MODULE 6

RANDOM VECTOR AND ITS JOINT DISTRIBUTION

LECTURES 25-36

Topics

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LECTURE 25

Topics

6.1 MULTIVARIATE DISTRIBUTIONS

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A (univariate) random variable describes a numerical characteristic of a typical outcome of a random experiment. In many situations we may be interested in simultaneously studying two or more numerical characteristics of outcomes of a random experiment. To make the above discussion more clear consider the following example.

Example 1.1

Two distinguishable dice (labeled as D_1 and D_2) are thrown simultaneously. Here the sample space is $\Omega = \{(i,j): i,j \in \{1,\dots,6\}\}$, where the outcome $(i,j) \in \Omega$ indicates that i number of dots are observed on the uppermost face of die D_1 and j number of dots are observed on uppermost face of die D_2. For $(i,j) \in \Omega$, define

$X_1((i,j)) = i + j =$ sum of number of dots on uppermost faces of two dice

and

$X_2((i,j)) = |i - j| =$ absolute difference of number of dots on uppermost faces of two dice.

It may be of interest to study numerical characteristics X_1 and X_2 simultaneously. This amounts to the study of the function $\underline{X} = (X_1, X_2) : \Omega \longrightarrow \mathbb{R}$ defined on the sample space Ω. ■

Throughout $\mathbb{R}^p = \{\underline{x} = (x_1, \dots, x_p): -\infty < x_i < \infty, i = 1, \dots, p\}$ will denote the p-dimensional Euclidean space and, for a set $B \subseteq \mathbb{R}^p$ and a function $\underline{X} = (X_1, X_2, \dots, X_p): \Omega \longrightarrow \mathbb{R}^p$,

$$\underline{X}^{-1}(B) \stackrel{\text{def}}{=} \{\omega \in \Omega: \underline{X}(\omega) = (X_1(\omega), X_2(\omega), \dots, X_p(\omega)) \in B\}.$$

Let $(\Omega, \mathcal{F}, P)$ be a given probability space.

Definition 1.1

A function $\underline{X} = (X_1, \dots, X_p): \Omega \longrightarrow \mathbb{R}^p$ is called a p-dimensional *random vector* (or simple a random vector) if $\underline{X}^{-1}((-\underline{\infty}, \underline{a}]) \in \mathcal{F}, \forall\, \underline{a} = (a_1, \dots, a_p) \in \mathbb{R}^p$; here $(-\underline{\infty}, \underline{a}] = (-\infty, a_1] \times \cdots \times (-\infty, a_p]$. ■

A 1-dimensional random vector will simply be referred to as a random variable. Clearly, a function $\underline{X} = (X_1, \dots, X_p): \Omega \longrightarrow \mathbb{R}^p$ is a random vector if

$$\{\omega \in \Omega : X_1(\omega) \leq a_1, \dots, X_p(\omega) \leq a_p\} \in \mathcal{F}, \ \ \forall\, \underline{a} = (a_1, \dots, a_p) \in \mathbb{R}^p.$$

For $\underline{a} = (a_1, \dots, a_p) \in \mathbb{R}^p$, $\underline{b} = (b_1, \dots, b_p) \in \mathbb{R}^p$, and $a_i < b_i, i = 1, \dots, p$, define

$$(\underline{a}, \underline{b}) = (a_1, b_1) \times \cdots \times (a_p, b_p) \equiv \prod_{i=1}^{p} (a_i, b_i),$$

$$(\underline{a}, \underline{b}] = (a_1, b_1] \times \cdots \times (a_p, b_p] \equiv \prod_{i=1}^{p} (a_i, b_i],$$

$$[\underline{a}, \underline{b}) = [a_1, b_1) \times \cdots \times [a_p, b_p) \equiv \prod_{i=1}^{p} [a_i, b_i),$$

$$[\underline{a}, \underline{b}] = [a_1, b_1] \times \cdots \times [a_p, b_p] \equiv \prod_{i=1}^{p} [a_i, b_i],$$

$$(-\underline{\infty}, \underline{b}) = (-\infty, b_1) \times \cdots \times (-\infty, b_p) \equiv \prod_{i=1}^{p} (-\infty, b_i),$$

$$(\underline{a}, \underline{\infty}) = (a_1, \infty) \times \cdots \times (a_p, \infty) \equiv \prod_{i=1}^{p} (a_i, \infty),$$

and

$$[\underline{a}, \underline{\infty}) = [a_1, \infty) \times \cdots \times [a_p, \infty) \equiv \prod_{i=1}^{p} [a_i, \infty).$$

Further define

$$\mathcal{C}_0 = \{(-\underline{\infty}, \underline{b}] : \underline{b} \in \mathbb{R}^p\},$$

$$\mathcal{C}_1 = \{(\underline{a}, \underline{b}) : \underline{a}, \underline{b} \in \mathbb{R}^p,\ a_i < b_i, i = 1, \dots, p\},$$

$$\mathcal{C}_2 = \{(\underline{a}, \underline{b}] : \underline{a}, \underline{b} \in \mathbb{R}^p,\ \ a_i < b_i, i = 1, \dots, p\},$$

$$\mathcal{C}_3 = \{[\underline{a}, \underline{b}) : \underline{a}, \underline{b} \in \mathbb{R}^p,\ a_i < b_i, i = 1, \dots, p\},$$

$$\mathcal{C}_4 = \{[\underline{a}, \underline{b}] : \underline{a}, \underline{b} \in \mathbb{R}^p,\ \ a_i < b_i, i = 1, \dots, p\},$$

$$\mathcal{C}_5 = \{(-\underline{\infty}, \underline{b}) : \underline{b} \in \mathbb{R}^p\},$$

$$\mathcal{C}_6 = \{(\underline{a}, \underline{\infty}) : \underline{a} \in \mathbb{R}^p\},$$

and

$$\mathcal{C}_7 = \{[\underline{a}, \underline{\infty}) : \underline{a} \in \mathbb{R}^p\}.$$

As in the case of $p = 1$ it can be shown that

(i) $\mathcal{B}_p =$ the Borel σ-field in $\mathbb{R}^p = \sigma(\mathcal{C}_i), i = 0, 1, \dots, 7$;

(ii) $\{\underline{a}\} \in \mathcal{B}_p,\ \forall\, \underline{a} \in \mathbb{R}^p,$ i.e., $\mathcal{B}_p$ contains all singleton subsets of $\mathbb{R}^p$;

(iii) If $B \subseteq \mathbb{R}^p$ is countable then $B \in \mathcal{B}_p$;

(iv) There exists a set $A \subseteq \mathbb{R}^p$ such that $A \notin \mathcal{B}_p$;

(v) $\underline{X}: \Omega \longrightarrow \mathbb{R}^p$ is a p-dimensional random vector if, and only if one of the following equivalent conditions hold:

a) $\underline{X}^{-1}(B) \in \mathcal{F},\ \ \forall\, B \in \mathcal{C}_1$;

b) $\underline{X}^{-1}(B) \in \mathcal{F},\ \ \forall\, B \in \mathcal{C}_2$;

c) $\underline{X}^{-1}(B) \in \mathcal{F},\ \forall\, B \in \mathcal{C}_3$;

d) $\underline{X}^{-1}(B) \in \mathcal{F},\ \forall\, B \in \mathcal{C}_4$;

e) $\underline{X}^{-1}(B) \in \mathcal{F},\ \forall\, B \in \mathcal{C}_5$;

f) $\underline{X}^{-1}(B) \in \mathcal{F},\ \forall\, B \in \mathcal{C}_6$;

g) $\underline{X}^{-1}(B) \in \mathcal{F},\ \forall\, B \in \mathcal{C}_7$;

h) $\underline{X}^{-1}(B) \in \mathcal{F},\ \forall\, B \in \mathcal{B}_p$.

(vi) If $\underline{X} = (X_1, \dots, X_p)$ is a p-dimensional random vector and $g_i: \mathbb{R}^p \to \mathbb{R}, i = 1, \dots, k$, are Borel functions $\left(\text{i.e.}, \ g_i^{-1}(B) \in \mathcal{F}, \forall\, B \in \mathcal{B}_p, i = 1, \dots, k\right)$ then $\left(g_1(\underline{X}), \dots, g_k(\underline{X})\right)$ is a k-dimensional random vector.

(vii) If $\underline{X}: \Omega \to \mathbb{R}^p$ is a p-dimensional random vector then
$\underline{X}^{-1}(\{\underline{a}\}) = \{\underline{\omega} \in \Omega: X_1(\omega) = a_1, \dots, X_p(\omega) = a_p\} \in \mathcal{F}, \forall\, \underline{a} = (a_1, \dots, a_p) \in \mathbb{R}^p$;

(viii) The function $P_{\underline{X}}: \mathcal{B}_p \to \mathbb{R}$ given by,

$$P_{\underline{X}}(B) = P\left(\underline{X}^{-1}(B)\right), B \in \mathcal{B}_p,$$

is a probability measure on $\mathcal{B}_p$ $\left(\text{i.e.}, (\mathbb{R}^p, \mathcal{B}_p, P_X)\right.$ is a probability space), called the *probability measure induced* by $\underline{X}$.

Example 1.2

Let $A, B \subseteq \Omega$. Define $\underline{X} = (X_1, X_2): \Omega \to \mathbb{R}^2$ by

$$X_1(\omega) = I_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases};$$

and

$$X_2(\omega) = I_B(\omega) = \begin{cases} 1 & \text{if } \omega \in B \\ 0, & \text{if } \omega \notin B \end{cases}.$$

Then, for $\underline{a} = (a_1, a_2) \in \mathbb{R}^2$,

$$\underline{X}^{-1}\left((-\underline{\infty}, \underline{a}]\right) = \{\omega \in \Omega: X_1(\omega) \le a_1, X_2(\omega) \le a_2\}$$

$$= \begin{cases} \phi, & \text{if } a_1 < 0 \text{ or } a_2 < 0 \\ A^c \cap B^c, & \text{if } 0 \le a_1 < 1, 0 \le a_2 < 1 \\ A^c, & \text{if } 0 \le a_1 < 1, a_2 \ge 1 \\ B^c, & \text{if } a_1 \ge 1, 0 \le a_2 < 1 \\ \Omega, & \text{if } a_1 \ge ,1,\ a_2 \ge 1 \end{cases}.$$

Thus

$$\underline{X} \text{ is a random vector} \Leftrightarrow \underline{X}^{-1}\left((-\underline{\infty}, \underline{a}]\right) \in \mathcal{F}, \forall \underline{a} \in \mathbb{R}^2$$

$$\Leftrightarrow A^c, B^c \in \mathcal{F}$$
$$\Leftrightarrow A, B \in \mathcal{F}$$

Thus $\underline{X}$ is a random vector if, and only if, $A, B \in \mathcal{F}$. ■

Theorem 1.1

Let $\underline{X} = (X_1, \dots, X_p): \Omega \to \mathbb{R}^p$ be a given function. Then $\underline{X}$ is a random vector if, and only if $X_1, \dots, X_p$ $(X_i: \Omega \to \mathbb{R}, i = 1, \dots, p)$ are random variables.

Proof. First suppose that $\underline{X} = (X_1, \dots, X_p)$ is a random vector. Then, for $a \in \mathbb{R}$, and for fixed $i \in \{1, \dots, p\}$

$$X_i^{-1}((-\infty, a]) = \bigcap_{n=1}^{\infty} \underbrace{\underline{X}^{-1}((-\infty, n] \times \cdots \times (-\infty, n] \times (-\infty, a] \times (-\infty, n] \times \cdots \times (-\infty, n])}_{\in \mathcal{F}, \forall n=1,2,\dots},$$

$$\in \mathcal{F}$$

i.e., X_i is a random variable.

Conversely suppose that $X_1, \dots, X_p$ are random variables. Then, for $\underline{a} = (a_1, \dots, a_p) \in \mathbb{R}^p$,

$$\underline{X}^{-1}\left((-\underline{\infty}, \underline{a}]\right) = \{\omega \in \Omega: X_i(\omega) \le a_i, i = 1, \dots, p\}$$

$$= \bigcap_{i=1}^{p} \{\omega \in \Omega: X_i(\omega) \le a_i\}$$

$$= \bigcap_{i=1}^{p} \underbrace{X_i^{-1}((-\infty, a_i])}_{\in \mathcal{F}}$$

$$\in \mathcal{F}$$

i.e., $\underline{X}$ is a random vector. ■

Remark 1.1

When Ω is countable we have $\mathcal{F} = \mathcal{P}(\Omega)$ and, therefore, any function $\underline{X} = (X_1, \dots, X_p): \Omega \to \mathbb{R}^p$ is a random vector. ■

Definition 1.2

(i) The *joint distribution function* of a p-dimensional random vector $\underline{X} = (X_1, \dots, X_p): \Omega \to \mathbb{R}^p$ is defined by

$$F_{\underline{X}}(x_1, \dots, x_p) = P(\{\omega \in \Omega: X_1(\omega) \le x_1, \dots, X_p(\omega) \le x_p\}), \qquad \underline{x} = (x_1, \dots, x_p) \in \mathbb{R}^p.$$

(ii) The joint distribution function of any subset of random variables $X_1, \dots, X_p$ is called a *marginal distribution function* of $F_{\underline{X}}(\cdot)$. ■

Remark 1.2

(i) If $F_{\underline{X}}(\cdot)$ is the distribution function of a p-dimensional random vector $\underline{X} = (X_1, \dots, X_p)$ then

$$\begin{aligned} F_{\underline{X}}(\underline{x}) &= P(\{X_i \le x_i, i = 1, \dots, p\}) \\ &= P\left(\underline{X}^{-1}\left((-\underline{\infty}, \underline{x}]\right)\right) \\ &= P_{\underline{X}}\left((-\underline{\infty}, \underline{x}]\right) \\ &= P\left(\bigcap_{i=1}^{p} \{X_i \le x_i\}\right) \\ &= P\left(\bigcap_{i=1}^{p} X_i^{-1}\left((-\infty, x_i]\right)\right), \quad \underline{x} = (x_1, \dots, x_p) \in \mathbb{R}^p. \end{aligned}$$

(ii) Let $F_{X_1,\dots,X_p}(\cdot)$ be the distribution function of a random vector $\underline{X} = (X_1, \dots, X_p)$ and let $\underline{\beta} = (\beta_1, \dots, \beta_p)$ be a permutation of $(1, \dots, p)$. Then

$$\begin{aligned} F_{X_1,\dots,X_p}(x_1, \dots, x_p) &= P\left(\bigcap_{i=1}^{p} \{X_i \le x_i\}\right) \\ &= P\left(\bigcap_{i=1}^{p} \{X_{\beta_i} \le x_{\beta_i}\}\right) \\ &= F_{X_{\beta_1},\dots,X_{\beta_p}}\left(x_{\beta_1}, \dots, x_{\beta_p}\right), \quad \underline{x} = (x_1, \dots, x_p) \in \mathbb{R}^p. \end{aligned}$$

It follows that the distribution function of $\left(X_{\beta_1}, \dots, X_{\beta_p}\right)$ is given by

$$F_{X_{\beta_1},\dots,X_{\beta_p}}(y_1, \dots, y_p) = F_{X_1,\dots,X_p}\left(y_{\gamma_1}, \dots, y_{\gamma_p}\right), \quad \underline{y} = (y_1, \dots, y_p) \in \mathbb{R}^p,$$

where $\underline{\gamma} = (\gamma_1, \dots, \gamma_p)$ is the inverse permutation of $\underline{\beta} = (\beta_1, \dots, \beta_p)$. To illustrate this point, consider $p = 3$ and $\underline{\beta} = (\beta_1, \beta_2, \beta_3) = (2, 3, 1)$. Then the inverse permutation of $\underline{\beta} = (\beta_1, \beta_2, \beta_3)$ is $\underline{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = (3, 1, 2)$, and therefore, for $\underline{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$,

$$\begin{aligned}
F_{X_{\beta_1}, X_{\beta_2}, X_{\beta_3}}(y_1, y_2, y_3) &= F_{X_2, X_3, X_1}(y_1, y_2, y_3) \\
&= P(\{X_2 \le y_1, X_3 \le y_2, X_1 \le y_3\}) \\
&= P(\{X_1 \le y_3, X_2 \le y_1, X_3 \le y_2\}) \\
&= F_{X_1, X_2, X_3}(y_3, y_1, y_2) \\
&= F_{X_1, X_2, X_3}\left(y_{\gamma_1}, y_{\gamma_2}, y_{\gamma_3}\right).
\end{aligned}$$

(iii) Note that a distribution function $F_{X_1, \dots, X_p}(x_1, \dots, x_p)$ is increasing in each argument when other arguments are kept fixed.

We recall the following results from the theory of multivariable calculus.

Lemma 1.1

Let $D \subseteq \mathbb{R}^p$ and let $g: D \to \mathbb{R}$ be a function such that:

(i) g is bounded above, i.e., there exists a real constant M such that $g(\underline{x}) \le M, \forall \underline{x} \in D$;

(ii) for every fixed $i \in \{1, \dots, p\}$ and fixed $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_p) \in \mathbb{R}^{p-1}$, $g(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_p)$ is non decreasing in $t \in D_i = \{y \in \mathbb{R}: (x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_p) \in D\}$. Then $\lim_{\underline{x} \to \underline{\infty}} g(\underline{x})$ exists and, for any permutation $\underline{\beta} = (\beta_1, \dots, \beta_p)$ of $(1, \dots, p)$,

$$\lim_{x_{\beta_p} \to \infty} \cdots \lim_{x_{\beta_1} \to \infty} g(x_1, \dots, x_p) = \lim_{\underline{x} \to \underline{\infty}} g(\underline{x}) \cdot$$

In particular all iterated limits

$$\lim_{x_{\beta_p} \to \infty} \cdots \lim_{x_{\beta_1} \to \infty} g(x_1, \dots, x_p), \quad (\beta_1, \dots, \beta_p) \in S_p,$$

exist and are equal, where S_p denotes the set of all permutations of $(1, \dots, p)$. We denote the common value of all iterated limits by

$$\lim_{\substack{x_i \to \infty \\ i = 1, \dots, p}} g(\underline{x}) \cdot \blacksquare$$

Note that if $F_{\underline{X}}(\cdot)$ is a distribution function in $\mathbb{R}^p$ $(p \geq 2)$ then, for a fixed $k \in \{1, \dots, p-1\}$ and fixed $\left(x_{k+1}, \dots, x_p\right) \in \mathbb{R}^{p-k}$, the function $g: \mathbb{R}^k \to \mathbb{R}$, given by

$$g(x_1, \dots, x_k) = F_{\underline{X}}\left(x_1, \dots, x_k, x_{k+1}, \dots, x_p\right),$$

satisfies properties (i) and (ii) stated in Lemma 1.1. Therefore, for fixed $\left(x_{k+1}, \dots, x_p\right) \in \mathbb{R}^{p-k}$

$$\lim_{\underline{x}^* \to \underline{\infty}} F_{\underline{X}}\left(x_1, \dots, x_k, x_{k+1}, \dots, x_p\right) = \lim_{\substack{x_i \to \infty \\ i=1,\cdots,k}} F_{\underline{X}}\left(x_1, \dots, x_k, x_{k+1}, \dots, x_p\right),$$

where $\underline{x}^* = (x_1, \dots, x_k)$.

Lemma 1.2

Let $F_{\underline{X}}(\cdot)$ be the distribution function of a p-dimensional $(p \geq 2)$ random vector $\underline{X} = \left(X_1, \dots, X_p\right)$. For a fixed positive integer $k \in \{1, \dots, p-1\}$, let $\underline{Y} = (X_1, \dots, X_k)$ and let $\underline{Z} = \left(X_{k+1}, \dots, X_p\right)$ so that $\underline{X} = (\underline{Y}, \underline{Z})$. Then the marginal distribution function of $\underline{Y} = (Y_1, \dots, Y_k)$ is given by

$$F_{\underline{Y}}(x_1, \dots, x_k) = \lim_{\substack{x_i \to \infty \\ i=k+1,\dots,p}} F_{\underline{X}}\left(x_1, \dots, x_k, x_{k+1}, \dots, x_p\right), \quad (x_1, \dots, x_k) \in \mathbb{R}^k.$$

Proof. For fixed $x_1, \dots, x_{p-1} \in \mathbb{R}$

$$\lim_{x_p \to \infty} F_{\underline{X}}\left(x_1, \dots, x_p\right) = \lim_{x_p \to \infty} P\left(\bigcap_{i=1}^{p} X_i^{-1}\left((-\infty, x_i]\right)\right)$$

$$= \lim_{n \to \infty} P\left(\underbrace{\left(\bigcap_{i=1}^{p-1} X_i^{-1}\left((-\infty, x_i]\right)\right) \bigcap X_p^{-1}\left((-\infty, n]\right)}_{=A_n \uparrow}\right)$$

$$= P\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$= P\left(\bigcap_{i=1}^{p-1} X_i^{-1}\left((-\infty, x_i]\right)\right) \quad \text{since} \left(\bigcup_{n=1}^{\infty} X_p^{-1}\left((-\infty, n]\right) = \Omega\right)$$

$$= F_{X_1,\ldots,X_{p-1}}\left(x_1,\ldots,x_{p-1}\right). \tag{1.1}$$

Now the assertion follows on recursively using (1.1). ■

Remark 1.3

Let $\underline{X} = \left(X_1,\ldots,X_p\right)$ be a random vector and let $\underline{\beta} = \left(\beta_1,\ldots,\beta_p\right) \in S_p$, the set of all permutations of $(1,\ldots,p)$. If $\underline{\gamma} = \left(\gamma_1,\ldots,\gamma_p\right)$ is the inverse permutation of $(\beta_1,\ldots,\beta_p)$ then, for a fixed $k \in \{1,\ldots,p-1\}$, the marginal distribution function of $\left(X_{\beta_1},\ldots,X_{\beta_k}\right)$ is given by

$$F_{X_{\beta_1},\ldots,X_{\beta_k}}(x_1,\ldots,x_k) = \lim_{\substack{x_j\to\infty \\ j=k+1,\cdots,p}} F_{X_{\beta_1},\ldots,X_{\beta_p}}\left(x_1,\ldots,x_p\right)$$

$$= \lim_{\substack{x_j\to\infty \\ j=k+1,\ldots,p}} F_{X_1,\ldots,X_p}\left(x_{\gamma_1},\ldots,x_{\gamma_p}\right). \blacksquare$$

Let $\underline{X} = \left(X_1,\ldots,X_p\right)$ be a random vector and let $\underline{a} = \left(a_1,\ldots,a_p\right)$, $\underline{b} = \left(b_1,\ldots,b_p\right) \in \mathbb{R}^p$. Then

$$P(\{a_1 < X_1 \le b_1\}) = P(\{X_1 \le b_1\}) - P(\{X_1 \le a_1\})$$

$$= F_{X_1}(b_1) - F_{X_1}(a_1). \tag{1.2}$$

Also

$$P(\{a_1 < X_1 \le b_1, a_2 < X_2 \le b_2\}) = P(\{a_1 < X_1 \le b_1, X_2 \le b_2\})$$

$$- P(\{a_1 < X_1 \le b_1, X_2 \le a_2\})$$

$$= [P(\{X_1 \le b_1, X_2 \le b_2\}) - P(\{X_1 \le a_1, X_2 \le b_2\})]$$

$$-[P(\{X_1 \le b_1, X_2 \le a_2\}) - P(\{X_1 \le a_1, X_2 \le a_2\})]$$

$$= F_{X_1,X_2}(b_1,b_2) - \left[F_{X_1,X_2}(a_1,b_2) + F_{X_1,X_2}(b_1,a_2)\right]$$

$$+ F_{X_1,X_2}(a_1,a_2). \tag{1.3}$$

To write the expression of $P(\{a_i < X_i \le b_i, i = 1,\ldots,p\})$ in a closed form define, for $k \in \{0,1,\ldots,p\}$,

$$\Delta_{k,p} \equiv \Delta_{k,p}\left(\left(\underline{a},\underline{b}\right]\right) = \left\{\underline{z} \in \mathbb{R}^p : z_i \in \{a_i,b_i\}, i = 1,\ldots,p, \text{and } k \text{ of } z_1,\ldots,z_p \text{ are } a_j\text{'s}\right\}. \tag{1.4}$$

Note that the set $\Delta_{k,p}$ has $\binom{p}{k}$ elements. From (1.2) and (1.3) we have

$$P(\{a_1 < X_1 \leq b_1\}) = F_{X_1}(b_1) - F_{X_1}(a_1) = \sum_{k=0}^{1}(-1)^k \sum_{z \in \Delta_{k,1}} F_{X_1}(z) \tag{1.5}$$

and

$$P(\{a_i < X_i \leq b_i, i = 1, 2\}) = \sum_{k=0}^{2}(-1)^k \sum_{(z_1,z_2) \in \Delta_{k,2}} F_{X_1,X_2}(z_1, z_2) \tag{1.6}$$

Lemma 1.3

Let $\underline{X} = (X_1, \dots, X_p): \Omega \to \mathbb{R}^p$ be a random vector and let $\underline{a} = (a_1, \dots, a_p)$, $\underline{b} = (b_1, \dots, b_p) \in \mathbb{R}^p$. Let $\Delta_{k,p} \equiv \Delta_{k,p}\left((\underline{a}, \underline{b}]\right), k = 0, 1, \dots, p$ be as defined in (1.4). Then

$$P(\{a_i < X_i \leq b_i, i = 1, \dots, p\}) = \sum_{\mathrm{k}=0}^{\mathrm{p}}(-1)^{\mathrm{k}} \sum_{\underline{z} \in \Delta_{k,p}((\underline{a},\underline{b}])} F_{\underline{X}}(\underline{z}). \tag{1.7}$$

Proof. From (1.5) and (1.6) it is clear that the result is true for $p = 1$ and $p = 2$. Now suppose that (1.7) holds for general p-dimensional random vectors. For simplicity assume that $P(\{a_{p+1} < X_{p+1} \leq b_{p+1}\}) > 0$. Then, for $(X_1, \dots, X_p, X_{p+1}): \Omega \to \mathbb{R}^{p+1}, \underline{a} = (a_1, \dots, a_p) \in \mathbb{R}^p$, $\underline{b} = (b_1, \dots, b_p) \in \mathbb{R}^p$, $\underline{a}^* = (a_1, \dots, a_p, a_{p+1}) \in \mathbb{R}^{p+1}$ and $\underline{b}^* = (b_1, \dots, b_p, b_{p+1}) \in \mathbb{R}^{p+1}$.

$P(\{a_i < X_i \leq b_i, i = 1, \dots, p+1\})$

$= P(\{a_i < X_i \leq b_i, i = 1, \dots, p\} | \{a_{p+1} < X_{p+1} \leq b_{p+1}\}) P(\{a_{p+1} < X_{p+1} \leq b_{p+1}\})$

$$= \sum_{k=0}^{p}(-1)^k \sum_{\underline{z} \in \Delta_{k,p}((\underline{a},\underline{b}])} P(\{X_i \leq z_i, i = 1, \dots, p\} | \{a_{p+1} < X_{p+1} \leq b_{p+1}\}) P(\{a_{p+1} < X_{p+1} \leq b_{p+1}\})$$

$$= \sum_{k=0}^{p}(-1)^k \sum_{\underline{z} \in \Delta_{k,p}((\underline{a},\underline{b}])} P(\{X_i \leq z_i, i = 1, \dots, p, a_{p+1} < X_{p+1} \leq b_{p+1}\})$$

$$= \sum_{k=0}^{p}(-1)^k \sum_{\underline{z} \in \Delta_{k,p}((\underline{a},\underline{b}])} \left[P(\{X_1 \leq z_1, \dots, X_p \leq z_p, X_{p+1} \leq b_{p+1}\})\right.$$

$$\left. - P(\{X_1 \leq z_1, \dots, X_p \leq z_p, X_{p+1} \leq a_{p+1}\})\right].$$

It is easy to verify that

$$\sum_{k=0}^{p}(-1)^k \sum_{\underline{z}\in\Delta_{k,p}((\underline{a},\underline{b}])} \left[P\left(\{X_1 \le z_1, \dots, X_p \le z_p, X_{p+1} \le b_{p+1}\}\right)\right.$$

$$\left. -P\left(\{X_1 \le z_1, \dots, X_p \le z_p, X_{p+1} \le a_{p+1}\}\right)\right]$$

$$= \sum_{k=0}^{p+1}(-1)^k \sum_{\underline{t}\in\Delta_{k,p+1}((\underline{a}^*,\underline{b}^*])} F_{X_1,\cdots,X_{p+1}}(t_1, \dots, t_{p+1}),$$

and therefore the assertion follows by principle of mathematical induction. ■

Theorem 1.2

Let $F_{\underline{X}}(\cdot)$ be the distribution of a p-dimensional random vector $\underline{X} = (X_1, \dots, X_p)$. Then

(i) $\lim\limits_{\substack{x_i\to\infty \\ i=1,\dots,p}} F_{\underline{X}}(x_1, \dots, x_p) = 1;$

(ii) For each fixed $i \in \{1, \dots, p\}$ and fixed $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_p) \in \mathbb{R}^{p-1}$,

$$\lim_{y\to-\infty} F_{\underline{X}}(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_p) = 0\,;$$

(iii) $F_{\underline{X}}(x_1, \dots, x_p)$ is right continuous in each argument (keeping other arguments fixed);

(iv) For each rectangle $(\underline{a}, \underline{b}] \in \mathbb{R}^p$

$$\sum_{k=0}^{p}(-1)^k \sum_{\underline{z}\in\Delta_{k,p}((\underline{a},\underline{b}])} F_{\underline{X}}(\underline{z}) \ge 0.$$

Proof. Note that, for $(\underline{a}, \underline{b}] \in \mathbb{R}^p$,

$$\sum_{k=0}^{p}(-1)^k \sum_{\underline{z}\in\Delta_{k,p}((\underline{a},\underline{b}])} F_{\underline{X}}(\underline{z}) = P\left(\underline{X} \in (\underline{a}, \underline{b}]\right) \ge 0. \qquad \text{(using Lemma 1.3)}$$

This proves (iv).

For notational convenience we will provide the proofs of (i) - (iii) for only $p = 2$.

(i) For fixed $x_1 \in \mathbb{R}$

$$\lim_{x_2\to\infty} F_{X_1,X_2}(x_1, x_2) = \lim_{n\to\infty} F_{X_1,X_2}(x_1, n)$$

$$= \lim_{n\to\infty} P(\underbrace{X_1^{-1}((-\infty, x_1]) \cap X_2^{-1}((-\infty, n])}_{=A_n\uparrow})$$

$$= P\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$= P(X_1^{-1}\,(-\infty, x_1]).$$

Therefore,

$$\lim_{x_1\to\infty}\lim_{x_2\to\infty} F_{X_1,X_2}(x_1,x_2) = \lim_{x_1\to\infty} P(X_1^{-1}\,(-\infty, x_1])$$

$$= \lim_{n\to\infty} P\left(\underbrace{X_1^{-1}((-\infty, n])}_{=B_n\uparrow}\right)$$

$$= P\left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$= P(\Omega)$$

$$= 1.$$

(ii) Fix $x_2 \in \mathbb{R}$. Then

$$\lim_{x_1\to-\infty} F_{X_1,X_2}(x_1,x_2) = \lim_{n\to\infty} P(\underbrace{X_1^{-1}(-\infty,-n]\ \cap X_2^{-1}((-\infty, x_2])}_{=B_n\downarrow})$$

$$= P\left(\bigcap_{n=1}^{\infty} B_n\right)$$

$$= P(\phi)$$

$$= 0.$$

Similarly, for fixed $x_1 \in \mathbb{R}$

$$\lim_{x_2\to-\infty} F_{X_1,X_2}(x_1,x_2) = 0.$$

(iii) Let $\underline{x} = (x_1, x_2) \in \mathbb{R}^2$. Then

$$\lim_{h\downarrow 0} F_{X_1,X_2}(x_1+h, x_2) = \lim_{n\to\infty} F_{X_1,X_2}\left(x_1+\frac{1}{n}, x_2\right)$$

$$= \lim_{n\to\infty} P(\underbrace{X_1^{-1}\left((-\infty, x_1+\frac{1}{n}]\right)\ \cap\ X_2^{-1}((-\infty, x_2])}_{=C_n\downarrow})$$

$$= P\left(\bigcap_{n=1}^{\infty} C_n\right)$$

$$= P\left(X_1^{-1}((-\infty, x_1]) \cap X_2^{-1}((-\infty, x_2])\right)$$

$$= F_{X_1,X_2}(x_1, x_2),$$

i.e., for every fixed $x_2 \in \mathbb{R}$, $F_{X_1,X_2}(x_1, x_2)$ is right continuous in x_1. Similarly, for every fixed $x_1 \in \mathbb{R}$, $F_{X_1,X_2}(x_1, x_2)$ is right continuous in x_2. ■

Remark 1.4

(i) Let $\Delta_p = \cup_{k=0}^{p} \Delta_{k,p}$. Then Δ_p is the set of 2^p vertices of the rectangle $\left(\underline{a}, \underline{b}\right] \in \mathbb{R}^p$. For example, for $p = 1$, $\left(\underline{a}, \underline{b}\right] = (a_1, b_1]$, $\Delta_1 = \{a_1, b_1\}$ and, for $p = 2$, $\left(\underline{a}, \underline{b}\right] = (a_1, b_1] \times (a_2, b_2]$, $\Delta_2 = \{(b_1, b_2), (b_1, a_2), (a_1, b_2), (a_1, a_2)\}$.

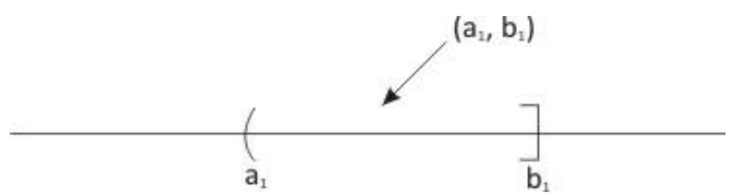

Figure 1.1

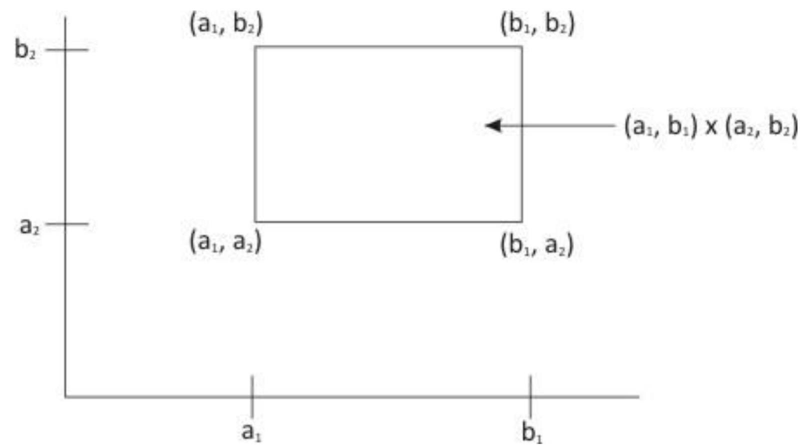

Figure 1.2

(ii) Note that, for $p = 1$, the assertion (iv) of Theorem 1.2 reduces to $F_X(b) \geq F_X(a), \forall -\infty < a \leq b < \infty$ i.e., F_X is non-decreasing.