Analysis of Variance and Design of Experiments-II

MODULE VI

LECTURE - 27

SPLIT-PLOT AND STRIP-PLOT DESIGNS

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Analysis of covariance with one split-plot covariate

There are various possibilities for the analysis of covariance in a split-plot experiment, e.g., there can be a covariate for the whole-plots and not for the split-plots, a covariate for the split-plots and not for the whole-plots, or to have different covariate for the whole and split-plots. The adjustments to the treatment means can be messy and so choose the model carefully. There is no simple unique way to use and adjust for covariates in split-plot experiments.

Assume that the covariates are part of the experimental units rather than responses to the treatments applied. This means that treatments do not affect the covariates and so the covariates are available to the experimenter at planning or execution stage. The covariates are observable constants in the model.

Development of model for one covariate at the split-plot level

Consider a very basic model for a split-plot experiment with whole-plots arranged in an RBD and one covariate that is associated with the split-plot experimental units,

$$y_{hij} = \mu + r_h + w_i + \varepsilon(1)_{hi} + s_j + (w \times s)_{ij} + \beta x_{hij} + \varepsilon(2)_{hij},$$

where $h = 1,\ldots,r,\ i = 1,\ldots,t$ and $j = 1,\ldots,s$ and the covariate is x_{hij}. Assume that the whole-and split-plot treatments are fixed effects, implying that $\overline{w}_o = \overline{s}_o = \overline{(w \times s)}_{oj} = \overline{(w \times s)}_{io} = 0.$ The x_{hij}'s are observed constants. We further assume that $\varepsilon(1)_{hi}$ and $\varepsilon(2)_{hij}$ are identically and independently normally distributed, each with mean 0 and variances σ_1^2 and σ_2^2, respectively. Moreover, they are mutually independent also.

Rewrite the model to isolate the covariate's contribution to bias and variance. Since there are two types of experimental units, two sources of error, first to split x_{hij} into $\overline{x}_{hio}$, (corresponding to the whole-plots) and $(x_{hij} - \overline{x}_{hio})$ (for the split-plots).

The model is generalized to allow different regression coefficients by introducing β^w for the whole-plot part of the analysis and β^s for the split-plot part. The model then becomes

$$y_{hij} = \mu + r_h + w_i + \beta^w \bar{x}_{hio} + \varepsilon(1)_{hi} + s_j + (w \times s)_{ij} + \beta^s(\bar{x}_{hij} - \bar{x}_{hio}) + \varepsilon(2)_{hij}.$$

Now write the model in a form that explicitly shows how the covariate contributes to the bias of the estimated whole-and split-plot factor effects and the variance components. Using the identities

$$\bar{x}_{hio} = \bar{x}_{ooo} + (\bar{x}_{hoo} - \bar{x}_{ooo}) + (x_{oio} - \bar{x}_{ooo}) + (\bar{x}_{hio} - \bar{x}_{hoo} - \bar{x}_{oio} + \bar{x}_{ooo})$$

and

$$(x_{hij} - \bar{x}_{hio}) = (\bar{x}_{ooj} - \bar{x}_{ooo}) + (\bar{x}_{oij} - \bar{x}_{oio} - \bar{x}_{ooj} + \bar{x}_{ooo}) + (x_{hij} - \bar{x}_{hio} - \bar{x}_{oij} + \bar{x}_{oio}),$$

the model is rewritten as

$$\begin{aligned} y_{hij} &= \mu + \beta^w \bar{x}_{ooo} + r_h + \beta^w(\bar{x}_{hoo} - \bar{x}_{ooo}) + w_i + \beta^w(\bar{x}_{oio} - \bar{x}_{ooo}) + \beta^w(\bar{x}_{hio} - \bar{x}_{hoo} - \bar{x}_{oio} + \bar{x}_{ooo}) + \varepsilon(1)_{hi} + s_j \\ &\quad + \beta^s(\bar{x}_{ooj} - \bar{x}_{ooo}) + (w \times s)_{ij} + \beta^s(\bar{x}_{oij} - \bar{x}_{oio} - \bar{x}_{ooj} + \bar{x}_{ooo}) + \beta^s(x_{hij} - \bar{x}_{hio} - \bar{x}_{oij} + \bar{x}_{oio}) + \varepsilon(2)_{hij} \qquad (2) \\ &= \mu^* + r_h^* + w_i^* + \beta^w(\bar{x}_{hio} - \bar{x}_{hoo} - \bar{x}_{oio} + \bar{x}_{ooo}) + \varepsilon(1)_{hi} + s_j^* + (w \times s)_{ij}^* + \beta^s(x_{hij} - \bar{x}_{hio} - \bar{x}_{oij} + \bar{x}_{oio}) + \varepsilon(2)_{hij}, \end{aligned}$$

where

$$\mu^* = \mu + \beta^w \bar{x}_{ooo}$$

$$r_h^* = r_h + \beta^w(\bar{x}_{hoo} - \bar{x}_{ooo})$$

$$w_i^* = w_i + \beta^w(\bar{x}_{oio} - \bar{x}_{ooo})$$

$$s_j^* = s_j + \beta^s(\bar{x}_{ooj} - \bar{x}_{ooo})$$

$$(w \times s)_{jk}^* = (w \times s)_{ij} + \beta^s(\bar{x}_{oij} - \bar{x}_{oio} - \bar{x}_{ooj} + \bar{x}_{ooo}).$$

The extra terms in $\mu^*, r_h^*, w_i^*, s_j^*$ and $(w \times s)_{ij}^*$ represent the contributions to the bias from the experimental units via the covariate and $\beta^w(\overline{x}_{hio} - \overline{x}_{hoo} - \overline{x}_{oio} + \overline{x}_{ooo})$ and $\beta^s(x_{hij} - \overline{x}_{hio} - \overline{x}_{oij} + \overline{x}_{oio})$ the contributions to variance. The analysis of covariance provides adjustments to remove all of these.

Analysis of covariance table

Once model has been constructed, the whole-plot part of the design matrix is orthogonal to the split-plot part and it is possible to do the analysis of covariance and estimate all the parameters. The first step is to construct a compact analysis of covariance table as follows:

Analysis of covariance table

Source	$y-$variable	Whole-Plot	Covariate	Split-plot	Covariate
Mean	M_{yy}	$M_{x_w y}$	$M_{x_w x_w}$		
Blocks	B_{yy}	$B_{x_w y}$	$B_{x_w x_w}$		
W	W_{yy}	$W_{x_w y}$	$W_{x_w x_w}$		
$Error(1)$	$E(1)_{yy}$	$E(1)_{x_w y}$	$E(1)_{x_w x_w}$		
S	S_{yy}			$S_{x_s y}$	$S_{x_s x_s}$
$W \times S$	$W \times S_{yy}$			$W \times S_{x_s y}$	$W \times S_{x_s x_s}$
$Error(2)$	$E(2)_{yy}$			$E(2)_{x_s y}$	$E(2)_{x_s x_s}$
Total	T_{yy}				

The quantities in the column labeled "*y*-variable" are the usual analysis of variance sums of squares. The columns under the heading "Whole-plot covariate" contains the sums of squares computed using the whole-plot covariate. The other columns contains the sums of cross-products involving the *y*-variable and then whole-plot covariate. Similarly, the two columns under the "Split-Plot Covariate" heading contain the sums of squares and the cross-products involving the split-plot covariate. The x_w and x_s subscript identify terms computed using the whole-plot and split-plot covariates, respectively . For example

$$E(1)_{x_w x_w} = s\sum_h \sum_i (\overline{x}_{hio} - \overline{x}_{oi} - \overline{x}_{hoo} + \overline{x}_{ooo})^2$$

and

$$E(2)_{x_s x_s} = \sum_h \sum_i \sum_j (x_{hij} - \overline{x}_{hio} - \overline{x}_{oij} + \overline{x}_{oio})^2.$$

The expected values are as follows:

$$E\left[E(1)_{yy}\right] = (r-1)(t-1)(\sigma_2^2 + s\sigma_1^2) + s(\beta^w)^2 \sum_h \sum_i (\overline{x}_{hio} - \overline{x}_{oio} - \overline{x}_{hoo} + \overline{x}_{ooo})^2$$

$$E\left[E(1)x_{wy}\right] = s\beta^w \sum_h \sum_i (\overline{x}_{hio} - \overline{x}_{oio} - \overline{x}_{hoo} + \overline{x}_{ooo})^2$$

$$E\left[(E(1)x_{wy})^2\right] = s^2(\beta^w)^2 \left[\sum_h \sum_i (\overline{x}_{hio} - \overline{x}_{oio} - \overline{x}_{hoo} + \overline{x}_{ooo})^2\right]^2 + s(\sigma_2^2 + s\sigma_1^2)\sum_h \sum_i (\overline{x}_{hio} - \overline{x}_{oio} - \overline{x}_{hoo} + \overline{x}_{ooo})^2$$

$$E\left[E(2)_{yy}\right] = (r-1)t(s-1)\sigma_2^2 + (\beta^s)^2 \sum_h \sum_i \sum_j \left(x_{hij} - \overline{x}_{hio} - \overline{x}_{oij} + \overline{x}_{oio}\right)^2$$

$$E\left[E(2)_{x_s y}\right] = \beta^s \sum_h \sum_i \sum_j \left(x_{hij} - \overline{x}_{hio} - \overline{x}_{oij} + \overline{x}_{oio}\right)^2$$

$$E\left[(E(2)_{x_s y})^2\right] = \left(\beta^s\right)^2 \left[\sum_h \sum_i \sum_j \left(x_{hij} - \overline{x}_{hio} - \overline{x}_{oij} + \overline{x}_{oio}\right)^2\right]^2 + \sigma_2^2 \sum_h \sum_i \sum_j \left(x_{hij} - \overline{x}_{hio} - \overline{x}_{oij} + \overline{x}_{oio}\right)^2.$$

Then

$$\hat{\beta}^{w} = \frac{E(1)_{x_w y}}{E(1)_{x_w x_w}}$$

$$\hat{\beta}^{s} = \frac{E(2)_{x_s y}}{E(2)_{x_s x_s}}$$

$$E(\hat{\beta}^{w}) = \beta^{w}$$

$$E(\hat{\beta}^{s}) = \beta^{s}$$

$$Var[\hat{\beta}_w] = \frac{(\sigma_2^2 + s\sigma_1^2)}{s\sum_h \sum_i (\overline{x}_{hio} - \overline{x}_{oio} - \overline{x}_{hoo} + \overline{x}_{ooo})^2}$$

$$= \frac{(\sigma_2^2 + s\sigma_1^2)}{E(1)_{x_w x_w}}$$

$$Var[\hat{\beta}_s] = \frac{\sigma_2^2}{\sum_h \sum_i \sum_j (x_{hij} - \overline{x}_{hio} - \overline{x}_{hoj} + \overline{x}_{hoo})^2}$$

$$= \frac{\sigma_2^2}{E(2)_{x_s x_s}}.$$

Note that there is no wasted degree of freedom if there really is one covariate in the whole-plot stratum and another in the split-plot stratum and the two effects are additive in the whole-plot stratum.

Adjusting the whole-plot error for the covariate gives

$$MSE(1)^a = \frac{1}{\left[(r-1)(t-1)-1\right]}\left(E(1)_{yy} - \frac{E(1)^2_{x_w y}}{E(1)_{x_w x_w}}\right)$$

with $(r-1)(t-1)-1$ degrees of freedom. Notice the superscript "a" indicates a mean square adjusted for the covariate.
$E\left[MSE(1)^a\right] = \sigma_2^2 + s\sigma_1^2$. Adjusting the split-plot error gives

$$MSE(2)^a = \frac{1}{\left[(r-1)t(s-1)-1\right]}\left(E(2)_{yy} - \frac{E(2)^2_{x_s y}}{E(1)_{x_s x_s}}\right)$$

with $(r-1)t(s-1)-1$ degrees of freedom.

$$E\left[MSE(2)^a\right] = \sigma_2^2$$

The treatment and interaction sums of squares must also be adjusted for the covariates to provide proper tests of hypotheses. In the whole-plot stratum, the adjusted whole-plot treatment sum of squares with (t – 1) degrees of freedom is

$$MSE^a = \frac{1}{(t-1)}\left(W_{yy} - \frac{\left[W_{x_w y} + E(1)^2_{x_w y}\right]^2}{\left[W_{x_w y_w} + E(1)^2_{x_w y_w}\right]}\right).$$

Similarly, in the split-plot stratum,

$$MSS^a = \frac{1}{(s-1)}\left(S_{yy} - \frac{\left[S_{x_w y} + E(2)_{x_s y}\right]^2}{\left[S_{x_s x_{sw}} + E(2)_{x_s x_s}\right]}\right)$$

$$MS(W\times S^a) = \frac{1}{(t-1)(s-1)}\left((W\times S_{yy}) - \frac{\left[(W\times S_{x_s y}) + E(2)_{x_s y}\right]^2}{(W\times S_{x_s x_s}) + E(2)_{x_s x_s}} + \frac{E(2)^2_{x_s y}}{E(2)_{x_s x_s}}\right).$$

Tests of hypotheses are performed using the adjusted mean squares. For whiole-plot treatments, use

$$F = \frac{MSW^{a}}{MSE(1)^{a}}$$

with t -1 and $(r-1)(t-1)-1$ degrees of freedom. In the split-plot stratum, test the split-plot treatment using

$$F = \frac{MSS^{a}}{MSE(2)^{a}}$$

with s -1 and $(r-1)t(s-1)-1$ degrees of freedom and the interaction of whole-plot and split-plot treatments is tested using

$$F = \frac{MS(W \times S^{a})}{MSE(2)^{a}}$$

with $(t-1)(s-1)$ and $(r-1)t(s-1)-1$ degrees of freedom.