Analysis of Variance and Design of Experiments-II

MODULE I

LECTURE - 4

INCOMPLETE BLOCK DESIGNS

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Theorem: $E(Q) = C\tau$

$Var(Q) = \sigma^2 C$

Proof: $Q_j = V_j - \left[\frac{n_{1j}B_1}{k_1} + \ldots + \frac{n_{bj}B_b}{k_b}\right]$

$$= V_j - \sum_{i=1}^{b} \frac{n_{ij}B_i}{k_i}$$

$$E(Q_j) = E(V_j) - \sum_{i=1}^{b} \frac{n_{ij}}{k_i} E(B_i)$$

$$E(V_j) = \sum_i \sum_m E(y_{ijm})$$

$$= \sum_i \sum_m E(\mu + \beta_i + \tau_j + \varepsilon_{ijm})$$

$$= \mu \sum_i n_{ij} + \sum_i \beta_i n_{ij} + \tau_j \sum_i n_{ij}$$

$$= \mu r_j + \sum_i \beta_i n_{ij} + \tau_j r_j$$

$$E(B_i) = \sum_j \sum_m E(y_{ijm})$$

$$= \sum_j \sum_m E(\mu + \beta_i + \tau_j + \varepsilon_{ijm})$$

$$= \sum_j \sum_m (\mu + \beta_i + \tau_j)$$

$$= \mu k_i + \beta_i k_i + \sum_j \tau_j n_{ij}$$

$$\sum_{i=1}^{b}\frac{n_{ij}}{k_i}E(B_i)=\sum_{i=1}^{b}\frac{n_{ij}}{k_i}\left[\mu k_i+\beta_i k_i+\sum_j \tau_j n_{ij}\right]$$
$$=\mu r_j+\sum_i \beta_i n_{ij}+\sum_i \frac{n_{ij}}{k_i}(\sum_j \tau_j n_{ij}).$$

Thus substituting these expressions in $E(Q_j),$ we have

$$E(Q_j)=r_j\tau_j-\sum_i\frac{n_{ij}}{k_i}\left(\sum_j \tau_j n_{ij}\right)$$
$$=\left(r_i-\sum_i\frac{n_{ij}^2}{k_i}\right)\tau_j-\sum_i\frac{n_{ij}}{k_i}\sum_{j(\neq\ell)} n_{i\ell}\tau_\ell$$
$$=c_{jj}\tau_j+\sum_{j'(\neq\ell)} c_{\ell j'}\tau_{j'}$$

Further, substituting $E(Q_j)$ in $E(Q)=(E(Q_1),E(Q_2),...,E(Q_b))',$ we get

$$E(Q)=C\tau.$$

Next,

$$Var(Q)=\begin{pmatrix} Var(Q_1) & Cov(Q_1,Q_2) & \dots & Cov(Q_1,Q_v)\\ Cov(Q_2,Q_1) & Var(Q_2) & \dots & Cov(Q_2,Q_v)\\ \vdots & \vdots & \ddots & \vdots\\ Cov(Q_v,Q_1) & Cov(Q_v,Q_2) & \dots & Var(Q_v)\end{pmatrix}$$

$$Var(Q_j)=Var[V_j-\sum_i\frac{n_{ij}}{k_i}B_i]$$
$$=Var(V_j)+\sum_i\left(\frac{n_{ij}}{k_i}\right)^2 Var(B_i)-2\sum_i\frac{n_{ij}}{k_i}Cov(V_j,B_i).$$

Note that

$$Var(V_j) = Var\left(\sum_i \sum_m y_{ijm}\right) = r_j\sigma^2$$

$$Var(B_i) = Var\left(\sum_j \sum_m y_{ijm}\right) = k_i\sigma^2$$

$$Cov(V_j, B_i) = Cov\left(\sum_i \sum_m y_{ijm}, \sum_j \sum_m y_{ijm}\right)$$

$$= n_{ij}\sigma^2$$

$$Var(Q_j) = r_j\sigma^2 + \sum_i \left(\frac{n_{ij}}{k_i}\right)^2 k_i\sigma^2 - 2\sum_i \left(\frac{n_{ij}}{k_i}\right) n_{ij}\sigma^2$$

$$= r_j\sigma^2 - \sum_i \frac{n_{ij}^2}{k_i}\sigma^2 - 2\sum_i \left(\frac{n_{ij}^2}{k_i}\right)\sigma^2$$

$$= r_j\sigma^2 - \sum_i \left(\frac{n_{ij}^2}{k_i}\right)\sigma^2$$

$$= c_{jj}\sigma^2.$$

$$Cov(Q_j, Q_\ell) = Cov\left[V_j - \sum_i \frac{n_{ij}}{k_i} B_i,\, V_\ell - \sum_i \frac{n_{i\ell}}{k_i} B_i\right]$$

$$= Cov(V_j, V_\ell) - \sum_i \frac{n_{i\ell}}{k_i} Cov(V_j, B_i) - \sum_i \frac{n_{ij}}{k_i} Cov(B_i, V_\ell) + \sum_i \frac{n_{ij} n_{i\ell}}{k_i^2} Cov(B_i, B_i)$$

$$= 0 - \sum_i \frac{n_{i\ell}}{k_i} Cov(V_j, B_i) - \sum_i \frac{n_{ij}}{k_i} Cov(B_i, V_\ell) + \sum_i \frac{n_{ij} n_{i\ell}}{k_i^2} Var(B_i)$$

$$= -\sum_i \left(\frac{n_{i\ell} n_{ij}}{k_i}\right)\sigma^2 - \sum_i \left(\frac{n_{ij} n_{i\ell}}{k_i}\right)\sigma^2 + \sum_i \frac{n_{ij} n_{i\ell}}{k_i^2} k_i \sigma^2$$

$$= c_{j\ell}\sigma^2 \ .$$

Substituting the terms of

$Var(Q_j) = c_{jj}\sigma^2$ and $Cov(Q_j, Q_l) = c_{jl}\sigma^2$ in $Var(Q),$ we get

$Var(Q) = C\sigma^2.$

Hence proved.

[Note: We will prove this result using matrix approach later].

Covariance matrix of adjusted treatment totals

Consider

$$Z = \begin{pmatrix} V \\ B \end{pmatrix} \quad \text{with } b+v \text{ variable.}$$

We can express

$$\begin{aligned} Q &= V - N'K^{-1}B \\ &= [I \quad -N'K^{-1}] \begin{bmatrix} V \\ B \end{bmatrix} \\ &= [I \quad -N'K^{-1}]Z. \end{aligned}$$

So

$$Cov(Q) = [I \quad -N'K^{-1}]\, Cov(Z) \begin{bmatrix} I' \\ (-N'K^{-1})' \end{bmatrix}.$$

Now we find

$$Cov(Z) = \begin{pmatrix} Var(V) & Cov(V,B) \\ Cov(B,V) & Var(B) \end{pmatrix}$$

Since B_i and V_j have n_{ij} observations in common and the observations are mutually independent, so

$$\begin{aligned} Cov(B_i, V_j) &= n_{ij}\sigma^2 \\ Var(B_i) &= k_i\sigma^2 \\ Var(V_j) &= r_j\sigma^2. \end{aligned}$$

Thus

$$Cov(Z) = \begin{pmatrix} R & N' \\ N & K \end{pmatrix} \sigma^2$$

$$\begin{aligned} Cov(Q) &= [I \quad -N'K^{-1}] \begin{bmatrix} R & N' \\ N & K \end{bmatrix} \begin{bmatrix} I \\ -K^{-1}N \end{bmatrix} \sigma^2 \\ &= [R - N'K^{-1}N \quad N' - N'] \begin{bmatrix} I \\ -K^{-1}N \end{bmatrix} \sigma^2 \\ &= (R - N'K^{-1}N)\sigma^2 \\ &= C\sigma^2. \end{aligned}$$

Next we show that $Cov(B,Q) = 0$

$$\begin{aligned} Cov(B,Q) &= Cov(B,V) - Cov(B,V - N'K^{-1}B) \\ &= Cov(B,V) - Var(B)K^{-1}N \\ &= N\sigma^2 - KK^{-1}N\sigma^2 \\ &= 0. \end{aligned}$$

Alternative approach to find/ prove $E(Q)=C\tau,\ D(Q)=C\sigma^2$

Now we illustrate another approach to find the expectations, etc. in the set up of an incomplete block design. We have now learnt three approaches- the classical approach based on summations, the approach based on matrix theory and this new approach which is also based on the matrix theory. We can choose any of the approaches. The objective here is to let the reader know these different approaches.

Rewrite the linear model

$$y_{ijm}=\mu+\beta_i+\tau_j+\varepsilon_{ijm},\quad i=1,2,...,b;\ j=1,2,...,v; m=0,1,..,n_{ij}.$$

as

$$y=\mu E_{n1}+D_1^{'}\tau+D_2^{'}\beta+\varepsilon$$

where $\tau=(\tau_1,\tau_2,...,\tau_v)',\quad \beta=(\beta_1,\beta_2,...,\beta_b)'.$

Since B_i and V_j have n_{ij} observations in common and the observations are mutually independent, so denote

$D_1: v\times n$ matrix of treatment effect versus N, i.e., $(i,\ j)^{th}$ element of this matrix is given by

$$D_1=\begin{cases}1 & \text{if } j^{th} \text{ observation comes from } i^{th} \text{ treatment}\\ 0 & \text{otherwise}\end{cases}$$

$D_2: b\times n$ matrix of treatment effect versus N, i.e., $(i,\ j)^{th}$ element of this matrix is given by

$$D_2=\begin{cases}1 & \text{if } j^{th} \text{ observation comes from } i^{th} \text{ block}\\ 0 & \text{otherwise.}\end{cases}$$

Following results can be verified:

$$D_1D_1' = R = diag(r_1, r_2, ..., r_v),$$

$$D_2D_2' = K = diag(k_1, k_2, ..., k_b),$$

$$D_2D_1' = N \text{ or } D_1D_2' = N'$$

$$D_1E_{n1} = (r_1, r_2, ..., r_v)'$$

$$D_2E_{n1} = (k_1, k_2, ..., k_b)'$$

$$D_1'E_{v1} = E_{n1} = D_2'E_{b1}.$$

In earlier notations,

$$V = (V_1, V_2, ..., V_v)' = D_1 y$$
$$B = (B_1, B_2, ..., B_b)' = D_2 y$$

Express Q in terms of D_1 and D_2 as

$$\begin{aligned} Q &= V - N'K^{-1}B \\ &= \left[D_1 - D_1D_2'(D_2D_2')^{-1}D_2 \right] y \end{aligned}$$

$$
\begin{aligned}
E(Q) &= \left[D_1 - D_1 D_2^{'} (D_2 D_2^{'})^{-1} D_2 \right] E(y) \\
&= \left[D_1 - D_1 D_2^{'} (D_2 D_2^{'})^{-1} D_2 \right] (\mu E_{n1} + D_1^{'} \tau + D_2^{'} \beta) \\
&= \left[D_1 E_{n1} - D_1 D_2^{'} (D_2 D_2^{'})^{-1} D_2 E_{n1} \right] \mu + \left[D_1 D_1^{'} - D_1 D_2^{'} (D_2 D_2^{'})^{-1} D_2 D_1^{'} \right] \tau + \left[D_1 D_2^{'} - D_1 D_2^{'} (D_2 D_2^{'})^{-1} D_2 D_2^{'} \right] \beta \\
&= \left[(r_1, r_2, ..., r_v)' - N' K^{-1} (k_1, ..., k_b)' \right] \mu + \left[R - N' K^{-1} N \right] \tau + \left[N' - N' K^{-1} K \right] \beta.
\end{aligned}
$$

Since

$$
\begin{aligned}
N' K^{-1} (k_1, ..., k_b)' &= N' diag\left(\frac{1}{k_1}, \frac{1}{k_2}, ..., \frac{1}{k_b} \right) \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_b \end{pmatrix} \\
&= N' \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n_{11} & n_{21} \dots n_{b1} \\ n_{12} & n_{22} \dots n_{b2} \\ \vdots & \vdots \ \ddots \vdots \\ n_{iv} & n_{2v} \dots n_{bv} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \\
&= \left(\sum_{i=1}^{b} n_{i1}, \sum_{i=1}^{b} n_{i2}, ..., \sum_{i=1}^{b} n_{iv} \right)^{'} \\
&= (r_1, r_2, ..., r_v)'.
\end{aligned}
$$

Thus

$$N' - N'K^{-1}K = \left(r_1, r_2, \ldots, r_v\right)' - N'K^{-1}(k_1, \ldots, k_b) = 0$$

and so

$$E(Q) = \left[R - N'K^{-1}N\right]\tau$$
$$= C\tau.$$

Next

$$Var(Q) = D_1\left[I - D_2'(D_2D_2')^{-1}D_2\right]Var(y)\left[I - D_2'(D_2D_2')^{-1}D_2\right]D_1'$$

$$= \sigma^2 D_1\left[I - D_2'(D_2D_2')^{-1}D_2\right]D_1'$$

$$= \sigma^2\left[D_1D_1' - D_1D_2'(D_2D_2')^{-1}D_2D_1'\right]$$

$$= \sigma^2\left[R - N'K^{-1}N\right]$$

$$= \sigma^2 C.$$

Note that $\left[I - D_2'(D_2D_2')^{-1}D_2\right]$ is an idempotent matrix.

Similarly, we can also express

$$P = B - NR^{-1}V$$
$$= [D_2 - D_2D_1'R^{-1}D_1]y.$$