Analysis of Variance and Design of Experiments-II

MODULE VI

LECTURE – 26

SPLIT-PLOT AND STRIP-PLOT DESIGNS

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Split-plot experiment with whole-plots in a Latin square

Now we consider the split-plot experiment with whole-plot in a Latin-square.

Statistical Model

The whole-plots are organized into a Latin square with *t* rows and *t* columns. The experiment is conducted with two strata and with two randomizations. The model for the whole-plot stratum now has terms for row and column effects. The model in case of Latin square design is given by

$$y_{ijk} = \mu + r_i + c_j + \tau_{d(i,j)} + \varepsilon_{ij}, \quad i = 1, 2, ..., t; \; j = 1, 2, ...t; d(i, j) = 1, 2, ..., t; \;,$$

where ε_{ij}'s are identically and independently distributed following normal distribution with mean 0 and variance σ^2 and $\tau_{d(i,j)}$ indicates the effect of treatment assigned to (*i*, *j*) th cell and incorporate the split-plot feature as in the following model:

$$y_{ijk} = \mu + r_i + c_j + \tau_{d(i,j)} + \varepsilon(1)_{ij} + s_k + (\tau \times s)_{d(i,j)k} + \varepsilon(2)_{ijk}.$$

where $i = 1, 2, ..., t; \; j = 1, 2, ...t; d(i, j) = 1, 2, ..., t,$ and $k = 1, ..., s.$ Also, the r_i's denote the row effects, c_j's the column effects, $\tau_{d(i,j)}$'s denote the whole-plot treatment effects, s_k's denote the split-plot treatment effects, $(\tau \times s)_{d(i,j)k}$'s are the interaction effect of whole-plot treatment and split-plot treatment, $\varepsilon(1)_{ij}$'s are the whole-plot errors and $\varepsilon(2)_{ijk}$'s are the split-plot errors. The analysis of variance and standard errors for treatment comparisons follow the usual pattern.

The basic analysis of variance is detailed in the following Table.

Expected mean squares and *F*-ratios for a split-plot experiment with whole-plots arranged in a Latin square

Source	Degrees of freedom	$E[MS]$	$F-$ratio
Rows	$(t-1)$		
Columns	$(t-1)$		
W	$(t-1)$	$\sigma_2^2 + s\sigma_1^2 + rs\phi_w$	$\dfrac{MSW}{MSE(1)}$
$Error(1)$	$(t-1)(t-2)$	$\sigma_2^2 + s\sigma_1^2$	
S	$(s-1)$	$\sigma_2^2 + t^2\phi_s$	$\dfrac{MSS}{MSE(2)}$
$W \times S$	$(t-1)(s-1)$	$\sigma_2^2 + t\phi_{w\times s}$	$\dfrac{MS(W \times S)}{MSE(2)}$
$Error(2)$	$t(t-1)(s-1)$	σ_2^2	
Total			

Strip-plot experiments

The **strip-block experiment** is a variation of split-plot experiment. The following two examples are from opted from Giesbrecht and Gumpertz (2004):

Example 1

Consider a field experiment in agriculture. A large rectangular plot is available for experimentation. The two treatments planned are the *s* modes of seedbed preparation with a large piece of equipment replicated r_c times and seeding *v* varieties of some crop with a large mechanical planer replicated r_r times. To conduct the experiment, the field is divided into $r_c s$ strips in one direction. The modes of seedbed preparation are randomly assigned to the strips as in the following figure:

Variety	**Seedbed Preparation**							
	3	1	4	3	1	2	4	2
3								
2								
2								
1								
3								
1								

Each seedbed preparation mode is assigned to r_c strips, termed as columns. This is one randomization. It defines one stratum in the experiment.

Next, the field is divided into vr_r strips at right angles to the original.

The varieties of the crop are randomly assigned to these strips called as rows. The r_r rows are assigned at random to the each variety. This establishes a stratum for variety.

In addition, it establishes a third stratum for the interaction of seedbed preparation and varieties. Both randomizations affect the assignment in this stratum.

If fertility gradients are suspected, then the strips (either one or both) can be grouped into sets, i.e., the blocking factors can be introduced in one or both the directions.

This is an example of a strip-plot or strip-block experiment where the stripping is dictated by the nature of the experimental treatments.

It is a convenient way to organize things if one needs to use large pieces of equipment. Note that eventually the experimenter harvests the subplots defined by the intersection of the row and column strips.

A feature of this design is that it provides most information on the interaction.

Example 2

Consider a cake baking study. It involves a slightly different experimental procedure, but the design principles are the same.

Suppose a food product developer wants to develop a new cake recipe. There are '*a*' similar recipes and '*b*' different baking regimes which are to be tested.

This can be achieved by first mixing the cake batters and use one batch for each recipe.

Make each batch large enough to provide '*b*' cakes.

Assign one cake from each batch of the ovens and each oven to a baking regime.

Assume that the individual ovens are large enough to hold '*a*' cakes. The recipes form rows and the baking regimes form columns that are the two strata. In addition, there is the third stratum - the interaction. Observations are collected on individual cakes. This entire procedure is repeated, i.e., replicated, *r* times.

An important advantage in this experimental designs is that the amount of work is reduced.

The alternative approach is to mix and bake '*ab*' cakes individually. This will constitute one complete replicate. This will involve much work. The best information is at the interaction level. In many cases this is a good thing since interactions are often very important. This experiment allows the experimenter to check the robustness of the recipes to variations in baking routine.

Statistical model for the strip-plot

Consider the following linear statistical model

$$y_{hij} = \mu + rep_h + a_i + \varepsilon(r)_{hi} + b_j + \varepsilon(c)_{hj} + (a \times b)_{ij} + \varepsilon_{hij}$$

Where $\varepsilon(r)_{hj}, \varepsilon(c)_{hj}$ and ε_{hij} are identically and independently distributed, each with mean 0 and variance σ_1^2, σ_2^2 and σ_ε^2, respectively. Moreover, they are mutually independent of each other for all $h = 1, \ldots, r,\ i = 1, \ldots, a,$ and $j = 1, \ldots, c.$ The replicate effects rep_h, effects a_i's and effects b_j's are measured as deviations from a mean and that the interaction effects are defined to sum to zero in both directions.

The replicate effects can be assumed to be fixed or random .

Analysis of variance and standard errors

The analysis of variance based on this model is given in the following Table.

Analysis of variance table for a strip-plot experiment

Source	Degrees of freedom	Sum of squares	Mean squares	$E[MS]$
Replication	$r-1$	$ab\sum_{h}^{r}(\bar{y}_{hoo}-\bar{y}_{ooo})^2$	$MS_{rep.}$	---
A	$a-1$	$rs\sum_{i}^{a}(\bar{y}_{oio}-\bar{y}_{ooo})^2$	MSA	$\sigma_{\varepsilon}^2+b\sigma_r^2+rb\phi_{\alpha}$
$MSE(r)$	$(r-1)(a-1)$	$b\sum_{h}^{r}\sum_{i}^{a}(\bar{y}_{hio}-\bar{y}_{hoo}-\bar{y}_{oio}+\bar{y}_{ooo})^2$	$MSE(r)$	$\sigma_{\varepsilon}^2+b\sigma_r^2$
B	$b-1$	$ra\sum_{j}^{b}(\bar{y}_{ooj}-\bar{y}_{ooo})^2$	MSB	$\sigma_{\varepsilon}^2+a\sigma_c^2+ra\phi_b$
$MSE(c)$	$(r-1)(b-1)$	$a\sum_{h}^{r}\sum_{j}^{b}(\bar{y}_{hoj}-\bar{y}_{hoo}-\bar{y}_{ooj}+\bar{y}_{ooo})^2$	$MSE(c)$	$\sigma_{\varepsilon}^2+a\sigma_c^2$
$A\times B$	$(a-1)(b-1)$	$r\sum_{i}^{a}\sum_{j}^{b}(\bar{y}_{oio}-\bar{y}_{oio}-\bar{y}_{ooj}+\bar{y}_{ooo})^2$	$MS(AB)$	$\sigma_{\varepsilon}^2+r\phi_{ab}$
$MSE(\varepsilon)$	$(r-1)(a-1)(b-1)$	by subtraction	$MSE(\varepsilon)$	σ_{ε}^2
Total (corrected)	$rab-1$	$\sum_{h}^{r}\sum_{i}^{a}\sum_{j}^{b}(\bar{y}_{hik}-\bar{y}_{ooo})^2$		

The standard errors for the row (treatment *A*) and for the column (treatment *B*) comparisons are

$$s.e(A) = \sqrt{\frac{\sum_i c_i^2 [MSE(r)]}{rb}}$$

and

$$s.e(B) = \sqrt{\frac{\sum_i c_j^2 [MSE(c)]}{ra}},$$

respectively where c_i and c_j are the sets of arbitrary constants that sum to zero. The standard errors for interaction contrasts are more complex. The general form of these contrasts is $\sum_i \sum_j c_{ij} \bar{y}_{oij}$, where c_{ij} 's are arbitrary constant coefficients that sum to zero. For example, one can select:

1. To compare the two *B* treatments at a given level of *A*, take $c_{ij} = 1$, $c_{ij} = -1$, for some i and some $j \neq j'$ and all the other coefficients equal to zero.

2. To compare the two *A* treatments at a given *B* level, take $c_{ij} = 1$, $c_{ij} = -1$, for some $i \neq i'$ and a specific j and all the other coefficients equal to zero.

3. To compare the two means for different *A* and *B* levels, take $c_{ij} = 1$, $c_{ij} = -1$, for specific $i \neq i'$ and $j \neq j'$ and all the other coefficients equal to zero.

The variance of the general contrast form is

$$Var\left[\sum_i\sum_j c_{ij}\bar{y}_{oij}\right]=Var\left[\sum_i\sum_j c_{ij}\sum_h \varepsilon(r)_{hi}/r\right]+Var\left[\sum_i\sum_j c_{ij}\sum_h \varepsilon(c)_{hj}/r\right]+Var\left[\sum_i\sum_j c_{ij}\sum_h \varepsilon_{hij}/r\right]$$

where all other cross-product terms are zero due to the assumption of independence of errors. Now we must look at special cases. When $c_{ij} = 1$, $c_{ij} = -1$, for some i and $j \neq j'$, $\sum_j c_{ij} = 0$ and then we have

$$Var\left[\sum_i\sum_j c_{ij}\bar{y}_{oij}\right]=Var\left[\frac{\sum_i\sum_j c_{ij}\sum_h \varepsilon(c)_{hi}}{r}\right]+Var\left[\frac{\sum_i\sum_j c_{ij}\sum_h \varepsilon_{hij}}{r}\right]$$

$$=\sum_i\sum_j \frac{c_{ij}^2\left[\sigma_c^2+\sigma_\varepsilon^2\right]}{r}$$

$$\frac{E\left[MSE(c)+(a-1)MSE(\varepsilon)\right]}{a}=\sigma_c^2+\sigma_\varepsilon^2.$$

The exact degrees of freedom for conducting the tests and constructing the confidence intervals are difficult to obtain in this case. We use the approximate degrees of freedom which can be obtained following the Saitterthwaite approach. In this case, choose $a_2 = a-1, m=2,\ a_1 = 1,\ MSE(1) = MSE(c),\ df_1 = (r-1)(b-1), MSE(2) = MSE(\varepsilon)$ and $df_2 = (r-1)(a-1)(b-1)$.

In case 2 with $c_{ij}=1, c_{i'j}=-1$ for some $i \neq i'$ and specific j with all the other coefficients zero, we have

$$Var\left[\sum_i \sum_j c_{ij}\bar{y}_{oij}\right] = \frac{1}{r}\sum_i \sum_j \frac{c_{ij}^2\left[\sigma_r^2+\sigma_\varepsilon^2\right]}{r}$$

$$E\left[\frac{MSE(r)+(b-1)MSE(\varepsilon)}{b}\right] = \sigma_r^2+\sigma_\varepsilon^2.$$

The approximate degrees of freedom are obtained using Saitterthwaite approach with

$m=2,\ a_1=1,\ MSE(1)=MSE(r),\ df_1=(r-1)(a-1),\ a_2=b-1, MSE(2)=MSE(\varepsilon)$ and $df_2=(r-1)(a-1)(b-1).$

In case 3 with $c_{ij}=1,\ c_{ij'}=-1$ for specific $i \neq i',\ j \neq j'$ with all the other coefficients zero, we have

$$Var\left[\sum_i \sum_j c_{ij}\bar{y}_{oij}\right] = \frac{1}{r}\sum_i \sum_j c_{ij}^2\left[\sigma_r^2+\sigma_c^2+\sigma_\varepsilon^2\right]$$

$$W=\frac{\left[aMSE(r)+bMSE(c)+(ab-a-b)MSE(\varepsilon)\right]}{ab} = \sigma_r^2+\sigma_c^2+\sigma_\varepsilon^2$$

The approximate degrees of freedom are obtained using Satterwhaite’s approximation with $m=3,\ a_1=a,\ MSE(1)=MSE(r),$

$df_1=(r-1)(a-1),\ a_2=b, MSE(2)=MSE(c),\ df_2=(r-1)(b-1),\ a_3=(ab-a-b),\ MSE(3)=MSE(\varepsilon)$ and $df_3=(r-1)(a-1)(b-1).$