Analysis of Variance and Design of Experiments-II

MODULE - III

LECTURE - 18

PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN (PBIBD)

Dr. Shalabh
Department of Mathematics & Statistics
Indian Institute of Technology Kanpur

Intrablock analysis of PBIBD with two associates

Consider a PBIBD under two associates scheme. The parameters of this scheme are

$b,\ v,\ r,\ k,\ \lambda_1,\ \lambda_2,\ n_1,\ n_2,\ p_{11}^1,\ p_{22}^1,\ p_{12}^1,\ p_{11}^2,\ p_{22}^2$ and p_{12}^2. The linear model involving block and treatment effects is

$$y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij};\ i = 1,2,...,b,\ \ j = 1,2,...,v,$$

where μ is the general mean effect;

β_i is the fixed additive i^{th} block effect satisfying $\sum_i \beta_i = 0;$

τ_j is the fixed additive j^{th} treatment effect satisfying $\sum_{j=1}^{r} \tau_j = 0$ and

ε_{ijm} is the i.i.d. random error with $\varepsilon_{ijm} \sim N(0,\sigma^2).$

The PBIBD is a binary, proper and equireplicate design. So in this case, the values of the parameters become $n_{ij} = 0$ or $1, k_i = k$ for all $i = 1,2,...,b$ and $r_j = r$ for all $j = 1,2,...,v$. There can be two types of null hypothesis which can be considered – one for the equality of treatment effects and another for the equality of block effects. We are considering here the intrablock analysis, so we consider the null hypothesis related to the treatment effects only. As done earlier in the case of BIBD, the block effects can be considered under the interblock analysis of PBIBD and the recovery of interblock information.

The null hypothesis of interest is

$H_0 : \tau_1 = \tau_2 = ... = \tau_v$ against alternative hypothesis

$H_1 :$ at least one pair of τ_j is different.

The null hypothesis related to the block effects is of not much practical relevance and can be treated similarly.

This was illustrated earlier in the case of BIBD. In order to obtain the least squares estimates of μ, β_i and τ_j, we minimize the sum of squares due to residuals

$$\sum_{i=1}^{b}\sum_{j=1}^{v}(y_{ij}-\mu-\beta_i-\tau_j)^2$$

with respect to μ, β_i and τ_j. This results in the three normal equations, which can be unified using the matrix notations. The set of reduced normal equations in matrix notation after eliminating the block effects are expressed as

$$Q = C\tau$$

where

$$C = R - N'K^{-1}N,$$

$$Q = V - N'K^{-1}B,$$

$$R = rI_v,$$

$$K = kI_b.$$

Then the diagonal elements of C are obtained as

$$c_{jj} = r - \frac{\sum_{i=1}^{b} n_{ij}^2}{k} = \frac{r(k-1)}{k}, \quad (j = 1, 2, ..., v),$$

the off-diagonal elements of C are obtained as

$$c_{jj'} = -\frac{1}{k}\sum_{i=1}^{b} n_{ij}n_{ij'} = \begin{cases} -\dfrac{\lambda_1}{k} & \text{if treatment } j \text{ and } j' \text{ are the first associates} \\ -\dfrac{\lambda_2}{k} & \text{if treatment } j \text{ and } j' \text{ are the second associates } (j \neq j' = 1, 2, ..., v) \end{cases}$$

and the j^{th} value in Q is

$$Q_j = V_j - \frac{1}{k}[\text{Sum of block totals in which } j^{th} \text{ treatment occurs}]$$

$$= \frac{1}{k}\left[r(k-1)\tau_j - \sum_i \sum_{j'(j\neq j')} n_{ij} n_{ij'} \tau_j \right].$$

Next, we attempt to simplify the expression of Q_j.

Let S_{j1} be the sum of all treatments which are the first associates of j^{th} treatment and S_{j2} be the sum of all treatments which are the second associates of j^{th} treatment. Then

$$\tau_j + S_{j1} + S_{j2} = \sum_{j=1}^{v} \tau_j.$$

Thus the equation in Q_j using this relationship for $j = 1, 2, .., v$ becomes

$$kQ_j = \left[r(k-1)\tau_j - (\lambda_1 S_{j1} + \lambda_2 S_{j2}) \right]$$

$$= r(k-1)\tau_j - \lambda_1 S_{j1} - \lambda_2 \left[\sum_{j=1}^{v} \tau_j - \tau_j - S_{j1} \right]$$

$$= \left[r(k-1) + \lambda_2 \right] \tau_j + (\lambda_2 - \lambda_1) S_{j1} - \lambda_2 \sum_{j=1}^{v} \tau_j.$$

Imposing the side condition $\sum_{j=1}^{v} \tau_j = 0,$ we have

$$kQ_j = \left[r(k-1)\tau_j + \lambda_2 \right] \tau_j + (\lambda_2 - \lambda_1) S_{j1}$$

$$= a_{12}^* \tau_j + b_{12}^* S_{j1}, \quad j = 1, 2, ..., v$$

where $a_{12}^* = r(k-1) + \lambda_2$ and $a_{12}^* = \lambda_2 - \lambda_1.$

These equations are used to obtain the adjusted treatment sum of squares.

Let Q_{j1} denote the adjusted sum of Q_j's over the set of those treatment,s which are the first associate of j^{th} treatment. We note that when we add the terms S_{j1} for all j, then j occurs n_1 times in the sum, every first associate of j occurs p_{11}^1 times in the sum and every second associate of j occurs p_{11}^2 times in the sum with $p_{11}^2 + p_{12}^2 = n_1$. Then using $K = kI_b$ and $\sum_{j=1}^{v} \tau_j = 0$, we have

$$kQ_{j1} = k\sum_{j \in S_{j1}} Q_j = \text{Sum of } Q_j\text{'s which are the first associates of treatment } j$$

$$= \left[r(k-1)+\lambda_2\right]S_{j1} + (\lambda_2-\lambda_1)\left[n_1\tau_j + p_{11}^1 S_{j1} + p_{11}^2 S_{j2}\right]$$

$$= \left[r(k-1)+\lambda_2+(\lambda_2-\lambda_1)(p_{11}^1 - p_{11}^2\right]S_{j1} + (\lambda_2-\lambda_1)p_{12}^2\tau_j$$

$$= b_{22}^* S_{j1} + a_{22}^*\tau_j$$

where

$$a_{22}^* = (\lambda_2-\lambda_1)p_{12}^2$$

$$b_{22}^* = r(k-1)+\lambda_2+(\lambda_2-\lambda_1)(p_{11}^1 - p_{11}^2).$$

Now we have following two equations in kQ_j and kQ_{j1}:

$$kQ_j = a_{12}^*\tau_j + b_{12}^* S_{j1}$$

$$kQ_{j1} = a_{22}^*\tau_j + b_{22}^* S_{j1}.$$

Now solving these two equations, the estimate of $\hat{\tau}_j$ is obtained as

$$\hat{\tau}_j = \frac{k[b_{22}^* Q_j - b_{12}^* Q_{j1}]}{a_{12}^* b_{12}^* - a_{22}^* b_{12}^*}, \quad (j = 1, 2, \ldots, v).$$

We see that

$$\sum_{j=1}^{v} Q_j = \sum_{j=1}^{v} Q_{j1} = 0,$$

so

$$\sum_{j=1}^{v} \hat{\tau}_j = 0.$$

Thus $\hat{\tau}_j$ is a solution of reduced normal equation.

The analysis of variance can be carried out by obtaining the unadjusted block sum of squares as

$$SS_{Block(unadj)} = \sum_{i=1}^{b} \frac{B_i^2}{k} - \frac{G^2}{bk},$$

the adjusted sum of squares due to treatment as

$$SS_{Treat(adj)} = \sum_{j=1}^{v} \hat{\tau}_j Q_j$$

where $G = \sum_{i=1}^{b} \sum_{j}^{v} y_{ij}.$

The sum of squares due to error as

$$SS_{Error(t)} = SS_{Total} - SS_{Block(unadj)} - SS_{Treat(adj)}$$

where

$$SS_{Total} = \sum_{i=1}^{b} \sum_{j=1}^{v} y_{ij}^2 - \frac{G^2}{bk}.$$

A test for $H_0 : \tau_1 = \tau_2 = ... = \tau_v$ is then based on the statistic

$$F_{Tr} = \frac{SS_{Treat(adj)} /(v-1)}{SS_{Error(unadj)} /(bk-b-v+1)}.$$

If $F_{Tr} > F_{1-\alpha; v-1, bk-v-b+1}$ then H_0 is rejected.

The intrablock analysis of variance for testing the significance of treatment effects is given in the following table:

Source	Sum of squares	Degrees of freedom	Mean squares	*F*
Between treatments (adjusted)	$SS_{Treat(adj)} = \sum_{j=1}^{v} \hat{\tau}_j Q_j$	$df_{Treat} = v-1$	$MS_{Treat} = \frac{SS_{Treat(adj)}}{df_{Treat}}$	$\frac{MS_{Treat}}{MSE}$
Between blocks (unadjusted)	$SS_{Block(unadj)}$ $\sum_{i=1}^{b} \frac{B^2}{k} - \frac{G^2}{bk}$	$df_{Block} = b-1$		
Intrablock error	$SS_{Error(t)}$ (By substraction)	$df_{ET} = bk-b-v+1$	$MSE = \frac{SS_{Error}}{df_{ET}}$	
Total	$SS_{Total} = \sum_{i=1}^{b}\sum_{j=1}^{v} y_{ij}^2 - \frac{G^2}{bk}$	$df_T = bk-1$		

Note that in the intrablock analysis of PBIBD analysis, at the step where we obtained S_{j1} and eliminated S_{j2} to obtain the following equation

$$kQ_j = \left[r(k-1)+\lambda_2\right]\tau_j + (\lambda_2-\lambda_1)S_{j1} - \lambda_2\sum_{j=1}^{v}\tau_j,$$

another possibility is to eliminate S_{j1} instead of S_{j2}. If we eliminate S_{j2} instead of S_{j1} (as we approached), then the solution has less work involved in the summing of Q_{j1} if $n_1 < n_2$.

If $n_1 > n_2$, then eliminating S_{j1} will involve less work in obtaining in obtaining Q_{j2} where Q_{j2} denotes the adjusted sum of Q_j's is over the set of those treatments which are the second associate of j^{th} treatment. When we do so, the following estimate of the treatment effect is obtained :

$$\hat{\tau}_j^* = \frac{k\left[b_{21}^* Q_j - b_{11}^* Q_{j2}\right]}{a_{11}^* b_{21}^* - a_{21}^* b_{11}^*}$$

where

$$a_{11}^* = r(k-1)+\lambda_1$$

$$b_{11}^* = \lambda_1 - \lambda_2$$

$$a_{21}^* = (\lambda_1-\lambda_2)\,p_{12}^1$$

$$b_{21}^* = r(k-1)+\lambda_1+(\lambda_1-\lambda_2)(p_{22}^2 - p_{22}^1).$$

The analysis of variance is then based on $\hat{\tau}_j^*$ and can be carried out similarly.

The variance of the elementary contrasts of estimates of treatments (in case of $n_1 < n_2$)

$$\hat{\tau}_j - \hat{\tau}_{j'} = \frac{b_{22}^*(kQ_j - kQ_{j'}) - b_{12}^*(kQ_{j1} - kQ_{j'1})}{a_{12}^* b_{22}^* - a_{22}^* b_{12}^*}$$

is

$$Var(\hat{\tau}_j - \hat{\tau}_{j'}) = \begin{cases} \dfrac{2k(b_{22}^* + b_{12}^*)}{a_{12}^* b_{22}^* - a_{22}^* b_{12}^*} & \text{if treatment } j \text{ and } j' \text{ are the first associates} \\ \dfrac{2kb_{12}^*}{a_{12}^* b_{22}^* - a_{22}^* b_{12}^*} & \text{if treatment } j \text{ and } j' \text{ are the second associates.} \end{cases}$$

We observe that the variance of $\hat{\tau}_j - \hat{\tau}_{j'}$ depends on the nature of j and j' in the sense that whether they are the first or second associates. So design is not (variance) balanced. But variance of any elementary contrast are equal under a given order of association, viz., first or second. That is why the design is said to be partially balanced in this sense.