NPTEL Video Course

**An Introduction to Riemann Surfaces and Algebraic Curves:
Complex 1-Tori and Elliptic Curves**

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Lecture 2: Simple Examples of Riemann Surfaces

2.1 Recall

A topological space X is equipped with the structure of a Riemann surface if X is given a collection of complex coordinate charts $\{(U_\alpha, \phi_\alpha) \ : \ \alpha \ \epsilon \ I\}$ such that

- the U_α cover X and each U_α is an open subset of X;
- $\phi_\alpha \ : \ U_\alpha \longrightarrow \mathbb{C}$ is a homeomorphism of U_α onto an open subset V_α of $\mathbb{C}$;
- whenever U_{α_1} and U_{α_2} intersect, the charts $(U_{\alpha_1}, \phi_{\alpha_1})$ and $(U_{\alpha_2}, \phi_{\alpha_2})$ have to be compatible. Compatibility requires that the so-called transition function $g_{12} \ : \ V_{\alpha_{21}} \longrightarrow V_{\alpha_{12}}$, defined by
$$g_{12} \ = \ (\phi_{\alpha_1}|_{U_{\alpha_1} \bigcap U_{\alpha_2}}) \circ (\phi_{\alpha_2}^{-1}|_{V_{\alpha_{21}}}) \ .$$
must be holomorphic. This would imply that g_{12}^{-1} is also holomorphic.

Definition 1 *A collection of compatible charts that covers X is called a complex atlas for X.*

2.2 Examples of Complex Atlases

Example 1

Let the space $X \ = \ \mathbb{R}^2$ and the atlas for $X = \{(U, \phi)\}$, where $U \ = \ X \ = \ \mathbb{R}^2$ and
$$\phi \ : \ U \longrightarrow \mathbb{C}, \quad \phi(x, y) \ = \ z \ = \ x \ + \ iy.$$
Obviously, ϕ is a homeomorphism. We need not check for compatibility of charts as there is just one chart in the atlas. With this atlas, $\mathbb{R}^2$ becomes a Riemann surface. What is this Riemann surface? It is the complex plane! For the holomorphic functions f on X are simply the holomorphic functions on $\mathbb{C}$:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\phi"] \arrow[d, "f(x,y)"'] & \mathbb{C} \arrow[dl, "f\circ\phi^{-1}=f(z)"] \\ \mathbb{C} & \end{tikzcd}$$

Hence the complex plane $\mathbb{C}$ is a Riemann surface structure on $\mathbb{R}^2$ given by the natural usual identification of $\mathbb{R}^2$ with $\mathbb{C}$.

Example 2

Again let $X = \mathbb{R}^2$ with the atlas for $X = \{(U, \phi)\}$ where $U = X = \mathbb{R}^2$ and

$$\phi : U \longrightarrow \mathbb{C} , \quad \phi(x,y) = \frac{z}{1+|z|} = \frac{x}{1+\sqrt{x^2+y^2}} + i\frac{y}{1+\sqrt{x^2+y^2}} .$$

We note that the image of ϕ is Δ = unit disc $= \{z : |z| < 1\}$. Further the map $\psi : \Delta \longrightarrow U$ given by

$$\psi(z) = \frac{z}{1-|z|} = \frac{x}{1-\sqrt{x^2+y^2}} + i\frac{y}{1-\sqrt{x^2+y^2}} ,$$

is the inverse of ϕ. Clearly, both ϕ and ψ are continuous. Therefore ϕ is a homeomorphism of $\mathbb{R}^2$ onto Δ. So now we have another Riemann surface structure on $\mathbb{R}^2$. When is a function f on X holomorphic?

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\phi"] \arrow[d, "f(x,y)"'] & \mathbb{C} \arrow[dl, "f\circ\phi^{-1}"] \\
\mathbb{C} &
\end{tikzcd}$$

By our definition, f is holomorphic $\Leftrightarrow f \circ \phi^{-1}$ is holomorphic.

$$(f \circ \phi^{-1})(z) = f(\frac{z}{1-|z|}) = f(\frac{x}{1-\sqrt{x^2+y^2}}, \frac{y}{1-\sqrt{x^2+y^2}})$$

Now let us test with f = the usual identification $f(x,y) = z = x+iy$. Then

$$f(\frac{z}{1-|z|}) = \frac{z}{1-|z|}, \text{ which is not holomorphic!}$$

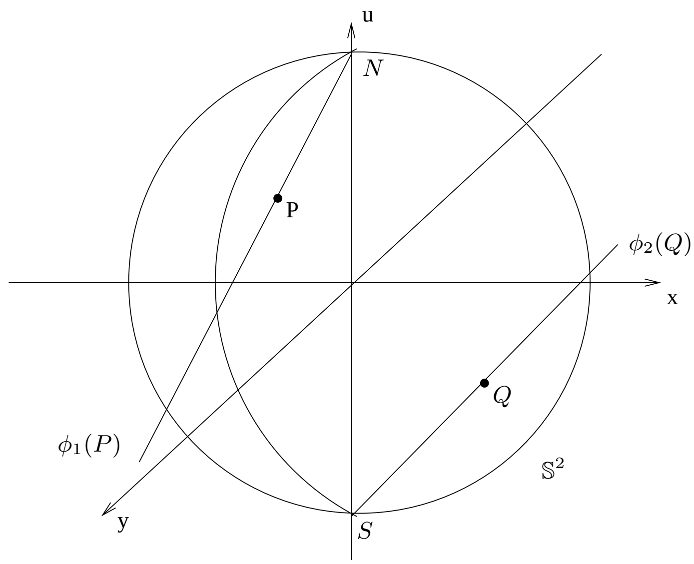
So on $\mathbb{R}^2$ we have a Riemann surface structure such that the natural identification map is not holomorphic! Here the Riemann surface structure on $\mathbb{R}^2$ is one which is isomorphic to the natural one on the unit disc.

We now state the *Uniformisation theorem*, which is the starting point for solving the problem of classification of Riemann surfaces up to isomorphism.

Uniformisation Theorem 1 *Any simply connected non-compact Riemann surface structure on a surface has to be isomorphic to either the unit disc or the whole complex plane.*

Example 3

We want to make the unit 2-sphere (centred at the origin) $X = \mathbb{S}^2 \subset \mathbb{R}^3$ into a Riemann surface. We try the "atlas" to be $\{(U_1, \phi_1), (U_2, \phi_2)\}$; here $U_1 = \mathbb{S}^2 \setminus \{N\}$, $U_2 = \mathbb{S}^2 \setminus \{S\}$, and S, N respectively denote the south pole and north pole with coordinates $(0,0,-1)$ and $(0,0,1)$; further $\phi_1 : U_1 \longrightarrow \mathbb{C}$



is the stereographic projection from N and $\phi_2 \;:\; U_2 \longrightarrow \mathbb{C}$ is the stereographic projection from S. Thus $\phi_1(P)$ is the point of intersection of the line NP with the XY plane and $\phi_2(Q)$ is the point of intersection of the line SQ with the XY plane. We know that ϕ_1 and ϕ_2 are homeomorphisms and that

$$S^2 \;=\; U_1 \;\cup\; U_2 \;.$$

Then $U_1 \cap U_2 \;=\; S^2 \backslash \{N, S\}$, $\phi_1(U_1 \cap U_2) \;=\; \mathbb{C}\backslash\{0\}$ and $\phi_2(U_1 \cap U_2) \;=\; \mathbb{C}\backslash\{0\}$. So we get a transition function from $\mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$ and we can see that this transition function is given by $z \mapsto\; 1/\overline{z}$ which as we know is not holomorphic. So the natural choice of maps fails to make the transition function holomorphic! In other words, what we have specified above is NOT an atlas!

However we can rectify the situation by replacing the map ϕ_2 with the map $\phi_3 \;:\; U_2 \longrightarrow \; \mathbb{C}$ defined as the stereographic projection from S followed by a conjugation. Now with respect to the atlas $\{(U_1, \phi_1)\;,\;(U_2, \phi_3)\}$ the transition function becomes $z \;\mapsto\; 1/z$ which is holomorphic. Thereby with this atlas we have a Riemann Surface structure on $\mathbb{S}^2$ which is called the *Riemann sphere*.

We now state the Uniformisation theorem for simply connected and compact Riemann surfaces.

Uniformisation Theorem 2 *Any simply connected compact Riemann surface structure on a surface has to be isomorphic to the Riemann sphere.*