MODULE 6

RANDOM VECTOR AND ITS JOINT DISTRIBUTION

LECTURE 35

Topics

6.10.2 Transformation of Variables Technique
6.10.3 Moment Generating Function Technique

Example 10.2.11

(i) Let X_1 and X_2 be independent random variables such that $X_i \sim G(\alpha_i, \theta), \alpha_i > 0, \theta > 0,\ i = 1, 2$. Define $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$. Show that Y_1 and Y_2 are independently distributed with

$$Y_1 \sim G(\alpha_1 + \alpha_2, \theta) \text{ and } Y_2 \sim \text{Be}(\alpha_1, \alpha_2).$$

(ii) If $X_1 \sim \text{Exp}(\theta)$ and $X_2 \sim \text{Exp}(\theta)$ are independently distributed then show that $Y = \frac{X_1}{X_1+X_2} \sim U(0,1)$.

Solution.

(i) The p.d.f.s of X_i and $\underline{X} = (X_1, X_2)$ are given by

$$f_{X_i}(x) = \frac{1}{\Gamma(\alpha_i)\theta^{\alpha_i}} x^{\alpha_i - 1} e^{-\frac{x}{\theta}} I_{(0,\infty)}(x),\ \ i = 1,2,$$

and

$$f_{\underline{X}}(x_1, x_2) = \prod_{i=1}^{2} f_{X_i}(x_i) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ \theta^{\alpha_1+\alpha_2}} {x_1}^{\alpha_1-1} {x_2}^{\alpha_2-1}\, e^{-\frac{x_1+x_2}{\theta}} I_{(0,\infty)^2}(\underline{x}),$$

respectively.

Clearly $S_{\underline{X}} = \{\underline{x} \in \mathbb{R}^2 : f_{\underline{X}}(x_1, x_2) > 0\} = (0, \infty)^2$. Consider the transformation $\underline{h} = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$h_1(x_1,x_2) = x_1 + x_2 \text{ and } h_2(x_1,x_2) = \begin{cases} \dfrac{x_1}{x_1+x_2}, \text{if } x_1+x_2 \neq 0 \\ 0, \qquad\quad \text{if } x_1+x_2 = 0 \end{cases}.$$

Then $P\left(\left\{(Y_1,Y_2) = \left(h_1(X_1,X_2), h_2(X_1,X_2)\right)\right\}\right) = 1$ and therefore

$$(Y_1,Y_2) \overset{d}{=} \left(h_1(X_1,X_2), h_2(X_1,X_2)\right).$$

Also the transformation $\underline{h} = (h_1,h_2): S_{\underline{X}} \to \mathbb{R}^2$ is one-to-one with inverse transformation $\underline{h}^{-1} = (h_1^{-1}, h_2^{-1})$, where for $(y_1,y_2) \in \underline{h}\left(S_{\underline{X}}\right)$,

$$h_1^{-1}(y_1,y_2) = y_1y_2 \text{ and } h_2^{-1}(y_1,y_2) = y_1(1-y_2).$$

The Jacobian determinant of the transformation is

$$\mathrm{J} = \begin{vmatrix} \dfrac{\partial h_1^{-1}}{\partial y_1} & \dfrac{\partial h_1^{-1}}{\partial y_2} \\ \dfrac{\partial h_2^{-1}}{\partial y_1} & \dfrac{\partial h_2^{-1}}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = -y_1.$$

Also

$$\underline{y} = (y_1,y_2) \in \underline{h}\left(S_{\underline{X}}\right) \Leftrightarrow \left(h_1^{-1}\left(\underline{y}\right), h_2^{-1}\left(\underline{y}\right)\right) \in S_{\underline{X}}$$

$$\Leftrightarrow y_1y_2 > 0, \;\; y_1(1-y_2) > 0$$

$$\Leftrightarrow y_1 > 0, \;\; 0 < y_2 < 1.$$

Therefore $\underline{h}\left(S_{\underline{X}}\right) = (0,\infty) \times (0,1)$ and the joint p.d.f. of $\underline{Y}$ is given by

$$f_{\underline{Y}}(y_1,y_2) = f_{\underline{X}}\left(h_1^{-1}(y), h_2^{-1}(y)\right)|J| I_{\underline{h}(S_{\underline{X}})}\left(\underline{y}\right)$$

$$= f_{\underline{X}}\left(y_1y_2, y_1(1-y_2)\right)|-y_1| I_{(0,\infty)\times(0,1)}(y_1,y_2)$$

$$= \left(\frac{y_1^{\alpha_1+\alpha_2-1} e^{-\frac{y_1}{\theta}}}{\Gamma(\alpha_1+\alpha_2)} I_{(0,\infty)}(y_1)\right)\left(\frac{1}{B(\alpha_1,\alpha_2)}\right) y_2^{\alpha_1-1}(1-y_2)^{\alpha_2-1} I_{(0,1)}(y_2).$$

It follows that Y_1 and Y_2 are independent random variables, $Y_1 \sim G(\alpha_1+\alpha_2,\theta)$ and $Y_2 \sim \mathrm{Be}(\alpha_1,\alpha_2)$.

(ii) Follows from (a) by taking $\alpha_1 = \alpha_2 = 1.$ ■

Example 10.2.12

(i) Let $\underline{X} = (X_1, X_2)$ be a random vector of absolutely continuous type with joint p.d.f.

$$f_{\underline{X}}(x_1, x_2) = g\left(\sqrt{x_1^2 + x_2^2}\right), \ \underline{x} = (x_1, x_2) \in \mathbb{R}^2,$$

where $g: [0, \infty) \to \mathbb{R}$ is a non-negative function such that

$$\int_0^\infty x g(x) dx = \frac{1}{2\pi}.$$

Let (R, Θ) be the polar coordinate of the point $\underline{X} = (X_1, X_2)$in the Cartesian plane, so that, $X_1 = R\cos\Theta, X_2 = R\,\mathrm{Sin}\Theta, R > 0, \Theta \in [0, 2\pi), R = \sqrt{X_1^2 + X_1^2}$ and one may take

$$\Theta = \begin{cases} 0, & \text{if } X_1 = 0, X_2 = 0 \\ \dfrac{\pi}{2}, & \text{if } X_1 = 0, X_2 > 0 \\ \dfrac{3\pi}{2}, & \text{if } X_1 = 0, X_2 < 0 \\ \tan^{-1}\left(\dfrac{X_2}{X_1}\right), & \text{if } X_1 > 0, X_2 \geq 0 \\ \pi + \tan^{-1}\left(\dfrac{X_2}{X_1}\right), & \text{if } X_1 < 0 \\ 2\pi + \tan^{-1}\left(\dfrac{X_2}{X_1}\right), & \text{if } X_1 > 0, X_2 < 0 \end{cases}$$

where $\tan^{-1}\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ denotes the principal value. Show that R and Θ are independently distributed with p.d.f.s

$$f_R(r) = 2\pi r g(r) I_{(0,\infty)}(r)$$

and

$$f_\Theta(\theta) = \frac{1}{2\pi} I_{(0,2\pi)}(\theta),$$

respectively.

(ii) Let X_1 and X_2 be independent and identically distributed $N(0,1)$ random variables. Show that the distribution of random variable $Y = \frac{X_2}{X_1}$ has p.d.f.

$$f_Y(y) = \frac{1}{\pi} \cdot \frac{1}{1 + y^2}, -\infty < y < \infty.$$

(iii) Let $\underline{X} = (X_1, X_2)$ have the joint p.d.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \frac{1}{\pi}, & \text{if } 0 < x_1^2 + x_2^2 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find $E\left(\sqrt{X_1^2 + X_2^2}\right)$ and $E(X_1 + X_2)$.

Solution.

(i) Let $S_{\underline{X}} = \{\underline{x} \in \mathbb{R}^2 : f_{\underline{X}}(\underline{x}) > 0\} = \left\{\underline{x} \in \mathbb{R}^2 : g\left(\sqrt{x_1^2 + x_2^2}\right) > 0\right\}$. Consider the transformation $\underline{h} = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $h_1(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$ and`

$$h_2(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 = 0, x_2 = 0 \\ \frac{\pi}{2}, & \text{if } x_1 = 0, x_2 > 0 \\ \frac{3\pi}{2}, & \text{if } x_1 = 0, x_2 < 0 \\ \tan^{-1}\left(\frac{X_2}{X_1}\right), & \text{if } x_1 > 0, x_2 \geq 0 \\ \pi + \tan^{-1}\left(\frac{X_2}{X_1}\right), & \text{if } x_1 < 0 \\ 2\pi + \tan^{-1}\left(\frac{X_2}{X_1}\right), & \text{if } x_1 > 0, x_2 < 0 \end{cases}.$$

Then $(R, \Theta) = \left(h_1(x_1, x_2), h_2(x_1, x_2)\right)$. The transformation $\underline{h} = (h_1, h_2) : S_{\underline{X}} \to \mathbb{R}^2$ is one-to-one with inverse transformation $\underline{h}^{-1}(y_1, y_2) = \left(h_1^{-1}(y_1, y_2), h_2^{-1}(y_1, y_2)\right)$, where for $(r, \theta) \in \underline{h}(S_{\underline{X}})$,

$$h_1^{-1}(r, \theta) = r\cos\theta \text{ and } h_2^{-1}(r, \theta) = r\sin\theta.$$

The Jacobian determinant of the transformation is

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial r} & \frac{\partial h_1^{-1}}{\partial \theta} \\ \frac{\partial h_2^{-1}}{\partial r} & \frac{\partial h_2^{-1}}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Also $\underline{h}(S_{\underline{X}}) = \{(r, \theta) \in \mathbb{R}^2 : r \in [0, \infty), \theta \in [0, 2\pi) \text{ and } g(r) > 0\} = A_1 \times A_2$, where $A_1 = \{r \in [0, \infty) : g(r) > 0\}$ and $A_2 = [0, 2\pi)$. The joint p.d.f. of (R, Θ) is given by

$$f_{R,\Theta}(r, \theta) = f_{\underline{X}}\left(h_1^{-1}(r, \theta), h_2^{-1}(r, \theta)\right)|J| I_{\underline{h}(S_{\underline{X}})}(r, \theta)$$

$$= f_{\underline{X}}(r\cos\theta, r\sin\theta)|r| I_{A_1 \times A_2}(r, \theta)$$

$$= \left(rg(r)I_{A_1}(r)\right)\left(I_{A_2}(\theta)\right)$$

$$= \left(2\pi r I_{A_1}(r)\right)\left(\frac{1}{2\pi} I_{(0,2\pi)}(\theta)\right)$$

$$= \left(2\pi r I_{(0,\infty)}(r)\right)\left(\frac{1}{2\pi} I_{(0,2\pi)}(\theta)\right).$$

It follows that R and Θ are independent random variables with respective p.d.f.s

$$f_R(r) = 2\pi r g(r) I_{(0,\infty)}(r)$$

and

$$f_\Theta(\theta) = \frac{1}{2\pi} I_{(0,2\pi)}(\theta).$$

(ii) Note that $Y = \frac{X_2}{X_1}$ is not defined if $X_1 = 0$. However $P(\{X_1 = 0\}) = 0$ $(\text{i.e.}, P(\{X_1 \neq 0\}) = 1)$ and therefore $Y = \frac{X_2}{X_1}$ is well defined with probability one. In fact, since $\underline{X} = (X_1, X_2)$ is of absolutely continuous type, we may, without loss of generality, take $S_{\underline{X}} = \mathbb{R}^2 - \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$. Define

$$Z = \begin{cases} Y, & \text{if } (x_1, x_2) \in S_{\underline{X}} \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \tan\Theta, & \text{if } \Theta \in [0, 2\pi) - \left\{0, \frac{\pi}{2}, \frac{3\pi}{2}\right\} \\ 0, & \text{otherwise} \end{cases}.$$

Then $P(\{Z = Y\}) = 1$ and therefore $Y \overset{d}{=} Z$. Thus we will find the distribution of random variable Z.

$$Z = \begin{cases} \tan\Theta, & \text{if } \Theta \in [0, 2\pi) - \left\{0, \frac{\pi}{2}, \frac{3\pi}{2}\right\} \\ 0, & \text{otherwise} \end{cases}.$$

The p.d.f. of Θ is given by

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & \text{if } 0 \leq \theta \leq 2\pi \\ 0, & \text{otherwise} \end{cases}.$$

Consider the transformation $h: \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \tan x, & \text{if } x \in [0, 2\pi) - \left\{0, \frac{\pi}{2}, \frac{3\pi}{2}\right\} \\ 0, & \text{otherwise} \end{cases}.$$

Note that the transformation $h: \mathbb{R} \to \mathbb{R}$ is not one-to–one. Since Θ is of absolutely continuous type we may, without loss of generality, take

$$\begin{aligned} S_\Theta &= \{\theta \in \mathbb{R}: f_\Theta(\theta) > 0\} \\ &= [0, 2\pi) - \left\{0, \frac{\pi}{2}, \frac{3\pi}{2}\right\} \\ &= S_1 \cup S_2 \cup S_3, \quad \text{say,} \end{aligned}$$

where $S_1 = \left(0, \frac{\pi}{2}\right), S_2 = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and $S_3 = \left(\frac{3\pi}{2}, 2\pi\right)$. On each of the sets S_1, S_2 and S_3, h is strictly increasing with inverse transformations

$$h_1^{-1}(z) = \tan^{-1} z, \qquad z \in (0, \infty),$$

$$h_2^{-1}(z) = \pi + \tan^{-1} z, \quad z \in (-\infty, \infty)$$

and

$$h_3^{-1}(z) = 2\pi + \tan^{-1} z, \quad z \in (-\infty, 0).$$

Also $h(S_1) = (0, \infty), h(S_2) = (-\infty, \infty)$ and $h(S_3) = (-\infty, 0)$. Therefore the p.d.f. of Z is given by

$$\begin{aligned} f_Z(z) &= \sum_{j=1}^{3} f_\Theta\left(h_j^{-1}(z)\right) \left| \frac{d}{dz} h_j^{-1}(z) \right| I_{h(S_j)}(z) \\ &= f_\Theta(\tan^{-1} z) \left| \frac{1}{1+z^2} \right| I_{(0,\infty)}(z) + f_\Theta(\pi + \tan^{-1} z) \left| \frac{1}{1+z^2} \right| I_{(-\infty,\infty)}(z) \\ &\quad + f_\Theta(2\pi + \tan^{-1} z) \left| \frac{1}{1+z^2} \right| I_{(-\infty,0)}(z) \\ &= \frac{1}{2\pi} \cdot \frac{1}{1+z^2} I_{(0,\infty)}(z) + \frac{1}{2\pi} \cdot \frac{1}{1+z^2} I_{(-\infty,\infty)}(z) + \frac{1}{2\pi} \cdot \frac{1}{1+z^2} I_{(-\infty,0)}(z) \\ &= \begin{cases} \frac{1}{\pi} . \frac{1}{1+z^2}, & \text{if } z \in \mathbb{R} - \{0\} \\ \frac{1}{2\pi}, & \text{if } z = 0 \end{cases}. \end{aligned}$$

Since the random variable Z is of absolutely continuous type we may take the p.d.f. of Z as

$$f_Z(z) = \frac{1}{\pi} . \frac{1}{1+z^2}, \qquad -\infty < z < \infty.$$

It follows that the random variable Z $\left(\text{and hence } Y = \frac{X_2}{X_1}\right)$ has the Cauchy distribution (see Definition 11.1 (ii)).

(iii) We have

$$E\left(\sqrt{X_1^2 + X_2^2}\right) = E(R)$$

$$\begin{aligned} E(X_1 + X_2) &= E\big(R(\cos\Theta + \sin\Theta)\big) \\ &= E(R)E(\cos\Theta + \sin\Theta) \quad \text{(since } R \text{ and } \Theta \text{ are independent).} \end{aligned}$$

Under the notation of (i), we have

$$g(x) = \begin{cases} \frac{1}{\pi}, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Moreover

$$f_R(r) = \begin{cases} 2r, & \text{if } 0 < r < 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_\Theta(\theta) = \frac{1}{2\pi} I_{(0,2\pi)}(\theta).$$

Therefore

$$E\left(\sqrt{X_1^2 + X_2^2}\right) = E(R) = \int_0^1 2r^2 dr = \frac{2}{3},$$

and

$$\begin{aligned} E(X_1 + X_2) &= E(R)E(\cos\Theta + \sin\Theta) \\ &= \frac{2}{3}\int_0^{2\pi} \frac{\cos\theta + \sin\theta}{2\pi} d\theta \\ &= 0. \blacksquare \end{aligned}$$

6.10.3 Moment Generating Function Technique

Let $\underline{X} = (X_1, \dots, X_p)$ be a random vector with p.d.f./p.m.f. $f_{\underline{X}}(\cdot)$ and let $\underline{g}: \mathbb{R}^p \to \mathbb{R}^q$ be a Borel function. Suppose that we seek the probability distribution of $\underline{Y} = \underline{g}(\underline{X})$. Under the m.g.f. technique we try to identify the m.g.f. $M_{\underline{Y}}(t)$ of random vector $\underline{Y}$ with the m.g.f. of some known distribution. Then the uniqueness of m.g.f.s (Theorem 7.3) ascertains that the random vector $\underline{Y}$ has that known distribution. Various usages of this technique are illustrated in Examples 7.1, 7.2, 7.3, 7.4, 7.5 and 7.6.

Theorem 10.3.1

Let $X_1, \dots, X_n (n \geq 2)$ be a random sample from $N(\mu, \sigma^2)$ distribution, where $\mu \in (-\infty, \infty)$ and $\sigma > 0$. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$ denote the sample mean and the sample variance respectively. Then

(i) $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$;

(ii) $\overline{X}$ and S^2 are independent random variables;

(iii) $\frac{(n-1)}{\sigma^2}S^2 \sim \chi^2_{n-1}$;

(iv) $E(S^2) = \sigma^2,\ \mathrm{Var}(S^2) = \frac{2\sigma^4}{n-1}$ and $E(S) = \sqrt{\frac{2}{n-1}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\sigma$.

Solution.

(i) Follows from Example 7.1.

(ii) Let $Y_i = X_i - \overline{X}, i = 1, \dots, n$ and let $\underline{Y} = (Y_1, \dots, Y_n)$. Then $\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} X_i - n\overline{X} = 0$ and $(n-1)S^2 = \sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 = \sum_{i=1}^{n} Y_i^2$, a function of $\underline{Y}$. The joint m.g.f. of $\left(\underline{Y}, \overline{X}\right)$ is given by

$$M_{\underline{Y},\overline{X}}\left(\underline{u}, v\right) = E\left(e^{\sum_{i=1}^{n} u_i Y_i + v\overline{X}}\right),\ \ \underline{u} = (u_1, \dots, u_n) \in \mathbb{R}^n, v \in \mathbb{R}.$$

Let us fix $\underline{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ and $v \in \mathbb{R}$. Then

$$\sum_{i=1}^{n} u_i Y_i + v\,\overline{X} = \sum_{i=1}^{n} u_i\left(X_i - \overline{X}\right) + v\overline{X}$$

$$= \sum_{j=1}^{n} u_j X_j + \frac{(v - \sum_{i=1}^{n} u_i)}{n}\sum_{j=1}^{n} X_j$$

$$= \sum_{j=1}^{n} \left(u_j - \overline{u} + \frac{v}{n}\right) X_j$$

$$= \sum_{j=1}^{n} t_j X_j \,,$$

where $\overline{u} = \frac{1}{n}\sum_{i=1}^{n} u_i$ and $t_j = u_j - \overline{u} + \frac{v}{n}, j = 1, \dots, n$. Note that $\sum_{j=1}^{n}(u_j - \overline{u}) = 0$, and therefore,

$$\sum_{j=1}^{n} t_j = \sum_{j=1}^{n} \left(u_j - \overline{u} + \frac{v}{n}\right) = v,$$

and

$$\sum_{j=1}^{n} t_j^2 = \sum_{j=1}^{n} \left(u_j - \overline{u} + \frac{v}{n}\right)^2 = \sum_{j=1}^{n} (u_j - \overline{u})^2 + \frac{v^2}{n}.$$

Consequently,

$$M_{\underline{Y},\overline{X}}(\underline{u}, v) = E\left(e^{\sum_{j=1}^{n} t_j X_j}\right)$$

$$= \prod_{j=1}^{n} E(e^{t_j X_j})$$

$$= \prod_{j=1}^{n} M_{X_j}(t_j)$$

$$= \prod_{j=1}^{n} e^{\mu t_j + \frac{\sigma^2 t_j{}^2}{2}}$$

$$= e^{\mu} \sum_{j=1}^{n} t_j + \frac{\sigma^2}{2} \sum_{j=1}^{n} t_j{}^2$$

$$= e^{\mu v + \frac{\sigma^2}{2}\left\{\sum_{j=1}^{n}(u_j - \overline{u})^2 + \frac{v^2}{n}\right\}}$$

$$= e^{\mu v + \frac{\sigma^2 v^2}{2n}} e^{\frac{\sigma^2 \sum_{j=1}^{n}\left(u_j - \overline{u}\right)^2}{2}}, \underline{u} \in \mathbb{R}^n, v \in \mathbb{R}.$$

The joint m.g.f. of $\underline{Y} = (Y_1, \ldots, Y_n)$ is given by

$$M_{\underline{Y}}(\underline{u}) = M_{\underline{Y},\overline{X}}(\underline{u}, 0) = e^{\frac{\sigma^2}{2}\sum_{j=1}^{n}(u_j - \overline{u})^2}, \;\; \underline{u} \in \mathbb{R}^n,$$

and the m.g.f. of $\overline{X}$ is given by

$$M_{\overline{X}}(v) = M_{\underline{Y},\overline{X}}(0, v) = e^{\mu v + \frac{\sigma^2 v^2}{2n}}, \;\; v \in \mathbb{R}.$$

Clearly

$$M_{\underline{Y},\overline{X}}(\underline{u}, v) = M_{\underline{Y}}(\underline{u}) M_{\overline{X}}(v), \;\; \forall(\underline{u}, v) \in \mathbb{R}^{n+1}.$$

Now using Theorem 7.4 it follows that $\underline{Y} = \left(X_1 - \overline{X}, \ldots, X_n - \overline{X}\right)$ and $\overline{X}$ are independent. This in turn implies that, for any Borel functions $\Psi_1(\cdot)$ and $\Psi_2(\cdot), \Psi_1(\underline{Y})$ and $\Psi_2(\overline{X})$ are independent. In particular, it follows that S^2 (a function of $\underline{Y}$) and $\overline{X}$ are independent.

(iii) Let $Z_i = \frac{X_i - \mu}{\sigma}, i = 1, \ldots, n.$ Then $Z_1, \ldots, Z_n$ are independent and identically distributed $N(0,1)$ random variables. Furthermore, by (i) and Theorem 4.1 (i)-(b), Module 5,$Z = \frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \sim N(0,1)$. Let $W = Z^2 = \frac{n(\overline{X} - \mu)^2}{\sigma^2}$ and $Y = \frac{(n-1)S^2}{\sigma^2}$. Then, by (ii), W and Y are independent random variables. Also, by Example 7.6 (ii), $W \sim \chi_1^2$ and $T = \sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$. Thus the m.g.f.s of W and T are

$$M_W(t) = (1 - 2t)^{-\frac{1}{2}}, \;\; t < \frac{1}{2},$$

and

$$M_T(t) = (1 - 2t)^{-\frac{n}{2}}, \;\; t < \frac{1}{2}.$$

Also

$$\begin{aligned} T &= \sum_{i=1}^{n} Z_i^2 \\ &= \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} \end{aligned}$$

$$= \sum_{i=1}^{n} \frac{\left(X_i - \overline{X} + \overline{X} - \mu\right)^2}{\sigma^2}$$

$$= \sum_{i=1}^{n} \frac{\left(X_i - \overline{X}\right)^2}{\sigma^2} + \frac{n\left(\overline{X} - \mu\right)^2}{\sigma^2}$$

$$= Y + W.$$

Since Y and W are independent random variables, we have

$$M_T(t) = M_Y(t) M_W(t)$$

$$\Rightarrow M_Y(t) = \frac{M_T(t)}{M_W(t)}$$

$$= \frac{(1-2t)^{-\frac{n}{2}}}{(1-2t)^{-\frac{1}{2}}}$$

$$= (1-2t)^{-\frac{n-1}{2}}, \ \ t < \frac{1}{2},$$

which is the m.g.f. of χ^2_{n-1} distribution. Now, by uniqueness of m.g.f.s it follows that $Y = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.

(iv) We have $Y = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. Therefore

$$E(S^r) = \frac{\sigma^r}{(n-1)^{\frac{r}{2}}} E\left(Y^{\frac{r}{2}}\right)$$

$$= \frac{\sigma^r}{(n-1)^{\frac{r}{2}}} \int_0^\infty y^{\frac{r}{2}} \frac{e^{-\frac{y}{2}} y^{\frac{n-1}{2}-1}}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} dy$$

$$= \frac{\sigma^r}{(n-1)^{\frac{r}{2}}} \frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} \int_0^\infty e^{-\frac{y}{2}} y^{\frac{n-1+r}{2}-1} dy$$

$$= \frac{2^{\frac{n-1+r}{2}}}{2^{\frac{n-1}{2}}} \frac{\Gamma(\frac{n-1+r}{2})}{\Gamma(\frac{n-1}{2})} \frac{\sigma^r}{(n-1)^{\frac{r}{2}}}, \ \ r > -(n-1)$$

$$= \left(\frac{2}{n-1}\right)^{\frac{r}{2}} \frac{\Gamma(\frac{n-1+r}{2})}{\Gamma(\frac{n-1}{2})} \sigma^r, \ \ r > -(n-1).$$

Therefore

$$E(S) = \sqrt{\frac{2}{n-1}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \sigma,$$

$$E(S^2) = \frac{2}{n-1} \frac{\Gamma(\frac{n-1}{2}+1)}{\Gamma(\frac{n-1}{2})} \sigma^2 = \sigma^2,$$

$$E(S^4) = \left(\frac{2}{n-1}\right)^2 \frac{\Gamma(\frac{n-1}{2}+2)}{\Gamma(\frac{n-1}{2})} \sigma^4 = \frac{n+1}{n-1} \sigma^4$$

and

$$\mathrm{Var}(S^2) = E(S^4) - \left(E(S^2)\right)^2 = \frac{2\sigma^4}{n-1} \cdot \blacksquare$$