Analysis of Variance and Design of Experiments-II

MODULE - II
LECTURE - 10

BALANCED INCOMPLETE BLOCK DESIGN (BIBD)

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Interpretation of conditions of BIBD

Interpretation of (I) *bk = vr*

This condition is related to the total number of plots in an experiment. In our settings, there are *k* plots is each block and there are *b* blocks. So total number of plots are *bk*. Further, there are *v* treatments and each treatment is replicated *r* times such that each treatment occurs atmost in one block. So the total number of plots containing all the treatments is *vr*. Since both the statements count the total number of plots, hence *bk = vr*

Interpretation of (II)

Each block has *k* plots. Thus the total pairs of plots in a block $= \binom{k}{2} = \frac{k(k-1)}{2}$.

There are *b* blocks. Thus the total pairs of plots such that each pair consists of plots within a block $= b\frac{k(k-1)}{2}$.

There are *v* treatments, thus the total number of pairs of treatment $= \binom{v}{2} = \frac{v(v-1)}{2}$.

Each pair of treatment is replicated λ times, i.e., each pair of treatment occurs in λ blocks.

Thus the total number of pairs of plots within blocks must be $= \lambda\frac{v(v-1)}{2}$.

Hence $b\frac{k(k-1)}{2} = \lambda\frac{v(v-1)}{2}$

Using $bk = vr$ in this relation, we get $r(k-1) = \lambda(v-1)$.

Proof of (III) was given by Fisher but quite long, so not needed here.

Balancing in designs

There are two type of balancing –

- variance balanced and
- efficiency balanced.

We discuss the variance balancing now and the efficiency balancing later.

Balanced design (Variance balanced)

A connected design is said to be balanced (variance balanced) if all the elementary contrasts of the treatment effects can be estimated with the same precision. This definition does not hold for the disconnected design, as all the elementary contrasts are not estimable in this design.

Proper design

An incomplete block design with $k_1 = k_2 = ... = k_b = k$ is called a proper design.

Symmetric BIBD

A BIBD is called symmetrical if the number of blocks = the number of treatments, i.e., $b = v$.

Since $b = v$ so from $bk = vr$

$$\Rightarrow k = r.$$

Thus the number of pairs of treatments common between any two blocks = λ .

The determinant of $N'N$ is

$$|N'N| = [r + \lambda(v-1)](r-\lambda)^{v-1}$$

$$= [r + r(k-1)](r-\lambda)^{v-1}$$

$$= rk(r-\lambda)^{v-1}.$$

When BIBD is symmetric, $b = v$ and then using $bk = vr$, we have $k = r$. Thus

$$|N'N| = |N|^2 = r^2(r-\lambda)^{v-1},$$

so

$$|N| = \pm\, r(r-\lambda)^{\frac{v-1}{2}}.$$

Since $|N|$ is an integer, hence when v is an even number, $(r-\lambda)$ must be a perfect square. So

$$|N'N| = (r-\lambda)I + \lambda E_{v1}E_{v1}',$$

$$(N'N)^{-1} = N^{-1}N'^{-1}$$

$$= \frac{1}{r-\lambda}\left[I - \frac{\lambda}{r^2}E_{v1}E_{v1}'\right],$$

$$N'^{-1} = \frac{1}{r-\lambda}\left[I - \frac{\lambda}{r^2}E_{v1}E_{v1}'\right].$$

Postmultiplying both sides by N', we get

$$NN' = (r-\lambda)I + \lambda E_{v1} E'_{v1} = N'N.$$

Hence in the case of a symmetric BIBD, any two blocks have λ treatments in common.

Since BIBD is an incomplete block design, so every pair of treatment can occur at most once is a block, we must have $v \geq k$.

If $v = k$, then it means that each treatment occurs once in every block and this occurs in the case of RBD.

So in BIBD, always assume $v > k$.

Similarly $\lambda < r$.

[If $\lambda = r$ then $\lambda(v-1) = r(k-1) \Rightarrow v = k \Rightarrow$ which means that the design is RBD]

Resolvable design

A block design of

- *b* blocks in which

- each of *v* treatments is replicated *r* times

is said to be resolvable if *b* blocks can be divided into *r* sets of $\dfrac{b}{r}$ blocks each, such that every treatment appears in each set precisely once. Obviously, in a resolvable design, *b* is a multiple of *r*.

Theorem If in a BIBD $D(v,b,r,k,\lambda)$, b is divisible by r, then $b \geq v+r-1$.

Proof Let $b = nr$ (where $n > 1$ is a positive integer.)

For a BIBD, $\lambda(v-1) = r(k-1)$

or $r = \dfrac{\lambda(v-1)}{(k-1)}$ $\qquad \left[\text{because } vr = bk \text{ or } vr = nrk \text{ or } v = nk\right]$

$\qquad = \dfrac{\lambda(nk-1)}{(k-1)}$

$\qquad = \lambda\left(\dfrac{n-1}{k-1}\right) + \lambda n.$

Since $n > 1$ and $k > 1$, so $\lambda n > 1$ is an integer. Since r has to be an integer

$\Rightarrow \lambda\dfrac{(n-1)}{k-1}$ is also a positive integer.

Now, if possible, let

$\qquad\qquad b < v+r-1$

$\Rightarrow\qquad nr < v+r-1$

or $\qquad r(n-1) < v-1$

or $\qquad r(n-1) < \dfrac{r(k-1)}{\lambda}$ $\qquad\qquad$ (because $v-1 = \dfrac{r(k-1)}{\lambda}$)

$\Rightarrow\qquad \dfrac{\lambda(n-1)}{k-1} < 1$ which is a contradiction as integer cannot be less than one

$\Rightarrow\quad b < v+r-1$ is impossible. Thus the opposite is true.

$\Rightarrow\quad b \geq v+r-1$ holds correct.