Analysis of Variance and Design of Experiments-II

MODULE I

LECTURE - 5

INCOMPLETE BLOCK DESIGNS

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Theorem $E(P) = D\beta, \quad Var(P) = \sigma^2 D$

Proof

$$D = K - NR^{-1}N'$$

$$P = B - NR^{-1}V$$

$$= D_2[I - D_1 R^{-1} D_1]y$$

$$= D_2[I - D_1^{'}(D_1 D_1^{'})^{-1} D_1]y$$

$$E(P) = D_2[I_1 - D_1^{'}(D_1 D_1^{'})^{-1} D_1](\mu E_{n1} + D_1^{'}\tau + D_2^{'}\beta)$$

$$= [D_2 E_{n1} - D_2 D_1^{'} R^{-1} D_1 E_{n1}]\mu + [D_2 D_1^{'} - D_2 D_1^{'} R^{-1} D_1 D_1^{'}]\tau + [D_2 D_1^{'} - D_2 D_1^{'} R^{-1} D_1 D_2^{'}]\beta$$

$$= [(k_1, k_2, \ldots, k_b)' - NR^{-1}(r_1, r_2, \ldots, r_v)']\mu + [N - NR^{-1}R]\tau + [K - NR^{-1}N']\beta$$

$$= [(k_1, k_2, \ldots, k_b)' - NE_{v1}]\mu + 0 + D\beta$$

$$= [(k_1, k_2, \ldots, k_b)' - (k_1, k_2, \ldots, k_b)']\mu + D\beta$$

$$= D\beta.$$

Next

$$Var(P) = \sigma^2 D_2[I - D_1'(D_1D_1')^{-1}D_1]D_2'$$

$$= \sigma^2[D_2D_2' - D_2D_1'(D_1D_1')^{-1}D_1D_2']$$

$$= \sigma^2[K - NR^{-1}N'] = \sigma^2 D.$$

Note that $[I - D_1'(D_1D_1')^{-1}D_1]$ is an idempotent matrix.

Alternatively, we can also find *Var*(*P*) as follows:

$$P = (I \quad -NR^{-1})\begin{pmatrix} B \\ V \end{pmatrix} = (I \quad -NR^{-1})Z$$

where $Z = (B, \quad V)'$

$$Var(P) = (I \quad -NR^{-1})Cov(Z)\begin{pmatrix} I \\ -R^{-1}N' \end{pmatrix}$$

$$= (I \quad -NR^{-1})\begin{pmatrix} K & N \\ N' & R \end{pmatrix}\begin{pmatrix} I \\ -R^{-1}N' \end{pmatrix}\sigma^2$$

$$= (K - NR^{-1}N' \quad N - NR^{-1}R)\begin{pmatrix} I \\ -R^{-1}N' \end{pmatrix}\sigma^2$$

$$= \left(K - NR^{-1}N'\right)\sigma^2$$

$$= D\sigma^2.$$

Now we consider some properties of incomplete block designs.

Lemma $b + rank(C) = v + rank(D)$.

Proof Consider $(b + v)$x$(b + v)$ matrix

$$A = \begin{bmatrix} K & N \\ N' & R \end{bmatrix}$$

Note that A is a submatrix of C.

Using the result that the rank of a matrix does not changes by the pre-multiplication of nonsingular matrix, consider the following matrices:

$$M = \begin{bmatrix} I_b & 0 \\ -N'K^{-1} & I_v \end{bmatrix} \text{ and } S = \begin{bmatrix} I_b & 0 \\ -R^{-1}N' & I_v \end{bmatrix}.$$

M and S are nonsingular, so we have

$$rank(A) = rank(MA) = rank(AS).$$

Now

$$MA = \begin{bmatrix} I_b & 0 \\ -N'K^{-1} & I_v \end{bmatrix} \begin{bmatrix} K & N \\ N' & R \end{bmatrix} = \begin{bmatrix} K & N \\ 0 & C \end{bmatrix}$$

$$AS = \begin{bmatrix} D & N \\ 0 & R \end{bmatrix}$$

Thus

$$rank \begin{bmatrix} K & N \\ 0 & C \end{bmatrix} = rank \begin{bmatrix} D & N \\ 0 & R \end{bmatrix}$$

or

$$rank(K) + rank(C) = rank(D) + rank(R)$$

or

$$b + rank(C) = v + rank(D).$$

Remark

$C: v \times v$ and $D: b \times b$ are symmetric matrices.

One can verify that

$CE_{v1} = 0$ and $DE_{b1} = 0$

Thus $rank(C) \leq v - 1$

and $rank(D) \leq b - 1.$

Lemma

If $rank(C) = v - 1$, then all blocks and treatment contrasts are estimable.

Proof

If $rank(C) = v - 1$, it is obvious that all the treatment contrasts are estimable.

Using the result from the lemma $b + rank(C) = v + rank(D)$, we have

$$
\begin{aligned}
rank(D) + v &= rank(C) + b \\
&= v - 1 + b
\end{aligned}
$$

Thus

$$rank(D) = b - 1.$$

Thus all the block contrasts are also estimable.

Orthogonality of *Q* and *P*

Now we explore the conditions under which *Q* and *P* can be orthogonal.

$$Q = V - N'K^{-1}B \quad = (D_1 - D_1D_2^{'}K^{-1}D_2)y$$

$$P = B - NR^{-1}V \quad = (D_2 - D_2D_1^{'}R^{-1}D_1)y$$

$$Cov(Q,P) = (D_1 - D_1D_2^{'}K^{-1}D_2)(D_2 - D_2D_1^{'}R^{-1}D_1)\sigma^2$$

$$= (D_1D_2^{'} - D_1D_2^{'}R^{-1}D_1D_2^{'} - D_1D_2^{'}K^{-1}D_2D_2^{'} + D_1D_2^{'}K^{-1}D_2D_1^{'}R^{-1}D_1D_2^{'})\sigma^2$$

$$=(N' - RR^{-1}N' - N'K^{-1}K + N'K^{-1}NR^{-1}N')\sigma^2$$

$$=(N'K^{-1}NR^{-1}N' - N')\sigma^2$$

Q and *P* (or equivalently Q_i and P_j) are orthogonal when

$$Cov(Q,P) = 0$$

or $N'K^{-1}N\ R^{-1}N' - N' = 0$ (i)

$\Rightarrow (R-C)R^{-1}N' - N' = 0$ (Using $C = R - N'K^{-1}N$)

$\Rightarrow CR^{-1}N' = 0$ (ii)

or equivalently

$$N'K^{-1}NR^{-1}N' - N' = 0$$

$\Rightarrow\ N'K^{-1}(K-D) - N' = 0$ (Using $D = K - NR^{-1}N'$)

$\Rightarrow\ N'K^{-1}D = 0.$ (iii)

Thus Q_i and P_j are orthogonal if $NR^{-1}N'K^{-1}N = N$

or equivalently $NR^{-1}C = 0$

or equivalently $DK^{-1}N = 0.$

Orthogonal block design

A block design is said to be orthogonal if Q_i 's and P_j 's are orthogonal for all i and j. Thus the condition for the orthogonality of a design is

$$NR^{-1}N'K^{-1}N = N,$$

$$NR^{-1}C = 0$$

or

$$DK^{-1}N = 0.$$

Lemma If $\frac{n_{ij}}{r_j}$ is constant for all *j*, then $\frac{n_{ij}}{k_i}$ is constant for all *i* and vice versa. In this case, we have $n_{ij} = \frac{k_i r_j}{n}$.

Proof If $\frac{n_{ij}}{r_j}$ is constant for all *j* then $\frac{n_{ij}}{r_j} = a_i$, say.

$$\Rightarrow n_{ij} = a_i r_j$$

or $\sum_j n_{ij} = \sum_j a_i r_j = a_i \sum_j r_j = a_i n$

or $k_i = a_i n$

or $a_i = \frac{k_i}{n}$

Thus

$$\frac{n_{ij}}{r_j} = \frac{k_i}{n}$$

or $n_{ij} = \frac{k_i r_j}{n}$

So $\frac{n_{ij}}{k_j} = \frac{r_i}{n}$: independent of i.

Hence proved.