Analysis of Variance and Design of Experiments-II

MODULE VI

LECTURE - 24

SPLIT-PLOT AND STRIP-PLOT DESIGNS

Dr. Shalabh
Department of Mathematics & Statistics
Indian Institute of Technology Kanpur

An example to motivate the use of split plot design is as follows.

Suppose we wish to study two factors, say methods of cultivation and varieties of wheat. Suppose, the first factor has *t* levels and the second factor consists of *s* varieties.

The first factor requires the use of a large complex equipment and consequently, relatively large plots of land are needed. This will require higher cost and puts a restriction on the number of plots to be used. Because of the nature of the equipment used for planting the wheat, the second factor can be accommodated in much smaller plots. To achieve this, the large plots are split into smaller plots at the planting stage.

This means that since the plots are close together, so less variability is expected among the plots and in turn, more plots and less variability among plots is expected, which implies that the contrasts will have more information in terms of smaller standard errors.

This suggests that the experiment can be conducted with two strata. The whole-plot stratum consists of large plots in which the plots can be assigned as per any standard design, e.g. CRD, RBD, or Latin square design.

Next stratum is the split-plot stratum which consists of the split-plots. There are the smaller plots that are obtained by splitting each of the large plots into *s* parts. The treatments assigned to the large whole-plots are replicated *r* times, and treatments assigned to the split-plots are replicated *rt* times.

Now much more information on the split-plot factor is available because of the extra replication, and in turn, a smaller split-plot-to-split-plot variance is expected. The interaction contrasts between whole- and split-plot treatment also fall into the split-plot stratum and benefit due to smaller variance.

There are two distinct randomizations in the split plot designs:

i. The first randomization takes place in stratum 1, when the levels of the whole-plot treatment are randomly assigned to the whole–plots.

ii. The second randomization takes place in stratum 2 where the levels in the split-plot treatment are randomly assigned in the split-plot.

Many split-plot plans can easily be modified to become strip-plot experiments. These have their own advantages and disadvantages.

Examples

Following examples have been opted from Giesbrecht and Gumpertz (2004).

"Consider a hypothetical cake baking study in an industry. Assume that there are *r* recipes and *c* baking conditions are to be studied. A simple split- plot experiment with the recipes as a whole-plot factor and the baking condition as a split-plot factor can be set up if cake batters are made up using recipes in a random order. Each batch of batter is then split into *c* portions. The portion sare then baked under the *c* conditions. A new random baking order is selected for each batter. Replication is provided by repeating recipes.

Another option is to make up enough batter to make one cake from each of the *r* recipes. All cakes based on *r* recipes are then baked at one time in an oven at one of the *c* conditions. Now we have an experiment with baking conditions as a whole-plot factor and recipe as a split-plot factor.

In case of a strip-plot design, the experimenter would make up batches of each of the batters large enough. Then partition each batch into *c* cakes and then bake the cakes in sets, with one cake of each recipe in each set. In terms of row-column structure of design, the rows represent recipes and the columns represent baking conditions. The advantage here is that in the absence of replication, only *r* batches of batter need be mixed and the oven need only be set up *c* times.

In another example of a split-plot experiment in industrial quality research is as follows:

The object of the project is to develop a packaging material that would give a better seal under the wide range of possible sealing process conditions used by potential customers. The package manufacturer identifies a number of factors which can affect the quality of the seal. In the whole-plot part of the experiment, the sample lots of eight different packaging materials are produced. These lots of material are then sent to a customer’s plant, where each of them is subdivided into six subplots. The subplots are used in six different sealing processes. This constitutes the split-plot part of the study.

Statistical analysis of split-plot experiments

Split-plot experiment with whole-plots in a CRD

Statistical Model

The statistical model for a split-plot consists of the two randomization steps in the split-plot experiment, one in each stratum. So it is a model with two terms. We consider an experiment with whole-plots arranged in a CRD. Suppose *W* represents the whole-plot treatment and *S* represents the split-plot treatment, then the linear statistical model is written as

$$y_{ijk} = \mu + w_i + \varepsilon(1)_{ij} + s_k + (w \times s)_{ik} + \varepsilon(2)_{ijk},$$

where $\varepsilon(1)_{ij}$'s and $\varepsilon'(2)_{ijk}$ are identically and independently distributed random errors, each with mean 0 but different variances σ_1^2 and σ_2^2, respectively, $i = 1, 2, \ldots, t;\ j = 1, 2, \ldots, s$ and $k = 1, 2, \ldots, s$. Moreover, $\varepsilon(1)_{ij}$'s and $\varepsilon(2)_{ij}$'s are mutually independent.

The whole-plot stratum of the model contains the whole-plot treatment effects w_i and the whole-plot error terms $\varepsilon(1)_{ij}$. If we include the mean μ , this part of the model is similar to the case in one way model for CRD. The split-plot stratum contains the split-plot treatment effects s_k, the interaction effect of *w* and *s* as $(w \times s)_{ik}$ and the experimental error associated with individual split-plots $\varepsilon(2)_{ijk}$. All the terms on the right-hand side of the model (except μ) are assumed to have observations measured as the deviation from the respective mean.

Analysis of variance

The analysis of variance for the split-plot experiment in the CRD is like an extended analysis for the CRD. This can be considered as two separate analysis of variance for each of two strata with two separate error terms. This is illustrated in the following table.

ANOVA table for a split-plot experiment with whole-plots arranged in a CRD

Source	Degrees of freedom	Sum of squares	Mean squares	$E(MS)$	F - ratio
W	$t-1$	$rs\sum_{i}^{t}(\bar{y}_{ioo}-\bar{y}_{ooo})^2$	MSW	$\sigma_2^2+s\sigma_1^2+rs\phi_w$	$\frac{MSW}{MSE(1)}$
Error(1)	$t(r-1)$	$s\sum_{i}^{t}\sum_{j}^{r}(\bar{y}_{ijo}-\bar{y}_{ioo})^2$	$MSE(1)$	$\sigma_2^2+s\sigma_1^2$	
S	$s-1$	$rt\sum_{k}^{s}(\bar{y}_{ook}-\bar{y}_{ooo})^2$	MSS	$\sigma_2^2+rt\phi_s$	$\frac{MSS}{MSE(2)}$
$W\times S$	$(t-1)(s-1)$	$r\sum_{i}^{t}\sum_{k}^{s}(\bar{y}_{iok}-\bar{y}_{ioo}-\bar{y}_{ook}+\bar{y}_{ooo})^2$	$MS(W\times S)$	$\sigma_2^2+rs\phi_{w\times s}$	
Error(2)	$t(r-1)(s-1)$	$\sum_{i}^{t}\sum_{j}^{r}\sum_{k}^{s}(\bar{y}_{ijk}-\bar{y}_{ijo}-\bar{y}_{iok}+\bar{y}_{ioo})^2$	$MSE(2)$	σ_2^2	$\frac{MS(W\times S)}{MSE(2)}$
Total (corrected)	$rts-1$	$\sum_{i}^{t}\sum_{j}^{r}\sum_{k}^{s}(\bar{y}_{ijk}-\bar{y}_{ooo})^2$			

Note that the sum of squares due to *W x S* is $\sum_{i}^{t}\sum_{k}^{s} r\bar{y}_{iok}^2 - rst\bar{y}_{ooo}^2 - SSW - SSS,$ where *SSW* is the whole-plot treatment sum of squares and *SSS* is the split-plot treatment sum of squares. The error(2) sum of squares is obtained by subtraction. The mean squares are obtained by dividing the sum of squares entries by respective degrees of freedom.

The quantities $\phi_w, \phi_s,$ and $\phi_{w\times s}$ represent quadratic forms as follows:

$$\phi_w = \frac{\sum_j w_j^2}{t-1}$$

$$\phi_s = \frac{\sum_k s_k^2}{s-1}$$

$$\phi_{w\times s} = \frac{\sum_j \sum_k (W \times S)_{jk}^2}{(t-1)(s-1)}.$$

These quadratic forms will be zero under the appropriate null hypotheses. It is clear from the expected mean squares that

- error (1) is used to test the hypothesis of no whole-plot treatment effect and error and
- error (2) is used to test the hypotheses of no interaction or split–plot treatment effects.

The test for interaction is performed first, otherwise other tests of hypothesis are doubtful.

Note that since all the levels of each factor are tested in combination with every level of the other factor, so the analysis in the whole-plot stratum and the split-plot stratum are orthogonal. The estimates of interactions between whole-and split-unit factors are the contrasts that are orthogonal to both whole-plot and split-plot treatment contrasts.

Standard errors of main-effect contrasts

The standard errors in the split-plot are more complex than in other designs. We first consider the contrasts among levels of the split-plot treatment. We write the general form of a split-plot contrast as $\sum_k c_k \bar{y}_{ook}$.

Then

$$E\left[\sum_k c_k \bar{y}_{ook}\right] = \sum_k c_k E\left[\bar{y}_{ook}\right]$$
$$= \sum_k c_k s_k$$

and

$$Var\left[\sum_k c_k \bar{y}_{ook}\right] = Var\left[\sum_k c_k \left(\sum_i \sum_j \varepsilon(2)_{ijk} / rt\right)\right]$$
$$= \sum_k c_k^2 \sigma_2^2 / rt.$$

Since $E\left[MSE(2)\right] = \sigma^2$, it follows that the estimated standard error (*s.e*) of a split-plot treatment contrast is of the form

$$s.e\left(\sum_k c_k \bar{y}_{ook}\right) = \sqrt{\frac{\sum_k c_k^2 MSE(2)}{rt}}.$$

For the case $c_k = 1$ and $c_k = -1$, the contrast is $\bar{y}_{ook} - \bar{y}_{ook'}$ where $k \neq k'$. It follows that

$$s.e(\bar{y}_{ook} - \bar{y}_{ook'}) = \sqrt{\frac{2MSE(2)}{rt}}.$$

The confidence intervals computed for the contrasts are based on $t(r-1)(s-1)$ degrees of freedom.

The general form of a whole-plot treatment contrast takes the form $\sum_i c_i \bar{y}_{ioo}$. We have

$$E\left[\sum_i c_i \bar{y}_{ioo}\right] = \sum_i c_i E\left[\bar{y}_{ioo}\right] = \sum_i c_i w_i.$$

$$Var\left[\sum_i c_i \bar{y}_{ioo}\right] = Var\left[\sum_i c_i \left(\frac{\sum_j \varepsilon(1)_{ij}}{r}\right) + \sum_i c_i \left(\frac{\sum_j \sum_k \varepsilon(2)_{ijk}}{rs}\right)\right] = \sum_i \frac{c_i^2(\sigma_2^2 + s\sigma_1^2)}{rs}.$$

Since $E\left[MSE(1)\right] = \sigma_2^2 + s\sigma_1^2$, it follows that the estimated standard error of a whole-plot treatment contrast of the form is

$$s.e\left(\sum_k c_i \bar{y}_{ioo}\right) = \sqrt{\sum_i \frac{c_i^2 MSE(1)}{rs}}.$$

For the whole-plot treatment difference, the estimate of standard error is

$$s.e\left(\bar{y}_{ioo} - \bar{y}_{i'oo}\right) = \sqrt{\frac{MSE(1)}{rs}}.$$