MODULE 6

RANDOM VECTOR AND ITS JOINT DISTRIBUTION

LECTURE 36

Topics

6.11 DISTRIBUTIONS BASED ON SAMPLING FROM A NORMAL DISTRIBUTION

6.11 DISTRIBUTIONS BASED ON SAMPLING FROM A NORMAL DISTRIBUTION

First we will introduce two new probability distributions, called the *Student t-distribution* and the *Snedecor* F*-distribution*, which arise as probability distributions of various statistics based on a random sample from normal distribution.

Definition 11.1

(i) For a given positive integer m, a random variable X is said to have the Student t-distribution with m degrees of freedom (written as $X \sim t_m$) if the p.d.f. of X is given by

$$f_X(x) = \frac{\Gamma(\frac{m+1}{2})}{\sqrt{m\pi}\,\Gamma(\frac{m}{2})} \frac{1}{\left(1+\frac{x^2}{m}\right)^{\frac{m+1}{2}}}, \quad -\infty < x < \infty.$$

(ii) The Student t-distribution with 1 degree of freedom is also called the *standard Cauchy distribution.*

(iii) For positive integers n_1 and n_2, a random variable X is said to have the Snedecor F distribution with (n_1, n_2) degrees of freedom (written as $X \sim F_{n_1,n_2}$) if the p.d.f. of X is given by

$$f_X(x) = \frac{\left(\frac{n_1}{n_2}\right)}{B\left(\frac{n_1}{2},\frac{n_2}{2}\right)} \frac{\left(\frac{n_1}{n_2}x\right)^{\frac{n_1}{2}-1}}{\left(1+\frac{n_1}{n_2}x\right)^{\frac{n_1+n_2}{2}}} I_{(0,\infty)}(x).$$

Remark 11.1

The following observations are obvious:

(i) If $X \sim t_m$, then $X \overset{d}{=} -X$(since $f_X(x) = f_X(-x), \forall x \in \mathbb{R}$), i.e., the distribution of $X \sim t_m$ is symmetric about 0;

(ii) The p.d.f. of $X \sim t_1$ is given by

$$f_Y(y) = \frac{1}{\pi} \cdot \frac{1}{1+y^2}, -\infty < y < \infty,$$

Which is the p.d.f. of Cauchy distribution (see Example 3.4, Module 3). By Example 3.4, Module 3, if the random variable X has the Cauchy distribution (i.e., if $X \sim t_1$) then $E(X)$ is not finite;

(iii) If $X \sim F_{n_1,n_2}$, then $Y = \frac{\frac{n_1 X}{n_2}}{1+\frac{n_1 X}{n_2}} \sim \mathrm{Be}\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$, the beta distribution with shape parameter $\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$ (see, Definition 3.2, Module 5);

(iv) Let Z_1 and Z_2 be independent and identically distributed $N(0,1)$ random variables and let $Z = Z_1/Z_2$. Then, by Example 10.2.12 (ii), the distribution of Z is Cauchy $(\text{i.e.}, Z \sim t_1)$. ■

The following theorem provides representations of the Student t and the Snedecor F random variables in terms of normal and chi-squared random variables.

Theorem 11.1

(i) Let $Z \sim N(0,1)$ and $Y \sim \chi_m^2$ (where $m \in \{1, 2, \ldots\}$) be independent random variables. Then

$$T = \frac{Z}{\sqrt{\frac{Y}{m}}} \sim t_m.$$

(ii) For positive integers n_1 and n_2, let $X_1 \sim \chi_{n_1}^2$ and $X_2 \sim \chi_{n_2}^2$ be independent random variables. Then

$$U = \frac{X_1/n_1}{X_2/n_2} \sim F_{n_1,n_2}.$$

(iii) Let m and r be positive integers and let $X \sim t_m$. Then $E(X^r)$ is not finite if $r \in \{m, m+1, \ldots\}$. For $r \in \{1, 2, \ldots, m-1\}$ and $m \geq r+1$

$$E(X^r) = \begin{cases} 0, & \text{if } r \text{ is odd} \\ \dfrac{m^{\frac{r}{2}} r!\ \Gamma(\frac{m-r}{2})}{2^r \left(\frac{r}{2}\right)!\ \Gamma(\frac{m}{2})}, & \text{if } r \text{ is even} \end{cases}.$$

(iv) If $X \sim t_m$, then

$$\mu_1^{'} = E(X) = 0, \quad \text{for } m \in \{2, 3, \dots\}$$

$$\mu_2 = \mathrm{Var}(X) = \frac{m}{m-2}, \quad \text{for } m \in \{3, 4, \dots\}$$

$$\beta_1 = \text{coefficient of skewness} = 0, \quad \text{for } m \in \{4, 5, \dots\}$$

and $\gamma_1 = \text{kurtosis} = \frac{3(m-2)}{m-4}$, for $m \in \{5, 6, \dots\}$.

(v) Let n_1, n_2 and r be positive integers and let $X \sim F_{n_1,n_2}$. Then, for $n_2 \in \{1, 2, \dots, 2r\}$ and $r \geq \frac{n_2}{2}$, $E(X^r)$ is not finite. For $n_2 \in \{2r+1, 2r+2, \dots\}$

$$E(X^r) = \left(\frac{n_2}{n_1}\right)^r \prod_{i=1}^{r} \left(\frac{n_1 + 2(i-1)}{n_2 - 2i}\right).$$

(vi) If $X \sim F_{n_1,n_2}$ then

$$\mu_1^{'} = E(X) = \frac{n_2}{n_2 - 2}, \quad \text{if } n_2 \in \{3, 4, \dots\}$$

$$\mu_2 = \mathrm{Var}(X) = \frac{2n_2^2(n_1 + n_2 - 2)}{n_1(n_2-2)^2(n_2-4)}, \quad \text{if } n_2 \in \{5, 6, \dots\}$$

$$\beta_1 = \text{coefficient of skewness} = \frac{2(2n_1 + n_2 - 2)}{n_2 - 6} \sqrt{\frac{2(n_2 - 4)}{n_1(n_1 + n_2 - 2)}}, \quad \text{if } n_2 \in \{7, 8, \dots\}$$

and

$$\gamma_1 = \text{kurtosis} = \frac{12[(n_2-2)^2(n_2-4) + n_1(n_1+n_2-2)(5n_2-22)]}{n_1(n_2-6)(n_2-8)(n_1+n_2-2)} + 3, \text{if } n_2 \in \{9, 10, \dots\}.$$

Proof.

(i) The joint p.d.f. of (Y, Z) is given by

$$f_{Y,Z}(y,z) = f_Y(y) f_Z(z) = \begin{cases} \dfrac{1}{2^{\frac{m+1}{2}} \sqrt{\pi}} e^{-\frac{(y+z^2)}{2}} y^{\frac{m}{2}-1}, & \text{if } (y,z) \in (0,\infty) \times \mathbb{R} \\ 0, & \text{otherwise} \end{cases}.$$

Clearly $S_{Y,Z} = \{(y,z) \in \mathbb{R}^2 : f_{Y,Z}(y,z) > 0\} = (0,\infty) \times \mathbb{R}$. Consider the transformation $\underline{h} = (h_1, h_2): S_{Y,Z} \longrightarrow \mathbb{R}^2$ defined by $h_1(y,z) = \frac{z}{\sqrt{\frac{y}{m}}}$ and $h_2(y,z) = \sqrt{\frac{y}{m}}$. Then $T = h_1(Y,Z) = \frac{Z}{\sqrt{\frac{Y}{m}}}$. Let $U = h_2(Y,Z) = \sqrt{\frac{Y}{m}}$. Clearly the transformation $\underline{h} = (h_1, h_2): S_{Y,Z} \longrightarrow \mathbb{R}^2$ is one-to-one with inverse transformation $\underline{h}^{-1} = (h_1^{-1}, h_2^{-1})$, where for $(t,u) \in \underline{h}(S_{Y,Z})$,

$$h_1^{-1}(t,u) = mu^2 \text{ and } h_2^{-1}(t,u) = tu.$$

The Jacobian determinant is

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial t} & \frac{\partial h_1^{-1}}{\partial u} \\ \frac{\partial h_2^{-1}}{\partial t} & \frac{\partial h_2^{-1}}{\partial u} \end{vmatrix} = \begin{vmatrix} 0 & 2mu \\ u & t \end{vmatrix} = -2mu^2.$$

Also

$$\begin{aligned} \underline{h}(S_{Y,Z}) &= \{(t,u) \in \mathbb{R}^2 : (h_1^{-1}(t,u), h_2^{-1}(t,u)) \in S_{Y,Z}\} \\ &= \{(t,u) \in \mathbb{R}^2 : mu^2 \in (0,\infty), u > 0, tu \in \mathbb{R}\} \\ &= \{(t,u) \in \mathbb{R}^2 : t \in \mathbb{R}, u > 0\} \\ &= \mathbb{R} \times (0,\infty) \\ &= A, \quad \text{say.} \end{aligned}$$

Therefore the joint p.d.f. of (T,U) is given by

$$\begin{aligned} f_{T,U}(t,u) &= f_{Y,z}\left(h_1^{-1}(t,u), h_2^{-1}(t,u)\right)|J| I_{\underline{h}(S_{Y,z})}(t,u) \\ &= f_{Y,z}(mu^2, tu)|-2mu^2| I_A(t,u) \\ &= \begin{cases} \dfrac{m^{m/2}}{\sqrt{\pi} 2^{\frac{m-1}{2}} \Gamma\left(\frac{m}{2}\right)} u^m e^{-\frac{(m+t^2)u^2}{2}}, & \text{if } (t,u) \in \mathbb{R} \times (0,\infty) \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

Consequently the p.d.f. of T is given by

$$f_T(t) = \int_{-\infty}^{\infty} f_{T,U}(t,u) du$$

$$= \frac{m^{m/2}}{\sqrt{\pi} 2^{\frac{m-1}{2}} \Gamma\left(\frac{m}{2}\right)} \int_0^{\infty} u^m \, e^{-\frac{(m+t^2)u^2}{2}} du, \quad t \in \mathbb{R}$$

$$= \frac{1}{\sqrt{m\pi}\, \Gamma\left(\frac{m}{2}\right) \left(1 + \frac{t^2}{m}\right)^{\frac{m+1}{2}}} \int_0^{\infty} y^{\frac{m-1}{2}} e^{-y} dy$$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{m\pi}\, \Gamma\left(\frac{m}{2}\right)} \cdot \frac{1}{\left(1 + \frac{t^2}{m}\right)^{\frac{m+1}{2}}}, \quad t \in \mathbb{R},$$

which is the p.d.f. of Student's t-distribution with m degrees of freedom.

(ii) The joint p.d.f. of $\underline{X} = (X_1, X_2)$ is given by

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$$

$$= \frac{1}{2^{\frac{n_1+n_2}{2}} \Gamma\left(\frac{n_1}{2}\right) \Gamma\left(\frac{n_2}{2}\right)} e^{-\frac{(x_1+x_2)}{2}} x_1^{\frac{n_1}{2}-1} x_2^{\frac{n_2}{2}-1} I_{(0,\infty)^2}(x_1, x_2).$$

We have $S_{X_1,X_2} = \left\{(x_1, x_2) \in \mathbb{R}^2 : f_{X_1,X_2}(x_1, x_2) > 0\right\} = (0, \infty)^2$. Consider the one-to-one transformation $\underline{h} = (h_1, h_2): S_{X_1,X_2} \to \mathbb{R}^2$ given by

$$h_1(x_1, x_2) = \frac{n_2}{n_1} \frac{x_1}{x_2} \quad \text{and} \quad h_2(x_1, x_2) = \frac{x_2}{n_2}.$$

Define $U = h_1(X_1, X_2) = \frac{X_1/n_1}{X_2/n_2}$ and $V = h_2(X_1, X_2) = \frac{X_2}{n_2}$. Then the inverse of transformation $\underline{h} = (h_1, h_2): S_{X_1,X_2} \to \mathbb{R}$ is $\underline{h}^{-1} = (h_1^{-1}, h_2^{-1})$, where for $(u, v) \in \underline{h}\left(S_{X_1,X_2}\right)$,

$$h_1^{-1}(u, v) = n_1 uv \quad \text{and} \quad h_2^{-1}(u, v) = n_2 v.$$

The Jacobian determinant is

$$J = \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial u} & \frac{\partial h_1^{-1}}{\partial v} \\ \frac{\partial h_2^{-1}}{\partial u} & \frac{\partial h_2^{-1}}{\partial v} \end{vmatrix} = \begin{vmatrix} n_1 v & n_1 u \\ 0 & n_2 \end{vmatrix} = n_1 n_2 v.$$

Also,

$$\underline{h}\left(S_{X_1,X_2}\right) = \left\{(u, v) \in \mathbb{R}^2 : \left(h_1^{-1}(u, v), h_2^{-1}(u, v)\right) \in S_{X_1,X_2}\right\}$$

$$= \{(u,v) \in \mathbb{R}^2 : n_1 uv > 0, n_2 v > 0\}$$

$$= (0,\infty)^2,$$

and therefore, the joint p.d.f. of (U,V) is given by

$$f_{U,V}(u,v) = f_{X_1,X_2}\left(h_1^{-1}(u,v), h_2^{-1}(u,v)\right)|J| I_{\underline{h}(S_{X_1,X_2})}(u,v)$$

$$= f_{X_1,X_2}(n_1 uv, n_2 v)|n_1 n_2 v| I_{(0,\infty)^2}(u,v)$$

$$= \frac{n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}}}{2^{\frac{n_1+n_2}{2}}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} u^{\frac{n_1}{2}-1} v^{\frac{n_1+n_2}{2}-1} e^{-\frac{(n_2+n_1 u)v}{2}} I_{(0,\infty)^2}(u,v).$$

The p.d.f. of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)dv.$$

Clearly $f_U(u) = 0$, if $u \leq 0$. For $u > 0$

$$f_U(u) = \frac{n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}}}{2^{\frac{n_1+n_2}{2}}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} u^{\frac{n_1}{2}-1} \int_0^{\infty} v^{\frac{n_1+n_2}{2}-1} e^{-\frac{(n_2+n_1 u)v}{2}} dv$$

$$= \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)\frac{n_1}{n_2}}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \frac{\left(\frac{n_1}{n_2}u\right)^{\frac{n_1}{2}-1}}{\left(1+\frac{n_1}{n_2}u\right)^{\frac{n_1+n_2}{2}}}, \quad 0 < u < \infty.$$

Therefore

$$U = \frac{X_1/n_1}{X_2/n_2} \sim F_{n_1,n_2}.$$

(iii) Fix $m \in \{1, 2, \ldots\}$. By (i)

$$X \stackrel{d}{=} \frac{Z}{\sqrt{\frac{Y}{m}}},$$

where $Z \sim N(0,1)$ and $Y \sim \chi_m^2$ are independent random variables. Thus, for $m \in \{1, 2, \ldots\}$ and $r > 0$,

$$E(X^r) = m^{\frac{r}{2}} E\left(Z^r Y^{-\frac{r}{2}}\right) = m^{\frac{r}{2}} E(Z^r) E\left(Y^{-\frac{r}{2}}\right), \quad \text{(since } Y \text{ and } Z \text{ are independent)}$$

provided the expectations are finite. We have, from the proof of Theorem 4.2 (iii), Module 5,

$$E(Z^r) = \begin{cases} 0, & \text{if } r \text{ is odd} \\ \dfrac{r!}{2^{\frac{r}{2}}\left(\frac{r}{2}\right)!}, & \text{if } r \text{ is even} \end{cases}.$$

Moreover, for $r \in \{1, 2, \dots\}$,

$$E\left(Y^{-\frac{r}{2}}\right) = \frac{1}{2^{\frac{m}{2}}\Gamma\left(\frac{m}{2}\right)} \int_0^\infty y^{\frac{m-r}{2}} e^{-\frac{y}{2}} dy,$$

which is finite if, and only if, $m > r$ (see Section 2, Module 5). Also, for $m > r$

$$E\left(Y^{-\frac{r}{2}}\right) = \frac{2^{\frac{m-r}{2}}\Gamma\left(\frac{m-r}{2}\right)}{2^{\frac{m}{2}}\,\Gamma\left(\frac{m}{2}\right)} = \frac{\Gamma\left(\frac{m-r}{2}\right)}{2^{\frac{r}{2}}\,\Gamma\left(\frac{m}{2}\right)}.$$

Thus $E(X^r)$ is finite if $r \in \{1, 2, \dots, m-1\}$. For $r \in \{1, 2, \dots, m-1\}$ and $m \geq r+1$

$$E(X^r) = \begin{cases} 0, & \text{if } r \text{ is odd} \\ \dfrac{m^{\frac{r}{2}} r!\,\Gamma\left(\frac{m-r}{2}\right)}{2^r \left(\frac{r}{2}\right)!\,\Gamma\left(\frac{m}{2}\right)}, & \text{if } r \text{ is even} \end{cases}.$$

(iv) Using (iii), we have

$$\mu_1' = E(X) = 0, \quad \text{if } m \in \{2, 3, \dots\}$$

$$\mu_2 = \mu_2' = E(X^2) = \frac{m}{m-2}, \quad \text{if } m \in \{3, 4, \dots\}$$

$$\mu_3 = \mu_3' = E(X^3) = 0, \quad \text{if } m \in \{4, 5, \dots\}$$

and

$$\mu_4 = \mu_4' = E(X^4) = \frac{3\,m^2}{(m-2)(m-4)}, \quad \text{if } m \in \{5, 6, \dots\}.$$

Consequently

$$\beta_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = 0, \quad \text{if } m \in \{4, 5, \dots\}$$

and

$$\gamma_1 = \frac{\mu_4}{{\mu_2}^2} = \frac{3(m-2)}{m-4}, \ \text{ if } m \in \{5, 6, \dots\}.$$

(v) Using (ii), we have

$$X \stackrel{d}{=} \frac{n_2}{n_1}\frac{X_1}{X_2},$$

where $X_1 \sim \chi^2_{n_1}$ and $X_2 \sim \chi^2_{n_2}$ are independent random variables. Fix $r \in \{1, 2, \dots\}$. Then

$$E(X^r) = \left(\frac{n_2}{n_1}\right)^r E(X_1^r X_2^{-r}) = \left(\frac{n_2}{n_1}\right)^r E(X_1^r)E(X_2^{-r}), \ \ (X_1 \text{ and } X_2 \text{ are independent})$$

provided the expectations are finite. Since $X_1 \sim \chi^2_{n_1}, E(X_1^r)$ is finite for any $r > 0$ and

$$E(X_1^r) = \frac{1}{2^{\frac{n_1}{2}}\,\Gamma\left(\frac{n_1}{2}\right)}\int_0^\infty x^{\frac{n_1}{2}+r-1}\, e^{-\frac{x}{2}}dx$$

$$= \frac{2^{\frac{n_1}{2}+r}\,\Gamma\left(\frac{n_1}{2}+r\right)}{2^{\frac{n_1}{2}}\,\Gamma\left(\frac{n_1}{2}\right)}$$

$$= 2^r\left(\frac{n_1}{2}+r-1\right)\left(\frac{n_1}{2}+r-2\right)\cdots\frac{n_1}{2}$$

$$= \big(n_1 + 2(r-1)\big)\big(n_1 + 2(r-2)\big)\cdots n_1$$

$$= \prod_{i=1}^{r}\big(n_1 + 2(i-1)\big), \ \ r \in \{1, 2, \dots\}.$$

Since $X_2 \sim \chi^2_{n_2}$, $E(X_2^{-r})$ is finite if, and only if, $n_2 > 2r$. For $n_2 > 2r$

$$E(X_2^{-r}) = \frac{2^{\frac{n_2}{2}-r}\,\Gamma\left(\frac{n_2}{2}-r\right)}{2^{\frac{n_2}{2}}\,\Gamma\left(\frac{n_2}{2}\right)} = \frac{1}{\prod_{i=1}^{r}(n_2 - 2i)}.$$

It follows that, for $n_2 \in \{1, 2, \dots, 2r\}$ and $r \geq \frac{n_2}{2}$, $E(X^r)$ is not finite. For $n_2 \in \{2r+1, 2r+2, \dots\}$

$$E(X^r) = \left(\frac{n_2}{n_1}\right)^r \prod_{i=1}^{r}\left(\frac{n_1 + 2(i-1)}{n_2 - 2i}\right).$$

(vi) Follows on using (v) after some tedious calculations. ■

Corollary 11.1

Let $X_1, \ldots, X_n (n \geq 2)$ be a random sample from $N(\mu, \sigma^2)$ distribution, where $\mu \in (-\infty, \infty)$ and $\sigma > 0$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ denote the sample mean and the sample variance respectively. Then

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim t_{n-1}.$$

Proof. By Theorem 10.3.1, $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ and $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ are independent random variables. This in turn implies that $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$ and $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ are independent random variables. Now by virtue of Theorem 11.1 (i)

$$\frac{\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}}{\sqrt{\frac{(n-1)S^2/\sigma^2}{n-1}}} \sim t_{n-1},$$

i.e.,

$$\frac{\sqrt{n}(\bar{X} - \mu)}{S} \sim t_{n-1}. \ ■$$

Corollary 11.2

Let $X_1, \ldots, X_m (m \geq 2)$ be a random sample from $N(\mu_1, \sigma_1^2)$ distribution and let $Y_1, \ldots, Y_n (n \geq 2)$ be a random sample from $N(\mu_2, \sigma_2^2)$ distribution, where $-\infty < \mu_i < \infty$ and $\sigma_i > 0, i = 1,2$. Further suppose that $\underline{X} = (X_1, \ldots, X_m)$ and $\underline{Y} = (Y_1, \ldots, Y_n)$ are independent. Let $S_1^2 = \frac{1}{m-1}\sum_{i=1}^{m}(X_i - \bar{X})^2$ and $S_2^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ be the sample variances based on random sample $\underline{X} = (X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$, respectively; here $\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i$ and $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ are the sample means based on two random samples. Then

$$\frac{\sigma_2^2}{\sigma_1^2}\frac{S_1^2}{S_2^2} \sim F_{m-1,n-1}.$$

Proof. By virtue of Theorem 10.3.1 (iii) we have

$$\frac{(m-1)S_1^2}{\sigma_1^2} \sim \chi^2_{m-1} \text{ and } \frac{(n-1)S_2^2}{\sigma_2^2} \sim \chi^2_{n-1}.$$

Also the independence of $\underline{X}$ and $\underline{Y}$ implies that $\frac{(m-1)S_1^2}{\sigma_1^2}$ (a function of $\underline{X}$ alone) and $\frac{(n-1)S_2^2}{\sigma_2^2}$ (a function of $\underline{Y}$ alone) are independent. Now use of Theorem 11.1 (ii) yields

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{m-1,n-1}$$

i. e.,

$$\frac{\sigma_2^2}{\sigma_1^2}\frac{S_1^2}{S_2^2} \sim F_{m-1,n-1}. \blacksquare$$

Remark 11.2

(i) Suppose that $X \sim t_m$. Then, by Theorem 11.1 (i),

$$X \stackrel{d}{=} \frac{Z}{\sqrt{\frac{Y}{m}}},$$

where $Z \sim N(0,1)$ and $Y \sim \chi^2_m$ are independent random variables. Therefore

$$X^2 \stackrel{d}{=} \frac{Z^2}{Y/m}.$$

Since $Z \sim N(0,1)$, by Theorem 4.1 (i), Module 5, $Z^2 \sim \chi^2_1$. It follows that $Z^2 \sim \chi^2_1$ and $Y \sim \chi^2_m$ are independent random variables. Consequently

$$X^2 \stackrel{d}{=} \frac{Z^2/1}{Y/m} \sim F_{1,m}.$$

Thus if $X \sim t_m$, then $X^2 \sim F_{1,m}$.

(ii) Suppose that $X \sim F_{n_1,n_2}$. Then, by Theorem 11.1 (ii),

$$X \stackrel{d}{=} \frac{X_1/n_1}{X_2/n_2},$$

where $X_1 \sim \chi^2_{n_1}$ and $X_2 \sim \chi^2_{n_2}$ are independent random variables. Then

$$\frac{1}{X} \stackrel{d}{=} \frac{X_2/n_2}{X_1/n_1},$$

where $X_2 \sim \chi^2_{n_2}$ and $X_1 \sim \chi^2_{n_1}$ are independent random variables. Now again using Theorem 11.1 (ii) it follows that

$$\frac{1}{X} \stackrel{d}{=} \frac{X_2/n_2}{X_1/n_1} \sim F_{n_2,n_1}.$$

Thus if $X \sim F_{n_1,n_2}$, then $\frac{1}{X} \sim F_{n_2,n_1}$. ■

Note that if $X \sim t_m$ then, by Remark 11.1 (i), the distribution of X is symmetric about 0 and, by Theorem 11.1 (iv), its kurtosis is

$$v_1 = \frac{3(m-2)}{m-4} > 3, \quad \text{provided } m > 4.$$

Thus a t-distribution with m (> 4) degrees of freedom is symmetric and leptokurtic (i.e., it has shaper peak and longer fatter tails). Note that the kurtosis v_1 decreases as m increases and $v_1 \to 3$, as $m \to \infty$. This suggests that, for large degrees of freedom, Student's t-distribution behaves like $N(0,1)$ distribution. A rigorous proof of this observation will be provided in the next module.

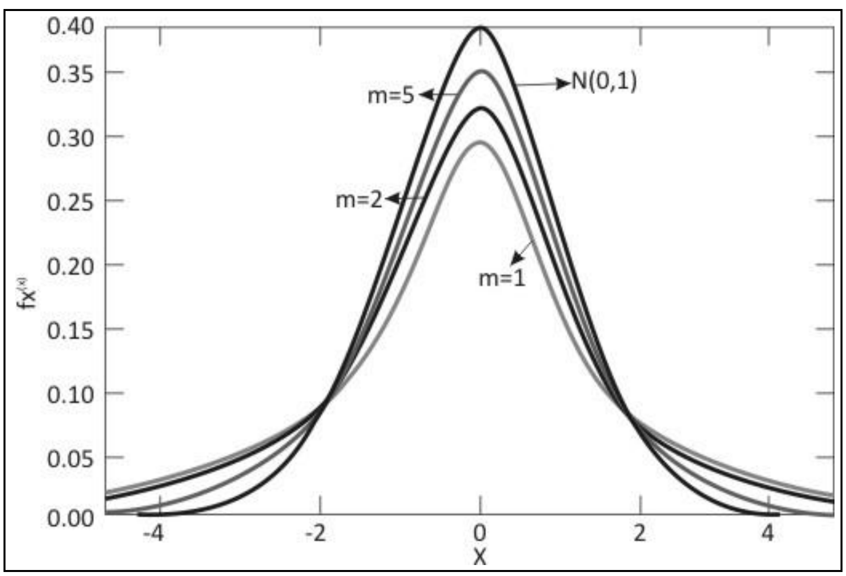

Figure 11.1 : Plot of p.d.f.s of $X \sim t_m$

Suppose that $X \sim t_m$ and, for a fixed $\alpha \in (0,1)$, let $t_{m,\alpha}$ be the $(1-\alpha)$-th quantile of X, i.e.,

$$F_X(t_{m,\alpha}) = P(\{X \le t_{m,\alpha}\}) = 1 - \alpha.$$

Then

$$F_X(-t_{m,\alpha}) = 1 - F_X(t_{m,\alpha}) = \alpha \quad \left(\text{since } X \stackrel{d}{=} -X\right).$$

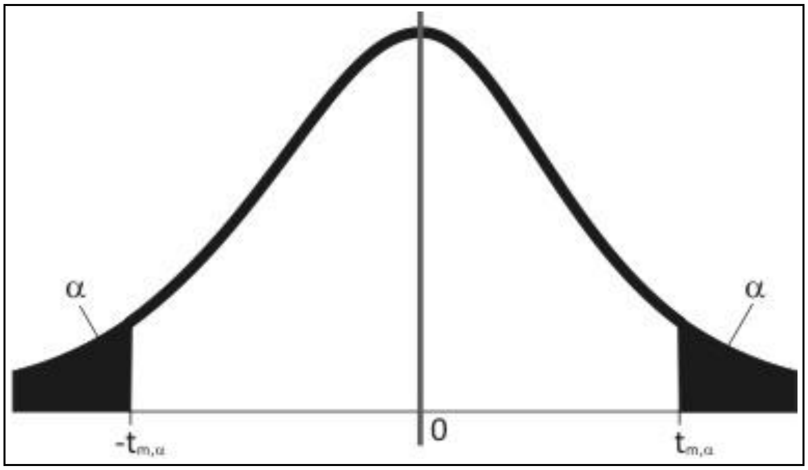

Figure 11.2 : $(1-\alpha)$-th quantile of $X \sim t_m$ $\left(P(\{X \le t_{m,\alpha}\}) = 1 - \alpha\right)$

Now suppose that $X \sim F_{n_1,n_2}$ and, for a fixed $\alpha \in (0,1)$, let $f_{n_1,n_2,\alpha}$ be the $(1-\alpha)$-th quantile of X, i.e.,

$$F_X(f_{n_1,n_2,\alpha}) = P(\{X \le f_{n_1,n_2,\alpha}\}) = 1 - \alpha.$$

Since $\frac{1}{X} \sim F_{n_2,n_1}$ and $P(\{X > 0\}) = 1$, it follows that

$$P\left(\left\{\frac{1}{X} \ge \frac{1}{f_{n_1,n_2,\alpha}}\right\}\right) = 1 - \alpha$$

$$\Rightarrow P\left(\left\{\frac{1}{X} \le \frac{1}{f_{n_1,n_2,\alpha}}\right\}\right) = \alpha = 1 - (1 - \alpha)$$

$$\Rightarrow f_{n_2,n_1,1-\alpha} = \frac{1}{f_{n_1,n_2,\alpha}}.$$

i.e., $$f_{n_1,n_2,\alpha} \times f_{n_2,n_1,1-\alpha} = 1.$$

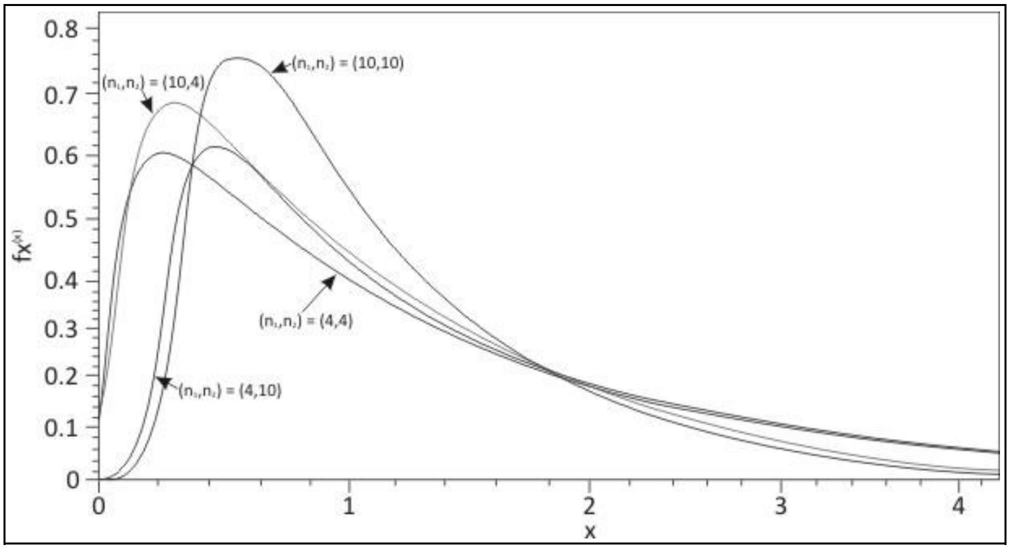

Figure11.3: Plots of p.d.f.s of $X\sim F_{n_1,n_2}$

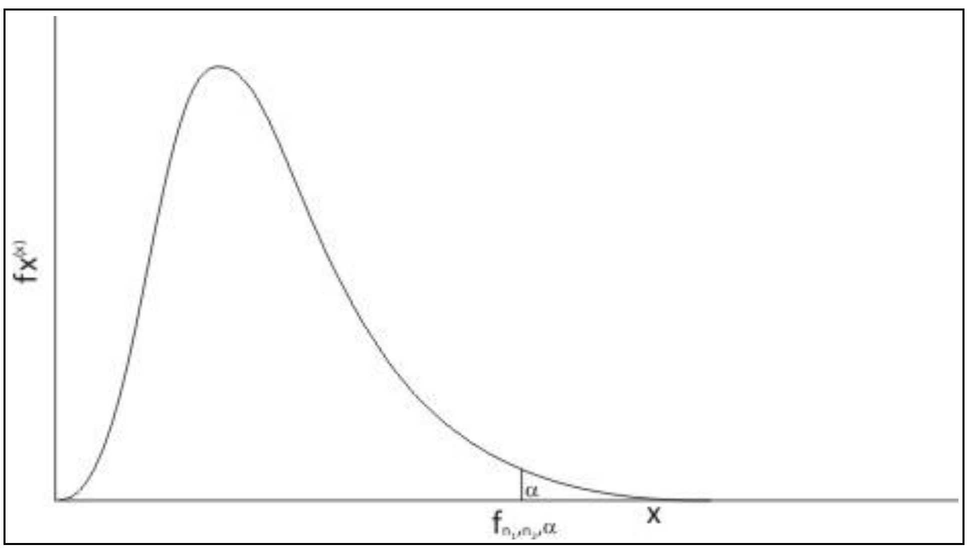

Figure 11.4: $(1-\alpha)$-th quantile of $X \sim F_{n_1,n_2}\left(P\left(\left\{X \leq f_{n_1,n_2,\alpha}\right\} = 1-\alpha\right)\right)$

Example 11.1

Let $Z_1, \ldots, Z_n$ be independent and identically distributed $N(0,1)$ random variables and let $a_1, \ldots, a_n,\ b_1, \ldots, b_n$ be real numbers such that $\sum_{i=1}^{n} a_i^2 > 0, \sum_{i=1}^{n} b_i^2 > 0$, and $\sum_{i=1}^{n} a_i b_i = 0$. Show that:

(i) $\quad Y_1 = \sqrt{\frac{\sum_{i=1}^n b_i^2}{\sum_{i=1}^n a_i^2}} \cdot \frac{\sum_{i=1}^n a_i Z_i}{|\sum_{i=1}^n b_i Z_i|} \sim t_1;$

(ii) $\quad Y_2 = \frac{\sum_{i=1}^n b_i^2}{\sum_{i=1}^n a_i^2} \cdot \left(\frac{\sum_{i=1}^n a_i Z_i}{\sum_{i=1}^n b_i Z_i}\right)^2 \sim F_{1,1};$

(iii) $\quad Y_3 = \sqrt{\frac{\sum_{i=1}^n b_i^2}{\sum_{i=1}^n a_i^2}} \cdot \frac{\sum_{i=1}^n a_i Z_i}{\sum_{i=1}^n b_i Z_i} \sim t_1.$

Solution. Let $T_1 = \sum_{i=1}^n a_i Z_i$ and $T_2 = \sum_{i=1}^n b_i Z_i$. For $c_1, c_2 \in \mathbb{R}$,

$$c_1 T_1 + c_2 T_2 = \sum_{i=1}^n (c_1 a_i + c_2 b_i) Z_i.$$

Since $Z_1, \dots, Z_n$ are independent, by Example 7.1,

$$c_1 T_1 + c_2 T_2 \sim N\left(0, \sum_{i=1}^n (c_1 a_i + c_2 b_i)^2\right)\cdot$$

Now using Theorem 9.1 (v) it follows that $\underline{T} = (T_1, T_2) \sim N_2(0, 0, \sum_{i=1}^n a_i^2, \sum_{i=1}^n b_i^2, 0)$ (since $E(T_1) = 0 = E(T_2)$, $\mathrm{Var}(T_1) = \sum_{i=1}^n a_i^2$, $\mathrm{Var}\,(T_2) = \sum_{i=1}^n b_i^2$ and $\mathrm{Cov}(T_1, T_2) = \sum_{i=1}^n a_i b_i = 0$). Since correlation between T_1 and T_2 is 0 and $\underline{T} = (T_1, T_2) \sim N_2(0, 0, \sum_{i=1}^n a_i^2, \sum_{i=1}^n b_i^2, 0)$, it follows that $T_1 \sim N(0, \sum_{i=1}^n a_i^2)$ and $T_2 \sim N(0, \sum_{i=1}^n b_i^2)$ are independent (see Theorem 9.1).

Thus

$$S_1 = \frac{T_1}{\sqrt{\sum_{i=1}^n {a_i}^2}} = \frac{\sum_{i=1}^n a_i Z_i}{\sqrt{\sum_{i=1}^n {a_i}^2}} \text{ and } S_2 = \frac{T_2}{\sqrt{\sum_{i=1}^n {b_i}^2}} = \frac{\sum_{i=1}^n b_i Z_i}{\sqrt{\sum_{i=1}^n {b_i}^2}},$$

are independent and identically distributed $N(0,1)$ random variables. This implies that $S_1 \sim N(0,1)$ $(S_1^2 \sim \chi_1^2)$ and $S_2 \sim N(0,1)$ $(S_2^2 \sim \chi_1^2)$ are independent random variables. Consequently

$$Y_1 = \frac{S_1}{\sqrt{S_2^2}} \sim t_1 \quad \text{(see Theorem 11.1 (i))}$$

$$Y_2 = \frac{S_1^2/1}{S_2^2/1} \sim F_{1,1} \quad \text{(see Theorem 11.1 (ii))}$$

and

$$Y_3 = \frac{S_1}{S_2} \sim t_1. \quad \text{(see Example 10.2.12 (ii)and Remark 11.1 (ii))}$$

Table 11.1: $(1-\alpha)$-th quantiles of $X \sim t_m$ $\left(P\left(\{X \leq t_{m,\alpha}\}\right) = 1-\alpha\right)$

	α						
m	.25	.1	.05	.025	.01	.005	.001
1	1.000	3.078	6.314	12.71	31.82	63.66	318.3
2	0.816	1.886	2.920	4.303	6.965	9.925	22.33
3	0.765	1.638	2.353	3.182	4.541	5.841	10.21
4	0.741	1.533	2.132	2.776	3.747	4.604	7.173
5	0.727	1.476	2.015	2.571	3.365	4.032	5.893
6	0.718	1.440	1.943	2.447	3.143	3.707	5.208
7	0.711	1.415	1.895	2.365	2.998	3.499	4.785
8	0.706	1.397	1.860	2.306	2.896	3.355	4.501
9	0.703	1.383	1.833	2.262	2.821	3.250	4.297
10	0.700	1.372	1.812	2.228	2.764	3.169	4.144
11	0.697	1.363	1.796	2.201	2.718	3.106	4.025
12	0.695	1.356	1.782	2.179	2.681	3.055	3.930
13	0.694	1.350	1.771	2.160	2.650	3.012	3.852
14	0.692	1.345	1.761	2.145	2.624	2.977	3.787
15	0.691	1.341	1.753	2.131	2.602	2.947	3.733
26	0.690	1.337	1.746	2.120	2.583	2.921	3.686
17	0.689	1.333	1.740	2.110	2.567	2.898	3.646
18	0.688	1.330	1.734	2.101	2.552	2.878	3.610
19	0.688	1.328	1.729	2.093	2.539	2.861	3.579
20	0.687	1.325	1.725	2.086	2.528	2.845	3.552
21	0.686	1.323	1.721	2.080	2.518	2.831	3.527
22	0.686	1.321	1.717	2.074	2.508	2.819	3.505
23	0.685	1.319	1.714	2.069	2.500	2.807	3.485
24	0.685	1.318	1.711	2.064	2.492	2.797	3.467
25	0.684	1.316	1.708	2.060	2.485	2.787	3.450
26	0.684	1.315	1.706	2.056	2.479	2.779	3.435
27	0.684	1.314	1.703	2.052	2.473	2.771	3.421
28	0.683	1.313	1.701	2.048	2.467	2.763	3.408
29	0.683	1.311	1.699	2.045	2.462	2.756	3.396
30	0.683	1.310	1.697	2.042	2.457	2.750	3.385
35	0.682	1.306	1.690	2.030	2.438	2.724	3.340
40	0.681	1.303	1.684	2.021	2.423	2.704	3.307
50	0.679	1.299	1.676	2.009	2.403	2.678	3.261
100	0.677	1.290	1.660	1.984	2.364	2.626	3.174
∞	0.674	1.282	1.645	1.960	2.326	2.576	3.090

Table11.2: $(1-\alpha)$-th quantiles of $X \sim F_{n_1,n_2}$ $\left(P\left(\left\{X \le f_{n_1,n_2,\alpha}\right\}\right) = 1-\alpha\right)$, $\alpha = 0.10$

	n_1								
n_2	1	2	3	4	5	6	7	8	9
1	39.86	49.5	53.59	53.83	57.24	58.2	58.91	59.44	59.86
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44
10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.3	2.27
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96
21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93
23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91
25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88
27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87
29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85
40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74
120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68
∞	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63

Table11.2: $(1-\alpha)$-th quantiles of $X \sim F_{n_1,n_2}$ $\left(P\left(\left\{X \le f_{n_1,n_2}\alpha\right\}\right) = 1-\alpha\right),\ \alpha = 0.10$

	n_1									
n_2	10	12	15	20	24	30	40	60	120	∞
1	60.19	60.71	61.22	61.74	62	62.26	62.93	62.79	63.06	63.33
2	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.48	9.49
3	5.23	5.22	5.20	5.18	5.80	5.17	5.16	5.15	5.14	5.13
4	3.92	4.90	3.87	3.84	3.83	3.82	3.80	3.79	3.78	3.76
5	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14	3.12	3.10
6	2.94	3.90	2.87	2.84	2.82	2.80	2.78	2.76	2.74	2.72
7	2.70	3.67	2.63	2.59	2.58	2.56	2.54	2.51	2.49	2.47
8	2.54	3.50	2.46	2.42	2.40	2.38	2.36	2.34	2.32	2.29
9	2.42	3.38	2.34	2.30	2.28	2.25	2.23	2.21	2.18	2.16
10	2.32	2.28	2.24	2.20	2.18	2.16	2.13	2.11	2.08	2.06
11	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03	2.00	1.97
12	2.19	2.15	2.10	2.06	2.04	2.01	1.99	1.96	1.93	1.90
13	2.40	2.10	2.05	2.01	1.98	1.96	1.93	1.90	1.88	1.85
14	2.10	2.05	2.01	1.96	1.94	1.91	1.89	1.86	1.83	1.80
15	2.06	2.02	1.97	1.92	1.90	1.87	1.85	1.82	1.79	1.76
16	2.03	1.99	1.94	1.89	1.87	1.84	1.81	1.78	1.75	1.72
17	2.00	1.96	1.91	1.86	1.84	1.81	1.78	1.75	1.72	1.69
18	1.98	1.93	1.89	1.84	1.81	1.78	1.75	1.72	1.69	1.66
19	1.96	1.91	1.86	1.81	1.79	1.76	1.73	1.70	1.67	1.63
20	1.94	1.89	1.84	1.79	1.77	1.74	1.71	1.68	1.64	1.61
21	1.92	1.87	1.83	1.78	1.75	1.72	1.69	1.66	1.62	1.59
22	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.64	1.60	1.57
23	1.89	1.84	1.80	1.74	1.72	1.69	1.66	1.62	1.59	1.55
24	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.61	1.57	1.53
25	1.87	1.82	1.77	1.72	1.69	1.66	1.63	1.59	1.56	1.52
26	1.86	1.81	1.76	1.71	1.80	1.65	1.61	1.58	1.54	1.50
27	1.85	1.80	1.75	1.70	1.67	1.64	1.60	1.57	1.53	1.49
28	1.84	1.79	1.74	1.69	1.66	1.63	1.59	1.56	1.52	1.48
29	1.83	1.78	1.73	1.68	1.65	1.62	1.58	1.55	1.51	1.47
30	1.82	1.77	1.72	1.67	1.64	1.61	1.57	1.54	1.50	1.46
40	1.76	1.71	1.66	1.61	1.57	1.54	1.51	1.47	1.42	1.38
60	1.71	1.66	1.60	1.54	1.51	1.48	1.44	1.40	1.35	.129
120	1.65	1.60	1.55	1.48	1.45	1.41	1.37	1.32	1.26	1.19
∞	1.60	1.55	1.49	1.42	1.38	1.34	1.30	1.24	1.17	1.00

Table11.2: $(1-\alpha)$-th quantiles of $X \sim F_{n_1,n_2}$ $\left(P\left(\left\{X \le f_{n_1,n_2}\alpha\right\}\right) = 1-\alpha\right),\ \alpha = 0.05$

	n_1								
n_2	1	2	3	4	5	6	7	8	9
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	940.5
2	18.51	19.00	19.16	19.25	19.3	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88

Table11.2: $(1-\alpha)$-th quantiles of $X \sim F_{n_1,n_2}$ $\left(P\left(\left\{X \leq f_{n_1,n_2}\alpha\right\}\right) = 1-\alpha\right)$, $\alpha = 0.05$

					n_1					
n_2	10	12	15	20	24	30	40	60	120	∞
1	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	19.4	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.5
3	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	2.41	2.64	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	2.16	2.09	2.01	1.3	1.89	1.84	1.79	1.74	1.68	162
40	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	1.91	1.83	1.75	1.66	1.10	1.55	1.50	1.43	1.35	1.25
∞	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Table 11.2: $(1-\alpha)$-th quantiles of $X \sim F_{n_1,n_2}$ $(P|\{X \le f_{n_1,n_2,\alpha}\}) = 1-\alpha)$, $\alpha = 0.10$

n_2	1	2	3	4	5	6	7	8	9
1	4052	4999.5	5403	5625	5764	5859	5928	5982	6022
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
3	34.12	30.82	29.46	28.71	2824	27.91	27.67	27.49	27.35
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
10	10.04	7.56	6.55	5.99	5.64	5.39	5.2	5.06	4.94
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.14
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
29	7.60	5.42	4.54	4.4	3.73	3.50	3.33	3.20	3.09
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41

Table 11.2: $(1-\alpha)$-th quantiles of $X \sim F_{n_1,n_2}$ $(P|\{X \le f_{n_1,n_2,\alpha}\}) = 1-\alpha),\ \alpha = 0.01$

	n_1									
n_2	10	12	15	20	24	30	40	60	120	∞
1	6056	6106	6157	6209	6235	6261	6287	6313	6339	6366
2	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00