Analysis of Variance and Design of Experiments-II

MODULE V

LECTURE - 23

FRACTIONAL REPLICATIONS

Dr. Shalabh
Department of Mathematics & Statistics
Indian Institute of Technology Kanpur

Determination of alias structure

The alias structure is determined by using the defining relation. Multiplying any column (or effect) by the defining relation yields the aliases for that column (or effect). For example, in this case, the defining relation is $I = ABC$. Now multiply the factors on both sides of $I = ABC$ yields

$$A \times I = (A) \times (ABC) = A^2BC = BC,$$
$$B \times I = (B) \times (ABC) = AB^2C = AC,$$
$$C \times I = (C) \times (ABC) = ABC^2 = AB.$$

The systematic rule to find aliases is to write down all the effects of a $2^{3-1} = 2^2$ factorial in standard order and multiply each factor by the defining contrast.

Alternate or complementary one-half fraction

We have considered upto now the one-half fraction corresponding to + signs of treatment combinations in *ABC* column in the table. Now suppose we choose other one-half fraction, i.e., treatment combinations with – signs in *ABC* column in the table. This is called **alternate or complementary one-half fraction.** In this case, the effects are estimated as

$$A = ab + ac - bc - (1),$$
$$B = ab - ac + bc - (1),$$
$$C = -ab + ac + bc - (1),$$
$$AB = ab - ac - bc + (1),$$
$$AC = -ab + ac - bc + (1),$$
$$BC = -ab - ac + bc + (1),$$

In this case, we notice that $A=-BC,\ B=-AC,\ C=-AB,$ so the same factors remain aliases again which are aliases in the one-half fraction with + sign in *ABC*. If we consider the setup of complete 2^3 factorial experiment, then in case of complete fractional

$$A=-(1)+a-b+ab-c+ac-bc+abc$$
$$BC=(1)+a-b-ab-c-ac+bc+abc,$$

we observe that *A - BC* in the complete 2^3 factorial experiment is the same as *A* or *BC* in the one half fractional with I = *-ABC* (ignoring the common multiplier). In order to find the relationship between the estimates under this one-half fraction and a complete factorial, we find that what we estimate in the one-half fraction with – sign in *ABC* is the same as of estimating *A - BC* from a complete 2^3 factorial experiment.

Similarly, using *B - AC* in the complete 2^3 factorial is the same as using *B* or *AC* in one half fraction with I = *ABC.*

Using *C - AB* in the complete 2^3 factorial experiment is the same as using *C* or *AB* in the one half fraction with I = *ABC* (ignoring the common multiplier).

Now there are two one-half fractions corresponding to + and – signs of treatment combinations in *ABC*. Based on that, there are now two sets of treatment combinations. A question arises that which one to use?

In practice, it does not matter which fraction is actually used. Both the one-half fractions belong to the same family of 2^3 factorial experiment. Moreover, the difference of negative signs in aliases of both the halves becomes positive while obtaining the sum of squares in analysis of variance.

Use of more than one defining relations

Further, suppose we want to have $1/2^2$ fraction of 2^3 factorial experiment with one more defining relation, say $I = BC$ along with $I = ABC$.

So the one-half fraction with + signs of *ABC* can further be divided into two halves. In this case, each half fraction will contain two treatments corresponding to

- + sign of *BC,* (viz., *a* and *abc*) and
- - sign of *BC,* (viz., *b* and *c*).

These two halves will constitute the one-fourth fraction of 2^3 factorial experiment.

Similarly, we can consider the other one-half fraction corresponding to – sign of *ABC*. Now we look for + and – sign corresponding to $I = BC$ which constitute the two one-half fractions consisting of the treatments

- (1), *bc* and
- *ab, ac.*

This will again constitute the one-fourth fraction of 2^3 factorial experiment.

Example in 2^6 factorial experiment

In order to have more understanding of the fractional factorial, we consider the setup of 2^6 factorial experiment . Since the highest order interaction in this case is *ABCDEF*, so we construct the one-half fraction using $I = ABCDEF$ as defining relation. Then we write all the factors of $2^{6-1} = 2^5$ factorial experiment in the standard order. Then multiply all the factors with the defining relation. For example

$$\begin{aligned} I \times A &= ABCDEF \times A \\ &= A^2BCDEF \\ \text{or} \quad A &= BCDEF \end{aligned}$$

$$\begin{aligned} I \times ABC &= ABCDEF \times ABC \\ &= A^2B^2CDEF \\ \text{or} \quad ABC &= CDEF \end{aligned}$$ etc.

All such operations are illustrated in following table .

One half fraction of 2^6 factorial experiment using $I = ABCDEF$ as defining relation			
$I = ABCDEF$	$D = ABCEF$	$E = ABCDF$	$DE = ABCF$
$A = BCDEF$	$AD = BCEF$	$AE = BCDF$	$ADE = BCF$
$B = ACDEF$	$BD = ACEF$	$BE = ACDF$	$BDE = ACF$
$AB = CDEF$	$ABD = CEF$	$ABE = CDF$	$ABDE = CF$
$C = ABDEF$	$CD = ABEF$	$CE = ABDF$	$CDE = ABF$
$AC = BDEF$	$ACD = BEF$	$ACE = BDF$	$ACDE = BF$
$BC = ADEF$	$BCD = AEF$	$BCE = ADF$	$BCDE = AF$
$ABC = DEF$	$ABCD = EF$	$ABCE = DF$	$ABCDE = F$

In this case, we observe that

- all the main effects have 5 factor interactions as aliases,
- all the 2 factor interactions have 4 factor interactions as aliases and
- all the 3 factor interactions have 3 factor interactions as aliases.

Suppose a completely randomized design is adopted with blocks of size 16. There are 32 treatments and *abcdef* is chosen as the defining contrast for half replicate. Now all the 32 treatments cannot be accommodated. Only 16 treatments can be accommodated. So the treatments are to be divided and allocated into two blocks of size 16 each.

This is equivalent to saying that one factorial effect (and its alias) are confounded with blocks. Suppose we decide that the three factor interactions and their aliases (which are also three factors interactions in this case) are to be used as error. So choose one of the three-factor interaction, say *ABC* (and its alias *DEF)* to be confounded. Now one of the block contains all the treatment combinations having an even number of letters *a, b* or *c.* These blocks are constructed in the following table:

One half replicate of 2^6 factorial experiment in the blocks of size 16

Block 1	Block 2
(1)	*ab*
de	*ae*
df	*af*
ef	*bd*
ab	*be*
ac	*bf*
bc	*cd*
abde	*ce*
abdf	*cf*
abef	*adef*
acde	*bdef*
acdf	*cdef*
acef	*abcd*
bcde	*abce*
bcdf	*abcf*
bcef	*abcdef*

There are total 31 degrees of freedom, out of which 6 degrees of freedom are used by the main effects, 15 degrees of freedom are used by the two-factor interactions and 9 degrees of freedom are used by the error (from three factor interactions). Additionally, one more division of degree of freedom arises in this case which is due to blocks. So the degree of freedom carried by blocks is 1. That is why the error degrees of freedom are 9 (and not 10) because one degree of freedom goes to block.

Suppose the block size is to be further reduced and we want to have blocks of size 8 in the same setup. This can be achieved by $1/2^2$ fraction of 2^6 factorial experiment. In terms of confounding setup, this is equivalent to saying that the two factorial effects are to be confounded. Suppose we choose *ABD* (and its alias *CEF)* in addition to *ABC* (and its alias *DEF*).

When we confound two effects, then their generalized interaction also gets confounded. So the interaction

$ABC \times ABD = A^2B^2CD = CD$ (or $DEF \times CEF = CDE^2F^2 = CD$) and its alias *ABEF* also get confounded.

One may note that a two factor interaction is getting confounded in this case, which is not a good strategy. A good strategy in such cases where an important factor is getting confounded is to choose the least important two-factor interaction.

The blocks arising with this plan are described in the following Table. These blocks are derived by dividing each block of earlier table of one half replicate of 2^6 factorial experiment in the blocks of size 16 into two halves.

These halves contain respectively an odd and even number of the letters *c* and *d*.

Block 1	Block 2	Block 3	Block 4
(1)	*de*	*ae*	*ad*
ef	*df*	*af*	*bd*
ab	*ac*	*be*	*ce*
abef	*bc*	*bf*	*cf*
acde	*abde*	*cd*	*abce*
acdf	*abdf*	*abcd*	*abcf*
bcde	*acef*	*cdef*	*adef*
bcdf	*bcef*	*abcdef*	*bdef*

The total degrees of freedom in this case are 31 which are divided as follows:

- the blocks carry 3 degrees of freedom,
- the main effect carry 6 degrees of freedom.
- the two factor interactions carry 14 degrees of freedom and
- the error carry 8 degrees of freedom.

The analysis of variance in case of fractional factorial experiments is conducted in the usual way as in the case of any factorial experiment. The sums of squares for blocks, main effects and two factor interactions are computed in the usual way.

Resolution

The criterion of resolution is used to compare the fractional factorial designs for overall quality of the statistical inferences which can be drawn. It is defined as the length of the shortest word (or order of the lowest-order effect) aliased with "*I*" in the generating relationship.

A fractional factorial design with greater resolution is considered to be better than a design with lower reticulation. An important objective in the designs is to find a fractional factorial design that has greatest resolution for a given number of runs and numbers of factors. The resolution of a design is generally denoted by a subscripted roman numeral. For example, a fractional factorial design constructed by using "$I = ABCD = \pm ABEF (= \pm CDEF)$" is denoted as 2_{IV}^{6-2} fractional factorial plan. In practice, the designs with resolution III, IV and V are used in practice.

When the design is of **resolution II**, it means that, e.g., "$I = +AB$". It means that in this case "$A = +B$" which indicates that at least some pairs of main effects are aliased.

When the design is of **resolution III**, the generating relation is like e.g., "$I = +ABC$". In this case "$A = +BC = \ldots$" This means that the main effects will not be aliased with each other but some of them will be aliased with two factor interaction. Thus such design can estimate all main effects if all interactions are absent.

When design is of **resolution IV**, then the generating relationship is like "$I = +ABCD$". Then the main effects will not be aliased with each other or with two factor interactions but some will get aliased with three factor interaction, e.g., "$A = +BCD$". Some pairs of two-factor interactions will also get aliased, e.g., "$AB = +CD =\ldots$". So this type of design unbiasedly estimates all the main effects even when two factor interactions are present.

Similarly, the generating relations like , "$I = +ABCDE$" are used in **resolution V** designs. In this case, all main effects can be estimated unbiasedly in the absence of all interactions of order less them five. The two factor interactions can be estimated if no effects of higher order are present. So resolution V design provides complete estimation of second order model.

The designs of **resolution II** or **higher than resolution V** are not used in practice. Reason being that resolution II design cannot separate the influence of main effects and design of resolution VI or higher require large number of units which may not be feasible all the times.