MODULE 3

FUNCTION OF A RANDOM VARIABLE AND ITS DISTRIBUTION

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MODULE 3

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LECTURE 12

Topics

3.1 FUNCTION OF A RANDOM VARIABLE

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let X be random variable defined on $(\Omega, \mathcal{F}, P)$. Further let $h: \mathbb{R} \to \mathbb{R}$ be a given function and let $Z: \Omega \to \mathbb{R}$be a function of random variable X, defined by $Z(\omega) = h\big(X(\omega)\big), \omega \in \Omega$. In many situations it may be of interest to study the probabilistic properties of Z, which is a function of random variable X. Since the variable Z takes values in $\mathbb{R}$, to study the probabilistic properties of Z, it is necessary that $Z^{-1}(B) \in \mathcal{F}, \forall B \in \mathcal{B}_1$, i.e., Z is a random variable. Throughout, for a positive integer $k, \mathbb{R}^k$ will denote the k-dimensional Euclidean space and $\mathcal{B}_k$ will denote the Borel sigma-field in $\mathbb{R}^k$.

Definition 1.1

Let k and m be positive integers. A function $h: \mathbb{R}^k \to \mathbb{R}^m$ is said to be a Borel function if $h^{-1}(B) \in \mathcal{B}_k, \forall B \in \mathcal{B}_m$. ■

The following lemma will be useful in deriving conditions on the function $h: \mathbb{R} \to \mathbb{R}$ so that $Z: \Omega \to \mathbb{R}$, defined by $Z(\omega) = h\big(X(\omega)\big), \omega \in \Omega$, is a random variable. Recall that, for a function $\Psi: D_1 \to D_2$ and $A \subseteq D_2, \Psi^{-1}(A) = \{\omega \in D_1: \Psi(\omega) \in A\}$.

Lemma 1.1

Let $X: \Omega \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$ be given functions. Define $Z: \Omega \to \mathbb{R}$ by $Z(\omega) = h\big(X(\omega)\big), \omega \in \Omega$. Then, for any $B \subseteq \mathbb{R}$,

$$Z^{-1}(B) = X^{-1}\big(h^{-1}(B)\big).$$

Proof. Fix $B \subseteq \mathbb{R}$. Note that $h^{-1}(B) = \{x \in \mathbb{R}: \ h(x) \in B\}$. Clearly

$$h\big(X(\omega)\big) \in B \Leftrightarrow X(\omega) \in h^{-1}(B).$$

Therefore,

$$\begin{aligned} Z^{-1}(B) &= \{\omega \in \Omega: Z(\omega) \in B\} \\ &= \big\{\omega \in \Omega: h\big(X(\omega)\big) \in B\big\} \\ &= \{\omega \in \Omega: X(\omega) \in h^{-1}(B)\} \\ &= X^{-1}\big(h^{-1}(B)\big). \ \blacksquare \end{aligned}$$

Theorem 1.1

Let X be a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and let $h: \mathbb{R} \to \mathbb{R}$ be a Borel function. Then the function $Z: \Omega \to \mathbb{R}$, defined by $Z(\omega) = h\big(X(\omega)\big), \omega \in \Omega$, is a random variable.

Proof. Fix $B \in \mathcal{B}_1$. Since h is a Borel function, we have $h^{-1}(B) \in \mathcal{B}_1$. Now using the fact that X is a random variable it follows that

$$Z^{-1}(B) = X^{-1}\big(h^{-1}(B)\big) \in \mathcal{F}.$$

This proves the result. $\blacksquare$

Remark 1.1

(i) Let $h: \mathbb{R} \to \mathbb{R}$ be a continuous function. According to a standard result in calculus inverse image of any open interval $(a, b), -\infty \leq a < b \leq \infty$, under continuous function h is a countable union of disjoint open intervals. Since $\mathcal{B}_1$ contains all open intervals and is closed under countable unions it follows that $h^{-1}\big((a, b)\big) \in \mathcal{B}_1$, whenever $-\infty \leq a < b \leq \infty$. Now on employing the arguments similar to the one used in proving Theorem 1.1, Module 2 (also see Theorem 1.2, Module 2) we conclude that $h^{-1}(B) \in \mathcal{B}_1, \forall B \in \mathcal{B}_1$. It follows that any continuous function $h: \mathbb{R} \to \mathbb{R}$ is a Borel function and thus, in view of Theorem 1.1, any continuous function of a random variable is a random variable. In particular if X is a random variable then $X^2, |X|, \max(X, 0)$, $\sin X$ and $\cos X$ are random variables.

(ii) Let $h: \mathbb{R} \to \mathbb{R}$ be a strictly monotone function. Then, for $-\infty \leq a < b \leq \infty$, $h^{-1}(a, b)$ is a countable union of intervals and therefore $h^{-1}(a, b) \in \mathcal{B}_1$, i.e., h is a Borel function. It follows that if X is a random variable and if $h: \mathbb{R} \to \mathbb{R}$ is strictly monotone then $h(X)$ is a random variable. ■

A random variable X takes values in various Borel sets according to some probability law called the probability distribution of random variable X. Clearly the probability distribution of a random variable of absolutely continuous/discrete type is described by its distribution function (d.f.) and/or by its probability density function/probability mass function (p.d.f/p.m.f.). For a given Borel function $h: \mathbb{R} \to \mathbb{R}$, in the following section, we will derive probability distribution of $h(X)$ using the probability distribution of random variable X. ■

3.2 PROBABILITY DISTRIBUTION OF A FUNCTION OF A RANDOM VARIABLE

In our future discussions when we refer to a random variable, unless otherwise stated, it will be either of discrete type or of absolutely continuous type. The probability distribution of a discrete type random variable will be referred to as a discrete (probability) distribution and the probability distribution of a random variable of absolutely continuous type will be referred to as an absolutely continuous (probability) distribution.

The following theorem deals with discrete probability distributions.

Theorem 2.1

Let X be a random variable of discrete type with support S_X and p.m.f. $f_X(\cdot)$. Let $h: \mathbb{R} \to \mathbb{R}$ be a Borel function and let $Z: \Omega \to \mathbb{R}$ be defined by $Z(\omega) = h\big(X(\omega)\big), \omega \in \Omega$. Then Z is a random variable of discrete type with support $S_Z = \{h(x): x \in S_X\}$ and p.m.f.

$$f_Z(z) = \begin{cases} \sum_{x \in A_z} f_X(x), & \text{if } z \in S_Z \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} P(\{X \in A_z\}), & \text{if } z \in S_Z \\ 0, & \text{otherwise}' \end{cases}$$

where $A_z = \{x \in S_X : h(x) = z\}$.

Proof. Since h is a Borel function, using Theorem 1.1, it follows that Z is a random variable. Also X is of discrete implies that S_X is countable which further implies that S_Z is countable. Fix $z_0 \in S_Z$, so that $z_0 = h(x_0)$ for some $x_0 \in S_X$.
Then

$$\begin{aligned}\{X = x_0\} = \{\omega \in \Omega : X(\omega) = x_0\} &\subseteq \{\omega \in \Omega : h(X(\omega)) = h(x_0)\} \\ &= \{h(X) = h(x_0)\} \\ &= \{Z = z_0\},\end{aligned}$$

and

$$\begin{aligned}\{X \in S_X\} = \{\omega \in \Omega : X(\omega) \in S_X\} &\subseteq \{\omega \in \Omega : h(X(\omega)) \in S_Z\} \\ &= \{h(X) \in S_Z\} \\ &= \{Z \in S_Z\}.\end{aligned}$$

Therefore,

$P(\{Z = z_0\}) \geq P(\{X \in x_0\}) > 0$, (since $x_0 \in S_X$),

and $P(\{Z \in S_Z\}) \geq P(\{X \in S_X\}) = 1$.

It follows that S_Z is countable, $P(\{Z = z\}) > 0, \forall z \in S_Z$ and $P(\{Z \in S_Z\}) = 1,$ i.e., Z is a discrete type random variable with support S_Z.
Moreover, for $z \in S_Z$,

$$\begin{aligned}P(\{Z = z\}) &= P(\{\omega \in \Omega : h(X(\omega)) = z\}) \\ &= \sum_{x \in A_z} P(\{X = x\}) \\ &= \sum_{x \in A_z} f_X(x) \\ &= P(\{X \in A_z\}).\end{aligned}$$

Hence the result follows. ■

The following corollary is an immediate consequence of the above theorem.

Corollary 2.1

Under the notation and assumptions of Theorem 2.1, suppose that $h: \mathbb{R} \to \mathbb{R}$ is one-one with inverse function $h^{-1}: D \to \mathbb{R}$, where $D = \{h(x): x \in \mathbb{R}\}$. Then Z is a discrete type random variable with support $S_z = \{h(x): x \in S_X\}$ and p.m.f.

$$f_Z(z) = \begin{cases} f_X\left(h^{-1}(z)\right), & \text{if } z \in S_Z \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} P(\{X = h^{-1}(z)\}), & \text{if } z \in S_Z \\ 0, & \text{otherwise} \end{cases}. \blacksquare$$

Example 2.1

Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \frac{1}{7}, & \text{if } x \in \{-2, -1, 0, 1\} \\ \frac{3}{14}, & \text{if } x \in \{2, 3\} \\ 0, & \text{otherwise} \end{cases}.$$

Show that $Z = X^2$ is a random variable. Find its p.m.f. and distribution function.

Solution. Since $h(x) = x^2, x \in \mathbb{R}$, is a continuous function and X is a random variable, using Remark 1.1 (i) it follows that $Z = h(X) = X^2$ is a random variable. Clearly $S_X = \{-2, -1, 0, 1, 2, 3\}$ and $S_Z = \{0, 1, 4, 9\}$. Moreover,

$$P(\{Z = 0\}) = P(\{X^2 = 0\}) = P(\{X = 0\}) = \frac{1}{7},$$

$$P(\{Z = 1\}) = P(\{X^2 = 1\}) = P(X \in \{-1, 1\}) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7},$$

$$P(\{Z = 4\}) = P(\{X^2 = 4\}) = P(X \in \{-2, 2\}) = \frac{1}{7} + \frac{3}{14} = \frac{5}{14},$$

and $$P(\{Z = 9\}) = P(\{X^2 = 9\}) = P\left(\{X \in \{-3, 3\}\}\right) = 0 + \frac{3}{14} = \frac{3}{14}.$$

Therefore the p.m.f. of Z is

$$f_Z(z) = \begin{cases} \frac{1}{7}, & \text{if } z = 0 \\ \frac{2}{7}, & \text{if } z = 1 \\ \frac{5}{14}, & \text{if } z = 4 \\ \frac{3}{14}, & \text{if } z = 9 \\ 0, & \text{otherwise} \end{cases},$$

and the distribution function of Z is

$$F_Z(z) = \begin{cases} 0, & \text{if } z < 0 \\ \frac{1}{7}, & \text{if } 0 \leq z < 1 \\ \frac{3}{7}, & \text{if } 1 \leq z < 4 \\ \frac{11}{14}, & \text{if } 4 \leq z < 9 \\ 1, & \text{if } z \geq 9 \end{cases}. \blacksquare$$

Example 2.2

Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \frac{|x|}{2550}, & \text{if } x \in \{\pm 1, \pm 2, \cdots, \pm 50\} \\ 0, & \text{otherwise} \end{cases}.$$

Show that $Z = |X|$ is a random variable. Find its p.m.f., and distribution function.

Solution. As $h(x) = |x|, x \in \mathbb{R}$, is a continuous function and X is a random variable, using Remark 1.1 (i), $Z = |X|$ is a random variable. We have $S_X = \{\pm 1, \pm 2, \cdots, \pm 50\}$ and $S_Z = \{1, 2, \cdots, 50\}$. Moreover, for $z \in S_Z$,

$$P(\{Z = z\}) = P(\{|X| = z\}) = P\big(\{X \in \{-z, z\}\}\big) = \frac{|-z|}{2550} + \frac{|z|}{2550} = \frac{z}{1275}.$$

Therefore the p.m.f. of Z is

$$f_Z(z) = \begin{cases} \frac{z}{1275}, & \text{if } z \in \{1, 2, \cdots, 50\} \\ 0, & \text{otherwise} \end{cases},$$

and the distribution function of Z is

$$F_Z(z) = \begin{cases} 0, & \text{if } z < 1 \\ \frac{1}{1275}, & \text{if } 1 \le z < 2 \\ \frac{i(i+1)}{2550}, & \text{if } i \le z < i+1, i = 2, 3, \cdots, 49 \\ 1, & \text{if } z \ge 50 \end{cases}. \blacksquare$$

Example 2.3

Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & \text{if } x \in \{0, 1, \cdots, n\} \\ 0, & \text{otherwise} \end{cases},$$

where n is a positive integer and $p \in (0,1)$. Show that $Y = n - X$ is a random variable. Find its p.m.f. and distribution function.

Solution. Note that $S_X = S_Y = \{0, 1, \cdots, n\}$ and $h(x) = n - x, x \in \mathbb{R}$, is a continuous function. Therefore $Y = n - X$ is a random variable. For $y \in S_Y$

$$P(\{Y = y\}) = P(\{X = n - y\}) = \binom{n}{n-y} p^{n-y} (1-p)^y = \binom{n}{y} (1-p)^y p^{n-y}.$$

Thus the p.m.f. of Y is

$$f_Y(y) = \begin{cases} \binom{n}{y} (1-p)^y p^{n-y}, & \text{if } y \in \{0, 1, \cdots, n\} \\ 0, & \text{otherwise} \end{cases},$$

and the distribution function of Y is

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0 \\ p^n, & \text{if } 0 \le y < 1 \\ \sum_{j=0}^{i} \binom{n}{j} (1-p)^j p^{n-j}, & \text{if } i \le y < i+1, i = 1,2, \cdots, n-1 \\ 1, & \text{if } y \ge n \end{cases}. \blacksquare$$

The following theorem deals with probability distribution of absolutely continuous type random variables.

Theorem 2.2

Let X be a random variable of absolutely continuous type with p.d.f. $f_X(\cdot)$ and support S_X. Let $S_1, S_2, \cdots, S_k$, be open intervals in $\mathbb{R}$ such that $S_i \cap S_j = \phi$, if $i \neq j$ and $\bigcup_{i=1}^{k} S_i = S_X$. Let $h: \mathbb{R} \to \mathbb{R}$ be a Borel function such that, on each $S_i (i = 1, \dots, k)$, $h: S_i \to \mathbb{R}$ is strictly monotone and continuously differentiable with inverse function $h_i^{-1}(\cdot)$. Let $h(S_j) = \{h(x): x \in S_j\}$ so that $h(S_j)(j = 1, \dots, k)$ is an open interval in $\mathbb{R}$. Then the random variable $T = h(X)$ is of absolutely continuous type with p.d.f.

$$f_T(t) = \sum_{j=1}^{k} f_X\left(h_j^{-1}(t)\right)\left|\frac{d}{dt}h_j^{-1}(t)\right| I_{h(S_j)}(t).$$

Proof. We will provide an outline of the proof which may not be rigorous. Let $F_T(\cdot)$ be the distribution function of T. For $t \in \mathbb{R}$ and $\Delta > 0$,

$$\frac{F_T(t+\Delta) - F_T(t)}{\Delta} = \frac{P(\{t < h(X) \le t + \Delta\})}{\Delta}$$

$$= \sum_{j=1}^{k} \frac{P\left(\{t < h(X) \le t + \Delta, X \in S_j\}\right)}{\Delta}.$$

Fix $j \in \{1, \dots, k\}$. First suppose that $h_j(\cdot)$ is strictly decreasing on S_j. Note that $\{X \in S_j\} = \{h(X) \in h(S_j)\}$ and $h(S_j)$ is an open interval. Thus, for t belonging to the exterior of $h(S_j)$ and sufficiently small $\Delta > 0$, we have $P\left(\{t < h(X) \le t + \Delta, X \in S_j\}\right) = 0$. Also, for $t \in h(S_j)$ and sufficiently small $\Delta > 0$,

$$P\left(\{t < h(X) \le t + \Delta, X \in S_j\}\right) = P\left(\{h_j^{-1}(t+\Delta) \le X < h_j^{-1}(t)\}\right).$$

Thus, for all $t \in \mathbb{R}$, we have

$$\frac{P\left(\{t < h(X) \le t + \Delta, X \in S_j\}\right)}{\Delta} = \frac{P\left(\{h_j^{-1}(t+\Delta) \le X < h_j^{-1}(t)\}\right) I_{h(S_j)}(t)}{\Delta}$$

$$= \frac{1}{\Delta}\left[\int_{h_j^{-1}(t+\Delta)}^{h_j^{-1}(t)} f_X(z)dz\right] I_{h(S_j)}(t)$$

$$\xrightarrow{\Delta \downarrow 0} -f_X\left(h_j^{-1}(t)\right)\left(\frac{d}{dt}h_j^{-1}(t)\right) I_{h(S_j)}(t). \qquad (2.1)$$

Similarly if h_j is strictly increasing on S_j then, for all $t \in \mathbb{R}$, we have

$$\frac{P\left(\left\{t < h(X) \le t+\Delta, X \in S_j\right\}\right)}{\Delta} = \frac{P\left(\left\{h_j^{-1}(t) < X \le h_j^{-1}(t+\Delta)\right\}\right) I_{h(S_j)}(t)}{\Delta}$$

$$= \frac{1}{\Delta}\left[\int_{h_j^{-1}(t)}^{h_j^{-1}(t+\Delta)} f_X(z)dz\right] I_{h(S_j)}(t)$$

$$\xrightarrow{\Delta\downarrow 0} f_X\left(h_j^{-1}(t)\right)\left(\frac{d}{dt}h_j^{-1}(t)\right) I_{h(S_j)}(t). \qquad (2.2)$$

Note that if h is strictly decreasing (increasing) on S_j then $\frac{d}{dt}h_j^{-1}(t) < (>)0$ on S_j. Now on combining (2.1) and (2.2) we get, for all $t \in \mathbb{R}$,

$$\frac{P\left(\left\{t < h(X) \le t+\Delta, X \in S_j\right\}\right)}{\Delta} \xrightarrow{\Delta\downarrow 0} f_X\left(h_j^{-1}(t)\right)\left|\frac{d}{dt}h_j^{-1}(t)\right| I_{h(S_j)}(t),$$

$$\Rightarrow \frac{F_T(t+\Delta) - F_T(t)}{\Delta} \xrightarrow{\Delta\downarrow 0} \sum_{j=1}^{k} f_X\left(h_j^{-1}(t)\right)\left|\frac{d}{dt}h_j^{-1}(t)\right| I_{h(S_j)}(t).$$

Similarly one can show that, for all $t \in \mathbb{R}$,

$$\lim_{\Delta\uparrow 0}\frac{F_T(t+\Delta) - F_T(t)}{\Delta} = \sum_{j=1}^{k} f_X\left(h_j^{-1}(t)\right)\left|\frac{d}{dt}h_j^{-1}(t)\right| I_{h(S_j)}(t). \qquad (2.3)$$

It follows that the distribution function of T is differentiable everywhere on $\mathbb{R}$ except possibly at a finite number of points (on boundaries of intervals $h(S_1), \cdots, h(S_k)$ of S_T). Now the result follows from Remark 4.2 (vii) of Module 2 and using (2.3). ■

The following corollary to the above theorem is immediate.

Corollary 2.2

Let X be a random variable of absolutely continuous type with p.d.f. $f_X(\cdot)$ and support S_X. Suppose that S_X is a finite union of disjoint open intervals in $\mathbb{R}$ and let $h: \mathbb{R} \to \mathbb{R}$ be a Borel function such that h is differentiable and strictly monotone on S_X (i.e., either $h'(x) < 0, \forall x \in S_X$ or $h'(x) > 0, \forall x \in S_X$). Let $S_T = \{h(x): x \in S_X\}$. Then $T = h(X)$ is a random variable of absolutely continuous type with p.d.f.

$$f_T(t) = \begin{cases} f_X\left(h^{-1}(t)\right)\left|\frac{d}{dt}h^{-1}(t)\right|, & \text{if } t \in S_T \\ 0, & \text{otherwise} \end{cases}. \blacksquare$$

It may be worth mentioning here that, in view of Remark 4.2 (vii) of Module 2, Theorem 2.2 and Corollary 2.2 can be applied even in situations where the function h is differentiable everywhere on S_X except possibly at a finite number of points.

Example 2.4

Let X be random variable with p.d.f.

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases},$$

and let $T = X^2$

(i) Show that T is a random variable of absolutely continuous type;
(ii) Find the distribution function of T and hence find its p.d.f.;
(iii) Find the p.d.f. of T directly (i.e., without finding the distribution function of T).

Solution. (i) and (iii). Clearly $T = X^2$ is a random variable (being a continuous function of random variable X). We have $S_X = S_T = (0, \infty)$. Also $h(x) = x^2, x \in S_X$, is strictly increasing on S_X with inverse function $h^{-1}(x) = \sqrt{x}, x \in S_T$. Using Corollary 2.1 it follows that $T = X^2$ is a random variable of absolutely continuous type with p.d.f.

$$f_T(t) = \begin{cases} f_X(\sqrt{t}) \left| \frac{d}{dt}(\sqrt{t}) \right|, & \text{if } t > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{e^{-\sqrt{t}}}{2\sqrt{t}}, & \text{if } t > 0 \\ 0, & \text{otherwise} \end{cases}.$$

(ii) We have $F_T(t) = P(\{X^2 \leq t\}), t \in \mathbb{R}$. Clearly, for $t < 0, F_T(t) = 0$. For $t \geq 0$,

$$F_T(t) = P(\{-\sqrt{t} \leq X \leq \sqrt{t}\})$$

$$= \int_{-\sqrt{t}}^{\sqrt{t}} f_X(x)dx$$

$$= \int_0^{\sqrt{t}} e^{-x}dx$$

$$= 1 - e^{-\sqrt{t}}.$$

Therefore the distribution function of T is

$$F_T(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1 - e^{-\sqrt{t}}, & \text{if } t \geq 0 \end{cases}.$$

Clearly F_T is differentiable everywhere except at $t = 0$. Therefore, using Remark 4.2 (vii) of Module 2, we conclude that the random variable T is of absolutely continuous type with p.d.f. $f_T(t) = F_T^{'}(t)$, if $t \neq 0$. At $t = 0$ we may assign any arbitrary non-negative value to $f_T(0)$. Thus a p.d.f. of T is

$$f_T(t) = \begin{cases} \frac{e^{-\sqrt{t}}}{2\sqrt{t}}, & \text{if } t > 0 \\ 0, & \text{otherwise} \end{cases}. \blacksquare$$

Example 2.5

Let X be a random variable with p.d.f.

$$f_X(x) = \begin{cases} \frac{|x|}{2}, & \text{if} -1 < x < 1 \\ \frac{x}{3}, & \text{if } 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases},$$

and let $T = X^2$

(i) Show that T is a random variable of absolutely continuous type;

(ii) Find the distribution function of T and hence find its p.d.f;

(iii) Find the p.d.f. of T directly (i.e., without finding the distribution function of T).

Solution. (i) and (iii). Clearly $T = X^2$ is a random variable (being a continuous function of random variable X). We have $S_X = (-1, 0) \cup (0, 2) = S_1 \cup S_2$, say. Also $h(x) = x^2, x \in S_X$, is strictly decreasing in $S_1 = (-1, 0)$ with inverse function $h_1^{-1}(t) = -\sqrt{t}$; $h(x) = x^2, x \in S_X$, is strictly increasing in $S_2 = (0, 2)$, with inverse function $h_2^{-1}(t) = \sqrt{t}$; $h(S_1) = (0, 1)$ and $h(S_2) = (0, 4)$. Using Theorem 2.2 it follows that $T = X^2$ is a random variable of absolutely continuous type with p.d.f.

$$f_T(t) = f_X(-\sqrt{t}) \left| \frac{d}{dt}(-\sqrt{t}) \right| I_{(0,1)}^{(t)} + f_X(\sqrt{t}) \left| \frac{d}{dt}(\sqrt{t}) \right| I_{(0,4)}^{(t)}$$

$$= \begin{cases} \frac{1}{2}, & \text{if } 0 < t < 1 \\ \frac{1}{6}, & \text{if } 1 < t < 4 \\ 0, & \text{otherwise} \end{cases}.$$

(ii) We have $F_T(t) = P(\{X^2 \leq t\}), t \in \mathbb{R}$. Since $P(\{X \in (-1, 2)\}) = 1$, we have $P(\{T \in (0, 4)\}) = 1$.

Therefore, for $t < 0$, $F_T(t) = P(\{T \leq t\}) = 0$ and, for $t \geq 4$, $F_T(t) = P(\{T \leq t\}) = 1$. For $t \in [0,4)$, we have

$$F_T(t) = P\left(\left\{-\sqrt{t} \leq X \leq \sqrt{t}\right\}\right)$$
$$= \int_{-\sqrt{t}}^{\sqrt{t}} f_X(x)\, dx$$
$$= \begin{cases} \displaystyle\int_{-\sqrt{t}}^{\sqrt{t}} \frac{|x|}{2} dx, & \text{if } 0 \leq t < 1 \\ \displaystyle\int_{-1}^{1} \frac{|x|}{2} dx + \int_{1}^{\sqrt{t}} \frac{x}{3} dx, & \text{if } 1 \leq t < 4 \end{cases}.$$

Therefore, the distribution function of T is

$$F_T(t) = \begin{cases} 0, & \text{if } t < 0 \\ \dfrac{t}{2}, & \text{if } 0 \leq t < 1 \\ \dfrac{t+2}{6}, & \text{if } 1 \leq t < 4 \\ 1, & \text{if } t \geq 4 \end{cases}.$$

Clearly F_T is differentiable everywhere except at points $0, 1$ and 4. Using Remark 4.2 (vii) of Module 2 it follows that the random variable T is of absolutely continuous type with a p.d.f.

$$f_T(t) = \begin{cases} \dfrac{1}{2}, & \text{if } 0 < t < 1 \\ \dfrac{1}{6}, & \text{if } 1 < t < 4 \\ 0, & \text{otherwise} \end{cases}. \blacksquare$$

Note that a Borel function of a discrete type random variable is a random variable of discrete type (see Theorem 1.1). Theorem 2.2 provides sufficient conditions under which a Borel function of an absolutely continuous type random variable is of absolutely continuous type. The following example illustrates that, in general, a Borel function of an absolutely continuous type random variable may not be of absolutely continuous type.

Example 2.6

Let X be a random variable of absolutely continuous type with p.d.f.

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & s\text{otherwise} \end{cases}'$$

and let $T = [x]$, where, for $x \in \mathbb{R}, [x]$ denotes the largest integer not exceeding x. Show that T is a random variable of discrete type and find its p.m.f.

Solution. For $a \in \mathbb{R}$, we have

$$T^{-1}\big((-\infty, a]\big) = (-\infty, [a] + 1) \in \mathcal{B}_1.$$

It follows that T is a random variable. Also $S_X = (0, \infty)$. Since $P(\{X \in S_X\}) = 1$, we have $P(T \in \{0, 1, 2, \cdots\}) = 1$. Also, for $i \in \{0, 1, 2, ...\}$.

$$\begin{aligned} P(\{T = i\}) &= P(\{i \le X < i + 1\}) \\ &= \int_i^{i+1} f_X(x) dx \\ &= \int_i^{i+1} e^{-x} dx \\ &= (1 - e^{-1})e^{-i} \\ &> 0. \end{aligned}$$

Consequently the random variable T is of discrete type with support $S_T = \{0, 1, 2, ...\}$ and p.m.f.

$$f_T(t) = P(\{T = t\}) = \begin{cases} (1 - e^{-1})e^{-t}, & \text{if } t \in \{0, 1, 2, ...\} \\ 0\,, & \text{otherwise} \end{cases}. \blacksquare$$