MODULE 6

RANDOM VECTOR AND ITS JOINT DISTRIBUTION

LECTURE 30

Topics

6.5 EXPECTATIONS AND MOMENTS

6.6 JOINT MOMENT GENERATING FUNCTION

Theorem 5.5

Under the above notations

(i) $E\left(E(\psi(\underline{Y})|\underline{Z})\right) = E\left(\psi(\underline{Y})\right)$;

(ii) $\mathrm{Var}\left(E(\psi(\underline{Y})|\underline{Z})\right) + E\left(\mathrm{Var}(\psi(\underline{Y})|\underline{Z})\right) = \mathrm{Var}\left(\psi(\underline{Y})\right)$.

Proof. We will provide the proof for the absolutely continuous case. The proof for the discrete case follows in the similar fashion.

(i) Note that

$$E\left(E(\psi(\underline{Y})|\underline{Z})\right) = E\left(\psi^*(\underline{Z})\right),$$

where $\psi^*(\cdot)$ is defined by (5.2) and (5.3). Therefore

$$\begin{aligned}
E\left(E(\psi(\underline{Y})|\underline{Z})\right) &= \int_{\mathbb{R}^{p_2}} \psi^*(\underline{z}) f_{\underline{Z}}(\underline{z})\, d\underline{z} \\
&= \int_{\mathbb{R}^{p_2}} \left(\int_{\mathbb{R}^{p_1}} \psi\left(\underline{y}\right) f_{\underline{Y}|\underline{Z}}\left(\underline{y}|\underline{z}\right) d\underline{y} \right) f_{\underline{Z}}(\underline{z}) d\underline{z} \\
&= \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_1}} \psi\left(\underline{y}\right) f_{\underline{Y},\underline{Z}}\left(\underline{y},\underline{z}\right) d\underline{y} d\underline{z} \\
&= E(\psi(\underline{Y})).
\end{aligned}$$

(ii) Let $\psi^*(\underline{Z}) = E(\psi(\underline{Y})|\underline{Z})$. Then, by (i),

$$\text{Var}\left(\psi(\underline{Y})\right) = E\left(\left(\psi(\underline{Y}) - E(\psi(\underline{Y}))\right)^2\right)$$

$$= E\left(E\left(\left(\psi(\underline{Y}) - E\left(\psi(\underline{Y})\right)\right)^2 \Big| \underline{Z}\right)\right) \qquad (5.4)$$

$$E\left(\left(\psi(\underline{Y}) - E(\psi(\underline{Y}))\right)^2|\underline{Z}\right) = E\left(\left(\psi(\underline{Y}) - E(\psi(\underline{Y})|\underline{Z}) + E(\psi(\underline{Y})|\underline{Z}) - E\left(\psi(\underline{Y})\right)\right)^2 |\underline{Z}\right)$$

$$= E\left(\left(\psi(\underline{Y}) - E(\psi(\underline{Y})|\underline{Z})\right)^2 |\underline{Z}\right) + \left(E(\psi(\underline{Y})|\underline{Z}) - E\left(\psi(\underline{Y})\right)\right)^2$$

$$+2\left[E(\psi(\underline{Y})|\underline{Z}) - E\left(\psi(\underline{Y})\right)\right] E(\psi(\underline{Y}) - E(\psi(\underline{Y})|\underline{Z})|\underline{Z})$$

$$= \text{Var}\left(\psi(\underline{Y})|\underline{Z}\right) + \left(E(\psi(\underline{Y})|\underline{Z}) - E\left(E(\psi(\underline{Y})|\underline{Z})\right)\right)^2. \qquad (5.5)$$

Combining (5.4) and (5.5), we get

$$\text{Var}\left(\psi(\underline{Y})\right) = E\left(\text{Var}(\psi(\underline{Y})|\underline{Z})\right) + E\left(E(\psi(\underline{Y})|\underline{Z}) - E\left(E\left(\psi(\underline{Y})|\underline{Z}\right)\right)\right)^2$$

$$= E\left(\text{Var}(\psi(\underline{Y})|\underline{Z})\right) + \text{Var}\left(E(\psi(\underline{Y})|\underline{Z})\right). \blacksquare$$

Remark 5.1

If $\underline{Y}$ and $\underline{Z}$ are independent then

$$E(\psi(\underline{Y})|\underline{Z}) = E\left(\psi(\underline{Y})\right) \text{ and } \text{Var}(\psi(\underline{Y})|\underline{Z}) = \text{Var}\left(\psi(\underline{Y})\right).$$

Example 5.1

Let $\underline{X} = (X_1, X_2, X_3)$ be a discrete type random vector with p.m.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} \dfrac{x_1 x_2 x_3}{72}, & \text{if } (x_1, x_2, x_3) \in \{1,2\} \times \{1,2,3\} \times \{1,3\} \\ 0, & \text{otherwise} \end{cases}.$$

(i) Let $Y_1 = 2X_1 - X_2 + 3X_3$ and $Y_2 = X_1 - 2X_2 + X_3$. Find the correlation coefficient between Y_1 and Y_2 ;

(ii) For a fixed $x_2 \in \{1,2,3\}$, find $E(Y|X_2 = x_2)$ and $\text{Var}(Y|X_2 = x_2)$, where $Y = X_1 X_3$.

Solution.

(i) From Example 4.1 (i) we know that X_1, X_2 and X_3 are independent. Therefore $\mathrm{Cov}(X_1, X_2) = \mathrm{Cov}(X_1, X_3) = \mathrm{Cov}(X_2, X_3) = 0$. Also $\mathrm{Cov}(X_i, X_i) = \mathrm{Var}(X_i), i = 1, 2, 3$. Using Theorem 5.2 (ii) we have

$$\mathrm{Cov}(Y_1, Y_2) = 2\,\mathrm{Var}(X_1) - 5\,\mathrm{Cov}(X_1, X_2) + 2\,\mathrm{Var}(X_2) + 5\,\mathrm{Cov}(X_1, X_3)$$

$$+3\mathrm{Var}(X_3) - 7\mathrm{Cov}(X_2, X_3)$$

$$= 2\mathrm{Var}(X_1) + 2\mathrm{Var}(X_2) + 3\mathrm{Var}(X_3).$$

From the solution of Example 4.1 (ii) we have

$$f_{X_1}(x_1) = \begin{cases} \frac{x_1}{3}, & \text{if } x_1 \in \{1, 2\} \\ 0, & \text{otherwise} \end{cases}, \quad f_{X_2}(x_2) = \begin{cases} \frac{x_2}{6}, & \text{if } x_2 \in \{1, 2, 3\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_{X_3}(x_3) = \begin{cases} \frac{x_3}{4}, & \text{if } x_3 \in \{1, 3\} \\ 0, & \text{otherwise} \end{cases}.$$

Therefore

$$E(X_1) = \sum_{x_1 \in S_{X_1}} x_1 f_{X_1}(x_1) = \sum_{x_1 \in \{1,2\}} \frac{x_1^2}{3} = \frac{(1^2 + 2^2)}{3} = \frac{5}{3}$$

$$E(X_1^2) = \sum_{x_1 \in S_{X_1}} x_1^2 f_{X_1}(x_1) = \sum_{x_1 \in \{1,2\}} \frac{x_1^3}{3} = \frac{(1^3 + 2^3)}{3} = 3$$

$$E(X_2) = \sum_{x_2 \in S_{X_2}} x_2 f_{X_2}(x_2) = \sum_{x_2 \in \{1,2,3\}} \frac{x_2^2}{6} = \frac{(1^2 + 2^2 + 3^2)}{6} = \frac{7}{3}$$

$$E(X_2^2) = \sum_{x_2 \in S_{X_2}} x_2^2 f_{X_2}(x_2) = \sum_{x_2 \in \{1,2,3\}} \frac{x_2^3}{6} = \frac{(1^3 + 2^3 + 3^3)}{6} = 6$$

$$E(X_3) = \sum_{x_3 \in S_{X_3}} x_3 f_{X_3}(x_3) = \sum_{x_3 \in \{1,3\}} \frac{x_3^2}{4} = \frac{(1^2 + 3^2)}{4} = \frac{5}{2}$$

$$E(X_3^2) = \sum_{x_3 \in S_{X_3}} x_3^2 f_{X_3}(x_3) = \sum_{x_3 \in \{1,3\}} \frac{x_3^3}{4} = \frac{(1^3 + 3^3)}{4} = 7$$

$$\text{Var}(X_1) = E(X_1^2) - (E(X_1))^2 = \frac{2}{9}$$

$$\text{Var}(X_2) = E(X_2^2) - (E(X_2))^2 = \frac{5}{9}$$

and

$$\text{Var}(X_3) = E(X_3^2) - \left(E(X_3)\right)^2 = \frac{3}{4}.$$

Therefore,

$$\text{Cov}(Y_1, Y_2) = \frac{4}{9} + \frac{10}{9} + \frac{9}{4} = \frac{137}{36}.$$

Also, by Corollary 5.1,

$$\begin{aligned}
\text{Var}(Y_1) &= \text{Var}(2X_1 - X_2 + 3X_3) \\
&= 4\,\text{Var}(X_1) + \text{Var}(X_2) + 9\,\text{Var}(X_3) \\
&= \frac{8}{9} + \frac{5}{9} + \frac{27}{4} \\
&= \frac{295}{36}
\end{aligned}$$

and

$$\begin{aligned}
\text{Var}(Y_2) &= \text{Var}(X_1 - 2X_2 + X_3) \\
&= \text{Var}(X_1) + 4\text{Var}(X_2) + \text{Var}(X_3) \\
&= \frac{2}{9} + \frac{20}{9} + \frac{3}{4} \\
&= \frac{115}{36}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\rho(Y_1, Y_2) &= \frac{\text{Cov}(Y_1, Y_2)}{\sqrt{\text{Var}(Y_1)\text{Var}(Y_2)}} \\
&= \frac{137}{\sqrt{295}\sqrt{115}}
\end{aligned}$$

$$= 0.7438\cdots$$

(ii) Since X_1, X_2 and X_3 are independent it follows that (X_1, X_3) and X_2 are independent. This in turn implies that $Y = X_1, X_3$ and X_2 are independent. Therefore $E(Y|X_2 = x_2) = E(Y)$ and $\text{Var}(Y|X_2 = x_2) = \text{Var}(Y)$. Now

$$\begin{aligned} E(Y) &= E(X_1 X_3) \\ &= E(X_1)E(X_3) \qquad \text{(using Theorem (5.3))} \\ &= \frac{25}{6}. \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(X_1 X_3) \\ &= \text{Var}(E(X_1 X_3|X_3) + E(\text{Var}(X_1 X_3|X_3)) \\ &= \text{Var}(X_3 E(X_1|X_3)) + E(X_3^2 \text{Var}(X_1|X_3)) \\ &= \text{Var}\big(X_3 E(X_1)\big) + E\big(X_3^2 \text{Var}(X_1)\big) \qquad \text{(Remark 5.1)} \\ &= \text{Var}\left(\frac{5}{3} X_3\right) + E\left(\frac{2}{9} X_3^2\right) \\ &= \frac{25}{9}\text{Var}(X_3) + \frac{2}{9} E(X_3^2) \\ &= \frac{75}{36} + \frac{14}{9} \\ &= \frac{131}{36}. \blacksquare \end{aligned}$$

Example 5.2

Let $\underline{X} = (X_1, X_2, X_3)$ be an absolutely continuous type random vector with p.d.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} \dfrac{1}{x_1 x_2}, & \text{if } 0 < x_3 < x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

(i) Let $Y_1 = 2X_1 - X_2 + 3X_3$ and $Y_2 = X_1 - 2X_2 + X_3$. Find $\rho(Y_1, Y_2)$;
(ii) For a fixed $x_1 \in (0,1)$ find $E(Y|X_1 = x_1)$ and $\text{Var}(Y|X_1 = x_1)$, where $Y = X_1 X_2 X_3$.

Solution.

(i) As in Example 5.1 (i)

$$\mathrm{Cov}(Y_1, Y_2) = 2\,\mathrm{Var}(X_1) + 2\,\mathrm{Var}(X_2) + 3\,\mathrm{Var}(X_3) - 5\,\mathrm{Cov}(X_1, X_2)$$
$$+5\,\mathrm{Cov}(X_1, X_3) - 7\,\mathrm{Cov}(X_2, X_3).$$

$$E(X_1) = \int_{\mathbb{R}^3} x_1 f_{\underline{X}}(\underline{x})d\underline{x} = \int_0^1\int_0^{x_1}\int_0^{x_2} \frac{1}{x_2}\,dx_3\,dx_2\,dx_1 = \frac{1}{2}$$

$$E(X_1^2) = \int_0^1\int_0^{x_1}\int_0^{x_2} \frac{x_1}{x_2}\,dx_3\,dx_2\,dx_1 = \frac{1}{3}$$

$$E(X_2) = \int_0^1\int_0^{x_1}\int_0^{x_2} \frac{1}{x_1}\,dx_3\,dx_2\,dx_1 = \frac{1}{4}$$

$$E(X_2^2) = \int_0^1\int_0^{x_1}\int_0^{x_2} \frac{x_2}{x_1}\,dx_3\,dx_2\,dx_1 = \frac{1}{9}$$

$$E(X_3) = \int_0^1\int_0^{x_1}\int_0^{x_2} \frac{x_3}{x_1\,x_2}\,dx_3\,dx_2\,dx_1 = \frac{1}{8}$$

$$E(X_3^2) = \int_0^1\int_0^{x_1}\int_0^{x_2} \frac{x_3^2}{x_1\,x_2}\,dx_3\,dx_2\,dx_1 = \frac{1}{27}$$

$$E(X_1X_2) = \int_0^1\int_0^{x_1}\int_0^{x_2} dx_3\,dx_2\,dx_1 = \frac{1}{6}$$

$$E(X_1X_3) = \int_0^1\int_0^{x_1}\int_0^{x_2} \frac{x_3}{x_2}\,dx_3\,dx_2\,dx_1 = \frac{1}{12}$$

$$E(X_2X_3) = \int_0^1\int_0^{x_1}\int_0^{x_2} \frac{x_3}{x_1}\,dx_3\,dx_2\,dx_1 = \frac{1}{18}$$

$$\mathrm{Var}(X_1) = E(X_1^2) - (E(X_1))^2 = \frac{1}{12}$$

$$\mathrm{Var}(X_2) = E(X_2^2) - (E(X_2))^2 = \frac{7}{144}$$

$$\mathrm{Var}(X_3) = E(X_3^2) - (E(X_3))^2 = \frac{37}{1728}$$

$$\mathrm{Cov}(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = \frac{1}{24}$$

$$\mathrm{Cov}(X_1, X_3) = E(X_1X_3) - E(X_1)E(X_3) = \frac{1}{48}$$

$$\mathrm{Cov}(X_2, X_3) = E(X_2X_3) - E(X_2)E(X_3) = \frac{7}{288}.$$

Therefore,

$$\mathrm{Cov}(Y_1, Y_2) = \frac{1}{6} + \frac{7}{72} + \frac{37}{576} - \frac{5}{24} + \frac{5}{48} - \frac{49}{288} = \frac{31}{576}.$$

Also,

$$\mathrm{Var}(Y_1) = 4\,\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 9\,\mathrm{Var}(X_3) - 4\,\mathrm{Cov}(X_1, X_2) + 12\,\mathrm{Cov}(X_1, X_3) - 6\,\mathrm{Cov}(X_2, X_3)$$

$$= \frac{1}{3} + \frac{7}{144} + \frac{37}{192} - \frac{1}{6} + \frac{1}{4} - \frac{7}{48}$$

$$= \frac{295}{576}.$$

$$\mathrm{Var}(Y_2) = \mathrm{Var}(X_1) + 4\mathrm{Var}(X_2) + \mathrm{Var}(X_3) - 4\mathrm{Cov}(X_1, X_2) + 2\mathrm{Cov}(X_1, X_3) - 4\mathrm{Cov}(X_2, X_3)$$

$$= \frac{1}{12} + \frac{7}{36} + \frac{37}{1728} - \frac{1}{6} + \frac{1}{24} - \frac{7}{72}$$

$$= \frac{133}{1728}.$$

Therefore

$$\rho(Y_1, Y_2) = \frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)\mathrm{Var}(Y_2)}} = 0.2710\cdots$$

(ii) Clearly, for a fixed $x_1 \in (0, 1)$,

$$f_{X_2,X_3|X_1}(x_2, x_3|x_1) = c_1(x_1) f_{X_1,X_2,X_3}(x_1, x_2, x_3)$$

$$= \begin{cases} \frac{c_2(x_1)}{x_2}, & \text{if } 0 < x_3 < x_2 < x_1 \\ 0, & \text{otherwise} \end{cases}.$$

Since

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_2, X_3 | X_1}(x_2, x_3 | x_1)\, dx_2 dx_3 = 1,$$

we have

$$c_2(x_1) \int_0^{x_1} \int_0^{x_2} \frac{1}{x_2} dx_3 dx_2 = 1,$$

$$\text{i.e.,} \qquad c_2(x_1) = \frac{1}{x_1}.$$

Also

$$E(Y|X_1 = x_1) = E(X_1 X_2 X_3 | X_1 = x_1)$$

$$= x_1 E(X_2 X_3 | X_1 = x_1)$$

$$= x_1 \int_0^{x_1} \int_0^{x_2} x_2 x_3 \frac{1}{x_1 x_2} dx_3 dx_2$$

$$= \frac{x_1^3}{6}.$$

$$E(Y^2|X_1 = x_1) = E(X_1^2 X_2^2 X_3^2 | X_1 = x_1)$$

$$= x_1^2 E(X_2^2 X_3^2 | X_1 = x_1)$$

$$= x_1^2 \int_0^{x_1} \int_0^{x_2} x_2^2 x_3^2 \frac{1}{x_1 x_2} dx_3 dx_2$$

$$= \frac{x_1^6}{15}.$$

Therefore

$$\mathrm{Var}(Y|X_1 = x_1) = E(Y^2|X_1 = x_1) - (E(Y|X_1 = x_1))^2$$

$$= \frac{x_1^6}{15} - \frac{x_1^6}{36}$$

$$= \frac{7}{180} x_1^6. \blacksquare$$

6.6 JOINT MOMENT GENERATING FUNCTION

Let $\underline{X} = (X_1, \dots, X_p)$ be a p-dimensional random vector defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $A = \{\underline{t} = (t_1, t_2, \dots, t_p) \in \mathbb{R}^p : E\left(\left|e^{\sum_{i=1}^p t_i X_i}\right|\right) = E\left(e^{\sum_{i=1}^p t_i X_i}\right) \text{is finite}\}$. Define the function $M_{\underline{X}} : A \to \mathbb{R}$ by

$$M_{\underline{X}}(\underline{t}) = E\left(e^{\sum_{i=1}^p t_i X_i}\right), \quad \underline{t} = (t_1, t_2, \dots, t_p) \in A. \qquad (6.1)$$

Definition 6.1

(i) The function $M_{\underline{X}} : A \to \mathbb{R}$, defined by (6.1), is called the *joint moment generating function* (m.g.f.) of random vector $\underline{X}$.

(ii) We say that the joint m.g.f. of $\underline{X}$ exists if it is finite in a rectangle $(-\underline{a}, \underline{a}) \subseteq \mathbb{R}^p$, for some $\underline{a} = (a_1, a_2, \dots, a_p) \in \mathbb{R}^p$; here $-\underline{a} = (-a_1, -a_2, \dots, -a_p)$ and $(-\underline{a}, \underline{a}) = \{\underline{t} \in \mathbb{R}^p : -a_i < t_i < a_i,\ i = 1, 2, \dots, p\}$. $\blacksquare$

As in the one- dimensional case many properties of probability distribution of $\underline{X}$ can be studied through joint m.g.f. of $\underline{X}$. Some of the results, which may be useful in this direction, are provided below without their proofs. Note that $M_{\underline{X}}(\underline{0}) = 1$. Also if $X_1, \dots, X_p$ are independent then

$$M_{\underline{X}}(\underline{t}) = E\left(e^{\sum_{i=1}^p t_i X_i}\right) = E\left(\prod_{i=1}^p e^{t_i X_i}\right) = \prod_{i=1}^p E\left(e^{t_i X_i}\right) = \prod_{i=1}^p M_{X_i}(t_i), \quad \underline{t} \in \mathbb{R}^p.$$

Theorem 6.1

Suppose that $M_{\underline{X}}(\underline{t})$ exists in a rectangle $(-\underline{a}, \underline{a}) \subseteq \mathbb{R}^p$. Then $M_{\underline{X}}(\underline{t})$ possesses partial derivatives of all orders in $(-\underline{a}, \underline{a})$. Furthermore, for positive integers $k_1, \dots, k_p$,

$$E\left(X_1^{k_1}X_2^{k_2}\cdots X_p^{k_p}\right) = \left[\frac{\partial^{k_1+k_2+k_3\cdots+k_p}}{\partial t_1^{k_1}\cdots\partial t_p^{k_p}} M_{\underline{X}}\left(\underline{t}\right)\right]_{(t_1,t_2,\ldots,t_p)=(0,\ldots,0)} .$$

Under the assumptions of Theorem 6.1, note that, for $\psi_{\underline{X}}\left(\underline{t}\right) = \ln M_{\underline{X}}\left(\underline{t}\right), \underline{t} \in A$,

$$E(X_i) = \left[\frac{\partial}{\partial t_i} M_{\underline{X}}\left(\underline{t}\right)\right]_{\underline{t}=\underline{0}} = \left[\frac{\partial}{\partial t_i} \psi_{\underline{X}}\left(\underline{t}\right)\right]_{\underline{t}=\underline{0}}, \quad i = 1, \ldots, p$$

$$E({X_i}^m) = \left[\frac{\partial^m}{\partial {t_i}^m} M_{\underline{X}}\left(\underline{t}\right)\right]_{\underline{t}=\underline{0}}, \quad i = 1, \ldots, p$$

$$\begin{aligned}\mathrm{Var}(X_i) &= \left[\frac{\partial^2}{\partial {t_i}^2} M_{\underline{X}}\left(\underline{t}\right)\right]_{\underline{t}=\underline{0}} - \left(\left[\frac{\partial}{\partial t_i} M_{\underline{X}}\left(\underline{t}\right)\right]_{\underline{t}=\underline{0}}\right)^2 \\ &= \left[\frac{\partial^2}{\partial {t_i}^2} \psi_{\underline{X}}\left(\underline{t}\right)\right]_{\underline{t}=\underline{0}}, \quad i = 1, \ldots, p,\end{aligned}$$

and, for $i, j \in \{1, \ldots, p\}, i \neq j$,

$$\begin{aligned}\mathrm{Cov}\left(X_i, X_j\right) &= E(X_i X_j) - E(X_i)E\left(X_j\right) \\ &= \left[\frac{\partial^2}{\partial t_i \partial t_j} M_{\underline{X}}\left(\underline{t}\right)\right]_{\underline{t}=\underline{0}} - \left[\frac{\partial}{\partial t_i} M_{\underline{X}}\left(\underline{t}\right)\right]_{\underline{t}=\underline{0}} \left[\frac{\partial}{\partial t_j} M_{\underline{X}}\left(\underline{t}\right)\right]_{\underline{t}=\underline{0}} \\ &= \left[\frac{\partial^2}{\partial t_i \partial t_j} \psi_{\underline{X}}\left(\underline{t}\right)\right]_{\underline{t}=\underline{0}}.\end{aligned}$$

Also note that

$M_{\underline{X}}(0, \ldots, 0, t_i, 0, \ldots, 0) = E(e^{t_i X_i}) = M_{X_i}(t_i)$, $i = 1, 2, \ldots, p$.

and $M_{\underline{X}}\left(0, \ldots, 0, t_i, 0, \ldots, 0, t_j, 0, \ldots, 0\right) = E(e^{t_i X_i + t_j X_j}) = M_{X_i, X_j}\left(t_i, t_j\right)$, $i, j \in \{1, \ldots, p\}$,

provided the involved expectations are finite.