Analysis of Variance and Design of Experiments-II

MODULE - III

LECTURE - 15

PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN (PBIBD)

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The balanced incomplete block designs have several advantages. They are connected designs as well as the block sizes are also equal. A restriction on using the BIBD is that they are not available for all parameter combinations. They exist only for certain parameters. Sometimes, they require large number of replications also. This hampers the utility of the BIBDs. For example, if there are v = 8 treatments and block size is k = 3 (i.e., 3 plots is each block) then the total number of required blocks are $b = \binom{8}{3} = 56$ and so using the relationship $bk = vr$, the total number of required replicates is

$$r = \frac{bk}{v} = 21.$$

Another important property of the BIBD is that it is efficiency balanced. This means that all the treatment differences are estimated with the same accuracy. The partially balanced incomplete block designs (PBIBD) compromise on this property upto some extent and help in reducing the number of replications. In simple words, the pairs of treatments can be arranged in different sets such that the difference between the treatment effects of a pair, for all pairs in a set, is estimated with the same accuracy. The partially balanced incomplete block designs remain connected like BIBD but no more balanced. Rather they are partially balanced in the sense that some pairs of treatments have same efficiency whereas some other pairs of treatments have the same efficiency but different from the efficiency of earlier pairs of treatments. This will be illustrated more clearly in the further discussion.

Before describing the set up of PBIBD, first we need to understand the concept of **"Association Scheme".** Instead of explaining the theory related to the association schemes, we consider here some examples and then understand the concept of association scheme. Let there be a set of v treatments. These treatments are denoted by the symbols 1, 2,…,v.

Partially balanced association schemes

A relationship satisfying the following three conditions is called a partially balanced association scheme with m - associate classes.

i.   Any two symbols are either first, second,…, or m^{th} associates and the relation of associations is symmetrical, i.e., if the treatment A is the ith associate of treatment B, then B is also the ith associate of treatment A.

ii. Each treatment A in the set has exactly n_i treatments in the set which are the ith associate and the number $n_i (i = 1, 2,…, m)$ does not depend on the treatment A.

iii. If any two treatments A and B are the ith associates, then the number of treatments which are both j^{th} associate and k^{th} associate of B is p_{jk}^{i} and is independent of the pair of ith associates A and B.

The numbers $v, n_1, n_2,..., n_m, p_{jk}^{i} \ (i, j, k = 1, 2,..., m)$ are called the **parameters of m-associate partially balanced scheme**.

We consider now the examples based on rectangular and triangular association schemes to understand the conditions stated in the partially balanced association scheme.

Rectangular association scheme

Consider an example of $m = 3$ associate classes. Let there be six treatments denoted as 1, 2, 3, 4, 5 and 6. Suppose these treatments are arranged as follows:

1	2	3
4	5	6

Under this arrangement, with respect to each symbol, the

- two other symbols in the same row are the first associates.

- One another symbol in the same column is the second associate and

- remaining two symbols are in the other row are the third associates.

For example, with respect to treatment 1,

- treatments 2 and 3 occur in the same row, so they are the first associates of treatment 1,

- treatment 4 occurs in the same column, so it is the second associate of treatment 1 and

- the remaining treatments 5 and 6 are the third associates of treatment 1 as they occur in the other (second) row.

Similarly, for treatment 5,

- treatments 4 and 6 occur in the same row, so they are the first associates of treatment 5,

- treatment 2 occurs in the same column, so it is the second associate of treatment 5 and

- remaining treatments 1 and 3 are in the other (second) row, so they are the third associates of treatment 5.

The table below describes the first, second and third associates of all the six treatments.

Treatment number	First associates	Second associates	Third associates
1	2, 3	4	5, 6
2	1, 3	5	4, 6
3	1, 2	6	4, 5
4	5, 6	1	2, 3
5	4, 6	2	1, 3
6	4, 5	3	1, 2

Further, we observe that for the treatment 1, the

❖ number of first associates (n_1) = 2,

❖ number of second associates (n_2) = 1, and

❖ number of third associates (n_3) = 2.

The same values of n_1, n_2, and n_3 hold true for other treatments also.

Now we understand the condition (iii) of definition of partially balanced association schemes related to p^i_{jk}.

Consider the treatments 1 and 2. They are the first associates (which means $i = 1$), i.e., treatments 1 and 2 are the first associate of each other; treatment 6 is the third associate (which means $j = 3$) of treatment 1 and also the third associate (which means $k = 3$) of treatment 2. Thus the number of treatments which are both, i.e., the jth ($j = 3$) associate of treatment A (here $A \equiv 1$) and k^{th} associate of treatment B (here $B \equiv 2$) are i^{th} (i.e., $i = 1$) associate is $p^i_{jk} = p^1_{33} = 1$.

Similarly, consider the treatments 2 and 3 which are the first associate (which means $i = 1$); treatment 4 is the third (which means $j = 3$) associate of treatment 2 and treatment 4 is also the third (which means $k = 3$) associate of treatment 3. Thus $p^1_{33} = 1$. Other values of p^i_{jk} $(i, j, k = 1, 2, 3)$ can also be obtained similarly.

Remark: This method can be used to generate 3-class association scheme in general for m x n treatments (symbols) by arranging them in m-row and n-columns.

Triangular association scheme

The triangular association scheme gives rise to a 2-class association scheme. Let there be a set of v treatments which are denoted as 1, 2,…, v. The treatments in this scheme are arranged in q rows and q columns where.

$$v = \binom{q}{2} = \frac{q(q-1)}{2}.$$

These symbols are arranged as follows:

a)   Positions in leading diagonals are left blank (or crossed).

b)   The $\binom{q}{2}$ positions are filled up in the positions above the principal diagonal by the treatment numbers 1, 2,.., v corresponding to the symbols.

a)   Fill the positions below the principal diagonal symmetrically.

This assignment is shown in the following table:

Rows → / Columns	1	2	3	4		q - 1	q
1	×	1	2	3	• • •	q - 2	q - 1
2	1	×	q	q + 1	• • •	2q - 2	2q - 1
3	2	q	×	• • •	• • •	• • •	• • •
4	3	q + 1	• • •	• • •	• • •	• • •	• • •
⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮
q - 1	q - 2	2q - 2	• • •	• • •	• • •	×	q(q - 1)/2
q	q - 1	2q - 1	• • •	• • •	• • •	q(q - 1)/2	×

Now based on this arrangement, we define the first and second associates of the treatments as follows. The symbols entering in the same column $i(i = 1, 2, ..., q)$ are the first associates of i and rest are the second associates.

Thus two treatments in the same row or in the same column are the first associates of treatment i. Two treatments which do not occur in the same row or in the same column are the second associates of treatment i.

Now we illustrate this arrangement by the following example:

Let there be 10 treatments. Then $q = 5$ as $v = \binom{5}{2} = 10$. The ten treatments denoted as 1, 2,…, 10 are arranged under the triangular association scheme as follows:

Rows $\longrightarrow$	1	2	3	4	5
Columns $\downarrow$					
1	$\times$	1	2	3	4
2	1	$\times$	5	6	7
3	2	5	$\times$	8	9
4	3	6	8	$\times$	10
5	4	7	9	10	$\times$

For example,

- for treatment 1,

    o the treatments 2, 3 and 4 occur in the same row (or same column) and

    o treatments 5, 6 and 7 occur in the same column (or same row).

    So the treatments 2, 3, 4, 5, 6 and 7 are the first associates of treatment.

- Then rest of the treatments 8, 9 and 10 are the second associates of treatment 1.

The first and second associates of the other treatments are stated in the following table.

Treatment number	First associates		Second associates
1	2, 3, 4	5, 6, 7	8, 9, 10
2	1, 3, 4	5, 8, 9	6, 7, 10
3	1, 2, 4	6, 8, 10	5, 7, 9
4	1, 2, 3	7, 9, 10	5, 6, 8
5	1, 6, 7	2, 8, 9	3, 4, 10
6	1, 5, 7	3, 8, 10	2, 4, 9
7	1, 5, 6	4, 9, 10	2, 3, 8
8	2, 5, 9	3, 6, 10	1, 4, 7
9	2, 5, 8	4, 7, 10	1, 3, 6
10	3, 6, 8	4, 7, 9	1, 2, 5

We observe from this table that the number of first and second associates of each of the 10 treatments $(v = 10)$ is the same with $n_1 = 6$, $n_2 = 3$ and $n_1 + n_2 = 9 = v - 1$. For example, the treatment 2 in the column of first associates occurs six times, viz., in first, third, fourth, fifth, eighth and ninth rows. Similarly, the treatment 2 in the column of second associates occurs three times, viz., in the sixth, seventh and tenth rows. Similar conclusions can be verified for other treatments also.

There are six parameters, viz., $p_{11}^1, p_{22}^1, p_{12}^1$ (or p_{21}^1), p_{11}^2, p_{22}^2 and p_{12}^2 (or p_{21}^2) which can be arranged in symmetric matrices P_1 and P_2 as follows:

$$P_1 = \begin{bmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^2 & p_{22}^2 \end{bmatrix}.$$

[Note: We would like to caution the reader not to read p_{11}^2 as squares of p_{11} but 2 in p_{11}^2 is only a superscript.]

$$P_1 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}.$$