MODULE 7

LIMITING DISTRIBUTIONS

LECTURE 41

Topics

7.3 SOME PRESERVATION RESULTS

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In this section, we will discuss the algebraic operations under which convergence in probability and/or convergence in distribution is preserved.

Theorem 3.1

Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be sequences of random variables and let X be another random variable.

(i) Let $g: \mathbb{R} \to \mathbb{R}$ be continuous at $c \in \mathbb{R}$ and let $X_n \xrightarrow{p} c$, as $n \to \infty$. Then $g(X_n) \xrightarrow{p} g(c)$, as $n \to \infty$.

(ii) Let $h: \mathbb{R}^2 \to \mathbb{R}$ be continuous at $(c_1, c_2), \in \mathbb{R}^2$ and let $X_n \xrightarrow{p} c_1, Y_n \xrightarrow{p} c_2$, as $n \to \infty$. Then $h(X_n, Y_n) \xrightarrow{p} h(c_1, c_2)$, as $n \to \infty$.

(iii) Let $g: \mathbb{R} \to \mathbb{R}$ be continuous on a support S_X of X and let $X_n \xrightarrow{d} X$, as $n \to \infty$. Then $g(X_n) \xrightarrow{d} g(X)$, as $n \to \infty$.

(iv) Let $h: \mathbb{R}^2 \to \mathbb{R}$ be continuous at all points in $D = \{(x, b): x \in S_X\}$, where b is a fixed real constant and S_X is a support of X. If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} b$, as $n \to \infty$, then $h(X_n, Y_n) \xrightarrow{d} h(X, b)$, as $n \to \infty$.

Proof. We shall not attempt to prove assertions (iii) and (iv) here as their proofs are slightly involved.

(i) Fix $\varepsilon > 0$. Since $g: \mathbb{R} \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$, there exists a $\delta \equiv \delta(\varepsilon, c)$ such that

$$|x - c| < \delta \Rightarrow |g(x) - g(c)| < \varepsilon \cdot$$

or equivalently

$$|g(x) - g(c)| \geq \varepsilon \Rightarrow |x - c| \geq \delta.$$

Therefore,

$$0 \leq P(\{|g(X_n) - g(c)| \geq \varepsilon\}) \leq P(\{|X_n - c| \geq \delta\}) \xrightarrow{n\to\infty} 0 \qquad (\text{since } X_n \xrightarrow{p} c)$$

$$\Rightarrow \lim_{n\to\infty} P(\{|g(X_n) - g(c)| \geq \varepsilon\}) = 0$$

$$\Rightarrow \; g(X_n) \xrightarrow{p} g(c), \qquad \text{as } n \to \infty.$$

(ii) Fix $\varepsilon > 0$. Since $h: \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(c_1, c_2) \in \mathbb{R}^2$, there exists a $\delta = \delta(\varepsilon, c_1, c_2)$ such that

$$|x - c_1| < \delta \text{ and } |y - c_2| < \delta \Rightarrow |h(x, y) - h(c_1, c_2)| < \varepsilon,$$

or equivalently

$$|h(x, y) - h(c_1, c_2)| \geq \varepsilon \Rightarrow |x - c_1| \geq \delta \text{or } |y - c_2| \geq \delta.$$

Therefore,

$$P(\{|h(X_n, Y_n) - h(c_1, c_2)| \geq \varepsilon\}) \leq P(\{|X_n - c_1| \geq \delta\} \cup \{|Y_n - c_2| \geq \delta\})$$

$$\leq P(\{|X_n - c_1| \geq \delta\} + P\{|Y_n - c_2| \geq \delta\}) \quad \text{(using Boole's inequality)}$$

$$\xrightarrow{n\to\infty} 0 + 0 = 0 \left(\text{since } X_n \xrightarrow{p} c_1 \text{ and } Y_n \xrightarrow{p} c_2\right)$$

$$\Rightarrow \lim_{n\to\infty} P\,(\{|h(X_n, Y_n) - h(c_1, c_2)| \geq \varepsilon\}) = 0$$

$$\Rightarrow h(X_n, Y_n) \xrightarrow{p} h(c_1, c_2), \text{as } n \to \infty.$$ ■

Throughout, we shall use the following convention. If, for a real constant c, we write $X_n \xrightarrow{d} c,$ as $n \to \infty$, then it would mean that X_n converges in distribution, as $n \to \infty$, to a random variable degenerate at c $(\text{i.e.}, X_n \xrightarrow{p} c, \text{as } n \to \infty)$. Similarly, for a random variable $X, 0 \times X$ will be treated as a random variable degenerate at 0.

Now we provide the following useful lemma whose proof, being straight forward, is left as an exercise.

Lemma 3.1

(i) Let X and Y be random variables and let c be a real constant. If $P(\{Y=c\})=1$ then $X+Y \stackrel{d}{=} X+c$ and $XY \stackrel{d}{=} cX$, where $0\times X$ is treated as a random variable degenerate at 0.

(ii) Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be sequences of real numbers such that $X_n \stackrel{d}{=} Y_n, n=1,2,....$ If, for some real constant $c, X_n \xrightarrow{p} c$, as $n\to\infty$, then $Y_n \xrightarrow{p} c$, as $n\to\infty$.

(iii) Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be sequences of real numbers such that $X_n \stackrel{d}{=} Y_n, n=1,2,....$ If, for some random variable X, $X_n \xrightarrow{d} X$, as $n\to\infty$, then $Y_n \xrightarrow{d} X$, as $n\to\infty$.

(iv) Let $\{a_n\}_{n\geq 1}$ be sequence of real numbers such that $\lim_{n\to\infty} a_n = a \in \mathbb{R}$ and let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that X_n is degenerate at $a_n, n=1,2,....$ Then $X_n \xrightarrow{p} a$, as $n\to\infty$. ■

Theorem 3.2

Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be sequences of random variables and let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be sequences of real numbers such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

(i) Suppose that, for some real constants c_1 and c_2, $X_n \xrightarrow{p} c_1$ and $Y_n \xrightarrow{p} c_2$, as $n\to\infty$. Then, as $n\to\infty$, $X_n+Y_n \xrightarrow{p} c_1+c_2, X_n-Y_n \xrightarrow{p} c_1-c_2$ and $X_nY_n \xrightarrow{p} c_1c_2$. Moreover, if $c_2\neq 0$, then $\frac{X_n}{Y_n} \xrightarrow{p} \frac{c_1}{c_2}$, as $n\to\infty$.

(ii) Suppose that, for a real constant c and a random variable $X, X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$, as $n\to\infty$. Then, as $n\to\infty$, $X_n+Y_n \xrightarrow{d} X+c$, $X_n-Y_n \xrightarrow{d} X-c$ and $X_nY_n \xrightarrow{d} c\,X$. Moreover, if $c\neq 0$, then $\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c}$, as $n\to\infty$.

(iii) Suppose that, for a real constant c, $X_n \xrightarrow{p} c$, as $n\to\infty$.Then $a_nX_n + b_n \xrightarrow{p} ac+b$, as $n\to\infty$.

(iv) Suppose that, for a random variable X, $X_n \xrightarrow{d} X$, as $n\to\infty$. Then $a_nX_n + b_n \xrightarrow{d} a\,X+b$, as $n\to\infty$.

Proof. (i) and (ii) follow from Theorem 3.1 (ii) and (iv) as $h_1(x,y)=x+y, h_2(x,y)=x-y$ and $h_3(x,y)=xy$ are continuous functions on $\mathbb{R}^2$, and $h_4(x,y)=\frac{x}{y}$ is continuous on $D=\{(s,t)\in\mathbb{R}^2 : t\neq 0\}$.

(iii) Let Y_n be a random variable that is degenerate at a_n and let Z_n be a random variable that is degenerate at $b_n, n=1,2,....$ Then $Y_n \xrightarrow{p} a$ and $Z_n \xrightarrow{p} b$, as $n\to\infty$

(Lemma 3.1 (iv)). Now using (i) we get $X_nY_n + Z_n \xrightarrow{p} ac + b, \text{as } n \to \infty$. Since $a_nX_n + b_n \stackrel{d}{=} X_nY_n + Z_n,\ n = 1,2,\ldots$, (Lemma 3.1 (i)), the assertion follows on using Lemma 3.1 (ii).

(iv) Let Y_n and Z_n be as defined in (iii). Then $Y_n \xrightarrow{p} a$ and $Z_n \xrightarrow{p} b, \text{as } n \to \infty$. Using (ii) we get $X_nY_n + Z_n \xrightarrow{d} aX + b, \text{as } n \to \infty$. Since $a_nX_n + b_n \stackrel{d}{=} X_nY_n + Z_n,\ n = 1, 2, \ldots$, the assertion follows on using Lemma 3.1(iii). ■

Remark 3.1

The CLT asserts that if $X_1, X_2, \ldots$ are i.i.d. random variables with mean μ and finite variance $\sigma^2 > 0$, then

$$Z_n \stackrel{\text{def}}{=} \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1), \text{as } n \to \infty,$$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Since $\frac{\sigma}{\sqrt{n}} \to 0, \text{as } n \to \infty$, using Theorem 3.2 (iv) we get

$$\bar{X}_n - \mu = \frac{\sigma}{\sqrt{n}} Z_n \xrightarrow{d} 0 \times Z, \text{as } n \to \infty.$$

Note that $0 \times Z$ is a random variable degenerate at 0. Thus it follows that

$$\begin{aligned} &\bar{X}_n - \mu \xrightarrow{d} 0, \quad \text{as } n \to \infty \\ \Leftrightarrow\ &\bar{X}_n - \mu \xrightarrow{p} 0, \quad \text{as } n \to \infty \\ \Leftrightarrow\ &\bar{X}_n \xrightarrow{p} \mu, \quad \text{as } n \to \infty. \end{aligned}$$

The above discussion suggests that, under the finiteness of second moment (or variance), the CLT is a stronger result than the WLLN. ■

Example 3.1

Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be sequences of random variables.

(i) If $X_n \xrightarrow{p} \ln 4$ and $Y_n \xrightarrow{p} 2, \text{as } n \to \infty$, show that $X_n + \ln Y_n \xrightarrow{p} \ln 8$ and $e^{X_n} \ln Y_n \xrightarrow{p} \ln 16, \text{as } n \to \infty$;

(ii) If $X_n \xrightarrow{d} Z \sim N(0,1)$, as $n \to \infty$, show that $X_n^2 \xrightarrow{d} Q_1 \sim \chi_1^2$ (the chi-square distribution with one degree of freedom), as $n \to \infty$.

(iii) If $X_n \xrightarrow{d} Z \sim N(0,1)$, and $Y_n \xrightarrow{p} 3$, as $n \to \infty$, show that $X_nY_n \xrightarrow{d} V \sim N(0, 9)$ and $2X_n + 3Y_n \xrightarrow{d} Q_2 \sim N(9, 4), \text{as } n \to \infty$.

(iv) For a given $\theta > 0$, if $X_1, X_2, \ldots$ are i.i.d. $U(0,\theta)$ random variables and $X_{n:n} = \max\{X_1, \ldots, X_n\}, n = 1, 2, \ldots$, show that $e^{X_{n:n}} \xrightarrow{p} e^{\theta}, X_{n:n}^2 + X_{n:n} + 1 \xrightarrow{p} \theta^2 + \theta + 1$ and $e^{-\frac{n(\theta - X_{n:n})}{\theta}} \xrightarrow{d} U \sim U(0,1)$, as $n \to \infty$.

Solution.

(i) Since $h_1(x) = \ln x, x \in (0,\infty)$ is a continuous function, using Theorem 3.1 (i) it follows that $\ln Y_n \xrightarrow{p} \ln 2$, as $n \to \infty$. Now on using Theorem 3.2 (i) we get $X_n + \ln Y_n \xrightarrow{p} \ln 4 + \ln 2 = \ln 8$, as $n \to \infty$. Also, since $h_2(x) = e^x, x \in \mathbb{R}$, is a continuous function on $\mathbb{R}$, on using Theorem 3.1 (i), we get $e^{X_n} \xrightarrow{p} e^{\ln 4} = 4$, as $n \to \infty$. Now on using Theorem 3.2(i) it follows that $e^{X_n} \ln Y_n \xrightarrow{p} 4 \ln 2 = \ln 16$, as $n \to \infty$.

(ii) Since $h_3(x) = x^2, x \in \mathbb{R}$, is a continuous function on $\mathbb{R}$, using Theorem 3.1 (iii) we get $X_n^2 \xrightarrow{d} Z^2$, as $n \to \infty$. Let $Q_1 = Z^2$. Since $Z \sim N(0,1)$, we have $Q_1 \sim \chi_1^2$ (Theorem 4.1 (ii), Module 5). Consequently $X_n^2 \xrightarrow{d} Q_1 \sim \chi_1^2$, as $n \to \infty$.

(iii) Using Theorem 3.2 (ii) we get $X_n Y_n \xrightarrow{d} 3Z$, as $n \to \infty$. Let $V = 3Z$. Since $Z \sim N(0,1)$ we have $V = 3Z \sim N(0,9)$ (Theorem 4.2 (ii) Module 5) and, therefore, $X_n Y_n \xrightarrow{d} V \sim N(0,9)$, as $n \to \infty$. Using theorem 3.2 (iii) and (iv) we get $2X_n \xrightarrow{d} 2Z$ and $3Y_n \xrightarrow{p} 9$, as $n \to \infty$. Now using Theorem 3.2 (ii) we also conclude that $2X_n + 3Y_n \xrightarrow{d} 2Z + 9$, as $n \to \infty$. Let $Q_2 = 2Z + 9$. Since $Z \sim N(0,1)$, we have $Q_2 \sim N(9,4)$ (Theorem 4.2 (ii), Module 5).

(iv) From Example 1.4 we have $X_{n:n} \xrightarrow{p} \theta$, as $n \to \infty$, and $Y_n = n(\theta - X_{n:n}) \xrightarrow{d} Y \sim \text{Exp}(\theta)$, as $n \to \infty$. Since $h_4(x) = e^x, x \in \mathbb{R}$, $h_5(x) = x^2 + x + 1$, $x \in \mathbb{R}$, and $h_6(x) = e^{-\frac{x}{\theta}}$, $x \in \mathbb{R}$, are continuous functions on $\mathbb{R}$, using Theorem 3.1 (i) and (ii), we get $e^{X_{n:n}} \xrightarrow{p} e^{\theta}, X_{n:n}^2 + X_{n:n} + 1 \xrightarrow{p} \theta^2 + \theta + 1$ and $e^{-\frac{Y_n}{\theta}} \xrightarrow{d} e^{-\frac{Y}{\theta}}$, as $n \to \infty$. Let $U = e^{-\frac{Y}{\theta}}$. Since $Y \sim \text{Exp}(\theta)$, it is easy to verify that $U \sim U(0,1)$. Consequently, $e^{-\frac{n(\theta - X_{n:n})}{\theta}} = e^{-\frac{Y_n}{\theta}} \xrightarrow{d} U \sim U(0,1)$, as $n \to \infty$. ■