Analysis of Variance and Design of Experiments-II

MODULE - III

LECTURE - 17

PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN (PBIBD)

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Group divisible association scheme

Let there be v treatments which can be represented as $v = pq$. Now divide the v treatments into p groups with each group having q treatments such that any two treatments in the same group are the first associates and the two treatments in different groups are the second associates. This is called the group divisible type scheme. The scheme simply amounts to arrange the $v = pq$ treatments in a (p x q) rectangle and then the association scheme can be exhibited. The columns in the (p x q) rectangle will form the groups.

Under this association scheme,

$$n_1 = q - 1$$
$$n_2 = q(p-1),$$

hence

$$(q-1)\lambda_1 + q(p-1)\lambda_2 = r(k-1)$$

and the parameters of second kind are uniquely determined by p and q. In this case

$$P_1 = \begin{pmatrix} q-2 & 0 \\ 0 & q(p-1) \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & q-1 \\ q-1 & q(p-2) \end{pmatrix},$$

For every group divisible design,

$$r \geq \lambda_1,$$
$$rk - v\lambda_2 \geq 0.$$

If $r = \lambda_1$, then the group group divisible design is said to be **singular.** Such singular group divisible design can always be derived from a corresponding BIBD. To obtain this, just replace each treatment by a group of *q* treatments. In general, if a BIBD has parameters $b^*, v^*, r^*, k^*, \lambda^*,$ then a divisible group divisible design is obtained which has following parameters

$$b = b^*, \quad v = qv^*, \quad r = r^*, \quad k = qk^*, \quad \lambda_1 = r, \quad \lambda_2 = \lambda^*, \quad n_1 = p, \quad n_2 = q.$$

A group divisible design is **nonsingular** if $r \neq \lambda_1$. Nonsingular group divisible designs can be divided into two classes – semi-regular and regular.

A group divisible design is said to be **semi-regular** if $r > \lambda_1$ and $rk - v\lambda_2 = 0$. For this design

$$b \geq v - p + 1.$$

Also, each block contains the same number of treatments from each group so that *k* must be divisible by *p*.

A group divisible design is said to be **regular** if $r > \lambda_1$ and $rk - v\lambda_2 > 0$. For this design

$$b \geq v.$$

Latin square type association scheme

Let L_i denotes the Latin square type PBIBD with *i* constraints. In this PBIBD, the number of treatments are expressible as $v = q^2$. The treatments may be arranged in a (q x q) square. For the case of $i = 2$, the two treatments are the first associates if they occur in the same row or the same column, and second associates otherwise. For the general case, we take a set of (i - 2) mutually orthogonal Latin squares, provided it exists. Then two treatments are the first associates if they occur in the same row or the same column, or corresponding to the same letter of one of the Latin squares. Otherwise they are second associates.

Under this association scheme,

$$v = q^2,$$

$$n_1 = i(q-1),$$

$$n_2 = (q-1)(q-i+1),$$

$$P_1 = \begin{pmatrix} (i-1)(i-2)+q-2 & (q-i+1)(i-1) \\ (q-i+1)(i-1) & (q-i+1)(q-i) \end{pmatrix},$$

$$P_2 = \begin{pmatrix} i(i-1) & i(q-i) \\ i(q-i) & (q-i)(q-i-1)+q-2 \end{pmatrix}.$$

Cyclic type association scheme

Let there be v treatments and they are denoted by integers $1,2,\ldots,v$. In a cyclic type PBIBD, the first associates of treatment i are

$$i+d_1,\ i+d_2,\ldots,i+d_{n1}\ (\text{mod } v),$$

where the d's satisfy the following conditions:

i. the d's are all different and $0<d_j<v(j=1,2,\ldots,n_1)$;

ii. among the $n_1(n_1-1)$ differences $d_j-d_{j'},\ (j,j'=1,2,\ldots,n_1,\ j\neq j')$ reduced (*mod v*), each of numbers $d_1,d_2,\ldots,d_n$ occurs α times, whereas each of the numbers $e_1,e_2,\ldots,e_{n2}$ occurs β times, where $d_1,d_2,\ldots,d_{n1},e_1,e_2,\ldots,e_{n2}$ are all the different $v-1$ numbers $1,2,\ldots,v-1$.

[**Note:** To reduce an integer mod *v*, we have to subtract from it a suitable multiple of v so that the reduced integer lies between 1 and *v*. For example, 17 when reduced *mod* 13 is 4.]

For this scheme,

$$n_1\alpha+n_2\beta=n_1(n_1-1),$$

$$P_1=\begin{pmatrix}\alpha & n_1-\alpha-1\\ n_1-\alpha-1 & n_2-n_1+\alpha+1\end{pmatrix},$$

$$P_2=\begin{pmatrix}\beta & n_1-\beta\\ n_1-\beta & n_2-n_1+\beta-1\end{pmatrix}.$$

Singly linked block association scheme

Consider a BIBD D with parameters $b^{**}, v^{**}, r^{**}, k^{**}, \lambda^{**}=1$ and $b^{**}> v^{**}$.

Let the block numbers of this design be treated as treatments, i.e., $v=b^{**}$. The first and second associates in the singly linked block association scheme with two classes are determined as follows: Define two block numbers of D to be the first associates if they have exactly one treatment in common and second associates otherwise.

Under this association scheme,

$$
\begin{aligned}
& v=b^{**}, \\
& n_1=k^{**}(r^{**}-1), \\
& n_2=b^{**}-1-n_1, \\
& P_1=\begin{pmatrix} r^{**}-2+(k^{**}-1)^2 & n_1-r^{**}-(k^{**}-1)^2+1 \\ n_1-r^{**}-(k^{**}-1)^2+1 & n_2-n_1+r^{**}+(k^{**}-1)^2-1 \end{pmatrix}, \\
& P_2=\begin{pmatrix} k^{**2} & n_1-k^{**2} \\ n_1-k^{**2} & n_2-n_1+k^{**2}-1 \end{pmatrix}.
\end{aligned}
$$

General theory of PBIBD

A PBIBD with m-associate classes is defined as follows. Let there be v treatments, b blocks and size of each block is k, i.e., there are k plots in each block. Then the v treatments are arranged in b blocks according to an m - associate partially balanced association scheme such that

a) every treatment occurs at most once in a block,

b) every treatment occurs exactly in r blocks and

c) if two treatments are the i^{th} associates of each other then they occur together exactly in $\lambda_i (i = 1, 2, ..., m)$ blocks.

The number λ_i is independent of the particular pair of i^{th} associate chosen. It is not necessary that λ_i should all be different and some of the λ_i's may be zero.

If v treatments can be arranged in such a scheme then we have a PBIBD. Note that here two treatments which are the i^{th} associates, occur together in λ_i blocks.

The parameters $b, v, r, k, \lambda_1, \lambda_2, ..., \lambda_m, n_1, n_2, ..., n_m$ are termed as the **parameters of first kind** and p^i_{jk} are termed as the **parameters of second kind**. It may be noted that $n_1, n_2, ..., n_m$ and all p^i_{jk} of the design are obtained from the association scheme under consideration. Only $\lambda_1, \lambda_2, ..., \lambda_m$, occur in the definition of PBIBD.

If $\lambda_i = \lambda$ for all $i = 1,2,\dots,m$ then PBIBD reduces to BIBD. So BIBD is essentially a PBIBD with one associate class.

Conditions for PBIBD

The parameters of a PBIBD are chosen such that they satisfy the following relations:

$$(i) \quad bk = vr$$

$$(ii) \quad \sum_{i=1}^{m} n_i = v - 1$$

$$(iii) \sum_{i=1}^{m} n_i \lambda_i = r(k-1)$$

$$(iv) \quad n_k p_{ij}^{k} = n_i p_{jk}^{i} = n_j p_{ki}^{j}$$

$$(v) \quad \sum_{k=1}^{m} p_{jk}^{i} = \begin{cases} n_j - 1 & \text{if } i = j \\ n_j & \text{if } i \neq j. \end{cases}$$

It follows from these conditions that there are only $m(m^2-1)/6$ independent parameters of the second kind.

Interpretations of conditions of PBIBD

The interpretations of conditions (i)-(v) are as follows.

(i) $bk = vr$

This condition is related to the total number of plots in an experiment. In our settings, there are k plots in each block and there are b blocks. So total number of plots are bk. Further, there are v treatments and each treatment is replicated r times such that each treatment occurs atmost in one block. So the total number of plots containing all the treatments is vr. Since both the statements counts the total number of blocks, hence $bk = vr$.

(ii) $\sum_{i=1}^{m} n_i = v-1$

This condition interprets as follows:

Since with respect to each treatment, the remaining (v - 1) treatments are classified as first, second,…, or m^{th} associates and each treatment has n_i associates, so the sum of all n_i's is the same as the total number of treatments except the treatment under consideration.

(iii) $\sum_{i=1}^{m} n_i \lambda_i = r(k-1)$

Consider *r* blocks in which a particular treatment *A* occurs. In these r blocks, $r(k-1)$ pairs of treatments can be found, each having *A* as one of its members. Among these pairs, the i^{th} associate of *A* must occur λ_i times and there are n_i associates, so $\sum_i n_i \lambda_i = r(k-1)$.

(iv) $n_k p^k_{ij} = n_i p^i_{jk} = n_j p^j_{ki}$

Let G_i be the set of i^{th} associates, i = 1, 2,…,m of a treatment A. For $i \neq j$, each treatment in G_i has exactly p^i_{jk} numbers of k^{th} associates in G_j. Thus the number of pairs of k^{th} associates that can be obtained by taking one treatment from G_i and another treatment from G_j on the one hand is $n_i p^i_{jk}$ and on the another hand is $n_j p^i_{jk}$.

(v) $\sum_{k=1}^{m} p^i_{jk} = n_j - 1$ if $i = j$ and $\sum_{k=1}^{m} p^i_{jk} = n_j$ if $i \neq j.$

Let the treatments A and B be i^{th} associates. The k^{th} associate of $A(k = 1, 2, ..., m)$ should contain all the n_j number of j^{th} associates of $B(j \neq i)$. When $j = i,\ A$ itself will be one of the j^{th} associate of B. Hence k^{th} associate of $A,\ (k = 1, 2, ..., m)$ should contain all the $(n_j - 1)$ numbers of j^{th} associate of B. Thus the condition holds.