MODULE 6

RANDOM VECTOR AND ITS JOINT DISTRIBUTION

LECTURE 26

Topics

6.2 TYPES OF RANDOM VARIABLES

Now we state the following theorem without providing its proof. This theorem states that properties (i) - (iv) described in Theorem 1.2 characterize distribution functions.

Theorem 1.3

Let $G: \mathbb{R}^p \to \mathbb{R}$ be a function such that

(i) $\lim_{\substack{x_i \to \infty \\ i=1,\ldots,p}} G(x_1, \ldots, x_p) = 1;$

(ii) for each fixed $i \in \{1, \ldots, p\}$ and each fixed $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p) \in \mathbb{R}^{p-1}$, $\lim_{y \to -\infty} G(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_p) = 0;$

(iii) $G(x_1, \ldots, x_p)$ is right continuous in each argument when other arguments are kept fixed;

(iv) for each rectangle $(\underline{a}, \underline{b}] \subseteq \mathbb{R}^p$

$$\sum_{k=0}^{p} (-1)^k \sum_{\underline{z} \in \Delta_{k,p}((\underline{a}, \underline{b}])} G(\underline{z}) \geq 0.$$

Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a random vector $\underline{X} = (X_1, \ldots, X_p)$ defined on $(\Omega, \mathcal{F}, P)$ such that G is the distribution function of $\underline{X}$ $\left(\text{i.e.}, F_{\underline{X}}(\underline{x}) = G(\underline{x}),\ \forall \underline{x} \in \mathbb{R}^p\right)$. ■

Remark 1.5

(i) As in the one dimensional case it can be shown that the probability measure $P_{\underline{X}}(\cdot)$, induced by a random vector $\underline{X}$, is completely determined by its distribution function $F_{\underline{X}}(\cdot)$. Thus, to study the induced probability measure $P_{\underline{X}}(\cdot)$, it is enough to study the distribution function $F_{\underline{X}}$.

(ii) The properties (i)-(iv) given in Theorem 1.3 are key properties of a distribution function. Let $\underline{a} = (a_1, a_2, \dots, a_p)$ and $\underline{b} = (a_1 + h, b_2, \dots, b_p)$, where $h > 0$. If $G: \mathbb{R}^{\boldsymbol{p}} \to \mathbb{R}$ is any function which satisfies properties (ii) and (iv) of Theorem 1.3, then

$$\sum_{k=0}^{p} (-1)^k \sum_{\underline{z} \in \Delta_{k,p}((\underline{a}, \underline{b}])} G(\underline{z}) \geq 0 \qquad \text{(using property (iv))}$$

$$\Rightarrow \lim_{\substack{a_i \to -\infty \\ i=2,\dots,p}} (-1)^k \sum_{k=0}^{p} \sum_{\underline{z} \in \Delta_{k,p}((\underline{a}, \underline{b}])} G(\underline{z}) \geq 0$$

$$\Rightarrow G\left(a_1 + h, b_2, \dots, b_p\right) - G\left(a_1, b_2, \dots, b_p\right) \geq 0, \qquad \text{(using property (ii))}$$

i.e., $G(\cdot)$ is non-decreasing in each argument when other arguments are kept fixed.It follows that if $G: \mathbb{R}^{\boldsymbol{p}} \to \mathbb{R}$ is a distribution function then the property that it is non-decreasing in each argument (when other arguments are kept fixed) is not one of its key characteristics and it is a consequence of properties (ii)-(iv) given in Theorem 1.3.

Example 1.3

Consider the function $G: \mathbb{R}^{\mathbf{2}} \to \mathbb{R}$ defined by

$$G(x, y) = \begin{cases} xy^2, & \text{if } 0 \leq x < 1, 0 \leq y < 1 \\ x, & \text{if } 0 \leq x < 1, \ y \geq 1 \\ y^2, & \text{if } x \geq 1, \ 0 \leq y < 1 \\ 1, & \text{if } x \geq 1, \ y \geq 1 \\ 0, & \text{otherwise} \end{cases}.$$

(i) Show that G is a distribution function of some two-dimensional random vector, say (X, Y).

(ii) Find marginal distribution functions of X and Y.

Solution. (i) Note that, for $x \geq 1$, $y \geq 1$, $G(x, y) = 1$. Therefore $\lim_{\substack{x \to \infty \\ y \to \infty}} G(x, y) = 1$.Also, for $x < 0$ or $y < 0$, $G(x, y) = 0$. Therefore, for each fixed $x \in \mathbb{R}$, $\lim_{y \to -\infty} G(x, y) = 0$ and, for each fixed $y \in \mathbb{R}$, $\lim_{x \to -\infty} G(x, y) = 0$.

Note that, $G(x, y) = 0, \forall\, x \in \mathbb{R}$ if $y < 0$, (1.8)

$$G(x,y)=\begin{cases}0, & \text{if } x<0\\ xy^2, & \text{if } 0\le x<1\\ y^2, & \text{if } x\ge 1\end{cases}, \quad \text{if } y\in[0,1) \tag{1.9}$$

and

$$G(x,y)=\begin{cases}0, & \text{if } x<0\\ x, & \text{if } 0\le x<1\\ 1, & \text{if } x\ge 1\end{cases}, \quad \text{if } y\in[1,\infty). \tag{1.10}$$

From (1.8) - (1.10) it is evident that, for each fixed value of $y \in \mathbb{R}, G(x,y)$ is a continuous (and hence right continuous) function of x. Similarly, for each fixed value of $x \in \mathbb{R}, G(x,y)$ is a continuous function of y.

From (1.8) - (1.10) it is also clear that, for each fixed value of $y \in \mathbb{R}$, $G(x,y)$ is a non-decreasing function of $x \in \mathbb{R}$. Similarly, for each fixed value of $x \in \mathbb{R}$, $G(x,y)$ is a non-decreasing function of $y \in \mathbb{R}$.

Now let
$-\infty < a_1 < b_1 < \infty$, $-\infty < a_2 < b_2 < \infty$, $\underline{a} = (a_1, a_2), \underline{b} = (b_1, b_2)$ and $(\underline{a}, \underline{b}] = (a_1, b_1] \times (a_2, b_2]$. Then

$$\Delta = \sum_{k=0}^{2} (-1)^k \sum_{\underline{z}\, \in \Delta_{k,2}\left((\underline{a},\underline{b}]\right)} G(z_1, z_2)$$

$$= G(b_1,b_2) - G(b_1,a_2) - G(a_1,b_2) + G(a_1,a_2).$$

The following cases arise:

Case I. $a_1 < 0$

In this case

$$\Delta = G(b_1,b_2) - G(b_1,a_2) \ge 0,$$

since, for a fixed $b_1 \in \mathbb{R}$, $G(b_1, y)$ is a non-decreasing function of y;

Case II. $a_2 < 0$

$$\Delta = G(b_1,b_2) - G(a_1,b_2) \ge 0,$$

since, for a fixed $b_2 \in \mathbb{R}$, $G(x, b_2)$ is a non-decreasing function of x;

Case III. $0 \le a_1 < 1, 0 \le a_2 < 1, 0 \le b_1 < 1, 0 \le b_2 < 1$

$$\Delta = b_1 b_2^2 - b_1 a_2^2 - a_1 b_2^2 + a_1 a_2^2$$

$$= (b_1 - a_1)(b_2^2 - a_2^2) \geq 0;$$

Case IV. $0 \leq a_1 < 1,\ 0 \leq a_2 < 1,\ 0 \leq b_1 < 1,\ b_2 \geq 1$

$$\Delta = b_1 - b_1 a_2^2 - a_1 + a_1 a_2^2$$

$$= (b_1 - a_1)(1 - a_2^2) \geq 0;$$

Case V. $0 \leq a_1 < 1, 0 \leq a_2 < 1, b_1 \geq 1, 0 \leq b_2 < 1$

$$\Delta = b_2^2 - a_2^2 - a_1 b_2^2 + a_1 a_2^2$$

$$= (1 - a_1)(b_2^2 - a_2^2) \geq 0;$$

Case VI. $0 \leq a_1 < 1, 0 \leq a_2 < 1,\ b_1 \geq 1,\ b_2 \geq 1$

$$\Delta = 1 - a_2^2 - a_1 + a_1 a_2^2$$

$$= (1 - a_1)(1 - a_2^2) \geq 0;$$

Case VII. $0 \leq a_1 < 1, a_2 \geq 1, 0 \leq b_1 < 1,\ b_2 \geq 1$

$$\Delta = b_1 - b_1 - a_1 + a_1 = 0;$$

Case VIII. $0 \leq a_1 < 1,\ a_2 \geq 1,\ b_1 \geq 1,\ b_2 \geq 1$

$$\Delta = 1 - 1 - a_1 + a_1 = 0;$$

Case IX. $a_1 \geq 1, 0 \leq a_2 < 1,\ b_1 \geq 1,\ 0 \leq b_2 < 1$

$$\Delta = b_2^2 - a_2^2 - b_2^2 + a_2^2 = 0;$$

Case X. $a_1 \geq 1,\ 0 \leq a_2 < 1,\ b_1 \geq 1,\ b_2 \geq 1$

$$\Delta = 1 - a_2^2 - 1 + a_2^2 = 0;$$

Case XI. $a_1 \geq 1,\ a_2 \geq 1,\ b_1 \geq 1,\ b_2 \geq 1$

$$\Delta = 1 - 1 - 1 + 1 = 0.$$

Combining Case I- Case XI it follows that

$$\sum_{k=0}^{2} (-1)^k \sum_{\underline{z} \in \Delta_{k,2}\left((\underline{a}, \underline{b}]\right)} G(z_1, z_2) \geq 0,\ \ \forall\ (\underline{a}, \underline{b}] \subseteq \mathbb{R}^2.$$

Now using Theorem 1.3 it follows that $G(x_1, x_2)$ is a distribution function of some two-dimensional random vector $(X, Y) \in \mathbb{R}^2$.

(ii) Using Lemma 1.2, we have

$$F_X(x) = \lim_{y \to \infty} G(x, y) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x < 1. \\ 1, & \text{if } x \geq 1 \end{cases}$$

Also using Lemma 1.2 and Remark 1.3 we have

$$F_Y(y) = \lim_{x \to \infty} G(x, y) = \begin{cases} 0, & \text{if } y < 0 \\ y^2, & \text{if } 0 \leq y < 1. \\ 1, & \text{if } x \geq 1 \end{cases} \blacksquare$$

Example 1.4

Let $G: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$G(x, y) = \begin{cases} x, & \text{if } 0 \leq x < 1, \ y \geq 1 \\ y^2, & \text{if } x \geq 1, \ 0 \leq y < 1 \\ 1, & \text{if } x \geq 1, \ y \geq 1 \\ 0, & \text{otherwise} \end{cases}.$$

Show that G is not a distribution function of any random vector (X, Y).

Solution. Note that $G(x, y)$ is non-decreasing in each argument when the other argument is kept fixed. Let $a_1 \in [0, 1)$, $a_2 \in [0, 1)$, $b_1 \in [1, \infty)$, $b_2 \in [1, \infty)$ $a_2^2 + a_1 > 1$, $\underline{a} = (a_1, a_2)$, $\underline{b} = (b_1, b_2)$ and $(\underline{a}, \underline{b}] = (a_1, b_1] \times (a_2, b_2]$. Then

$$\sum_{k=0}^{2} (-1)^k \sum_{\underline{z} \in \Delta_{k,2}\left((\underline{a}, \underline{b}]\right)} G(z_1, z_2) = G(b_1, b_2) - G(b_1, a_2) - G(a_1, b_2) + G(a_1, a_2)$$

$$= 1 - a_2^2 - a_1 < 0.$$

Thus G is not a distribution function of any random vector. $\blacksquare$

6.2 TYPES OF RANDOM VECTORS

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\underline{X} = \left(X_1, \dots, X_p\right): \Omega \to \mathbb{R}^p$ be a random vector with distribution function $F_{\underline{X}}\left(x_1, \dots, x_p\right)$.

Definition 2.1

(i) $\underline{X}$ is said to a *random vector of discrete type* if there exists a non-empty countable set $S_{\underline{X}} \subseteq \mathbb{R}^p$ such that $P\left(\{\underline{X} = \underline{x}\}\right) > 0$, $\forall \underline{x} \in S_{\underline{X}}$ and $P\left(\{\underline{X} \in S_{\underline{X}}\}\right) = \sum_{\underline{x} \in S_{\underline{X}}} P\left(\{\underline{X} = \underline{x}\}\right) = 1$. The set $S_{\underline{X}}$ is called the support of the discrete type random vector $\underline{X}$ (or simply the support of the probability distribution of $\underline{X}$) and the function

$$f_{\underline{X}}(\underline{x}) = P(\{\underline{X} = \underline{x}\}),\ \ \underline{x} \in \mathbb{R}^p,$$

which is such that $f_{\underline{X}}(\underline{x}) > 0,\ \forall \underline{x} \in S_{\underline{X}},\ f_{\underline{X}}(\underline{x}) = 0,\ \forall \underline{x} \in S_{\underline{X}}^c$ (see Remark 2.1 (i) later) and $\sum_{\underline{x} \in S_{\underline{X}}} f_{\underline{X}}(\underline{x}) = 1$, is called the *joint probability mass function* (p.m.f.) of $\underline{X}$.

(ii) $\underline{X}$ is said to be a *random vector of continuous type* if $F_{\underline{X}}(\underline{x})$ is continuous at every $\underline{x} \in \mathbb{R}^p$;

(iii) $\underline{X}$ is said to be a *random vector of absolutely continuous type* if there exists a non-negative function $f_{\underline{X}}: \mathbb{R}^p \to \mathbb{R}$ such that

$$F_{\underline{X}}(\underline{x}) = \int_{(-\underline{\infty}, \underline{x}]} f_{\underline{X}}(\underline{y})\, d\underline{y},\ \ \underline{x} = (x_1, \dots, x_p) \in \mathbb{R}^p,$$

where $(-\underline{\infty}, \underline{x}] = (-\infty, x_1] \times \cdots \times (-\infty, x_p]$, $\underline{y} = (y_1, \dots, y_p)$ and $d\underline{y} = dy_1 \cdots dy_p$.

The function $f_{\underline{X}}(\cdot)$, which is non-negative and is such that

$$\int_{\mathbb{R}^p} f_{\underline{X}}(x_1, \dots, x_p)\, d\,\underline{x} = \lim_{\substack{y_i \to \infty \\ i=1,\cdots,p}} F_{\underline{X}}(y_1, \dots, y_p) = 1,$$

is called the *joint probability density function* (p.d.f.) of $\underline{X}$. The set $S_{\underline{X}} = \{\underline{x} \in \mathbb{R}^p : f_{\underline{X}}(\underline{x}) > 0\}$ is called a support of the p.d.f. $f_{\underline{X}}$. ■

Remark 2.1

(i) If $\underline{X}$ is of discrete type with support $S_{\underline{X}}$ then $P(\{\underline{X} \in S_{\underline{X}}\}) = 1$ and, therefore, $P(\{\underline{X} \in S_{\underline{X}}^c\}) = 0$. In particular $f_{\underline{X}}(\underline{x}) = P(\{\underline{X} = \underline{x}\}) = 0,\ \forall \underline{x} \in S_{\underline{X}}^c$.

(ii) Let $\underline{X}$ be a random vector of discrete type with support $S_{\underline{X}}$ and p.m.f. $f_{\underline{X}}(\cdot)$. Then we know that $S_{\underline{X}}$ is countable, $f_{\underline{X}}(\underline{x}) \geq 0,\ \forall \underline{x} \in \mathbb{R}^p, f_{\underline{X}}(\underline{x}) > 0,\ \forall \underline{x} \in S_{\underline{X}}$ and $\sum_{\underline{x} \in S_{\underline{X}}} f_{\underline{X}}(\underline{x}) = 1$. As in the one-dimensional case ($p = 1$) it can be shown that if $g: \mathbb{R}^p \to \mathbb{R}$ is any function such that $g(\underline{x}) \geq 0,\ \forall \underline{x} \in \mathbb{R}^p, g(\underline{x}) > 0, \forall \underline{x} \in D$ and $\sum_{\underline{x} \in D} g(\underline{x}) = 1$, for some non-empty countable set $D \subseteq \mathbb{R}^p$, then $g(\cdot)$ is a joint p.m.f. of a random vector of discrete type.

(iii) Let $\underline{X}$ be a random vector of absolutely continuous type with joint and p.d.f. $f_{\underline{X}}(\cdot)$. Then $f_{\underline{X}}(\underline{x}) \geq 0,\ \forall \underline{x} \in \mathbb{R}^p$ and

$$\int_{\mathbb{R}^p} f_{\underline{X}}(\underline{x})\, d\underline{x} = 1,$$

where $\underline{x} = (x_1, \dots, x_p)$ and $d\underline{x} = dx_1 \cdots dx_p$. Conversely if $h: \mathbb{R}^p \to \mathbb{R}$ is any function such that $h(\underline{x}) \geq 0,\ \forall \underline{x} \in \mathbb{R}^p$, and

$$\int_{\mathbb{R}^p} h(\underline{x}) d\, \underline{x} = 1,$$

then it can be shown that $h(\cdot)$ is a joint p.d.f. of some random vector of absolutely continuous type.

(iv) Let $(\underline{a}, \underline{b}] \subseteq \mathbb{R}^p$ and let $\Psi: (\underline{a}, \underline{b}] \to \mathbb{R}$ be a non-negative function. Let $D = D_1 \times \cdots \times D_p$, where each $D_i, i = 1, \dots, p$, is countable. Then, provided the integral (or sum)

$$\int_{(\underline{a},\underline{b}]} \Psi(\underline{x})\, d\underline{x} \quad \left(\text{or} \sum_{\underline{x} \in D} \Psi(\underline{x})\right)$$

is finite, we know that the order in which (section wise) integral (or sum) is carried out is immaterial. In particular if $h: \mathbb{R}^p \to \mathbb{R}$ is a joint p.d.f. (or joint p.m.f.), then

$$\int_{(\underline{a},\underline{b}]} h(\underline{x})\, dx_1 \cdots dx_p = \int_{a_{\beta_p}}^{b_{\beta_p}} \cdots \int_{a_{\beta_1}}^{b_{\beta_1}} h(\underline{x})\, dx_{\beta_1} \cdots bx_{\beta_p}$$

$$\text{or}\left(\sum_{\underline{x} \in D} h(\underline{x}) = \sum_{x_{\beta_1} \in D_{\beta_1}} \cdots \sum_{x_{\beta_p} \in D_{\beta_p}} h(\underline{x})\right).$$

(v) Let $\underline{X}$ be a p-dimensional random vector with distribution function $F_{\underline{X}}$. For $\underline{a} = (a_1, \dots, a_p) \in \mathbb{R}^p$, define $\underline{a}_n = \left(a_1 - \frac{1}{n}, \dots, a_p - \frac{1}{n}\right), n = 1, 2, \dots$. Then

$$\{\underline{X} = \underline{a}\} = \underline{X}^{-1}(\{\underline{a}\})$$

$$= \underline{X}^{-1}\left(\bigcap_{n=1}^{\infty} (\underline{a}_n, \underline{a}]\right)$$

$$= \bigcap_{n=1}^{\infty} \underline{X}^{-1}\left((\underline{a}_n, \underline{a}]\right)$$

$$\Rightarrow P(\{\underline{X} = \underline{a}\}) = P\left(\bigcap_{n=1}^{\infty} \underbrace{\underline{X}^{-1}\left((\underline{a}_n, \underline{a}]\right)}_{=A_n \downarrow}\right)$$

$$= \lim_{n\to\infty} P\left(\underline{X}^{-1}\left((\underline{a}_n, \underline{a}]\right)\right)$$

$$= \lim_{n\to\infty} \sum_{k=0}^{p} (-1)^k \sum_{\underline{z}_n \in \Delta_{k,p}\left((\underline{a}_n, \underline{a}]\right)} F_{\underline{X}}\left(\underline{z}_n\right).$$

(vi) Let $\underline{X}$ be a p-dimensional random vector with distribution function $F_{\underline{X}}$ that is continuous at $\underline{a} \in \mathbb{R}^p$. Let $\underline{a}_n,\ n = 1, 2 \ldots$ be as defined in (v) above. Then, for $\underline{z}_n \in \Delta_{k,p}\left(\left(\underline{a}_n, \underline{a}\right]\right), n = 1, 2, \ldots$ (so that, as $n \to \infty$, $\underline{z}_n \to \underline{a}$), $F_{\underline{X}}\left(\underline{z}_n\right) \to F_{\underline{X}}(\underline{a})$ as $n \to \infty$. Therefore

$$P(\{\underline{X} = \underline{a}\}) = \lim_{n\to\infty} \sum_{k=0}^{p} (-1)^k \sum_{\underline{z}_n \in \Delta_{k,p}\left((\underline{a}_n, \underline{a}]\right)} F_{\underline{X}}\left(\underline{z}_n\right)$$

$$= \sum_{k=0}^{p} (-1)^k \binom{p}{k} F_{\underline{X}}(\underline{a})$$

$$= (1-1)^p F_{\underline{X}}(\underline{a})$$

$$= 0.$$

It follows that if the distribution functions $F_{\underline{X}}$ of a p-dimensional random vector $\underline{X}$ is continuous at $\underline{a} \in \mathbb{R}^p$ then

$$P(\{\underline{X} = \underline{a}\}) = 0.$$

(vii) Let $\underline{X}$ be a p-dimensional random vector of continuous type so that its distribution function $F_{\underline{X}}(\cdot)$ is continuous at every $\underline{x} \in \mathbb{R}^p$. Then, by (vi),

$$P(\{\underline{X} = \underline{a}\}) = 0, \forall \underline{a} \in \mathbb{R}^p.$$

Consequently, for any countable set $S \subseteq \mathbb{R}^p$,

$$P(\{\underline{X} \in S\}) = P\left(\left\{\bigcup_{\underline{a} \in S} \{\underline{X} = \underline{a}\}\right\}\right)$$

$$= \sum_{\underline{a} \in S} P(\{\underline{X} = \underline{a}\})$$

$$= 0.$$

(viii) Suppose that $\underline{X}$ is a p-dimensional random vector of absolutely continuous type with p.d.f. $F_{\underline{X}}(\cdot)$. Then it can be shown that its distribution function

$$F_{\underline{X}}(\underline{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_p} f_{\underline{X}}\left(\underline{y}\right) dy_p \cdots dy_1, \;\; \underline{x} \in \mathbb{R}^p,$$

is continuous at every $\underline{x} \in \mathbb{R}^p$. Thus a random vector of absolutely continuous type is also continuous. Moreover if $\underline{X}$ is of absolutely continuous type then

$$P(\{\underline{X} = \underline{a}\}) = 0, \forall \underline{a} \in \mathbb{R}^p \text{ and } P(\{\underline{X} \in S\}) = 0,$$

for any countable set S.

(ix) Let $\underline{X}$ be a p-dimensional random vector of discrete type with joint p.m.f. $f_{\underline{X}}(\cdot)$ and support $S_{\underline{X}}$. Then, for any $A \in \mathcal{B}_p$,

$$P(\{\underline{X} \in A\}) = P(\{\underline{X} \in A \cap S_{\underline{X}}\}) \qquad \left(\text{since } P(\{\underline{X} \in S_{\underline{X}}\}) = 1\right)$$

$$= P\left(\bigcup_{\underline{x} \in A \cap S_{\underline{X}}} \{\underline{X} = \underline{x}\}\right)$$

$$= \sum_{\underline{x} \in A \cap S_{\underline{X}}} P(\{\underline{X} = \underline{x}\}) \qquad \left(A \cap S_{\underline{X}} \subseteq S_{\underline{X}} \text{ is countable}\right)$$

$$= \sum_{\underline{x} \in A \cap S_{\underline{X}}} f_{\underline{X}}(\underline{x})$$

$$= \sum_{\underline{x} \in S_{\underline{X}}} f_{\underline{X}}(\underline{x}) I_A(\underline{x}).$$

(x) Let $\underline{X}$ be a p-dimensional random vector of absolutely continuous type with joint p.d.f. $f_{\underline{X}}(\cdot)$ and let $\underline{a}, \underline{b} \in \mathbb{R}^p,\ a_i < b_i, i = 1, \dots, p$. Then, using the idea of the proof of Lemma 1.3, it can be shown that

$$\int\limits_{(\underline{a},\underline{b}]} f_{\underline{X}}(\underline{x}) d\underline{x} = \int\limits_{a_1}^{b_1} \cdots \int\limits_{a_p}^{b_p} f_{\underline{X}}(\underline{x}) dx_p \cdots dx_1$$

$$= \sum_{k=0}^{p} (-1)^k \sum_{\underline{z} \in \Delta_{k,p}\left((\underline{a},\underline{b}]\right)} \int\limits_{-\infty}^{z_1} \cdots \int\limits_{-\infty}^{z_p} f_{\underline{X}}(\underline{x})\, dx_p \cdots dx_1$$

$$= \sum_{k=0}^{p} (-1)^k \sum_{\underline{z} \in \Delta_{k,p}\left((\underline{a},\underline{b}]\right)} F_{\underline{X}}(\underline{z})$$

$$= P(\{a_i < X_i \le b_i, i = 1, \dots, p\})$$

$$= P\left(\left\{\underline{X} \in (\underline{a}, \underline{b}]\right\}\right).$$

It follows that

$$P\left(\left\{\underline{X} \in (\underline{a}, \underline{b}]\right\}\right) = P(\{a_i < X_i \le b_i, i = 1, \dots, p\})$$

$$= \int\limits_{a_1}^{b_1} \cdots \int\limits_{a_p}^{b_p} f_{\underline{X}}(\underline{x}) dx_p \cdots dx_1$$

$$= \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} f_{\underline{X}}(\underline{x})\, I_{\left((\underline{a},\underline{b}]\right)}(\underline{x}) dx_p \cdots dx_1$$

$$= \int\limits_{\mathbb{R}^p} f_{\underline{X}}(\underline{x})\, I_{\left((\underline{a},\underline{b}]\right)}(\underline{x}) d\underline{x}.$$

In general, for any set $A \in \mathcal{B}_p$, if can be shown that

$$P\left(\left\{\underline{X} \in A\right\}\right) = \int\limits_{\mathbb{R}^p} f_{\underline{X}}(\underline{x})\, I_A(\underline{x}) d\underline{x}.$$

Consequently if A comprises of a countable number of curves then

$$P(\{\underline{X} \in A\}) = \int_{\mathbb{R}^p} f_{\underline{X}}(\underline{x})\, I_A(\underline{x}) d\underline{x} = 0.$$

In particular $P(\{X_i = X_j\}) = 0, \forall i \neq j$.

(xi) Let $\underline{X}$ be a p-dimensional random vector of discrete type with joint distribution function $F_{\underline{X}}(\cdot)$, joint p.m.f. $f_{\underline{X}}(\cdot)$ and support $S_{\underline{X}}$. Then, using (ix),

$$F_{\underline{X}}(\underline{x}) = P(\{\underline{X} \in (-\underline{\infty}, \underline{x}]\})$$

$$= \sum_{\underline{x} \in ((-\underline{\infty}, \underline{x}]) \cap S_{\underline{X}}} f_{\underline{X}}(\underline{x}), \quad \underline{x} \in \mathbb{R}^p \tag{2.1}$$

Also, using (v),

$$f_{\underline{X}}(\underline{x}) = P(\{\underline{X} = \underline{x}\}) = \lim_{n \to \infty} \sum_{k=0}^{p} (-1)^k \sum_{\underline{z}_n \in \Delta_{k,p}((\underline{x}_n, \underline{x}])} F_{\underline{X}}(\underline{z}_n), \tag{2.2}$$

where $\underline{x}_n = \left(x_1 - \frac{1}{n}, \dots, x_p - \frac{1}{n}\right), n = 1, 2, \dots$.

Using (2.1) and (2.2) we conclude that the joint distribution function of a discrete type random vector is determined by its joint p.m.f. and vice-versa. Thus to study the probability measure $P_{\underline{X}}(\cdot)$ induced by a discrete type random vector $\underline{X}$ it is enough to study its p.m.f. (also see Remark 1.5 (i)).

(xii) If $\underline{X}$ is a random vector of absolutely continuous type then its joint p.d.f. is not unique and there are different versions of joint p.d.f. . In fact if the values of the joint p.d.f. $f_{\underline{X}}(\cdot)$ of a random vector $\underline{X}$ of absolutely continuous type are changed at a countable number of curves with other non-negative values then the resulting function is again a p.d.f. of $\underline{X}$.

(xiii) As in the one-dimensional case it can be shown that if $\underline{X}$ is a p-dimensional random vector with distribution function $F_{\underline{X}}(\cdot)$ such that

$$\frac{\partial^p}{\partial x_1 \cdots \partial x_p} F_{\underline{X}}(x_1, \dots, x_p),$$

exists everywhere except (possibly) on a set C comprising of countable number of curves and

$$\int_{\mathbb{R}^p} \frac{\partial^p}{\partial x_1 \cdots \partial x_p} F_{\underline{X}}(x_1, \dots, x_p)\, I_{C^c}(\underline{x}) d\underline{x} = 1.$$

Then $\underline{X}$ is of absolutely continuous type with a p.d.f.

$$f_{\underline{X}}(\underline{x}) = \begin{cases} \frac{\partial^p}{\partial x_1 \cdots \partial x_p} F_{\underline{X}}(x_1, \dots, x_p), & \text{if } \underline{x} \notin C \\ a_{\underline{x}}, & \text{if } \underline{x} \in C \end{cases},$$

here $a_{\underline{x}}, \underline{x} \in C$, are arbitrary non-negative constants.

(xiv) Let $\underline{X}$ be a p-dimensional random vector of absolutely continuous type with joint distribution function $F_{\underline{X}}(\cdot)$ and joint p.d.f. $f_{\underline{X}}(\cdot)$. Then

$$F_{\underline{X}}(\underline{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_p} f_{\underline{X}}\left(\underline{y}\right) dy_p \cdots dy_1, \;\; \underline{x} \in \mathbb{R}^p.$$

Clearly the joint distribution function of an absolutely continuous type random vector $\underline{X}$ is determined by its joint p.d.f. $f_{\underline{X}}(\cdot)$. Thus to study the probability measure $P_{\underline{X}}(\cdot)$ induced by an absolutely continuous type random vector $\underline{X}$ it is enough to study its joint p.d.f. $f_{\underline{X}}(\cdot)$.

Using Remark 1.2 (ii) and using (v) above it follows that if $f_{\underline{X}}(\underline{x}),\; \underline{x} \in \mathbb{R}^p$, is the p.m.f. (a p.d.f.) of p-dimensional random vector $\underline{X} = (X_1, \dots, X_p)$ then, for any permutation $(\beta_1, \dots, \beta_p)$ of $(1, \dots, p)$ with inverse permutation $(\gamma_1, \dots, \gamma_p)$ the joint p.m.f. (joint p.d.f.) of $\left(X_{\beta_1}, \dots, X_{\beta_p}\right)$ is $f_{X_{\beta_1},\dots,X_{\beta_p}}(x_1, \dots, x_p) = f_{X_1,\dots,X_p}\left(x_{\gamma_1}, \dots, x_{\gamma_p}\right),\; \underline{x} \in \mathbb{R}^p.$ ■