MODULE 3

FUNCTION OF A RANDOM VARIABLE AND ITS DISTRIBUTION

LECTURE 13

Topics

3.3 EXPECTATION AND MOMENTS OF A RANDOM VARIABLE

Suppose that X is a discrete type random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ associated with a random experiment. Let S_X and f_X denote, respectively, the support and the p.m.f. of X. Suppose that the random experiment is repeated a large number of times. Let $f_n(x), x \in S_X$, denote the frequency of the event $\{X = x\}$ in the first n repetitions of the random experiment. Then, according to the relative frequency approach to the probability,

$$P(\{X = x\}) = \lim_{n\to\infty} \frac{f_n(x)}{n}, x \in S_X.$$

Note that $\sum_{x \in S_X} \frac{x f_n(x)}{n}$ represents the mean observed value (or expected value) of random variable X in the first n repetitions of the random experiment. Therefore, in line with axiomatic approach to probability, one may define the mean value (or expected value) of random variable X as

$$E(X) = \lim_{n\to\infty} \sum_{x \in S_X} \frac{x f_n(x)}{n}$$

$$= \sum_{x \in S_X} x \lim_{n\to\infty} \frac{f_n(x)}{n}$$

$$= \sum_{x \in S_X} x\, P(\{X = x\})$$

$$= \sum_{x \in S_X} x \, f_X(x),$$

provided the involved limits exist and the interchange of signs of summation and limit is admissible. A similar discussion can be provided for defining the expected value of an absolutely continuous type random variable, having p.d.f. f_X, as

$$E(X) = \int_{-\infty}^{\infty} x \, f_X(x) dx,$$

provided the integral is defined. ■

The above discussion leads to the following definitions.

Definition 3.1

(i) Let X be a discrete type random variable with p.m.f. f_X and support S_X. We say that the expected value of X (denoted by $E(X)$) is finite and equals

$$E(X) = \sum_{x \in S_X} x \, f_X(x),$$

provided $$\sum_{x \in S_X} |x| \, f_X(x) < \infty.$$

(ii) Let X be an absolutely continuous type random variable with p.d.f. f_X. We say that the expected value of X (denoted by $E(X)$) is finite and equals

$$E(X) = \int_{-\infty}^{\infty} x \, f_X(x) dx,$$

provided $$\int_{-\infty}^{\infty} |x| \, f_X(x) dx < \infty.$$ ■

The following observations to above definitions are immediate.

Remark 3.1

(i) Since

$$\left| \sum_{x \in S_X} x \, f_X(x) \right| \leq \sum_{x \in S_X} |x \, f_X(x)| = \sum_{x \in S_X} |x| \, f_X(x)$$

and

$$\left|\int_{-\infty}^{\infty} x\, f_X(x) dx\right| \leq \int_{-\infty}^{\infty} |x\, f_X(x)| dx = \int_{-\infty}^{\infty} |x|\, f_X(x) dx,$$

it follows that if the expected value of a random variable X is finite then $|E(X)| < \infty$.

(ii) If X is a random variable of discrete type with finite support S_X, then

$$\sum_{x \in S_X} |x|\, f_X(x) < \infty.$$

Consequently the expected value of X is finite.

(iii) Support that X is a random variable of absolutely continuous type with p.d.f. f_X and support $S_X \subseteq [-a, a]$, for some $a \in (0, \infty)$. Then

$$\int_{-\infty}^{\infty} |x|\, f_X(x) dx = \int_{-a}^{a} |x|\, f_X(x) dx \leq a \int_{-a}^{a} f_X(x)\, dx = a < \infty.$$

Consequently the expected value of X is finite. ■

Example 3.1

Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \left(\frac{1}{2}\right)^x, & \text{if } x \in \{1, 2, 3, \cdots\} \\ 0, & \text{otherwise} \end{cases}.$$

Show that the expected value of X is finite and find its value.

Solution. We have $S_X = \{1, 2, 3, \cdots\}$ and

$$\sum_{x \in S_X} |x|\, f_X(x) = \sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} a_n \text{, say.}$$

Clearly $a_n = \frac{n}{2^n} > 0, n = 1, 2, \cdots$ and

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2n} \to \frac{1}{2} < 1, \text{as } n \to \infty.$$

By the ratio test,

$$\sum_{x \in S_X} |x| \, f_X(x) = \sum_{n=1}^{\infty} a_n < \infty,$$

and therefore the expected value of X is finite.

Moreover,

$$E(X) = \sum_{x \in S_X} x \, f_X(x) = \sum_{j=1}^{\infty} \frac{j}{2^j} = \lim_{n \to \infty} S_n,$$

where

$$S_n = \sum_{j=1}^{n} \frac{j}{2^j} \tag{3.1}$$

$$\Rightarrow \frac{S_n}{2} = \sum_{j=1}^{n} \frac{j}{2^{j+1}}$$

$$= \sum_{j=2}^{n+1} \frac{j-1}{2^j}. \tag{3.2}$$

On subtracting (3.2) from (3.1), we get

$$\frac{S_n}{2} = \sum_{j=1}^{n} \frac{1}{2^j} - \frac{n}{2^{n+1}}$$

$$= 1 - \left(\frac{1}{2}\right)^n - \frac{n}{2^{n+1}}$$

$$\Rightarrow S_n = 2\left[1 - \left(\frac{1}{2}\right)^n - \frac{n}{2^{n+1}}\right]$$

$$\Rightarrow E(X) = \lim_{n \to \infty} S_n = 2 \cdot \blacksquare$$

Example 3.2

Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \frac{3}{\pi^2 x^2}, & \text{if } x \in \{\pm 1, \pm 2, \pm 3, \cdots\} \\ 0, & \text{otherwise} \end{cases}.$$

Show that the expected value of X is not finite.

Solution. We have $S_X = \{\pm 1, \pm 2, \pm 3, \cdots\}$ and

$$\sum_{x \in S_X} |x|\, f_X(x) = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty \cdot$$

Thus the expected value of X is not finite. ■

Example 3.3

Let X be random variable with p.d.f.

$$f_X(x) = \frac{e^{-|x|}}{2}, -\infty < x < \infty.$$

Show that the expected value of X is finite and find its value.

Solution. We have

$$\int_{-\infty}^{\infty} |x|\, f_X(x) dx = \int_{-\infty}^{\infty} |x| \frac{e^{-|x|}}{2} dx$$

$$= \int_{0}^{\infty} x\, e^{-x} dx$$

$$= 1.$$

Thus the expected value of X is finite and

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x) dx$$

$$= \int_{-\infty}^{\infty} x \frac{e^{-|x|}}{2} dx$$

$$= 0. \blacksquare$$

Example 3.4

Let X be random variable with p.d.f.

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, -\infty < x < \infty.$$

Show that the expected value of X is not finite. (The above p.d.f. is called the p.d.f. of the Cauchy distribution).

Solution. We have

$$\int_{-\infty}^{\infty} |x|\, f_X(x)dx = \frac{1}{\pi} \int_{-\infty}^{\infty} |x| \frac{1}{1+x^2}\, dx = \frac{2}{\pi} \int_{0}^{\infty} \frac{1}{1+x^2}\, dx = \infty.$$

Therefore the expected value of X is not finite. ■

Theorem 3.1

Let X be a random variable of absolutely continuous or discrete type with finite expected value. Then

(i) $E(X) = \int_0^\infty P(\{X > t\})\, dt - \int_{-\infty}^0 P(\{X < t\})\, dt$;

(ii) $E(X) = \int_0^\infty P(\{X > t\})\, dt$, provided $P(\{X \geq 0\}) = 1$;

(iii) $E(X) = \sum_{n=1}^{\infty} P(\{X \geq n\}) - \sum_{n=1}^{\infty} P(\{X \leq -n\})$, provided $P(\{X \in \{0, \pm 1, \pm 2, \cdots\}) = 1$;

(iv) $E(X) = \sum_{n=1}^{\infty} P(\{X \geq n\})$, provided $P(\{X \in \{0, 1, 2, \cdots\}) = 1$.

Proof. (i)

Case I. X is of absolutely continuous type

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x\, f_X(x)dx \\ &= \int_{-\infty}^{0} x\, f_X(x)dx + \int_{0}^{\infty} x\, f_X(x)dx \\ &= -\int_{-\infty}^{0}\int_{x}^{0} f_X(x)dt\, dx + \int_{0}^{\infty}\int_{0}^{x} f_X(x)dtdx \cdot \end{aligned}$$

On changing the order of integration in the two integrals above, we get

$$E(X) = -\int_{-\infty}^{0}\int_{-\infty}^{t} f_X(x)dx\, dt + \int_{0}^{\infty}\int_{t}^{\infty} f_X(x)dxdt$$

$$= -\int_{-\infty}^{0} P(\{X < t\})\, dt + \int_{0}^{\infty} P(\{X > t\})\, dt.$$

Case II. X is of discrete type

We will illustrate the idea of the proof by considering a special case where $S_X = \{x_1, x_2, \cdots\}, -\infty < x_1 < x_2 < \cdots < x_i < 0 < x_{i+1} < x_{i+2} < \cdots < \infty$ and $\lim_{n\to\infty} x_n = \infty$. Under the above situation

$$\int_{0}^{\infty} P\,(\{X > t\})dt = \int_{0}^{x_{i+1}} P(X > t)\, dt + \sum_{j=i+1}^{\infty} \int_{x_j}^{x_{j+1}} P(\{X > t\})\, dt$$

$$= \int_{0}^{x_{i+1}} P\,(\{X \geq x_{i+1}\})dt + \sum_{j=i+1}^{\infty} \int_{x_j}^{x_{j+1}} P\left(\{X \geq x_{j+1}\}\right) dt$$

$$= x_{i+1} P(\{X \geq x_{i+1}\}) + \sum_{j=i+1}^{\infty} \left(x_{j+1} - x_j\right) P\left(\{X \geq x_{j+1}\}\right)$$

$$= x_{i+1} P(\{X \geq x_{i+1}\}) + \sum_{j=i+1}^{\infty} x_{j+1} P\left(\{X \geq x_{j+1}\}\right) - \sum_{j=i+1}^{\infty} x_j P\left(\{X \geq x_{j+1}\}\right)$$

$$= x_{i+1} P(\{X \geq x_{i+1}\}) + \sum_{j=i+2}^{\infty} x_j P\left(\{X \geq x_j\}\right) - \sum_{j=i+1}^{\infty} x_j P\left(\{X \geq x_{j+1}\}\right)$$

$$= x_{i+1}[P(\{X \geq x_{i+1}\}) - P(\{X \geq x_{i+2}\})] + \sum_{j=i+2}^{\infty} x_j\ [P\left(\{X \geq x_j\}\right) - P\left(\{X \geq x_{j+1}\}\right)]$$

$$= \sum_{j=i+1}^{\infty} x_j P\left(\{X = x_j\}\right) \cdot$$

Also,

$$\int_{-\infty}^{0} P(\{X < t\})\, dt = \int_{-\infty}^{x_1} P(\{X < t\})\, dt + \sum_{j=1}^{i-1} \int_{x_j}^{x_{j+1}} P(\{X < t\})\, dt + \int_{x_i}^{0} P(\{X < t\})\, dt$$

$$= 0 + \sum_{j=1}^{i-1} \int_{x_j}^{x_{j+1}} P(\{X \le x_j\})\, dt + \int_{x_i}^{0} P(\{X \le x_i\})\, dt$$

$$= \sum_{j=1}^{i-1} (x_{j+1} - x_j)\, P(\{X \le x_j\}) - x_i P(\{X \le x_i\})$$

$$= \sum_{j=1}^{i-1} x_{j+1}\, P(\{X \le x_j\}) - \sum_{j=1}^{i-1} x_j\, P(\{X \le x_j\}) - x_i P(\{X \le x_i\})$$

$$= \sum_{j=2}^{i} x_j\, P(\{X \le x_{j-1}\}) - \sum_{j=1}^{i-1} x_j\, P(\{X \le x_j\}) - x_i P(\{X \le x_i\})$$

$$= -\sum_{j=1}^{i} x_j\, P(\{X = x_j\}) \cdot$$

Therefore,

$$\int_0^\infty P(\{X > t\})dt - \int_{-\infty}^{0} P(\{X < t\})dt = \sum_{j=1}^{\infty} x_j\, P(\{X = x_j\}) = E(X).$$

(ii) Suppose that $P(\{X \ge 0\}) = 1$. Then $P(\{X < t\}) = 0, \forall t \ge 0$, and therefore

$$E(X) = \int_0^\infty P(\{X > t\})dt - \int_{-\infty}^{0} P(\{X < t\})dt = \int_0^\infty P(\{X > t\})dt.$$

(iii) Suppose that $P(\{X \in \{0, \pm 1, \pm 2, \cdots\})$. Then, for $m \in Z$ (the set of integers) and $m - 1 \le t \le m$, we have

$$P(\{X > t\}) = P(\{X \ge m\}) \text{ and } P(\{X < t\}) = P(\{X \le m - 1\}) \cdot$$

Therefore,

$$\int_0^\infty P(\{X > t\})dt = \sum_{n=1}^{\infty} \int_{n-1}^{n} P(\{X > t\})\, dt$$

$$= \sum_{n=1}^{\infty} \int_{n-1}^{n} P\left(\{X \geq n\}\right) dt$$

$$= \sum_{n=1}^{\infty} P(\{X \geq n\}),$$

$$\int_{-\infty}^{0} P\left(\{X < t\}\right) dt = \sum_{n=1}^{\infty} \int_{-n}^{-n+1} P\left(\{X < t\}\right) dt$$

$$= \sum_{n=1}^{\infty} \int_{-n}^{-n+1} P\left(\{X \leq -n\}\right) dt$$

$$= \sum_{n=1}^{\infty} P(\{X \leq -n\}),$$

and the assertion follows on using (i).

(iv) Suppose that $P\left(\{X \in \{0, 1, 2, \cdots\}\}\right) = 1$. Then $P(\{X \leq -n\}) = 0, \forall n \in \{1, 2, \cdots\}$ and the result follows from (iii).

Theorem 3.2

(i) Let X be a random variable of discrete type with support S_X and p.m.f. f_X. Let $h: \mathbb{R} \to \mathbb{R}$ be a Borel function and let $T = h(X)$. Then

$$E(T) = \sum_{x \in S_X} h(x)\, f_X(x),$$

provided it is finite.

(ii) Let X be a random variable of absolutely continuous type with p.d.f. f_X. Let $h: \mathbb{R} \to \mathbb{R}$ be a Borel function and let $T = h(X)$. Then

$$E(T) = \int_{-\infty}^{\infty} h(x)\, f_X(x) dx,$$

provided it is finite.

Proof.

(i) By Theorem 2.1, $T = h(X)$ is a random variable of discrete type with support $S_T = \{h(x): x \in S_X\}$ and p.m.f.

$$f_T(t) = P(\{T = t\}) = \begin{cases} \sum_{x \in A_t} P(\{X = x\}), & \text{if } t \in S_T \\ 0, & \text{otherwise} \end{cases},$$

where $A_t = \{x \in S_X : h(x) = t\}, t \in S_T$, so that $\{A_t : t \in S_T\}$ forms a partition of $S_X \left(A_s \cap A_t = \phi, \text{if } s \neq t, \text{and } \bigcup_{t \in S_T} A_t = S_X\right)$. Therefore,

$$\begin{aligned} E(T) &= \sum_{t \in S_T} t P(\{T = t\}) \\ &= \sum_{t \in S_T} t \left\{ \sum_{x \in A_t} P(\{X = x\}) \right\} \\ &= \sum_{t \in S_T} \sum_{x \in A_t} t\, P(\{X = x\}) \\ &= \sum_{t \in S_T} \sum_{x \in A_t} h(x)\, P(\{X = x\}) \qquad (\text{since for } x \in A_t, t = h(x)) \\ &= \sum_{x \in \bigcup_{t \in S_T} A_t} h(x)\, P(\{X = x\}) \qquad (\text{since } A_s \cap A_t = \phi, \text{if } s \neq t) \\ &= \sum_{x \in S_X} h(x)\, P(\{X = x\}) \qquad \left(\text{since} \bigcup_{t \in S_T} A_t = S_X\right) \\ &= \sum_{x \in S_X} h(x)\, f_X(x). \end{aligned}$$

(ii) Define $A_t = \{x \in S_X : h(x) > t\}, t \geq 0$, and $B_s = \{x \in S_X : h(x) < s\}, s \leq 0$. For simplicity we assume that, for every $t \geq 0$ and $s \leq 0$, A_t and B_s are intervals. Then, using Theorem 3.1 (i),

$$\begin{aligned} E(T) &= \int_0^\infty P(\{T > t\}) dt - \int_{-\infty}^0 P(\{T < s\}) ds \\ &= \int_0^\infty \int_{A_t} f_X(x) dx\, dt - \int_{-\infty}^0 \int_{B_s} f_X(x) dx\, ds \\ &= \int_{A_0} \int_0^{h(x)} f_X(x) dt\, dx - \int_{B_0} \int_{h(x)}^0 f_X(x) ds\, dx, \end{aligned}$$

on interchanging the order of integration in the above two integrals and using the following two observations: (a) $t \in (0, \infty), x \in A_t \Leftrightarrow x \in A_0$ and $t \in \big(0, h(x)\big)$; (b) $s \in (-\infty, 0), x \in B_s \Leftrightarrow x \in B_0$ and $s \in (h(x), 0)$. Therefore,

$$E(T) = \int_{A_0} h\,(x) f_X(x) dx + \int_{B_0} h\,(x) f_X(x) dx$$

$$= \int_{S_X} h\,(x) f_X(x) dx$$

$$= \int_{-\infty}^{\infty} h\,(x) f_X(x) dx,$$

Since $A_0 \cap B_0 = \phi$ and $S_X = A_0 \cup B_0 \cup \{x \in S_X : h(x) = 0\}$. ■

Remark 3.2

Recall that probability density function of absolutely continuous type random variable is not unique. However the distribution function of any random variable is unique. Theorem 3.1 (i) implies that the expected value

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x) dx = \int_0^{\infty} \big(1 - F_X(t)\big) dt - \int_{-\infty}^{0} F_X(t\,-) dt\,,$$

of an absolutely continuous type random variable is unique (i.e., it does not depend on the version of p.d.f. used) although the probability density function f_X may not be unique. ■

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