LINEAR REGRESSION ANALYSIS

MODULE – XV

Lecture - 42

Generalized Linear Models

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The usual linear regression model assumes normal distribution of study variables whereas nonlinear logistic and Poison regressions are based on Bernoulli and Poisson distributions respectively of study variables. Similar to as in logistic and Poisson regressions, the study variable can follow different probability distributions like exponential, gamma, inverse normal etc. One such family of distribution is described by **exponential family of distributions**. The generalized linear model is based on this distribution and unifies linear and nonlinear regression models. It assumes that the distribution of study variable is a member of exponential family of distribution.

Exponential family of distribution

A random variable *X* belongs to exponential family with single parameter θ has a probability density function

$$f(X,\theta) = \exp\left[a(X)b(\theta) + c(\theta) + d(X)\right]$$

where $a(X), b(\theta), c(\theta)$ and $d(X)$ are all known function.

If $a(X) = X,$ the distribution in said to be in **canonical form**. The function $b(\theta)$ is called the **natural parameter** of the distribution. The parameter θ is of interest and all other parameters which are not of interest are called **nuisance parameters**.

Examples:

(1) Normal distribution

$$f_x(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2\sigma^2}(x-\mu)^2\right]; -\infty < x < \infty; -\infty < \mu < \infty; \sigma^2 > 0$$

$$= \exp\left[x\left(\frac{\mu}{\sigma^2}\right) + \left(-\frac{\mu^2}{2\sigma^2} - \frac{1}{2}\ln 2\pi\sigma^2\right) - \frac{x^2}{2\sigma^2}\right].$$

Here

$$a(x) = x,\ b(\theta) = \frac{\mu}{\sigma^2}.$$

(2) Binomial distribution

$$f(x,p)=\binom{n}{x}p^x(1-p)^{n-x},\ 0<p<1,\ x=0,1,\dots,n$$

$$=\exp\left[x\ln\left(\frac{p}{1-p}\right)+n\ln(1-p)+\ln\binom{n}{x}\right].$$

Here

$$a(x)=x,\ b(\theta)=\ln\left(\frac{p}{1-p}\right).$$

Expected values and variance of *a*(*X*)

The exponential family of distribution for a random variables X and parameter of interest θ is

$$f(X,\theta)=\exp\left[a(X)b(\theta)+c(\theta)+d(X)\right]$$

$$L=\ln f(X,\theta)=a(X)b(\theta)+c(\theta)+d(X).$$

Let $U=\dfrac{dL}{d\theta}$

then for any distribution

$$E(U)=0$$

$$Var(U)=E(U^2)=E(-U')$$

where $U'=\dfrac{dU}{d\theta}$. The function U is called **score** and *Var*(*U*) is called **information.**

The log-likelihood function is

$$L = \ln\left[f(X,\theta)\right] = a(X)b(\theta) + c(\theta) + d(y)$$

and then

$$U = \frac{dL}{d\theta} = a(X)b'(\theta) + c'(\theta)$$

$$U' = \frac{d^2L}{d\theta^2} = a(X)b''(\theta) + c''(\theta)$$

where $b'(\theta) = \frac{db(\theta)}{d\theta}$, $b''(\theta) = \frac{d^2b(\theta)}{d\theta^2}$, $c'(\theta) = \frac{dc(\theta)}{d\theta}$ and $c''(\theta) = \frac{d^2c(\theta)}{d\theta^2}$.

Since $E(U) = 0$, so

$$E(U) = b'(\theta)E\left[a(X)\right] + c'(\theta)$$

$$0 = b'(\theta)E\left[a(X)\right] + c'(\theta)$$

$$\Rightarrow E\left[a(X)\right] = -\frac{c'(\theta)}{b'(\theta)}.$$

Since

$$Var(U) = \left[b'(\theta)\right]^2 Var\left[a(X)\right],$$

$$E(-U') = -b''(\theta)E\left[a(X)\right] - c''(\theta)$$

$$Var(U) = E(-U')$$

$$\Rightarrow Var\left[a(X)\right] = \frac{-b''(\theta)E\left[a(X)\right] - c''(\theta)}{\left[b'(\theta)\right]^2}$$

$$= \frac{b''(\theta)c'(\theta) - c''(\theta)b'(\theta)}{\left[b'(\theta)\right]^3}.$$

Now we consider two examples which illustrate how other distribution and their properties can be obtained as particular cases:

Example: Binomial distribution

Consider X follows a Binomial distribution with parameters n and π , i.e. $X \sim Bin(n,\pi)$. Then in the framework of exponential class of family

$$
\begin{aligned}
f(x,\pi) &= \binom{n}{x}\pi^{x}(1-\pi)^{n-x} \\
&= \exp\left[x\ln\pi - x\ln(1-\pi) + n\ln(1-\pi) + \ln\binom{n}{x}\right].
\end{aligned}
$$

Here
$$a(x)=x,\ \theta=\pi,\ b(\theta)=\ln\frac{\pi}{1-\pi},\ c(\theta)=n\ln(1-\pi),\ \ d(x)=\ln\binom{n}{x}$$

$$L=\ln\ f(x,\pi)=x\ln\pi - x\ln(1-\pi)+n\ln(1-\pi)+\ln\binom{n}{x}.$$

It is the canonical form of $f(x,\pi)$ with natural parameter $\ln\pi$.

$$
\begin{aligned}
U &= \frac{dL}{d\pi} = \frac{x}{\pi} + \frac{x}{1-\pi} - \frac{n}{1-\pi} \\
&= \frac{x}{\pi(1-\pi)} - \frac{n}{1-\pi} \\
&= \frac{x-n\pi}{\pi(1-\pi)}
\end{aligned}
$$

$$
\begin{aligned}
E(U) &= \frac{E(x)-n\pi}{\pi(1-\pi)} = \frac{n\pi-n\pi}{\pi(1-\pi)} \\
&= 0.
\end{aligned}
$$

$$Var(U) = \frac{Var(x)}{\pi^2(1-\pi)^2}$$

$$= \frac{n\pi(1-\pi)}{\pi^2(1-\pi)^2}$$

$$= \frac{n}{\pi(1-\pi)^2}$$

$$E(-U') = E\left[-\frac{(-n)}{\pi(1-\pi)^2}\right]$$

$$= \frac{n}{\pi(1-\pi)}$$

$$b'(\theta) = \frac{1}{\pi(1-\pi)} = b'(\pi)$$

$$b''(\theta) = \frac{2\pi-1}{\left[\pi(1-\pi)\right]^2} = b''(\pi)$$

$$c'(\theta) = -\frac{n}{1-\pi} = c'(\pi).$$

$$c''(\theta) = -\frac{n}{(1-\pi)^2} = c''(\pi).$$

Thus

$$E\left[a(X)\right] = E(X) = -\frac{c'(\pi)}{b'(\pi)} = \pi$$

$$Var\left[a(X)\right] = Var(X)$$

$$= \frac{b''(\pi)c'(\pi) - c''(\pi)b'(\pi)}{\left[b'(\pi)\right]^3} = n\pi(1-\pi).$$

Example: Poisson distribution

Let that the random variable *X* follows a poisson distribution with parameter λ, i.e., $X \sim P(\lambda)$. Then

$$f(x,\lambda) = \frac{\exp(-\lambda)\lambda^x}{x!}$$
$$= \exp\left[x \ln \lambda - \lambda - \ln x!\right]$$
$$L = \ln f(x,\lambda) = x \ln \lambda - \lambda - \ln x! \ .$$

It is the canonical form of $f(x,\lambda)$ and $\ln \lambda$ is the natural parameter. Here

$$a(X) = X,\ b(\theta) = \ln \lambda,\ c(\theta) = -\lambda,\ d(X) = -\ln X!$$

$$U = \frac{dL}{d\lambda} = \frac{x}{\lambda} - 1$$

$$E(U) = \frac{E(x)}{\lambda} - 1$$
$$= \frac{\lambda}{\lambda} - 1$$
$$= 0$$

$$Var(U) = \frac{Var(x)}{\lambda^2}$$
$$= \frac{\lambda}{\lambda^2}$$
$$= \frac{1}{\lambda}$$

$$E(-U') = E\left[-\left\{\frac{d}{d\lambda}\left(\frac{x}{\lambda}-1\right)\right\}\right]$$

$$= E\left(\frac{x}{\lambda^2}\right)$$

$$= \frac{\lambda}{\lambda^2}$$

$$= \frac{1}{\lambda}$$

$$b'(\theta) = \frac{1}{\lambda} = b'(\lambda)$$

$$b''(\theta) = -\frac{1}{\lambda^2} = b''(\lambda)$$

$$c'(\theta) = -1 = c'(\lambda)$$

$$c''(\theta) = 0 = c''(\lambda)$$

$$E\left[a(X)\right] = E(X) = -\frac{c'(\lambda)}{b'(\lambda)} = \lambda$$

$$Var\left[a(X)\right] = \frac{b''(\lambda)c'(\lambda) - c''(\lambda)b'(\lambda)}{\left[b'(\lambda)\right]^3}$$

$$= \frac{\frac{1}{\lambda^2} - 0}{\frac{1}{\lambda^3}}$$

$$= \lambda.$$