

Taylor's Theorem for functions of two real variables

Notes for MOOC Course: Calculus of several real variables.

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In the class we wrote down the expression for the Taylor's expansion for functions of ~~more than~~ two variables. Here we provide the proof as given in the book titled: "Calculus for the Ambitious" by T.W. Körner, who is a Professor of Fourier Analysis at the University of Cambridge.

Körner approaches the Taylor's expansion of ~~two~~ functions of two variables using the Taylor's expansion of ~~two~~ functions of one variable. The key is to use polar coordinates.

Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a given function which is twice continuously differentiable. Let $x = r \cos \theta$, $y = r \sin \theta$. For a fixed θ , set

$$f_{\theta}(r) = g(r \cos \theta, r \sin \theta)$$

Then using the Taylor's expansion upto second order we have

$$f_{\theta}(r) = f_{\theta}(0) + \frac{f'_{\theta}(0)}{1!} r + \frac{f''_{\theta}(0)}{2!} r^2 + o(r^2)$$

where $\frac{o(r^2)}{r^2} \rightarrow 0$ as $r \rightarrow 0$. We need to now compute f'_{θ} & f''_{θ} in terms of the partial derivatives of g . But we will use the definition of the derivative of a function of two variables.

Note that

$$\begin{aligned} f_{\theta}(r+h) &= g((r+h) \cos \theta, (r+h) \sin \theta) \\ &= g(r \cos \theta + h \cos \theta, r \sin \theta + h \sin \theta) \\ &= g(r \cos \theta, r \sin \theta) + \frac{\partial}{\partial x} g(r \cos \theta, r \sin \theta) h \cos \theta \\ &\quad + \frac{\partial}{\partial y} g(r \cos \theta, r \sin \theta) h \sin \theta + o(\sqrt{h^2 \cos^2 \theta + h^2 \sin^2 \theta}) \end{aligned}$$

$$f(r+h) = f_{\theta}(r) + \left[\frac{\partial g(r \cos \theta, r \sin \theta)}{\partial x} \cos \theta + \frac{\partial g(r \cos \theta, r \sin \theta)}{\partial y} \sin \theta \right] h + o(h)$$

Thus by the definition of derivatives of one variable, and noting that $\frac{o(h)}{h} \rightarrow 0$ as $h \rightarrow 0$, we have

$$f'_{\theta}(r) = \lim_{h \rightarrow 0} \frac{f(r+h) - f_{\theta}(r)}{h} = \left[\frac{\partial g(r \cos \theta, r \sin \theta)}{\partial x} \cos \theta + \frac{\partial g(r \cos \theta, r \sin \theta)}{\partial y} \sin \theta \right]$$

$$f'_{\theta}(r) = \frac{\partial g(r \cos \theta, r \sin \theta)}{\partial x} \cos \theta + \frac{\partial g(r \cos \theta, r \sin \theta)}{\partial y} \sin \theta \rightarrow (*)$$

$$\text{Let } G_1(x, y) = \frac{\partial g(x, y)}{\partial x}, \quad G_2(x, y) = \frac{\partial g(x, y)}{\partial y}$$

$$\text{Let } G_{1,\theta}(r) = G_1(r \cos \theta, r \sin \theta), \quad G_{2,\theta}(r) = G_2(r \cos \theta, r \sin \theta)$$

$$\therefore \cancel{f'_{\theta}(r) = G_{1,\theta}(r \cos \theta, r \sin \theta) \cos \theta}$$

$$f'_{\theta}(r) = G_{1,\theta}(r) \cos \theta + G_{2,\theta}(r) \sin \theta$$

$$\therefore f''_{\theta}(r) = G'_{1,\theta}(r) \cos \theta + G'_{2,\theta}(r) \sin \theta$$

Now using the expression (*) we can write that

$$G'_{1,\theta}(r) = \frac{\partial G_1(r \cos \theta, r \sin \theta)}{\partial x} \cos \theta + \frac{\partial G_1(r \cos \theta, r \sin \theta)}{\partial y} \sin \theta$$

$$= \frac{\partial^2 g}{\partial x^2}(r \cos \theta, r \sin \theta) \cos \theta + \frac{\partial^2 g}{\partial y \partial x}(r \cos \theta, r \sin \theta) \sin \theta$$

(The last line comes by using the definition of $G_1(x, y)$)

$$G'_{2,\theta}(r) = \frac{\partial G_2(r \cos \theta, r \sin \theta)}{\partial x} \cos \theta + \frac{\partial G_2(r \cos \theta, r \sin \theta)}{\partial y} \sin \theta$$

$$= \frac{\partial^2 g}{\partial x \partial y}(r \cos \theta, r \sin \theta) \cos \theta + \frac{\partial^2 g}{\partial y^2}(r \cos \theta, r \sin \theta) \sin \theta$$

$$f''_{\theta}(r) = G'_{1,\theta}(r) \cos \theta + G'_{2,\theta}(r) \sin \theta$$

$$= \left[\frac{\partial^2 g}{\partial x^2}(r \cos \theta, r \sin \theta) \cos \theta + \frac{\partial^2 g}{\partial y \partial x}(r \cos \theta, r \sin \theta) \sin \theta \right] \cos \theta$$

$$+ \left[\frac{\partial^2 g}{\partial x \partial y}(r \cos \theta, r \sin \theta) \cos \theta + \frac{\partial^2 g}{\partial y^2}(r \cos \theta, r \sin \theta) \sin \theta \right] \sin \theta$$

$$= \left[\frac{\partial^2 g}{\partial x^2}(r \cos \theta, r \sin \theta) \cos^2 \theta + 2 \frac{\partial^2 g}{\partial y \partial x}(r \cos \theta, r \sin \theta) \sin \theta \cos \theta + \frac{\partial^2 g}{\partial y^2}(r \cos \theta, r \sin \theta) \sin^2 \theta \right]$$

Let us choose r and θ in such a way that $h = r \cos \theta$, $y = r \sin \theta$.

$$g(h, k) = g(r \cos \theta, r \sin \theta)$$

$$= f_{\theta}(r) = f_{\theta}(0) + \frac{f'_{\theta}(0)}{1!} r + \frac{f''_{\theta}(0)}{2!} r^2 + o(r^2)$$

$$= g(0,0) + \left[\frac{\partial g}{\partial x}(0,0) \cos \theta + \frac{\partial g}{\partial y}(0,0) \sin \theta \right] r$$

$$+ \frac{1}{2!} \left[\frac{\partial^2 g}{\partial x^2}(0,0) \cos^2 \theta + 2 \frac{\partial^2 g}{\partial x \partial y}(0,0) \sin \theta \cos \theta + \frac{\partial^2 g}{\partial y^2}(0,0) \sin^2 \theta \right] r^2$$

$$+ o(h^2 + k^2)$$

$$= g(0,0) + \left[\frac{\partial g}{\partial x}(0,0) h + \frac{\partial g}{\partial y}(0,0) k \right]$$

$$+ \frac{1}{2!} \left[\frac{\partial^2 g}{\partial x^2}(0,0) h^2 + 2 \frac{\partial^2 g}{\partial x \partial y}(0,0) h k + \frac{\partial^2 g}{\partial y^2}(0,0) k^2 \right]$$

$$+ o(h^2 + k^2)$$

The last expression comes from observing that

$$h = r \cos \theta, \quad y = r \sin \theta$$

— x —
Voilà!

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