Analysis of Variance and Design of Experiments-II

MODULE - II

LECTURE - 14

BALANCED INCOMPLETE BLOCK DESIGN (BIBD)

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The effective variance can be approximately estimated by

$$\hat{\sigma}_E^2 = MSE\left[1+(v-k)\omega^*\right]$$

where MSE is the mean square due to error obtained from the intrablock analysis as

$$MSE = \frac{SS_{Error(t)}}{bk-b-v+1}$$

and

$$\omega^* = \frac{\omega_1 - \omega_2}{v(k-1)\omega_1 + k(v-k)\omega_2}.$$

The quantity ω^* depends upon the unknown σ^2 and σ_β^2. To obtain an estimate of ω^*, we can obtain the unbiased estimates of σ^2 and σ_β^2 and then substitute them back in place of σ^2 and σ_β^2 in ω^*. To do this, we proceed as follows.

An estimate of ω_1 can be obtained by estimating σ^2 from the intrablock analysis of variance as

$$\hat{\omega}_1 = \frac{1}{\hat{\sigma}_2} = [MSE]^{-1}$$

The estimate of ω_2 depends on $\hat{\sigma}^2$ and $\hat{\sigma}_\beta^2$. To obtain an unbiased estimator of σ_β^2, consider

$$SS_{Block(adj)} = SS_{Treat(adj)} + SS_{Block(unadj)} - SS_{Treat(unadj)}$$

for which

$$E(SS_{Block(adj)}) = (bk-v)\sigma_\beta^2 + (b-1)\sigma^2.$$

Thus an unbiased estimator of σ_β^2 is

$$\begin{aligned}
\hat{\sigma}_\beta^2 &= \frac{1}{bk-v}\left[SS_{Block(adj)} - (b-1)\hat{\sigma}^2\right] \\
&= \frac{1}{bk-v}\left[SS_{Block(adj)} - (b-1)MSE\right] \\
&= \frac{b-1}{bk-v}\left[MS_{Block(adj)} - MSE\right] \\
&= \frac{b-1}{v(r-1)}\left[MS_{Block(adj)} - MSE\right]
\end{aligned}$$

where

$$MS_{Block(adj)} = \frac{SS_{Block(adj)}}{b-1}.$$

Thus

$$\begin{aligned}
\hat{\omega}_2 &= \frac{1}{k\hat{\sigma}^2 + \hat{\sigma}_\beta^2} \\
&= \frac{1}{v(r-1)\left[k(b-1)SS_{Block(adj)} - (v-k)SS_{Error(t)}\right]}.
\end{aligned}$$

Recall that our main objective is to develop a test of hypothesis for $H_0: \tau_1 = \tau_2 = ... = \tau_v$ and we now want to develop it using the information based on both interblock and intrablock analysis.

To test the hypothesis related to treatment effects based on the pooled estimate, we proceed as follows.

Consider the adjusted treatment totals based on the intrablock and the interblock estimates as

$$T_j^* = T_j + \omega^* W_j^*;\ j = 1, 2, \ldots, v$$

and use it as usual treatment total as in earlier cases.

The sum of squares due to T_j^* is

$$S_{T^*}^2 = \sum_{j=1}^{v} T_j^{*2} - \frac{\left(\sum_{j=1}^{v} T_j^*\right)^2}{v}.$$

Note that in the usual analysis of variance technique, the test statistic for such hull hypothesis is developed by taking the ratio of the sum of squares due to treatment divided by its degrees of freedom and the sum of squares due to error divided by its degrees of freedom. Following the same idea, we define the statistics

$$F^* = \frac{S_{T^*}^2 / [(v-1)r]}{MSE[1+(v-k)\hat{\omega}^*]}$$

where $\hat{\omega}^*$ is an estimator of ω^*. It may be noted that F^* depends on $\hat{\omega}^*$. Also, $\hat{\omega}^*$ itself depends on the estimated variances $\hat{\sigma}^2$ and $\hat{\sigma}_f^2$. So it cannot be ascertained that the statistic F^* necessary follows the F distribution. Since the construction of F^* is based on the earlier approaches where the statistic was found to follow the exact F-distribution, so based on this idea, the distribution of F^* can be considered to be approximately F distributed. Thus the approximate distribution of F^* is considered as F distribution with $(v-1)$ and $(bk-b-v+1)$ degrees of freedom. Also, $\hat{\omega}^*$ is an estimator of ω^* which is obtained by substituting the unbiased estimators of ω_1 and ω_2.

An approximate best pooled estimator of $\sum_{j=1} l_j \tau_j$ is

$$\sum_{j=1}^{v} l_j \frac{V_j + \hat{\xi} W_j}{r}$$

and its variance is approximately estimated by

$$\frac{k \sum_j l_j^2}{\lambda v \hat{\omega}_1 + (r-\lambda) k \hat{\omega}_2}.$$

In case of a resolvable BIBD, $\hat{\sigma}_\beta^2$ can be obtained by using the adjusted block with replications sum of squares from the intrablock analysis of variance. If sum of squares due to such block total is SS_{Block}^* and corresponding mean square is

$$MS_{Block}^* = \frac{SS_{Block}^*}{b-r}$$

and then

$$\begin{aligned} E(MS_{Block}^*) &= \sigma^2 + \frac{(v-k)(r-1)}{b-r}\sigma_\beta^2 \\ &= \sigma^2 + \frac{(r-1)k}{r}\sigma_\beta^2 \end{aligned}$$

and $k(b-r) = r(v-k)$ for a resolvable design. Thus

$$E\left[rMS_{Block}^* - MSE\right] = (r-1)(\sigma^2 + k\sigma_\beta^2)$$

and hence

$$\hat{\omega}_2 = \left[\frac{rMS_{block}^* - MSE}{r-1}\right]^{-1},$$

$$\hat{\omega}_1 = \left[MSE\right]^{-1}.$$

The analysis of variance table for recovery of interblock information in BIBD is described in the following table:

Source	Sum of squares	Degrees of freedom	Mean square	F^*
Between treatments (unadjusted)	$S^2_{T^*}$	$v-1$		$F^* = \dfrac{MS_{Blocks(adj)}}{MSE}$
Between blocks (adjusted)	$SS_{Block(adj)} = SS_{Treat(adj)} + SS_{Block(unadj)} - SS_{Treat(unadj)}$	$b-1$	$MS_{Blocks(adj)} = \dfrac{SS_{Block(adj)}}{b-1}$	
Intrablock error	$SS_{Error(t)}$ (by substraction)	$bk-b-v+1$	$MSE = \dfrac{SS_{Error(t)}}{bk-b-v+1}$	
Total	SS_{Total}	$bk-1$		

The increase in the precision using the interblock analysis as compared to the intrablock analysis is

$$\frac{Var(\hat{\tau})}{Var(\tau^*)} - 1$$

$$= \frac{\lambda v \omega_1 + \omega_2 k(r-\lambda)}{\lambda v \omega_1} - 1$$

$$= \frac{\omega_2 (r-\lambda)k}{\lambda v \omega_1}.$$

Such an increase may be estimated by

$$\frac{\hat{\omega}_2 (r-\lambda)k}{\lambda v \hat{\omega}_1}$$

Although $\omega_1 > \omega_2$ but this may not hold true for $\hat{\omega}_1$ and $\hat{\omega}_2$. The estimates $\hat{\omega}_1$ and $\hat{\omega}_2$ may be negative also and in that case we take $\hat{\omega}_1 = \hat{\omega}_2$.