Analysis of Variance and Design of Experiments-II

MODULE - II

LECTURE - 11

BALANCED INCOMPLETE BLOCK DESIGN (BIBD)

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Intrablock analysis of BIBD

Consider the model

$$y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij};\ i = 1, 2, ..., b;\ j = 1, 2, ..., v,$$

where

μ is the general mean effect;

β_i is the fixed additive i^{th} block effect;

τ_j is the fixed additive j^{th} treatment effect and

ε_{ij} is the i.i.d. random error with $\varepsilon_{ij} \sim N(0, \sigma^2)$.

We don't need to develop the analysis of BIBD in the beginning. Since BIBD is also an incomplete block design and the analysis of incomplete block design has already been presented in the earlier module, so we implement those derived expressions directly under the set up and conditions of BIBD.

Using the same notations, we represent the

- blocks totals by $B_i = \sum_{j=1}^{v} y_{ij}$
,
- treatment totals by $V_j = \sum_{i=1}^{b} y_{ij}$,
- adjusted treatment totals by Q_j and
- grand total by $G = \sum_{i=1}^{b} \sum_{j=1}^{v} y_{ij}$.

The normal equations are obtained by differentiating the error sum of squares. Then the block effects are eliminated from the normal equations and the normal equations are solved for the treatment effects. The resulting intrablock equations of treatment effects in matrix notations are expressible as

$$Q = C\hat{\tau}.$$

Now we obtain the forms of C and Q in the case of BIBD.

The diagonal elements of C are given by

$$\begin{aligned} c_{jj} &= r - \frac{\sum_{i=1}^{b} n_{ij}^2}{k} \quad (j = 1, 2, \ldots, v) \\ &= r - \frac{r}{k}. \end{aligned}$$

The off-diagonal elements of C are given by

$$\begin{aligned} c_{jj'} &= -\frac{1}{k}\sum_{i=1}^{b} n_{ij} n_{ij'} \quad (j \neq j'; j, j' = 1, 2, \ldots, v) \\ &= -\frac{\lambda}{k}. \end{aligned}$$

The adjusted treatment totals are obtained as

$$Q_j = V_j - \frac{1}{k}\sum_{i=1}^{b} n_{ij} B_i \quad (j \neq 1, 2, ..., v)$$

$$= V_j - \frac{1}{k}\sum_{i(j)} B_i$$

where $\sum_{i(j)}$ denotes the sum over those blocks containing j^{th} treatment.

Denote

$$T_j = \sum_{i(j)} B_i, \text{ then}$$

$$Q_j = V_j - \frac{T_j}{k}.$$

The C matrix is simplified as follows:

$$C = rI - \frac{N'N}{k}$$

$$= rI - \frac{1}{k}\left[(r-\lambda)I + \lambda E_{v1}E_{v1}^{'}\right]$$

$$= r\left(\frac{k-1}{k}\right) + \frac{\lambda}{k}(I - E_{v1}E_{vi}^{'})$$

$$= \lambda\left(\frac{v-1}{k}\right) + \frac{\lambda}{k}(I - E_{v1}E_{vi}^{'})$$

$$= \frac{\lambda V}{k}\left(I - \frac{E_{v1}E_{v1}^{'}}{v}\right).$$

Since C is not a full rank matrix, so its unique inverse does not exist. The generalized inverse of C, denoted as C^{-}, is which is obtained as

$$C^{-} = \left(C + \frac{E_{v1}E_{v1}^{'}}{v}\right)^{-1}.$$

Since

$$C = \frac{\lambda v}{k}\left(I_v - \frac{E_{v1}E_{v1}^{'}}{v}\right)$$

or $$\frac{kC}{\lambda v} = I_v - \frac{E_{v1}E_{v1}^{'}}{v},$$

the generalized inverse of $\frac{k}{\lambda v}C$ is

$$\begin{aligned}\left(\frac{k}{\lambda v}\right)^{-1} C^{-} &= \left[C + \frac{E_{v1}E_{v1}^{'}}{v},\right]^{-1} \\ &= \left[I_v - \frac{E_{v1}E_{v1}^{'}}{v} + \frac{E_{v1}E_{v1}^{'}}{v}\right]^{-1} \\ &= I_v.\end{aligned}$$

Thus $C^{-} = \frac{\lambda v}{k} I_v.$

Thus an estimate of τ is obtained from $Q = C\tau$ as

$$\begin{aligned}\hat{\tau} &= C^{-}Q \\ &= \frac{\lambda v}{k} Q.\end{aligned}$$

The null hypothesis of our interest is $H_0 : \tau_1 = \tau_2 = ... = \tau_v$ against the alternative hypothesis $H_1 :$ at least one pair of τ_j's is different. Now we obtain the various sum of squares involved in the development of analysis of variance as follows.

The adjusted treatment sum of squares is

$$\begin{aligned} SS_{Treat(adj)} &= \hat{\tau}' Q \\ &= \frac{k}{\lambda v} Q' Q \\ &= \frac{k}{\lambda v} \sum_{j=1}^{v} Q_j^2. \end{aligned}$$

The unadjusted block sum of squares is

$$SS_{Block(unadj)} = \sum_{i=1}^{b} \frac{B_i^2}{k} - \frac{G^2}{bk}.$$

The total sum of squares is

$$SS_{Total} = \sum_{i=1}^{b} \sum_{j=1}^{v} y_{ij}^2 - \frac{G^2}{bk}.$$

The residual sum of squares is obtained by

$$SS_{Error(t)} = SS_{Total} - SS_{Block(unadj)} - SS_{Treat(adj)}.$$

A test for $H_0 : \tau_1 = \tau_2 = ... = \tau_v$ is then based on the statistic

$$F_{Tr} = \frac{SS_{Treat(adj)} / (v-1)}{SS_{Error(t)} / (bk-b-v+1)}$$

$$= \frac{k}{\lambda v} . \frac{bk-b-v+1}{v-1} . \frac{\sum_{j=1}^{v} Q_j^2}{SS_{Error(t)}}$$

If $F_{Tr} > F_{1-\alpha, v-1, bk-b-v+1;}$ then $H_{0(t)}$ is rejected.

This completes the analysis of variance test and is termed as intrablock analysis of variance.

This analysis can be compiled into the intrablock analysis of variance table for testing the significance of treatment effect given as follows

Intrablock analysis of variance table of BIBD for $H_0 : \tau_1 = \tau_2 = ... = \tau_v$

Source	**Sum of squares**	**Degrees of freedom**	**Mean squares**	***F***
Between treatments (adjusted)	$SS_{Treat(adj)}$	$v - 1$	$MS_{treat} = \frac{SS_{Treat(adj)}}{v-1}$	$\frac{MS_{Treat}}{MS_E}$
Between blocks (unadjusted)	$SS_{Block(unadj)}$	$b - 1$	$MS_E = \frac{SS_{Error(t)}}{bk-b-v+1}$	
Intrablock error	$SS_{Error(t)}$ (by substraction)	$bk - b - v + 1$		
Total	$SS_{Total} = \sum_i \sum_j y_{ij}^2 - \frac{G^2}{bk}$	$bk - 1$		

In case, the null hypothesis is rejected, then we go for pair-wise comparison of the treatments.

For that, we need an expression for the variance of difference of two treatment effects.