

Analysis of Variance and Design of Experiments-II

MODULE IV

LECTURE - 20

PARTIAL CONFOUNDING

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Similarly, when B is estimated from the replicates 1 and 3 separately, then the individual estimates of B are given by

$$B_{rep1} = \frac{\left(\sum_{i=1}^r \ell'_{B1} y_{*i} \right)_{rep1}}{2r}$$

and

$$B_{rep3} = \frac{\left(\sum_{i=1}^r \ell'_{B3} y_{*i} \right)_{rep3}}{2r}.$$

Both the estimators are combined as arithmetic mean and the estimator of B based on partial confounding is

$$\begin{aligned} B_{pc} &= \frac{B_{rep1} + B_{rep3}}{2} \\ &= \frac{\left(\sum_{i=1}^r \ell'_{B1} y_{*i} \right)_{rep1} + \left(\sum_{i=1}^r \ell'_{B3} y_{*i} \right)_{rep3}}{4r} \\ &= \frac{\left(\sum_{i=1}^r \ell_{B}^{*'} y_{*i} \right)}{4r} \end{aligned}$$

where the (8×1) vector

$$\ell_B^{*' } = (\ell_{B1}, \ell_{B3})$$

has 8 elements. The sum of squares due to B_{pc} is obtained as

$$SS_{B_{pc}} = \frac{\left(\sum_{i=1}^r \ell_{B}^{*'} y_{*i} \right)^2}{r \ell_B^{*' } \ell_B^{*}} = \frac{\left(\sum_{i=1}^r \ell_{B}^{*'} y_{*i} \right)^2}{8r}.$$

Assuming that y_{ij} 's are independent and $\text{Var}(y_{ij}) = \sigma^2$, the variance of B_{pc} is

$$\begin{aligned}\text{Var}(B_{pc}) &= \left(\frac{1}{4r}\right)^2 \text{Var}\left(\sum_{i=1}^r \ell_{B}^{*i} y_{*i}\right) \\ &= \frac{\sigma^2}{2r}.\end{aligned}$$

When AB is estimated from the replicates 1 and 2 separately, then its estimators based on the observations available from replicates 1 and 2 are

$$AB_{rep1} = \frac{\left(\sum_{i=1}^r \ell'_{AB} y_{*i}\right)_{rep1}}{2r}$$

and

$$AB_{rep2} = \frac{\left(\sum_{i=1}^r \ell'_{AB2} y_{*i}\right)_{rep2}}{2r},$$

respectively.

Both the estimators are combined as arithmetic mean and the estimator of AB is given by

$$\begin{aligned}AB_{pc} &= \frac{AB_{rep1} + AB_{rep2}}{2} \\ &= \frac{\left(\sum_{i=1}^r \ell'_{AB1} y_{*i}\right)_{rep1} + \left(\sum_{i=1}^r \ell'_{AB2} y_{*i}\right)_{rep2}}{4r} \\ &= \frac{\sum_{i=1}^r \ell_{AB}^{*i} y_{*i}}{4r}\end{aligned}$$

where the (8×1) vector

$$\ell_{AB}^{*'} = (\ell_{AB1}, \ell_{AB2})$$

consists of 8 elements.

The sum of squares due to AB_{pc} is

$$\begin{aligned} SS_{AB_{pc}} &= \frac{\left(\sum_{i=1}^r \ell_{AB}^{*'} y_{*i} \right)^2}{r \ell_{AB}^* \ell_{AB}^*} \\ &= \frac{\left(\sum_{i=1}^r \ell_{AB}^{*'} y_{*i} \right)^2}{8r} \end{aligned}$$

and the variance of AB_{pc} under the assumption that y_{ij} 's are independent and $Var(y_{ij}) = \sigma^2$ is given by

$$\begin{aligned} Var(AB_{pc}) &= \left(\frac{1}{4r} \right)^2 Var \left(\sum_{i=1}^r \ell_{AB}^{*'} y_{*i} \right) \\ &= \frac{\sigma^2}{2r}. \end{aligned}$$

Block sum of squares

Note that in case of partial confounding, the block sum of squares will have two components – due to replicates and within replicates. So the usual sum of squares due to blocks need to be divided into two components based on these two variants.

Now we illustrate how the sum of squares due to blocks are adjusted under the partial confounding. We consider the setup as in the earlier example.

There are 6 blocks (2 blocks under each replicate 1, 2 and 3), each repeated r times. So there are total $(6r - 1)$ degrees of freedom associated with the sum of squares due to blocks.

The sum of squares due to blocks is divided into two parts-

- the sum of squares due to replicates with $(3r - 1)$ degrees of freedom and
- the sum of squares due to within replicates with $3r$ degrees of freedom.

Now, denoting

- B_i to be the total of i^{th} block and
- R_i to be the total due to i^{th} replicate,

the sum of squares due to blocks is

$$\begin{aligned}
 SS_{\text{Block}(pc)} &= \frac{1}{\text{Total number of treatment}} \sum_{i=1}^{\text{Total number of blocks}} B_i^2 - \frac{G^2}{N} \\
 &= \frac{1}{2^2} \sum_{i=1}^{3r} B_i^2 - \frac{G^2}{12r}; \quad (N = 12r) \\
 &= \frac{1}{2^2} \sum_{i=1}^{3r} (B_i^2 - R_i^2 + R_i^2) - \frac{G^2}{12r} \\
 &= \frac{1}{2^2} \sum_{i=1}^{3r} (B_i^2 - R_i^2) + \left(\frac{1}{2^2} \sum_{i=1}^{3r} R_i^2 - \frac{G^2}{12r} \right) \\
 &= \frac{1}{2^2} \sum_{i=1}^{3r} \left(\frac{B_{1i}^2 + B_{2i}^2}{2} - R_i^2 \right) + \left(\frac{1}{2^2} \sum_{i=1}^{3r} R_i^2 - \frac{G^2}{12r} \right)
 \end{aligned}$$

where B_{ji} denotes the total of j^{th} block in i^{th} replicate, ($j = 1, 2$).

The sum of squares due to blocks within replications (wr) is

$$SS_{Block(wr)} = \frac{1}{2^2} \sum_{i=1}^{3r} \left(\frac{B_{1i}^2 + B_{2i}^2}{2} - R_i^2 \right).$$

The sum of squares due to replications is

$$SS_{Block(r)} = \frac{1}{2^2} \sum_{i=1}^{3r} R_i^2 - \frac{G^2}{12r}.$$

So we have in case of partial confounding

$$SS_{Block} = SS_{Block(wr)} + SS_{Block(r)}.$$

The total sum of squares remains same as usual and is given by

$$SS_{Total(pc)} = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{G^2}{N}; \quad (N = 12r).$$

The analysis of variance table in this case of partial confounding is given in the following table. The test of hypothesis can be carried out in a usual way as in the case of factorial experiments.

Source	Sum of squares	Degrees of freedom	Mean squares
Replicates	$SS_{Block(r)}$	$3r(=r^*)$	$MS_{Block(r)}$
Blocks within replicates		$3r-1(=r^*-1)$	$MS_{Block(wr)}$
Factor <i>A</i>	$SS_{A_{pc}}$	1	$MS_{A(pc)}$
Factor <i>B</i>	$SS_{B_{pc}}$	1	$MS_{B(pc)}$
<i>AB</i>	$SS_{AB_{pc}}$	1	$MS_{AB(pc)}$
Error	By subtraction	$6r-3(=2r^*-3)$	$MS_{E(pc)}$
Total	$SS_{Total(pc)}$	$12r-1(=4r^*-1)$	