NPTEL Video Course

An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-Tori and Elliptic Curves

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Lecture 1: The Idea of a Riemann Surface

1.1 Introduction

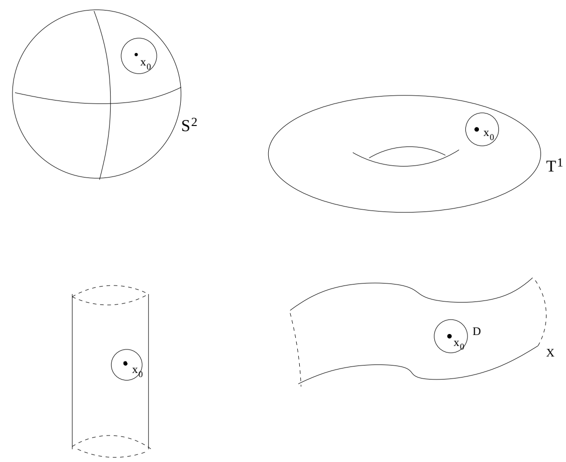
Recall the following from a first course in Complex Analysis (Functions of One Complex Variable): $f(z)$ is said to be analytic or holomorphic at z_0 if one of the following three equivalent conditions holds:

1. Write $w = u + iv$, $u = \mathrm{Re}(f)$, $v = \mathrm{Im}(f)$. We want the first partial derivatives
$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad v_x = \frac{\partial v}{\partial x}, \quad v_y = \frac{\partial v}{\partial y},$$
to exist and be continuous and further satisfy the Cauchy-Riemann equations
$$u_x = v_y \ , \quad u_y = -v_x \ ,$$
$\forall\ z$ in a neighbourhood of z_0 .

2. The limit
$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \ ,$$
exists $\forall\ z$ in a neighbourhood of z_0 .

3. $\exists$ a power series of the form
$$\sum_{n \geq 0} a_n (z - z_0)^n \ ,$$
which is convergent to $f(z)$ for each point z in a neighbourhood of z_0 .

Also recall that an injective holomorphic map is a holomorphic isomorphism: if
$$f : U \longrightarrow \mathbb{C}, \quad U \subset \mathbb{C}, \quad U \text{ being an open subset.}$$
is holomorphic and injective, then $f(U)$ is open (in fact any non-constant holomorphic map is an open map) and $f^{-1} : f(U) \longrightarrow U$ is also holomorphic.



1.2 The idea of a Riemann surface

Start with a surface, like the sphere or the torus or the cylinder that one can visualise in 3-space. Our aim in giving a Riemann surface structure to the surface is to do Complex Analysis i.e., to define and study holomorphic or analytic functions on the surface. Suppose that we are given a point x_0 on the surface and a small neighbourhood around the point that looks like the disc D, and suppose that we are also given a complex-valued function

$$f : D \longrightarrow \mathbb{C}.$$

We want to formulate a set of conditions that will define when f is holomorphic at x_0. One way to do this is to identify D with an open subset, say the unit disc $\Delta = \{z \epsilon \mathbb{C} : |z| = 1\}$, by choosing a homeomorphism $\phi : D \longrightarrow \Delta$ and then requiring that $f \circ \phi^{-1}$ is holomorphic at $\phi(x_0)$:

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "\phi", "\cong"'] & \mathbb{C} \\ \Delta \arrow[ur, "f \circ \phi^{-1}"'] & \end{tikzcd}$$

We can extend this definition to all points of D, and in the same way we can say that f is holomorphic on D if $f \circ \phi^{-1}$ is holomorphic on $\phi(D)$. We call the pair (D, ϕ) a *complex coordinate chart.* More generally a complex coordinate chart is a pair (U, ϕ) where U is an open subset of X and $\phi : U \longrightarrow V$ is a homeomorphism of U onto an open subset V of $\mathbb{C}$.

1.3 Preliminary definition of a Riemann surface

A surface X covered by a collection of charts:

$$\{(U_\alpha, \phi_\alpha) \mid \alpha \epsilon I\} \quad \text{such that} \quad X = \bigcup_\alpha U_\alpha \, ,$$

could be preliminarily called a Riemann surface. But with this definition we run into problems immediately as follows. A given point may occur in more than one chart and one gets as many definitions of holomorphicity of a function at that point as there are such charts! Suppose U_{α_1} and U_{α_2} both contain the point x_0. Consider a function $f : U_{\alpha_1} \bigcap U_{\alpha_2} \longrightarrow \mathbb{C}$. How do we require that f is holomorphic at x_0?

1. One way is to require that $f \circ \phi_{\alpha_1}^{-1}$ is holomorphic at $\phi_{\alpha_1}(x_0) = z_1$.
2. The other way is to require that $f \circ \phi_{\alpha_2}^{-1}$ is holomorphic at $\phi_{\alpha_2}(x_0) = z_2$.

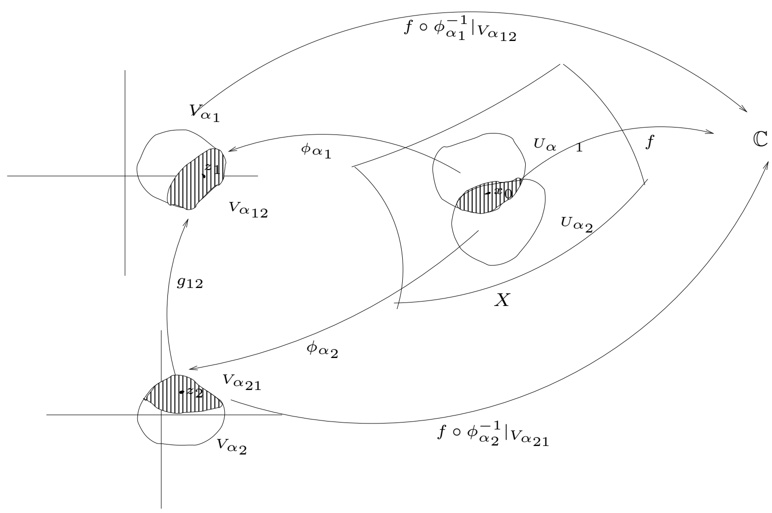
It may happen that the function is holomorphic with respect to one of the charts say $(U_{\alpha_1}, \phi_{\alpha_1})$ and not with the other: $(U_{\alpha_2}, \phi_{\alpha_2})$. If this happens, we do not have a proper definition of holomorphicity. So we need the charts to be *compatible* in the following way. We further require that the function

$$g_{12} : V_{\alpha_{21}} \longrightarrow V_{\alpha_{12}}, \; g_{12} = (\phi_{\alpha_1}|_{U_{\alpha_1} \bigcap U_{\alpha_2}}) \circ (\phi_{\alpha_2}^{-1}|_{V_{\alpha_{21}}}),$$

is holomorphic. Since g_{12} is a homeomorphism, it is injective. So if g_{12} is holomorphic, g_{12} would become an open map and g_{12}^{-1} holomorphic. In other words g_{12} would then become a holomorphic isomorphism. Thus if we require g_{12} to be holomorphic, we notice that

$$f \circ \phi_{\alpha_1}^{-1} \circ g_{12} = f \circ \phi_{\alpha_2}^{-1},$$

so that $f \circ \phi_{\alpha_1}^{-1}$ and $f \circ \phi_{\alpha_2}^{-1}$ differ by a holomorphic isomorphism. It follows that $f \circ \phi_{\alpha_1}^{-1}$ would be holomorphic if and only if $f \circ \phi_{\alpha_2}^{-1}$ is holomorphic (because g_{12} has an inverse). In conclusion: if we require that functions such as g_{12}, called *transition functions* are holomorphic whenever $U_{\alpha_1} \bigcap U_{\alpha_2}$ is nonempty, we would get a *compatible* collection of charts that cover X and this would give us a Riemann surface structure on X.


$f \circ \phi_{\alpha_1}^{-1}|_{V_{\alpha_{12}}}$
V_{α_1}
ϕ_{α_1}
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f
$\mathbb{C}$
z_1
z_0
$V_{\alpha_{12}}$
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$f \circ \phi_{\alpha_2}^{-1}|_{V_{\alpha_{21}}}$