MODULE 2

RANDOM VARIABLE AND ITS DISTRIBUTION

LECTURE 10

Topics

2.4 TYPES OF RANDOM VARIABLES: DISCRETE, CONTINUOUS AND ABSOLUTELY CONTINUOUS

Definition 4.3

(i) A random variable X is said to be of *continuous type* if its distribution function F_X is continuous everywhere.

(ii) A random variable X with distribution function F_X is said to be of *absolutely continuous* type if there exists an integrable function $f_X: \mathbb{R} \to \mathbb{R}$ such that $f_X(x) \geq 0, \forall x \in \mathbb{R}$, and

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt,\ x \in \mathbb{R}.$$

The function f_X is called the *probability density function* (p.d.f.) of random variable X and the set $S_X = \{x \in \mathbb{R}: f_X(x) > 0\}$ is called a *support* of random variable X (or of p.d.f. f_X). ■

Note that if f_X is p.d.f. of an absolutely continuous type r.v. X then $f_X(x) \geq 0, \forall x \in \mathbb{R}$ and $\int_{-\infty}^{\infty} f_X(\mathrm{t})dt = F_X(\infty) = 1$, where $F_X(\infty) = \lim_{x\to\infty} F_X(x)$.

Example 4.5

Let X be a r.v. having the d.f.

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \geq 0 \end{cases}.$$

Clearly F_X is continuous at every $x \in \mathbb{R}$ and therefore X is of continuous type. Also

$$F_X(x) = \int_{-\infty}^{x} f_X(\mathrm{t})dt,\ \ x \in \mathbb{R},$$

where $f_X: \mathbb{R} \to [0, \infty)$ is given by

$$f_X(t) = \begin{cases} 0, & \text{if} \quad t < 0 \\ e^{-t}, & \text{if} \quad t \geq 0 \end{cases}. \tag{4.5}$$

It follows that X is also if absolutely continuous type with p.d.f. given by (4.5).

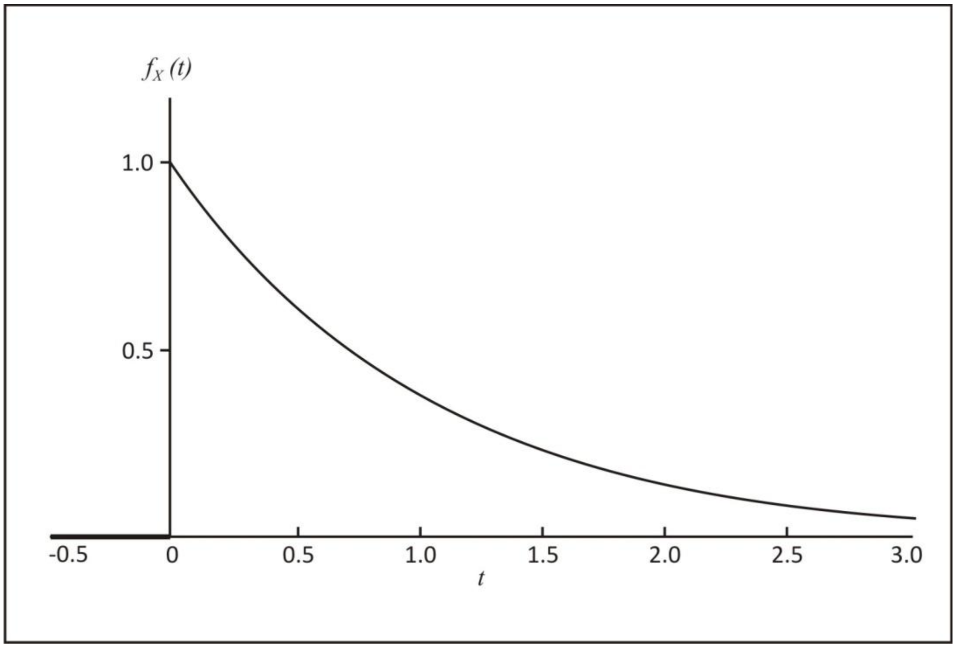

Figure 4.3. Plot of p.d.f. $f_X(t)$

Remark 4.2

(i) Suppose that $X$is a r.v. of continuous type. Then $P_X(\{x\}) = P(\{X = x\}) = F_X(x) - F_X(x-) = 0, \forall x \in \mathbb{R}$. In general, for any countable set $C, P(\{X \in C\}) = P_X(C) = P_X(\cup_{x\in C}\{x\}) = \sum_{x\in C} P_X(\{x\}) = 0.$

(ii) Since the p.d.f. f_X of an absolutely continuous type r.v. X determines its d.f., using Remark 3.2, it follows that we may study the induced probability space $(\mathbb{R}, \mathcal{B}_1, P_X)$ through the study of p.d.f. f_X.

(iii) Suppose that X is a r.v. of absolutely continuous type. Then, for $x \in \mathbb{R}$ and $h > 0$,

$$\begin{aligned} |F_X(x-h) - F_X(x)| &= F_X(x) - F_X(x-h) \\ &= \int_{-\infty}^{x} f_X(\mathrm{t})\, dt - \int_{-\infty}^{x-h} f_X(\mathrm{t})\, dt \\ &= \int_{x-h}^{x} f_X(\mathrm{t})\, dt \end{aligned}$$

$$\to 0, \quad \text{as } h \downarrow 0,$$

i.e., F_X is also left continuous on $\mathbb{R}$. It follows that if X is an absolutely continuous type r.v. then its d.f. F_X is continuous everywhere on $\mathbb{R}$ and hence X is of continuous type.

(iv) Let X be a r.v. of absolutely continuous type. Then X is also of continuous type (see (iii) above) and therefore $P(\{X = x\}) = 0, \ \forall x \in \mathbb{R}$. Consequently,

$$P(\{X < x\}) = P(\{X \le x\}) = F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt\,, x \in \mathbb{R}$$

$$P(\{X \ge x\}) = 1 - P(\{X < x\}) = \int_{x}^{\infty} f_X(t)\, dt\,, x \in \mathbb{R}, \quad \left(\text{since } \int_{-\infty}^{\infty} f_X(t)\, dt = 1\right)$$

and, for $-\infty < a < b < \infty$,

$$\begin{aligned} P(\{a < X \le b\}) = P(\{a < X < b\}) = P(\{a \le X < b\}) &= P(\{a \le X \le b\}) \\ &= F_X(b) - F_X(a) \\ &= \int_{-\infty}^{b} f_X(t)\, dt - \int_{-\infty}^{a} f_X(t)\, dt \\ &= \int_{a}^{b} f_X(t)\, dt \\ &= \int_{-\infty}^{\infty} f_X(t) I_{(a,b)}(t) dt, \end{aligned}$$

where, for a set $A \subseteq \mathbb{R}, I_A$ denotes its indicator function, i.e.,

$$I_A(t) = \begin{cases} 1, & \text{if } t \in A \\ 0, & \text{otherwise} \end{cases}.$$

In general, for any $B \in \mathcal{B}_1$, it can be shown that

$$P(\{X \in B\}) = \int_{-\infty}^{\infty} f_X(\mathrm{t})\, I_B(\mathrm{t})\, dt.$$

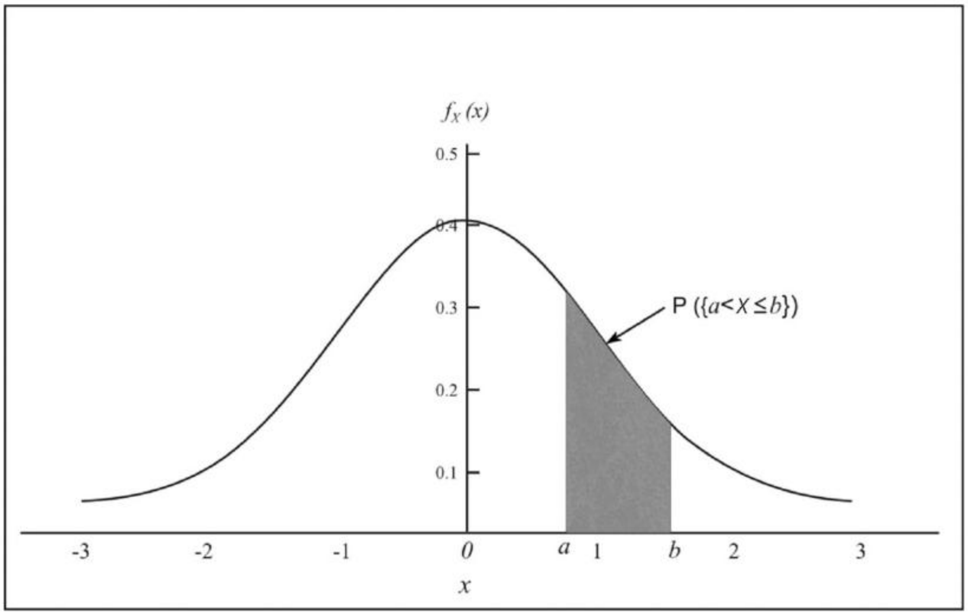

Figure 4.4. Probability of a region

(v) We know that if $h: \mathbb{R} \to \mathbb{R}$ is a non-negative integrable function then, for any countable set $D(\neq \phi)$ in $\mathbb{R}$, and for $-\infty \leq a < b \leq \infty$,

$$\begin{aligned}\int_a^b h(t)\, I_D(\mathrm{t})\, dt &= \int_a^b h(\mathrm{t})\, I_{\bigcup_{x\in D}\{x\}}(t)\, dt \\ &= \int_a^b h(t)\left(\sum_{x\in D} I_{\{x\}}(\mathrm{t})\right) dt \\ &= \sum_{x\in D} \int_a^{\mathrm{b}} h(\mathrm{t})\, I_{\{x\}}(\mathrm{t}) dt \\ &= 0, \qquad (4.6)\end{aligned}$$

since $\int_a^b h(t) I_{\{x\}}(\mathrm{t})\, dt = 0, \forall x \in \mathbb{R}$.

Now let X be a r.v. of absolutely continuous type with p.d.f. f_X and d.f. F_X so that

$$F_X(x) = \int_{-\infty}^{x} f_X(\mathrm{t})\, dt, x \in \mathbb{R}.$$

Let E be any countable set and let $g: \mathbb{R} \to [0, \infty)$ be any non-negative function such that $g(x) = f_X(x), \forall x \in E^C = \mathbb{R} - E$ and $g(x) \neq f_X(x), \forall x \in E$. Then, for $x \in \mathbb{R}$,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

$$= \int_{-\infty}^{x} f_X(t)[I_E(\mathrm{t}) + I_{E^C}(t)]dt$$

$$= \int_{-\infty}^{x} f_X(t)I_{E^C}(t)dt \qquad \text{(using(4.6))}$$

$$= \int_{-\infty}^{x} g(t)I_{E^C}(t)dt \quad (\text{since } f_X(t)\, I_{E^C}(t) = g(t)I_{E^C}(t))$$

$$= \int_{-\infty}^{x} g(t)I_{E^C}(t)dt + \int_{-\infty}^{x} g(t)I_E(t)dt \qquad \text{(using (4.6))}$$

$$= \int_{-\infty}^{x} g(t)\, dt,$$

i.e., g is also a p.d.f. of r.v. X. Note that $g(x) = f_X(x), \forall x \in E^C$ and $g(x) \neq f_X(x), \forall x \in E$. It follows that the p.d.f. of a r.v. of absolutely continuous type is not unique. In fact if the values of the p.d.f. f_X of a r.v.X of absolutely continuous type are changed at a countable number of points with some other non-negative values then the resulting function is again a p.d.f. of X. In other words a r.v. of absolutely continuous type has different versions of p.d.f.s. Consequently the support of an absolutely continuous type r.v. is also not unique and it depends upon the version of p.d.f. chosen. However it is worth mentioning here that the d.f. of any r.v. is unique.

(vi) Suppose that the d.f. F_X of a r.v. X is differentiable at every $x \in \mathbb{R}$. Then

$$F_X(x) = \int_{-\infty}^{x} F_X^{'}(t)dt, x \in \mathbb{R}.$$

It follows that if the d.f. F_X is differentiable everywhere then the r.v. X is of absolutely continuous type and one may take its p.d.f. to be $f_X(x) = F_X^{'}(x)$, $x \in \mathbb{R}$.

(vii) Suppose that the d.f. of a r.v. X is differentiable everywhere except on countable set D. Further suppose that

$$\int_{-\infty}^{\infty} F_X^{'}(t)I_{D^c}(t)dt = 1.$$

Then, using a standard result in advanced calculus, it follows that the random variable X is of absolutely continuous type with a p.d.f.

$$f_X(x) = \begin{cases} F_X'(x), & \text{if } x \notin D \\ a_x, & \text{if } x \in D \end{cases},$$

where $a_x, x \in D$ are arbitrary nonnegative constants. Here, note that

$$\int_{-\infty}^{\infty} F_X'(t) I_{D^c}(t)dt = \int_{-\infty}^{\infty} f_X(t)dt = 1$$

and

$$F_X(x) = \int_{-\infty}^{x} F_X'(t) I_{D^c}(t)dt = \int_{-\infty}^{x} f_X(t)dt, \quad x \in \mathbb{R}.$$

(viii) There are random variables that are neither of discrete type nor of continuous type (and hence also nor of absolutely continuous type). To see this let us consider a r.v. X having the d.f. F_X (see Example 3.2 (ii)) given by

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{4}, & \text{if } 0 \le x < 1 \\ \frac{x}{3}, & \text{if } 1 \le x < 2 \\ \frac{3x}{8}, & \text{if } 2 \le x < \frac{5}{2} \\ 1, & \text{if } x \ge \frac{5}{2} \end{cases}.$$

The set of discontinuity points of F_X is $D_X = \left\{1, 2, \frac{5}{2}\right\}$. Since $D_X \neq \phi$ the r.v. is not of continuous type. Moreover

$$\begin{aligned} P(\{X \in D_X\}) &= P(\{X = 1\}) + P(\{X = 2\}) + P\left(\left\{X = \frac{5}{2}\right\}\right) \\ &= [F_X(1) - F_X(1-)] + [F_X(2) - F_X(2-)] + \left[F_X\left(\frac{5}{2}\right) - F_X\left(\frac{5}{2}-\right)\right] \\ &= \frac{11}{48} < 1, \end{aligned}$$

implying that the r.v. X is also not of discrete type.

(ix) There are random variables which are of continuous type but not of absolutely continuous type. These random variables are normally difficult to study. ■

Example 4.6

Consider a r.v. X having the d.f. F_X (see Example 4.5) given by

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \ge 0 \end{cases}.$$

Clearly F_X is differentiable everywhere except at $x \in D = \{0\}$. Also

$$\int_{-\infty}^{\infty} F_X'(t) I_{D^c}(t) dt = \int_0^{\infty} e^{-t} dt = 1.$$

Using Remark 4.2 (vii) it follows that the r.v. X is of absolutely continuous type and one may take

$$f_X(x) = \begin{cases} 0, & \text{if} \quad x < 0 \\ a, & \text{if} \quad x = 0 \\ e^{-x}, & \text{if} \quad x > 0 \end{cases},$$

as a p.d.f. of X; here a is an arbitrary non-negative constant. In particular one may take

$$f_X(x) = \begin{cases} 0, & \text{if} \quad x \le 0 \\ e^{-x}, & \text{if} \quad x > 0 \end{cases}$$

as a p.d.f. of X. ■

Note that the p.d.f. f_X of a r.v. X of absolutely continuous type satisfies the following two properties:

(i) $f_X(x) \ge 0, \ \forall x \in \mathbb{R}$;

(ii) $\int_{-\infty}^{\infty} f_X(\mathrm{t})\, dt = \lim\limits_{x \to \infty} F_X(x) = 1.$