Analysis of Variance and Design of Experiments-II

MODULE - II

LECTURE - 13

BALANCED INCOMPLETE BLOCK DESIGN (BIBD)

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The variance of $l'\hat{\tau}$ is obtained as

$$Var(l'\hat{\tau}) = \left(\frac{k}{\lambda v}\right) Var\left(\sum_j l_j Q_j\right)$$
$$= \left(\frac{k}{\lambda v}\right)^2 \left[\sum_j l_j^2 Var(Q_j) + 2\sum_j \sum_{j'(\neq j)} l_j l_{j'} Cov(Q_j, Q_{j'})\right].$$

Since

$$Var(Q_j) = r\left(1 - \frac{1}{k}\right)\sigma^2,$$
$$Cov(Q_j, Q_{j'}) = -\frac{\lambda}{k}\sigma^2, \quad (j \neq j')$$

so

$$Var(l'\hat{\tau}) = \left(\frac{k}{\lambda v}\right)^2 \left[r\left(1 - \frac{1}{k}\right)\sigma^2 \sum_j l_j^2 - \frac{\lambda}{k}\left\{\left(\sum_j l_j\right)^2 - \sum_j l_j^2\right\}\sigma^2\right]$$
$$= \left(\frac{k}{\lambda v}\right)^2 \left[\frac{r(k-1)}{k}\sum_j l_j^2 + \frac{\lambda}{k}\sum_j l_j^2\right]\sigma^2 \text{ (since } \sum_j \ell_j = 0 \text{ being contrast)}$$
$$= \left(\frac{k}{\lambda v}\right)^2 \frac{1}{k}\left[\lambda(v-1) + \lambda\right]\sum_j l_j^2 \qquad \text{(using } r(k-1) = \lambda(v-1))$$
$$= \left(\frac{k}{\lambda v}\right)\sigma^2 \sum_j l_j^2.$$

Similarly, the variance of $\ell'\hat{\tau}$ is obtained as

$$
\begin{aligned}
Var(l'\tilde{\tau}) &= \left(\frac{1}{r-\lambda}\right)^2 \left[\sum_j l_j^2 Var(T_j) + 2\sum_j \sum_{j'(\neq j)} l_j l_{j'} Cov(T_j, T_{j'})\right] \\
&= \left(\frac{1}{r-\lambda}\right)^2 \left[r\sigma_f^2 \sum_j l_j^2 + \lambda\sigma_f^2 \left\{\left(\sum_j l_j\right)^2 - \sum_j l_j^2\right\}\right] \\
&= \frac{\sigma_f^2}{r-\lambda} \sum_j l_j^2.
\end{aligned}
$$

The information on $\ell'\hat{\tau}$ and $\ell'\hat{\tau}$ can be used together to obtain a more efficient estimator of $\ell'\tau$ by considering the weighted arithmetic mean of $\ell'\hat{\tau}$ and $\ell'\tilde{\tau}$.

This will be the minimum variance unbiased and estimator of $\ell'\tau$ when the weights of the corresponding estimates are chosen such that they are inversely proportional to the respective variances of the estimators.

Thus, the weights to be assigned to intrablock and interblock estimates are reciprocal to their variances as $\lambda v/(k\sigma^2)$ and $(r-\lambda)/\sigma_f^2$, respectively.

Then the pooled mean of these two estimators is

$$L^* = \frac{\dfrac{\lambda v}{k\sigma^2} l'\hat{\tau} + \dfrac{r-\lambda}{\sigma_f^2} l'\hat{\tau}}{\dfrac{\lambda v}{k\sigma^2} + \dfrac{r-\lambda}{\sigma_f^2}}$$

$$= \frac{\dfrac{\lambda v}{k\sigma^2} \sum_j l_j \hat{\tau}_j + \dfrac{r-\lambda}{\sigma_f^2} \sum_j l_j \tilde{\tau}_j}{\dfrac{\lambda v}{k\sigma^2} + \dfrac{r-\lambda}{\sigma_f^2}}$$

$$= \frac{\dfrac{\lambda v \omega_1}{k} \sum_j l_j \hat{\tau}_j + (r-\lambda)\omega_2 \sum_j l_j \tilde{\tau}_j}{\dfrac{\lambda v}{k}\omega_1 + (r-\lambda)\omega_2}$$

$$= \frac{\lambda v \omega_1 \sum_j l_j \hat{\tau}_j + k(r-\lambda)\omega_2 \sum_j l_j \tilde{\tau}_j}{\lambda v \omega_1 + k(r-\lambda)\omega_2}$$

$$= \sum_j l_j \left[\frac{\lambda v \omega_1 \hat{\tau}_j + k(r-\lambda)\omega_2 \tilde{\tau}_j}{\lambda v \omega_1 + k(r-\lambda)\omega_2} \right]$$

$$= \sum_j l_j \tau_j^*$$

where

$$\tau_j^* = \frac{\lambda v \omega_1 \hat{\tau}_j + k(r-\lambda)\omega_2 \hat{\tau}_j}{\lambda v \omega_1 + k(r-\lambda)\omega_2}$$

$$\omega_1 = \frac{1}{\sigma^2}, \quad \omega_2 = \frac{1}{\sigma_f^2}.$$

Now we simplify the expression of τ_j^* so that it becomes more compatible in further analysis.

Since

$$\hat{\tau}_j = (k/\lambda v)Q_j$$

and

$$\tilde{\tau}_j = T_j/(r-\lambda),$$

so the numerator of τ_j^* can be expressed as

$$\omega_1 \lambda v \hat{\tau}_j + \omega_2 k(r-\lambda)\tilde{\tau}_j = \omega_1 k Q_j + \omega_2 k T_j.$$

Similarly, the denominator of τ_j^* can be expressed as

$$\omega_1 \lambda v + \omega_2 k(r-\lambda) = \omega_1 \left[\frac{vr(k-1)}{v-1}\right] + \omega_2 \left[k\left(r - \frac{r(k-1)}{v-1}\right)\right] \qquad \text{(using } \lambda(v-1) = r(k-1)\text{)}$$

$$= \frac{1}{v-1}\left[\omega_1 vr(k-1) + \omega_2 kr(v-k)\right].$$

Let $W_j^* = (v-k)V_j - (v-1)T_j + (k-1)G$ where $\sum_j W_j^* = 0.$

Using these results we have

$$\tau_j^* = \frac{(v-1)\left[\omega_1 kQ_j + \omega_2 kT_j\right]}{\omega_1 rv(k-1) + \omega_2 kr(v-k)}$$

$$= \frac{(v-1)\left[\omega_1(kV_j - T_j) + \omega_2 kT_j\right]}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]} \qquad \text{(using } Q_j = V_j - \frac{T_j}{k}\text{)}$$

$$= \frac{\omega_1 k(v-1)V_j + (k\omega_2 - \omega_1)(v-1)T_j}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]}$$

$$= \frac{\omega_1 k(v-1)V_j + (\omega_1 - k\omega_2)\left[W_j^* - (v-k)V_j - (k-1)G\right]}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]}$$

$$= \frac{\left[\omega_1 k(v-1) - (\omega_1 - k\omega_2)(v-k)\right]V_j + (\omega_1 - k\omega_2)\left[W_j^* - (k-1)G\right]}{r\left[\omega_1 v(k-1) + \omega_2 k(v-k)\right]}$$

$$= \frac{1}{r}\left[V_j + \frac{\omega_1 - k\omega_2}{\omega_1 v(k-1) + \omega_2 k(v-k)}\left\{W_j^* - (k-1)G\right\}\right]$$

$$= \frac{1}{r}\left[V_j + \xi\left\{W_j^* - (k-1)G\right\}\right]$$

where

$$\xi = \frac{\omega_1 - k\omega_2}{\omega_1 v(k-1) + \omega_2 k(v-k)}$$

$$\omega_1 = \frac{1}{\sigma^2}, \quad \omega_2 = \frac{1}{\sigma_f^2}.$$

Thus the pooled estimate of the contrast $l'\tau$ is

$$l'\tau^* = \sum_j l_j \tau_j^*$$
$$= \frac{1}{r}\sum_j l_j (V_j + \xi W_j^*) \quad \text{(since } \sum_j l_j = 0 \text{ being contrast).}$$

The variance of $l'\tau^*$ is

$$Var(l'\tau^*) = \frac{k}{\lambda v \omega_1 + k(r-\lambda)\omega_2} \sum_j l_j^2$$
$$= \frac{k(v-1)}{r\left[v(k-1\omega_1 + k(v-k)\omega_2\right]} \sum_j l_j^2 \qquad \text{(using } \lambda(v-1) = r(k-1)\text{)}$$
$$= \sigma_E^2 \frac{\sum_j l_j^2}{r}$$

where

$$\sigma_E^2 = \frac{k(v-1)}{v(k-1)\omega_1 + k(v-k)\omega_2}$$

is called as the **effective variance.**

Note that the variance of any elementary contrast based on the pooled estimates of the treatment effects is

$$Var(\tau_i^* - \tau_j^*) = \frac{2}{r}\sigma_E^2.$$