MODULE 2

RANDOM VARIABLE AND ITS DISTRIBUTION

LECTURE 11

Topics

2.4 TYPES OF RANDOM VARIABLES: DISCRETE, CONTINUOUS AND ABSOLUTELY CONTINUOUS

The following theorem establishes that any function $g: \mathbb{R} \to [0, \infty)$ satisfying the above two properties is a p.d.f. of some r.v. of absolutely continuous type.

Theorem 4.3

Suppose that there exists a non-negative function $g: \mathbb{R} \to \mathbb{R}$ satisfying:

(i) $g(x) \geq 0, \forall x \in \mathbb{R}$;

(ii) $\int_{-\infty}^{\infty} g(t) dt = 1$.

Then there exists an absolutely continuous type random variable X on some probability space $(\Omega, \mathcal{B}_1, P)$ such that the p.d.f. X is g.

Proof. Define the set function $P: \mathcal{B}_1 \to \mathbb{R}$ by

$$P(B) = \int_{-\infty}^{\infty} g(t) I_B(t)\, dt, B \in \mathcal{B}_1.$$

It is easy to verify that P is a probability measure on $\mathcal{B}_1$, i.e., $(\mathbb{R}, \mathcal{B}_1, P)$ is a probability space. Define $X: \mathbb{R} \to \mathbb{R}$ by $X(\omega) = \omega,\ \omega \in \mathbb{R}$. Clearly X is a random variable on the probability space $(\mathbb{R}, \mathcal{B}_1, P)$. The space $(\mathbb{R}, \mathcal{B}_1, P)$ is also the probability space induced by X. Clearly, for $x \in \mathbb{R}$,

$$\begin{aligned} F_X(x) &= P_X((-\infty, x]) \\ &= P((-\infty, x]) \\ &= \int_{-\infty}^{\infty} g(\mathrm{t})\, I_{(-\infty, x]}(\mathrm{t})\, dt \end{aligned}$$

$$= \int_{-\infty}^{x} g(t)dt.$$

It follows that X is of absolutely continuous type and g is the p.d.f. of X. ■

Example 4.7

Let X be r.v. with the d.f.

$$F_X(x) = \begin{cases} 0, & \text{if} \quad x < 0 \\ \frac{x^2}{2}, & \text{if} \quad 0 \le x < 1 \\ \frac{x}{2}, & \text{if} \quad 1 \le x < 2 \\ 1, & \text{if} \quad x \ge 2 \end{cases}.$$

Show that the r.v. X is of absolutely continuous type and find the p.d.f. of X.

Solution. Clearly F_X is differentiable everywhere except at points 1 and 2. Let $D = \{1, 2\}$, so that

$$\int_{-\infty}^{\infty} F_X'(t) I_{D^c}(t) dt = \int_0^1 t dt + \int_1^2 \frac{1}{2} dt = 1.$$

Using Remark 4.2 (vii) it follows that the r.v. X is of absolutely continuous type with a p.d.f.

$$f_X(x) = \begin{cases} x, & \text{if} \quad 0 \le x < 1 \\ a, & \text{if} \quad x = 1 \\ \frac{1}{2}, & \text{if} \quad 1 < x < 2 \\ b, & \text{if} \quad x = 2 \\ 0, & \text{otherwise} \end{cases},$$

where a and b are arbitrary nonnegative constants. In particular a p.d.f. of X is

$$f_X(x) = \begin{cases} x, & \text{if} \quad 0 < x < 1 \\ \frac{1}{2}, & \text{if} \quad 1 < x < 2. \\ 0, & \text{otherwise} \end{cases}$$ ■

Example 4.8

Let X be an absolutely continuous type r.v. with p.d.f.

$$f_X(x) = \begin{cases} k - |x|, & \text{if} \quad |x| < \frac{1}{2}, \\ 0, & \text{otherwise} \end{cases}$$

where $k \in \mathbb{R}$.

(i) Find the value of constant k;

(ii) Evaluate: $P(\{X<0\}), P(\{X \le 0\}), P\left(\left\{0<X \le \frac{1}{4}\right\}\right)$, $P\left(\left\{0 \le X<\frac{1}{4}\right\}\right)$ and $P\left(\left\{-\frac{1}{8} \le X \le \frac{1}{4}\right\}\right)$;

(iii) Find the d.f. of X.

Solution.

(i) Since f_X is a p.d.f.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\Rightarrow \int_{-1/2}^{1/2} (k-|x|)\,dx = 1$$

$$\Rightarrow k = \frac{5}{4}.$$

Also, for $k=\frac{5}{4}, f_X(x) \ge 0, \forall x \in \mathbb{R}$.

(ii) Since the r.v. X is of absolutely continuous type, $P(\{X=x\})=0,\ \forall x \in \mathbb{R}$ (see Remark 4.2 (iv)). Therefore

$$P(\{X<0\}) = P(\{X \le 0\}) = \int_{-\infty}^{0} f_X(x)dx = \int_{-1/2}^{0}\left(\frac{5}{4}+x\right)dx = \frac{1}{2},$$

$$P\left(\left\{0<X \le \frac{1}{4}\right\}\right) = P\left(\left\{0 \le X<\frac{1}{4}\right\}\right) = \int_{0}^{1/4} f_X(x)dx = \int_{0}^{1/4}\left(\frac{5}{4}-x\right)dx = \frac{9}{32},$$

and

$$P\left(-\frac{1}{8} \le X \le \frac{1}{4}\right) = \int_{-1/8}^{1/4} f_X(x)dx$$

$$= \int_{-1/8}^{0}\left(\frac{5}{4}+x\right)dx + \int_{0}^{1/4}\left(\frac{5}{4}-x\right)dx$$

$$= \frac{55}{128}\cdot$$

(iii) Clearly, for $x < -\frac{1}{2}, F_X(x) = 0$ and, for $x \geq \frac{1}{2}, F_X(x) = 1$. For $-\frac{1}{2} \leq x < 0$,

$$F_X(x) = \int_{-\infty}^{x} f_X(\mathrm{t})\, dt$$

$$F_X(x) = \int_{-1/2}^{x} \left(\frac{5}{4} + \mathrm{t}\right) dt$$

$$= \frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2},$$

and, for $0 \leq x < \frac{1}{2}$,

$$F_X(x) = \int_{-\infty}^{x} f_X(\mathrm{t})\, dt$$

$$= \int_{-\frac{1}{2}}^{0} \left(\frac{5}{4} + \mathrm{t}\right) dt + \int_{0}^{x} \left(\frac{5}{4} - \mathrm{t}\right) dt$$

$$= -\frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2}\cdot$$

Therefore the d.f. of X is

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < -\frac{1}{2} \\ -\frac{x|x|}{2} + \frac{5}{4}x + \frac{1}{2}, & \text{if} \quad -\frac{1}{2} \leq x < \frac{1}{2}. \blacksquare \\ 1, & \text{if} \quad x \geq \frac{1}{2} \end{cases}$$

Theorem 4.4

Let F_X be the distribution function of a random variable X. Then F_X can be decomposed as $F_X(x) = \alpha F_d(x) + (1-\alpha)\, F_c(x),\ x \in \mathbb{R}$, where $\alpha \in [0,1]$, F_d is a distribution function of some random variable of discrete type and F_c is a distribution function of some random variables of continuous type.

Proof. Let D_X denote the set of discontinuity points of F_X. We will prove the result for the case when D_X is finite. The idea of the proof for the case when D_X is countably infinite is similar but slightly involved. First suppose that $D_X = \phi$. In this case the result follows trivially by taking $\alpha = 0$ and $F_c \equiv F_X$. Now suppose that $D_X = \{a_1, a_2, \dots, a_n\}$ for some $n \in \mathbb{N}$. Without loss of generality let $-\infty < a_1 < a_2 < \cdots < a_n < \infty$.

Define

$$p_i = P(\{X = a_i\}) = F_x(a_i) - F_X(a_i -), \quad i = 1, 2, \dots, n,$$

so that $p_i > 0, \ i = 1, \dots, n$.

Let $\alpha = \sum_{i=1}^{n} p_i$ so that $\alpha \in (0, 1]$. Define $F_d \colon \mathbb{R} \to \mathbb{R}$ by

$$F_d(x) = \begin{cases} 0, & \text{if} \quad x < 0 \\ \dfrac{\sum_{j=1}^{i} p_j}{\alpha} & \text{if} \quad a_i \le x < a_{i+1}, \ \ i = 1, \dots, n-1. \\ 1, & \text{if} \quad x \ge a_n \end{cases}$$

Clearly F_d is non-decreasing, right continuous $F_d(-\infty) = 0$ and $F_d(\infty) = 1$. The set of discontinuity points of F_d is $\{a_1, \dots, a_n\}$ and

$$\begin{aligned} \sum_{i=1}^{n} [F_d(a_i) - F_d(a_i -)] &= \sum_{i=1}^{n} \left\{ \frac{\sum_{j=1}^{i} p_{\mathrm{j}}}{\alpha} - \frac{\sum_{j=1}^{i-1} p_j}{\alpha} \right\} \\ &= \frac{1}{\alpha} \sum_{i=1}^{n} p_i \\ &= 1. \end{aligned}$$

It follows that F_d is a d.f. of some r.v. of discrete type. If $\alpha = 1$ then the result follows on taking $F_d \equiv F_X$. Now suppose that $\alpha \in (0, 1)$.

Define $F_c \colon \mathbb{R} \to \mathbb{R}$ by

$$F_C(x) = \frac{F_X(x) - \alpha F_d(x)}{1 - \alpha}, \quad x \in \mathbb{R}.$$

For $A \subseteq \mathbb{R}$, let $S(A) = \{i \in \{1, \dots, n\} \colon a_i \in A\}$. Then, for $-\infty < x < y < \infty$,

$$\begin{aligned} F_d(y) - F_d(x) &= \sum_{i \in S((-\infty, y])} \frac{p_i}{\alpha} - \sum_{i \in S((-\infty, x])} \frac{p_i}{\alpha} \\ &= \sum_{i \in S((x, y])} \frac{p_i}{\alpha}, \end{aligned}$$

$$F_X(y) - F_X(x) = P(\{x < X \le y\})$$

$$\geq \sum_{i \in S((x,y])} p_i$$

$$= \alpha\big(F_d(y) - F_d(x)\big),$$

where, for $A \subseteq \mathbb{R}$, $\sum_{i \in S(A)} p_i = 0$, if $S(A) = \phi$.

Therefore, for $-\infty < x < y < \infty$,

$$F_c(y) - F_c(x) = \frac{F_X(y) - F_X(x) - \alpha\big(F_d(y) - F_d(x)\big)}{1-\alpha}$$

$$\geq 0,$$

i.e., F_c is non-decreasing. Note that $F_X(a_i) - F_X(a_i -) = \alpha\big(F_d(a_i) - F_d(a_i -)\big) = p_i, i = 1, \ldots, n$ and $F_X(x) - F_X(x -) = 0$, if $x \notin \{a_1, \ldots, a_n\}$. It follows that

$$F_c(x) - F_c(x -) = \frac{F_X(x) - F_X(x -) - \alpha\big(F_d(x) - F_d(x -)\big)}{1-\alpha}$$

$$= 0, \qquad \forall x \in \mathbb{R},$$

i.e., F_c is continuous everywhere. Since $F_X(-\infty) = F_d(-\infty) = 0$ and $F_X(\infty) = F_d(\infty) = 1$ we also have $F_c(-\infty) = 0$ and $F_c(\infty) = 1$. Therefore F_c is a d.f. of some r.v. of continuous type. Hence the result follows. ■

Example 4.9

Let X be a r.v. having the d.f. F_X (see Example 3.2 (iii)) given by

$$F_X(x) = \begin{cases} 0, & \text{if} \quad x < 0 \\ \frac{x}{4}, & \text{if} \quad 0 \leq x < 1 \\ \frac{x}{3}, & \text{if} \quad 1 \leq x < 2 \\ \frac{3x}{8}, & \text{if} \quad 2 \leq x < \frac{5}{2} \\ 1, & \text{if} \quad x \geq \frac{5}{2} \end{cases}.$$

Decompose F_X as $F_X(x) = \alpha H_d(x) + (1-\alpha) H_c(x), x \in \mathbb{R}$, where $\alpha \in [0,1]$, H_d is a d.f. of some r.v. X_d of discrete type and H_c is a d.f. of some r.v. X_c of continuous type.

Solution. The set of discontinuity points F_X is $D_X = \{1, 2, 5/2\}$ with

$$p_1 = P\{X = 1\} = F_X(1) - F_X(1 -) = \frac{1}{12},$$

$$p_2 = P\{X = 2\} = F_X(2) - F_X(2-) = \frac{1}{12},$$

and

$$p_3 = P\left(\left\{X = \frac{5}{2}\right\}\right) = F_X\left(\frac{5}{2}\right) - F_X\left(\frac{5}{2}-\right) = \frac{1}{16}.$$

Thus,

$$\alpha = p_1 + p_2 + p_3 = \frac{11}{48},$$

$$P(\{X_d = 1\}) = \frac{p_1}{\alpha} = \frac{4}{11}, \qquad P(\{X_d = 2\}) = \frac{p_2}{\alpha} = \frac{4}{11}, \qquad P\left(\left\{X_d = \frac{5}{2}\right\}\right) = \frac{p_3}{\alpha} = \frac{3}{11},$$

$$H_d(x) = \begin{cases} 0, & \text{if} \quad x < 1 \\ \frac{4}{11}, & \text{if} \quad 1 \le x < 2 \\ \frac{8}{11}, & \text{if} \quad 2 \le x < \frac{5}{2} \\ 1, & \text{if} \quad x \ge \frac{5}{2} \end{cases}$$

and

$$H_C(x) = \frac{H(x) - \alpha H_d(x)}{1 - \alpha}$$

$$= \begin{cases} 0, & \text{if} \quad x < 0 \\ \frac{12}{37}x, & \text{if} \quad 0 \le x < 1 \\ \frac{4(4x-1)}{37}, & \text{if} \quad 1 \le x < 2 \\ \frac{2(9x-4)}{37}, & \text{if} \quad 2 \le x < \frac{5}{2} \\ 1, & \text{if} \quad x \ge \frac{5}{2} \end{cases}.$$

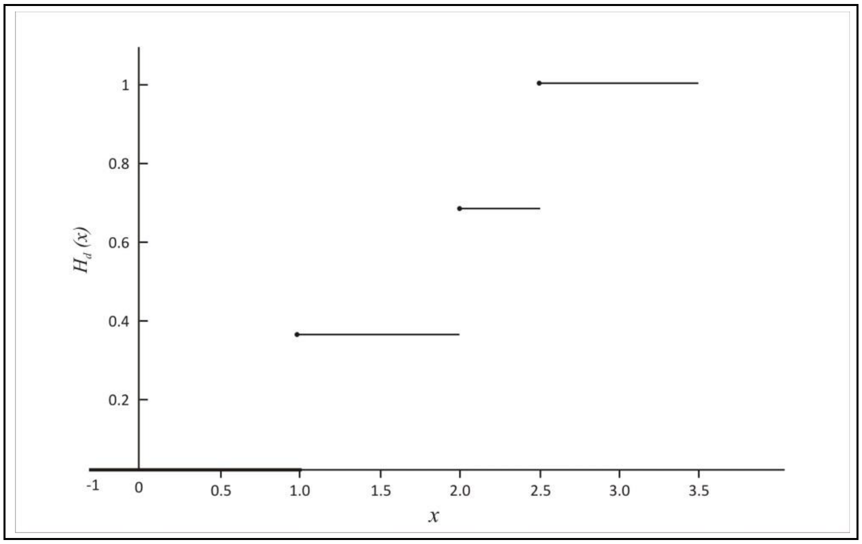

Figure 4.5. Plot of distribution function $H_d\,(x)$

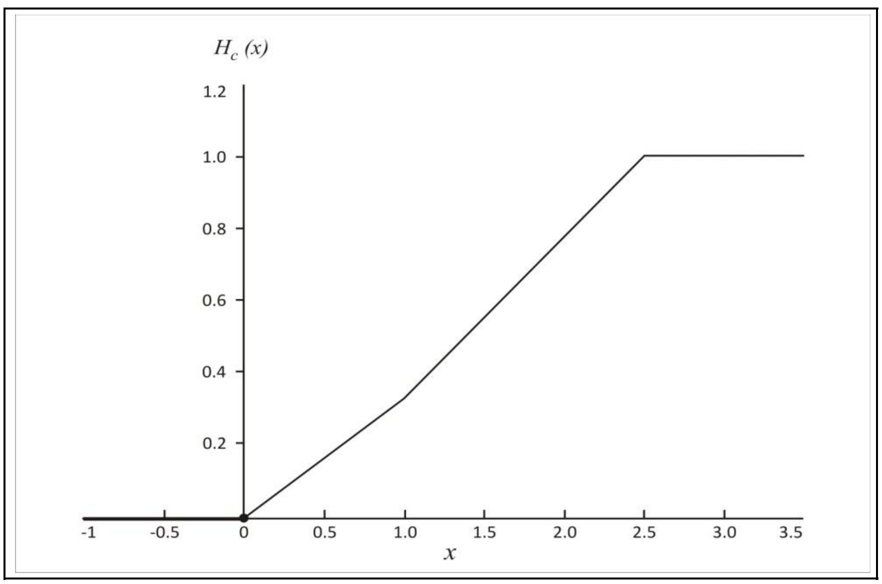

Figure 4.6. Plot of distribution function $H_c\,(x)$