MODULE 6

RANDOM VECTOR AND ITS JOINT DISTRIBUTION

LECTURE 34

Topics

6.10.2 Transformation of Variables Technique

6.10.2.1 Distribution of Order Statistics

6.10.2.2 Distribution of Normalized Spacing's of Exponential Distribution

For finding the probability distributions of functions of a random vector of absolutely continuous type we have the following theorem.

Theorem 10.2.2

Let $\underline{X} = (X_1, \dots, X_p)$ be a random vector of absolutely continuous type with a joint p.d.f. $f_{\underline{X}}(\cdot)$ and support $S_{\underline{X}} = \{\underline{x} \in \mathbb{R}^p : f_{\underline{X}}(\underline{x}) > 0\}$. Let $S_1, \dots, S_k$ be open subset of $\mathbb{R}^p$ such that $S_i \cap S_j = \phi$, if $i \neq j$, and $\bigcup_{i=1}^k S_i = S_{\underline{X}}$. Suppose that $h_j : \mathbb{R}^p \to \mathbb{R}$, $j = 1, \dots, p$, are p Borel functions such that on each S_i, $\underline{h} = (h_1, \dots, h_p): S_i \to \mathbb{R}^p$ is one-to-one with inverse transformation $h_i^{-1}(\underline{t}) = \left(h_{1,i}^{-1}(\underline{t}), \dots, h_{p,i}^{-1}(\underline{t})\right)$ (say), $i = 1, \dots, k$. Further suppose that $h_{j,i}^{-1}(\underline{t})$, $j = 1, \dots, p, i = 1, \dots, k$, have continuous partial derivatives and the Jacobian determinants

$$J_i = \begin{vmatrix} \dfrac{\partial h_{1,i}^{-1}(\underline{t})}{\partial t_1} & \cdots & \dfrac{\partial h_{1,i}^{-1}(\underline{t})}{\partial t_p} \\ \dfrac{\partial h_{2,i}^{-1}(\underline{t})}{\partial t_1} & \cdots & \dfrac{\partial h_{2,i}^{-1}(\underline{t})}{\partial t_p} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial h_{p,i}^{-1}(\underline{t})}{\partial t_1} & \cdots & \dfrac{\partial h_{p,i}^{-1}(\underline{t})}{\partial t_p} \end{vmatrix} \neq 0, i = 1, \dots, p.$$

Define $\underline{h}(S_j) = \left\{\underline{h}(\underline{x}) = \left(h_1(\underline{x}), \dots, h_p(\underline{x})\right) \in \mathbb{R}^p : \underline{x} \in S_j\right\}$, $j = 1, \dots, p$ and $T_j = h_j(X_1, \dots, X_p)$, $j = 1, \dots, p$. Then the random vector $\underline{T} = (T_1, \dots, T_p)$ is of absolutely continuous type with joint p.d.f.

$$f_{\underline{T}}(\underline{t}) = \sum_{j=1}^{k} f_{\underline{X}}\left(h_{1,j}^{-1}(\underline{t}), \dots, h_{p,j}^{-1}(\underline{t})\right) |J_j| \, I_{\underline{h}(s_j)}(\underline{t}). \blacksquare$$

We shall not provide the proof of the above theorem. The idea of the proof of the above theorem is similar to that of Theorem 2.2, Module 3. In the proof of the theorem, the joint distribution function of $\underline{T}$ is written in the form of multiple integrals which are simplified by making change of variables using change of variable Theorem of multivariable calculus.

The following corollary is immediate from Theorem 10.2.2.

Corollary 10.2.1

Let $\underline{X} = (X_1, \dots, X_p)$ be a random vector of absolutely continuous type with a joint p.d.f. $f_{\underline{X}}(\cdot)$ and support $S_{\underline{X}} = \{\underline{x} \in \mathbb{R}^p : f_{\underline{X}}(\underline{x}) > 0\}$, an open set in $\mathbb{R}^p$. Suppose that $h_j : \mathbb{R}^p \to \mathbb{R}, j = 1, \dots, p$, are p Borel functions such that $\underline{h} = (h_1, \dots, h_p) : S_{\underline{X}} \to \mathbb{R}^p$ is one-to-one with inverse transformation $\underline{h}^{-1}(\underline{t}) = (h_1^{-1}(\underline{t}), \dots, h_p^{-1}(\underline{t}))$ (say). Further suppose that $h_i^{-1}, i = 1, \dots, p$, have continuous partial derivatives and the Jacobian determinant

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_1^{-1}(\underline{t})}{\partial t_p} \\ \frac{\partial h_2^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_2^{-1}(\underline{t})}{\partial t_p} \\ \vdots & & \\ \frac{\partial h_p^{-1}(\underline{t})}{\partial t_1} & \cdots & \frac{\partial h_p^{-1}(\underline{t})}{\partial t_p} \end{vmatrix} \neq 0.$$

Define $\underline{h}(S_{\underline{X}}) = \{\underline{h}(\underline{x}) = (h_1(\underline{x}), \cdots, h_p(\underline{x})) \in \mathbb{R}^p : \underline{x} \in S_{\underline{X}}\}$ and $T_j = h_j(X_1, \dots, X_p), j = 1, \dots, p$. Then the random vector $\underline{T} = (T_1, \dots, T_p)$ is of absolutely continuous type with joint p.d.f.

$$f_{\underline{T}}(t) = f_{\underline{X}}\left(h_1^{-1}(\underline{t}), \cdots, h_p^{-1}(\underline{t})\right) \; |J| I_{\underline{h}(S_{\underline{X}})}(\underline{t}). \blacksquare$$

Remark 10.2.1

Let $\underline{X} = (X_1, \dots, X_p)$ be a random vector of absolutely continuous type with joint p.d.f. $f_{\underline{X}}$ and let $S_{\underline{X}} = \{\underline{x} \in \mathbb{R}^p : f_{\underline{X}}(\underline{x}) > 0\}$.Suppose that we are interested in finding the joint probability distribution of random vector $\underline{T} = (T_1, \dots, T_k) = (h_1(\underline{X}), \dots, h_k(\underline{X}))$, where

$k \in \{1, \dots, p\}$ and $h_i: \mathbb{R}^p \to \mathbb{R},\ i = 1, \dots, k$, are some Borel functions. For this we shall define $p - k$ additional auxiliary Borel functions $h_i: \mathbb{R}^p \to \mathbb{R},\ i = k+1, \dots, p$, such that the transformation $\underline{h} = (h_1, \dots, h_p): S_{\underline{X}} \to \mathbb{R}^p$, satisfies the assumptions of Theorem 10.2.2/Corollary 10.2.1. Then an application of Theorem 10.2.2/Corollary 10.2.1 will provide the joint p.d.f. $f_{\underline{T}}(t_1, \dots, t_p)$ of $\underline{T} = (T_1, \dots, T_p)$ from which marginal joint p.d.f. of $\underline{U} = (T_1, \dots, T_k)$ is obtained by integrating out unwanted variables $t_{k+1}, \dots, t_p$ in $f_{\underline{T}}(u_1, \dots, u_k, t_{k+1}, \dots, t_p)$.

Example 10.2.8

Let X_1 and X_2 be independent and identically distributed random variables with common p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if} -2 < x < -1 \\ \frac{1}{6}, & \text{if } 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

Find the p.d.f. of $Y_1 = |X_1| + |X_2|$.

Solution. The joint p.d.f. of $\underline{X} = (X_1, X_2)$ is given by

$$f_{\underline{X}}(x_1, x_2) = f(x_1)f(x_2)$$

$$= \begin{cases} \frac{1}{4}, & \text{if } (x_1, x_2) \in (-2,-1) \times (-2,-1) \\ \frac{1}{12}, & \text{if } (x_1, x_2) \in \big((-2,-1) \times (0,3)\big) \cup \big((0,3) \times (-2,-1)\big) \\ \frac{1}{36}, & \text{if } (x_1, x_2) \in (0,3) \times (0,3) \\ 0, & \text{otherwise} \end{cases}.$$

Define the auxiliary random variable $Y_2 = |X_1|$. We have

$$S_{\underline{X}} = \{\underline{x} = (x_1, x_2) \in \mathbb{R}^2 : f_{\underline{X}}(x_1, x_2) > 0\}$$

$$= S_1 \cup S_2 \cup S_3 \cup S_4,$$

where $S_1 = (-2,-1)^2, S_2 = (-2,-1) \times (0,3), S_3 = (0,3) \times (-2,-1)$ and $S_4 = (0,3)^2$. Let $\underline{h} = (h_1, h_2): \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$h_1(x_1, x_2) = |x_1| + |x_2| \text{ and } h_2(x_1, x_2) = |x_1|,\ \underline{x} = (x_1, x_2) \in \mathbb{R}^2.$$

Then $Y_1 = h_1(X_1, X_2), Y_2 = h_2(X_1, X_2), S_i \cap S_j = \phi, i \neq j$ and on each $S_i, i = 1, 2, 3, 4$, $\underline{h} = (h_1, h_2): S_i \to \mathbb{R}^2$ is one-to-one. Under the notation of Theorem 10.2.2 we have

$$h_{1,1}^{-1}(\underline{t}) = -t_2, h_{2,1}^{-1}(\underline{t}) = t_2 - t_1, J_1 = \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} = -1;$$

$$h_{1,2}^{-1}(\underline{t}) = -t_2, h_{2,2}^{-1}(\underline{t}) = t_1 - t_2, \qquad J_2 = \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = 1;$$

$$h_{1,3}^{-1}(\underline{t}) = t_2, h_{2,3}^{-1}(\underline{t}) = t_2 - t_1, J_3 = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1;$$

$$h_{1,4}^{-1}(\underline{t}) = t_2, h_{2,4}^{-1}(\underline{t}) = t_1 - t_2, J_4 = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1;$$

$$\underline{h}(S_1) = \{(t_1, t_2) \in \mathbb{R}^2: -2 < -t_2 < -1, -2 < t_2 - t_1 < -1\}$$

$$= \{(t_1, t_2) \in \mathbb{R}^2: t_2 + 1 < t_1 < t_2 + 2, 1 < t_2 < 2\};$$

$$\underline{h}(S_2) = \{(t_1, t_2) \in \mathbb{R}^2: -2 < -t_2 < -1, 0 < t_1 - t_2 < 3\}$$

$$= \{(t_1, t_2) \in \mathbb{R}^2: t_2 < t_1 < t_2 + 3, 1 < t_2 < 2\};$$

$$\underline{h}(S_3) = \{(t_1, t_2) \in \mathbb{R}^2: 0 < t_2 < 3, -2 < t_2 - t_1 < -1\}$$

$$= \{(t_1, t_2) \in \mathbb{R}^2: t_2 + 1 < t_1 < t_2 + 2, 0 < t_2 < 3\}$$

and

$$\underline{h}(S_4) = \{(t_1, t_2) \in \mathbb{R}^2: 0 < t_2 < 3, 0 < t_1 - t_2 < 3\}$$

$$= \{(t_1, t_2) \in \mathbb{R}^2: t_2 < t_1 < t_2 + 3, 0 < t_2 < 3\}.$$

Consequently the joint p.d.f. of $\underline{Y} = (Y_1, Y_2)$ is given by

$$f_{\underline{Y}}(t_1, t_2) = \sum_{j=1}^{4} f_{\underline{X}}\left(h_{1,j}^{-1}(\underline{t}), h_{2,j}^{-1}(\underline{t})\right)|J_j| I_{\underline{h}(S_j)}(\underline{t})$$

$$= f_{\underline{X}}(-t_2, t_2 - t_1) I_{\underline{h}(S_1)}(\underline{t}) + f_{\underline{X}}(-t_2, t_1 - t_2) I_{\underline{h}(S_2)}(\underline{t})$$

$$+ f_{\underline{X}}(t_2, t_2 - t_1) I_{\underline{h}(S_3)}(\underline{t}) + f_{\underline{X}}(t_2, t_1 - t_2) I_{\underline{h}(S_4)}(\underline{t})$$

$$= \begin{cases} \frac{1}{36}, & \text{if } t_2 < t_1 < t_2 + 1, 0 < t_2 < 1 \\ \frac{1}{12} + \frac{1}{36}, & \text{if } t_2 + 1 < t_1 < t_2 + 2, 0 < t_2 < 1 \\ \frac{1}{36}, & \text{if } t_2 + 2 < t_1 < t_2 + 3, 0 < t_2 < 1 \\ \frac{1}{12} + \frac{1}{36}, & \text{if } t_2 < t_1 < t_2 + 1, 1 < t_2 < 2 \\ \frac{1}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{36}, & \text{if } t_2 + 1 < t_1 < t_2 + 2, 1 < t_2 < 2 \\ \frac{1}{12} + \frac{1}{36}, & \text{if } t_2 + 2 < t_1 < t_2 + 3, 1 < t_2 < 2 \\ \frac{1}{36}, & \text{if } t_2 < t_1 < t_2 + 1, 2 < t_2 < 3 \\ \frac{1}{12} + \frac{1}{36}, & \text{if } t_2 + 1 < t_1 < t_2 + 2, 2 < t_2 < 3 \\ \frac{1}{36}, & \text{if } t_2 + 2 < t_1 < t_2 + 3, 2 < t_2 < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{36}, & \text{if } 0 < t_1 < 2, \max\{0, t_1 - 1\} < t_2 < \min\{1, t_1\} \\ & \text{or} \\ & 2 < t_1 < 4, \max\{0, t_1 - 3\} < t_2 < \min\{1, t_1 - 2\} \\ & \text{or} \\ & 2 < t_1 < 4, \max\{2, t_1 - 1\} < t_2 < \min\{3, t_1\} \\ & \text{or} \\ & 4 < t_1 < 6, \max\{2, t_1 - 3\} < t_2 < \min\{3, t_1 - 2\} \\ \frac{1}{9}, & \text{if } 1 < t_1 < 3, \max\{0, t_1 - 2\} < t_2 < \min\{1, t_1 - 1\} \\ & \text{or} \\ & 1 < t_1 < 3, \max\{1, t_1 - 1\} < t_2 < \min\{2, t_1\} \\ & \text{or} \\ & 3 < t_1 < 5, \max\{1, t_1 - 3\} < t_2 < \min\{2, t_1 - 2\} \\ & \text{or } 3 < t_1 < 5, \max\{2, t_1 - 2\} < t_2 < \min\{3, t_1 - 1\} \\ \frac{4}{9}, & \text{if } 2 < t_1 < 4, \max\{1, t_1 - 2\} < t_2 < \min\{2, t_1 - 1\} \\ 0, & \text{otherwise} \end{cases}.$$

Then the marginal p.d.f. of Y_1 is given by

$$f_{Y_1}(t_1) = \int_{-\infty}^{\infty} f_{\underline{Y}}(t_1, t_2) dt_2, \quad t_1 \in \mathbb{R}.$$

For $t_1 \in (0,1)$

$$f_{Y_1}(t_1) = \frac{\min\{1, t_1\} - \max\{0, t_1 - 1\}}{36}$$

$$= \frac{t_1}{36};$$

for $t_1 \in (1,2)$

$$f_{Y_1}(t_1) = \frac{\min\{1, t_1\} - \max\{0, t_1 - 1\}}{36} + \frac{\min\{1, t_1 - 1\} - \max\{0, t_1 - 2\}}{9}$$

$$+ \frac{\min\{2, t_1\} - \max\{1, t_1 - 1\}}{9}$$

$$= \frac{7t_1 - 6}{36};$$

for $t_1 \in (2,3)$

$$f_{Y_1}(t_1) = \frac{\min\{1, t_1 - 2\} - \max\{0, t_1 - 3\}}{36} + \frac{\min\{3, t_1\} - \max\{2, t_1 - 1\}}{36}$$

$$+ \frac{\min\{1, t_1 - 1\} - \max\{0, t_1 - 2\}}{9} + \frac{\min\{2, t_1\} - \max\{1, t_1 - 1\}}{9}$$

$$+ \frac{4[\min\{2, t_1 - 1\} - \max\{1, t_1 - 2\}]}{9}$$

$$= \frac{5t_1 - 6}{18};$$

for $t_1 \in (3,4)$

$$f_{Y_1}(t_1) = \frac{\min\{1, t_1 - 2\} - \max\{0, t_1 - 3\}}{36} + \frac{\min\{3, t_1\} - \max\{2, t_1 - 1\}}{36}$$

$$+ \frac{\min\{2, t_1 - 2\} - \max\{1, t_1 - 3\}}{9} + \frac{\min\{3, t_1 - 1\} - \max\{2, t_1 - 2\}}{9}$$

$$+ \frac{4[\min\{2, t_1 - 1\} - \max\{1, t_1 - 2\}]}{9}$$

$$= \frac{24 - 5t_1}{18};$$

for $t_1 \in (4, 5)$

$$f_{Y_1}(t_1) = \frac{\min\{3, t_1 - 2\} - \max\{2, t_1 - 3\}}{36} + \frac{\min\{2, t_1 - 2\} - \max\{1, t_1 - 3\}}{9}$$

$$+ \frac{\min\{3, t_1 - 1\} - \max\{2, t_1 - 2\}}{9}$$

$$= \frac{36 - 7t_1}{36};$$

and, for $t_1 \in (5, 6)$

$$f_{Y_1}(t_1) = \frac{\min\{3, t_1 - 2\} - \max\{2, t_1 - 3\}}{36}$$

$$= \frac{6 - t_1}{36}.$$

Therefore the p.d.f. of $Y_1 = |X_1| + |X_2|$ is given by

$$f_{Y_1}(t_1) = \begin{cases} \frac{t_1}{36}, & \text{if } 0 < t_1 < 1 \\ \frac{7t_1 - 6}{36}, & \text{if } 1 < t_1 < 2 \\ \frac{5t_1 - 6}{18}, & \text{if } 2 < t_1 < 3 \\ \frac{24 - 5t_1}{18}, & \text{if } 3 < t_1 < 4 \\ \frac{36 - 7t_1}{36}, & \text{if } 4 < t_1 < 5 \\ \frac{6 - t_1}{36}, & \text{if } 5 < t_1 < 6 \\ 0, & \text{otherwise} \end{cases} . \blacksquare$$

6.10.2.1 Distribution of Order Statistics

Example 10.2.9

Let $X_1, \ldots, X_n$ be a random sample of absolutely continuous type random variables having a common p.d.f. $f(\cdot)$, the common distribution function $F(\cdot)$ and a common support $S = \{x \in \mathbb{R}: f(x) > 0\}$, an open set in $\mathbb{R}$. Let $X_{1:n}, \ldots, X_{n:n}$ denote the order statistics of

$X_1, \ldots, X_n$, i.e.,$X_{r:n}$ = r-th smallest of $X_1, \ldots, X_n, r = 1, \ldots, n$. For notational convenience, let $Y_r = X_{r:n},\ r = 1, \ldots, n$.

(i) Find an expression for the joint distribution function of $\underline{Y} = (Y_1, \ldots, Y_n)$. Hence find the joint p.d.f. of $\underline{Y}$;

(ii) Find the joint p.d.f. of $\underline{Y}$ directly using Theorem 10.2.2;

(iii) Using (ii), find the marginal p.d.f. of $Y_r, r = 1, \ldots, n$;

(iv) Using (ii), find the marginal joint p.d.f. of (Y_r, Y_s), where $1 \le r < s \le n$.

Proof. The joint p.d.f. of $\underline{X} = (X_1, \ldots, X_n)$ is given by

$$f_{\underline{X}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) = \prod_{i=1}^{n} f(x_i),\ \ \underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Let $S_n = \left\{\underline{\Pi}_1, \cdots, \underline{\Pi}_{n!}\right\}$ denote the set of all permutations of $(1, \ldots, n)$; here for $i \in \{1, \ldots, n!\}, \underline{\Pi}_i = \left(\Pi_{i,(1)}, \ldots, \Pi_{i,(n)}\right)$ is a permutation of $(1, \ldots, n)$.

(i) Since $X_1, \ldots, X_n$ is a random sample we have

$$(X_1, \ldots, X_n) \stackrel{d}{=} \left(X_{\Pi_{i,(1)}}, \ldots, X_{\Pi_{i,(n)}}\right),\ \ i \in \{1, \ldots, n!\}. \qquad (10.2.1)$$

Also since $\underline{X} = (X_1, \ldots, X_n)$ is of absolutely continuous type (as $X_1, \ldots, X_n$ are of absolutely continuous type) we have $P\left(\{X_i = X_j, \text{for some } i \neq j\}\right) = 0$. Therefore

$$\sum_{i=1}^{n!} P\left(\left\{X_{\Pi_{i,(1)}} < \cdots < X_{\Pi_{i,(n)}}\right\}\right) = 1.$$

Then, for $\underline{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$\begin{aligned}
F_Y(y) &= P(\{X_{1:n} \le y_1, \ldots, X_{n:n} \le y_n\}) \\
&= \sum_{i=1}^{n!} P\left(\left\{X_{1:n} \le y_1, \ldots, X_{n:n} \le y_n, X_{\Pi_{i,(1)}} < \cdots < X_{\Pi_{i,(n)}}\right\}\right) \\
&= \sum_{i=1}^{n!} P\left(\left\{X_{\Pi_{i,(1)}} \le y_1, \ldots, X_{\Pi_{i,(n)}} \le y_n, X_{\Pi_{i,(1)}} < \cdots < X_{\Pi_{i,(n)}}\right\}\right) \\
&= \sum_{i=1}^{n!} P\left(\{X_1 \le y_1, \ldots, X_n \le y_n, X_1 < \cdots < X_n\}\right) \qquad \text{(using (10.2.1))}
\end{aligned}$$

$$= n!\,P(\{X_1 \le y_1, \dots, X_n \le y_n, X_1 < \cdots < X_n\})$$

$$= \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_n} n!\left(\prod_{i=1}^{n} f(x_i)\right) I_A(\underline{x})\,dx_n \cdots dx_1,$$

where $A = \{\underline{x} \in \mathbb{R}^n : -\infty < x_1 < \cdots < x_n < \infty\}$.

It follows that $\underline{Y}$ is of absolutely continuous type with p.d.f.

$$f_{\underline{Y}}(\underline{y}) = n!\left(\prod_{i=1}^{n} f(y_i)\right) I_A(\underline{y})$$

$$= \begin{cases} n!\left(\prod_{i=1}^{n} f(y_i)\right), & \text{if } -\infty < y_1 < \cdots < y_n < \infty \\ 0, & \text{otherwise} \end{cases}.$$

(ii) Since $\underline{X}$ is of absolutely continuous type we may, without loss of generality, take $S_{\underline{X}} \subseteq \left\{\underline{x} \in \mathbb{R}^n : x_i \ne x_j, \forall i \ne j, i, j \in \{1, \dots, n\}\right\}$. Then $S_X = \bigcup_{i=1}^{n!} S_i$, where $S_i = \left\{\underline{x} \in S_{\underline{X}} : x_{\Pi_{i,(1)}} < \cdots < x_{\Pi_{i,(n)}}\right\}, i = 1, \dots, n!$. Define $h_i : \mathbb{R}^n \to \mathbb{R}$ by $h_i(\underline{x}) = i$-th smallest of $x_1, \dots, x_n, i = 1, \dots, n$ and $\underline{h} = (h_1, \dots, h_n)$. Then $\underline{h} : S_{\underline{X}} \to \mathbb{R}^n$ is not one-to-one (for each $\underline{y} \in \underline{h}(S_{\underline{X}}) = \{\underline{h}(\underline{t}) : \underline{t} \in S_{\underline{X}}\}$, there are $n!$ pre-images). However, on each $S_i, i = 1, \dots, n!$, $\underline{h} : S_i \to \mathbb{R}^n$ is one-to-one with inverse transformation $\underline{h}_i^{-1}(\underline{y}) = \left(\underline{h}_{1,i}^{-1}(\underline{y}), \dots, \underline{h}_{n,i}^{-1}(\underline{y})\right) = \left(y_{\Pi_{i,(1)}^{-1}}, \dots, y_{\Pi_{i,(n)}^{-1}}\right)$, where $\underline{\Pi}_i^{-1} = \left(\Pi_{i,(1)}^{-1}, \dots, \Pi_{i,(n)}^{-1}\right), i = 1, \dots, n!$ is the inverse permutation of $\underline{\Pi}_i$. Under the notation of Theorem 10.2.2 each row and each column of the Jacobian determinant J_i contains one, and only one, non-zero element which is 1. Therefore $J_i = \pm 1, i = 1, \dots, n!$. Also $\underline{h}(S_i) = \left\{\underline{y} \in S_{\underline{X}} : -\infty < y_1 < \cdots < y_n < \infty\right\} = B$, say, $i = 1, \dots, n$. Therefore the joint p.d.f. of $\underline{Y}$ is given by

$$f_{\underline{Y}}(\underline{y}) = \sum_{i=1}^{n!} f_{\underline{X}}\left(y_{\Pi_{i,(1)}^{-1}}, \dots, y_{\Pi_{i,(n)}^{-1}}\right) |J_i| I_{\underline{h}(S_i)}(\underline{y})$$

$$= \sum_{i=1}^{n!}\left(\prod_{l=1}^{n} f\left(y_{\Pi_{i,(l)}^{-1}}\right)\right) I_B(\underline{y}).$$

Since $\{\Pi_{i,(1)}^{-1}, \dots, \Pi_{i,(n)}^{-1}\} = \{1, \dots, n\}$, we have

$$\prod_{l=1}^{n} f\left(y_{\Pi_{i,(l)}^{-1}}\right) = \prod_{l=1}^{n} f(y_l), \forall \underline{y} \in B.$$

Consequently

$$f_{\underline{Y}}\left(\underline{y}\right) = \left(\sum_{i=1}^{n!}\left(\prod_{l=1}^{n} f(y_l)\right) I_B\left(\underline{y}\right)\right)$$

$$= n!\left(\prod_{l=1}^{n} f(y_l)\right) I_B\left(\underline{y}\right)$$

$$= \begin{cases} n!\left(\prod_{l=1}^{n} f(y_l)\right), & \text{if } -\infty < y_1 < \cdots < y_n < \infty \\ 0, & \text{otherwise} \end{cases}, \underline{y} \in S_{\underline{X}}.$$

(iii) The marginal p.d.f of $Y_r\,(r = 1, \dots, n)$ is given by

$$f_{Y_r}(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{\underline{Y}}\left(y_1, \dots, y_{r-1} y, y_{r+1,} \dots, y_n\right) \prod_{\substack{l=1 \\ l \neq r}}^{n} dy_l$$

$$= \int_{y}^{\infty} \int_{y}^{y_n} \cdots \int_{y}^{y_{r+2}} \int_{-\infty}^{y} \int_{-\infty}^{y_{r-1}} \cdots \int_{-\infty}^{y_2} n!\left(\prod_{\substack{l=1 \\ l \neq r}}^{n} f(y_l)\right) f(y) \prod_{\substack{l=1 \\ l \neq r}}^{n} dy_l$$

$$= \frac{n!}{(r-1)!\,(n-r)!}[F(y)]^{r-1}[1 - F(y)]^{n-r} f(y), \quad -\infty < y < \infty,$$

since $\int_{-\infty}^{a} f(t)\,dt = F(a)$ and $\int_{b}^{\infty} f(t)\,dt = 1 - F(b)$, $a, b \in \mathbb{R}$.

(iv) As in (iii) we have

$$f_{Y_r,Y_s}(x, y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} n!\, f_{\underline{Y}}(y_1, \dots, y_{r-1}, x, y_{r+1}, \dots y_{s-1}, y, y_{s+1}, \dots y_n) \prod_{\substack{l=1 \\ l \neq r,s}}^{n} dy_l$$

$$= \int_{y}^{\infty} \int_{y}^{y_n} \cdots \int_{y}^{y_{s+2}} \int_{x}^{y} \int_{x}^{y_{s-1}} \cdots \int_{x}^{y_{r+2}} \int_{-\infty}^{x} \int_{-\infty}^{y_{r-1}} \cdots \int_{-\infty}^{y_2} n!\left(\prod_{\substack{l=1 \\ l \neq r,s}}^{n} f(y_l)\right) f(x) f(y) \prod_{\substack{l=1 \\ l \neq r,s}}^{n} dy_l, \text{ if } x < y$$

$$= \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}[F(x)]^{r-1} \times$$

$$[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y), \quad \text{if} -\infty < x < y < \infty.$$

Clearly $f_{Y_r,Y_s}(x,y) = 0,$ if $x \geq y$.

Therefore,

$$f_{r,s}(x,y) = \begin{cases} \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y), & \text{if} -\infty < x < y < \infty \\ 0, & \text{otherwise} \end{cases} \blacksquare$$

6.10.2.2 Distribution of Normalized Spacing's of Exponential Distribution

Example 10.2.10

Let $X_1, \dots, X_n$ be a random sample from $\text{Exp}(\theta)$ distribution, where $\theta > 0$. Let $X_{1:n} \leq \dots \leq X_{n:n}$ denote the order statistics of $X_1, \dots, X_n$. Define $Z_1 = n\,X_{1:n},\ Z_i = (n-i+1)(X_{i:n} - X_{i-1:n}),\ i = 2, \dots, n$. Show that $Z_1, \dots, Z_n$ are independent and identically distributed $\text{Exp}(\theta)$ random variables.

Solution. The common p.d.f. of random variables $X_1, \dots, X_n$ is

$$f(y) = \begin{cases} \frac{1}{\theta} e^{-\frac{y}{\theta}}, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases}.$$

For notational convenience, let $Y_r = X_{r:n}, r = 1, \dots, n$. Then, by Example 10.2.9, a joint p.d.f. of $\underline{Y} = (Y_1, \dots, Y_n)$ is

$$f_{\underline{Y}}\left(\underline{y}\right) = \begin{cases} n!\left(\prod_{i=1}^{n} f(y_i)\right), & \text{if } 0 < y_1 < y_2 < \cdots < y_n < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{n!}{\theta^n} e^{-\frac{\sum_{1=1}^{n} y_i}{\theta}}, & \text{if } 0 < y_1 < y_2 < \cdots < y_n < \infty \\ 0, & \text{otherwise} \end{cases}.$$

The support of $f_{\underline{Y}}(\cdot)$ is $S_{\underline{Y}} = \left\{\underline{y} \in \mathbb{R}^n : 0 < y_1 < y_2 < \cdots < Y_n < \infty\right\}$. Consider the transformation $\underline{h} = (h_1, \dots, h_n): \mathbb{R}^n \to \mathbb{R}^n$, where $h_1\left(\underline{y}\right) = ny_1, h_i\left(\underline{y}\right) = (n-i+$

$1)(y_i - y_{i-1}), i = 2, \dots, n$. Then $Z_1 = h_1(\underline{Y})$ and $Z_i = h_i(\underline{Y}),\ i = 2, \dots, n$. Clearly the transformation $\underline{h}: S_{\underline{Y}} \to \mathbb{R}^n$ is one-to-one with inverse transformation $\underline{h}^{-1} = (h_1^{-1}, \dots, h_n^{-1})$, where for $\underline{z} \in \underline{h}(S_{\underline{Y}})$,

$$h_1^{-1}(\underline{z}) = \frac{z_1}{n}$$

$$h_2^{-1}(\underline{z}) = \frac{z_1}{n} + \frac{z_2}{n-1}$$

$$\vdots$$

$$h_i^{-1}(\underline{z}) = \frac{z_1}{n} + \frac{z_2}{n-1} + \cdots + \frac{z_i}{n-i+1} = \sum_{j=1}^{i} \frac{z_j}{n-j+1}$$

$$\vdots$$

$$h_n^{-1}(\underline{z}) = \frac{z_1}{n} + \frac{z_2}{n-1} + \cdots + \frac{z_{n-1}}{2} + \frac{z_n}{1} = \sum_{j=1}^{n} \frac{z_j}{n-j+1}.$$

The Jacobian determinant of the transformation is

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial z_1} & \frac{\partial h_1^{-1}}{\partial z_2} & \cdots & \frac{\partial h_1^{-1}}{\partial z_n} \\ \frac{\partial h_2^{-1}}{\partial z_1} & \frac{\partial h_2^{-1}}{\partial z_2} & \cdots & \frac{\partial h_2^{-1}}{\partial z_n} \\ \vdots & & & \\ \frac{\partial h_n^{-1}}{\partial z_1} & \frac{\partial h_n^{-1}}{\partial z_2} & \cdots & \frac{\partial h_n^{-1}}{\partial z_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{n} & 0 & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 0 \\ \vdots & & & & \\ \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & 1 \end{vmatrix}$$

$$= \frac{1}{n!}.$$

Also

$$\underline{z} = (z_1, z_2, \dots, z_n) \in h(S_{\underline{Y}}) \Leftrightarrow \left(h_1^{-1}(\underline{z}), \dots, h_n^{-1}(\underline{z})\right) \in S_{\underline{Y}}$$

$$\Leftrightarrow 0 < \frac{z_1}{n} < \frac{z_1}{n} + \frac{z_2}{n} < \cdots < \frac{z_1}{n} + \frac{z_2}{n} + \cdots + \frac{z_n}{n} < \infty$$

$$\Leftrightarrow z_i > 0, \;\; i = 1, \dots n.$$

Therefore $\underline{h}(S_{\underline{Y}}) = (0, \infty)^n$ and the joint p.d.f. of $\underline{Z}$ is given by

$$f_{\underline{Z}}(\underline{z}) = f_{\underline{Y}}\left(h_1^{-1}(\underline{z}), \dots, h_n^{-1}(\underline{z})\right) |J| I_{\underline{h}(S_{\underline{Y}})}(\underline{z})$$

$$= \frac{n!}{\theta^n} e^{-\frac{1}{\theta}\sum_{1=1}^{n} h_i^{-1}(\underline{z})} \times \frac{1}{n!} \times I_{(0,\infty)^n}(\underline{z}).$$

We have, for $\underline{z} \in (0, \infty)^n$,

$$\sum_{i=1}^{n} h_i^{-1}(\underline{z}) = \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{z_j}{n-j+1}$$

$$= \sum_{j=1}^{n} \sum_{i=j}^{n} \frac{z_j}{n-j+1}$$

$$= \sum_{j=1}^{n} z_j .$$

Since $I_{(0,\infty)^n}(\underline{z}) = \prod_{i=1}^{n} I_{(0,\infty)}(z_i)$, we have

$$f_{\underline{Z}}(\underline{z}) = \prod_{i=1}^{n} \left(\frac{1}{\theta} e^{-\frac{z_i}{\theta}} I_{(0,\infty)}(z_i)\right).$$

It follows that $Z_1, \dots, Z_n$ are independent and identically distributed $\text{Exp}(\theta)$ random variables. ■