MODULE 6

RANDOM VECTOR AND ITS JOINT DISTRIBUTION

LECTURE 33

Topics

6.10.2 Transformation Of Variables Technique

6.10.2 Transformation Of Variables Technique

The following theorem, whose proof is similar to that of Theorem 2.1, Module 3, deals with joint probability distribution of functions of a discrete type random vector $\underline{X}$.

Theorem 10.2.1

Let $\underline{X} = (X_1, \dots, X_p)$ be a discrete type random vector with support $S_{\underline{X}}$ and p.m.f. $f_{\underline{X}}(\cdot)$. Let $g_i: \mathbb{R}^k \to \mathbb{R}, i = 1, \dots, k$ be k Borel functions and let $Y_i = g_i(\underline{X}), i = 1, \dots, k$. Define, for $\underline{y} = (y_1, \dots, y_k) \in \mathbb{R}^k$, $A_{\underline{y}} = \{\underline{x} = (x_1, \dots, x_p) \in S_{\underline{X}}: \ g_1(\underline{x}) \le y_1, \dots, g_k(\underline{x}) \le y_k\}$ and $B_{\underline{y}} = \{\underline{x} \in S_{\underline{X}}: g_1(\underline{x}) = y_1, \dots, g_k(\underline{x}) = y_k\}$ .Then the random vector $\underline{Y} = (Y_1, \dots, Y_k)$ is of discrete type with distribution function

$$F_{\underline{Y}}(\underline{y}) = \sum_{\underline{x} \in A_{\underline{y}}} f_{\underline{X}}(\underline{x}), \quad \underline{y} \in \mathbb{R}^k$$

and the p.m.f.

$$f_{\underline{Y}}(\underline{y}) = \sum_{\underline{x} \in B_{\underline{y}}} f_{\underline{X}}(\underline{x}), \quad \underline{y} \in \mathbb{R}^k. \ \blacksquare$$

We will denote the Cartesian product of sets $A_1, \dots, A_m$ by $\prod_{i=1}^m A_i = \{(x_1, \dots, x_m): x_i \in A_i, i = 1, \dots, m\}$.

Example 10.2.1

Let $X_1, \dots, X_p$ be independent random variables with $X_i \sim \text{Bin}(n_i, \theta)$, where $n_i \in \mathbb{N}, i = 1, \dots, p$ and $\theta \in (0, 1)$. Without using the m.g.f. of $Y = \sum_{i=1}^p X_i$, find the p.m.f. of Y.

Solution. For finding the probability distribution of Y using the uniqueness of m.g.f. see Example 7.2. The joint p.m.f. of $\underline{X} = (X_1, \dots, X_p)$ is given by

$$
\begin{aligned}
f_{\underline{X}}(\underline{x}) &= \prod_{i=1}^{p} f_{X_i}(x_i) \\
&= \begin{cases} \prod_{i=1}^{p} \binom{n_i}{x_i} \theta^{x_i} (1-\theta)^{n_i - x_i}, & \text{if } \underline{x} \in \prod_{i=1}^{p} \{0, 1, \dots, n_i\} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \left(\prod_{i=1}^{p} \binom{n_i}{x_i} \right) \theta^{\sum_{i=1}^{p} x_i} (1-\theta)^{n - \sum_{i=1}^{p} x_i}, & \text{if } \underline{x} \in \prod_{i=1}^{p} \{0, 1, \dots, n_i\}, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

where $n = \sum_{i=1}^{p} n_i$. By Theorem 10.2.1, we have

$$
f_Y(y) = \sum_{\underline{x} \in B_y} f_{\underline{X}}(\underline{x}), \quad y \in \mathbb{R},
$$

where, for $y \in \mathbb{R}, B_y = \{\underline{x} \in S_{\underline{X}} : x_1 + \cdots + x_p = y\}$. Clearly, for $y \notin \{0, 1, \cdots, n\}$, $B_y = \phi$ and therefore $f_Y(y) = 0$. Also, for $y \in \{0, 1, \dots, n\}$,

$$
\begin{aligned}
f_Y(y) &= \underset{x_1 + \cdots + x_p = y}{\sum_{x_1=0}^{n_1} \cdots \sum_{x_p=0}^{n_p}} f_{\underline{X}}(\underline{x}) \\
&= \underset{x_1 + \cdots + x_p = y}{\sum_{x_1=0}^{n_1} \cdots \sum_{x_p=0}^{n_p}} \left(\prod_{i=1}^{p} \binom{n_i}{x_i} \right) \theta^{\sum_{i=1}^{p} x_i} (1-\theta)^{n - \sum_{i=1}^{p} x_i} \\
&= \left(\underset{x_1 + \cdots + x_p = y}{\sum_{x_1=0}^{n_1} \cdots \sum_{x_p=0}^{n_p}} \left(\prod_{i=1}^{p} \binom{n_i}{x_i} \right) \right) \theta^{y} (1-\theta)^{n-y} \\
&= \binom{n}{y} \theta^{y} (1-\theta)^{n-y}.
\end{aligned}
$$

Therefore

$$f_Y(y) = \begin{cases} \binom{n}{y} \theta^y (1-\theta)^{n-y}, & \text{if } y \in \{0, 1, \ldots, n\} \\ 0, & \text{otherwise} \end{cases},$$

i.e. $Y \sim \text{Bin}(n, \theta)$. ■

Example 10.2.2

Let $X_1, \ldots, X_p$ be independent random variables such that $X_i \sim \text{P}(\lambda_i), i = 1, \ldots, p$, where $\lambda_i > 0, i = 1, \ldots, p$. Without using the m.g.f. of $Y = \sum_{i=1}^{p} X_i$, find the probability distribution of Y.

Solution. For derivation of probability distribution of Y using the uniqueness of m.g.f. see Example 7.4. We have $S_{\underline{X}} = \{0, 1, \ldots\}^p$. The joint p.m.f. of $\underline{X} = (X_1, \ldots, X_p)$ is

$$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{p} f_{X_i}(x_i)$$

$$= \begin{cases} e^{-\sum_{i=1}^{p} \lambda_i} \left(\prod_{i=1}^{p} \frac{\lambda_i^{x_i}}{x_i!} \right), & \text{if } \underline{x} = (x_1, \ldots, x_p) \in \{0, 1, \ldots\}^p \\ 0, & \text{otherwise} \end{cases}.$$

Using Theorem 10.2.1, we have

$$f_Y(y) = \sum_{\underline{x} \in B_y} f_{\underline{X}}(\underline{x}), \quad y \in \mathbb{R},$$

where, for $y \in \mathbb{R}, B_y = \{\underline{x} \in S_{\underline{X}} : x_1 + \cdots + x_p = y\}$.

For $y \notin \{0, 1, \ldots\}, B_y = \phi$ and therefore $f_Y(y) = 0$.

For $y \in \{0, 1, \ldots\}$,

$$f_Y(y) = \sum_{\substack{x_1=0 \\ x_1+\cdots}}^{\infty} \cdots \sum_{\substack{x_p=0 \\ +x_p=y}}^{\infty} e^{-\sum_{i=1}^{p} \lambda_i} \left(\prod_{i=1}^{p} \frac{\lambda_i^{x_i}}{x_i!} \right)$$

$$= \frac{e^{-\sum_{i=1}^{p} \lambda_i}}{y!} \sum_{\substack{x_1=0 \\ x_1+\cdots}}^{\infty} \cdots \sum_{\substack{x_p=0 \\ +x_p=y}}^{\infty} \frac{(x_1 + \cdots + x_p)!}{x_1! \cdots x_p!} \lambda_1^{x_1} \cdots \lambda_p^{x_p}$$

$$= \frac{e^{-\sum_{i=1}^{p} \lambda_i} \left(\lambda_1 + \cdots + \lambda_p\right)^y}{y!}.$$

Therefore,

$$f_Y(y) = \begin{cases} \frac{e^{-\sum_{i=1}^{p} \lambda_i} \left(\sum_{i=1}^{p} \lambda_i\right)^y}{y!}, & \text{if } y \in \{0, 1, 2, \dots\} \\ 0, & \text{otherwise} \end{cases},$$

$$\text{i.e.}, Y \sim \mathrm{P}\left(\sum_{i=1}^{p} \lambda_i\right). \blacksquare$$

Example 10.2.3

Let $X_1, \dots, X_p$ be independent random variables such that $X_i \sim NB(r_i, \theta), \theta \in (0,1), r_i \in \{1, 2, \dots\}, i = 1, \dots, p$. Without using the m.g.f. of $Y = \sum_{i=1}^{p} X_i$ show that $Y \sim NB\left(\sum_{i=1}^{p} r_i, \theta\right)$.

Solution. For a solution utilizing the uniqueness of m.g.f. refer to Example 7.3.One can also provide a solution based on methods used in solving problems 10.2.1 and 10.2.2 by using the identity

$$\sum_{\substack{k_1=1 \\ k_1+\cdots}}^{\infty} \cdots \sum_{\substack{k_p=1 \\ +k_p=y}}^{\infty} \binom{r_1 + k_1 - 1}{r_1} \cdots \binom{r_p + k_p - 1}{r_p} = \binom{\sum_{i=1}^{p} r_i + y - 1}{y}, \quad y \in \{0, 1, 2, \dots\}. \blacksquare$$

Example 10.2.4

Let X_1 and X_2 be independent and identically distributed random variables with $X_1 \sim \mathrm{NB}(1, p)$, where $p \in (0, 1)$. Find the distribution function of $Y = X_1 + X_2$. Hence find the p.m.f. of Y (also see Examples 7.3 and 10.2.3).

Solution. Since X_1 and X_2 have the common support $S = \{0, 1, 2, \dots\}$, we have $F_Y(y) = 0$, if $y < 0$. Moreover, for $y \in [k, k + 1), k \in \{0, 1, 2, \dots\}$

$$F_Y(y) = P(\{X_1 + X_2 \le y\})$$

$$= P(\{X_1 + X_2 \le k\})$$

$$= \sum_{j=0}^{\infty} P\left(\{X_1 + X_2 \le k, X_2 = j\}\right)$$

$$= \sum_{j=0}^{\infty} P\left(\{X_1 \le k - j, X_2 = j\}\right)$$

$$= \sum_{j=0}^{\infty} P\left(\{X_1 \le k - j\}\right) P(\{X_2 = j\}).$$

We have $P(\{X_1 \le l\}) = 0,$ if $l \in \{-1, \ -2, \dots\}$ and, for $l \in \{0, 1, 2, \dots\}$

$$P(\{X_1 \le l\}) = \sum_{j=0}^{l} P\left(\{X_1 = j\}\right)$$

$$= \sum_{j=0}^{l} (1-p)^j p$$

$$= 1 - (1-p)^{l+1}.$$

It follows that, for $y \in [k, k+1), k \in \{0, 1, 2, \dots\}$

$$F_Y(y) = \sum_{j=0}^{k} P(\{X_1 \le k - j\})\, P(\{X_2 = j\})$$

$$= \sum_{j=0}^{k} \left(1 - (1-p)^{k-j+1}\right) p(1-p)^j$$

$$= 1 - (1-p)^{k+1} - (k+1)p(1-p)^{k+1}.$$

Consequently

$$F_Y(y) \ = \begin{cases} 0, & \text{if } y < 0 \\ 1 - (1-p)^{k+1} - (k+1)p(1-p)^{k+1}, & \text{if } k \le y < k+1, k \in \{0, 1, \dots\} \end{cases}.$$

Clearly Y is a discrete type random variable with support $S_Y = \{0, 1, 2, \dots\}$ and for $k \in S_Y$,

$$P(\{Y = k\}) = F_Y(k) - F_Y(k-)$$

$$= \ F_Y(k) - F_Y(k-1)$$

$$= (k+1)p^2\,(1-p)^k.$$

Therefore the p.m.f of Y is given by

$$f_Y(y) = \begin{cases} (y+1)p^2\,(1-p)^y, & \text{if } y \in \{0,1,2,\dots\} \\ 0, & \text{otherwise} \end{cases}. \blacksquare$$

Example 10.2.5

Let X_1 and X_2 be independent and identically distributed random variables with common p.m.f.

$$f(x) = \begin{cases} \theta(1-\theta)^{x-1}, & \text{if } x \in \{1,2,\dots\} \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in (0,1)$. Let $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\} - \min\{X_1, X_2\}$.

(i) Find the marginal p.m.f. of Y_1 without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$;
(ii) Find the marginal p.m.f. of Y_2 without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$;
(iii) Find the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$;
(iv) Are Y_1 and Y_2 independent?
(v) Using (iii) find the marginal p.m.f.s of Y_1 and Y_2.

Solution. The joint p.m.f of $\underline{X} = (X_1, X_2)$ is given by

$$f_{\underline{X}}(x_1, x_2) = f(x_1)f(x_2) = \begin{cases} \theta^2\,(1-\theta)^{x_1+x_2-2}, & \text{if } (x_1, x_2) \in \mathbb{N} \times \mathbb{N} \\ 0, & \text{otherwise} \end{cases},$$

where $\mathbb{N} = \{1, 2, \dots\}$. Clearly $S_{\underline{X}} = \mathbb{N} \times \mathbb{N}$.

(i) By Theorem 10.2.1

$$f_{Y_1}(y) = \sum_{\underline{x} \in B_y} f_{\underline{X}}\,(x_1, x_2), \quad y \in \mathbb{R},$$

where, for $y \in \mathbb{R}$, $B_y = \{(x_1, x_2) \in S_{\underline{X}}\colon \min(x_1, x_2) = y\}$. Clearly, for $y \notin \{1, 2, \dots\} = \mathbb{N}$, $B_y = \phi$ and therefore $f_{Y_1}(y) = 0$.

For $y \in \{1, 2, \dots\}$

$$B_y = \left\{(x_1, x_2) \in S_{\underline{X}}\colon x_1 = x_2 = y\right\} \cup \left\{(x_1, x_2) \in S_{\underline{X}}\colon x_2 = y, x_1 \in \{y+1, y+2, \dots\}\right\}$$
$$\cup \left\{(x_1, x_2) \in S_{\underline{X}}\colon x_1 = y, \quad x_2 \in \{y+1, y+2, \dots\}\right\}$$

$= B_{1,y} \cup B_{2,y} \cup B_{3,y}$, say.

Clearly, for $y \in \{1, 2, \dots\}, B_{1,y}, B_{2,y}$ and $B_{3,y}$, are pairwise disjoint sets. Therefore, for $y \in \mathbb{N}$,

$$f_{Y_1}(y) = \sum_{\underline{x} \in B_{1,y}} f_{\underline{X}}(x_1, x_2) + \sum_{\underline{x} \in B_{2,y}} f_{\underline{X}}(x_1, x_2) + \sum_{\underline{x} \in B_{3,y}} f_{\underline{X}}(x_1, x_2)$$

$$= \theta^2 (1-\theta)^{2y-2} + \sum_{x_1=y+1}^{\infty} \theta^2 (1-\theta)^{x_1+y-2} + \sum_{x_2=y+1}^{\infty} \theta^2 (1-\theta)^{x_2+y-2}$$

$$= \theta^2 (1-\theta)^{2y-2} + 2\theta^2 \sum_{x=y+1}^{\infty} (1-\theta)^{x+y-2}$$

$$= \theta^2 (1-\theta)^{2y-2} + 2\theta(1-\theta)^{2y-1}$$

$$= \theta(2-\theta)(1-\theta)^{2y-2}.$$

Therefore,

$$f_{Y_1}(y) = \begin{cases} \theta(2-\theta)(1-\theta)^{2y-2}, & \text{if } y \in \{1, 2, \dots\} \\ 0, & \text{otherwise} \end{cases}.$$

(ii) We have

$$f_{Y_2}(y) = \sum_{\underline{x} \in B_y} f_{\underline{X}}(x_1, x_2), \quad y \in \mathbb{R},$$

where, for $y \in \mathbb{R}$,

$$B_y = \{(x_1, x_2) \in S_{\underline{X}} : \max\{x_1, x_2\} - \min\{x_1, x_2\} = y\}.$$

Clearly, for $y \notin \{0, 1, 2, \dots\}$, $B_y = \phi$, and therefore $f_{Y_2}(y) = 0$.

For $y = 0$, $B_y = \{(x_1, x_2) \in S_{\underline{X}} : x_1 = x_2\} = \{(x, x) : x \in \{1, 2, \dots\}\}$, and therefore

$$f_{Y_2}(y) = \sum_{x=1}^{\infty} \theta^2 (1-\theta)^{2x-2}$$

$$= \frac{\theta}{2-\theta}.$$

For $y \in \{1, 2, \dots\}$,

$$B_y = \left\{(x_1, x_2) \in S_{\underline{X}}: \max\{x_1, x_2\} - \min\{x_1, x_2\} = y\right\}$$

$$= \left\{(x, x+y): x \in \{1, 2, \dots\}\right\} \cup \left\{(x+y, x): x \in \{1, 2, \dots\}\right\}$$

$$= B_{1,y} \cup B_{2,y}, \qquad \text{say.}$$

Since $B_{1,y} \cap B_{2,y} = \phi,\ y \in \{1, 2, \dots\}$, we have for $y \in \{1, 2, \dots\}$,

$$f_{Y_2}(y) = \sum_{\underline{x} \in B_{1,y}} f_{\underline{X}}(x_1, x_2) + \sum_{\underline{x} \in B_{2,y}} f_{\underline{X}}(x_1, x_2)$$

$$= \sum_{x=1}^{\infty} \theta^2 (1-\theta)^{2x+y-2} + \sum_{x=1}^{\infty} \theta^2 (1-\theta)^{2x+y-2}$$

$$= \frac{2\theta(1-\theta)^y}{2-\theta}.$$

Therefore

$$f_{Y_2}(y) = \begin{cases} \dfrac{\theta}{2-\theta}, & \text{if } y = 0 \\ \dfrac{2\theta(1-\theta)^y}{2-\theta}, & \text{if } y \in \{1, 2, \dots\}. \\ 0, & \text{otherwise} \end{cases}$$

(iii) We have, for $\underline{y} = (y_1, y_2) \in \mathbb{R}^2$,

$$f_{\underline{Y}}(y_1, y_2) = P\left(\left\{\min\{X_1, X_2\} = y_1, \max\{X_1, X_2\} - \min\{X_1, X_2\} = y_2\right\}\right)$$

$$= P\left(\left\{\min\{X_1, X_2\} = y_1, \max\{X_1, X_2\} = y_1 + y_2\right\}\right)$$

$$= \sum_{\underline{x} \in B_{\underline{y}}} f_{\underline{X}}(x_1, x_2),$$

where, for $\underline{y} = (y_1, y_2) \in \mathbb{R}^2$,

$$B_{\underline{y}} = \left\{(x_1, x_2) \in S_{\underline{X}}: \min\{x_1, x_2\} = y_1, \max\{x_1, x_2\} = y_1 + y_2\right\}.$$

Note that, for $\underline{y} = (y_1, y_2) \notin \mathbb{N} \times \{0, 1, 2, \dots\}$, $B_{\underline{y}} = \phi$ and therefore

$$f_{\underline{Y}}(y_1, y_2) = 0.$$

For $\underline{y} = (y_1, y_2) \in \mathbb{N} \times \{0\}$, $B_{\underline{y}} = \{(y_1, y_1)\}$ and therefore

$$f_{\underline{Y}}(y_1, y_2) = \theta^2\,(1-\theta)^{2y_1-2}.$$

Also, for $\underline{y} = (y_1, y_2) \in \mathbb{N} \times \mathbb{N}, B_{\underline{y}} = \{(y_1, y_1 + y_2), (y_1 + y_2, y_1)\}$ and therefore

$$f_{\underline{Y}}(y_1, y_2) = 2\theta^2\,(1-\theta)^{2y_1+y_2-2}.$$

It follows that

$$f_{\underline{Y}}(y_1, y_2) = \begin{cases} \theta^2(1-\theta)^{2y_1-2}, & \text{if } (y_1, y_2) \in \mathbb{N} \times \{0\} \\ 2\theta^2\,(1-\theta)^{2y_1+y_2-2}, & \text{if } (y_1, y_2) \in \mathbb{N} \times \mathbb{N} \\ 0, & \text{otherwise} \end{cases}.$$

(iv) By (i)−(iii) we have

$$f_{\underline{Y}}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2), \forall \underline{y} = (y_1, y_2) \in \mathbb{R}^2.$$

Consequently Y_1 and Y_2 are independent random variables.

(v) From (iii) we have $S_{\underline{Y}} = \mathbb{N} \times \{0, 1, 2, \dots\}$. Therefore

$$S_{Y_1} = \{y_1 \in \mathbb{R}: (y_1, y_2) \in S_{\underline{Y}} \text{ for some } y_2 \in \mathbb{R}\} = \mathbb{N}$$

and

$$S_{Y_2} = \{y_2 \in \mathbb{R}: (y_1, y_2) \in S_{\underline{Y}} \text{ for some } y_1 \in \mathbb{R}\} = \{0, 1, 2, \dots\}.$$

Also

$$f_{Y_1}(y_1) = \begin{cases} \sum_{y_2 \in R_{y_1}} f_{\underline{Y}}(y_1, y_2), & \text{if } y_1 \in S_{Y_1} \\ 0, & \text{otherwise} \end{cases},$$

where, for $y_1 \in S_{Y_1}$, $R_{y_1} = \{y_2 \in \mathbb{R}: (y_1, y_2) \in S_{\underline{Y}}\} = \{0, 1, 2, \dots\}$.

Thus, for $y_1 \in S_{Y_1} = \{1, 2, \dots\}$,

$$f_{Y_1}(y_1) = \sum_{y_2 \in R_{y_1}} f_{\underline{Y}}(y_1, y_2)$$

$$= \sum_{y_2=0}^{\infty} f_{\underline{Y}}(y_1, y_2)$$

$$= \theta^2 (1-\theta)^{2y_1-2} + \sum_{y_2=1}^{\infty} 2\theta^2 (1-\theta)^{2y_1+y_2-2}$$

$$= \theta(2-\theta)(1-\theta)^{2y_1-2}.$$

Therefore

$$f_{Y_1}(y_1) = \begin{cases} \theta(2-\theta)(1-\theta)^{2y_1-2}, & \text{if } y_1 \in \{1, 2, \ldots\} \\ 0, & \text{otheriwse} \end{cases}.$$

Similarly,

$$f_{Y_2}(y_2) = \begin{cases} \sum_{y_1 \in R_{y_2}} f_{\underline{Y}}(y_1, y_2), & \text{if } y_2 \in S_{Y_2} \\ 0, & \text{otherwise} \end{cases},$$

where, for $y_2 \in S_{Y_2}$, $R_{y_2} = \{y_1 \in \mathbb{R}: (y_1, y_2) \in S_{\underline{Y}}\} = \{1, 2, \ldots\}$. Therefore, for $y_2 = 0$

$$f_{Y_2}(y_2) = \sum_{y_1=1}^{\infty} \theta^2 (1-\theta)^{2y_1-2}$$

$$= \frac{\theta}{2-\theta},$$

and, for $y_2 \in \{1, 2, \ldots\}$

$$f_{Y_2}(y_2) = \sum_{y_1=1}^{\infty} 2\theta^2 (1-\theta)^{2y_1+y_2-2}$$

$$= \frac{2\theta(1-\theta)^{y_2}}{2-\theta}.$$

It follows that

$$f_{Y_2}(y_2) = \begin{cases} \frac{\theta}{2-\theta}, & \text{if } y_2 = 0 \\ \frac{2\theta(1-\theta)^{y_2}}{2-\theta}, & \text{if } y_2 \in \{1, 2, \ldots\} \\ 0, & \text{otherwise} \end{cases}. \blacksquare$$

Example10.2.6

Let $\underline{X} = (X_1, X_2, X_3)$ be a discrete type random vector with p.m.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \begin{cases} \frac{2}{9}, & \text{if } (x_1, x_2, x_3) \in \{(1,1,0), (1,0,1), (0,1,1)\} \\ \frac{1}{3}, & \text{if } (x_1, x_2, x_3) = (1,1,1) \\ 0, & \text{otherwise} \end{cases}.$$

Define $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$.

(i) Find the marginal p.m.f. of Y_1 without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$;
(ii) Find the marginal p.m.f. of Y_2 without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$;
(iii) Find the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$;
(iv) Are Y_1 and Y_2 independent?
(v) Using (iii) find the marginal p.m.f.s of Y_1 and Y_2.

Solution.

(i) We have

$$P(\{Y_1 = y\}) = P(\{X_1 + X_2 = y\}) = 0, \qquad \text{if } y \notin \{1,2\},$$

$$\begin{aligned} P(\{Y_1 = 1\}) &= P(\{X_1 + X_2 = 1\}) \\ &= P(\{(X_1, X_2, X_3) \in \{(1,0,1), (0,1,1)\}\}) \\ &= P(\{(X_1, X_2, X_3) = (1,0,1)\}) + P(\{(X_1, X_2, X_3) = (0,1,1)\}) \\ &= \frac{4}{9} \end{aligned}$$

and

$$\begin{aligned} P(\{Y_1 = 2\}) &= P(\{X_1 + X_2 = 2\}) \\ &= P(\{(X_1, X_2, X_3) = (1,1,0)\}) + P(\{(X_1, X_2, X_3) = (1,1,1)\}) \\ &= \frac{5}{9}. \end{aligned}$$

Therefore,

$$f_{Y_1}(y)=\begin{cases}\frac{4}{9}, & \text{if } y=1\\ \frac{5}{9}, & \text{if } y=2\\ 0, & \text{otherwise}\end{cases}.$$

(ii) By symmetry

$$f_{Y_2}(y)=\begin{cases}\frac{4}{9}, & \text{if } y=1\\ \frac{5}{9}, & \text{if } y=2\\ 0, & \text{otherwise}\end{cases}.$$

(iii) The joint p.m.f. of $\underline{Y}=(Y_1,Y_2)$ is

$$f_{\underline{Y}}(y_1,y_2)=P(\{X_1+X_2=y_1,X_1+X_3=y_2\})$$

$$=0,\quad \text{if}\quad (y_1,y_2)\notin\{(1,1),(1,2),(2,1),(2,2)\},$$

$$f_{\underline{Y}}(1,1)=P(\{X_1+X_2=1,X_2+X_3=1\})$$

$$=P(\{(X_1,X_2,X_3)=(1,0,1)\})$$

$$=\frac{2}{9},$$

$$f_{\underline{Y}}(1,2)=P(\{X_1+X_2=1,X_2+X_3=2\})$$

$$=P(\{(X_1,X_2,X_3)=(0,1,1)\})$$

$$=\frac{2}{9},$$

$$f_{\underline{Y}}(2,1)=P(\{X_1+X_2=2,X_2+X_3=1\})$$

$$=P(\{(X_1,X_2,X_3)=(1,1,0)\})$$

$$=\frac{2}{9}$$

and

$$f_{\underline{Y}}(2,2)=P(\{X_1+X_2=2,X_2+X_3=2\})$$

$$= P(\{(X_1, X_2, X_3) = (1,1,1)\})$$

$$= \frac{1}{3}.$$

Therefore

$$f_{\underline{Y}}(y_1, y_2) = \begin{cases} \frac{2}{9}, & \text{if } (y_1, y_2) \in \{(1,1), (1,2), (2,1)\} \\ \frac{1}{3}, & \text{if } (y_1, y_2) = (2,2) \\ 0, & \text{otherwise} \end{cases}.$$

(iv) Since

$$P(\{Y_1 = 1, Y_2 = 1\}) = \frac{2}{9}$$

$$\neq P(\{Y_1 = 1\})P(\{Y_2 = 1\})$$

$$= \frac{16}{81},$$

Y_1 and Y_2 are not independent.

(v) Using (iii) we have $S_{\underline{Y}} = \{(1,1), (1,2), (2,1), (2,2)\}$. Therefore

$$S_{Y_1} = \{y_1 \in \mathbb{R}: (y_1, y_2) \in S_{\underline{Y}} \text{ for some } y_2 \in \mathbb{R}\} = \{1,2\}$$

and

$$S_{Y_2} = \{y_2 \in \mathbb{R}: (y_1, y_2) \in S_{\underline{Y}} \text{ for some } y_1 \in \mathbb{R}\} = \{1,2\}.$$

Also

$$f_{Y_1}(y_1) = \begin{cases} \sum_{y_2 \in R_{y_1}} f_{\underline{Y}}(y_1, y_2), & \text{if } y_1 \in S_{Y_1} \\ 0, & \text{otherwise} \end{cases},$$

where, for $y_1 \in S_{Y_1}$, $R_{y_1} = \{y_2 \in \mathbb{R}: (y_1, y_2) \in S_{\underline{Y}}\} = \{1,2\}$.

Consequently, for $y_1 \in S_{Y_1} = \{1,2\}$,

$$f_{Y_1}(y_1) = \sum_{y_2 \in R_{y_1}} f_{\underline{Y}}(y_1, y_2) = \sum_{y_2=1}^{2} f_{\underline{Y}}(y_1, y_2),$$

$$f_{Y_1}(1) = f_{\underline{Y}}(1,1) + f_{\underline{Y}}(1,2) = \frac{4}{9}$$

and

$$f_{Y_1}(2) = f_{\underline{Y}}(2,1) + f_{\underline{Y}}(2,2) = \frac{5}{9}.$$

Therefore

$$f_{Y_1}(y_1) = \begin{cases} \frac{4}{9}, & \text{if } y_1 = 1 \\ \frac{5}{9}, & \text{if } y_1 = 2 \\ 0, & \text{otherwise} \end{cases}.$$

By symmetry

$$f_{Y_2}(y) = f_{Y_1}(y), \qquad \forall y \in \mathbb{R}. \blacksquare$$

Example 10.2.7

Let $\underline{X} = (X_1, X_2)$ be a discrete type random vector with p.m.f. given by

	$f_{X_1,X_2}(x_1,x_2)$	
x_1 / x_2	-1	1
0	$\frac{1}{4}$	$\frac{1}{2}$
2	$\frac{1}{16}$	$\frac{3}{16}$

Find the p.m.f. of $Y = |X_1 - 2X_2|$.

Solution. We have

(x_1, x_2)	$f_{X_1,X_2}(x_1, x_2)$	$y = \lvert x_1 - 2x_2 \rvert$
$(-1, 0)$	$\frac{1}{4}$	1
$(1, 0)$	$\frac{1}{2}$	1
$(-1, 2)$	$\frac{1}{16}$	5
$(1, 2)$	$\frac{3}{16}$	3

Therefore the p.m.f. of $Y = |X_1 - 2X_2|$ is given by

$$f_Y(y) = \begin{cases} \frac{3}{4}, & \text{if } y = 1 \\ \frac{3}{16}, & \text{if } y = 3 \\ \frac{1}{16}, & \text{if } y = 5 \\ 0, & \text{otherwise} \end{cases} . \blacksquare$$