Analysis of Variance and Design of Experiments-II

MODULE - III

LECTURE - 16

PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN (PBIBD)

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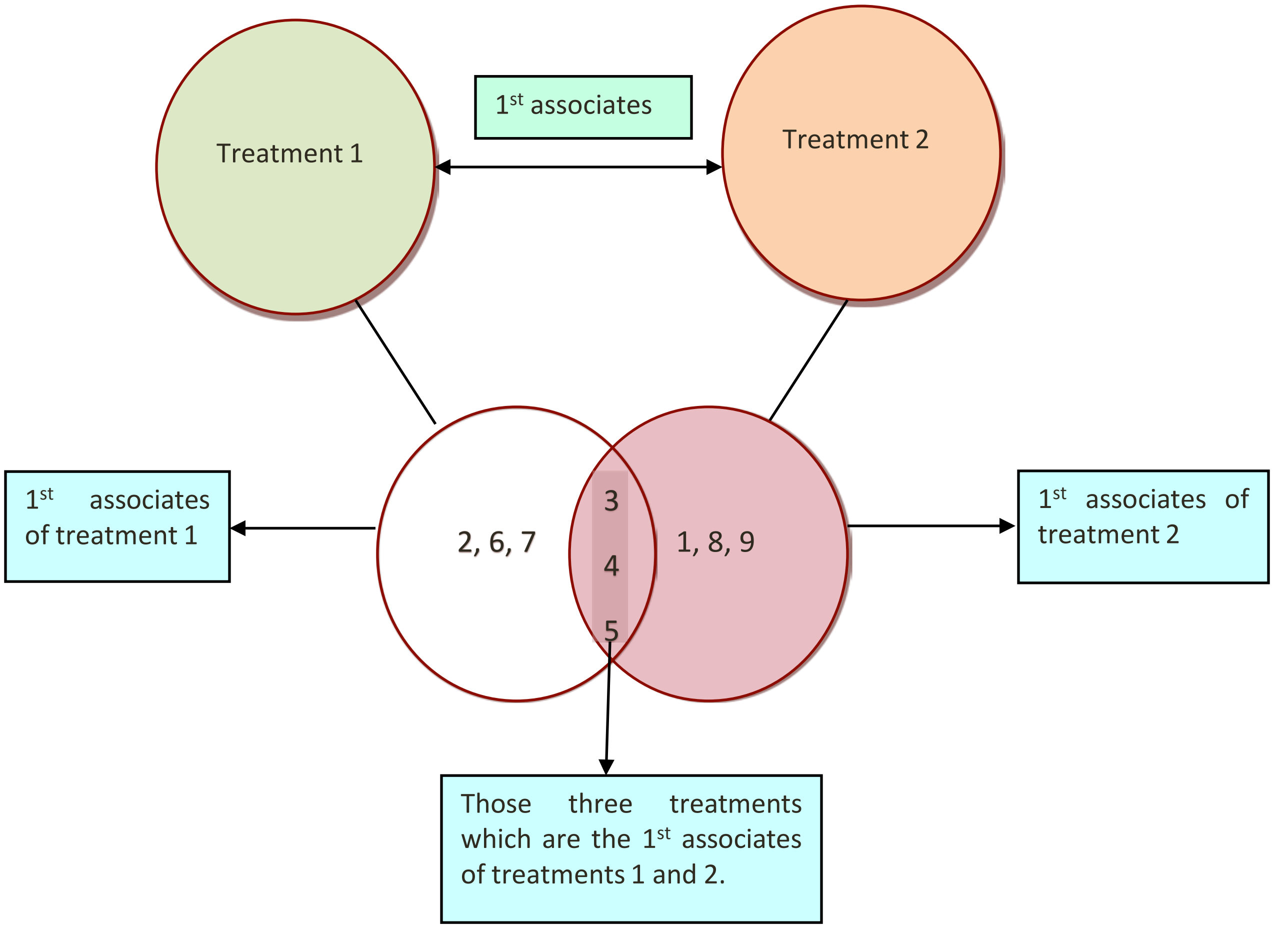
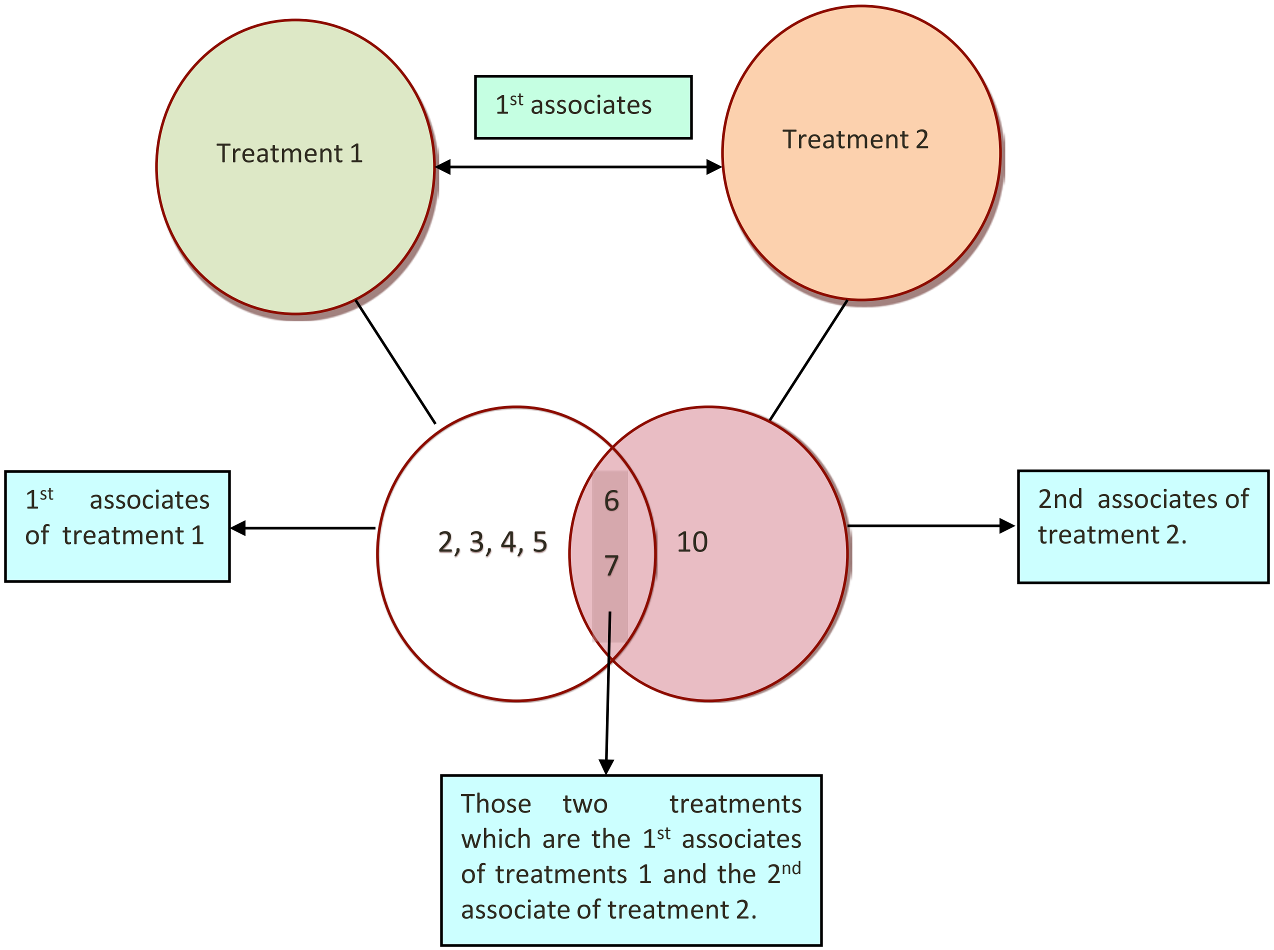
In order to learn how to write these matrices P_1 and P_2, we consider the treatments 1, 2 and 8. Note that the treatment 8 is the second associate of treatment 1. Consider only the rows corresponding to treatments 1, 2 and 8 and obtain the elements of P_1 and P_2 as follows:

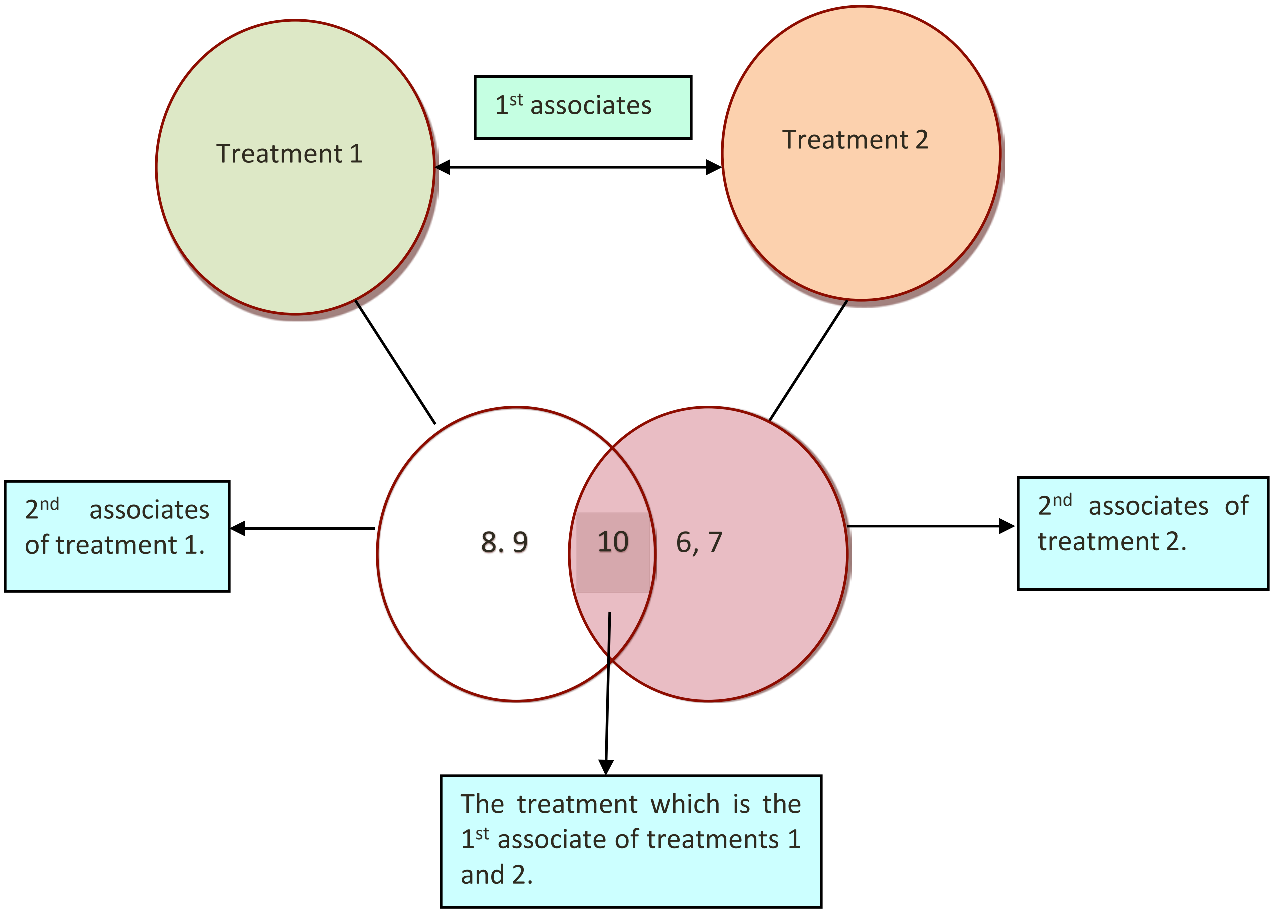
p_{11}^1: Treatments 1 and 2 are the first associates of each other. There are three common treatments (viz., 3, 4 and 5) between the first associates of treatment 1 and the first associates of treatment 2. So $p_{11}^1 = 3.$



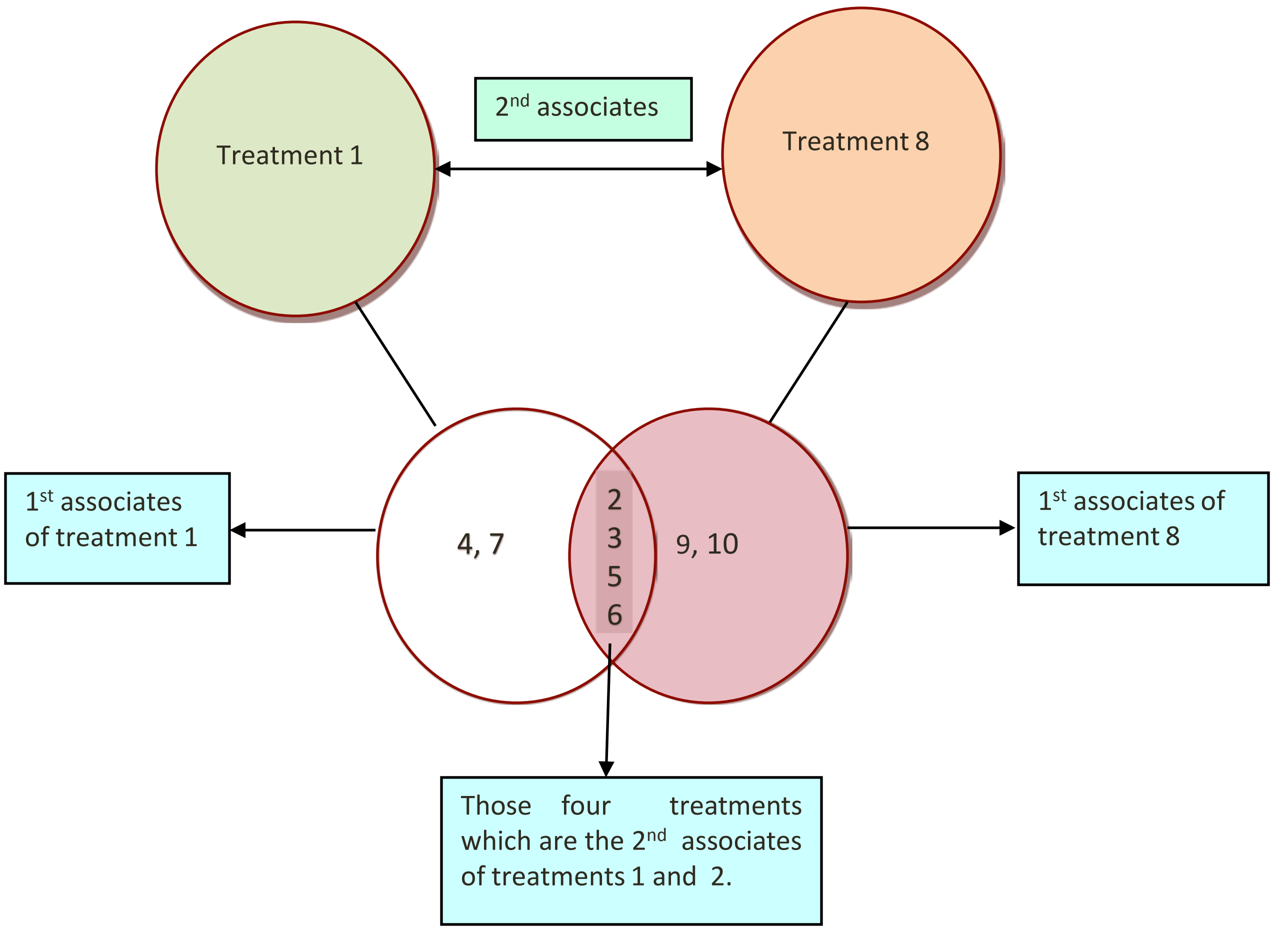
p_{12}^{1} and p_{21}^{1} : Treatments 1 and 2 are the first associates of each other. There are two treatments (viz., 6 and 7) which are common between the first associates of treatment 1 and the second associates of treatment 2. So $p_{12}^{1} = 2 = p_{21}^{1}$.



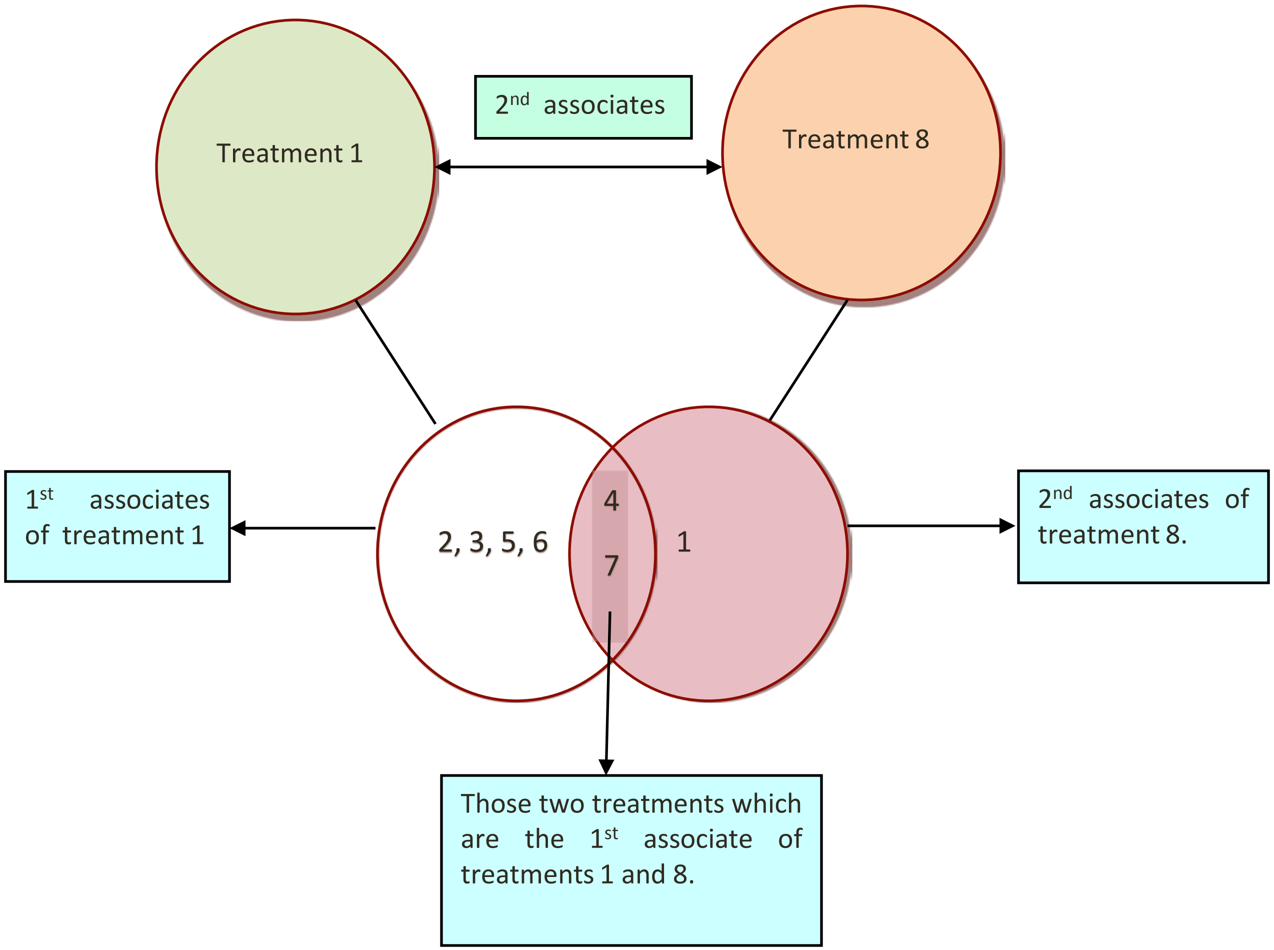
p_{22}^{1} : Treatments 1 and 2 are the first associates of each other. There is only one treatment (viz., treatment 10) which is common between the second associates of treatment 1 and the second associates of treatment 2. So $p_{22}^{1} = 1.$



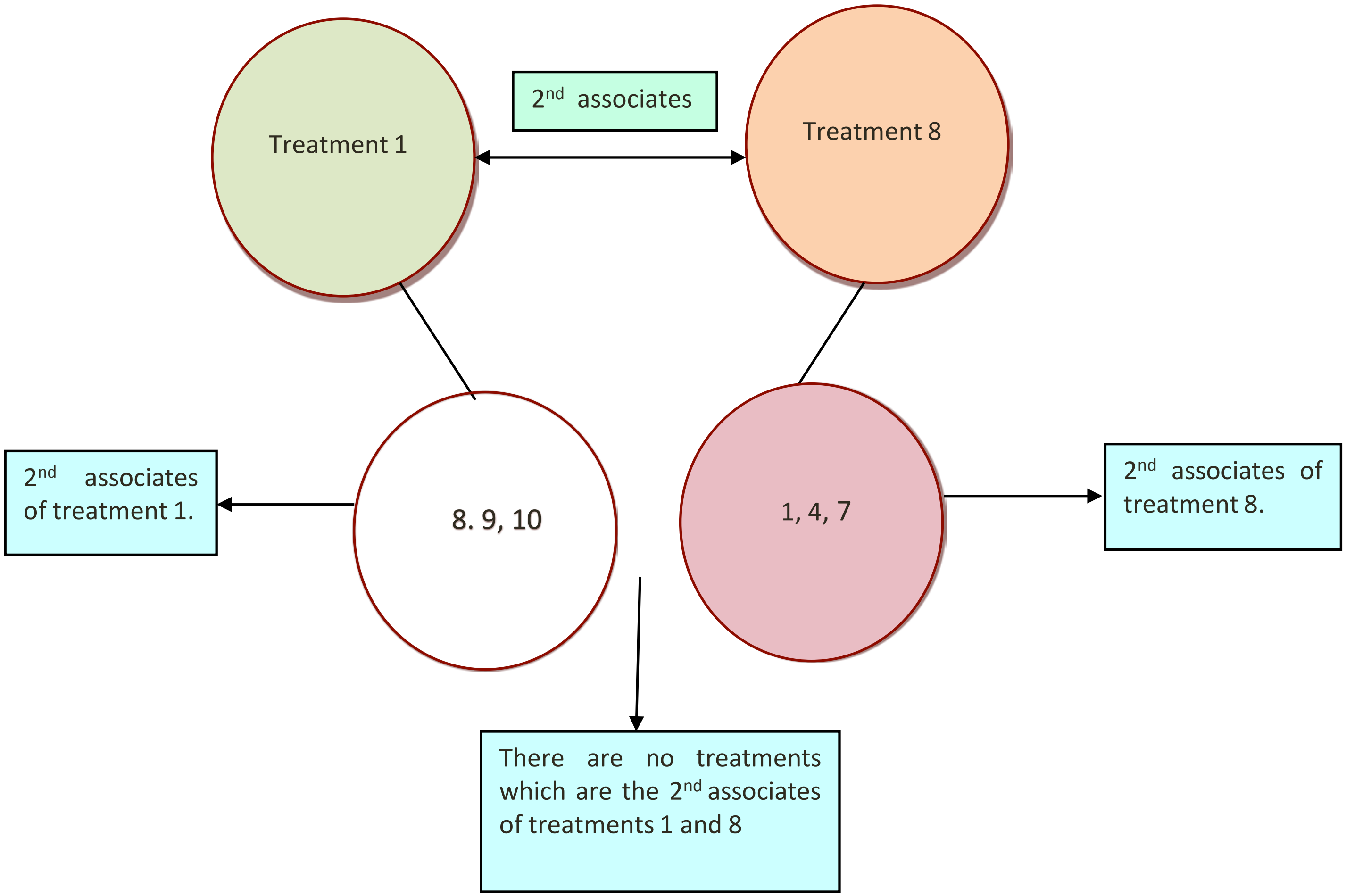
p_{11}^2 : Treatments 1 and 8 are the second associates of each other. There are four treatment (viz., 2,3,5 and 6) which are common between the first associates of treatment 1 and the first associates of treatment 8. So $p_{11}^1 = 4.$



p_{12}^{2} and p_{21}^{2} : Treatments 1 and 8 are the second associates of each other. There are two treatments (viz., 4 and 7) which are common between the first associates of treatment 1 and the second associates of treatment 8. So $p_{12}^{1} = 2 = p_{21}^{1}$.



p_{22}^{2} : Treatments 1 and 8 are the second associates of each other. There is no treatment which is common between the second associates of treatment 1 and the second associates of treatment 8. So $p_{22}^{2} = 0.$



In general, if q rows and q columns of a square are used, then for $q > 3$

$$v = \begin{pmatrix} q \\ 2 \end{pmatrix} = \frac{q(q-1)}{2},$$

$$n_1 = 2q-4,$$

$$n_2 = \frac{(q-2)(q-3)}{2},$$

$$P_1 = \begin{bmatrix} q-2 & q-3 \\ q-3 & \dfrac{(q-3)(q-4)}{2} \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 4 & 2q-8 \\ 2q-8 & \dfrac{(q-4)(q-5)}{2} \end{bmatrix}.$$

For $q = 3$, there are no second associates. This is a degenerate case where second associates do not exist and hence P_2 cannot be defined.

The graph theory techniques can be used for counting p^i_{jk}.

Further, it is easy to see that all the parameters in P_1 , P_2 etc. are not independent.

Construction of blocks of PBIBD under triangular association scheme

The blocks of a PBIBD can be obtained in different ways through an association scheme. Even different block structure can be obtained using the same association scheme. We now illustrate this by obtaining different block structures using the same triangular scheme. Consider the earlier illustration where $v = \binom{q}{2} = 10$ with $q = 5$ was considered and the first, second and third associates of treatments were obtained.

Approach 1 One way to obtain the treatments in a block is to consider the treatments in each row. This constitutes the set of treatments to be assigned in a block. When $q = 5$, the blocks of PBIBD are constructed by considering the rows of the following table

Rows $\longrightarrow$ / Columns $\downarrow$	1	2	3	4	5
1	×	1	2	3	4
2	1	×	5	6	7
3	2	5	×	8	9
4	3	6	8	×	10
5	4	7	9	10	×

From this arrangement, the treatments are assigned in different blocks and following blocks are obtained.

Blocks	Treatments
Block 1	1, 2, 3, 4
Block 2	1, 5, 6, 7
Block 3	2, 5, 8, 9
Block 4	3, 6, 8, 10
Block 5	4, 7, 9, 10

The parameters of such a design are $b=5,\ v=10,\ r=2,\ k=4,\ \lambda_1=1$ and $\lambda_2=0.$

Approach 2

Another approach to obtain the blocks of PBIBD from a triangular association scheme is as follows:

- Consider the pairwise columns of the triangular scheme.
- Then delete the common treatments between the chosen columns and retain others.
- The retained treatments will constitute the blocks.

Consider e.g., the triangular association scheme for q = 5. Then the first block under this approach is obtained by deleting the common treatments between columns 1 and 2 which results in a block containing the treatments 2, 3, 4, 5, 6 and 7. Similarly, considering the other pairs of columns, the other blocks can be obtained which are presented in the following table :

Blocks	Columns of association scheme	Treatments
Block 1	(1, 2)	2, 3, 4, 5, 6, 7
Block 2	(1, 3)	1, 3, 4, 5, 8, 9
Block 3	(1, 4)	1, 2, 4, 6, 8, 10
Block 4	(1, 5)	1, 2, 3, 7, 9, 10
Block 5	(2, 3)	1, 2, 6, 7, 8, 9
Block 6	(2, 4)	1, 3, 5, 7, 8, 10
Block 7	(2, 5)	1, 4, 5, 6, 9, 10
Block 8	(3, 4)	2, 3, 5, 6, 9, 10
Block 9	(3, 5)	2, 4, 5, 7, 8, 10
Block 10	(4, 5)	3, 4, 6, 7, 8, 9

The parameters of the PBIBD are $b=10,\ v=10,\ r=6, k=6,\ \lambda_1=3$ and $\lambda_2=4$.

Since both these PBIBDs are arising from same association scheme, so the values of n_1, n_2, P_1 and P_2 remain the same for both the designs. In this case, we have

$$n_1 = 6, \quad n_2 = 3, \quad P_1 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}.$$

Approach 3 Another approach to derive the blocks of PBIBD is to consider all the first associates of a given treatment in a block. For example, in case of q = 5, the first associates of treatment 1 are the treatments 2, 3, 4, 5, 6 and 7. So these treatments constitute one block. Similarly other blocks can also be found. This results in the same arrangement of treatments in blocks as considered by deleting the common treatments between the pair of columns.

The PBIBD with two associate classes are popular in practical applications and can be classified into following types depending on the association scheme,

(*Reference: Classification and analysis of partially balanced incomplete block designs with two associate classes, R. Bose and Shimamoto, 1952, Journal of American Statistical Association, 47, pp. 151-184*).

1. Triangular
2. Group divisible
3. Latin square with *i* constraints
4. Cyclic and
5. Singly linked blocks

The triangular association scheme has already been discussed. We now briefly present other types of association schemes.