LINEAR REGRESSION ANALYSIS

MODULE – XIV

Lecture - 40

Logistic and Poisson Regression Models

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Logistic regression model

In the linear regression model $y = X\beta + \varepsilon$ there are two types of variables – explanatory variables $X_1, X_2, .., X_k$ and study variable *y*. These variables can be measured on a continuous scale as well as like an indicator variables.

When the explanatory variables are qualitative, then their values are expressed as indicator variables and then dummy variable models are used.

When the study variable is qualitative variable, then its values can be expressed using an indicator variable taking only two possible values 0 and 1. In such a case, the logistic regression is used.

For example, *y* can denotes the values like success or failure, yes or no, like or dislike which can be denoted by two values 0 and 1.

Consider the model

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + ... + \beta_k x_{ik} + \varepsilon_i$$
$$= x_i'\beta + \varepsilon_i, \ i = 1, 2, ..., n$$

where

$$x_i' = [x_{i1}, x_{i2}, ..., x_{ik}],$$
$$\beta' = [\beta_1, \beta_2, ..., \beta_k].$$

Usually $x_{i1} = 1$ for all $i = 1, 2, \ldots, n$ which corresponds to the intercept turn in the model.

The study variable takes two values as $y_i =$ 0 or 1. Assume that y_i follows a Bernoulli distribution with parameter π_i so its probability distribution is

$$y_i = \begin{cases} 1 & \text{with } P(y_i = 1) = \pi_i \\ 0 & \text{with } P(y_i = 0) = 1 - \pi_i. \end{cases}$$

Assuming $E(\varepsilon_i) = 0,$

$$E(y_i) = 1.\pi_i + 0.(1 - \pi_i) = \pi_i.$$

From the model $y_i = x_i'\beta + \varepsilon_i,$ we have

$$\begin{aligned} & E(y_i) = x_i'\beta \\ \Rightarrow\ & E(y_i) = x_i'\beta = \pi_i \\ \Rightarrow\ & E(y_i) = P(y_i = 1). \end{aligned}$$

Thus response function $E(y_i)$ is simply the probability that $y_i = 1$.

Note that $\varepsilon_i = y_i - x_i'\beta,$ so

- when $y_i = 1,$ then $\varepsilon_i = 1 - x_i'\beta$
- when $y_i = 0,$ then $\varepsilon_i = -x_i'\beta.$

Recall that earlier ε_i was assumed to follow a normal distribution when y was not an indicator variable.

When y is an indicator variable, then ε_i takes only two values, so it cannot be assumed to follow a normal distribution.

In usual regression model, the errors are homoskedastic, i.e., $Var(\varepsilon_i) = \sigma^2$ and so $Var(y_i) = \sigma^2$. When y is an indicator variable, then

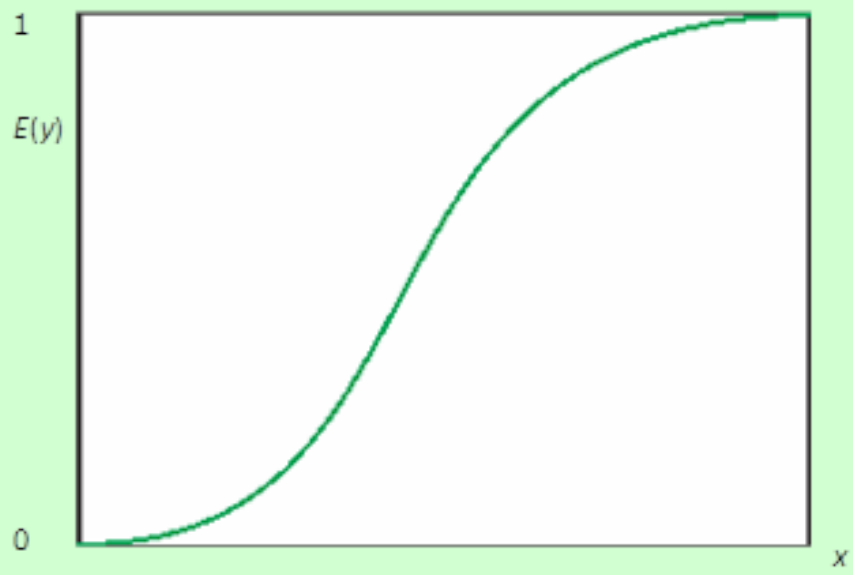
$$
\begin{aligned}
Var(y_i) &= E\left[y_i - E(y_i)\right]^2 \\
&= (1-\pi_i)^2 \pi_i + (0-\pi_i)^2 (1-\pi_i) \\
&= \pi_i (1-\pi_i)\left[1-\pi_i+\pi_i\right] \\
&= \pi_i (1-\pi_i) \\
&= E(y_i)\left[1-E(y_i)\right] \\
&= \sigma^2_{y_i} \ .
\end{aligned}
$$

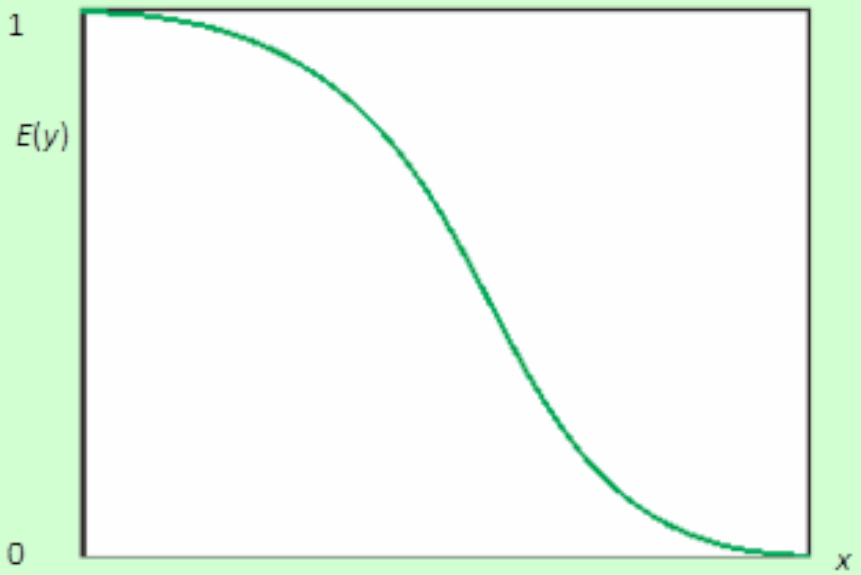
Thus $Var(y_i)$ depends on y_i and is a function mean of y_i. Moreover, since $E(y_i) = \pi_i$ and π_i is the probability, so $0 \le \pi_i \le 1$ and thus there is a constraint on $E(y_i)$ that

$$0 \le E(y_i) \le 1.$$

This puts a big constraint on the choice of linear response function. One cannot fit a model in which the predicted values lie outside the interval of 0 and 1.

When *y* is a dichotomous variable, then empirical evidences suggest that the function *E*(*y*) on the whole real line that can be mapped to [0, 1] has the sigmoid shape. It is a nonlinear *S*-shape like





A natural choice for *E*(*y*) would be the cumulative distribution function of a random variable. In particular, the logistic distribution, whose cumulative distribution function is the simplified logistic function yields a good link and is given by

$$E(y) = \frac{\exp(x'\beta)}{1+\exp(x'\beta)}$$

$$= \frac{1}{1+\exp(-x'\beta)}.$$

Linear predictor and link functions

The systematic component in *E*(*y*) is the linear predictor and is denoted as

$$\eta_i = \sum_j \beta_j x_{ij} = x_i^{'}\beta,\ i = 1,2,...,n,\ j = 1,2,...,k.$$

The link function in generalized linear model relates the linear predictor η_i to the mean response μ_i .

Thus

$$g(\mu_i) = \eta_i$$

or $$\mu_i = g^{-1}(\eta_i).$$

In the usual linear models based on the normally distributed study variable, the link $g(\mu_i) = \mu_i$ is used and is called as **identity link**. A link function maps the range of μ_i onto the whole real line, provides good empirical approximation and carries meaningful interpretations in real applications.

In case of logistic regression, the link function is defined as

$$\eta = \ln \frac{\pi}{1-\pi}.$$

This transformation is called as the **logit** transformation of probability π and $\frac{\pi}{1-\pi}$ is called as **odds**.

The link η is also called as **log-odds**. This link function is obtained as follows:

$$\pi = \frac{1}{1+\exp(-\eta)}$$

or $\pi\left[1+\exp(-\eta)\right]=1$

or $e^{-\eta} = \frac{1-\pi}{\pi}$

or $\eta = \ln\frac{\pi}{1-\pi}.$

Note: Similar to logit function, there are other functions also which have same shape as of logistic function. These functions can also be transformed through π.

There are two such popular functions – probit transformation and complementary log-log transformation. The probit transformation is based on the transformation of π using the cumulative distribution function of normal distribution and based on this is the **probit regression model.**

The **complementary log-log transformation of** π is $\ln\left[-\ln(1-\pi)\right]$.

Maximum likelihood estimation of parameters

Consider the general form of the logistic regression model

$$y_i = E(y_i) + \varepsilon_i$$

where y_i's are independent Bernoulli random variable with parameter π_i with

$$E(y_i) = \pi_i$$
$$= \frac{\exp(x_i^{'}\beta)}{1+\exp(x_i^{i}\beta)}.$$

The probability density function of y_i is

$$f_i(y_i) = \pi_i^{y_i}(1-\pi_i)^{1-y_i}, i = 1,2,...,n,\ y_i = 0 \text{ or } 1.$$

The likelihood function is

$$L(y_1, y_2, ..., y_n, \beta_1, \beta_2, ..., \beta_k) = L = \prod_{i=1}^{n} f_i(y_i)$$
$$= \prod_{i=1}^{n} f_i(y_i)(1-\pi_i)^{1-y_i}$$

$$\ln L = \sum_{i=1}^{n}\left[\ln \pi_i^{y_i} + \ln(1-\pi_i)^{1-y_i}\right]$$
$$= \sum_{i=1}^{n}\left[y_i \ln \pi_i + (1-y_i)\ln(1-\pi_i)\right]$$
$$= \sum_{i=1}^{n}\left[y_i \ln\left(\frac{\pi_i}{1-\pi_i}\right)\right] + \sum_{i=1}^{n}\left[\ln(1-\pi_i)\right].$$

Since

$$\pi_i = \frac{\exp(x_i'\beta)}{1+\exp(x_i'\beta)},$$

$$1-\pi_i = \frac{1}{1+\exp(x_i'\beta)},$$

$$\frac{\pi_i}{1-\pi_i} = \exp(x_i'\beta),$$

$$\ln\frac{\pi_i}{1-\pi_i} = \exp x_i'\beta \ ,$$

so

$$\ln L = \sum_{i=1}^{n} y_i x_i'\beta - \sum_{i=1}^{n} \ln\left[1+\exp(x_i'\beta)\right].$$

Suppose repeated observations are available at each level of the *x*-variables. Let y_i be the numbers of 1's observed for i^{th} observation and n_i be the number of trials at each observation. Then

$$\ln L = \sum_{i=1}^{n} y_i\pi_i + \sum_{i=1}^{n} n_i \ln(1-\pi_i) - \sum_{i=1}^{n} y_i \ln(1-\pi_i).$$

The maximum likelihood estimate $\hat{\beta}$ of β is obtained by the numerical maximization.

If $V(\varepsilon) = \Omega$ is known, then asymptotically

$$E(\hat{\beta}) = \beta$$

$$V(\hat{\beta}) = (X'\Omega^{-1}X)^{-1}.$$

After obtaining $\hat{\beta}$, the linear predictor is estimated by

$$\hat{\eta}_i = x_i'\beta.$$

The fitted value is

$$\begin{aligned}\hat{y}_i = \hat{\pi}_i &= \frac{\exp(\hat{\eta}_i)}{1+\exp(\hat{\eta}_i)} \\ &= \frac{1}{1+\exp(-\hat{\eta}_i)} \\ &= \frac{1}{1+\exp(-x_i'\hat{\beta})}.\end{aligned}$$