Analysis of Variance and Design of Experiments-II

MODULE V

LECTURE - 22

FRACTIONAL REPLICATIONS

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Consider the set up of complete factorial experiment, say 2^k . If there are four factors, then the total number of plots needed to conduct the experiment is $2^4 = 16$. When the number of factors increases to six, then the required number of plots to conduct the experiment becomes $2^6 = 64$ and so on. Moreover, the number of treatment combinations also become large when the number of factors increases.

Sometimes, it is so large that it becomes practically difficult to organize such a huge experiment. Also, the quantity of experimental material, time, manpower etc., also increase and sometimes even it may not be possible to have so much of resources to conduct a complete factorial experiment. The non-experimental type of errors also enters in the planning and conduct of the experiment. For example, there can be a slip in numbering the treatments or plots or they may be wrongly reported if they are too large in numbers

About the degree of freedoms, in the 2^6 factorial experiment there are $2^6 - 1 = 63$ degrees of freedom, which are divided as 6 for main effects, 15 for two-factor interactions and rest 42 for three or higher order interactions. In case, the higher order interactions are not of much use or much importance, then they can possibly be ignored. The information on main and lower order interaction effects can then be obtained by conducting a fraction of complete factorial experiments. Such experiments are called as **fractional factorial experiments.**

The utility of such experiments becomes more when the experimental process is more influenced and governed by the main and lower order interaction effects rather than the higher order interaction effects. The fractional factorial experiments need less number of plots and lesser experimental material than required in the complete factorial experiments. Hence it involves less cost, less manpower, less time etc.

It is possible to combine the runs of two or more fractional factorials to assemble sequentially a larger experiment to estimate the factor and interaction effects of interest.

To explain the fractional factorial experiment and its related concepts, we consider here examples in the set up of 2^k factorial experiments.

One half fraction of 2^3 factorial experiment with two levels

First we consider the set up of 2^3 factorial experiment and consider its one-half fraction. This is a very simple set up to understand the basics, definitions, terminologies and concepts related to the fractional factorials .

Consider the setup of 2^3 factorial experiment consisting of three factors, each at two levels. There are total 8 treatment combinations involved. So 8 plots are needed to run the complete factorial experiment.

Suppose the material needed to conduct the complete factorial experiment in 8 plots is not available or the cost of total experimental material is too high. The experimenter has material or money which is sufficient only for four plots. So the experimenter decides to have only four runs, i.e., 1/2 fraction of 2^3 factorial experiment. Such an experiment contains one-half fraction of a 2^3 experiment and is called 2^{3-1} factorial experiment. Similarly, $1/2^2$ fraction of 2^3 factorial experiment requires only 2 runs and contains $1/2^2$ fraction of 2^3 factorial experiment and is called as 2^{3-2} factorial experiment. In general, $1/2^p$ fraction of a 2^k factorial experiment requires only 2^{k-p} runs and is denoted as 2^{k-p} factorial experiment.

We consider the case of 1/2 fraction of 2^3 factorial experiment to describe the various issues involved and to develop the concepts. The first question is how to choose four out of eight treatment combinations for conducting the experiment. In order to decide this, first we have to choose an interaction factor which the experimenter feels can be ignored. Generally, this can be a higher order interaction which is usually difficult to interpret. We choose *ABC* in this case. Now we create the table of treatment combinations as in the following Table

Arrangement of treatment combinations for one-half fraction of 2^3 factorial experiment

Factors Treatment combinations	I	A	B	C	AB	AC	BC	ABC
a	+	+	-	-	-	-	+	+
b	+	-	+	-	-	+	-	+
c	+	-	-	+	+	-	-	+
abc	+	+	+	+	+	+	+	+
****************	*******	*******	*******	*******	*******	*******	*******	*******
ab	+	+	+	-	+	-	-	-
ac	+	+	-	+	-	+	-	-
bc	+	+	+	+	-	-	+	-
(1)	+	+	-	-	+	+	+	-

This table is obtained by the following steps

- Write down the factor to be ignored which is *ABC* in our case. We can express *ABC* as

$$ABC = (a+b+c+abc)-(ab+ac+bc+(1)).$$

- Collect the treatment combinations with plus (+) and minus (–) signs together; divide the eight treatment combinations into two groups with respect to the + and – signs. This is done in the last column corresponding to *ABC.*

- Write down the symbols + or – of the other factors *A, B, C, AB, AC* and *BC* corresponding to (*a, b, c, abc*) and (*ab, ac, bc,* (1)).

This provides the arrangement of treatments as given in the table. Now consider the column of *ABC*. The treatment combinations corresponding to + signs of treatment combinations in *ABC* provide one-half fraction of 2^3 factorial experiment.

The remaining treatment combinations corresponding to the – signs in *ABC* will constitute another one-half fractions of 2^3 factorial experiment. Here one of the one-half fractions corresponding to + signs contains the treatment combinations *a, b, c* and *abc*.

Another one-half fraction corresponding to - signs contains the treatment combinations *ab, ac, bc* and (1). Both the one-half fractions are separated by a starred line in the Table.

Generator

The factor which is used to generate the one-half fractions is called as the **generator.** For example, *ABC* is the generator of a fraction in the present case and we have two one-half fractions.

Defining relation

The defining relation for a fractional factorial is the set of all columns that are equal to the identity column *I.* The identity column *I* always contains all the + signs. So in our case, *I* = *ABC* is called the **defining relation** of this fractional factorial experiment.

The number of degrees of freedom associated with one-half fraction of 2^3 factorial experiment, i.e., 2^{3-1} factorial experiment is 3 which is essentially used to estimate the main effects.

Now consider the one-half fraction containing the treatment combinations *a, b, c* and *abc* (corresponding to + signs in the column of *ABC*).

The factors *A, B, C, AB, AC* and *BC* are now estimated from this block as follows:

$$A = a - b - c + abc,$$

$$B = -a + b - c + abc,$$

$$C = -a - b + c + abc,$$

$$AB = -a - b + c + abc,$$

$$AC = -a + b - c + abc,$$

$$BC = a - b - c + abc.$$

Aliases

We notice that the estimates of *A* and *BC* are the same. So it is not possible to differentiate between whether *A* is being estimated or *BC* is being estimated. As such, $A = BC$. Similarly, the estimates of *B* and of *AC* as well as the estimates of *C* and of *AB* are also same. We write this as $B = AC$ and $C = AB$. So it is not possible to differentiate between *B* and *AC* as well as between *C* and *AB* in the sense that which one is being estimated. Two or more effects that have this property are called **aliases**.

Thus

- *A* and *BC* are aliases,
- *B* and *AC* are aliases and
- *C* and *AB* are aliases.

Note that the estimates of *A*, *B*, *C*, *AB*, *BC*, *AC* and *ABC* are obtained is one-half fraction set up. These estimates can also be obtained from the complete factorial set up. A question arises that how the estimate of an effect in the two different set ups are related?

The answer is as follows:

In fact, when we estimate *A, B* and *C* in 2^{3-1} factorial experiment, then we are essentially estimating $A + BC$, $B + AC$ and $C + AB$, respectively in a complete 2^3 factorial experiment. To understand this, consider the setup of complete 2^3 factorial experiment in which *A* and *BC* are estimated by

$$A = -(1) + a - b + ab - c + ac - bc + abc,$$
$$BC = (1) + a - b - ab - c - ac + bc + abc.$$

Adding A and BC and ignoring the common multiplier, we have

$$A + BC = a - b - c + abc$$

which is the same as A or BC is a one-half fraction with $I = ABC$.

Similarly, considering the estimates of B and AC in 2^3 factorial experiment, adding them together and ignoring the common multiplier, we have

$$B = -(1) - a + b + ab - c - ac + bc + abc,$$
$$AC = (1) - a + b - ab + ac - bc + abc,$$
$$B + AC = -a + b - c + abc,$$

which is the same B or AC in one half fraction with $I = ABC$

The estimates of C and AB in 2^3 factorial experiment and their sum is as follows:

$$C = -(1) - a - b - ab + c + ac + bc + abc,$$
$$AB = (1) - a - b + ab + c - ac - bc + abc,$$
$$C + AB = -a - b - c + abc,$$

which is the same as C or AB in one half fraction with $I = ABC$.