MODULE 1

PROBABILITY

LECTURE 6

Topics

1.3 CONDITIONAL PROBABILITY AND INDEPENDENCE OF EVENTS

1.4 CONTINUITY OF PROBABILITY MEASURES

1.4.1 Continuity of Probability Measures

Theorem 3.5

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let A and B be independent events $(A, B \in \mathcal{F})$. Then

(i) A^c and B are independent events;

(ii) A and B^c are independent events;

(iii) A^c and B^c are independent events.

Proof. We have

$$P(A \cap B) = P(A)P(B).$$

(i) Since $B = (A \cap B) \cup (A^c \cap B)$ and $(A \cap B) \cap (A^c \cap B) = \phi$, we have

$$P(B) = P(A \cap B) + P(A^c \cap B)$$

$$\Rightarrow P(A^c \cap B) = P(B) - P(A \cap B)$$

$$= P(B) - P(A)P(B)$$
$$= \big(1 - P(A)\big)P(B)$$
$$= P(A^c)P(B),$$

i.e., A^c and B are independent events.

(ii) Follows from (i) by interchanging the roles of A and B.

(iii) Follows on using (i) and (ii) sequentially. ■

The following theorem strengthens the results of Theorem 3.5.

Theorem 3.6

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $F_1, \dots, F_n (n \in \mathbb{N}, n \geq 2)$ be independent events in $\mathcal{F}$. Then, for any $k \in \{1, 2, \dots, n-1\}$ and any permutation $(\alpha_1, \dots, \alpha_n)$ of $(1, \dots, n)$, the events $F_{\alpha_1}, \dots, F_{\alpha_k}, F_{\alpha_{k+1}}^c, \dots, F_{\alpha_n}^c$ are independent. Moreover the events $F_1^c, \dots, F_n^c$ are independent.

Proof. Since the notion of independence is symmetric in the events involved, it is enough to show that for any $k \in \{1, 2, \dots, n-1\}$ the events $F_1, \dots, F_k, F_{k+1}^c, \dots, F_n^c$ are independent. Using backward induction and symmetry in the notion of independence the above mentioned assertion would follow if, under the hypothesis of the theorem, we show that the events $F_1, \dots, F_{n-1}, F_n^c$ are independent. For this consider a subcollection $\{F_{i_1}, \dots, F_{i_m}, G\}$ of $F_1, \dots, F_{n-1}, F_n^c$ $(\{i_1, \dots, i_m\} \subseteq \{1, \dots, n-1\})$, where $G = F_n^c$ or $G = F_j$, for some $j \in \{1, \dots, n-1\} - \{i_1, \dots, i_m\}$, depending on whether or not F_n^c is a part of subcollection $\{F_{i_1}, \dots, F_{i_m}, G\}$. Thus the following two cases arise:

Case I. $G = F_n^c$

Since $F_1, \dots, F_n$ are independent, we have

$$P\left(\bigcap_{j=1}^{m} F_{i_j}\right) = \prod_{j=1}^{m} P\left(F_{i_j}\right),$$

and

$$P\left(\left(\bigcap_{j=1}^{m} F_{i_j}\right) \cap F_n\right) = \left[\prod_{j=1}^{m} P\left(F_{i_j}\right)\right] P(F_n)$$

$$= P\left(\bigcap_{j=1}^{m} F_{i_j}\right) P(F_n)$$

$\Rightarrow$ events $\bigcap_{j=1}^{m} F_{i_j}$ and F_n are independent

$\Rightarrow$ events $\cap_{j=1}^{m} F_{i_j}$ and F_n^c are independent (Theorem 3.5 (ii))

$$\Rightarrow P\left(\left(\bigcap_{j=1}^{m} F_{i_j}\right) \cap F_n^c\right) = P\left(\bigcap_{j=1}^{m} F_{i_j}\right) P(F_n^c)$$

$$= \left[\prod_{j=1}^{m} P\left(F_{i_j}\right) \right] P(F_n^c)$$

$$\Rightarrow P\left(F_{i_1} \cap \cdots \cap F_{i_m} \cap G\right) = \left[\prod_{j=1}^{m} P\left(F_{i_j}\right) \right] P(G).$$

Case II. $G = F_j$, for some $j \in \{1, \dots, n-1\} - \{i_1, \dots, i_m\}$.

In this case $\left\{F_{i_1}, \dots, F_{i_m}, G\right\}$ is a sub collection of independent events $F_1, \dots, F_n$ and therefore

$$P\left(F_{i_1} \cap \cdots \cap F_{i_m} \cap G\right) = \left[\prod_{j=1}^{m} F_{i_j} \right] P(G).$$

Now the result follows on combining the two cases. ■

When we say that two or more random experiments are independent (or that two or more random experiments are performed independently) it simply means that the events associated with the respective random experiments are independent.

1.4 CONTINUITY OF PROBABILITY MEASURES

We begin this section with the following definition.

Definition 4.1

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{A_n : n = 1, 2, \dots\}$ be a sequence of events in $\mathcal{F}$.

(i) We say that the sequence $\{A_n : n = 1, 2, \dots\}$ is *increasing* (written as $A_n \uparrow$) if $A_n \subseteq A_{n+1}, n = 1,2, \dots$;

(ii) We say that the sequence $\{A_n : n = 1, 2, \dots\}$ is *decreasing* (written as $A_n \downarrow$) if $A_{n+1} \subseteq A_n, n = 1,2, \dots$;

(iii) We say that the sequence $\{A_n : n = 1, 2, \dots\}$ is *monotone* if either $A_n \uparrow$ or $A_n \downarrow$;

(iv) If $A_n \uparrow$ we define the limit of the sequence $\{A_n : n = 1, 2, \dots\}$ as $\bigcup_{n=1}^{\infty} A_n$ and write $\mathrm{Lim}_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$;

(v) If $A_n \downarrow$ we define the limit of the sequence $\{A_n : n = 1, 2, \dots\}$ as $\bigcap_{n=1}^{\infty} A_n$ and write $\mathrm{Lim}_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$. ■

Throughout we will denote the limit of a monotone sequence $\{A_n : n = 1, 2, \dots\}$ of events by $\mathrm{Lim}_{n\to\infty} A_n$ and the limit of a sequence $\{a_n : n = 1, 2, \dots\}$ of real numbers (provided it exists) by $\lim_{n\to\infty} a_n$.

1.4.1 Continuity of Probability Measures

Theorem 4.1

Let $\{A_n : n = 1, 2, \dots\}$ be a sequence of monotone events in a probability space $(\Omega, \mathcal{F}, P)$. Then

$$P\left(\operatorname*{Lim}_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

Proof.

Case I. $A_n \uparrow$

In this case, $\mathrm{Lim}_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$. Define $B_1 = A_1,\ B_n = A_n - A_{n-1}, n = 2, 3, \dots$.

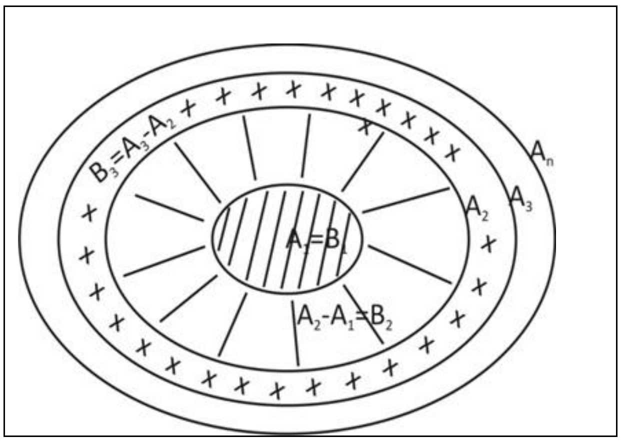

Figure 4.1

Then $B_n \in \mathcal{F}, n = 1, 2 \dots, B_n$s are mutually exclusive and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n = \mathrm{Lim}_{n\to\infty} A_n$. Therefore,

$$P\left(\operatorname*{Lim}_{n\to\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$= \sum_{n=1}^{\infty} P(B_n)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k)$$

$$= \lim_{n\to\infty} \left[P(A_1) + \sum_{k=2}^{n} P(A_k - A_{k-1}) \right]$$

$$= \lim_{n\to\infty} \left[P(A_1) + \sum_{k=2}^{n} \big(P(A_k) - P(A_{k-1}) \big) \right]$$

(using Theorem 2.1 (iv) since $A_{k-1} \subseteq A_k, k = 1, 2, \ldots$)

$$= \lim_{n\to\infty} \left[P(A_1) + \sum_{k=2}^{n} P\,(A_k) - \sum_{k=2}^{n} P\,(A_{k-1}) \right]$$

$$= \lim_{n\to\infty} [P(A_1) + P(A_n) - P(A_1)]$$

$$= \lim_{n\to\infty} P(A_n).$$

Case II. $A_n \downarrow$

In this case, $\operatorname{Lim}_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$ and $A_n^c \uparrow$. Therefore,

$$P\left(\operatorname{Lim}_{n\to\infty} A_n\right) = P\left(\bigcap_{n=1}^{\infty} A_n\right)$$

$$= 1 - P\left(\left(\bigcap_{n=1}^{\infty} A_n\right)^c\right)$$

$$= 1 - P\left(\bigcup_{n=1}^{\infty} A_n^c\right)$$

$$= 1 - P(\operatorname{Lim}_{n\to\infty} A_n^c)$$

$$= 1 - \lim_{n\to\infty} P(A_n^c) \qquad \text{(using Case I, since } A_n^c \uparrow)$$

$$= 1 - \lim_{n\to\infty} \big(1 - P(A_n)\big)$$

$$= \lim_{n\to\infty} P(A_n). \blacksquare$$

Remark 4.1

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{E_i: i = 1, 2, \ldots\}$ be a countably infinite collection of events in $\mathcal{F}$. Define

$$B_n = \bigcup_{i=1}^{n} E_i \text{ and } \quad C_n = \bigcap_{i=1}^{n} E_i \,, n = 1,2, \ldots$$

Then $B_n \uparrow, C_n \downarrow$, $\operatorname*{Lim}_{n\to\infty} B_n = \bigcup_{n=1}^{\infty} B_n = \bigcup_{i=1}^{\infty} E_i$ and $\operatorname*{Lim}_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n = \bigcap_{i=1}^{\infty} E_i$. Therefore

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = P\left(\operatorname*{Lim}_{n\to\infty} B_n\right)$$

$$= \lim_{n\to\infty} P(B_n) \qquad \text{(using Theorem 4.1)}$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} E_i\right)$$

$$= \lim_{n\to\infty} \left[S_{1,n} + S_{2,n} + \cdots + S_{n,n}\right],$$

where $S_{k,n}s$ are as defined in Theorem 2.2.

Moreover,

$$P\left(\bigcap_{i=1}^{\infty} E_i\right) = P\left(\operatorname*{Lim}_{n\to\infty} C_n\right)$$

$$= \lim_{n\to\infty} P(C_n) \qquad \text{(using Theorem 4.1)}$$

$$= \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} E_i\right).$$

Similarly, if $\{E_i:\ i = 1, 2, \cdots\}$ is a collection of independent events, then

$$P\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} E_i\right)$$

$$= \lim_{n\to\infty} \left[\prod_{i=1}^{n} P\,(E_i)\right]$$

$$= \prod_{i=1}^{\infty} P\,(E_i). \ \blacksquare$$