NPTEL Video Course

An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-Tori and Elliptic Curves

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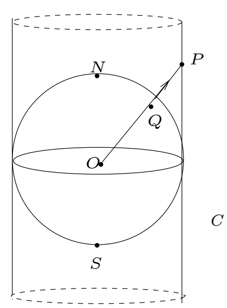
Lecture 6: Riemann Surface Structures on Cylinders and Tori via Covering Spaces

Consider the following four sets:

- The cylinder $\mathbb{C}/\mathbb{Z}$ where $\mathbb{Z}$ is thought of as the subgroup of translations by integer multiples of a fixed non-zero complex number;
- The punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$;
- The punctured unit disc $\Delta^* = \Delta \setminus \{0\}$, where, $\Delta = \{z \ : \ |z| < 1\}$ and
- The annulus $\Delta_r = \{z \ \in \ \mathbb{C} \mid r < |z| < 1\}$.

Each of the above topological spaces is homeomorphic to the others. It is easy to see that $\mathbb{C}^*$, Δ^* and Δ_r are homeomorphic to one another. One way to see that $\mathbb{C}^*$ is homemorphic to the cylinder $\mathbb{C}/\mathbb{Z}$ is the following. Consider the sphere



$\mathbb{S}^2$. We can see from the above image that there is a homeomorphism from the cylinder $\mathbb{C}/\mathbb{Z}$ to $\mathbb{S}^2 \setminus \{N, S\}$ where N, S denote respectively the "north-" and "south poles" of the sphere. This homeomorphism is obtained by sending each point P on the cylinder to the point Q on $\mathbb{S}^2$, where the line joining P with the centre of $\mathbb{S}^2$ pierces $\mathbb{S}^2$ at Q. Thereby $\mathbb{S}^2 \setminus \{N, S\}$ becomes homeomorphic to $\mathbb{C}/\mathbb{Z}$. We know that $\mathbb{S}^2 \setminus \{N\} \cong \mathbb{C}$ by the stereographic projection from N. So $\mathbb{S}^2 \setminus \{N, S\} \cong \mathbb{C} \setminus \{0\}$ since under this stereographic projection the south pole S corresponds to the origin. Hence we now have $\mathbb{C}/\mathbb{Z} \cong \mathbb{S}^2 \setminus \{N, S\} \cong \mathbb{C}^*$. Recall the following theorem that we stated in the previous lecture.

Theorem 1 *For $z_0 \neq 0$ in $\mathbb{C}$ consider the Riemann surface structure C_{z_0} on the cylinder $C = \mathbb{C}/\mathbb{Z}.T_{z_0}$ and the quotient map π_{z_0} which is holomorphic:*

$$\pi_{z_0} : \mathbb{C} \longrightarrow \mathbb{C}/\mathbb{Z}.T_{z_0} : z \mapsto \text{equivalence class}(z) = \{z + nz_0 : n \in \mathbb{Z}\}.$$

The set $\{C_w : w \in \mathbb{C}\setminus\{0\}\}$ mod isomorphism of Riemann surfaces is a singleton and is represented by the Riemann surface structure on $\mathbb{C}^*$.

We now note that it is impossible to find a non-constant (bi)holomorphic map from $\mathbb{C}^*$ to Δ^* or to Δ_r. For if we do have such a holomorphic map f from $\mathbb{C}^*$ to say Δ^*, then the map in a deleted neighbourhood of the origin would be bounded as the target set is bounded. By Riemann's theorem on removable singularities f would extend to a holomorphic map from all of $\mathbb{C}$. Recall that the *Riemann theorem on removable singularities* is the following. Let $D \subset \mathbb{C}$ be an open subset of the complex plane, $a \in D$ and f a holomorphic function defined on $D \setminus \{a\}$. The following are then equivalent:

- f is holomorphically extendable to a.
- f is continuously extendable to a.
- There exists a deleted neighborhood of a on which f is bounded.

To sum up we have an entire funtion f that is bounded, which by Liouville's theorem has to be constant. All this shows we cannot have a non-constant (bi)holomorphic map from $\mathbb{C}^*$ to Δ^*. A similar argument shows that there cannot be non-constant holomorphic maps from the punctured plane to Δ_r. Hence the natural Riemann surface structures on Δ^* and Δ_r induced from the complex plane are certainly going to be Riemann surface structures on the cylinder different from that induced on $\mathbb{C}^*$. In fact, for different values of r, the corresponding Riemann surfaces Δ_r are not biholomorphic to one another and moreover no Δ_r can be biholomorphic to Δ^*.

Theorem 2 *The set of isomorphism classes of all Riemann surface structures on the cylinder is given as the disjoint union of three sets, two of these being singletons and the third a one-real-parameter family as follows:*

$$\{[\mathbb{C}^*]\} \ \amalg \ \{[\Delta^*]\} \ \amalg \ \{\ [\Delta_r] \ : \ r \in (0,1)\}.$$

Here $[X]$ denotes the (biholomorphic or conformal or analytic) isomorphism class of the Riemann surface X.

How do we distinguish between the Riemann surfaces occurring in the above three sets? This is a question about classifying Riemann surfaces up to isomorphism and Covering Space Theory is the tool that is used to answer it.

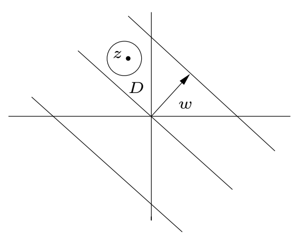
Definition 1 *Let X and $\widetilde{X}$ be topological spaces. Assume X and $\widetilde{X}$ are pathwise and locally pathwise connected. A map $p : \widetilde{X} \longrightarrow X$ is called a covering map if:*

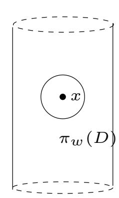
- *p is continuous and surjective, and*
- *given $x \in X$, $\exists$ an open set U, $x \in U$, such that $p^{-1}(U) = \amalg_{\alpha \in I} V_\alpha$, where $V_\alpha \subset \widetilde{X}$ is open and $p|_{V_\alpha} : V_\alpha \longrightarrow U$ is a homeomorphism.*

Observation:
We do not have non-constant holomorphic maps $\mathbb{C}^* \longrightarrow \Delta^*$ or $\Delta^* \longrightarrow \Delta_r$. However we do have holomorphic quotient maps $\mathbb{C} \longrightarrow \mathbb{C}^*$ or $\mathbb{C} \longrightarrow C_w$ which are *covering maps*.

$$\begin{tikzcd} \mathbb{C} \arrow[d, "\pi_w"'] \\ \mathbb{C}/\mathbb{Z}.T_w \end{tikzcd}$$

π_w is surjective and continuous. $\pi_w|_D : D \longrightarrow \pi_w(D)$ is a homeomorphism (refer to the following image). $\pi_w^{-1}(\pi_w(D)) = \bigcup_{n \in \mathbb{Z}}(D + nw)$ which is a





disjoint union of open sets and $\pi_w|_{D+nw} : D + nw \longrightarrow \pi_w(D + nw) = \pi_w(D)$ is a homeomorphism. Hence, π_w is a covering map. Consider next the case of the Riemann surface on a torus we obtained by fixing $w_1,\ w_2 \in \mathbb{C} \setminus \{0\}$ with $w_1/w_2 \notin \mathbb{R}$. The quotient map is

$$\pi_{w_1,w_2} : \mathbb{C} \longrightarrow \mathbb{C}/(\mathbb{Z}.T_{w_1} + \mathbb{Z}.T_{w_2}) = \mathbb{T}_{w_1,w_2}.$$

It is easy to check that π_{w_1,w_2} is a covering map as well.