NPTEL Video Course

An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-Tori and Elliptic Curves

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Lecture 7: Moebius Transformations Make Up Fundamental Groups of Riemann Surfaces

The study of general Riemann surfaces is facilitated by the study of covering spaces.

Basic Assumption : All spaces X in the forthcoming lectures are second countable and Hausdorff, unless stated otherwise.

Definition 1 *A map $p : \widetilde{X} \longrightarrow X$, (where $\widetilde{X}$, X are arcwise connected and locally arcwise connected) is called a covering map if :*

- *p is continuous and surjective;*
- *Every point $x \in X$ is contained in an admissible open neighbourhood U, i.e., $p^{-1}(U) = \amalg_{\alpha \in I} V_\alpha$, $V_\alpha \subset \widetilde{X}$ open, such that $p|_{V_\alpha} : V_\alpha \longrightarrow U$ is a homeomorphism.*

$\widetilde{X}$ is called the covering space and p is called the covering map.

7.1 Recall

$C_w =$ the cylinder $\simeq \mathbb{C}^* \simeq \Delta^* \simeq \Delta_r$, i.e. all these four sets are homeomorphic as topological spaces. We have the quotient map $\pi_w : \mathbb{C} \longrightarrow C_w = \mathbb{C}/\mathbb{Z}.T_w$, which is holomorphic and the different (non-zero) w-s provide the same Riemann surface structure upto isomorphism. We saw in the previous lecture that π_w has all the properties of a covering map. Also we have C_w as a quotient of $\mathbb{C}$ by a certain subgroup of automorphisms of $\mathbb{C}$ ($\mathbb{Z}.T_w$ are Möbius transformations). This is true in the case of a general universal covering space as well!

Examples of covering spaces :

1. Cylinders: $\pi_w : \mathbb{C} \longrightarrow C_w = \mathbb{C}/\mathbb{Z}.T_w$. Here the Riemann surface structure is isomorphic to that on $\mathbb{C}^*$.
2. Tori: $\pi_{w_1,w_2} : \mathbb{C} \longrightarrow \mathbb{T}_{w_1,w_2} = \mathbb{C}/(\mathbb{Z}.T_{w_1} \times \mathbb{Z}.T_{w_2})$. The set of isomorphism classes of such structures is bijective to $\mathbb{C}$, and this set naturally acquires a Riemann surface structure which is just (isomorphic to) the complex plane.

Covering space theory helps us distinguish (or classify) Riemann surface structures.

Suppose X is a Riemann surface. Then the underlying topological space is connected, arcwise connected, locally arcwise connected and locally simply connected. For such a space it can be shown that we can get a covering $p : \widetilde{X} \longrightarrow X$, with $\widetilde{X}$ simply connected. Such a covering space is called a *Universal covering space* for X and is uniquely determined upto isomorphism.

Any covering space (not necessarily universal) of a Riemann surface also inherits a Riemann surface structure, defined uniquely upto an isomorphism, such that the covering map becomes holomorphic. For, if X is any Riemann surface, $\widetilde{X}$ is any covering space and $p : \widetilde{X} \longrightarrow X$ is the covering map, then p being a local homeomorphism, given $x \in \widetilde{X}$, we have a neighbourhood V around it such that $p(V) = U$ an open subset of X, and $p|_V : V \longrightarrow U$ is a homeomorphism (we get this from the definition of covering space). Since X is a Riemann surface, X has charts locally, and since $\widetilde{X}$ is locally homeomorphic to X, we can transport these charts to $\widetilde{X}$. In this way, we can make $\widetilde{X}$ into a Riemann surface. So, if we have a covering space of a Riemann surface, we can make the covering space into a Riemann surface as well, and this Riemann surface structure is in fact determined uniquely upto an isomorphism.

What happened in the case of the cylinder $\pi_w : \mathbb{C} \longrightarrow C_w$ or the torus $\pi_{w_1,w_2} : \mathbb{C} \longrightarrow \mathbb{T}_{w_1,w_2}$? Both are covering maps, but to begin with C_w and $\mathbb{T}_{w_1,w_2}$ were not Riemann surfaces. We wanted to give the cylinder and the torus the structure of a Riemann surface, and we got the Riemann surface structure on the cylinder and the torus because of the covering space which is a Riemann surface; i.e. because the covering maps π_w and π_{w_1,w_2} are local homeomorphisms, we were able to get charts and define a Riemann surface structure on the target space for which the covering maps became holomorphic. So what really happens in these two cases is that we have a covering space situation as follows : on top is a Riemann surface ($\mathbb{C}$), and what we get below (as the Riemann surface we wanted) is a quotient of the space on top ($\mathbb{C}$) by a subgroup of automorphisms of the space on top (in this case, they are Möbius transformations, which are automorphisms of $\mathbb{C}$). That is, in the case of the cylinder or the torus, we are obtaining a Riemann surface structure on the space below. We have a topological space which is the base space of a covering, and since the space on top is already a Riemann surface, the space below also becomes a Riemann surface. What we saw earlier was the converse, i.e., if we start with a Riemann surface and we take any covering space of that Riemann surface, then the covering space becomes a Riemann surface so that the covering map becomes holomorphic.

Essentially what this means is that given a covering space situation, if we have a Riemann surface structure on the top, we can push it to the bottom (in nice cases); further the converse always holds.

Now in particular, if we take the universal covering space of the Riemann surface, then the universal covering space also becomes a Riemann surface. But the universal covering space is simply connected. Thus by the Uniformization Theorem, it has to be the complex plane $\mathbb{C}$ or the unit disc Δ (equivalently, the upper half plane $\mathbb{U}$) or the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$. This means that every Riemann surface is obtained from these three known Riemann surfaces by going modulo a certain group of automorphisms (Möbius transformations in this case).

We have been talking about the covering space. The next obvious question we ask is: where does the fundamental group come into the picture?

- The fundamental group of C_w is $\mathbb{Z}$. The fibre over any point in C_w is bijective to $\mathbb{Z}$. $\mathbb{Z}$ is also isomorphic to the group of translations going modulo which we get the Riemann surface below. $\mathbb{C}/\Pi_1(C_w) = \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}/\mathbb{Z}.T_w$.
- The fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$, and we have a similar situation as above.

The situation above is one that holds in general: in the case of a universal covering $p : \widetilde{X} \longrightarrow X$, the fibres $p^{-1}(x)$ are bijective to the (first) fundamental group $\Pi_1(X)$, which can also be identified with a subgroup of automorphisms of the covering space; moreover X is precisely the quotient of $\widetilde{X}$ by this subgroup. Therefore, $\tilde{X}/\Pi_1(X) \cong X$ as Riemann surfaces and the covering map $p : \widetilde{X} \longrightarrow X$ is just the quotient map $\widetilde{X} \longrightarrow \widetilde{X}/\Pi_1(X, x)$. Hence, any Riemann surface structure is a quotient of $\mathbb{C}$ or Δ or $\mathbb{P}^1_{\mathbb{C}}$ by a subgroup of automorphisms (Moebius transformations) isomorphic to the fundamental group of the Riemann surface.

In conclusion, to study any Riemann surface, we need to study subgroups of automorphisms of $\mathbb{C}$ or Δ or $\mathbb{P}^1_{\mathbb{C}}$ i.e., subgroups of Möbius transfomations.
We recall that:

- $Aut_{Hol}(\mathbb{P}^1_{\mathbb{C}}) = PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\mathbb{Z}_2$,
- $Aut_{Hol}(\mathbb{U}) = PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\mathbb{Z}_2$,
- $Aut_{Hol}(\mathbb{C}) = P\Delta(2, \mathbb{C})$, i.e. the upper triangular elements of $PSL(2, \mathbb{C})$.