Analysis of Variance and Design of Experiments-II

MODULE IX

NONPARAMETRIC ANALYSIS OF VARIANCE

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Multiple comparison test

If $H_0 : \theta_1 = \theta_2 = ... = \theta_p$ is rejected by Kruskel Wallis test, then the treatments are to be divided into groups such that the treatments having identical responses are in the same group and those with different responses belong to the different groups. Suppose the treatments under consideration consists of standard treatment (control) and (p - 1) new treatments. Now we want to find a group of new treatments which are better than the control treatment. To find such groups, we use Wilcoxon test to compare each new treatment with standard treatment. Suppose out of p treatments, the (p - 1) treatments are compared with standard treatment (control). Suppose i^{th} treatment has n_i ($i = 1, 2, …, p$) observations. To apply Wilcoxon test, we proceed as follows:

- Combine n_i observations (for the ith new one) with n_1 observations (for the control).
- Rank the observations.

Let

$\tilde{R}_i, i = 2, 3, ..., p$ be the sum of the ranks of n_i observations for the ith new treatment.

The ith new treatment is judged to be superior to the control if

$$\tilde{R}_i \geq C_i, \ i = 2, 3, ..., p$$

where the constant C_i depends on the level of significance α_i of the test . The $C_i's$ are determined from the table of Wilcoxon statistic.

If $n_2 = n_3 = ... = n_p = n$ (say), then we can choose $\alpha_2 = \alpha_3 = ... = \alpha_p = \alpha$ (say) and then

$$C_2 = C_2 = ... = C_p = C, \ (\text{say}).$$

Then the decision rule is to reject H_0 whenever

$$\tilde{R}_i \geq C$$

where C is a constant to be determined such that $P(R_i \geq C) = \alpha$ under the hypothesis that i^{th} new treatment is not supervisor to the control.

Experiments with two or more factors: (Two way classification)

We consider only the complete block design.

Let there be *J* treatments and *I* blocks. Consider the two way classification model where *J* different treatments are applied

to each of *I* different blocks so that each block contains exactly *J* treatment and each treatment is replicated exactly *n* times.

Let $Y_{ij}(i=1,2,...,J, j=1,2,...,J)$ denote the random variable associated with the yield/output of the jth treatment in the ith
block. Let Y_{ij} be independent and let

$$F_{ij}(y) = P(Y_{ij} \le y).$$

Recall in parametric case, we had

$$F_{ij}(y) = F(y - \mu - \beta_i - \tau_j) \text{ with } \sum_{i=1}^{I} \beta_i = 0, \sum_{i=1}^{J} \tau_j = 0,$$

where $F(x)$ is the distribution function of $N(0,\sigma^2)$ with $H_0 : \tau_1 = \tau_2 = ... = \tau_J = 0.$

We consider the methods of "*n*-ranking" and "aligning".

Method of "$n-$ ranking"

The method of n-ranking does not assumes

- the additive nature of model,

- that the F_{ij} differ only in location,

- normal distribution of F_{ij},

- homoscadasticity of random errors (equality of variance of random error).

Assume that F_{ij} are continuous. The null hypothesis of no treatment effect is formulated as

$$H_0 : F_{i1}(y) = F_{i2}(y) = ... = F_{im}(y) = F_i(m), \ i = 1, 2, ..., n.$$

The treatment j is stochastically

- better than treatment $j*$ if $P(Y_{ij} > Y_{ij*}) \geq \dfrac{1}{2}$ for all i and

- worse than treatment $j*$ if $P(Y_{ij} > Y_{ij*}) \leq \dfrac{1}{2}$ for all i

with the strict inequality holding for at least one i.

The alternative hypothesis is

H_1 **:** At least one of the treatments is stochastically better (or worse) than the rest.

In nonparametric model, the specification of null and alternative hypothesis becomes simpler if we assume that the treatment effects are additive in the sense that

$$F_{ij}(y) = F_i(y - \theta_j), \ i = 1, 2, ..., I, \ j = 1, 2, ..., J.$$

In such case, the null hypothesis becomes

$$H_0 : \theta_1 = \theta_2 = ... = \theta_J = 0$$

$H_1 : \theta_j \neq 0$ for at least one j.

Let R_{ij} = rank of Y_{ij} among $Y_{i1}, Y_{i2}, ..., Y_{iJ}$

Since F_{ij} are continuous by assumption, so the ties among the observations may be ignored in probability. Under H_0, the distribution of $(R_{i1}, R_{i2}, ..., R_{iJ})$ over $J!$ possible realizations is uniform irrespective of form of F_i provided that they are continuous.

Several tests based on R_{ij} have been proposed in the literature. Among them, the **Friedman test** is based on the statistic

$$Q = \frac{12}{IJ(J+1)} \sum_{j=1}^{J} \left(R_j - \frac{I(J+1)}{2} \right)^2$$

$$= \frac{12}{IJ(J+1)} \sum_{j=1}^{J} R_j^2 - 3I(J+1)$$

where $R_j = \sum_{i=1}^{I} R_{ij}$ is the sum of ranks for the treatment j.

The Friedman test rejects H_0 whenever

$$Q \geq C$$

where the constant C depends on the level of significance α of the test. Tables for C are available.

For large I and J, Q is distributed as Chi-square distribution with J - 1 degrees of freedom under H_0 and this distribution is reasonable whenever $IJ \geq 30.$

The Friedman test for $J = 2$ reduces to the sign test and the Wilcoxon's sign test is more effective for this. Friedman test is relatively less sensitive for low values of J. It is mainly due to the fact that a separate ranking in each block provides comparisons only to the responses within each block. A direct comparison of response in different blocks is not, in general, meaningful because some may give consistently low responses and others may give consistently high responses due to the variations between blocks. This difficulty can be eliminated by subtracting some estimate of the location in each block. Such an estimate can be the average of the observations of the block from the observations in each block or the median of these observations. This difficulty is overcome in the method of aligning.

Method of aligning

This method assumes the additively of the block effects and homoscadasticity of the random errors of different blocks. Assume

$$Y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij}, \ i = 1, 2, ..., I, \ j = 1, 2, ..., J$$

where μ is general mean, $\beta_1, \beta_2, ..., \beta_I$ are block effects satisfying $\sum_{i=1}^{I} \beta_i = 0$ (may be random or fixed), $\tau_1, \tau_2, ..., \tau_J$ are the treatment effects satisfying $\sum_{j=1}^{J} \tau_j = 0$ (fixed only) and ε_{ij} are random error components. Assume that the random vectors $\underset{\sim}{\varepsilon} = (\varepsilon_{i1}, \varepsilon_{i2}, ..., \varepsilon_{iJ}), \ i = 1, 2, ..., I$ are independent having continuous distribution function $F_i(y_1, y_2, ..., y_J)$, $i = 1, 2, ..., I$ such that each F_i is symmetric in each of its J arguments. Obviously, this includes as a special case the classical case of independent and identically distributed ε_{ij}.

Let $\tilde{Y}_i$ is a suitable estimator of the location of ith block.

Now eliminate the block effect β_i by taking

$$Z_{ij} = Y_{ij} - \tilde{Y}_i$$

We consider

$$\tilde{Y}_i = \overline{Y}_{io}$$

where $\overline{Y}_{io}$ is the average based on i^{th} block.

Now arrange $Z_{ij} = Y_{ij} - \bar{Y}_{io}$, $i = 1, 2, ..., I$, $j = 1, 2, ..., J$ in the order of magnitude. Let $\hat{R}_{ij}$ be the rank of Z_{ij} (or mid rank is case of tied Z_{ij}). The $\hat{R}_{ij}$ are called the **aligned rank** of Y_{ij}.

Under $H_0 : \theta_1 = \theta_2 = ... = \theta_m = 0$, (i.e., under H_0 of no treatment effect regardless of which position is a block were assigned to which treatment), then $J!$ possible arrangements in any given block will simply result in $J!$ permutations of the aligned rank in this block.

The total number of equally likely permutations within the I sets of aligned ranks in all the blocks is $(J!)^I$.

Therefore the distribution of $\hat{R}_{ij}$ under H_0 is

$$P(\hat{R}_{ij} = \hat{r}_{ij}) = \frac{1}{(J!)^I}, \ i = 1, 2, ..., I, \ j = 1, 2, .., J.$$

Let

$$\hat{R}_j = \sum_i R_{ij}$$

$$\bar{R}_{jo} = \frac{\hat{R}_{jo}}{n}.$$

Under H_0,

$$\sum_j \hat{R}_{jo} = \frac{J(IJ+1)}{2}.$$

A test statistic based on $\hat{R}_{ij}$, analogous to Friedman Q - test statistic is

$$Q^* = \frac{J-1}{\sum_{i=1}^{I}\sum_{j=1}^{J}(\hat{R}_{ij} - \hat{R}_{io})^2}\left[\sum_{j=1}^{J}\left(\hat{R}_j - \frac{I(IJ+1)}{2}\right)^2\right]$$

$$= \frac{J-1}{\sum_{i=1}^{I}\sum_{j=1}^{J}(\hat{R}_{ij} - \hat{R}_{io})^2}\left[\sum_{j=1}^{J}\hat{R}_j^2 - \frac{I^2 J(IJ+1)^2}{4}\right]$$

where $\bar{R}_{io} = \frac{1}{J}\sum_{j=1}^{J}\hat{R}_{ij}.$

The decision rule is to reject H_0 whenever

$$Q^* \geq C$$

where the constant C depends on the level of significance α of the test.

Note that

- $\displaystyle\sum_{i=1}^{I}\sum_{j=1}^{J}\hat{R}_{ij}^{2}$: sum of squares of all ranks or mid-ranks and is constant.

- The set of J ranks in a block is fixed (only the order being random). So $J\hat{R}_{io}$ is the sum of ranks of ith block is also constant.

- $\displaystyle\sum_{i=1}^{I}\sum_{j=1}^{J}(\hat{R}_{ij}-\hat{R}_{io})^{2}=\sum_{i=1}^{I}\sum_{j=1}^{J}\hat{R}_{ij}^{2}-\frac{1}{m}\sum_{i=1}^{I}(J\hat{R}_{io})^{2}$ is constant.

The distribution of Q^{*} under H_0 is

- complicated for small values of I and J.

- Chi-square with $(J-1)$ degrees of freedom for large values of I and J.

Friedman test in balanced incomplete block designs (BIBD)

Consider a BIBD $D(b, v, r, k, \lambda)$. If the jth treatment occurs in i^{th} block, denote R_{ij} to be the rank of the corresponding response among the k responses in the ith block. Otherwise, we let

$$R_{ij} = 0.$$

Let $\quad R_j = \sum_{i=1}^{I} R_{ij}, \quad j = 1, 2, ..., J (k \leq J).$

For testing H_0 of no treatment effects, the test statistic analogous to Friedman test is

$$\tilde{Q} = \frac{12}{J\lambda(k+1)} \sum_{j=1}^{J} \left[R_j - \frac{r(k+1)}{2} \right]^2.$$

The decision rule is to reject H_0 whenever

$$\tilde{Q} \geq C$$

where the constant C depends on the level of significance α of the test . Under H_0, $\tilde{Q}$ is distributed as Chi-square distribution with $(J-1)$ degrees of freedom when both I and J are large.