Analysis of Variance and Design of Experiments-II

MODULE VII

LECTURE - 30

CROSS-OVER DESIGNS

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Testing carry-over effects, i.e. , $H_0 : \lambda_1 = \lambda_2$

First we consider the test of equality of the carry-out effects λ_1 and λ_2. Since the difference of the carry-over effects $\lambda_d = \lambda_1 - \lambda_2$ is the aliased effect of the *treatment* x *period* interaction, so the following tests on main effects are valid only when $H_0 : \lambda_1 = \lambda_2$ is rejected.

The subject total Y_{1ok} of the *k*th subject in Group 1 is

$$Y_{1ok} = y_{11k} + y_{12k}$$

and

$$\begin{aligned} E(Y_{1ok}) &= E(y_{11k}) + E(y_{12k}) \\ &= (\mu + \pi_1 + \tau_1) + (\mu + \pi_2 + \tau_2 + \lambda_1) \\ &= 2\mu + \pi_1 + \pi_2 + \tau_1 + \tau_2 + \lambda_1. \end{aligned}$$

Similarly, the subject total Y_{2ok} of the *k*th subject in Group 2 (*BA*) is

$$Y_{2ok} = y_{21k} + y_{22k}$$

and

$$E(Y_{2ok}) = 2\mu + \pi_1 + \pi_2 + \tau_1 + \tau_2 + \lambda_2.$$

Under the null hypothesis,

$$H_0 : \lambda_1 = \lambda_2,$$

we have

$$E(Y_{1ok}) = E(Y_{2ok}) \text{ for all } k.$$

Now we can apply the two-sample *t*-test to the subject totals and define

$$\lambda_d = \lambda_1 - \lambda_2.$$

Then

$$\hat{\lambda}_d = \frac{Y_{1oo}}{n_1} - \frac{Y_{2oo}}{n_2} = 2(\bar{y}_{1oo} - \bar{y}_{2oo})$$

and

$$E(\hat{\lambda}_d) = \lambda_d.$$

Using

$$Y_{iok} - E(Y_{iok}) = 2s_{ik} + \varepsilon_{i1k} + \varepsilon_{i2k}$$

and

$$Var(Y_{iok}) = 4\sigma_s^2 + 2\sigma^2,$$

we get

$$Var\left(\frac{Y_{ioo}}{n_i}\right) = \frac{1}{n_i^2}\sum_{k=1}^{n_i} Var(Y_{iok}) = \frac{4\sigma_s^2 + 2\sigma^2}{n_i} \ (i = 1, 2).$$

Therefore

$$Var(\hat{\lambda}_d) = 2\left(2\sigma_s^2 + \sigma^2\right)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

$$= \sigma_d^2\left(\frac{n_1 + n_2}{n_1 n_2}\right)$$

where $\sigma_d^2 = 2(2\sigma_s^2 + \sigma^2).$

Let s_1^2 and s_2^2 be the sample variances of the response totals within the group where

$$s_1^2 = \frac{1}{n_1 - 1}\sum_{k=1}^{n_1}\left(Y_{1ok} - \frac{Y_{1oo}}{n_1}\right)^2 = \frac{1}{n_1 - 1}\left(\sum_{k=1}^{n_1} Y_{1ok}^2 - \frac{Y_{1oo}^2}{n_1}\right)$$

$$s_2^2 = \frac{1}{n_2 - 1}\sum_{k=1}^{n_2}\left(Y_{2ok} - \frac{Y_{2oo}}{n_2}\right)^2 = \frac{1}{n_2 - 1}\left(\sum_{k=1}^{n_2} Y_{2ok}^2 - \frac{Y_{2oo}^2}{n_2}\right).$$

The pooled sample variance is given as

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}.$$

Then the test statistic is given by

$$T_\lambda = \frac{\hat{\lambda}_d}{s}\sqrt{\frac{n_1 n_2}{n_1 + n_2}}$$

which follows the Student's t-distribution with $(n_1 + n_2 - 2)$ degrees of freedom under $H_0 : \lambda_1 = \lambda_2$.

In practice, we use the 1% level of significance to test this hypothesis. If this test results in the acceptance of H_0, then we proceed further to test the main effects.

Testing treatment effects (Given $\lambda_1 = \lambda_2 = \lambda$)

Assume that $\lambda_1 = \lambda_2 = \lambda$, then for the period differences

$$d_{1k} = y_{11k} - y_{12k} \text{ (for Group } 1, \; i.e., \; A - B),$$

$$d_{2k} = y_{21k} - y_{22k} \text{ (for Group } 2, \; i.e., B - A),$$

we have

$$E(d_{1k}) = \pi_1 - \pi_2 + \tau_1 - \tau_2 - \lambda,$$

$$E(d_{2k}) = \pi_1 - \pi_2 + \tau_2 - \tau_1 - \lambda.$$

The null hypothesis under consideration is

$$H_0 : \tau_1 = \tau_2.$$

Under the null hypothesis that there is no difference in the treatment effects, i.e. $H_0 : \tau_1 = \tau_{2,}$ we have $H_0, E\left(d_{1k}\right) = E\left(d_{2k}\right).$

The difference of the treatment effects is given by

$$\tau_d = \tau_1 - \tau_2,$$

which is estimated by

$$\hat{\tau}_d = \frac{1}{2}(\bar{d}_{1o} - \bar{d}_{2o}).$$

Further,

$$E(\hat{\tau}_d) = \tau_d,$$

so $\hat{\tau}_d$ is an unbiased estimator of τ_d and has variance

$$Var(\hat{\tau}_d) = \frac{2\sigma^2}{4}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

$$= \frac{\sigma_D^2}{4}\left(\frac{1}{n_1} + \frac{1}{n_2}\right),$$

where

$$\sigma_D^2 = 2\sigma^2.$$

We proceed as follows to obtain an estimate of σ_D^2. Consider the pooled estimate

$$s^2 = \frac{(n-1)s_1^2(n_2-1)s_2^2}{n_1+n_2-2}$$

and replace s_1^2 and s_2^2 by

$$s_{1D}^2 = \frac{1}{n_1-1}\sum_{k=1}^{n_1}(d_{1k}-d_{1o})^2$$

and

$$s_{2D}^2 = \frac{1}{n_2-1}\sum_{k=1}^{n_2}(d_{2k}-d_{2o})^2,$$

respectively.

Then s^2 is modified as s_D^2 given by

$$s_D^2 = \frac{(n_1-1)s_{1D}^2+(n_2-1)s_{2D}^2}{n_1+n_2-2}.$$

Under the null hypothesis $H_0:\tau_d=0$, the statistic

$$T_\tau = \frac{\hat{\tau}_d}{\frac{1}{2}s_D}\sqrt{\frac{n_1n_2}{n_1+n_2}},$$

follows the t-distribution with $(n_1 + n_2 - 2)$ degrees of freedom.

Testing period effects (Given $\lambda_1 + \lambda_2 = 0$)

The null hypothesis concerning the period effects is given as

$$H_0 : \pi_1 = \pi_2.$$

The "cross-over " differences are

$$c_{1k} = d_{1k}$$
$$c_{2k} = -d_{2k},$$

and

$$E(c_{1k}) = \pi_1 - \pi_2 + \tau_1 - \tau_2 - \lambda_1,$$
$$E(c_{2k}) = \pi_2 - \pi_1 + \tau_1 - \tau_2 + \lambda_1.$$

Under $H_0 : \pi_1 = \pi_2$ and the reparametrization $\lambda_1 + \lambda_2 = 0,$ we have $E(c_{1k}) = E(c_{2k})$.

Thus an unbiased estimator for the difference of the period effects $\pi_d = \pi_1 - \pi_2$ is given by

$$\hat{\pi}_d = \frac{1}{2}(c_{1o} - c_{2o})$$

and we get the test statistic with s_D from s_D^2 as

$$T_\pi = \frac{\hat{\pi}_d}{\frac{1}{2} s_D} \sqrt{\frac{n_1 n_2}{n_1 + n_2}},$$

which follows the t-distribution with $(n_1 + n_2 - 2)$ degrees of freedom.

Unequal carry-over effect

If the null hypothesis $\lambda_1 = \lambda_2$ is rejected, then $\hat{\tau}_d$ does not remains an unbiased estimator of τ_d. So this procedure for testing $\tau_1 = \tau_2$ should not be used. Given $\lambda_d = \lambda_1 - \lambda_2 \neq 0,$ we get

$$E(\hat{\tau}_d) = E\left(\frac{d_{1o} - d_{2o}}{2}\right) = \tau_d - \frac{\lambda_d}{2}.$$

With

$$\hat{\lambda}_d = \bar{y}_{11o} + \bar{y}_{12o} - \bar{y}_{21o} - \bar{y}_{22o}$$

and

$$\hat{\tau}_d = \frac{1}{2}(\bar{y}_{11o} - \bar{y}_{12o} - \bar{y}_{21o} + \bar{y}_{22o}),$$

an unbiased estimator $\hat{\tau}_{d|\lambda_d}$ of τ_d is given by

$$\begin{aligned}\hat{\tau}_{d|\lambda_d} &= \frac{1}{2}(\bar{y}_{11o} - \bar{y}_{12o} - \bar{y}_{21o} + \bar{y}_{22o}) + \frac{1}{2}(\bar{y}_{11o} + \bar{y}_{12o} - \bar{y}_{21o} - \bar{y}_{22o}) \\ &= \bar{y}_{11o} - \bar{y}_{21o}.\end{aligned}$$

The unbiased estimator of τ_d for $\lambda_d \neq 0$ is based on between-subject information of the first period and the measurements. Testing for $H_0 : \tau_d = 0$ is done following a two-sample *t* - test. In such a case, the measurements of the first period only are used to estimate the variance. Thus, it is possible that the sample size may become too small to get significant results for the treatment effect.

The estimator $\hat{\pi}_d$ still remains an unbiased estimator under the reparametrization

$$\lambda_1 + \lambda_2 = 0.$$

See as follows:

$$\begin{aligned}
E(\hat{\pi}_d) &= E\left(\frac{\bar{c}_{1o} - \bar{c}_{2o})}{2}\right) \\
&= \frac{1}{2}E\left(\frac{1}{n_1}\sum_{k=1}^{n_1} c_{1k} - \frac{1}{n_2}\sum_{k=1}^{n_2} c_{2k}\right) \\
&= \frac{1}{2}E\left(\frac{1}{n_1}\sum_{k=1}^{n_1} E(c_{1k}) - \frac{1}{n_2}\sum_{k=1}^{n_2} E(c_{2k})\right) \\
&= \frac{1}{2}(2\pi_1 - 2\pi_2 - (\lambda_1 + \lambda_2)) \\
&= \pi_d.
\end{aligned}$$

Thus $\hat{\pi}_d$ is unbiased even if $\lambda_d = \lambda_1 - \lambda_2 \neq 0$ but $\lambda_1 + \lambda_2 = 0.$