MODULE 7

LIMITING DISTRIBUTIONS

LECTURE 38

Topics

7.1 CONVERGENCE IN DISTRIBUTION AND PROBABILITY

Theorem 1.1

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that $X_n \xrightarrow{d} X$, as $n \to \infty$, for some random variable X. Let F_n and F denote the d.f.s of $X_n (n = 1, 2, \dots)$ and X, respectively. Then

$$\lim_{n\to\infty} F_n\,(x-) = F(x-) = F(x) = \lim_{n\to\infty} F_n\,(x), \forall x \in C_F,$$

where C_F is the set of continuity points of F.

Proof. We are given that

$$\lim_{n\to\infty} F_n\,(x) = F(x), \forall x \in C_F \left(\text{since } X_n \xrightarrow{d} X, \text{as } n \to \infty\right).$$

Moreover $F(x-) = F(x), \forall x \in C_F$ .Thus it suffices to show that $\lim_{n\to\infty} F_n\,(x-) = F(x-), \forall x \in C_F$. Let $d \in C_F$ so that $F(d-) = F(d)$. Fix $m \in \mathbb{N} = \{1, 2, \dots\}$. Since the set $C_F^c = \mathbb{R} - C_F$ of discontinuity points of F is countable and the interval $\left(d - \frac{1}{m}, d\right)$ is uncountable there exists a $d_m \in \left(d - \frac{1}{m}, d\right) \cap C_F$. Then we have $\lim_{n\to\infty} F_n\,(d_m) = F(d_m)$ and $\lim_{n\to\infty} F_n\,(d) = F(d)$. Moreover

$$F_n(d_m) \leq F_n(d-) \leq F_n(d), n = 1, 2, \dots$$

$$\Rightarrow \lim_{n\to\infty} F_n\,(d_m) \leq \lim_{n\to\infty} F_n\,(d-) \leq \lim_{n\to\infty} F_n\,(d)$$

$$\Rightarrow F(d_m) \leq \lim_{n\to\infty} F_n\,(d-) \leq F(d) = F(d-). \qquad (1.1)$$

Since $d_m \in \left(d - \frac{1}{m}, d\right)$, we have

$$\lim_{m\to\infty} F\,(d_m) = F(d-) = F(d). \qquad (1.2)$$

On taking $m \to \infty$ in (1.1) we get

$$\lim_{m\to\infty} F(d_m) \le \lim_{m\to\infty} F_n(d-) \le F(d-)$$

$$\Rightarrow F(d-) \le \lim_{n\to\infty} F_n(d-) \le F(d-) \qquad \text{(using (1.2))}$$

$$\Rightarrow \lim_{n\to\infty} F_n(d-) = F(d-). \blacksquare$$

Corollary 1.1

Let $\{X_n\}_{n\ge1}$ be a sequence of random variables with corresponding sequence of d.f.s as $\{F_n\}_{n\ge1}$. Further let X be another random variable having the d.f. F.

(i) If $X_n \xrightarrow{d} X$, as $n \to \infty$, and X is of continuous type then $\lim_{n\to\infty} F_n(x) = F(x), \forall x \in \mathbb{R}$ and $\lim_{n\to\infty} F_n(x-) = F(x-), \forall x \in \mathbb{R}$.

(ii) Suppose that $P(\{X_n \in \{0,1,2,\ldots\}\}) = P(\{X \in \{0,1,2,\ldots\}\}) = 1$ and $X_n \xrightarrow{d} X$, as $n \to \infty$. Then $\lim_{n\to\infty} F_n(x) = F(x)$, $\forall x \in \mathbb{R}$ and $\lim_{n\to\infty} F_n(x-) = F(x-), \forall x \in \mathbb{R}$.

(iii) Under the assumptions of (ii), let f and f_n be the p.m.f.s of X and X_n, respectively, $n = 1, 2, \ldots$. Then

$$X_n \xrightarrow{d} X, \text{ as } n \to \infty \Leftrightarrow \lim_{n\to\infty} f_n(x) = f(x), \forall x \in \{0,1,2,\ldots\}.$$

Proof.

(i) Since X is of continuous type we have $C_F = \mathbb{R}$, where C_F is the set of continuity points of F. The assertion now follows from Theorem 1.1.

(ii) Fix $x \in \mathbb{R}$. If $P(\{X = x\}) = 0$ then $x \in C_F$ and, therefore, by Theorem 1.1.

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \text{and} \quad \lim_{n\to\infty} F_n(x-) = F(x-).$$

Now suppose that $P(\{X = x\}) > 0$. Then $x \in \{0,1,2,\ldots\}$ and $P(\{X = x + 0.5\}) = P(\{X = x - 0.5\}) = 0$. Consequently $x \pm 0.5 \in C_F$,

$$F_n(x) = F_n(x+0.5) \text{ and } F_n(x-) = F_n(x-0.5), \qquad n = 1,2,\ldots$$

$$\Rightarrow \lim_{n\to\infty} F_n(x) = F(x+0.5) = F(x) \text{ and } \lim_{n\to\infty} F_n(x-) = F(x-0.5) = F(x-).$$

It follows that

$$\lim_{n\to\infty} F_n(x) = F(x) \text{ and } \lim_{n\to\infty} F_n(x-) = F(x-), \forall x \in \mathbb{R}.$$

(iii) First suppose that $X_n \xrightarrow{d} X,$ as $n \to \infty$. Then, for $x \in \{0, 1, 2, ...\}$,

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} P(\{X_n = x\}) \\
&= \lim_{n\to\infty} [F_n(x) - F_n(x-)] \\
&= F(x) - F(x-) \quad \text{(using (ii))} \\
&= P(\{X = x\}) \\
&= f(x).
\end{aligned}$$

Conversely suppose that $\lim_{n\to\infty} f_n(x) = f(x), \forall x \in \{0, 1, 2, ...\}$. Then, for $x \in \mathbb{R}$,

$$\begin{aligned}
F_n(x) &= P(\{X_n \le x\}) \\
&= \sum_{k=0}^{[x]} P(\{X_n = k\}) \\
&= \sum_{k=0}^{[x]} f_n(k) \\
&\xrightarrow{n\to\infty} \sum_{k=0}^{[x]} f(k) \\
&= F(x),
\end{aligned}$$

where $[x]$ denotes the largest integer not exceeding x. It follows that $X_n \xrightarrow{d} X,$ as $n \to \infty$.

■

For the random variables of absolutely continuous type we state the following theorem without providing its proof.

Theorem 1.2

Let $\{X_n\}_{n\ge1}$ be a sequence of random variables of absolutely continuous type with corresponding sequence of p.d.f.s as $\{f_n\}_{n\ge1}$. Further let X be another random variable of absolutely continuous type with p.d.f. f. Suppose that $\lim_{n\to\infty} f_n(x) = f(x), \forall x \in \mathbb{R}$. Then $X_n \xrightarrow{d} X$, as $n \to \infty$. ■

The following example demonstrates that if $X_n \xrightarrow{d} X$, as $n \to \infty$, then $\lim_{n\to\infty} F_n(x-) = F(x-)$ may not hold; here F_n and F are d.f.s of X_n $(n = 1, 2, ...)$ and X, respectively.

Example 1.5

Let $X_n \sim N\left(0, \frac{1}{n}\right), n = 1,2, \dots,$ and let X be a random variable degenerate at 0 (i.e., $P(\{X = 0\}) = 1$). Then, for $x \in \mathbb{R}$,

$$F(x) = P(\{X \le x\}) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \ge 0 \end{cases}$$

$$\begin{aligned} F_n(x) &= P(\{X_n \le x\}) \\ &= \Phi(\sqrt{n}x) \\ &\xrightarrow{n\to\infty} \begin{cases} 0, \text{ if } x < 0 \\ \frac{1}{2}, \text{ if } x = 0. \\ 1, \text{ if } x > 0 \end{cases} \end{aligned}$$

Clearly $\lim_{n\to\infty} F_n(x) = F(x), \forall x \in C_F = \mathbb{R} - \{0\}$ and, therefore, $X_n \xrightarrow{d} X$, (equivalently $X_n \xrightarrow{p} 0$) as $n \to \infty$. However $\lim_{n\to\infty} F_n(0-) = \lim_{n\to\infty} F_n(0) = \frac{1}{2} \neq F(0-) = 0$. ■

The following example illustrates that, in general, the limiting distribution cannot be obtained by taking the limit of p.m.f.s/p.d.f.s.

Example 1.6

Let $\{X_n\}_{n\ge1}$ be a sequence of random variables such that

$$P\left(\left\{X_n = \frac{1}{2n}\right\}\right) = P\left(\left\{X_n = \frac{1}{n}\right\}\right) = \frac{1}{2}, n = 1,2, \dots,$$

and let X be another random variable with $P(\{X = 0\}) = 1$. Then it is easy to verify that $X_n \xrightarrow{d} X$, as $n \to \infty$. The p.m.f. of X_n is

$$f_n(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in \left\{\frac{1}{2n}, \frac{1}{n}\right\}, \\ 0, & \text{otherwise} \end{cases}$$

and the p.m.f. of X is

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}.$$

We have

$$\lim_{n\to\infty} f_n(x) = 0 \neq f(x), \forall x \in \mathbb{R}. \blacksquare$$

The following theorem provides a characterization of convergence in probability.

Theorem 1.3

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables and let c be a real constant. Then

$$X_n \xrightarrow{p} c, \text{as } n \to \infty \Leftrightarrow \forall \varepsilon > 0, \lim_{n\to\infty} P\left(\{|X_n - c| \geq \varepsilon\}\right) = 0.$$

Proof. Let F_n denote the d.f. of X_n $(n = 1, 2, \ldots)$ and let F denote the d.f. of random variable degenerate at c. First suppose that $X_n \xrightarrow{p} c$, as $n \to \infty$. Then, for $x \in \mathbb{R} - \{c\}$,

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} P\left(\{X_n \leq x\}\right)$$

$$= \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x > c \end{cases} = F(x).$$

Fix $\varepsilon > 0$. Then $c \pm \varepsilon \in C_F$ and therefore, using Theorem 1.1,

$$\lim_{n\to\infty} P(\{|X_n - c| \geq \varepsilon\}) = \lim_{n\to\infty} [P(\{X_n \leq c - \varepsilon\}) + P(\{X_n \geq c + \varepsilon\})]$$

$$= \lim_{n\to\infty} \left[F_n(c - \varepsilon) + 1 - F_n\left((c + \varepsilon) -\right)\right] \qquad (1.3)$$

$$= [F(c - \varepsilon) + 1 - F(c + \varepsilon)]$$

$$= 0.$$

Conversely, suppose that

$$\lim_{n\to\infty} P(\{|X_n - c| \geq \varepsilon\}) = 0, \forall \varepsilon > 0.$$

Then, using (1.3),

$$\lim_{n\to\infty} \left[F_n(c - \varepsilon) + 1 - F_n\left((c + \varepsilon) -\right)\right] = 0, \forall \varepsilon > 0,$$

$$\Rightarrow \lim_{n\to\infty} F_n(c - \varepsilon) = \lim_{n\to\infty} \left[1 - F_n\left((c + \varepsilon) -\right)\right] = 0, \forall \varepsilon > 0$$
$$\left(\text{since } F_n(c - \varepsilon) \geq 0 \text{ and } 1 - F_n\left((c + \varepsilon) -\right) \geq 0, \forall n \geq 1\right)$$

$$\Rightarrow \lim_{n\to\infty} F_n(x) = 0, \forall x < c \text{ and} \lim_{n\to\infty} F_n\,(y-) = 1, \forall y > c$$

$$\Rightarrow \lim_{n\to\infty} F_n(x) = 0, \forall x < c \text{ and} \lim_{n\to\infty} F_n\,(y) = 1, \forall y > c$$
$$(\text{since } 1 \geq F_n(y) \geq F_n(y-), n = 1,2, \dots).$$

Thus, for all $x \in \mathbb{R} - \{c\}$,

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x > c \end{cases} = F(x)$$

$\Rightarrow X_n \xrightarrow{p} c, \text{as } n \to \infty.$ ■

In many situations the above theorem in conjunction with Markov's inequality (see Corollary 5.1, Module 3) turns out to be quite useful in proving convergence in probability.