

Example 0.1. Change the order of the integration $\int_0^a dx \int_0^x f(x, y) dy$.

Sol. The given region of integration may be thought of as consisting of lines parallel to the X -axis starting from $x = y$ to $x = a$.

$$\begin{aligned} I &= \int_0^a dx \int_0^x f(x, y) dy \\ &= \int_0^a dy \int_y^a f(x, y) dx \end{aligned}$$

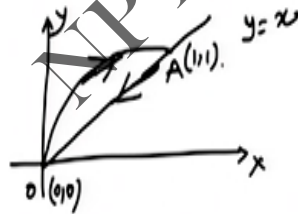
Green's theorem in $\mathbb{R}^2$. Let P, Q be two single valued functions such that $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ exist and are continuous on a simply connected domain bounded by a closed curve C . Then,

$$\oint_C (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

where, $\oint_C$ denotes that C is closed and it is described in the +ve direction.

Example 0.2. Evaluate by Green's theorem the integral $\int_C (x^2y dx + xy^2 dy)$ taken along the closed path formed by $y = x$ and $x^2 = y^3$ in the first quadrant.

Sol. The curves intersect at $O(0,0)$ and $A(1,1)$. The integral is of form $\int_C (Pdx + Qdy)$ where $P = x^2y$ and $Q = xy^2$.



$$\begin{aligned} I &= \int_C (x^2y dx + xy^2 dy) = \iint_D (y^2 - x^2) dx dy = \int_{x=0}^1 \int_{y=x}^{x^{\frac{2}{3}}} (y^2 - x^2) dx dy \\ \Rightarrow I &= \int_0^1 dx \int_x^{x^{\frac{2}{3}}} (y^2 - x^2) dy = \int_0^1 \left[\frac{1}{3}(y^3 - x^3) - x^2(y - x) \right] dx = \frac{1}{198} \end{aligned}$$

Example 0.3. Evaluate the integral $\int_C [(2xy - x^2) dx + (x + y)^2 dy]$, where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$

Change of variable in double integral.

Let a function $f(x, y)$ be continuous in a region D . Let the variables x, y in the double integral $\iint_D f(x, y) dx dy$ be changed to u, v by means of relation $x = \phi(u, v)$, $y = \psi(u, v)$ where ϕ and ψ have continuous first order partial derivative in a certain region D_1 in the uv -plane. Then, it can be shown that

$$\iint_D f(x, y) dx dy = \iint_{D_1} f(\phi(u, v), \psi(u, v)) |J| du dv,$$

where $|J| = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$, the Jacobian of the transformation.

Let $I = \iint_D f(x, y) dx dy$, changing to polar coordinate $x = r \cos \theta$, $y = r \sin \theta$, then integration converts to $I = \iint_{D_A} f(r \cos \theta, r \sin \theta) |J| dr d\theta$, where Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

This is how we convert integration of Cartesian coordinate to Polar coordinate.

Example 0.4. Evaluate $I = \iint \sin \pi(x^2 + y^2) dx dy$ over the interior of the circle $x^2 + y^2 = 1$.

Sol. let us take $x = r \cos \theta$, $y = r \sin \theta \Rightarrow x^2 + y^2 = r^2$. Now

$$\begin{aligned} I &= \iint \sin \pi(x^2 + y^2) dx dy \\ &= \int_{\theta=0}^{\pi} \int_0^1 \sin \pi r^2 r dr d\theta \\ &= \int_{\theta=0}^{\pi} d\theta \int_0^1 \frac{1}{2} (\sin \pi t) dt \quad (\text{taking } t = r^2) \\ &= \int_0^{2\pi} \left[-\frac{\cos \pi t}{2\pi} \right]_0^1 d\theta \\ &= 2 \end{aligned}$$

Example 0.5. Show that $\iint_D \sqrt{4a^2 - x^2 - y^2} dx dy = \frac{4}{9}(3\pi - 4)a^3$, taken over the upper half of the circle $x^2 + y^2 = 2ax$.

Sol. The given circle is $x^2 + y^2 = 2ax \Rightarrow (x - a)^2 + y^2 = a^2$. Center = $(a, 0)$ and radius = a . Putting $x = r \cos \theta$ and $y = r \sin \theta$ in $x^2 + y^2 = 2ax$ then,

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 2ar \cos \theta \Rightarrow r^2 = 2ar \cos \theta \Rightarrow r = 2a \cos \theta$$

Thus for the given upper half of the circle our θ varies from 0 to π and r varies from 0 to $2a \cos \theta$. Then

$$\begin{aligned} I &= \iint_D \sqrt{4a^2 - x^2 - y^2} dx dy \\ &= \int_0^{\pi} \int_0^{2a \cos \theta} \sqrt{4a^2 - r^2} r dr d\theta \quad (\text{Substitute } 4r^2 - a^2 = t \Rightarrow -2r dr = dt) \\ &= \int_0^{\pi} \left[\int_{4a^2}^{4a^2 \sin^2 \theta} \sqrt{t} \frac{dt}{-2} \right] d\theta \\ &= \frac{4}{9}(3\pi - 4)a^3 \end{aligned}$$

Example 0.6. Using the transformation $x+y = u$ and $y = uv$, find the value of $\int_0^1 dx \int_0^{1-x} e^{\frac{y}{x+y}} dy$.

Sol. $x + y = u, y = uv$, Now,

$$y = 1 - x \Rightarrow x + y = 1 \Rightarrow u = 1,$$

$$y = 0 \Rightarrow uv = 0 \Rightarrow u = 0, v = 0$$

Again, $x = 0 \Rightarrow y = u \Rightarrow u = uv \Rightarrow v = 1$ and $x = u - uv, y = uv$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

$$\begin{aligned} I &= \int_0^1 \int_0^1 e^{\frac{uv}{u}} |J| du dv = \int_0^1 \int_0^1 e^v u du dv \\ &= \int_0^1 u du \int_0^1 e^v dv = \frac{e-1}{2} \end{aligned}$$

Integration in $\mathbb{R}^3$. Consider a curve C in $\mathbb{R}^3 : x = f(t), y = g(t), z = h(t), a \leq t \leq b$. Let P, Q, R be three functions of x, y, z defined on domain $D \subset \mathbb{R}^3$ containing C . Let f, g, h possess continuous derivatives on (a, b) . Also suppose P, Q, R are continuous on D . Then the line integral $I = \int_C (P dx + Q dy + R dz)$ exists and $I = \int_C (P dx + Q dy + R dz) = \int_{t=a}^b (Pf' + Qg' + Rh') dt$

Alternatively $\vec{F} = (P, Q, R)$ and $\vec{r} = (x, y, z) \Rightarrow d\vec{r} = (dx, dy, dz)$ then line integral $I = \vec{F} \cdot d\vec{r} = \int_{t=a}^b \left(\vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt$

Example 0.7. Evaluate $I = \int_C (x^2 y^3 dx + dy + z dz)$ where C is the circle $x^2 + y^2 = R^2, z = 0$.

Sol. Putting $x = R \cos t, y = R \sin t, z = 0 \Rightarrow \frac{dx}{dt} = -R \sin t, \frac{dy}{dt} = R \cos t, \frac{dz}{dt} = 0$

$$\begin{aligned} I &= \int_{t=0}^{2\pi} [R^5 \cos^2 t \sin^2 t (-R \sin t) + R \cos t + 0 \cdot 0] dt \\ &= -R^6 \int_{t=0}^{2\pi} \cos^2 t \sin^4 t dt + \int_{t=0}^{2\pi} \cos t dt \\ &= -4R^6 \int_{t=0}^{\frac{\pi}{2}} \cos^2 t \sin^4 t dt \\ &= -4R^6 \times \frac{3 \times 1}{6 \times 4 \times 1} \times \frac{\pi}{2} \\ &= -\frac{\pi}{8} R^6 \end{aligned}$$

Example 0.8. Evaluate $I = \int_C (xy dx + yz dy + zx dz)$ where C is curve given by $x = t, y = t^2, z = t^3$, where $-1 \leq t \leq 1$

Sol.

$$\begin{aligned} I &= \int_C (xy dx + yz dy + zx dz) \\ &= \int_{-1}^1 (t \cdot t^2 \cdot 1 + t^2 \cdot t^3 \cdot 2t + t^3 \cdot t \cdot 3t^2) dt \\ &= \int_{-1}^1 (t^3 + 2t^6 + 3t^6) dt \\ &= \left[\frac{t^4}{4} + \frac{5t^7}{7} \right]_{-1}^1 = \end{aligned}$$

Integral of bounded function over a parallelopiped $R = [a, b] \times [c, d] \times [e, f]$.

Let P_1, P_2, P_3 are the partitions of three intervals,

$$U(P, f) = \sum_{t=1}^p \sum_{s=1}^n \sum_{r=1}^m M_{rst}(x_r - x_{r-1})(y_s - y_{s-1})(z_t - z_{t-1})$$

$$U(P, f) = \sum_{t=1}^p \sum_{s=1}^n \sum_{r=1}^m m_{rst}(x_r - x_{r-1})(y_s - y_{s-1})(z_t - z_{t-1})$$

where, M_{rst} = Upper bound of f on R and m_{rst} = Lower bound of f on R .

$\lim_{\|P\| \rightarrow 0} U(P, f) = \lim_{\|P\| \rightarrow 0} L(P, f)$. Then f is integrable over R .

$$\iiint_D f(x, y, z) dx dy dz = \int_a^b dx \int_c^d dy \int_e^f f(x, y, z) dz$$

Calculation of integral in a bounded domain bounded by D. If f is continuous on a domain D bounded by $z = \phi(x, y), z = \psi(x, y); y = g(x), y = h(x); x = a, x = b$. Then,

$$\iiint_D f(x, y, z) dx dy dz = \int_a^b \int_{g(x)}^{h(x)} \int_{\phi(x, y)}^{\psi(x, y)} f(x, y, z) dx dy dz$$

Example 0.9. Evaluate, $I = \iiint (x + y + z + 1)^2 dx dy dz$, over the region $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$.

Sol. The region is bounded by $z = 0, z = 1 - x - y$. It's projection on D in xy -plane a triangle bounded by $x = 0, y = 0, y = 1 - x$. Then

$$\begin{aligned} I &= \iiint (x + y + z + 1)^2 dx dy dz \\ &= \int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (x + y + z + 1)^2 dx dy dz \\ &= \frac{1}{3} \int_{x=0}^1 \int_{y=0}^{1-x} \left[(x + y + z + 1)^3 \right]_0^{1-x-y} dx dy \\ &= \frac{1}{3} \int_{x=0}^1 \int_{y=0}^{1-x} \left[8 - (x + y + 1)^3 \right] dx dy \\ &= \frac{1}{3} \int_{x=0}^1 \left[8y - \frac{1}{4}(x + y + 1)^4 \right]_0^{1-x} dx \\ &= \frac{1}{3} \int_{x=0}^1 \left[8(1-x) - \frac{1}{4}(16 - (x+1)^4) \right] dx \\ &= \frac{31}{60} \end{aligned}$$

Triple Integral(Change of variable).

$$\begin{aligned} \iiint_D f(x, y, z) dx dy dz &= \iiint_{D_1} f(x(u, v, w), y(u, v, w), z(u, v, w)) |J| du dv dw \\ &= \iiint_{D_1} f(u, v, w) |J| du dv dw, \end{aligned}$$

where $|J| = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$, the Jacobian of the transformation.

Example 0.10. Spherical polar co-ordinate: $x = r \cos \theta \sin \phi, y = r \sin \theta \sin \phi, z = r \cos \theta \Rightarrow |J| = r^2 \sin \theta$.

Example 0.11. Cyndrical polar co-ordinate: $x = r \cos \theta, y = r \sin \theta, z = z \Rightarrow |J| = r$

Example 0.12. Evaluate, $I = \iiint xyz \, dx \, dy \, dz$, over the region bounded by $x = 0, y = 0, z = 0, x + y + z = 1$.

Sol.

$$\begin{aligned}
 I &= \iiint xyz \, dx \, dy \, dz, \\
 &= \int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} xyz \, dz \, dy \, dx \\
 &= \int_{x=0}^1 x \, dx \int_{y=0}^{1-x} y \, dy \int_{z=0}^{1-x-y} z \, dz \\
 &= \int_{x=0}^1 x \, dx \int_{y=0}^{1-x} \frac{1}{2} y(1-x-y)^2 \, dy \\
 &= \frac{1}{2} \int_{x=0}^1 x \, dx \int_{y=0}^{1-x} y(1-x-y)^2 \, dy \\
 &= \frac{1}{2} \int_{x=0}^1 x \, dx \left[-\frac{1}{3} y(1-x-y)^3 \Big|_0^{1-x} + \frac{1}{3} \int_0^{1-x} (1-x-y)^3 \cdot 1 \cdot dy \right] \\
 &= \frac{1}{720}
 \end{aligned}$$

Example 0.13. Evaluate, $I = \iiint_D x^2 \, dx \, dy \, dz$, where D is the sphere $x^2 + y^2 + z^2 \leq 1$.

Sol. Changing the variables to spherical polar co-ordinate,

$$\begin{aligned}
 I &= \int_0^{2\pi} \int_0^\pi \int_0^1 r^2 \sin^2 \theta \cos^2 \phi (r^2 \sin \theta) \, dr \, d\theta \, d\phi \\
 &= \int_0^{2\pi} \cos^2 \phi \, d\phi \int_0^\pi \sin^2 \theta \, d\theta \int_0^1 r^4 \, dr \\
 &= \frac{1}{5} \int_0^{2\pi} \cos^2 \phi \, d\phi \int_0^\pi \sin^2 \theta \, d\theta \\
 &= \frac{1}{5} \times 4 \times 2 \int_0^{\frac{\pi}{2}} \cos^2 \phi \, d\phi \int_0^{\frac{\pi}{2}} \sin^2 \theta \, d\theta \\
 &= \frac{4\pi}{15}
 \end{aligned}$$

Example 0.14. Evaluate, $I = \iiint_D e^{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}} \, dx \, dy \, dz$, where D is the region $\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} \leq 1$.

Sol. Changing the variables to polar, $\frac{x}{a} = r \sin \theta \cos \phi, \frac{y}{b} = r \sin \theta \sin \phi, \frac{z}{c} = r \cos \theta$.

$$\begin{aligned}
 I &= \int_0^{2\pi} \int_0^\pi \int_0^1 e^r (abc r^2 \sin \theta) \, dr \, d\theta \, d\phi \\
 &= abc \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \, d\theta \int_0^1 r^2 e^{-r} \, dr \\
 &= 4\pi abc \int_0^1 r^2 e^{-r} \, dr = 4\pi abc \times (e - 2) = 4\pi abc(e - 2)
 \end{aligned}$$

(Doubt)

Volume of a solid by triple integral. $V = \iiint_D dx dy dz$ It can be shown that on a closed (region) domain D bounded below and above by the surface $z = \phi(x, y), z = \psi(x, y), \psi(x, y) \geq \phi(x, y)$.

$$\begin{aligned} V &= \iiint_D dx dy dz \\ &= \iiint_{\phi(x,y)}^{\psi(x,y)} dx dy dz \end{aligned}$$

Example 0.15. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Sol.

$$\begin{aligned} V &= \iiint_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx dy dz \\ &= 2c \iint \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dx dy \quad \left(\frac{x}{a} = r \cos \theta, \quad \frac{y}{b} = r \sin \theta, \quad |J| = abr \right) \\ &= 2c \int_0^\pi \int_0^1 \sqrt{1 - r^2} abr dr d\theta \\ &= \frac{4}{3} \pi abc \end{aligned}$$

Example 0.16. Evaluate the volume cut off from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$

Sol. The given region is bounded by the surfaces $z = \sqrt{a^2 - x^2 - y^2}$ and $z = -\sqrt{a^2 - x^2 - y^2}$ and its projection in the xy - plane is the circular domain $D : x^2 + y^2 = ax$.

$$\begin{aligned} V &= \iint_D dx dy \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz \\ &= \iint_D \sqrt{a^2 - x^2 - y^2} dx dy \\ &= 2 \int_0^\pi \int_0^{a \cos \theta} \sqrt{a^2 - r^2} r dr d\theta \\ &= -\frac{2}{3} \int_0^\pi (a^2 - r^2)^{\frac{3}{2}} \Big|_0^{a \cos \theta} d\theta \\ &= \frac{2}{3} a^3 \int_0^\pi (1 - \sin^2 \theta) d\theta \\ &= \frac{2}{3} a^3 \left(\pi - 2 \cdot \frac{2}{3} \right) = \frac{2}{3} a^3 \left(\pi - \frac{4}{3} \right) \end{aligned}$$

Example 0.17. Find the volume of the region above xy plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$

Sol. The given solid is bounded above by $z = x^2 + y^2$ and below by $x^2 + y^2 = a^2$. Moreover the solids are symmetrical, its volume V four times of the volume lying in first octant.

$$\begin{aligned} V &= 4 \iint_D z dx dy \\ &= 4 \iint_D (x^2 + y^2) dx dy \\ &= 4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^a r^2 r dr d\theta = 4 \times \frac{\pi}{2} \times \frac{a^4}{4} = \frac{\pi a^4}{2} \end{aligned}$$

Example 0.18. Find the volume of the region bounded by the paraboloid $z = (x-1)^2 + y^2$ and the plane $2x + z = 2$.

Sol. The required value is,

$$\begin{aligned} V &= 4 \iint_D (z_2 - z_1) dx dy \\ &= 4 \iint_D \left[(2 - 2x) - \left\{ (x-1)^2 + y^2 \right\} \right] dx dy \end{aligned}$$

Where D is a region given by eliminating z between two given surfaces, $(x-1)^2 + y^2 = 2 - 2x$. So,

$$\begin{aligned} V &= 4 \iint_D \left[1 - (x^2 + y^2) \right] dx dy \\ &= 4 \int_{\theta=0}^{2\pi} \int_0^1 (1 - r^2) dr d\theta = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{2\pi}{4} = \frac{\pi}{2} \end{aligned}$$

Surface area. The area of the smooth surface $x = f(u, v), y = g(u, v), z = h(u, v), (u, v) \in D$ defined as double integral,

$$\iint_D \sqrt{\left[\frac{\partial(y, z)}{\partial(u, v)} \right]^2 + \left[\frac{\partial(z, x)}{\partial(u, v)} \right]^2 + \left[\frac{\partial(x, y)}{\partial(u, v)} \right]^2} du dv$$

Given the surface $z = f(x, y)$ or $x = u, y = v$ and $z = f(u, v)$. If we take the surface area over the surface $z = f(x, y)$ having projection D on xy plane as S , then

$$S = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} dx dy$$

Example 0.19. Find the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ cut of by $x^2 + y^2 = ax$

Sol. We have $z = \sqrt{a^2 - x^2 - y^2}$ and the required surface area is

$$\begin{aligned} S &= 4 \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} dx dy \quad (\text{where } D \text{ is the projection of sphere in } xy \text{ plane}) \\ &= 4 \iint_D \frac{a dx dy}{\sqrt{a^2 - x^2 - y^2}} \\ &= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a \cos \theta} \frac{a r dr d\theta}{\sqrt{a^2 - r^2}} \quad (\because x^2 + y^2 = ax \Rightarrow r^2 = ar \cos \theta \Rightarrow r = a \cos \theta) \\ &= 4a \int_0^{\frac{\pi}{2}} \left[-\sqrt{a^2 - r^2} \right]_a^{a \cos \theta} d\theta \\ &= 4a^2 \int_0^{\frac{\pi}{2}} (1 - \sin \theta) d\theta = 4a^2 \left[\frac{\pi}{2} - 1 \right] = 2a^2(\pi - 2) \end{aligned}$$

Example 0.20. Find the area of the part of the surface of the cylinder $x^2 + y^2 = a^2$ which is cut of by the cylinder $x^2 + z^2 = a^2$.

Sol. The given cylinder is $y = \sqrt{a^2 - x^2}$. Then, $\sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} = \frac{az}{a^2 - x^2}$. The

required surface area is (doubt in integration)

$$\begin{aligned}
 S &= 4 \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy \\
 &= 2a \iint_D \frac{dx dz}{\sqrt{a^2 - x^2}} \\
 &= 2a \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} \int_0^{\sqrt{a^2 - x^2}} dz \\
 &= 8a \int_0^a \frac{dx}{\sqrt{a^2 - x^2}} \times \sqrt{a^2 - x^2} = 8a^2
 \end{aligned}$$

1 Differential Equation.

$$\begin{aligned}
 f(x, y, y') &= 0, y' = \frac{dy}{dx} \\
 f(x, y, y', y'', \dots y^n) &= 0 \\
 y'' + xy' + 2y &= 0 \text{ order 2, degree 1}
 \end{aligned}$$

Example 1.1. From the relation $y(x) = Ae^x + Be^{-x} + x^2$, find differential equation where A, B are arbitrary constants.

Sol.

$$\begin{aligned}
 y(x) &= Ae^x + Be^{-x} + x^2 \\
 \Rightarrow y'(x) &= Ae^x - Be^{-x} + 2x \\
 \Rightarrow y''(x) &= Ae^x + Be^{-x} + 2 \Rightarrow y'' - y = 2 - x^2
 \end{aligned}$$

It is the required differential equation.

Example 1.2. (Solution by Separation of variables) $x^2 \frac{dx}{dy} + y = 1$ (Doubt from slide)

Sol. Given,

$$\begin{aligned}
 x^2 \frac{dx}{dy} + y &= 1 \\
 \Rightarrow x^2 \frac{dx}{dy} &= 1 - y \\
 \Rightarrow \frac{dy}{1 - y} &= \frac{dx}{x^2} \\
 \Rightarrow - \int \frac{dy}{1 - y} &= \int \frac{dx}{x^2} + \log c \\
 \Rightarrow - \log_e(1 - y) &= -\frac{1}{x} + \log c \\
 \Rightarrow - \log_e \frac{1 - y}{c} &= \frac{1}{x} \\
 \Rightarrow \frac{1 - y}{c} &= e^{\frac{1}{x}} \\
 \Rightarrow (1 - y) &= ce^{\frac{1}{x}} \\
 \Rightarrow y &= 1 - ce^{\frac{1}{x}}
 \end{aligned}$$

It is the required solution of the given differential equation.

Example 1.3. Solve the differential equation $\sqrt{1 - x^2} dy + \sqrt{1 - y^2} dx = 0$. Also find a particular solution if $y(x = 0) = 1$.

Sol.

$$\begin{aligned}\frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} &= 0 \\ \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} + \int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} c \\ \Rightarrow \sin^{-1} x + \sin^{-1} y &= \sin^{-1} c \\ \Rightarrow \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) &= \sin^{-1} c \\ \Rightarrow x\sqrt{1-y^2} + y\sqrt{1-x^2} &= c\end{aligned}$$

Putting $x = 0$ in the above equation, we obtain

$$0 \cdot \sqrt{1-1} + 1 \cdot \sqrt{1-0} = c \Rightarrow c = 1$$

Thus,

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} = 1.$$

Exact 1st order ODE. If the differential equation $Mdx + Ndy = 0$ can be expressed as the form $du = 0$ where u is a function of x and y without multiplying by any factor then the equation is called to be an exact differential equation. In general, the solution is $u(x, y) = c$.

$$Mdx + Ndy = 0 \Rightarrow \int du = 0 \Rightarrow u(x, y) = c.$$

Example 1.4. $xdy + ydx = 0 \Rightarrow d(xy) = 0 \Rightarrow xy = c$.

Example 1.5. $\frac{x dy - y dx}{x^2} = 0 \Rightarrow d\left(\frac{y}{x}\right) = 0$.

Example 1.6. $\frac{x dy - y dx}{y^2} = 0 \Rightarrow d\left(\frac{x}{y}\right) = 0$.

Example 1.7. $\frac{x dy + y dx}{\sqrt{1-x^2}y^2} = 0 \Rightarrow d(\sin^{-1}xy) = 0$.

Theorem 1.1. The necessary sufficient condition for the ODE to be exact is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example 1.8. Verify whether the ODE $a^2 - 2xy - y^2 dx - (x + y)^2 dy = 0$ is exact or not.

Sol. Comparing with $Mdx + Ndy = 0$, we get $M(x, y) = a^2 - 2xy - y^2$ and $N(x, y) = -(x + y)^2$. For exactness we need to show $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\frac{\partial M}{\partial y} = -2x - 2y$ and $\frac{\partial N}{\partial x} = -2(x + y) \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ the given ODE is exact.

Example 1.9. Verify whether the ODE $(y^2 e^x + 2xy)dx - x^2 dy = 0$ is exact or not.

Sol. Comparing with $Mdx + Ndy = 0$, we get $M(x, y) = y^2 e^x + 2xy$ and $N(x, y) = -x^2$. For exactness we need to show $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\frac{\partial M}{\partial y} = 2ye^x + 2x$ and $\frac{\partial N}{\partial x} = -2x \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the given ODE is not exact.

Example 1.10. Verify whether the ODE $(x^2 + y^2 + 2x)dx + 2xdy = 0$ is exact or not.

Sol. Comparing with $Mdx + Ndy = 0$, we get $M(x, y) = (x^2 + y^2 + 2x)$ and $N(x, y) = 2y$. For exactness we need to show $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the given ODE is not exact.

Integrating factor. Sometimes to make $Mdx + Ndy = 0$ as exact ODE, we multiply it by a function of x and y . This function is called integrating factor.

Rule 1. If $Mx + Ny \neq 0$ and the equation is homogeneous, then $\frac{1}{Mx + Ny}$ is an integrating factor of $Mdx + Ndy = 0$

Example 1.11. Solve $(x^2y - 2xy^2) dx + (3x^2y - x^3) dy = 0$

Sol. Comparing with $Mdx + Ndy = 0$, we get $M(x, y) = (x^2y - 2xy^2)$ and $N(x, y) = (3x^2y - x^3)$. $\frac{\partial M}{\partial y} = x^2 - 4xy$ and $\frac{\partial N}{\partial x} = 6xy - 3x^2 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the given ODE is not exact.

Now, $Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y = x^2y^2 \neq 0$. The integrating factor is $\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$. Multiplying the given ODE by the IF, then

$$\begin{aligned} & \frac{dx}{y} - \frac{2}{x} dx + \frac{3}{y} dy - \frac{x}{y^2} dy = 0 \\ \Rightarrow & \frac{ydx - xdy}{y^2} - \frac{2}{x} dx + \frac{3}{y} dy = 0 \\ \Rightarrow & d\left(\frac{x}{y}\right) - 2d(\log x) + 3d(\log y) = 0 \\ \Rightarrow & d\left(\frac{x}{y} - 2\log x + 3\log y\right) = 0 \\ \Rightarrow & \frac{x}{y} - 2\log x + 3\log y = c \end{aligned}$$

Rule 2. If $Mx - Ny \neq 0$ and the equation can be written as $f(x, y)y dx + F(x, y)x dy = 0$, then $\frac{1}{Mx-Ny}$ is an IF of the equation $M dx + N dy = 0$.

Example 1.12. Solve $\frac{(xy \sin xy + \cos xy)}{x} dx + \frac{(xy \sin xy - \cos xy)}{y} dy = 0$

Sol. Given

$$\begin{aligned} & \frac{(xy \sin xy + \cos xy)}{x} dx + \frac{(xy \sin xy - \cos xy)}{y} dy = 0 \\ \Rightarrow & y(xy \sin xy + \cos xy) dx + x(xy \sin xy - \cos xy) dy = 0 \end{aligned} \quad (1.1)$$

Comparing with $Mdx + Ndy = 0$, we get $M(x, y) = y(xy \sin xy + \cos xy)$ and $N(x, y) = x(xy \sin xy - \cos xy)$. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the ODE is not exact.

Now, $Mx - Ny = y(xy \sin xy + \cos xy)x - x(xy \sin xy - \cos xy)y = 2xy \cos xy$. Multiplying the equation (1.1) by $\frac{1}{Mx-Ny} = \frac{1}{2xy \cos xy}$, we obtain

$$\begin{aligned} & \tan xy (y dx + x dy) + \frac{1}{x} dx - \frac{dy}{y} = 0 \\ \Rightarrow & \tan xy d(xy) + \frac{dx}{x} - \frac{dy}{y} = 0 \\ \Rightarrow & \log(\sec(xy)) + \log x - \log y = \log c \\ \Rightarrow & \frac{\sec(xy)x}{y} = c \\ \Rightarrow & cy = x \sec(xy) \end{aligned}$$

Rule 3. If $\frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$ is a function of x alone (say $f(x)$), then $e^{\int f(x) dx}$ is an integrating factor.

Rule 4. If $\frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$ is a function of y alone (say $\phi(y)$), then $e^{\int \phi(y) dy}$ is an integrating factor.

Example 1.13. Solve $(x^2 + y^2 + 2x) dx + 2y dy = 0$.

Sol.

$$M(x, y) = x^2 + y^2 + 2x, N(x, y) = 2y \Rightarrow \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Now, $f(x) = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{2y} (2y - 0) = 1$. Then, the required IF = $e^{\int f(x) dx} = e^x$.

Multiplying the given differential equation by e^x , we get

$$\begin{aligned}(x^2 + y^2 + 2x)e^x dx + 2ye^x dy &= 0 \\ \Rightarrow (x^2 + y^2)e^x dx + y^2 e^x dx + 2ye^x dy &= 0 \\ \Rightarrow d(x^2 e^x) + d(y^2 e^x) &= 0 \\ \Rightarrow (x^2 + y^2)e^x &= c\end{aligned}$$

Example 1.14. Solve $(3x^2y^4 + 2xy) dx + (2x^3y^3 - x^2) dy = 0$

Sol. Comparing with $Mdx + Ndy = 0$, we get $M(x, y) = (3x^2y^4 + 2xy)$ and $N(x, y) = (2x^3y^3 - x^2)$.

Here $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Now, $\phi(y) = \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = \frac{1}{xy(3xy^2+2)} (6x^2y^3 - 2x - 12x^2y^3 - 2x) = -\frac{2}{y}$.

The required IF is $e^{\int \phi(y) dy} = e^{-\int \frac{2}{y} dy} = e^{-2 \log y} = \frac{1}{y^2}$.

Multiplying the given differential equation by $\frac{1}{y^2}$, we obtain

$$\begin{aligned}\frac{1}{y^2} (3x^2y^4 + 2xy) dx + \frac{1}{y^2} (2x^3y^3 - x^2) dy &= 0 \\ \Rightarrow 3x^2y^2 dx + 2\frac{x}{y} dx + 2x^3y dy - \frac{x^2}{y^2} dy &= 0 \\ \Rightarrow d(x^3y^2) + \frac{2xy dx - x^2 dy}{y^2} &= 0 \\ \Rightarrow d(x^3y^2) + d\left(\frac{x^2}{y}\right) &= 0 \\ \Rightarrow x^3y^2 + \frac{x^2}{y} &= c\end{aligned}$$

Linear equations. Consider the following linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Here integrating factor is $\text{IF} = e^{\int P(x) dx}$, then the solution is given by

$$\int d(\phi(x)y) = \int Q(x)\phi(x) dx + c$$

Example 1.15. Solve $\frac{dy}{dx} + \frac{4x}{x^2+1}y = \frac{1}{(x^2+1)^3}$

Sol. Given $P(x) = \frac{4x}{x^2+1}$ and $Q(x) = \frac{1}{(x^2+1)^3}$. Integrating factor, $\phi(x) = e^{\int P(x) dx}$.

$$\phi(x) = e^{\int \frac{4x}{x^2+1} dx} = e^{2 \log(x^2+1)} = (x^2 + 1)^2.$$

Then,

$$\begin{aligned}(x^2 + 1)^2 \frac{dy}{dx} + (x^2 + 1)^2 \frac{4x}{x^2 + 1} y &= (x^2 + 1)^2 \frac{1}{(x^2 + 1)^3} \\ \Rightarrow d(y(x^2 + 1)^2) &= \frac{dx}{x^2 + 1} \\ \Rightarrow y(x^2 + 1)^2 &= \tan^{-1} + c\end{aligned}$$

Second order differential equation. Let $a(x)y'' + b(x)y' + c(x)y = 0$ be a second order differential equation, where $y(x)$ is the unknown, x is the independent variable, a, b, c are the functions of x . Let $a(x), b(x), c(x)$ are all constants say α, β, γ , respectively and $y = e^{mx}$ be an arbitrary solution, then $\alpha m^2 + \beta m + \gamma = 0$ becomes an auxillary equation.

Case 1. If the roots of the auxillary equations are distinct, then solution $y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$.

Case 2. If the roots of the auxillary equations are repeated, then solution $y(x) = (C_1 + C_2 x)e^{mx}$.

Case 3. If the roots of the auxillary equations are complex, then solution $y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$, where $m = \alpha \pm i\beta$.

Example 1.16. Solve the second order differential equation $4y'' + 4y' - 3y = 0$.

Sol. Let $y = e^{mx}$ be the arbitrary solution, then the auxillary equation is

$$4m^2 + 4m - 3 = 0 \Rightarrow (m - \frac{1}{2})(m + \frac{3}{2}) = 0 \Rightarrow m = \frac{1}{2}, -\frac{3}{2}$$

The general solution is $y(x) = C_1 e^{\frac{1}{2}x} + C_2 e^{-\frac{3}{2}x}$.

Example 1.17. Solve the second order ODE: $y'' - 24y' + 144y = 0$.

Sol. Let $y = e^{mx}$ be arbitrary solution, then the auxillary equation becomes $m^2 - 24m + 144 = 0 \Rightarrow (m - 12)^2 = 0 \Rightarrow m = 12, 12$. The general solution of the given equation is $y(x) = (C_1 + C_2 x)e^{12x}$.

Example 1.18. Solve the second order ODE: $y'' + 8y' + 25y = 0$.

Sol. Let $y = e^{mx}$ be the arbitrary solution. The auxillary equation is $m^2 + 8m + 25 = 0$. The roots are given by, $m = \frac{-8 \pm 6i}{2} = -4 \pm 3i$. The required solution is $y(x) = e^{-4x}(C_1 \cos 3x + C_2 \sin 3x)$.

Example 1.19. Solve the ODE: $2y''' - 7y'' + 7y' - 2y = 0$

Sol. Let $y = e^{mx}$ be the arbitrary solution. The auxillary equation is $2m^3 - 7m^2 + 7m - 2 = 0 \Rightarrow (m - 1)(m - 2)(m - \frac{1}{2}) = 0 \Rightarrow m = 1, 2, \frac{1}{2}$. The required general solution is $y(x) = C_1 e^x + C_2 e^{2x} + C_3 e^{\frac{x}{2}}$.

Example 1.20. Solve the ODE: $y'''' - y''' - 9y'' - 11y' - 4y = 0$.

Sol. Let $y = e^{mx}$ be the arbitrary solution. The auxillary equation is $m^4 - m^3 - 9m^2 - 11m - 4 = 0 \Rightarrow (m + 1)^3(m - 4) = 0 \Rightarrow m = -1, -1, -1, 4$. The required general solution is $y(x) = (C_1 + C_2 x + C_3 x^2)e^{-x} + C_4 e^{4x}$.

Example 1.21. Solve the ODE: $y'''' + a^4 y = 0$.

Let $y = e^{mx}$ be the arbitrary solution. The auxillary equation is $m^4 + a^4 = 0 \Rightarrow m = -\frac{a}{\sqrt{2}} \pm i \frac{a}{\sqrt{2}}$ and $m = -\frac{a}{\sqrt{2}} \pm i \frac{a}{\sqrt{2}}$. The required general solution is $y(x) = e^{-\frac{a}{\sqrt{2}}x} (C_1 \cos \frac{a}{\sqrt{2}}x + C_2 \sin \frac{a}{\sqrt{2}}x) + e^{\frac{a}{\sqrt{2}}x} (C_3 \cos \frac{a}{\sqrt{2}}x + C_4 \sin \frac{a}{\sqrt{2}}x)$.

Bisection method. Let us consider the given equation $f(x) = 0$. If $f(x) > 0$ and $f(x) < 0$ or $f(x) < 0$ and $f(x) > 0$. Either way $f(a) \cdot f(b) < 0$ To find mid point $x = \frac{a+b}{2}$. Check $f(x_1) > 0$, $[a, x_1]$ there exist a root in $[x_1, b]$. $x_2 = \frac{x_1+b}{2}$

Example 1.22. Find the positive root of the equation $x^3 - 9x + 1 = 0$, correct up to two decimal places.

Sol. Let $f(x) = x^3 - 9x + 1$, $f(0) = 1 > 0$ and $f(1) = -7 < 0$. There is a sign change in $[0, 1]$, so there exists a root between $x = 0$ to $x = 1$, i.e, in $[0, 1]$. Let $x_1 = \frac{0+1}{2} = \frac{1}{2}$. Then, $f(\frac{1}{2}) = -3.37 < 0$

n	a_n	b_n	x_n	$f(x_n)$
0	0	1	0.5	-3.37
1	0	0.5	0.25	-1.23
2	0	0.25	0.125	-0.123
3	0	0.125	0.0625	0.437
4	0.0625	0.125	0.09375	0.155
5	0.09375	0.125	0.119375	0.016933
6	0.109375	0.125	0.11718	-0.053

The required root is 0.11.

Iterative method/Fixed Point argument. Let us take $f(x) = 0 \Rightarrow x = \phi(x)$. Let us say our initial guess is x_0 . Then, we formulate a numerical scheme,

$$\begin{aligned} x_1 &= \phi(x_0) \\ x_2 &= \phi(x_1) \\ x_3 &= \phi(x_2) \\ &\dots \\ x_{n+1} &= \phi(x_n) \end{aligned}$$

Theorem 1.2. If $\phi(x)$ is continious function on $[a, b]$, which contains the root of $f(x) = 0$ and if $|\phi'(x)| < 1 \forall x \in (a, b)$, then for any choice of $x_0 \in [a, b]$, the sequence $(x_n)_n$, determined as $x_{n+1} = \phi(x_n) \forall n$ converges to the root of $x = \phi(x)$.

Example 1.23. In order to find the root of $x^3 - x - 1 = 0$ near $x = 1$ which one of the following scheme will be useful. i) $x = x^3 - 1$ ii) $x = \frac{x+1}{x^2}$ iii) $x = \sqrt{\frac{x+1}{x}}$

Sol. i) Given equation is $x^3 - x - 1 = 0 \Rightarrow x = x^3 - 1 = \phi(x) \Rightarrow \phi'(x) = 3x^2$ Clearly $|\phi'(x)| = 3x^2 > 1$ in nbd of 1. The scheme does not converge.

ii) Given equation is $x = \frac{x+1}{x^2} = \phi(x) \Rightarrow \phi'(x) = -\frac{1}{x^2} - \frac{2}{x^3} \Rightarrow |\phi'(x)| \leq \frac{1}{x^2} + \frac{2}{x^3} \leq \frac{1}{x^3} + \frac{2}{x^3} = \frac{3}{x^3} \geq 3 > 1$. The scheme does not converge.

iii) Given equation is $x = \sqrt{\frac{x+1}{x}} = \phi(x) \Rightarrow \phi'(x) = -\frac{1}{2x^2} \sqrt{\frac{x}{x+1}} \Rightarrow |\phi'(x)| = \left| -\frac{1}{x^2} \sqrt{\frac{x}{x+1}} \right| \leq \frac{1}{2} \cdot \frac{1}{x^3} \frac{1}{\sqrt{x+1}} < 1$. The scheme will converge.

Example 1.24. Let $f(x) = x^3 + x - 5 = 0$

Sol. since $f(1) = -3$ and $f(2) = 3$. So there is a root $[1, 2]$. $x = (5 - x)^{\frac{1}{3}} = \phi(x) \Rightarrow |\phi'(x)| = \left| -\frac{1}{3} \frac{1}{(5-x)^{\frac{2}{3}}} \right| \leq \frac{1}{3} \frac{1}{(5-x)^{\frac{2}{3}}} \leq 1$

Newton-Raphson Method. Let α_0 be the approximate value of a root of $f(x) = 0$. Let α be the exact root. Then, $\alpha = \alpha_0 + h$, where h is very small.

$$\begin{aligned} f(\alpha) = 0 &\Rightarrow f(\alpha_0 + h) = 0 \\ &\Rightarrow f(\alpha_0) + hf'(\alpha_0) + \frac{h^2}{2}f''(\alpha_0) + \cdots = 0 \end{aligned}$$

Since h is very small, we can neglect higher order of h .

$$\begin{aligned} &\Rightarrow f(\alpha_0) + hf'(\alpha_0) = 0 \\ &\Rightarrow h = -\frac{f(\alpha_0)}{f'(\alpha_0)} \end{aligned}$$

Therefore, $\alpha = \alpha_0 + h = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)} \Rightarrow \alpha_n = \alpha_{n-1} - \frac{f(\alpha_{n-1})}{f'(\alpha_{n-1})}, \quad n = 1, 2, \dots$

Example 1.25. Find the root of $2x^3 - 8x - 6 = 0$ by Newton-Raphson method.

Sol. $f(x) = 2x^3 - 8x - 6$. Now $f(1) = 2 - 8 - 6 = -7 < 0$. $f(2) = 2 \cdot 2^3 - 8 \cdot 2 - 6 = 4 > 0$. There exist a root in the interval $[1, 2]$. Let $x_0 = 2$. Also $f'(x) = 6x^2 - 8$. Now,

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{4}{21} = 1.809524 \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.784200 \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.783769 \\ x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = 1.783769 \end{aligned}$$

The required root is 1.783769 correct upto six decimal places.

Example 1.26. Find a positive root for $3x - \cos x - 1 = 0$.

Sol. Let $f(x) = 3x - \cos x - 1$. $f(0) = -2 < 0$ and $f(1) = 3 - \cos 1 - 1 \geq 1 > 0$. There is a root in $[0, 1]$. let $x_0 = 0.5$, therefore the Newton-Raphson scheme is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n \cos x_n - 1}{3 + \sin x_n} \\ \Rightarrow x_{n+1} &= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}, \quad n = 0, 1, 2, \dots \end{aligned}$$

System of linear equation.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Where, $A\vec{x} = \vec{b}$.

$$\begin{aligned} &\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ \vec{x} &= [x_1 \ x_2 \ x_3]^t, \quad \vec{b} = [b_1 \ b_2 \ b_3]^t \end{aligned}$$

1. Gauss-Seidal method. 2. Gauss- Jacobi method.

Gauss-Jacobi method.

$$\begin{aligned}a_1x_1 + b_1x_2 + c_1x_3 &= d_1 \\a_2x_1 + b_2x_2 + c_2x_3 &= d_2 \\a_3x_1 + b_3x_2 + c_3x_3 &= d_3\end{aligned}$$

Let us assume that, $|a_1| > |b_1| + |c_1|$, $|b_2| > |a_2| + |c_2|$, $|c_3| > |a_3| + |b_3|$.

The Jacobi method will converge if in each equation of the given system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients. This is also called diagonally dominant.

The iteration scheme is given as follows,

$$\left. \begin{aligned}x^{(1)} &= \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)}) \\y^{(1)} &= \frac{1}{b_2}(d_2 - a_2x^{(0)} - c_2z^{(0)}) \\z^{(1)} &= \frac{1}{c_3}(d_3 - a_3x^{(0)} - b_3y^{(0)})\end{aligned} \right\} \text{1st iteration}$$

$$\vdots$$

$$\left. \begin{aligned}x^{(n+1)} &= \frac{1}{a_1}(d_1 - b_1y^{(n)} - c_1z^{(n)}) \\y^{(n+1)} &= \frac{1}{b_2}(d_2 - a_2x^{(n)} - c_2z^{(n)}) \\z^{(n+1)} &= \frac{1}{c_3}(d_3 - a_3x^{(n)} - b_3y^{(n)})\end{aligned} \right\} n + 1 \text{ th iteration}$$

Gauss-seidal method. The iteration scheme is given as follows,

$$\left. \begin{aligned}x^{(1)} &= \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)}) \\y^{(1)} &= \frac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(0)}) \\z^{(1)} &= \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)})\end{aligned} \right\} \text{1st iteration}$$

$$\vdots$$

$$\left. \begin{aligned}x^{(n+1)} &= \frac{1}{a_1}(d_1 - b_1y^{(n)} - c_1z^{(n)}) \\y^{(n+1)} &= \frac{1}{b_2}(d_2 - a_2x^{(n+1)} - c_2z^{(n)}) \\z^{(n+1)} &= \frac{1}{c_3}(d_3 - a_3x^{(n+1)} - b_3y^{(n+1)})\end{aligned} \right\} n + 1 \text{ th iteration}$$

Example 1.27. Solve via Gauss-Jacobi and Gauss-Seidal methods:

$$\begin{aligned}10x - 5y - 2z &= 3 \\4x - 10y + 3z &= -3 \\x + 6y + 10z &= -3\end{aligned}$$

Sol. The coefficient matrix is, $\begin{bmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{bmatrix}$

For Gauss-Jacobi method,

$$\begin{aligned}x^{(n+1)} &= \frac{1}{10}(3 - 5y^{(n)} - 2z^{(n)}) \\y^{(n+1)} &= \frac{1}{10}(3 + 4x^{(n+1)} + 3z^{(n)}) \\z^{(n+1)} &= \frac{1}{10}(-3 - x^{(n+1)} - 6y^{(n)})\end{aligned}$$

Let us take, $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$. Then,

$$\begin{aligned}x^{(1)} &= 0.3 & y^{(1)} &= 0.3 & z^{(1)} &= -0.3 \\&\dots & & & & \\x^{(8)} &= 0.34 & y^{(8)} &= 0.285 & z^{(8)} &= -0.5051 \\x^{(9)} &= 0.34 & y^{(9)} &= 0.285 & z^{(9)} &= -0.5053\end{aligned}$$

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