MODULE 7

LIMITING DISTRIBUTIONS

LECTURE 39

Topics

7.1 CONVERGENCE IN DISTRIBUTION AND PROBABILITY

Theorem 1.4

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables with $E(X_n) = \mu_n \in (-\infty, \infty)$, and $\text{Var}(X_n) = \sigma_n^2 \in (0, \infty), n = 1, 2, \dots$. Suppose that $\lim_{n\to\infty} \mu_n = \mu \in \mathbb{R}$ and $\lim_{n\to\infty} \sigma_n^2 = 0$. Then $X_n \xrightarrow{p} \mu$, as $n \to \infty$.

Proof. Fix $\varepsilon > 0$. Using the Markov inequality we have

$$0 \leq P(\{|X_n - \mu| \geq \varepsilon\}) \leq \frac{E(|X_n - \mu|^2)}{\varepsilon^2} = \frac{E((X_n - \mu)^2)}{\varepsilon^2}.$$

Also,

$$\begin{aligned} E((X_n - \mu)^2) &= E((X_n - \mu_n + \mu_n - \mu)^2) \\ &= E((X_n - \mu_n)^2) + (\mu_n - \mu)^2 \\ &= \sigma_n^2 + (\mu_n - \mu)^2 \cdot \end{aligned}$$

Therefore,

$$\begin{aligned} 0 \leq P(\{|X_n - \mu| \geq \varepsilon\}) &\leq \frac{\sigma_n^2 + (\mu_n - \mu)^2}{\varepsilon^2} \\ &\xrightarrow{n\to\infty} 0. \end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} P(\{|X_n - \mu| \geq \varepsilon\}) = 0, \qquad \forall \varepsilon > 0.$$

$$\Rightarrow X_n \xrightarrow{p} \mu, \text{as } n \to \infty \qquad \text{(using Theorem 1.3).} \blacksquare$$

Example 1.7

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. $U(0, \theta)$ random variables, where $\theta > 0$. Let $X_{n:n} = \max\{X_1, X_2, \ldots, X_n\}, n = 1, 2, \ldots$. For any real constant s, show that $X_{n:n}^s \xrightarrow{p} \theta^s$, as $n \to \infty$.

Solution. It is easy to verify that a p.d.f. of $X_{n:n}$ is

$$f_n(x) = \begin{cases} \dfrac{nx^{n-1}}{\theta^n}, & \text{if } 0 < x < \theta \\ 0, & \text{otherwise} \end{cases}.$$

Then

$$E(X_{n:n}^s) = \frac{n}{n+s}\theta^s, \quad n > -s$$

$$\to \theta^s, \text{as } n \to \infty \cdot$$

Also,

$$\begin{aligned} \text{Var}(X_{n:n}^s) &= E(X_{n:n}^{2s}) - \left(E(X_{n:n}^s)\right)^2 \\ &= \frac{n}{n+2s}\theta^{2s} - \left(\frac{n}{n+s}\theta^s\right)^2, \quad n > \max(-s, -2s) \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Now, using Theorem 1.4, it follows that $X_{n:n}^s \xrightarrow{p} \theta^s$, as $n \to \infty$. ■

Example 1.8

Let $X_n \sim \text{Bin}(n, \theta), n = 1, 2, \ldots, \theta \in (0,1)$. If $Y_n = \frac{X_n}{n}, n = 1, 2, \ldots$, show that $Y_n \xrightarrow{p} \theta$, as $n \to \infty$.

Solution. We have

$$E(Y_n) = E\left(\frac{X_n}{n}\right) = \theta, n = 1, 2, \ldots,$$

and

$$\text{Var}(Y_n) = \text{Var}\left(\frac{X_n}{n}\right) = \frac{\text{Var}(X_n)}{n^2} = \frac{\theta(1-\theta)}{n} \to 0, \text{as } n \to \infty \cdot$$

Using Theorem 1.4 it follows that $Y_n \xrightarrow{p} \theta$, as $n \to \infty$. ■

Remark 1.2

Theorem 1.3 provides an interpretation of the concept of convergence in probability. Theorem 1.3 suggests that if $X_n \xrightarrow{p} c$, as $n \to \infty$, then X_n is stochastically (in probability) very close to c for large values of n. Such an interpretation does not hold for the concept of convergence in distribution. Specifically, if $X_n \xrightarrow{d} X$, as $n \to \infty$, (where X is some non-degenerate random variable) then it cannot be inferred that X_n is getting close to X, for large values of n, in any sense. All we know in that case is that, for large values of n, the distribution of X_n is getting close to that of X. ■

The following example demonstrates that convergence in probability may not imply convergence of moments.

Example 1.9

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables with

$$1 - P(\{X_n = 0\}) = P(\{X_n = n\}) = \frac{1}{n}, \ \ n = 1,2, \ldots.$$

Then the d.f. of X_n is

$$F_n(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - \dfrac{1}{n}, & \text{if } 0 \leq x < n, n = 1, 2, \ldots. \\ 1, & \text{if } x \geq n \end{cases}$$

$$\xrightarrow{n\to\infty} \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}.$$

Thus $X_n \xrightarrow{p} 0$, as $n \to \infty$. However, for $r \in \{1, 2, \ldots\}$

$$E(X_n^r) = E(|X_n|^r) = n^{r-1} \nrightarrow 0, \text{as } n \to \infty. \ ■$$

The following example illustrates that convergence in distribution to a non-degenerate random variable also does not imply convergence of moments.

Example 1.10

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables with p.m.f.s

$$f_n(x)=\begin{cases}\frac{1}{2}-\frac{1}{2n}, & \text{if } x\in\left\{0,\frac{1}{2}\right\}\\ \frac{1}{n}, & \text{if } x=n,\ \ n=1,2,\ldots\\ 0, & \text{otherwise}\end{cases},$$

and let X be a random variable with p.m.f.

$$f(x)=\begin{cases}\frac{1}{2}, & \text{if } x\in\left\{0,\frac{1}{2}\right\}\\ 0, & \text{otherwise}\end{cases}.$$

Then the distribution function of X is

$$F(x)=\begin{cases}0, & \text{if } x<0\\ \frac{1}{2}, & \text{if } 0\le x<\frac{1}{2}\\ 1, & \text{if } x\ge\frac{1}{2}\end{cases},$$

and the distribution function of X_n is

$$F_n(x)=\begin{cases}0, & \text{if } x<0\\ \frac{1}{2}-\frac{1}{2n}, & \text{if } 0\le x<\frac{1}{2}\\ 1-\frac{1}{n}, & \text{if } \frac{1}{2}\le x<n\\ 1, & \text{if } x\ge n\end{cases},\ n=1,2,\ldots$$

$$\xrightarrow{n\to\infty}\begin{cases}0, & \text{if } x<0\\ \frac{1}{2}, & \text{if } 0\le x<\frac{1}{2}\\ 1, & \text{if } x\ge\frac{1}{2}\end{cases}.$$

It follows that $X_n \xrightarrow{d} X$, as $n\to\infty$. Moreover $E(X)=\frac{1}{4}$ and

$$E(X_n)=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{2n}\right]+1 \xrightarrow{n\to\infty} \frac{5}{4}\neq E(X). \blacksquare$$

We know that, for a real constant c, $X_n \xrightarrow{p} c$, as $n\to\infty \Leftrightarrow X_n-c\xrightarrow{p} 0$, as $n\to\infty$. The following example illustrates that $X_n \xrightarrow{d} X$, as $n\to\infty$ may not imply that $X_n - X\xrightarrow{p} 0$, as $n \to \infty$ or, equivalently, $X_n \xrightarrow{d} X$, as $n \to \infty$, does not imply that $X_n - X$ will converge in distribution to a random variable degenerate at 0 (also see Remark 1.2).

Example 1.11

Let $\{X_n\}_{n\geq 1}$ and X be as defined in Example 1.10. Further suppose that, for each $n \in \{1, 2, ...\}, X_n$ and X are independent. Then $X_n \xrightarrow{d} X$, as $n \to \infty$. However, for $0 < \varepsilon < \frac{1}{2}$

$$P(\{|X_n - X| \geq \varepsilon\}) = \frac{1}{2}\left[P(\{|X_n| \geq \varepsilon\}) + P\left(\left\{\left|X_n - \frac{1}{2}\right| \geq \varepsilon\right\}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{2n}\right) + \frac{1}{n} + \left(\frac{1}{2} - \frac{1}{2n}\right) + \frac{1}{n}\right]$$

$$\xrightarrow{n\to\infty} \frac{1}{2},$$

implying that $X_n - X$ does not converge in distribution to a random variable degenerate at 0. ■

Definition 1.2

A sequence $\{X_n\}_{n\geq 1}$ of random variables is said to be *bounded in probability* if there exists a positive real constant M (not depending on n) such that

$$P\left(\bigcap_{n=1}^{\infty}\{|X_n| \leq M\}\right) = 1. \ ■$$

The following theorem relates convergence in distribution of a sequence $\{X_n\}_{n\geq 1}$ of random variables to the convergence of corresponding sequence of moment generating functions (m.g.f.s). We shall not provide the proof of the theorem as it is slightly involved.

Theorem1.5

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables and let X be another random variable. Suppose that there exists an $h > 0$ such that the m.g.f.s $M(\cdot), M_1(\cdot),\ M_2(\cdot)$, ...of $X,\ X_1$, X_2, ..., respectively, are finite on $(-h, h)$.

(i) If $\lim_{n\to\infty} M_n(t) = M(t),\ \forall t \in (-h, h)$, then $X_n \xrightarrow{d} X$, as $n \to \infty$;

(ii) If $X_1, X_2, ...$ are bounded in probability and $X_n \xrightarrow{d} X$, as $n \to \infty$, then $\lim_{n\to\infty} M_n(t) = M(t),\ \forall t \in (-h, h)$. ■

The following example demonstrates that the conclusion of Theorem 1.5 (ii) may not hold if $X_1, X_2, \ldots$ are not bounded in probability.

Example 1.12

Let $\{X_n\}_{n\geq 1}$ and X be as defined in Example 1.10. Then the m.g.f. of X is

$$M(t) = \frac{1+e^{\frac{t}{2}}}{2}, t \in \mathbb{R},$$

and the m.g.f. of X_n is

$$M_n(t) = \left(\frac{1}{2} - \frac{1}{2n}\right)\left(1 + e^{\frac{t}{2}}\right) + \frac{e^{nt}}{n}$$

$$\xrightarrow{n\to\infty} \begin{cases} \frac{1+e^{\frac{t}{2}}}{2}, & \text{if } t \leq 0 \\ \infty, & \text{if t} > 0 \end{cases}$$

$$\neq M(t), \qquad \forall t \in \mathbb{R}.$$

However, $X_n \xrightarrow{d} X$, as $n \to \infty$. ■