Analysis of Variance and Design of Experiments-II

MODULE VI

LECTURE - 25

SPLIT-PLOT AND STRIP-PLOT DESIGNS

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Standard errors of interaction contrasts

When the interaction between whole-plot treatment and split-plot treatment is significant, then first consider the standard errors for contrasts among the split-plot treatment levels at a given whole-plot treatment level. Such contrasts is $\sum_k c_k \bar{y}_{iok}$ for any specific choice, e.g., contrast of quadratic effect of split-plot factor *B* with whole-plot factor *A* at a specific level.

Then

$$
\begin{aligned}
E\left[\sum_k c_k \bar{y}_{iok}\right] &= \sum_k c_k E\left[\bar{y}_{iok}\right] \\
&= \sum_k c_k \left(s_k + (w \times s)_{ik}\right).
\end{aligned}
$$

and

$$
\begin{aligned}
Var\left[\sum_k c_k \bar{y}_{iok}\right] &= Var\left[\sum_k c_k \left(\frac{\sum_j \varepsilon(2)_{ijk}}{r}\right)\right] \\
&= \frac{\sum_k c_k^2 \sigma_2^2}{r}.
\end{aligned}
$$

It follows that, in general,

$$
s.e\left(\sum_k c_k \bar{y}_{iok}\right) = \sqrt{\frac{\sum_k c_k^2 MSE(2)}{rt}}
$$

and the standard error of the difference between two split-plot treatment means at a given whole-plot treatment level is given by

$$
s.e(\bar{y}_{iok} - \bar{y}_{iok'}) = \sqrt{\frac{2MSE(2)}{rt}}.
$$

Also, the contrasts among whole-plot treatment levels for the same split-plot treatment level or at different split-plot treatment levels can also be investigated. These contrasts are of the form $\sum_i \sum_k c_{ik} \bar{y}_{iok}$, e.g., contrast of the quadratic effect of whole-plot factor at a given level of split-plot factor, contrasts of the quadratic effects of whole-plot factor at two different split-plot factor levels. In this case

$$E\left[\sum_i \sum_k c_{ik} \bar{y}_{ik}\right] = \sum_i \sum_k c_{ik} w_k + \sum_i \sum_k c_{ik} (w \times s)_{ik}.$$

and

$$Var\left[\sum_i \sum_k c_{ik} \bar{y}_{iok}\right] = Var\left[\sum_i \sum_k c_{ik} \left(\frac{\sum_j \varepsilon(1)_{ij}}{r}\right) + \sum_i \sum_k c_{ik} \left(\frac{\sum_j \varepsilon(2)_{ijk}}{r}\right)\right]$$

$$= \frac{1}{r} \sum_i \sum_k c_{ik}^2 \sigma_1^2 + \frac{1}{r} \sum_i \sum_k c_{ik}^2 \sigma_2^2$$

$$= \frac{1}{r} \sum_i \sum_k c_{ik}^2 \left(\sigma_1^2 + \sigma_2^2\right).$$

Note that we have separate unbiased estimators of σ_1^2 and σ_2^2 but not for $(\sigma_1^2 + \sigma_2^2)$ in the analysis of variance.

However, if we consider the weighted mean of $MSE(2)$ and $MSE(1)$, then we have an unbiased estimator of $\sigma_1^2 + \sigma_2^2$ as follows:

$$E\left[\frac{(s-1)MSE(2)+MSE(1)}{s}\right] = \sigma_1^2 + \sigma_2^2$$

and then the standard error can be obtained as the positive square root of

$$\widehat{Var}\left[\sum_i \sum_k c_{ik}\bar{y}_{iok}\right] = \sum_i \sum_k \frac{c_{ik}^2}{r}\left[\frac{(s-1)MSE(2)+MSE(1)}{s}\right]. \qquad (1)$$

The estimate of standard errors of the differences between the two whole-plot treatment means at either the same split-plot treatment level or at the two different split-plot treatment levels, e.g., $\bar{y}_{1o1} - \bar{y}_{2o2}$, can be obtained as

$$\sqrt{\frac{2}{r}\left[\frac{(s-1)MSE(2)+MSE(1)}{s}\right]}$$

It is difficult to find the exact number of associated degrees of freedom in such an estimate which is obtained as the weighted mean of the two *MSE*s. The approximate degrees of freedom in such cases can be obtained through Satterthwaite's approach given as follows:

Satterthwaite's approach

Note that the expression (1) can be expressed in the general form

$$\sum_{i}^{m} a_i MSE(i).$$

This expression can be viewed as approximately distributed χ^2 random variable with

$$\frac{\left[\sum_{i}^{m} a_i MSE(i)\right]^2}{\sum_{i}^{m} \frac{\left[a_i MSE(i)\right]^2}{df_i}}$$

degrees of freedom where *MSE*(*i*) has df_i degrees of freedom. If we consider $m = 2, a_1 = (s-1), a_2 = 1, df_1 = (r-1)(t-1)-1,$ and $df_2 = t(r-1)(s-1)-1$ in this expression, we have the expression of variance in (1).

Variance components, variance of *y*, and variance of treatment means

The variance of *y* depends on the variance among whole-plots and can be expressed as $Var\left[y_{ijk}\right] = \sigma_1^2 + \sigma_2^2$ where σ_1^2 is the variance among whole-plots and σ_2^2 is the variance among split-plots within a whole-plot. An unbiased estimator of σ_1^2 is then given by

$$\hat{\sigma}_1^2 = \frac{MSE(1) - MSE(2)}{s}.$$

The variability of whole-plots also contributes to the variance of a treatment mean as

$$Var\left[\bar{y}_{iok}\right] = \frac{\sigma_1^2 + \sigma_2^2}{r}.$$

Error (2) bigger than Error (1)

It is observed from the model $y_{ijk} = \mu + w_i + \varepsilon(1)_{ij} + s_k + (w \times s)_{ik} + \varepsilon(2)_{ijk}$ and also from the $E(MS)$ that error(1) $\geq$ error (2). The split-plot experiment design is based on this basis. In some practical situations, it may happen that the estimate $MSE(2)$ is larger than the estimate $MSE(1)$.

A question arises how to handle such situation in the statistical analysis.

Various solutions are available in the literature and there is no unique answer. In some practical situations, one option may be to replace $MSE(1)$ by $MSE(2)$.

Another option is to just ignore the whole plots and utilize factorial experiment. This amounts to simply pooling the two error terms.

Both of these strategies entail a shift in the model. One model provides the basis for the construction of the plan and the randomizations. Then different model is used for the final statistical analysis.

A general acceptable view is to use the original model which is based on randomization and complete the analysis. Just accept the fact that estimated standard error of split-plot treatment differences are larger than the standard errors of whole-plot treatment differences. If this problem occurs frequently, then one needs to be concerned about this. There can be various reasons for this. It may be possible that any assumption is violated, the randomization is incorrect, there may be negative correlations within whole-plots, or some unknown interaction may be present, etc.

Split-Plot experiment with whole-plots in an RBD

Statistical Model

Now we consider the split-plot experiment with whole plots in the set up of a randomized block design.

The whole-plots are organized into *r* blocks. There are two strata and two randomizations. The model for the whole-plot stratum will now have block effects and the model is given as

$$y_{hik} = \mu + b_h + w_i + \varepsilon(1)_{hi} + s_k + (w \times s)_{ik} + \varepsilon(2)_{hik},$$

where b_h's denote the block effects as the differences among blocks. The block effects can be fixed as well as random. If the block effects are assumed to be random, then they are assumed to be identically and independently distributed with mean 0 and variance σ_h^2. It is also assumed that $b_h, \varepsilon(1)_{hi},$ and $\varepsilon(2)_{hik}$ are all mutually uncorrelated. We consider the case when the block effects are random.

Analysis of variance

The analysis of variance table in this case is shown in following table:

Analysis of variance for a split-plot experiment with whole-plots in an RBD

Source	Degrees of freedom	Sum of squares	Mean squares	$E(MS)$	F
Blocks	$r-1$	$st\sum_{h}^{r}(\bar{y}_{hoo}-\bar{y}_{ooo})^2$		$\sigma_2^2+s\sigma_1^2+st\sigma_b^2$	
W	$t-1$	$rs\sum_{i}^{t}(\bar{y}_{oio}-\bar{y}_{ooo})^2$	MSW	$\sigma_2^2+s\sigma_1^2+rs\phi_w$	$\frac{MSW}{MSE(1)}$
Error(1)	$(r-1)(t-1)$	$s\sum_{h}^{r}\sum_{i}^{t}(\bar{y}_{hio}-\bar{y}_{hoo}-\bar{y}_{oio}+\bar{y}_{ooo})^2$	$MSS(1)$	$\sigma_2^2+s\sigma_1^2$	
S	$s-1$	$rt\sum_{k}^{s}(\bar{y}_{ook}-\bar{y}_{ooo})^2$	MSS	$\sigma_2^2+rt\phi_s$	$\frac{MSS}{MSE(2)}$
$W\times S$	$(t-1)(s-1)$	$r\sum_{i}^{t}\sum_{k}^{s}(\bar{y}_{oik}-\bar{y}_{oio}-\bar{y}_{ook}+\bar{y}_{ooo})^2$	$MS(W\times S)$	$\sigma_2^2+rs\phi_{w\times s}$	$\frac{MS(W\times S)}{MSE(2)}$
Error(2)	$(r-1)t(s-1)$	$\sum_{h}^{r}\sum_{i}^{t}\sum_{k}^{s}(\bar{y}_{hik}-\bar{y}_{hio}-\bar{y}_{hok}+\bar{y}_{hoo})^2$	$MSE(2)$	σ_2^2	
Total (corrected)	$rts-1$	$\sum_{h}^{r}\sum_{i}^{t}\sum_{k}^{s}(\bar{y}_{hik}-\bar{y}_{ooo})^2$			

where

$$\phi_w = \frac{\sum_j w_j^2}{t-1}$$

$$\phi_s = \frac{\sum_k s_k^2}{s-1}$$

$$\phi_{w\times s} = \frac{\sum_j \sum_k (W \times S)_{jk}^2}{(t-1)(s-1)}.$$

Notice also that the error(1) sum of squares is usually calculated as

$$SSE(1) = s\sum_h^r \sum_i^t \bar{y}_{hio}^2 - rts\ \bar{y}_{ooo}^2 - SS\ blocks - SSW$$

and *SSE*(2) is calculated by subtraction.

In this case, the whole-plot factor is tested using *MSW / MSE*(1). The difference among the split-plot treatment means is tested using the *MSE*(2). Formally, the *SSE*(1) or whole-plot error sum of squares can be thought as the interaction of whole-plot treatments and blocks. The error is considered as the inability of the treatments to perform identically across blocks. The split-plot error is thought of comprising two parts the interaction of split-plot treatments and blocks with $(s-1)(r-1)$ degrees of freedom and the three-way interaction of whole-plots, split-plots, and blocks with $(t-1)\ (s-1)(r-1)$ degrees of freedom.

Standard errors of contrasts

The standard errors for treatment and interaction contrasts are the same as for the CRD as described earlier.

Split-plot model in the mixed model framework

The assignment of whole-plot factors to plots and split-plot factors to split plots within the whole-plots introduces a correlated error structure. This error structure is similar to that for randomized block experiments with random block effects. In a split-plot experiment where the whole-plots arranged as per CRD, under the model

$$y_{ijk} = \mu + w_i + \varepsilon(1)_{ij} + s_k + (w \times s)_{ik} + \varepsilon(2)_{ijk},$$

we have

$$\begin{aligned} Cov\left[y_{ijk}, y_{ijk'}\right] &= Cov\left[\varepsilon(1)_{ij} + \varepsilon(2)_{ijk}, \varepsilon(1)_{ij} + \varepsilon(2)_{ijk'}\right] \\ &= \sigma_1^2, \end{aligned}$$

which is the covariance for the two split-plot observations within the same whole-plot and two different split-plots are indicated by $k \neq k'$.

On the other hand, the correlation between the two split-plot observations within the same whole-plot is given by

$$Corr\left[y_{ijk}, y_{ijk'}\right] = \frac{\sigma^2}{\sigma_1^2 + \sigma_2^2}$$

which is based on the fact that the covariance between the two split-plot observations on different whole-plots is zero and the variance of an observation is $\sigma_1^2 + \sigma_2^2$.

This correlation structure underlies the form for the analysis of variance.

The random blocks of whole-plots an RBD also introduce the correlation structure in which the covariances are given by

$$Cov\left[y_{hik}, y_{h'i'k'}\right] = \begin{cases} 0 & \text{if } h \neq h' \\ \sigma_b^2 & \text{if } h = h' \text{ and } i \neq i' \\ \sigma_b^2 + \sigma_1^2 & \text{if } h = h' \text{ and } i = i'. \end{cases}$$

These covariance are based on the assumption that the observations in different blocks are uncorrelated and observations within blocks are correlated as well as the observations in the same whole-plot are more highly correlated. The expressions derived for $E(MS)$ and F remains valid when no observation is missing. If some observations are missing, then these forms are complex.