Analysis of Variance and Design of Experiments-II

MODULE I

LECTURE - 6

INCOMPLETE BLOCK DESIGNS

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Contrast

A linear function $\sum_{j=1}^{v} c_j \tau_j = C'\tau$ where $c_1, c_2, ..., c_v$ are given numbers such that $\sum_{j=1}^{v} c_j = 0$ is called a contrast of τ_j's.

Elementary contrast

A contrast $\sum_{j=1}^{v} c_j \tau_j = C'\tau$ with $C = (c_1, c_2, ..., c_v)'$ in the treatment effects $\tau = (\tau_1, \tau_2, ..., \tau_v)'$ is called an elementary contrast if C has only two non-zero components 1 and -1.

Elementary contrasts in the treatment effects involve all the differences of the form $\tau_i - \tau_j, \quad i \neq j.$

It is desirable to design experiments where all the elementary contrasts are estimable.

Connected design

A design where all the elementary contrasts are estimable is called a connected design otherwise it is called a

disconnected design.

The physical meaning of connectedness of a design is as follows:

Given any two treatment effects τ_{i1} and τ_{i2} , it is possible to have a chain of treatment effects like $\tau_{i1}, \tau_{1j}, \tau_{2j}, ..., \tau_{nj}, \tau_{i2},$

such that two adjoining treatments in this chain occur in the same block.

Example of connected design

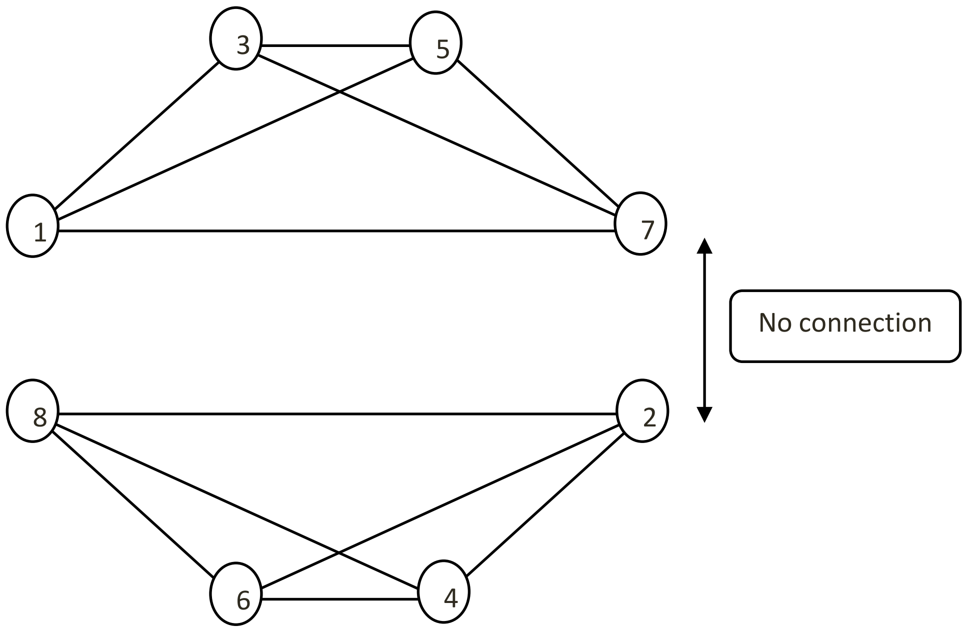
In a connected design, within every block, all the treatment contrasts are estimable and pair-wise comparison of estimators have similar variances .

Consider a disconnected incomplete block design as follows:

$b = 8$ (Block numbers: I, II,…,VIII), $k = 3$, $v = 8$ (treatment numbers: 1,2,…,8), $r = 3$

Blocks	Treatments
I	1 3 5
II	2 4 6
III	3 5 7
IV	4 6 8
V	5 7 1
VII	6 8 2
VII	7 1 3
VIII	8 2 4

The blocks of this design can be represented graphically as follows:



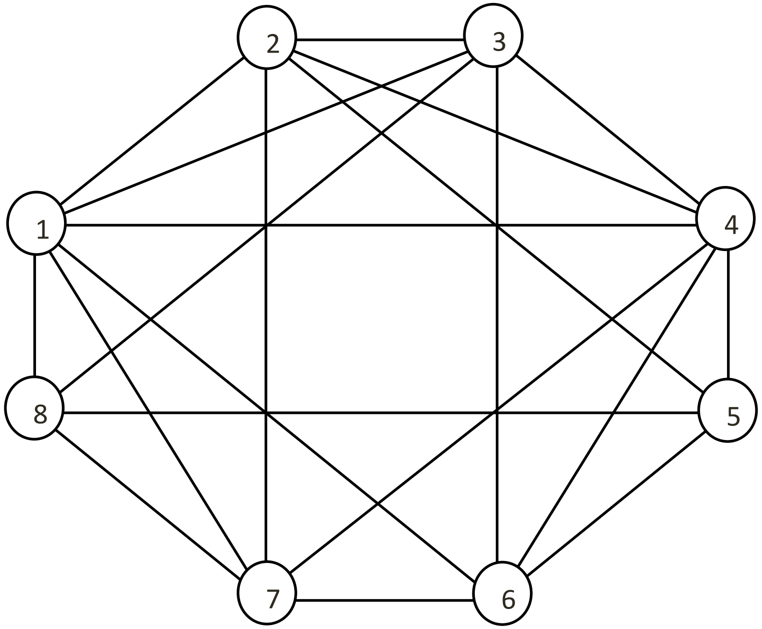
Note that it is not possible to reach the treatment, e.g., 7 from 2, 3 from 4 etc.

So the design is not connected

Moreover if the blocks of the design are given like in the following figure, then any treatment can be reached from any treatment. So the design in this case is connected.

For example, treatment 2 can be reached from treatment 6 through different routes like

6→5 → 4 → 3 → 2,   6 → 3 → 2,   6 → 7 → 8 → 1 → 2,   6 → 7 → 2 etc.



A design is connected if every treatment can be reached from every treatment via lines in the connectivity graph.

Theorem An incomplete block design with *v* treatments is connected if and only if *rank*(*C*) = *v* - 1.

Proof Let the design be connected. Consider a set of (*v* - 1) linearly independent contrasts (*j* = 2, 3,…,*v*). Let these contrasts be $C_2'\tau, C_3'\tau, ..., C_v'\tau$ where $\tau = (\tau_2, \tau_3, ...\tau_v)'$. Obviously, vectors $C_2, C_3, ..., C_v$ form the basis of vector space of dimension (*v* - 1). Thus any contrast $p'\tau$ is expressible as a linear combination of the contrasts $C_i'\tau (i = 2, 3, ..., v)$. Also, $p'\tau$ is estimable if and only if *p* belongs to the column space of *C*-matrix of the design.

Therefore, dimension of column space of *C* must be same as that of the vector space spanned by the vectors $C_i (i = 2, 3, ..., v)$, i.e., equal to (*v* - 1).

Thus *rank*(*C*) = *v* – 1.

Conversely , let *rank*(*C*) = *v* – 1 and let $\xi_1, \xi_2, ..., \xi_{v-1}$ be a set of orthonormal eigen vectors corresponding to the (not necessarily distinct) non-zero eigenvalues $\theta_1, \theta_2, ..., \theta_{v-1}$ of *C* .

Then $$E(\xi_1' Q) = \xi_i' C\tau$$
$$= \theta_i \xi_i' \tau.$$

Thus an unbiased estimator of $\xi_i'\tau$ is $\dfrac{\xi_i' Q}{\theta_i}$.

Also, since each ξ_i is orthogonal to E_{v1} and $\xi_i' s$ are mutually orthogonal, so any contrast $p'\tau$ belongs to the vector space spanned by $\{\xi_i, i = 1...v\}$, *i.e.*, $p = \sum_{i=1}^{v-1} a_i \xi_i$.

So $$E\left[\sum_{i=1}^{v-1} a_i \frac{\xi_i Q}{\theta_i}\right] = p'\tau.$$

Thus $p'\tau$ is estimable and this completes the proof.

Lemma For a connected block design $Cov(Q,P)=0$ if and only if $N'=\frac{rk'}{n}$.

Proof "if" part

When $N'=\frac{rk'}{n}$, we have

$$\frac{1}{\sigma^2}Cov(Q,P)=N'K^{-1}NR^{-1}N'-N'$$
$$=\frac{rk'K^{-1}kr'R^{-1}N'}{n^2}-N'$$

Since $k'K^{-1}k=(k_1,k_2,...,k_b)diag\left(\frac{1}{k_1},\frac{1}{k_2},...,\frac{1}{k_b}\right)\begin{pmatrix}k_1\\k_2\\\vdots\\k_b\end{pmatrix}$

$$=(1,...,1)\begin{pmatrix}k_1\\\vdots\\k_b\end{pmatrix}=\sum_{i=1}^{b}k_i=n$$

and

$$r'R^{-1}=(r_1,r_2,...,r_v)diag\left(\frac{1}{r_1},\frac{1}{r_2},...,\frac{1}{r_v}\right)=E_{1v}.$$

Then

$$\frac{1}{\sigma^2}Cov(Q,P)=\frac{rnE_{1v}N'}{n^2}-N'$$
$$=\frac{rE_{1v}N'}{n}-N'=\frac{rk'}{n}-N'$$
$$=N'-N'=0.$$

"Only if part"

Let $Cov(Q,P)=0$

$\Rightarrow N'K^{-1}NR^{-1}N' - N' = 0$ (Since $C = R - N'K^{-1}N$)

or $(R-C)R^{-1}N' - N' = 0$

or $CR^{-1}N' = 0.$

Let

$R^{-1}N' = A = (\underline{a}_1, \underline{a}_2, ..., \underline{a}_b)$

where $\underline{a}_1, \underline{a}_2, ..., \underline{a}_b$ are the columns of A.

Since the design is connected, so the columns of *A* are proportional to E_{v1} . Also all row / column sums of *C* are zero.

So $(CE_{v1}, CE_{v1}, ..., CE_{v1}) = 0$

and

$CA = 0$

or $C(\underline{a}_1, \underline{a}_2, ..., \underline{a}_b) = 0$

$\Rightarrow a_i \propto E_{v1}$

or $a_i = \alpha_i E_{v1}$; $i = 1, 2, ..., b$

where α_i are some scalars.

This gives

$A = R^{-1}N' = E_{v1}\alpha'$ where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_b)'$.

So we have

$$N' = RE_{v1}\alpha = (r_1, r_2, .., r_v)'\alpha'$$
$$= r\alpha' \quad \text{where } r = (r_1, r_2, .., r_v)'.$$

Premultiply by E_{1v} gives

$$E_{1v}N' = (k_1, k_2, ..., k_b)' = E_{1v}r\alpha' = n\alpha'$$

or $k = n\alpha'$

$$\Rightarrow \alpha' = \frac{k'}{n} \quad \text{where } k = (k_1, k_2, ..., k_b)'$$

Thus

$$N' = r\alpha' = \frac{rk'}{n}.$$

Hence proved.

Definition A **connected block design** is said to be **orthogonal** if and only if the incidence matrix of the design satisfies the condition

$$N' = \frac{rk'}{n}.$$

Design which do not satisfy this condition are called **non-orthogonal**. It is clear from this result that if atleast one entry of *N* is zero, the design cannot be orthogonal.

A block design with at least one zero entry in its incidence matrix is called an incomplete block design.

Theorem A sufficient condition for the orthogonality of a design is that $\frac{n_{ij}}{r_j}$ is constant for all *j*.

Conclusion It is obvious from the condition of orthogonality of a design that a design which is not connected and an incomplete design even though it may be connected cannot have an orthogonal structure.

Now we illustrate the general nature of the incomplete block design. We try to obtain the results for a randomized block design through the results of an incomplete block design.

Randomized block design

Randomized block design is an arrangement of v treatment in b blocks of v plots each, such that every treatment occurs in every block, one treatment in each plot.

The arrangement of treatment within a block is random and in terms of incidence matrix,

$$n_{ij} = 1 \text{ for all } i = 1, 2, \ldots, b; \; j = 1, 2, \ldots, v.$$

Thus we have

$$k_i = \sum_j n_{ij} = v \text{ for all } i$$

$$r_j = \sum_j n_{ij} = b \text{ for all } j.$$

We have $\dfrac{n_{ij}}{r_j} = \dfrac{1}{b}$ constant for all j.

$$C_{jj} = b - \frac{b}{v}$$

$$C_{jj'} = 1 - \frac{b}{v}$$

$$Q_j = V_j - \frac{G}{v}.$$

Normal equations for τ's are

$$\left(b-\frac{b}{v}\right)\tau_j-\frac{b}{v}\sum_{i\neq j'=1}^{v}\tau_{j'}=V_j-\frac{G}{v};\;\; j=1....v$$

$$\tau_1+\tau_2+....+\tau_v=0.$$

Thus

$$b\tau_j-\frac{b}{v}\sum_j\tau_j=V_j-\frac{G}{v}$$

or $$\hat{\tau}_j=\frac{1}{b}\left(V_j-\frac{G}{v}\right)=\overline{y}_{oj}-\overline{y}_{oo}.$$

The sum of squares due to treatments adjusted for blocks is $=\sum_j\hat{\tau}_jQ_j$

$$=\frac{1}{b}\sum_j\left(V_j-\frac{G}{v}\right)^2$$

$$=\frac{\sum_j V_j^2}{b}-\frac{G^2}{bv},$$

which is also the sum of squares due to treatments which are unadjusted for blocks because the design is orthogonal.

Sum of squares due to blocks $=\dfrac{\sum_i B_i^2}{v}-\dfrac{G^2}{bv}$

Sum of squares due to error $=\sum_i\sum_j\left(y_{ij}-\dfrac{B_i}{v}-\dfrac{V_j}{b}+\dfrac{G}{bv}\right)^2.$

These expressions are the same as obtained under the analysis of variance in the setup of a randomized block design.