Analysis of Variance and Design of Experiments-II

MODULE - II

LECTURE - 9

BALANCED INCOMPLETE BLOCK DESIGN (BIBD)

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The designs like CRD and RBD are the complete block designs. We now discuss the balanced incomplete block design (BIBD) and the partially balanced incomplete block design (PBIBD) which are the incomplete block designs.

A balanced incomplete block design (BIBD) is an incomplete block design in which

- b blocks have the same number k of plots each and
- every treatment is replicated r times in the design.
- Each treatment occurs at most once in a block, i.e., $n_{ij} = 0$ or 1 where n_{ij} is the number of times the j^{th} treatment occurs in i^{th} block, $i = 1, 2, \ldots, b$; $j = 1, 2, \ldots, v$.
- Every pair of treatments occurs together is λ of the b blocks .

Such design is denoted by 5 parameters $D(b, k, v, r; \lambda)$.

The parameters b, k, v, r and λ are not chosen arbitrarily.

They satisfy the following relations:

$(I) \quad bk = vr$

$(II) \quad \lambda(v-1) = r(k-1)$

(III) $b \geq v$ (and hence $r > k$).

Hence $\sum_j n_{ij} = k$ for all i,

$\sum_j n_{ij} = r$ for all j,

and $n_{1j}n_{ij'} + n_{2j}n_{ij'} + \ldots + n_{b_j}n_{b_{j'}} = \lambda$ for all $j \neq j' = 1, 2, \ldots, v.$ Obviously $\frac{n_{ij}}{r}$ cannot be a constant for all j. So the design is not orthogonal.

Example of BIBD

In the design $D(b,k;v,r;\lambda)$: consider $b=10$ (say, $B_1,...,B_{10}$), $v=6$ (say, $T_1,...,T_6$), $k=3, r=5, \lambda=2$

Blocks	Treatments
$B1$	1 2 3
$B2$	1 2 4
$B3$	1 3 5
$B4$	1 4 6
$B5$	1 5 6
$B6$	2 3 6
$B7$	2 4 5
$B8$	2 5 6
$B9$	3 4 5
$B10$	3 4 6

Now we see how the conditions of BIBD are satisfied.

(i) $bk = 10\times 3 = 30$ and $vr = 6\times 5 = 30$

$\Rightarrow bk = vr$

(ii) $\lambda(v-1) = 2\times 5 = 10$ and $r(k-1) = 5\times 2 = 10$

$\Rightarrow \lambda(v-1) = r(k-1)$

(iii) $b = 10 \geq 6$

Even if the parameters satisfy the relations, it is not always possible to arrange the treatments in blocks to get the corresponding design.

The necessary and sufficient conditions to be satisfied by the parameters for the existence of a BIBD are not known.

The conditions (I)-(III) are some necessary condition only. The construction of such design depends on the actual arrangement of the treatments into blocks and this problem is addressed in combinatorial mathematics. Tables are available giving all the designs involving at most 20 replications and their methods of construction.

Theorem

$(I)\quad bk = vr$

$(II)\quad \lambda(v-1) = r(k-1)$

$(III)\quad b \geq v.$

Proof: (I)

Let $N = (n_{ij}) : b \times v$ incidence matrix.

Observing that the quantities $E_{1b} N E_{v1}$ and $E_{1v} N' E_{b1}$ are the scalars and the transpose of each other, we find their values.

Consider

$$E_{1b} N E_{v1} = (1,1,\ldots,1) \begin{pmatrix} n_{11} & n_{21} \cdots n_{b1} \\ n_{12} & n_{22} \cdots n_{b2} \\ \vdots & \vdots \ \ddots \ \vdots \\ n_{1v} & n_{2v} \cdots n_{bv} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= (1,1,\ldots,1) \begin{pmatrix} \sum_j n_{1j} \\ \sum_j n_{2j} \\ \vdots \\ \sum_j n_{bj} \end{pmatrix}$$

$$= (1,1,\ldots,1)_{1\times b} \begin{pmatrix} k \\ k \\ \vdots \\ k \end{pmatrix}$$

$$= \ bk.$$

Similarly,

$$E_{1v}N'E_{b1} = (1,1,\ldots,1)\begin{pmatrix} n_{11} & n_{21} \cdots n_{b1} \\ n_{12} & n_{22} \cdots n_{b2} \\ \vdots & \vdots \quad \ddots \quad \vdots \\ n_{1v} & n_{2v} \cdots n_{bv} \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

$$=(1,1,\ldots,1)\begin{pmatrix} \sum_i n_i \\ \vdots \\ \sum_i n_{iv} \end{pmatrix}$$

$$= (1,1,\ldots,1)_{1\times v}\begin{pmatrix} r \\ \vdots \\ r \end{pmatrix}$$

$$= vr.$$

But $E_{1b}NE_{v1} = E_{1v}N'E_{b1}$ as both are scalars.

Thus $bk = vr.$

Proof (II)

Consider

$$N'N = \begin{pmatrix} n_{11} & n_{21} \cdots n_{b1} \\ n_{12} & n_{22} \cdots n_{b2} \\ \vdots & \vdots \ \ddots \ \vdots \\ n_{1v} & n_{2v} \cdots n_{bv} \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} \cdots n_{1v} \\ n_{21} & n_{22} \cdots n_{2v} \\ \vdots & \vdots \ \ddots \ \vdots \\ n_{b1} & n_{b2} \cdots n_{bv} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_i n_{i1}^2 & \sum_i n_{i1} n_{i2} \cdots \sum_i n_{i1} n_{iv} \\ \sum_i n_{i1} n_{i2} & \sum_i n_{i2}^2 \cdots \sum_i n_{i2} n_{iv} \\ \vdots & \vdots \quad \ddots \quad \vdots \\ \sum_i n_{iv} n_{i1} & \sum_i n_{iv} n_{i2} \cdots \sum_i n_{iv}^2 \end{pmatrix}$$

$$= \begin{pmatrix} r & \lambda \cdots \lambda \\ \lambda & r \cdots \lambda \\ \vdots & \vdots \ \ddots \ \vdots \\ r & \lambda \cdots \lambda \end{pmatrix}. \qquad (1)$$

Since $n_{ij}^2 = 1$ or 0 as $n_{ij} = 1$ or 0,

so $\sum_i n_{ij}^2 =$ Number of times τ_j occurs in the design

$= r$ for all $j = 1, 2, ..., v$ of times occurs in the design

and $\sum_i n_{ij} n_{ij'} =$ Number of blocks in which τ_j and $\tau_{j'}$ occurs together

$= \lambda$ for all $j \neq j'$.

$$N'NE_{v1} = \begin{pmatrix} r & \lambda \cdots \lambda \\ \lambda & r \cdots \lambda \\ \vdots & \vdots \ddots \vdots \\ \lambda & \lambda \cdots r \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} r+\lambda(v-1) \\ r+\lambda(v-1) \\ \vdots \\ r+\lambda(v-1) \end{pmatrix} = [r+\lambda(v-1)]E_{v1}. \qquad (2)$$

Also,

$$N'NE_{v1} = N' \left[\begin{pmatrix} n_{11} & n_{12} \cdots n_{1v} \\ n_{21} & n_{22} \cdots n_{2v} \\ \vdots & \vdots \ddots \vdots \\ n_{b1} & n_{b2} \cdots n_{bv} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right]$$

$$= N' \begin{pmatrix} \sum_j n_{1j} \\ \sum_j n_{2j} \\ \vdots \\ \sum_j n_{bj} \end{pmatrix}$$

$$= \begin{pmatrix} n_{11} & n_{21} \cdots n_{b1} \\ n_{12} & n_{22} \cdots n_{b2} \\ \vdots & \\ n_{iv} & n_{2v} \cdots n_{bv} \end{pmatrix} \begin{pmatrix} k \\ k \\ \vdots \\ k \end{pmatrix}_{b\times 1}$$

$$= k\begin{pmatrix} \sum_i n_{i1} \\ \sum_i n_{i2} \\ \vdots \\ \sum_i n_{iv} \end{pmatrix}$$

$$= k\begin{pmatrix} r \\ r \\ \vdots \\ r \end{pmatrix}$$

$$= krE_{v1} \qquad (3)$$

From (2) and (3)

$$[r + \lambda(v-1)]E_{v1} = krE_{v1}$$

or $\quad r + \lambda(v-1) = kr$

or $\quad \lambda(v-1) = r(k-1).$

Proof (III)

From (I), the determinant of $N'N$ is

$$\det\left|N'N\right| = [r+\lambda(v-1)](r-\lambda)^{v-1}$$
$$= [r+r(k-1)](r-\lambda)^{v-1}$$
$$= rk(r-\lambda)^{v-1}$$
$$\neq 0$$

because since if $r=\lambda \Rightarrow$ from (II) that $k = v$. This contradicts the incompleteness of the design.

Thus $N'N$ is a $v \times v$ nonsingular matrix.

Thus $rank(N'N) = v.$

We know from matrix theory result

$$rank(N) = rank(N'N)$$

so $rank(N) = v.$

But $rank(N) \leq b$, there being b rows in N.

Thus $v \leq b.$