MODULE 5

SOME SPECIAL ABSOLUTELY CONTINUOUS DISTRIBUTIONS

LECTURE 21

Topics

5.1 UNIFORM OR RECTANGULAR DISTRIBUTION

5.1.1 Quantile function and uniform distribution

5.2 GAMMA AND RELATED DISTRIBUTIONS

Lemma 1.1

Let X be a random variable having distribution function $F_X(\cdot)$ and quantile function $Q_X(\cdot)$. Let $x \in \mathbb{R}$, $p \in (0,1)$ and $0 < p_1 < p_2 < 1$. Then

(i) $Q_X\big(F_X(x)\big) \leq x$, provided $0 < F_X(x) < 1$;

(ii) $F_X\big(Q_X(p)\big) \geq p$;

(iii) $F_X\big(Q_X(p)\big) = p$, provided there exists an $x_0 \in \mathbb{R}$ such that $F_X(x_0) = p$. In particular if $F_X(\cdot)$ is continuous then $F_X\big(Q_X(p)\big) = p$;

(iv) $Q_X(p) \leq x \Leftrightarrow F_X(x) \geq p$;

(v) $Q_X(p) = F_X^{-1}(p)$, provided $F_X^{-1}(p)$ exists;

(vi) $Q_X(p_1) \leq Q_X(p_2)$.

Proof. For $p \in (0,1)$, define

$$S_p = \{s \in \mathbb{R}: F_X(s) \geq p\},$$

so that $Q_X(p) = \inf S_p$, $p \in (0,1)$.

(i) Let $x \in \mathbb{R}$ be such that $0 < F_X(x) < 1$. Then $x \in S_{F_X(x)} = \{s \in \mathbb{R}: F_X(s) \geq F_X(x)\}$ and, therefore, $x \geq \inf S_{F_X(x)} = Q(F_X(x))$, i.e., $Q_X\big(F_X(x)\big) \leq x$.

(ii) Let $p \in (0,1)$. Then $Q_X(p) = \inf S_p$. Thus there exists a sequence $\{t_n: n = 1, 2, \ldots\}$ in S_p such that $\lim_{n\to\infty} t_n = Q_X(p)$. Consequently $t_n \geq Q_X(p), n = 1, 2, \ldots$ and $F_X(t_n) \geq p, n = 1, 2, \ldots$. This implies that $\lim_{n\to\infty} F_X(t_n) \geq p$. Since $F_X(\cdot)$ is right continuous, $t_n \geq Q_X(p), n = 1, 2, \ldots$ and $\lim_{n\to\infty} t_n = Q_X(p)$, we get

$$F_X\big(Q_X(p)\big) = \lim_{n\to\infty} F_X(t_n) \geq p.$$

(iii) Let $x_0 \in \mathbb{R}$ be such that $F_X(x_0) = p$. Then

$$x_0 \in S_p = \{s \in \mathbb{R}: F_X(s) \geq p\}$$
$$\Rightarrow x_0 \geq \inf S_p = Q_X(p).$$

Now using (ii) and the fact that $F_X(\cdot)$ is non-decreasing, we get

$$p = F_X(x_0) \geq F_X(Q_X(p)) \geq p$$
$$\Rightarrow F_X(Q_X(p)) = p.$$

Note that $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to\infty} F_X(x) = 1$. Thus if $F_X(\cdot)$ is continuous then the intermediate value property of continuous functions implies that there exists an $x_0 \in \mathbb{R}$ such that $F_X(x_0) = p \in (0,1)$ and therefore $F_X(Q_X(p)) = p$.

(iv) First suppose that $Q_X(p) = \inf S_p \leq x$. Then, since $F_X(\cdot)$ is non-decreasing, we have

$$F_X\big(Q_X(p)\big) \leq F_X(x)$$
$$\Rightarrow p \leq F_X(x). \quad \text{(using (ii))}$$

Now suppose that $F_X(x) \geq p$. Then $x \in S_p = \{s \in \mathbb{R}: F_X(s) \geq p\}$ and, therefore,

$$x \geq \inf S_p = Q_X(p).$$

(v) Since $p_1 < p_2$, we have

$$S_{p_2} = \{s \in \mathbb{R}: F_X(s) \geq p_2\} \subseteq \{s \in \mathbb{R}: F_X(s) \geq p_1\} = S_{p_1}$$
$$\Rightarrow S_{p_2} \subseteq S_{p_1}$$
$$\Rightarrow Q_X(p_1) = \inf S_{p_1} \leq \inf S_{p_2} = Q_X(p_2). \blacksquare$$

Theorem 1.3

Let X be a random variable with distribution function $F_X(\cdot)$ and quantile function $Q_X(\cdot)$.

(i) ***(Probability Integral Transformation)*** If the random variable X is of continuous type then $Y \stackrel{\text{def}}{=} F_X(X) \sim U(0,1)$;

(ii) Let $U \sim U(0,1)$. Then $Z \stackrel{\text{def}}{=} Q_X(U) \stackrel{d}{=} X$.

Proof.

(i) Let $G(\cdot)$ be the d.f. of $Y \stackrel{\text{def}}{=} F_X(X)$, i.e.,

$$G(y) = P(\{F_X(X) \leq y\}), \quad y \in \mathbb{R}.$$

Clearly, for $y < 0$, $G(y) = 0$ and, for $y \geq 1, G(y) = 1$. Now suppose that $y \in (0,1)$. By Lemma 1.1 (iv) we have

$$\{s \in \mathbb{R}: F_X(s) \geq y\} = \{s \in \mathbb{R}: s \geq Q_X(y)\}$$

$$\Rightarrow P(\{F_X(X) \geq y\}) = P(\{X \geq Q_X(y)\})$$

$$\Rightarrow P(\{F_X(X) < y\}) = P(\{X < Q_X(y)\})$$

$$\Rightarrow P(\{F_X(X) < y\}) = P(\{X \leq Q_X(y)\}). \quad (\text{since } F_X(\cdot) \text{ is continuous}) \quad (1.4)$$

Since $F_X(\cdot)$ is continuous $\{x \in \mathbb{R}: F_X(x) = y\} = [x_1, x_2]$, for some real numbers x_1 and x_2 such that $-\infty < x_1 \leq x_2 < \infty$ (see Figures 1.5 (a) & (b)).

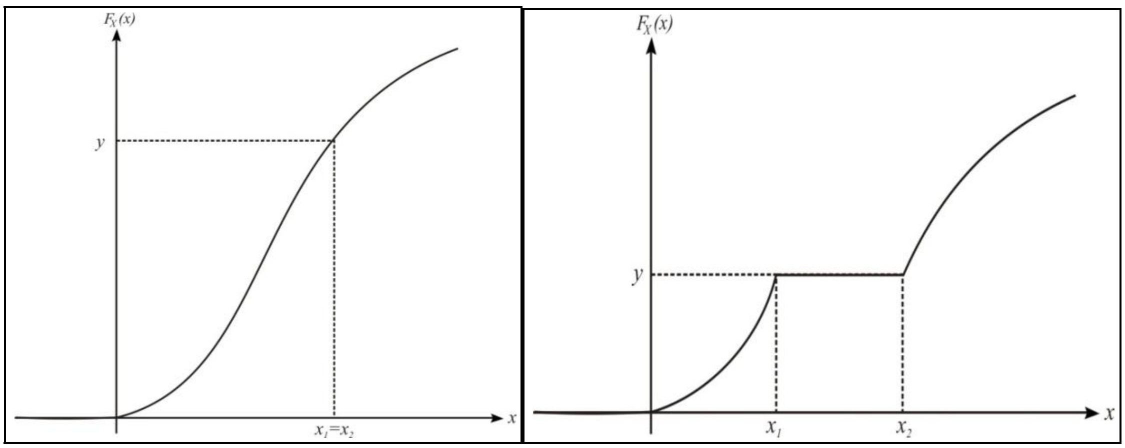

Figure 1.5 (a) **Figure 1.5 (b)**

Thus, for $y \in (0, 1)$,

$$\begin{aligned} P(\{F_X(X) = y\}) &= P(\{x_1 \leq X \leq x_2\}) \\ &= F_X(x_2) - F_X(x_1) \\ &= y - y = 0. \end{aligned} \quad (1.5)$$

Using (1.4), (1.5) and Lemma 1.1 (iii) we get, for $y \in (0, 1)$,

$$G(y) = P(\{F_X(X) \leq y\}) = P(\{F_X(X) < y\}) = P(\{X \leq Q_X(y)\}) = y.$$

Also right continuity of d.f. $G(\cdot)$ implies that

$$G(0) = \lim_{x \downarrow 0} G(x) = \lim_{x \downarrow 0} x = 0.$$

Therefore we have

$$G(y) = \begin{cases} 0, & \text{if } y < 0 \\ y, & \text{if } 0 \leq y < 1 \\ 1, & \text{if } y \geq 1 \end{cases},$$

i.e., $Y \stackrel{\text{def}}{=} F_X(X) \sim U(0, 1)$.

(ii) Let $U \sim U(0,1)$, so that $P(\{U \leq u\}) = u, \forall u \in [0,1]$ and $P(\{0 < U < 1\}) = 1$. Then the d.f. of $Z \stackrel{\text{def}}{=} Q_X(U)$ is

$$\begin{aligned} H(z) &= P(\{Z \leq z\}) \\ &= P(\{Q_X(U) \leq z\}) \\ &= P(\{Q_X(U) \leq z, 0 < U < 1\}) \text{ (since } P(\{0 < U < 1\}) = 1) \\ &= P(\{F_X(z) \geq U, 0 < U < 1\}) \text{ (using Lemma 1.1 (iv))} \\ &= P(\{U \leq F_X(z)\}) \\ &= F_X(z), \quad z \in \mathbb{R} \end{aligned}$$

$$\Rightarrow Z \stackrel{d}{=} X. \blacksquare$$

Remark 1.3
The above theorem provides a method to generate observations from any arbitrary distribution using observations from $U(0,1)$ distribution. Suppose that we require an observation X from a distribution having known d.f. $F(\cdot)$ and quantile function $Q(\cdot)$. To do so, the above theorem suggests that, generate an observation U from the $U(0,1)$ distribution and take $X = Q(U)$. $\blacksquare$

Example 1.2

Using a random observation $U \sim U(0,1)$, describe a method to generate a random observation X from the distribution having

(i) probability density function

$$f(x) = \frac{e^{-|x|}}{2}, -\infty < x < \infty;$$

(ii) probability mass function

$$g(x) = \begin{cases} \binom{n}{x} \theta^x (1-\theta)^{n-x}, & \text{if } x \in \{0, 1, \dots, n\} \\ 0, & \text{otherwise} \end{cases},$$

where $n \in \mathbb{N}$ and $\theta \in (0,1)$ are real constants.

Solution.

(i) For $x < 0$, we have

$$\begin{aligned} F(x) &= P(\{X \leq x\}) \\ &= \int_{-\infty}^{x} f_X(t)dt \end{aligned}$$

$$= \int_{-\infty}^{x} \frac{e^t}{2} dt$$
$$= \frac{e^x}{2},$$

and, for $x \geq 0$, we have

$$F(x) = P(\{X \leq x\})$$
$$= \int_{-\infty}^{x} f_X(t) dt$$
$$= \int_{-\infty}^{0} f_X(t) dt + \int_{0}^{x} f_X(t) dt$$
$$= \int_{-\infty}^{0} \frac{e^t}{2} dt + \int_{0}^{x} \frac{e^{-t}}{2} dt$$
$$= 1 - \frac{e^{-x}}{2}.$$

Thus the d.f. of X is given by

$$F(x) = \begin{cases} \frac{e^x}{2}, & \text{if } x < 0 \\ 1 - \frac{e^{-x}}{2}, & \text{if } x \geq 0 \end{cases},$$

and the q.f. of X is given by

$$Q(p) = F^{-1}(p) = \begin{cases} \ln(2p), & \text{if } 0 < p < \frac{1}{2} \\ -\ln\big(2(1-p)\big), & \text{if } \frac{1}{2} \leq p < 1 \end{cases}.$$

Using Theorem 1.3 (ii) the desired random observation is given by

$$X = Q(U) = \begin{cases} \ln(2U), & \text{if } 0 < U < \frac{1}{2} \\ -\ln\big(2(1-U)\big), & \text{if } \frac{1}{2} \leq U < 1 \end{cases}.$$

(ii) The distribution function of X is given by

$$G(x) = \begin{cases} 0, & \text{if } x < 0 \\ \sum_{j=0}^{k} \binom{n}{j} \theta^j (1-\theta)^{n-j}, & \text{if } k \leq x < k+1;\ \ k = 0, 1, \dots, n-1, \\ 1, & \text{if } x \geq n \end{cases}$$

and the quantile function of X is given by

$$Q(p) = \inf\{s \in \mathbb{R}: G(s) \geq p\}$$

$$= \begin{cases} 1, & \text{if } 0 < p \le (1-\theta)^n \\ k, & \text{if} \sum_{j=0}^{k-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < p \le \sum_{j=0}^{k} \binom{n}{j} \theta^j (1-\theta)^{n-j} \,; \\ & \qquad k = 0, 1, \dots, n-1 \\ n, & \text{if} \sum_{j=0}^{n-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < p < 1 \end{cases}.$$

Now, using Theorem 1.3 (ii), the desired random observation is given by

$$X = \begin{cases} 1, & \text{if } 0 < U \le (1-\theta)^n \\ k, & \text{if} \sum_{j=0}^{k-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < U \le \sum_{j=0}^{k} \binom{n}{j} \theta^j (1-\theta)^{n-j} \,; \\ & \qquad k = 0, 1, \dots, n-1 \\ n, & \text{if} \sum_{j=0}^{n-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < U < 1 \end{cases}.$$

5.2 GAMMA AND RELATED DISTRIBUTIONS

We begin this section with the definition of gamma function.

Definition 2.1

The function $\Gamma: (0,\infty) \to (0,\infty)$, defined by,

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt\,, \qquad \alpha > 0$$

is called the *gamma function*. ■

To examine convergence of the integral

$$\int_0^\infty e^{-t} t^{\alpha-1} dt\;, \qquad \alpha \in \mathbb{R}\,,$$

consider the following cases.

<u>**Case I**</u> $\alpha \le 0$

In this case the integral

$$\int_0^\infty e^{-t} t^{\alpha-1} dt$$

will converge if, and only if, both the integrals

$$\int_0^1 e^{-t} t^{\alpha-1} dt \text{ and } \int_1^\infty e^{-t} t^{\alpha-1} dt$$

converge. Note that, for $\alpha \le 0$,

$$e^{-t} t^{\alpha-1} \ge \frac{t^{\alpha-1}}{e}, \qquad \forall t \in (0,1)$$

and the integral

$$\int_0^1 t^{\alpha-1} dt$$

diverges. This implies that, for $\alpha \le 0$, the integral

$$\int_0^1 e^{-t} t^{\alpha-1} dt$$

diverges. Consequently the integral

$$\int_0^\infty e^{-t} t^{\alpha-1} dt$$

diverges for $\alpha \le 0$.

<u>Case II</u> $0 < \alpha < 1$

In this case again the integral

$$\int_0^\infty e^{-t} t^{\alpha-1} dt$$

will converge if, and only if, both the integrals

$$\int_0^1 e^{-t} t^{\alpha-1} dt \text{ and } \int_1^\infty e^{-t} t^{\alpha-1} dt$$

converge. Note that, for $\alpha > 0$,

$$0 \leq e^{-t} t^{\alpha-1} \leq t^{\alpha-1}, \qquad \forall t \in (0,1)$$

and the integral

$$\int_0^1 t^{\alpha-1} dt$$

is convergent. Therefore the integral

$$\int_0^1 e^{-t} t^{\alpha-1} dt$$

is convergent for any $\alpha > 0$.

Now let us examine the convergence of the integral

$$\int_1^\infty e^{-t} t^{\alpha-1} dt.$$

Fix $\alpha \in \mathbb{R}$ and choose $k_0 \in \mathbb{N}$ such that $k_0 > \alpha$. Then we know that

$$e^t \geq \frac{t^{k_0}}{k_0!}, \qquad \forall t > 0$$

$$\Rightarrow 0 \leq e^{-t} t^{\alpha-1} \leq \frac{k_0!}{t^{k_0-\alpha+1}}, \quad \forall t > 0.$$

Also $k_0 - \alpha + 1 > 1$ and, therefore, the integral

$$\int_1^\infty \frac{1}{t^{k_0-\alpha+1}} dt$$

converges. Consequently

$$\int_1^\infty e^{-t} t^{\alpha-1} dt$$

converges for any $\alpha \in \mathbb{R}$. From the above discussion it follows that the integral

$$\int_0^\infty e^{-t} t^{\alpha-1} dt$$

converges for $0 < \alpha < 1$.

<u>Case III</u> $\alpha \geq 1$

In this case the integral

$$\int_0^\infty e^{-t} t^{\alpha-1} dt$$

will converge if, and only if, the integral

$$\int_1^\infty e^{-t} t^{\alpha-1} \, dt$$

converges. We have seen in the Case II above that the integral

$$\int_1^\infty e^{-t} t^{\alpha-1} dt$$

converges for any $\alpha \in \mathbb{R}$.

On combining cases I – III we conclude that the integral

$$\int_0^\infty e^{-t} t^{\alpha-1} dt$$

converges if, and only if, $\alpha > 0$.

Using integration by parts, for $\alpha > 0$, we have

$$\Gamma(\alpha+1) = \int_0^\infty e^{-t} t^{\alpha} dt$$

$$= [-e^{-t} t^{\alpha}]_0^\infty + \alpha \int_0^\infty e^{-t} t^{\alpha-1} dt$$

$$= \alpha \int_0^\infty e^{-t} t^{\alpha-1} dt$$

i. e., $$\boxed{\Gamma(\alpha+1) = \alpha\, \Gamma(\alpha), \;\; \alpha > 0.} \qquad (2.1)$$

Note that

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1. \qquad (2.2)$$

For $n \in \mathbb{N}$, using (2.1) and (2.2), we have

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma\,(n-1) = \cdots = n(n-1)\cdots 3\cdot 2\cdot 1\,\Gamma(1) = n!. \quad (2.3)$$

On combining (2.1), (2.2) and (2.3) we get

$$\boxed{\Gamma(n) = (n-1)!\,, \quad n \in \mathbb{N},} \qquad (2.4)$$

with the convention that $0! = 1$.

We have

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-1/2} dt$$

$$= 2 \int_0^\infty e^{-x^2} dx$$

$$\Rightarrow \left(\Gamma(\frac{1}{2})\right)^2 = 4 \left[\int_0^\infty e^{-x^2} dx\right]\left[\int_0^\infty e^{-y^2} dy\right]$$

$$= 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dxdy.$$

On making the transformation $x = r\cos\theta$ and $y = r\sin\theta$ in the above integral (so that the Jacobian of the transformation is r), we have

$$\left(\Gamma(\frac{1}{2})\right)^2 = 4 \int_0^\infty \int_0^{\pi/2} re^{-r^2} d\theta\, dr$$

$$= 2\pi \int_0^\infty r e^{-r^2} dr$$

$$= \pi \int_0^\infty e^{-t}\, dt$$

$$= \pi.$$

Since

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{1/2-1} dt \geq 0,$$

we get

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}} \tag{2.5}$$

Also, using (2.1),

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2},$$

and

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1\cdot 3}{2^2}\sqrt{\pi},$$

In general

$$\boxed{\Gamma\left(\frac{2n+1}{2}\right) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\sqrt{\pi}, \qquad n \in \mathbb{N},} \tag{2.6}$$

i.e., for $n \in \mathbb{N}$,

$$\boxed{\Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n)!}{n!\,4^n}\sqrt{\pi}, \qquad n \in \mathbb{N}.} \tag{2.7}$$