MODULE 7

LIMITING DISTRIBUTIONS

LECTURES 37-42

Topics

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LECTURE 37

Topics

7.1 CONVERGENCE IN DISTRIBUTION AND PROBABILITY

Let $\underline{T} = (T_1, \dots, T_n)$ be a random vector having a probability density function/probability mass function (p.d.f./p.m.f.) $f_{\underline{T}}(\cdot)$ and let $h: \mathbb{R}^n \to \mathbb{R}$ be a Borel function. Suppose that the distribution of random variable $X_n = h(\underline{T})$ is desired. Very often it is not possible to derive the expression for distribution (i.e., p.d.f. or p.m.f.) of $X_n = h(\underline{T})$. To make this point clear let $T_1, \dots, T_n$ be a random sample from $\mathrm{Be}(a, b)$ distribution, where a and b are positive real constants, and suppose that the distribution (i.e., the distribution function or a p.d.f.) of the sample mean $\overline{T}_n = \frac{1}{n}\sum_{i=1}^{n} T_i$ is desired. The form of the p.d.f. (or distribution function) of $\overline{T}_n$ is so complicated (it involves multiple integrals which cannot be expressed in a closed form) that hardly anybody would be interested in using it. Therefore, it will be helpful if we can approximate the distribution of $\overline{T}_n$ by a distribution which is mathematically tractable. In this module we will develop a theory which will help us in approximating distributions of a sequence $\{X_n\}_{n\geq 1}$ of random variables for large values of n (say, as $n \to \infty$). Such approximations are quite useful in statistical inference problems.

7.1 CONVERGENCE IN DISTRIBUTION AND PROBABILITY

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables with corresponding sequence of distribution functions (d.f.s) as $\{F_n\}_{n\geq 1}$. Suppose that an approximation to the distribution of X_n (i.e., of F_n) is desired, for large values of n (say, as $n \to \infty$). It may be tempting to approximate $F_n(\cdot)$ by $F(x) = \lim_{n\to\infty} F_n(x)\,, x \in \mathbb{R}$. However, as the following examples illustrate, $F(x) = \lim_{n\to\infty} F_n(x)\,, x \in \mathbb{R}$, may not be a d.f..

Example 7.1

(i) Let $\{X_n\}_{n\geq 1}$ be sequence of random variables with $P(\{X_n = n\}) = 1,\ n = 1, 2, \ldots$. Then the d.f. of X_n is given by

$$F_n(x) = \begin{cases} 0, & \text{if } x < n \\ 1, & \text{if } x \geq n \end{cases}, \qquad n = 1, 2, \ldots.$$

We have $F(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} F_n(x) = 0, \forall x \in \mathbb{R}$. Clearly F is not a d.f..

(ii) Let $X_n \sim U(-n, n), n = 1, 2, \ldots$. Then the d.f. of X_n is

$$F_n(x) = \begin{cases} 0, & \text{if } x < -n \\ \dfrac{x+n}{2n}, & \text{if} -n \leq x < n, \\ 1, & \text{if } x \geq n \end{cases} \qquad n = 1, 2, \ldots.$$

Clearly $F(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} F_n(x) = \frac{1}{2}, \forall x \in \mathbb{R}$ and $F(\cdot)$ is not a d.f. ■

The above examples illustrate that a sequence $\{F_n\}_{n\geq 1}$ of d.f.s on $\mathbb{R}$ may converge, at all points, but the limiting function $F(x) = \lim_{n\to\infty} F_n(x)\,, x \in \mathbb{R}$, may not be a d.f..

The following example illustrates that if a sequence $\{F_n\}_{n\geq 1}$ of d.f.s converges at every point then it may be too restrictive to require that $\{F_n\}_{n\geq 1}$ converges to a d.f. F at all points (i.e., to require that $\lim_{n\to\infty} F_n(x) = F(x), \forall x \in \mathbb{R}$, for some d.f. F).

Example 7.2

Let $\{X_n\}_{n\geq 1}$be a sequence of random variables with $P\left(\left\{X_n = \frac{1}{n}\right\}\right) = 1,\ n = 1, 2, \ldots.$ Then the d.f. of X_n is

$$F_n(x) = \begin{cases} 0, & \text{if } x < \dfrac{1}{n} \\ 1, & \text{if } x \geq \dfrac{1}{n} \end{cases}, \qquad n = 1, 2, \ldots.$$

Clearly,

$$F(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} F_n(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}$$

is not a d.f. (it is not right continuous at $x = 0$). However, F can be converted into a distribution function

$$F^*(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases},$$

by changing its value at the point 0 (the point of discontinuity of F). Since $P\left(\left\{X_n = \frac{1}{n}\right\}\right) = 1, n = 1, 2, \ldots,$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, a natural approximation of F_n seems to be the distribution function of a random variable X that is degenerate at 0 (i.e., $P(\{X = 0\}) = 1$). Note that F^* is the d.f. of random variables X that is degenerate at 0. The above discussion suggests that it is too restrictive to require

$$\lim_{n\to\infty} F_n(x) = F^*(x), \forall x \in \mathbb{R},$$

and that exceptions may be permitted at the points of discontinuities of F^*. ■

Definition 1.1

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables and let F_n be the d.f. of $X_n, n = 1, 2, \ldots.$

(i) Let X be a random variable with d.f. F. The sequence $\{X_n\}_{n\geq 1}$ is said to *converge in distribution* to X, as $n \to \infty$ (written as $X_n \xrightarrow{d} X$, as $n \to \infty$) if $\lim_{n\to\infty} F_n(x) = F(x), \forall x \in C_F$, where C_F is the set of continuity points of F. The d.f. F (or the corresponding p.d.f/p.m.f.) is called the *limiting distribution* of X_n , as $n \to \infty$.

(ii) Let $c \in \mathbb{R}$. The sequence $\{X_n\}_{n\geq 1}$ is said to converge in probability to c, as $n \to \infty$ (written as $X_n \xrightarrow{p} c$, as $n \to \infty$) if $X_n \xrightarrow{d} X$, as $n \to \infty$, where X is a random variable that is degenerate at c. ■

Remark 1.1

(i) Suppose that $X_n \xrightarrow{d} X$, as $n \to \infty$. Since the set $D_F = C_F^c = \mathbb{R} - C_F$ of discontinuity points of limiting d.f. F is at most countable we have $\lim_{n\to\infty} F_n(x) = F(x)$ everywhere except, possibly, at a countable number of points.

(ii) Note that the distribution function of a random variable degenerate at point $c \in \mathbb{R}$ is given by

$$F(x) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x \geq c \end{cases}.$$

Thus we have

$$X_n \xrightarrow{p} c, \text{as } n \to \infty \Leftrightarrow \lim_{n\to\infty} F_n\,(x) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x \geq c \end{cases}.$$

(iii) Suppose that $X_n \xrightarrow{d} X$, as $n \to \infty$. If the random variable X is of continuous type (i.e., $C_F = \mathbb{R}$) then $\lim_{n\to\infty} F_n\,(x) = F(x), \forall x \in \mathbb{R}$.

(iv) Note that, for a real constant c, $X_n \xrightarrow{p} c$ if, and only if, $X_n - c \xrightarrow{p} 0$, as $n \to \infty$. ■

Example 1.3

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that $P(\{X_n = 0\}) = \frac{1}{n} = 1 - P\left(\left\{X_n = \frac{1}{n}\right\}\right)$, $n = 1, 2, \ldots$. Show that $X_n \xrightarrow{p} 0$, as $n \to \infty$.

Solution. Let F be the d.f. of a random variable degenerate at 0, i.e.,

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}.$$

Since F is continuous everywhere except at point 0 (i.e., $C_F = \mathbb{R} - \{0\}$), we need to show that $\lim_{n\to\infty} F_n(x) = F(x), \forall x \in \mathbb{R} - \{0\}$, where $F_n(\cdot)$ is the d.f. of $X_n, n = 1, 2, \ldots$.

We have

$$F_n(x) = \begin{cases} 0, & \text{if } x < 0 \\ \dfrac{1}{n}, & \text{if } 0 \leq x < \dfrac{1}{n}, \\ 1, & \text{if } x \geq \dfrac{1}{n} \end{cases} \qquad n = 1, 2, \ldots$$

$$\xrightarrow{n\to\infty} \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0 \end{cases}.$$

Clearly $\lim_{n\to\infty} F_n(x) = F(x), \forall x \in \mathbb{R} - \{0\}$. ■

Example 1.4

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) $U(0,\theta)$ random variables, where $\theta > 0$. Let $X_{n:n} = \max\{X_1, \ldots, X_n\}$ and let $Y_n = n(\theta - X_{n:n})$, $n = 1, 2, \ldots$.

(i) Show that $X_{n:n} \xrightarrow{p} \theta$, as $n \to \infty$;

(ii) Find the limiting distribution of $\{Y_n\}_{n\geq 1}$.

Solution.

(i) Let H_n be the d.f. of $X_{n:n}$, $n = 1, 2, \ldots$, and let

$$H(x) = \begin{cases} 0, & \text{if } x < \theta \\ 1, & \text{if } x \geq \theta \end{cases}$$

be the d.f. of random variable degenerate at θ. We need to show that $\lim_{n\to\infty} H_n(x) = H(x), \ \forall x \in \mathbb{R} - \{\theta\}$.

We have, for $x \in \mathbb{R}$,

$$\begin{aligned} H_n(x) &= P(\{X_{n:n} \leq x\}) \\ &= P(\{\max\{X_1, \ldots, X_n\} \leq x\}) \\ &= P(\{X_i \leq x, i = 1, \ldots, n\}) \\ &= \prod_{i=1}^{n} P(\{X_i \leq x\}) \qquad \text{(since } X_i\text{s are independent)} \\ &= [F(x)]^n, n = 1,2, \ldots, \qquad \text{(since } X_i\text{s are identically distributed),} \end{aligned}$$

where

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \dfrac{x}{\theta}, & \text{if } 0 \leq x < \theta \\ 1, & \text{if } x \geq \theta \end{cases}$$

is the common distribution function of $X_1, X_2, \ldots$.

Thus

$$H_n(x) = \begin{cases} 0, & \text{if } x < 0 \\ \left(\frac{x}{\theta}\right)^n, & \text{if } 0 \le x < \theta \\ 1, & \text{if } x \ge \theta \end{cases}$$

$$\xrightarrow{n\to\infty} \begin{cases} 0, & \text{if } x < \theta \\ 1, & \text{if } x \ge \theta \end{cases}$$

$$= H(x), \forall x \in \mathbb{R}.$$

(ii) For $y \in \mathbb{R}$, we have

$$F_{Y_n}(y) = P(\{Y_n \le y\})$$
$$= P\left(\left\{X_{n:n} \ge \theta - \frac{y}{n}\right\}\right)$$
$$= 1 - H_n\left(\left(\theta - \frac{y}{n}\right) -\right)$$
$$= 1 - H_n\left(\theta - \frac{y}{n}\right) \qquad \text{(since } H_n \text{ is continuous)}$$
$$= \begin{cases} 0, & \text{if } y \le 0 \\ 1 - \left(1 - \frac{y}{n\theta}\right)^n, & \text{if } 0 < y \le n\theta, \quad n = 1, 2, \ldots. \\ 1, & \text{if } y > n\theta \end{cases}$$

$$\xrightarrow{n\to\infty} \begin{cases} 0, & \text{if } y \le 0 \\ 1 - e^{-\frac{y}{\theta}}, & \text{if } y > 0 \end{cases}$$

$$= G(y), \text{ say.}$$

Note that $G(\cdot)$ is the d.f. of $\text{Exp}(\theta)$ random variable. Thus $Y_n \xrightarrow{d} Y \sim \text{Exp}(\theta)$, as $n \to \infty$. ■

In the above example we saw that $X_{n:n} \xrightarrow{p} \theta$, as $n \to \infty$, and $n(\theta - X_{n:n}) \xrightarrow{d} Y \sim \text{Exp}(\theta)$, as $n \to \infty$, i.e., the limiting distribution of $X_{n:n}$ is degenerate (at θ) and, to get a non-degenerate limiting distribution, we needed normalized version $Y_n = n(\theta - X_{n:n})$ of $X_{n:n}, n = 1, 2, \ldots.$ This phenomenon is observed quite commonly. Generally, we will have a sequence $\{X_n\}_{n\ge1}$ of random variables, such that $X_n \xrightarrow{p} c$, as $n \to \infty$ for some real constant c (i.e., the limiting distribution of X_n is degenerate at c). In order to get a non-degenerate limiting distribution a normalized version $Z_n = n^r(X_n - c)$ $\left(\text{or } Z_n = n^r(c - X_n)\right)$, $r > 0$, of X_n, $n = 1, 2, \ldots.$ is considered. Typically there is a choice of $r > 0$ such that the limiting distribution of Z_n is non-degenerate.