Analysis of Variance and Design of Experiments-II

MODULE VI

LECTURE - 28

SPLIT-PLOT AND STRIP-PLOT DESIGNS

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Treatment contrasts: Main effect

The usefulness of having covariate in the model provides more accurate and more precise estimates of treatment contrasts. Now adjust for the covariate to remove biases attributable to differences among experimental units. Now reduce the bias by adjusting treatment means and contrasts.

Consider the whole-plot part of the design. Consider the model given in equation (2) which is as follows:

$$\begin{aligned} y_{hij} &= \mu + \beta^w \bar{x}_{ooo} + r_h + \beta^w(\bar{x}_{hoo} - \bar{x}_{ooo}) + w_i + \beta^w(\bar{x}_{oio} - \bar{x}_{ooo}) + \beta^w(\bar{x}_{hio} - \bar{x}_{hoo} - \bar{x}_{oio} + \bar{x}_{ooo}) + \varepsilon(1)_{hi} + s_j \\ &\quad + \beta^s(\bar{x}_{ooj} - \bar{x}_{ooo}) + (w \times s)_{ij} + \beta^s(\bar{x}_{oij} - \bar{x}_{oio} - \bar{x}_{ooj} + \bar{x}_{ooo}) + \beta^s(x_{hij} - \bar{x}_{hio} - \bar{x}_{oij} + \bar{x}_{oio}) + \varepsilon(2)_{hij} \qquad (2) \\ &= \mu^* + r_h^* + w_i^* + \beta^w(\bar{x}_{hio} - \bar{x}_{hoo} - \bar{x}_{oio} + \bar{x}_{ooo}) + \varepsilon(1)_{hi} + s_j^* + (w \times s)_{ij}^* + \beta^s(x_{hij} - \bar{x}_{hio} - \bar{x}_{oij} + \bar{x}_{oio}) + \varepsilon(2)_{hij}, \end{aligned}$$

where

$$\mu* = \mu + \beta^w \bar{x}_{ooo}$$

$$r_h^* = r_h + \beta^w(\bar{x}_{hoo} - \bar{x}_{ooo})$$

$$w_i^* = w_i + \beta^w(\bar{x}_{oio} - \bar{x}_{ooo})$$

$$s_j^* = s_j + \beta^s(\bar{x}_{ooj} - \bar{x}_{ooo})$$

$$(w \times s)_{jk}^* = (w \times s)_{ij} + \beta^s(\bar{x}_{oij} - \bar{x}_{oio} - \bar{x}_{ooj} + \bar{x}_{ooo}).$$

The unbiased estimates of μ^* and w_i^* are obtained from this model as follows:

$$\bar{y}_{oio} = \hat{\mu}^* + \hat{w}_i^*$$

and

$$\sum_i c_i \bar{y}_{oio} = \sum_i c_i \hat{w}_i^*$$

with $\sum_i c_i = 0.$ But the unbiased estimates of contrasts of the form $\sum_i c_i w_i$ are needed instead of these contrasts. Using the definitions implied in equation (2), the unbiased estimates are given as

$$\bar{y}_{oio} = \hat{\mu} + \hat{\beta}^w \bar{x}_{ooo} + \hat{w}_i + \hat{\beta}^w (\bar{x}_{oio} - \bar{x}_{ooo})$$

and

$$\sum_i c_i \bar{y}_{oio} = \sum_i c_i \hat{w}_i + \sum_i c_i \hat{\beta}^w \bar{x}_{oioo}.$$

Since x_{hij}'s are the observed constants, so

$$\hat{\mu} + \hat{w}_i = \bar{y}_{oio} - \hat{\beta}^w \bar{x}_{oio}$$

and

$$\sum_i c_i \hat{w}_i = \sum_i c_i \bar{y}_{oio} - \sum_i c_i \hat{\beta}^w x_{oio}.$$

These are the required contrasts as the adjusted main effect contrasts.

Further,

$$Var\left[\sum_i c_i \hat{w}_i\right] = \left[\sum_i \frac{c_i^2}{r} + \frac{(\sum_i c_i \overline{x}_{oio})^2}{E(1)_{x_w x_w}}\right] \frac{(\sigma_2^2 + s\sigma_1^2)}{s}$$

$E\left[MSE(1)^a\right] = \sigma_2^2 + s\sigma_1^2$ and $MSE(1)^a$ has $(r-1)(t-1)-1$ degrees of freedom.

Comparisons among the split-plot treatment levels are similar. Then

$$\overline{y}_{ooj} = \hat{\mu}* + \hat{s}_j^*$$

and

$$\sum_j c_j \overline{y}_{ooj} = \sum_j c_j \hat{s}_j^*.$$

The unbiased estimates of the contrasts of the form $\sum_j c_j s_j$ are needed. Using the definitions in equation (2), the unbiased estimates are given by

$$\overline{y}_{ooj} = \hat{\mu} + \hat{\beta}^w \overline{x}_{ooo} + \hat{s}_j + \hat{\beta}^s (\overline{x}_{ooj} - \overline{x}_{ooo})$$

and

$$\sum_j c_j \overline{y}_{ooj} = \sum_j c_j \hat{s}_j + \sum_j c_j \hat{\beta}^s \overline{x}_{ooj}.$$

These expressions can be re-expressed as $\hat{\mu} + \hat{s}_j = \overline{y}_{ooj} - (\hat{\beta}_s \overline{x}_{ooj} - \overline{x}_{ooo}) - \hat{\beta}^w \hat{x}_{ooo}$ and $\sum_j c_j \hat{s}_j = \sum_j c_j \overline{y}_{ooj} - \sum_j c_j \hat{\beta}^s \overline{x}_{ooj}.$

Also

$$Var\left[\sum_j c_j \hat{s}_j\right] = \left[\sum_j \frac{c_j^2}{rt} + \frac{\left(\sum_j c_j \overline{x}_{ooj}\right)^2}{E(2)_{x_s x_s}}\right] \sigma_2^2.$$

$E\left[MSE(2)^a\right] = \sigma_2^2$ and $MSE(2)^a$ has $(r-1)t(s-1)-1$ degrees of freedom.

Interaction contrasts

Interaction contrasts are of the form $\sum_{ij} c_{ij}\overline{y}_{oij}$ with $\sum_{ij} c_{ij}=0.$ The standard errors of such contrasts depend on the nature of the contrasts.

Consider the contrasts among split-plot levels for a fixed whole-plot level. These are of the form $\sum_{ij} c_{ij}\overline{y}_{oij}$ with $\sum_{ij} c_{ij}=0$ with a specific *i* value. Then

$$E\left[\sum_j c_{ij}\overline{y}_{oij}\right]=\sum_j c_{ij}(s_j^*+(w\times s)_{ij}^*)$$
$$=\sum_j c_{ij}(s_j^*+(w\times s)_{ij}+\beta^s)(x_{oij}-\overline{x}_{oio})).$$

The adjusted contrast is

$$\sum_j c_{ij}(\hat{s}_j+(\widehat{w\times s})_{ij})=\sum_j c_{ij}\overline{y}_{oij}-\hat{\beta}^s(x_{oij}-\overline{x}_{oio})$$

and its variance is

$$\sum_j c_{ij}^2\left[\frac{1}{r}-\frac{(\overline{x}_{oij}-\overline{x}_{oio})^2}{E(2)_{x_sx_s}}\right]\sigma_2^2.$$

Next consider the contrasts among whole-plot treatment levels, either at the same split-plot treatment level or at the different split-plot treatment levels. These contrasts take the general form $\sum_i\sum_j c_{ij}\overline{y}_{oij}$, where $\sum_i\sum_j c_{ij}=0.$ Then

$$E\left[\sum_i\sum_j c_{ij}\overline{y}_{oij}\right]=\sum_i\sum_j c_{ij}(w_i^*+s_j^*+(w\times s)_{ij}^*)$$
$$=\sum_i\sum_j c_{ij}(w_i+\beta^w(\overline{x}_{oio}-\overline{x}_{ooo})+s_j+(w\times s)_{ij}+\beta^s(\overline{x}_{oij}-\overline{x}_{oio})).$$

The adjusted contrast is

$$\sum_i \sum_j c_{ij}(\hat{w}_i + \hat{s}_j + \widehat{(w \times s)}_{ij}) = \sum_i \sum_j c_{ij}(y_{oij} - \hat{\beta}^w(\overline{x}_{oio} - \overline{x}_{ooo}) - \hat{\beta}^s(\overline{x}_{oij} - \overline{x}_{oio}))$$

and has variance

$$\sum_i \sum_j c_{ij}^2 \left[\frac{(\sigma_1^2 + \sigma_2^2)}{r} + (\sigma_1^2 + s\sigma_2^2)\frac{(\overline{x}_{oio} - \overline{x}_{ooo})^2}{E(1)_{x_w x_w}} + \sigma_2^2 \frac{(\overline{x}_{oij} - \overline{x}_{oio})^2}{E(2)_{x_s x_s}} \right].$$

There is no “nice” estimate of this variance. A moderate simplification is obtained by splitting this in two cases. Consider first the whole-plot contrasts at one split-plot level. This implies that the c_{ij}'s are zero for all $j \neq j'$ and the general adjusted contrast then becomes

$$\sum_i c_{ij'}(\hat{w}_i + \hat{s}_{j'} + \widehat{(w \times s)}_{ij'}) = \sum_i c_{ij'}\left(\overline{y}_{oij'} - \hat{\beta}^s\left(x_{oij'} - \overline{x}_{oio}\right)\right),$$

and its variance becomes

$$\sum_i \sum_j c_{ij'}^2 \left[\frac{(\sigma_1^2 + \sigma_2^2)}{r} + \sigma_2^2 \frac{(\overline{x}_{oij'} - \overline{x}_{oio})^2}{E(2)_{x_s x_s}} \right] = \sum_i \sum_j c_{ij} \frac{(\sigma_1^2 + \sigma_2^2)}{r} + \frac{\sum_i c_{ij'}^2 (\overline{x}_{oij'} - \overline{x}_{oio})^2}{E(2)_{x_s x_s}} \sigma_2^2.$$

An unbiased estimate of this variance is given by

$$\sum_i \sum_j c_{ij'} \left[\frac{s-1}{sr} + \frac{(\overline{x}_{oij'} - \overline{x}_{oio})^2}{E(2)_{x_s x_s}} \right] MSE(2)^a + \sum_i c_{ij'}^2 \left(\frac{1}{sr} \right) MSE(1)^a.$$

It is difficult to find its exact degrees of freedom. The Scatterwaite’s formula is used to approximate the degrees of freedom which is used for the test of hypothesis and confidence interval estimation.

For the more general case, the unbiased estimate of the variance is given by

$$\sum_i \sum_j c_{ij}^2 \left[\frac{s-1}{sr} + \frac{s^2-1}{s}\frac{(\overline{x}_{oio} - \overline{x}_{ooo})^2}{E(1)_{x_w x_w}} + \frac{(\overline{x}_{oij} - \overline{x}_{oio})^2}{E(2)_{x_s x_s}} \right] MSE(2)^a + \sum_i \sum_j c_{ij}^2 \left[\frac{1}{sr} + \frac{(\overline{x}_{oio} - \overline{x}_{ooo})^2}{sE(1)_{x_w x_w}} \right] MSE(1)^a.$$

The exact degree of freedom are difficult to obtain. The approximate degrees of freedom can be obtained using Scatterthwaite’s approximation.

Analysis of covariance with one whole-plot covariate in RBD

We consider illustrate the analysis of covariance for the split-plot experiment with whole-plots in an RBD and one covariate associated with the whole-plots.

Developing the model

We begin with the model

$$y_{hij} = \mu + r_h + w_i + \beta x_{hi} + \varepsilon(1)_{hi} + s_j + (w\times s)_{ij} + \varepsilon(2)_{hij}, \quad h = 1,\ldots,r,\ i = 1,\ldots,t, \text{ and } j = 1,\ldots,s,$$

where $h = 1,2,\ldots,r$; $i = 1,2,\ldots,t$ and $j = 1,2,\ldots,s$, and the covariate is x_{hi}. Assume that both the whole-and split-plot treatments are fixed effects, implying that $\overline{w}_o = \overline{s}_o = \overline{(w\times s)}_{oj} = \overline{(w\times s)}_{io} = 0$. The x_{hi}'s are observed constants. The $\varepsilon(1)_{hi}$ and $\varepsilon(2)_{hij}$ are identically and independently normally distributed each with mean 0 and variances σ_1^2 and σ_2^2 respectively. Moreover they are mutually independent.

Rewrite the model to isolate the covariate's contribution to bias and variance. Using

$$x_{hio} = \overline{x}_{oo} + (\overline{x}_{ho} - \overline{x}_{oo}) + (\overline{x}_{oi} - \overline{x}_{oo}) + (x_{hi} - \overline{x}_{ho} - \overline{x}_{oi} + \overline{x}_{oo}),$$

the model is rewritten as

$$\begin{aligned} y_{hij} &= \mu + \beta\overline{x}_{oo} + r_h + \beta(\overline{x}_{ho} - \overline{x}_{oo}) + w_i + \beta(\overline{x}_{oi} - \overline{x}_{oo}) + \beta(x_{hi} - \overline{x}_{ho} - \overline{x}_{oi} + \overline{x}_{oo}) + \varepsilon(1)_{hi} + s_j + (w\times s)_{ij} + \varepsilon(2)_{hij} \\ &= \mu^* + r_h^* + w_i^* + \beta(x_{hi} - \overline{x}_{ho} - \overline{x}_{oi} + \overline{x}_{oo}) + \varepsilon(1)_{hi} + s_j + (w\times s)_{ij} + \varepsilon(2)_{hij} \end{aligned}$$

where

$$\mu^* = \mu + \beta\overline{x}_{oo},$$

$$r_h^* = r_h + \beta(\overline{x}_{ho} - \overline{x}_{oo}),$$

$$w_i^* = w_i + \beta(\overline{x}_{oi} - \overline{x}_{oo}).$$

The terms in μ^*, r_h^*, and w_i^* represent the contributions to bias from the experimental units via the covariate, and $\beta(x_{hi} - \overline{x}_{ho} - \overline{x}_{oi} + \overline{x})$ represents the contribution to the variance. The analysis of covariance provides adjustments to remove all of these.

Computations in the analysis of covariance

Once the model is constructed, the next step is to construct the compact analysis of covariance table as follows:

Analysis of covariance table for the split-plot with only a whole-plot covariate

Source	$y-$variable	Covariate	
Mean	M_{yy}	M_{xy}	M_{xx}
Blocks	B_{yy}	B_{xy}	B_{xx}
W	W_{yy}	W_{xy}	W_{xx}
$Error(1)$	$E(1)_{yy}$	$E(1)_{xy}$	$E(1)_{xx}$
S	S_{yy}		
$W \times S$	$W \times S_{yy}$		
$Error(2)$	$E(2)_{yy}$		
Total	T_{yy}	T_{xy}	T_{xx}

There are no covariance adjustments in the split-plot stratum. In the whole-plot stratum, we have

$$\hat{\beta} = \frac{E(1)_{xy}}{E(1)_{xx}}$$

$$Var(\hat{\beta}) = \frac{(\sigma_2^2 + s\sigma_1^2)}{E(1)_{xx}}$$

$$MSE(1)^a = \frac{1}{[(r-1)(t-1)-1]}\left(E(1)_{yy} - \frac{E^2(1)_{xy}}{E(1)_{xx}} \right)$$

$$MSW^a = \frac{1}{(t-1)}\left(W_{yy} - \frac{[W_{xy} + E(1)_{xy}]^2}{W_{xx} + E(1)_{xx}} + \frac{E(1)_{xy}^2}{E(1)_{xx}} \right)$$

Contrasts

Since there is only a whole-plot covariate, so only the whole-plot treatment contrasts are adjusted. Consider the contrast

$$\sum_i c_i \bar{y}_{oio} = \sum_i c_i \hat{w}_i + \hat{\beta} \sum_i c_i (\bar{x}_{oi} - \bar{x}_{oo})$$

with $\sum_i c_i = 0$. Rewrite this as

$$\sum_i c_i \hat{w}_i = \sum_i c_i \bar{y}_{oio} - \hat{\beta} \sum_i c_i \bar{x}_{oio}.$$

This contrast has variance

$$\left(\sum_i c_i^2 \frac{1}{rs} + \frac{\left(\sum_i c_i \bar{x}_{oi} \right)^2}{E(1)_{xx}} \right) \left(\sigma_2^2 + s\sigma_1^2 \right).$$

The estimate of $(\sigma_2^2 + s\sigma_1^2)$ with $(r-1)(t-1)-1$ degrees of freedom is given by $MSE(1)^a$.