MODULE 5

SOME SPECIAL ABSOLUTELY CONTINUOUS DISTRIBUTIONS

LECTURE 24

Topics

5.4 NORMAL DISTRIBUTION

5.4 NORMAL DISTRIBUTION

Recall that

$$\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$$

$$= 2\int_0^\infty e^{-x^2} dx$$

$$= \int_{-\infty}^\infty e^{-x^2} dx$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^\infty e^{-\frac{x^2}{2}} dx$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{x^2}{2}} dx = 1$$

$$\Rightarrow \boxed{\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1, \quad \forall \mu \in \mathbb{R}, \text{and } \sigma > 0.} \qquad (4.1)$$

Definition 4.1

(i) Let $\mu \in \mathbb{R}$ and $\sigma > 0$ be real constants. An absolutely continuous type random variable X is said to follow a *normal distribution* with parameters μ and σ^2 (written as $X \sim N(\mu, \sigma^2)$) if its probability density function is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

(ii) The $N(0,1)$ distribution is called the *standard normal distribution*. ■

The p.d.f. and the d.f. of $N(0,1)$ distributions will be denoted by $\phi(\cdot)$ and $\Phi(\cdot)$ respectively, i.e.,

$$\boxed{\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty,}$$

and

$$\boxed{\Phi(z) = \int_{-\infty}^{z} \phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx.}$$

Clearly if $X \sim N(\mu, \sigma^2)$ then

$$f_X(\mu - x) = f_X(\mu + x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \forall x \in \mathbb{R}$$

i.e., $\boxed{\text{if } X \sim N(\mu, \sigma^2) \text{ then } X - \mu \stackrel{d}{=} \mu - X.}$

Thus the distribution of $X \sim N(\mu, \sigma^2)$ is symmetric about μ. Since the p.d.f. $f_X(x)$ of $X \sim N(\mu, \sigma^2)$ is strictly increasing in $(-\infty, \mu)$ and strictly decreasing in (μ, ∞) the distribution of $X \sim N(\mu, \sigma^2)$ is unimodal with mode at μ.

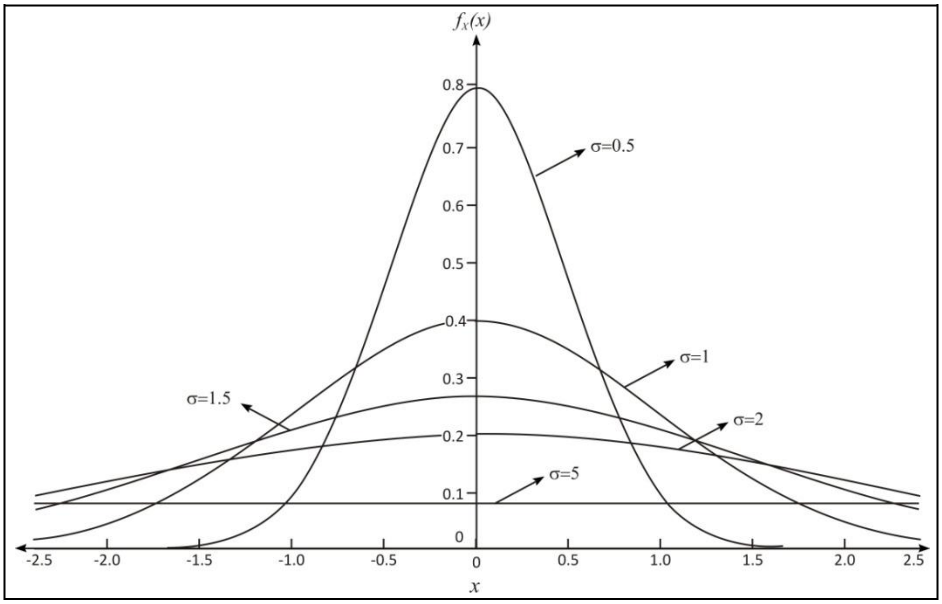

Figure 4.1. Plots of p.d.f.s of $N(0, \sigma^2)$ distributions.

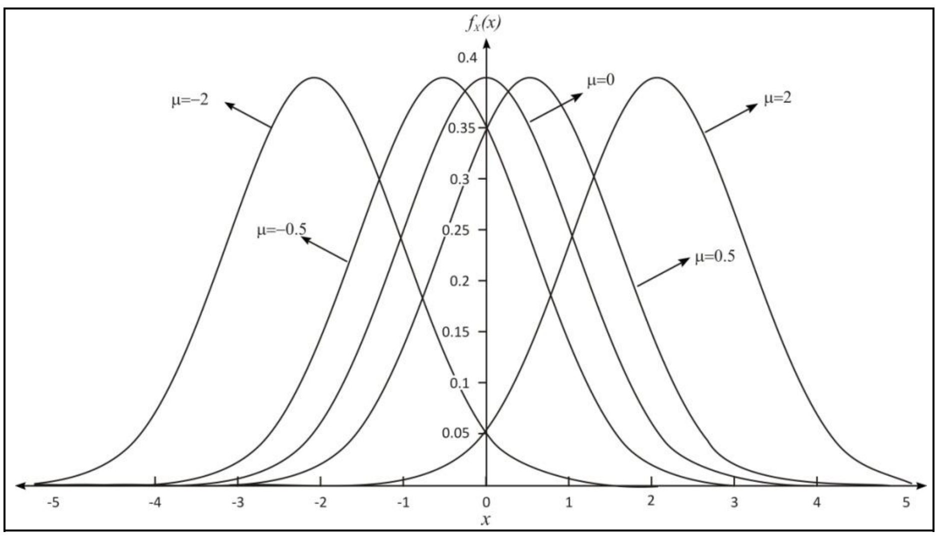

Figure 4.2. Plots of p.d.f.s of $N(\mu, 1)$ distributions

Since the distribution of $X \sim N(\mu, \sigma^2)$ is symmetric about μ (i.e., $X - \mu \stackrel{d}{=} \mu - X$) we have, for $x \in \mathbb{R}$,

$$P(\{X - \mu \leq -x\}) = P(\{\mu - X \leq -x\})$$

$$\Longrightarrow P(\{X \leq \mu - x\}) = P(\{X \geq \mu + x\})$$

$$\Longrightarrow P(\{X \leq \mu - x\}) = 1 - P(\{X \leq \mu + x\}).$$

Thus,

$$\boxed{X \sim N(\mu,\sigma^2) \Longrightarrow F_X(\mu - x) = 1 - F_X(\mu + x), \forall x \in \mathbb{R}, \text{ and } F_X(\mu) = \frac{1}{2}.}$$

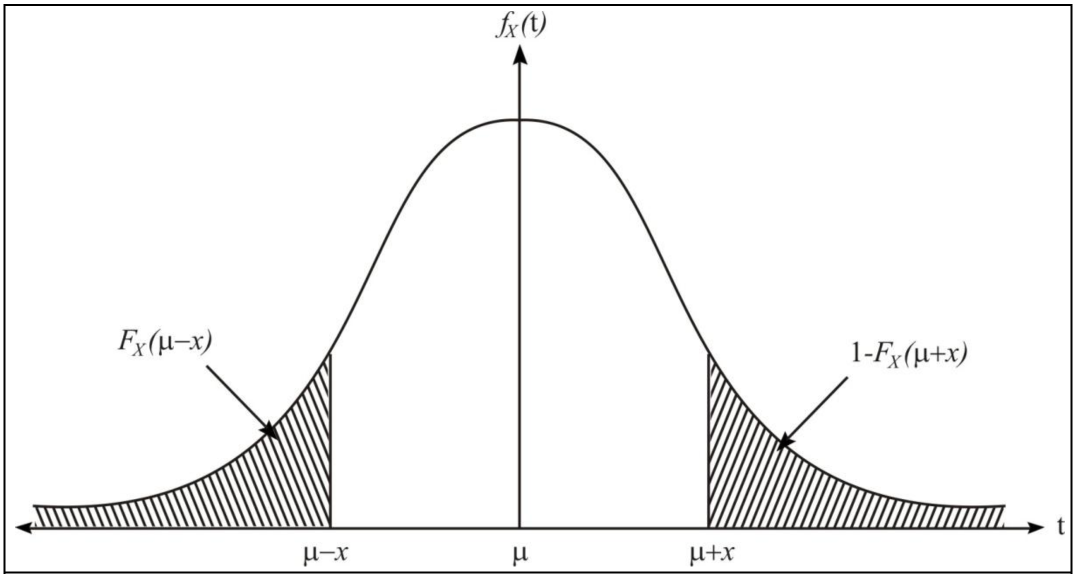

Figure 4.3

In particular

$$\boxed{\Phi(-z) = 1 - \Phi(z), \forall z \in \mathbb{R}, \text{ and } \Phi(0) = \frac{1}{2}.}$$

It follows that the median of $X \sim N(\mu, \sigma^2)$ is μ.

Suppose that $X \sim N(\mu, \sigma^2)$. Then the p.d.f. of $Z = \frac{X-\mu}{\sigma}$ is given by

$$f_Z(z) = f_X(\mu + \sigma z)|\sigma| I_{(-\infty,\infty)}(z)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \qquad -\infty < z < \infty.$$

i.e., $$\boxed{X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X - \mu}{\sigma} \sim N(0,1).}$$

Also if $Z \sim N(0,1)$ then the d.f. of $Y = Z^2$ is given by

$$F_Y(t) = P(\{Z^2 \leq t\}).$$

Clearly, for $t < 0$, $F_Y(t) = 0$, and, for $t \geq 0$,

$$F_Y(t) = P\left(\{-\sqrt{t} \le Z \le \sqrt{t}\}\right)$$

$$= \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{t}} e^{-\frac{z^2}{2}} dz$$

$$= \int_0^{t} \frac{1}{\sqrt{2\pi}} z^{\frac{1}{2}-1} e^{-\frac{z}{2}} dz.$$

Therefore, if $Z \sim N(0,1)$ then a p.d.f. of $Y = Z^2$ is given by

$$f_Y(t) = \begin{cases} \frac{1}{\sqrt{2\pi}} t^{\frac{1}{2}-1} e^{-\frac{t}{2}}, & \text{if } t > 0 \\ 0, & \text{otherwise} \end{cases},$$

which is a p.d.f. of a χ_1^2 distribution. Thus

$$\boxed{Z \sim N(0,1) \Rightarrow Y = Z^2 \sim \chi_1^2.}$$

We have the following result.

Theorem 4.1

(i) Let $X \sim N(\mu, \sigma^2)$, for some $\mu \in (-\infty, \infty)$ and $\sigma > 0$. Then

(a) $X - \mu \overset{d}{=} \mu - X$, i.e. , the distribution of X is symmetric about μ;

(b) $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$;

(ii) If $Z \sim N(0,1)$ then $Y = Z^2 \sim \chi_1^2$. ■

In the following theorem we derive some more properties of normal distribution.

Theorem 4.2

Let $X \sim N(\mu, \sigma^2)$, for some $\mu \in (-\infty, \infty)$ and $\sigma > 0$.

(i) Then the moment generating function of of X is given by

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}, \quad t \in \mathbb{R}.$$

(ii) Let $Y = aX + b$, where $a \in \mathbb{R} - \{0\}$ and $b \in \mathbb{R}$. Then $Y \sim N(a\mu + b, a^2\sigma^2)$.

(iii) Let $Z = \frac{X-\mu}{\sigma}$, so that $Z \sim N(0,1)$ (Theorem 4.1 (i)). Then

$$E(Z^r) = \begin{cases} 0\,, & \text{if } r \text{ is odd} \\ \dfrac{r!}{2^{\frac{r}{2}}\left(\frac{r}{2}\right)!}\,, & \text{if } r \text{ is even}\cdot \end{cases}$$

(iv) Then

$$\text{Mean} = \mu_1^{'} = E(X) = \mu,$$
$$\text{Variance} = \mu_2 = \text{Var}(X) = \sigma^2,$$
$$\text{Coefficient of skewness} = \beta_1 = 0,$$

and $$\text{Kurtosis} = \gamma_1 = 3.$$

Proof.

(i) For $t \in \mathbb{R}$

$$M_X(t) = E(e^{tX})$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-\frac{z^2}{2}} dz$$
$$= e^{\mu t+\frac{\sigma^2 t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-t\sigma)^2}{2}} dz$$
$$= e^{\mu t+\frac{\sigma^2 t^2}{2}}. \qquad \text{(using(4.1))}$$

(ii) The m.g.f. of $Y = aX + b$ is given by

$$M_Y(t) = E(e^{tY})$$
$$= E(e^{t(aX+b)})$$
$$= e^{tb} E(e^{atX})$$
$$= e^{tb} M_X(at)$$
$$= e^{tb} e^{\mu at+\frac{\sigma^2 a^2 t^2}{2}}, \quad t \in \mathbb{R}$$
$$= e^{(a\mu+b)t+\frac{\sigma^2 a^2 t^2}{2}}, \quad t \in \mathbb{R},$$

which is the m.g.f. of $N(a\mu+b, a^2\sigma^2)$ distribution. Thus by the uniqueness of m.g.f.s it follows that $Y \sim N(a\mu+b, a^2\sigma^2)$.

(iii) Let $Z=\frac{X-\mu}{\sigma}$. Then, by Theorem 4.1, $Z\sim N(0,1)$ and by (i) the m.g.f. of Z is

$$M_z(t)=e^{\frac{t^2}{2}},\quad t\in\mathbb{R}$$

$$=\sum_{k=0}^{\infty}\frac{t^{2k}}{2^k k!},\quad t\in\mathbb{R}.$$

For $r\in\{1,2,\dots\}$

$$E(Z^r)=\text{Coefficient of }\frac{t^r}{r!}\text{ in the expansion of }M_Z(t)$$

$$=\begin{cases}0, & \text{if } r \text{ is odd}\\ \dfrac{r!}{2^{\frac{r}{2}}\left(\frac{r}{2}\right)!}, & \text{if } r \text{ is even}\cdot\end{cases}$$

(iv) Let $Z=\frac{X-\mu}{\sigma}$. Then, by (iii), $E(Z)=0$ and $E(Z^2)=1$, i.e.,

$$E\left(\frac{X-\mu}{\sigma}\right)=0\quad\text{and } E\left(\left(\frac{X-\mu}{\sigma}\right)^2\right)=1$$

$$\Rightarrow E(X)=\mu \text{ and } E((X-\mu)^2)=\sigma^2$$

$$\Rightarrow E(X)=\mu \text{ and } \mu_2=\mathrm{Var}(X)=\sigma^2.$$

Also by (iii)

$$\mu_3=E((X-\mu)^3)=\sigma^3E(Z^3)=0$$

$$\Rightarrow \text{Skewness}=\beta_1=\frac{\mu_3}{\mu_2^{\frac{3}{2}}}=0.$$

Moreover

$$\mu_4=E((X-\mu)^4)=\sigma^4E(Z^4)=3\sigma^4$$

$$\Rightarrow \text{Kurtosis}=\gamma_1=\frac{\mu_4}{\mu_2^2}=3. \blacksquare$$

Remark 4.1

In the $N(\mu,\sigma^2)$ distribution the parameters $\mu\in\mathbb{R}$ and $\sigma^2(\sigma>0)$ are respectively the mean and the variance of the distribution. Also σ is the standard deviation of the distribution. Moreover, for $N(\mu, \sigma^2)$ distribution, the mean, the median and the mode coincide at μ. ■

If $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ and therefore

$$F_X(x) = P(\{X \leq x\})$$

$$= P\left(\left\{\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right\}\right)$$

$$= P\left(\left\{Z \leq \frac{x-\mu}{\sigma}\right\}\right)$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right), x \in \mathbb{R}.$$

Thus,

$$\boxed{X \sim N(\mu, \sigma^2) \Longrightarrow F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right), x \in \mathbb{R}.}$$

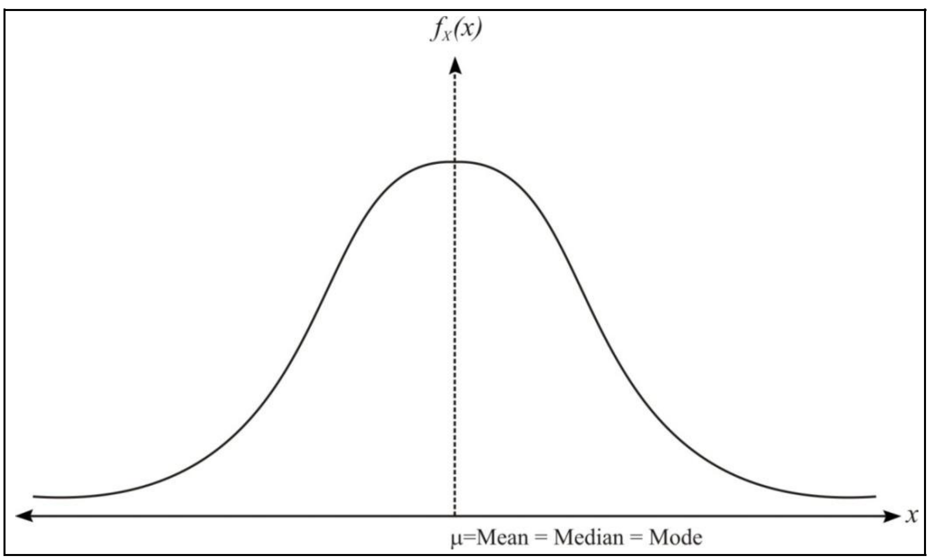

Figure 4.4. Plot of p.d.f. of $N(\mu, \sigma^2)$ distribution

Let τ_α be the $(1-\alpha)$ - th quantile of $N(0,1)$ distribution, i.e., let $\Phi(\tau_\alpha) = 1-\alpha$. Then clearly

$$\Phi(-\tau_\alpha) = 1 - \Phi(\tau_\alpha) = \alpha.$$

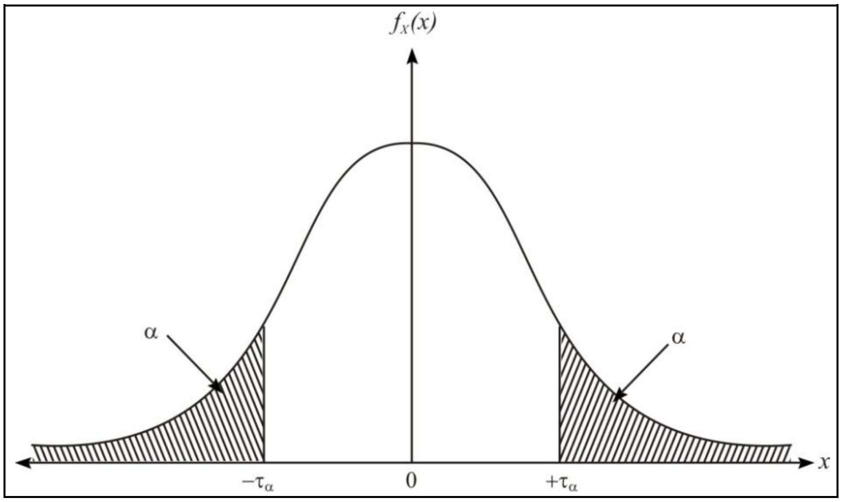

Figure 4.5. $(1-\alpha)$-th quantile of $N(0,1)$ distribution

The following table provides various quantiles of $N(0,1)$ distribution

Table 4.1.$(1-\alpha)$-th quantile of $N(0,1)$ distribution for selected values of α

α	.001	.005	.01	.025	.05	.1	.25
τ_α	3.092	2.5758	2.326	1.96	1.6499	1.282	.675

Values of $\Phi(x)$, for various values of x are tabulated in the following table.

Table 4.2 Values of $\Phi(x)$

$$\Phi(x) = P(\{X \leq x\}) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

x	9	8	7	6	5	4	3	2	1	0
-3.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
-3.80	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-3.70	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
-3.60	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0002	0.0002
-3.50	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
-3.40	0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
-3.30	0.0003	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0005	0.0005	0.0005
-3.20	0.0005	0.0005	0.0005	0.0006	0.0006	0.0006	0.0006	0.0006	0.0007	0.0007
-3.10	0.0007	0.0007	0.0008	0.0008	0.0008	0.0008	0.0009	0.0009	0.0009	0.0010
-3.00	0.0010	0.0010	0.0011	0.0011	0.0011	0.0012	0.0012	0.0013	0.0013	0.0013
-2.90	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
-2.80	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
-2.70	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
-2.60	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
-2.50	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
-2.40	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
-2.30	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
-2.20	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
-2.10	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
-2.00	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
-1.90	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
-1.80	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
-1.70	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
-1.60	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
-1.50	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
-1.40	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
-1.30	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
-1.20	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
-1.10	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
-1.00	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
-0.90	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
-0.80	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
-0.70	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
-0.60	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
-0.50	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
-0.40	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
-0.30	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
-0.20	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
-0.10	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
-0.00	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

Table 4.2 (Continued): Values of $\Phi(x)$

$$\Phi(x) = P(\{X \leq x\}) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

x	0	1	2	3	4	5	6	7	8	9
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.40	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.50	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.60	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.70	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.80	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.00	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.20	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.30	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.40	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.50	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.60	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.70	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.80	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.90	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.00	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.10	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.20	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.30	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.40	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.50	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.60	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.70	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.80	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Example 4.1

Let $X \sim N(2,4)$. Find $P(\{X \leq 0\}), P(\{|X| \geq 2\}),\ P(\{1 < X \leq 3\})$ and $P(\{X \leq 3\}|\{X > 1\})$.

Solution. We have

$$P(\{X \leq 0\}) = \Phi\left(\frac{0-2}{2}\right) = \Phi(-1) = .1587;$$

$$\begin{aligned} P(\{|X| \geq 2\}) &= P(\{X \leq -2\}) + P(\{X \geq 2\}) \\ &= \Phi\left(\frac{-2-2}{2}\right) + 1 - \Phi\left(\frac{2-2}{2}\right) \\ &= \Phi(-2) + 1 - \Phi(0) \\ &= .0228 + 0.5 \\ &= 0.5228; \end{aligned}$$

$$\begin{aligned} P(\{1 < X \leq 3\}) &= P(\{X \leq 3\}) - P(\{X \leq 1\}) \\ &= \Phi\left(\frac{3-2}{2}\right) - \Phi\left(\frac{1-2}{2}\right) \\ &= \Phi(0.5) - \Phi(-0.5) \\ &= 2\Phi(0.5) - 1 \qquad (\text{since } \Phi(x) + \Phi(-x) = 1, \forall x \in \mathbb{R}) \\ &= 2 \times .6915 - 1 \\ &= 0.383; \end{aligned}$$

and

$$\begin{aligned} P(\{X \leq 3\}|\{X > 1\}) &= \frac{P(\{1 < X \leq 3\})}{P(\{X > 1\})} \\ &= \frac{0.383}{1 - \Phi(-0.5)} \\ &= \frac{0.383}{1 - 0.3085} \\ &= .5539. \blacksquare \end{aligned}$$

Example 4.2

Let $Z \sim N(0,1)$ and let $Y = [|Z|]$, where, for a real number x, $[x]$ denotes the largest integer not exceeding x.

(i) Find $E(|Z|^r),\ r > -1$;
(ii) Show that $E(Y) = 2\sum_{i=1}^{\infty}[1-\Phi(i)]$.

Solution.

(i) Let $X = Z^2$. Then, by Theorem 4.1 (ii), $X \sim \chi_1^2$. Therefore

$$\begin{aligned} E(|Z|^r) &= E(X^{\frac{r}{2}}) \\ &= \int_0^{\infty} x^{\frac{r}{2}} \frac{e^{-\frac{x}{2}} x^{\frac{1}{2}-1}}{\sqrt{2\pi}}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x^{\frac{r+1}{2}-1} e^{-\frac{x}{2}} dx \\ &= \frac{2^{\frac{r+1}{2}}\,\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{2\pi}} \\ &= \frac{2^{\frac{r}{2}}\,\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi}}. \end{aligned}$$

(ii) Note that $P(Y \in \{0,1,2,\ldots\}) = 1$. Therefore by Theorem 3.1 (iv), Module 3,

$$\begin{aligned} E(Y) &= \sum_{n=1}^{\infty} P(\{Y \geq n\}) \\ &= \sum_{n=1}^{\infty} P(\{|Z| \geq n\}) \\ &= \sum_{n=1}^{\infty} [P(\{Z \leq -n\}) + P(\{Z \geq n\})] \\ &= \sum_{n=1}^{\infty} [\Phi(-n) + 1 - \Phi(n)] \\ &= 2\sum_{n=1}^{\infty} [1-\Phi(n)]. \blacksquare \end{aligned}$$