Analysis of Variance and Design of Experiments-II

MODULE - II

LECTURE - 12

BALANCED INCOMPLETE BLOCK DESIGN (BIBD)

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The variance of an elementary contrast $(\tau_j - \tau_{j'}, j \neq j')$ under the intrablock analysis is

$$\begin{aligned}
V^* &= Var(\hat{\tau}_j - \hat{\tau}_{j'}) \\
&= Var\left(\frac{k}{\lambda v}(Q_j - Q_{j'})\right) \\
&= \frac{k^2}{\lambda^2 v^2}[Var(Q_j) + Var(Q_j) - 2Cov(Q_j Q_{j'})] \\
&= \frac{k^2}{\lambda^2 v^2}(c_{jj} + c_{j'j'} - 2c_{jj'})\sigma^2 \\
&= \frac{k^2}{\lambda^2 v^2}\left[2r\left(1 - \frac{1}{k}\right) + \frac{2\lambda}{k}\right]\sigma^2 \\
&= \frac{2k}{\lambda v}\sigma^2.
\end{aligned}$$

This expression depends on σ^2 which is unknown. So it is unfit for use in the real data applications. One solution is to estimate σ^2 from the given data and use it is place of σ^2.

An unbiased estimator of σ^2 is given as

$$\hat{\sigma}^2 = \frac{SS_{Error(t)}}{bk - b - v + 1}.$$

Thus an unbiased estimator of V^* can be obtained by substituting $\hat{\sigma}^2$ in it as

$$\hat{V}^* = \frac{2k}{\lambda v} . \frac{SS_{Error(t)}}{bk - b - v + 1}.$$

If H_0 is rejected, then we make pairwise comparison and use the multiple comparison test. In order to test $H_0 : \tau_{j'} = \tau_{j'} (j \neq j'),$ a suitable statistic is

$$t = \frac{k(bk-b-v+1)}{\lambda v} . \frac{Q_j - Q_{j'}}{\sqrt{SS_{Error(t)}}}$$

which follows a t-distribution with $(bk - b - v + 1)$ degrees of freedom under H_0 .

A question arises that how a BIBD compares with an RBD. Note that BIBD is an incomplete block design whereas RBD is a complete block design. This point should be kept is mind while making such restrictive comparisons.

We now compare the efficiency of BIBD with a randomized block (complete) design with r replicates. The variance of an elementary contrast under a randomized block design (RBD) is

$$V_R^* = Var(\hat{\tau}_j^2 - \hat{\tau}_{j'})_{RBD} = \frac{2\sigma_*^2}{r}$$

where $Var(y_{ij}) = \sigma_*^2$ under RBD.

Thus the relative efficiency of BIBD relative to RBD is

$$\frac{Var(\hat{\tau}_j - \hat{\tau}_{j'})_{RBD}}{Var(\hat{\tau}_j - \hat{\tau}_{j'})_{BIBD}} = \frac{\left(\dfrac{2\sigma_*^2}{r}\right)}{\left(\dfrac{2k\sigma^2}{\lambda v}\right)} = \frac{\lambda v}{rk}\left(\frac{\sigma_*^2}{\sigma^2}\right).$$

The factor $\frac{\lambda v}{rk} = E$ (say) is termed as the **efficiency factor** of BIBD and

$$\begin{aligned} E &= \frac{\lambda v}{rk} \\ &= \frac{v}{k}\left(\frac{k-1}{v-1}\right) \\ &= \left(1-\frac{1}{k}\right)\left(1-\frac{1}{v}\right)^{-1} \\ &< 1 \text{ (since } v > k). \end{aligned}$$

The actual efficiency of BIBD over RBD not only depends on efficiency factor but also on the ratio of variances $\frac{\sigma_*^2}{\sigma^2}$.

So BIBD can be more efficient than RBD as σ_*^2 can be more than σ^2 because $k < v$.

Efficiency balanced design

A block design is said to be efficiency balanced if every contrast of the treatment effects is estimated through the design with the same efficiency factor.

If a block design satisfies any two of the following properties:

i. efficiency balanced,
ii. variance balanced and
iii. equal number of replications,

then the third property also holds true.

Missing observations in BIBD

The intrablock estimate of missing $(i, j)^{th}$ observation y_{ij} is

$$y_{ij} = \frac{vr(k-1)B_i - k(v-1)Q_j - (v-1)Q_j^{'}}{k(k-1)(bk-b-v+1)}$$

$Q_j^{'}$: sum of Q value for all other treatment (but not the j^{th} one) which are present in the i^{th} block.

All other procedures remain the same.

Interblock analysis and recovery of interblock information in BIBD

In the intrablock analysis of variance of an incomplete block design or BIBD, the treatment effects were estimated after eliminating the block effects from the normal equations.

In a way, the block effects were assumed to be not marked enough and so they were eliminated. It is possible in many situations that the block effects are influential and marked. In such situations, the block totals may carry information about the treatment combinations also.

This information can be used in estimating the treatment effects which may provide more efficient results. This is accomplished by an interblock analysis of BIBD and used further through recovery of interblock information.

So we first conduct the interblock analysis of BIBD. We do not derive the expressions a fresh but we use the assumptions and results from the interblock analysis of an incomplete block design. We additionally assume that the block effects are random with variance σ_β^2.

After estimating the treatment effects under interblock analysis, we use the results for the pooled estimation and recovery of interblock information in a BIBD.

In case of BIBD,

$$N'N = \begin{pmatrix} \sum_i n_{i1}^2 & \sum_i n_{i1}n_{i2} & \cdots \sum_i n_{i1}n_{iv} \\ \sum_i n_{i1}n_{i2} & \sum_i n_{i2}^2 & \cdots & \sum_i n_{i2}n_{iv} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_i n_{iv}n_{i1} & \sum_i n_{iv}n_{i2} & \cdots & \sum_i n_{iv}^2 \end{pmatrix} = \begin{pmatrix} r & \lambda \cdots \lambda \\ \lambda & r \cdots \lambda \\ \vdots & \vdots \ddots \vdots \\ \lambda & \lambda \cdots r \end{pmatrix}$$

$$= (r-\lambda)I_v + \lambda E_{v1}E_{v1}^{'}$$

$$(N'N)^{-1} = \frac{1}{r-\lambda}\left[I_v - \frac{\lambda E_{v1}E_{v1}^{'}}{rk}\right].$$

The interblock of τ can be obtained by substituting the expression $(N'N)^{-1}$ in the earlier obtained interblock estimate as

$$\tilde{\tau} = (N'N)^{-1}N'B - \frac{GE_{v1}}{bk}.$$

Our next objective is to use the intrablock and interblock estimates of treatment effects together to find an improved estimator of treatment effect.

In order to use the interblock and intrablock estimates of τ together through pooled estimate, we consider the interblock and intrablock estimates of the treatment contrast.

The intrablock estimate of treatment contrast $l'\tau$ is

$$\begin{aligned} l'\hat{\tau} &= l'C^{-}Q \\ &= \frac{k}{\lambda v} l'Q \\ &= \frac{k}{\lambda v} \sum_j l_j Q_j \\ &= \sum_j l_j \hat{\tau}_j, \text{ say.} \end{aligned}$$

The interblock estimate of treatment contrast $l'\tau$ is

$$\begin{aligned} l'\tilde{\tau} &= \frac{l'N'B}{r-\lambda} \quad \text{(since } l'E_{v1} = 0) \\ &= \frac{1}{r-\lambda} \sum_{j=1}^{v} l_j \left(\sum_{i=1}^{b} n_{ij} B_i \right) \\ &= \frac{1}{r-\lambda} \sum_{j=1}^{v} l_j T_j \\ &= \sum_{j=1}^{v} l_j \tilde{\tau}_j, \text{ say.} \end{aligned}$$