MODULE 1

PROBABILITY

LECTURE 4

Topics

1.3 CONDITIONAL PROBABILITY AND INDEPENDENCE OF EVENTS

1.3.1 Theorem of Total Probability

1.3 CONDITIONAL PROBABILITY AND INDEPENDENCE OF EVENTS

Let $(\Omega, \mathcal{F}, P)$ be a given probability space. In many situations we may not be interested in the whole space Ω. Rather we may be interested in a subset $B \in \mathcal{F}$ of the sample space Ω. This may happen, for example, when we know apriori that the outcome of the experiment has to be an element of $B \in \mathcal{F}$.

Example 3.1

Consider a random experiment of shuffling a deck of 52 cards in such a way that all 52! arrangements of cards (when looked from top to bottom) are equally likely. Here Ω comprises of all 52! permutations of cards, and $\mathcal{F} = \mathcal{P}(\Omega)$.

Now suppose that it is noticed that the bottom card is the king of heart. In the light of this information, sample space B comprises of $51!$ arrangements of 52 cards with bottom card as king of heart. Define the event

$$K\text{: top card is king.}$$

For $E \in \mathcal{F}$, define

$$P(E) = \text{probability of event } E \text{ under sample space } \Omega,$$

$$P_B(E) = \text{probability of event } E \text{ under sample space } B.$$

Clearly,

$$P_B(K) = \frac{3 \times 50!}{51!}.$$

Note that

$$P_B(K) = \frac{3 \times 50!}{51!} = \frac{\frac{3\times 50!}{52!}}{\frac{51!}{52!}} = \frac{P(K \cap B)}{P(B)}$$

$$\text{i.e.}, P_B(K) = \frac{P(K \cap B)}{P(B)}. \qquad (3.1)$$

We call $P_B(K)$ the conditional probability of event K given that the experiment will result in an outcome in B (i.e., the experiment will result in an outcome $\omega \in B$) and $P(K)$ the unconditional probability of event K. ■

Example 3.1 has laid ground for introduction of the concept of conditional probability.

Let $(\Omega, \mathcal{F}, P)$ be a given probability space. Suppose that we know in advance that the outcome of the experiment has to be an element of $B \in \mathcal{F}$, where $P(B) > 0$. In such situations the sample space is B and natural contenders for the membership of the event space are $\{A \cap B : A \in \mathcal{F}\}$. This raises the question whether $\mathcal{F}_B = \{A \cap B : A \in \mathcal{F}\}$ is an event space i.e., whether $\mathcal{F}_B = \{A \cap B : A \in \mathcal{F}\}$ is a sigma-field of subsets of B

Theorem 3.1

Let $\mathcal{F}$ be a σ-field of subsets Ω and let $B \in \mathcal{F}$. Define $\mathcal{F}_B = \{A \cap B : A \in \mathcal{F}\}$. Then $\mathcal{F}_B$ is a σ-field of subsets of B and $\mathcal{F}_B \subseteq \mathcal{F}$.

Proof. Since $B \in \mathcal{F}$ and $\mathcal{F}_B = \{A \cap B : A \in \mathcal{F}\}$ it is obvious that $\mathcal{F}_B \subseteq \mathcal{F}$. We have $\Omega \in \mathcal{F}$ and therefore

$$B = \Omega \cap B \in \mathcal{F}_B. \qquad (3.2)$$

Also,

$$C \in \mathcal{F}_B \Rightarrow \mathrm{C} = \mathrm{A} \cap B \text{ for some } A \in \mathcal{F}$$

$$\Rightarrow C^c = B - C = \underbrace{(\Omega - A)}_{\in \mathcal{F}} \cap B \qquad (\text{since } A \in \mathcal{F})$$

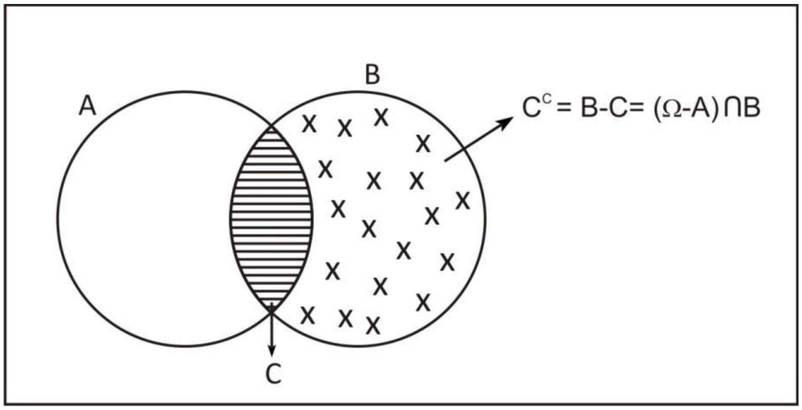

Figure 3.1

$$\Rightarrow C^C = B - C \in \mathcal{F}_B, \tag{3.3}$$

i.e., $\mathcal{F}_B$ is closed under complements with respect to B.

Now suppose that $C_i \in \mathcal{F}_B, i = 1,2, \ldots$. Then $C_i = A_i \cap B$, for some $A_i \in \mathcal{F}, i = 1,2, \ldots$. Therefore,

$$\bigcup_{i=1}^{\infty} C_i = \underbrace{\left(\bigcup_{i=1}^{\infty} A_i\right)}_{\in \mathcal{F}} \cap B \text{ (since } A_i \in \mathcal{F}, i = 1,2, \ldots)$$

$$\in \mathcal{F}_B, \tag{3.4}$$

i.e., $\mathcal{F}_B$ is closed under countable unions.

Now (3.2), (3.3) and (3.4) imply that $\mathcal{F}_B$ is a σ-field of subsets of B. ■

Equation (3.1) suggests considering the set function $P_B\colon \mathcal{F}_B \to \mathbb{R}$ defined by

$$P_B(C) = \frac{P(C)}{P(B)}, \qquad C \in \mathcal{F}_B = \{A \cap B : A \in \mathcal{F}\}.$$

Note that, for $C \in \mathcal{F}_B, P(C)$ is well defined as $\mathcal{F}_B \subseteq \mathcal{F}$.

Let us define another set function $P(\cdot\,|B) : \mathcal{F} \to \mathbb{R}$ by

$$\mathrm{P}(A|B) = P_B(A \cap B) = \frac{P(A \cap B)}{P(B)}, \qquad A \in \mathcal{F}.$$

Theorem 3.2

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $B \in \mathcal{F}$ be such that $P(B) > 0$. Then $(B, \mathcal{F}_B, P_B)$ and $\left(\Omega, \mathcal{F}, P(\cdot\,|B)\right)$ are probability spaces.

Proof. Clearly

$$P_B(C) = \frac{P(C)}{P(B)} \geq 0, \forall C \in \mathcal{F}_B.$$

Let $C_i \in \mathcal{F}_B, i = 1, 2, \ldots$ be mutually exclusive. Then $C_i \in \mathcal{F}, i = 1, 2, \ldots$ (since $\mathcal{F}_B \subseteq \mathcal{F}$), and

$$\begin{aligned} P_B\left(\bigcup_{i=1}^{\infty} C_i\right) &= \frac{P(\bigcup_{i=1}^{\infty} C_i)}{P(B)} \\ &= \frac{\sum_{i=1}^{\infty} P(C_i)}{P(B)} \\ &= \sum_{i=1}^{\infty} \frac{P(C_i)}{P(B)} \\ &= \sum_{i=1}^{\infty} P_B\,(C_i), \end{aligned} \tag{3.5}$$

i.e., P_B is countable additive on $\mathcal{F}_B$.

Also

$$P_B(B) = \frac{P(B)}{P(B)} = 1 \cdot$$

Thus P_B is a probability measure on $\mathcal{F}_B$.

Note that $P(A|B) \geq 0, \forall A \in \mathcal{F}$ and

$$P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 \cdot$$

Let $E_i \in \mathcal{F}, i = 1,2, \ldots$ be mutually exclusive. Then $C_i = E_i \cap B \in \mathcal{F}_B, i = 1, 2, \ldots$ are mutually exclusive and

$$P\left(\bigcup_{i=1}^{\infty} E_i|B\right) = P_B\left(\bigcup_{i=1}^{\infty} C_i\right) = \sum_{i=1}^{\infty} P_B(C_i) = \sum_{i=1}^{\infty} P_B\,(E_i \cap B) = \sum_{i=1}^{\infty} P\,(E_i|B). \quad \text{(using (3.5))}$$

It follows that $P(\cdot\,|B)$ is a probability measure on $\mathcal{F}$. ■

Note that domains of $P_B(\cdot)$ and $P(\cdot\,|B)$ are $\mathcal{F}_B$ and $\mathcal{F}$ respectively. Moreover,

$$P(A|B) = P_B(A \cap B) = \frac{P(A \cap B)}{P(B)}, \qquad A \in \mathcal{F}.$$

Definition 3.1

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $B \in \mathcal{F}$ be a fixed event such that $P(B) > 0$. Define the set function $P(\cdot\,|B): \mathcal{F} \to \mathbb{R}$ by

$$P(A|B) = P_B(A \cap B) = \frac{P(A \cap B)}{P(B)}, \qquad A \in \mathcal{F}.$$

We call $P(A|B)$ the *conditional probability* of event A given that the outcome of the experiment is in B or simply the conditional probability of A given B. ■

Example 3.2

Six cards are dealt at random (without replacement) from a deck of 52 cards. Find the probability of getting all cards of heart in a hand (event A) given that there are at least 5 cards of heart in the hand (event B).

Solution. We have,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Clearly,

$$P(A \cap B) = P(A) = \frac{\binom{13}{6}}{\binom{52}{6}},$$

and
$$P(B) = \frac{\binom{13}{5}\binom{39}{1} + \binom{13}{6}}{\binom{52}{6}}.$$

Therefore,

$$P(A|B) = \frac{\binom{13}{6}}{\binom{13}{5}\binom{39}{1} + \binom{13}{6}}. \ ■$$

Remark 3.1

For events $E_1, \dots, E_n \in \mathcal{F}$ $(n \geq 2)$,

$$P(E_1 \cap E_2) = P(E_1)P(E_2|E_1), \text{ if } P(E_1) > 0,$$

and

$$\begin{aligned} P(E_1 \cap E_2 \cap E_3) &= P\big((E_1 \cap E_2) \cap E_3\big) \\ &= P(E_1 \cap E_2)P(E_3|E_1 \cap E_2) \\ &= P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2), \end{aligned}$$

if $P(E_1 \cap E_2) > 0$ (which also guarantees that $P(E_1) > 0$, since $E_1 \cap E_2 \subseteq E_1$).

Using principle of mathematical induction it can be shown that

$$P\left(\bigcap_{i=1}^{n} E_i\right) = P(E_1)P(E_2|E_1)P(E_3|E_1 \cap E_2) \cdots P(E_n|E_1 \cap E_2 \cap \cdots \cap E_{n-1}),$$

provided $P(E_1 \cap E_2 \cap \cdots \cap E_{n-1}) > 0$ (which also guarantees that $P(E_1 \cap E_2 \cap \cdots \cap E_i) > 0,\ i = 1, 2, \cdots, n-1$). ■

Example 3.3

An urn contains four red and six black balls. Two balls are drawn successively, at random and without replacement, from the urn. Find the probability that the first draw resulted in a red ball and the second draw resulted in a black ball.

Solution. Define the events

A: first draw results in a red ball;

B: second draw results in a black ball.

Then,

$$\begin{aligned} \text{Required probability} &= P(A \cap B) \\ &= P(A)P(B|A) \\ &= \frac{4}{10} \times \frac{6}{9} = \frac{12}{45}. \ ■ \end{aligned}$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space. For a countable collection $\{E_i : i \in \Lambda\}$ of mutually exclusive and exhaustive events, the following theorem provides a relationship between marginal probability $P(E)$ of an event $E \in \mathcal{F}$ and joint probabilities $P(E \cap E_i)$ of events E and $E_i, i \in \Lambda$.

1.3.1 Theorem of Total Probability

Theorem 3.3

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{E_i : i \in \Lambda\}$ be a countable collection of mutually exclusive and exhaustive events (i.e., $E_i \cap E_j = \phi$, whenever $i \neq j$, and $P(\bigcup_{i \in \Lambda} E_i) = 1$) such that $P(E_i) > 0, \forall i \in \Lambda$. Then, for any event $E \in \mathcal{F}$,

$$P(E) = \sum_{i \in \Lambda} P(E \cap E_i) = \sum_{i \in \Lambda} P(E|E_i)\, P(E_i).$$

Proof. Let $F = \bigcup_{i \in \Lambda} E_i$. Then $P(F) = 1$ and $P(F^c) = 1 - P(F) = 0$. Therefore,

$$\begin{aligned}
P(E) &= P(E \cap F) + P(E \cap F^c) \\
&= P(E \cap F) \qquad (E \cap F^c \subseteq F^c \Rightarrow 0 \leq P(E \cap F^c) \leq P(F^c) = 0) \\
&= P\left(\bigcup_{i \in \Lambda} (E \cap E_i)\right) \\
&= \sum_{i \in \Lambda} P(E \cap E_i) \; (E_i s \text{ are disjoint } \Rightarrow E_i \cap E\text{s } (\subseteq E_i) \text{ are disjoint}) \\
&= \sum_{i \in \Lambda} P(E|E_i)\, P(E_i). \; \blacksquare
\end{aligned}$$

Example 3.4

Urn U_1 contains 4 white and 6 black balls and urn U_2 contains 6 white and 4 black balls. A fair die is cast and urn U_1 is selected if the upper face of die shows 5 or 6 dots, otherwise urn U_2 is selected. If a ball is drawn at random from the selected urn find the probability that the drawn ball is white.

Solution. Define the events:

$$\begin{aligned}
W &: \text{ drawn ball is white;} \\
E_1 &: \text{ urn } U_1 \text{ is selected;} \\
E_2 &: \text{ urn } U_2 \text{ is selected.}
\end{aligned}$$

Then $\{E_1, E_2\}$ is a collection of mutually exclusive and exhaustive events. Therefore

$$P(W) = P(E_1)P(W|E_1) + P(E_2)P(W|E_2)$$

$$= \frac{2}{6} \times \frac{4}{10} + \frac{4}{6} \times \frac{6}{10}$$

$$= \frac{8}{15}. \blacksquare$$