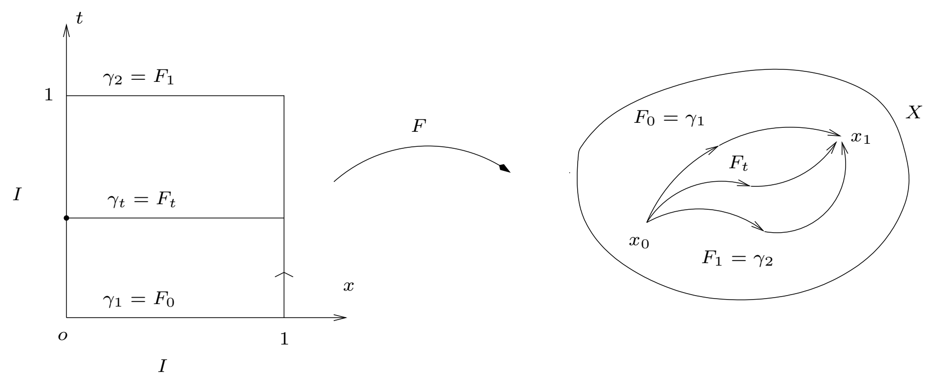


Figure 1: FEP Homotopy

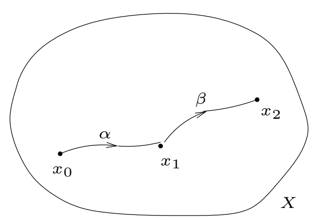


Figure 2: Concatenation

It is easy to check that concatenation and FEP-homotopy are well-behaved with respect to each other : if $\alpha \sim \alpha_1$ and $\beta \sim \beta_1$, then $\alpha\beta \sim \alpha_1\beta_1$.

We define the set $\Pi_1(X, x)$ to be the set of FEP-Homotopy equivalence classes of paths starting from and ending at x. Such paths are usually called loops "based" at x, and so $\Pi_1(X, x)$ is the set of FEP-homotopy equivalence classes of loops based at x. We get a binary operation

$$\Pi_1(X, x) \times \Pi_1(X, x) \longrightarrow \Pi_1(X, x) \text{ given by } ([\alpha], [\beta]) \mapsto [\alpha\beta],$$

where $[\gamma]$ denotes the FEP-homotopy equivalence class of the path γ starting at and ending with x.

8.1 Remarks

- With respect to the binary operation operation above, $\Pi_1(X, x)$ becomes a group. This group may not be commutative (abelian). The identity element is the class of the constant path at x, the associativity is easy to verify and the inverse of a path is the same path taken in the reverse direction.

- If X is arcwise (or pathwise) connected, then we have an isomorphism of groups $\Pi_1(X, x) \cong \Pi_1(X, x')$ for any two points $x, x' \in X$. An isomorphism $\gamma_\# : \Pi_1(X, x) \longrightarrow \Pi_1(X, x')$ is for example given by $[\alpha] \mapsto [\gamma^{-1}\alpha\gamma]$, where γ is a path from x to x'. In other words, if X is arcwise connected, the fundamental group $\Pi_1(X, x)$ is not dependent on the base point x. By this we mean that its isomorphism class is independent of the base point.
- $\Pi_1(X)$ is a topological invariant, i.e., if X and X' are homeomorphic, then their fundamental groups are isomorphic. This is useful in the following way: if we want to show that the real 2-torus is not homeomorphic to the real 2-sphere, the easiest way would be to show that the fundamental group of the torus is different from that of the sphere. Note that the fundamental group of the torus is non-trivial, since there are loops which cannot be shrunk to a point. But the fundamental group of the sphere is trivial.
- The fundamental group of any simply connected space is trivial. Here by a simply connected space we mean a space which is arcwise connected and which has the property that any closed loop at a point can be "shrunk to that point" i.e., it is homotopic to the constant path at that point.

8.2 Examples of Fundamental Groups

- $\Pi_1(\mathbb{S}^2) \cong \{1\}$
- $\Pi_1(\mathbb{C}) \cong \{1\}$
- $\Pi_1(\Delta) \cong \{1\}$
- $\Pi_1(\mathbb{U}) \cong \{1\}$
- $\Pi_1(\text{annulus}) \cong \mathbb{Z} = \Pi_1(\text{punctured plane}) = \Pi_1(\text{punctured disc})$
- $\Pi_1(\mathbb{S}^1) \cong \mathbb{Z}$
- $\Pi_1(\text{cylinder}) \cong \mathbb{Z}$
- $\Pi_1(\text{torus}) \cong \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$