Analysis of Variance and Design of Experiments-II

MODULE IV

LECTURE - 21

PARTIAL CONFOUNDING

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Example 2

Consider the setup of 2^3 factorial experiment. The block size is 2^2 and 4 replications are made as in the following figure.

Replicate 1 (AB confounded)		Replicate 2 (AC confounded)		Replicate 3 (BC confounded)		Replicate 4 (ABC confounded)	
Block 1	Block 2	Block 1	Block 2	Block 1	Block 2	Block 1	Block 2
(1)	a	(1)	a	(1)	b	(1)	a
ab	b	b	ab	a	c	ab	b
c	ac	ac	c	bc	ab	ac	c
abc	bc	abc	bc	abc	ac	bc	abc

The arrangement of treatments in the different blocks in various replicates is based on the fact that different interaction effects are confounded in the different replicates. The interaction effect *AB* is confounded in replicate 1, *AC* is confounded in replicate 2, *BC* is confounded in replicate 3 and *ABC* is confounded in replicate 4. Then the r replications of each block are obtained. There are total eight factors involved in this case including (1). Out of them, three factors, viz., *A, B* and *C* are unconfounded whereas *AB, BC, AC* and *ABC* are partially confounded. Our objective is to estimate all these factors. The unconfounded factors can be estimated from all the four replicates whereas partially confounded factors can be estimated from the following replicates:

- AB from the replicates 2, 3 and 4,
- AC from the replicates 1, 3 and 4,
- BC from the replicates 1, 2 and 4 and
- ABC from the replicates 1, 2 and 3.

We first consider the estimation of unconfounded factors A, B and C which are estimated from all the four replicates.

The estimation of factor A from ℓ^{th} replicate $(\ell = 1, 2, 3, 4)$ is as follows:

$$A_{rep\ j} = \frac{\sum_{i=1}^{r} \ell'_{Aj} y_{*i}}{4r}$$

$$\begin{aligned} A &= \frac{\sum_{j=1}^{4} \ell'_{Aj} A_{rep\ j}}{4} \\ &= \frac{\sum_{j=1}^{4} \sum_{i=1}^{r} \ell'_{Aj} y_{*i}}{16r} \\ &= \frac{\sum_{i=1}^{r} \ell^{*'}_{A} y_{*i}}{16r} \end{aligned}$$

where the (32 x 1) vector $\ell^{*'}_{A} = \left(\ell_{A1},\ \ell_{A2},\ \ell_{A3},\ \ell_{A4} \right)$ consists of 32 elements and each ℓ_{Aj} $(j = 1, 2, 3, 4)$ is having 8 elements in it. The sum of square due to A is now based on 32r elements as

$$SS_A = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{A} y_{*i} \right)^2}{r \ell^{*'}_{A} \ell^{*}_{A}} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{A} y_{*i} \right)^2}{32r}.$$

Assuming that y_{ij}'s are independent and $Var(y_{ij}) = \sigma^2$ for all *i* and *j*, the variance of *A* is obtained as

$$Var(A) = \left(\frac{1}{16r}\right)^2 Var\left(\sum_{i=1}^{r} \ell_A^{*'} y_{*i}\right)$$

$$= \left(\frac{1}{16r}\right)^2 \times 32r\sigma^2$$

$$= \frac{\sigma^2}{8r}.$$

Similarly, the factor *B* is estimated as an arithmetic mean of the estimates of *B* from each replicate as

$$B = \frac{\sum_{i=1}^{r} \ell_B^{*'} y_{*i}}{16r}$$

where the (32 x 1) vector

$$\ell_B^{*'} = \left(\ell_{B1},\ \ell_{B2},\ \ell_{B3},\ \ell_{B4}\right)$$

consists of 32 elements.

The sum of squares due to *B* is obtained on the similar lines as in the case of *A* as

$$SS_B = \frac{\left(\sum_{i=1}^{r} \ell_B^{*'} y_{*i}\right)^2}{32r}.$$

The variance of *B* is obtained on the similar lines as in the case of *A* as

$$Var(B) = \frac{\sigma^2}{8r}.$$

The unconfounded factor *C* is also estimated as the average of estimates of *C* from all the replicates as

$$C = \frac{\sum_{i=1}^{r} \ell_C^{*'} y_{*i}}{16r}$$

where the (32 x 1) vector

$$\ell_C^{*'} = \left(\ell_{C1},\ \ell_{C2},\ \ell_{C3},\ \ell_{C4}\right)$$

consists of 32 elements.

The sum of square due to *C* in this case is obtained as

$$SS_C = \frac{\left(\sum_{i=1}^{r} \ell_C^{*'} y_{*i}\right)^2}{32r}.$$

The variance of *C* is obtained as

$$Var(C) = \frac{\sigma^2}{8r}.$$

Next we consider the estimation of the confounded factor *AB.* This factor *AB* can be estimated from each of the replicates 2, 3 and 4 and the final estimate of *AB* can be obtained as the arithmetic mean of those three estimates as

$$\begin{aligned} AB_{pc} &= \frac{AB_{rep2}+AB_{rep3}+AB_{rep4}}{3} \\ &= \frac{1}{12r}\left(\left(\sum_{i=1}^{r}\ell'_{AB2}y_{*i})_{rep2}\right)+\left(\sum_{i=1}^{r}\ell'_{AB3}y_{*i})_{rep3}\right)+\left(\sum_{i=1}^{r}\ell'_{AB3}y_{*i})_{rep4}\right)\right) \\ &= \frac{\sum_{i=1}^{r}\ell'_{AB}y_{*i}}{12r} \end{aligned}$$

where the (24 x 1) vector $\ell^{*'}_{AB} = (\ell_{AB2},\ \ell_{AB3},\ \ell_{AB4})$ consists of 24 elements and each of the (8 x 1) vectors l_{AB2}, l_{AB3} and l_{AB4} is having 8 elements in it. The sum of squares due to AB_{pc} is then based on 24*r* elements given as

$$SS_{AB_{pc}} = \frac{\left(\sum_{i=1}^{r}\ell^{*}_{AB'}y_{*i}\right)^2}{r\ell^{*}_{AB'}\ell^{*}_{AB}} = \frac{\left(\sum_{i=1}^{r}\ell^{*}_{AB'}y_{*i}\right)^2}{24r}.$$

The variance of AB_{pc} in this case is obtained under the assumption that y_{ij}'s are independent and each has variance σ^2 as

$$\begin{aligned} Var(AB_{pc}) &= \left(\frac{1}{12r}\right)^2 Var\left(\sum_{i=1}^{r}\ell^{*'}_{AB}y_{*i}\right) \\ &= \left(\frac{1}{12r}\right)^2 Var\left(\left(\sum_{i=1}^{r}\ell^{*'}_{AB2}y_{*i}\right)_{rep2}+\left(\sum_{i=1}^{r}\ell^{*'}_{AB3}y_{*i}\right)_{rep3}+\left(\sum_{i=1}^{r}\ell^{*'}_{AB4}y_{*i}\right)_{rep4}\right) \\ &= \left(\frac{1}{12r}\right)^2 (8r\sigma^2+8r\sigma^2+8r\sigma^2) \\ &= \frac{\sigma^2}{6r}. \end{aligned}$$

The confounded effects AC is obtained as the average of the estimates of AC obtained from the replicates 1, 3 and 4 as

$$AC_{pc} = \frac{AC_{rep1} + AC_{rep3} + AC_{rep4}}{3}$$

$$= \frac{\sum_{i=1}^{r} \ell^{*'}_{AC} y_{*i}}{12r}$$

where the (24 x 1) vector

$$\ell^{*'}_{AC} = \left(\ell_{AB1},\ \ell_{AC3},\ \ell_{AC4}\right)$$

consists of 24 elements.

The sum of squares due to AC in this case is given by

$$SS_{AC_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell^{*}_{AC} y_{*i}\right)^2}{24r}.$$

The variance of AC in this case under the assumption that y_{ij}'s are independent and each has variance σ^2 is given by

$$Var(AC_{pc}) = \frac{\sigma^2}{6r}.$$

Similarly, the confounded effect *BC* is estimated as the average of the estimates of *BC* obtained from the replicates 1, 2 and 4 as

$$BC_{pc} = \frac{BC_{rep1} + BC_{rep2} + BC_{rep4}}{3}$$

$$= \frac{\sum_{i=1}^{r} \ell^{*'}_{BC} y_{*i}}{12r}$$

where the (24 x 1) vector

$$\ell^{*'}_{BC} = \left(\ell_{BC1}, \ \ell_{BC3}, \ \ell_{BC4}\right)$$

consists of 24 elements.

The sum of squares due to *BC* in this case is based on $24r$ elements and is given as

$$SS_{BC_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{BC} y_{*i}\right)^2}{24r}.$$

The variance of *BC* in this case is obtained under the assumption that y_{ij}'s are independent and each has variance σ^2 as

$$Var(BC_{pc}) = \frac{\sigma^2}{6r}.$$

Lastly, the confounded effect *ABC* can be estimated first from the replicates 1, 2 and 3 and then the estimate of *ABC* is obtained as the average of these three individual estimates as

$$ABC_{pc} = \frac{ABC_{rep1} + ABC_{rep2} + ABC_{rep3}}{3}.$$

where the vector

$$\ell^{*'}_{ABC} = \left(\ell_{ABC1},\ \ell_{ABC2},\ \ell_{ABC3}\right)$$

consists of 24 elements.

The sum of squares due to *ABC* in this case is based on 24*r* elements and is given by

$$SS_{ABC_{pc}} = \frac{\left(\sum_{i=1}^{r} \ell^{*'}_{ABC} y_{*i}\right)^2}{24r}.$$

The variance of *ABC* in this case assuming that y_{ij}'s are independent and each has variance σ^2 is given by

$$Var(ABC_{pc}) = \frac{\sigma^2}{6r}.$$

If an unconfounded design with $4r$ replication was used then the variance of each of the factors $A, B, C,\ AB,\ BC,\ AC$ and ABC is $\sigma^{*2}/8r$ where σ^{*2} is the error variance on blocks of size 8. So the relative efficiency of a confounded effect in the partially confounded design with respect to that of an unconfounded one in a comparable design is

$$\frac{6r/\sigma^2}{8r/\sigma^{*2}} = \frac{3}{4}\frac{\sigma^{*2}}{\sigma^2}.$$

So the information on a partially confounded effect relative to an unconfounded effect is $\frac{3}{4}$. If $\sigma^{*2} > 4\sigma^2/3,$ then partially confounded design gives more information than the unconfounded design.

Further, the sum of squares due to block can be divided into two components – within replicates and due to replications. So we can write

$$SS_{Block} = SS_{Block(wr)} + SS_{Block(r)}$$

where the sum of squares due to blocks within replications (wr) is

$$SS_{Block(wr)} = \frac{1}{2^3}\sum_{i=1}^{4r}\left(\frac{B_{1i}^2 + B_{2i}^2}{2} - R_i^2\right)$$

which carries $4r$ degrees of freedom and the sum of squares due to replication is

$$SS_{Block(r)} = \frac{1}{2^3}\sum_{i=1}^{4r} R_i^2 - \frac{G^2}{32r}$$

which carries ($4r$ -1) degrees of freedom. The total sum of squares is

$$SS_{Total(pc)} = \sum_i\sum_j\sum_k y_{ijk}^2 - \frac{G^2}{32r}.$$

The analysis of variance table in this case of 2^3 factorial under partial confounding is given as

Source	Sum of squares	Degrees of freedom	Mean squares
Replicates	$SS_{Block(r)}$	$4r - 1$	$MS_{Block(r)}$
Blocks within replicate	$SS_{Block(wr)}$	$4r$	$MS_{Block(wr)}$
Factor A	SS_A	1	MS_A
Factor B	SS_B	1	MS_B
Factor C	SS_C	1	MS_C
AB	$SS_{AB(pc)}$	1	$MS_{AB(pc)}$
AC	$SS_{AC(pc)}$	1	$MS_{AC(pc)}$
BC	$SS_{BC(pc)}$	1	$MS_{BC(pc)}$
ABC	$SS_{ABC(pc)}$	1	$MS_{ABC(pc)}$
Error	by subtraction	$24r - 7$	$MS_{E(pc)}$
Total	$SS_{Total(pc)}$	$32r – 1$	

The test of hypothesis can be carried out in the usual way as in the case of factorial experiment.