MODULE 7

LIMITING DISTRIBUTIONS

LECTURE 42

Topics

7.3 SOME PRESERVATION RESULTS

7.3.1 Normal Approximation to The Student-t Distribution

7.4 THE DELTA-METHOD

7.4.1 The Delta-Method

Theorem 3.3

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with finite mean μ. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$, $n = 2, 3, \ldots$, be sequences of sample means and sample variances, respectively. Define $T_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{S_n}$, $n = 2, 3, \ldots$.

(i) If $\sigma^2 = \mathrm{Var}(X_1) \in (0, \infty)$, then $S_n^2 \xrightarrow{p} \sigma^2$, $S_n \xrightarrow{p} \sigma$ and $T_n \xrightarrow{d} Z \sim N(0,1)$, as $n \to \infty$;

(ii) Suppose that the kurtosis $\gamma_1 = \frac{E\left((X_1-\mu)^4\right)}{\sigma^4} < \infty$. Then $\sqrt{n}(S_n^2 - \sigma^2) \xrightarrow{d} W \sim N(0, (\gamma_1 - 1)\sigma^4)$, as $n \to \infty$.

Proof.

(i) We have

$$\begin{aligned} S_n^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \\ &= \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{n}{n-1}\bar{X}_n^2 \\ &= \frac{n}{n-1}\cdot\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \frac{n}{n-1}\bar{X}_n^2, \qquad n = 2, 3, \ldots. \end{aligned}$$

Let $Y_i = X_i^2$, $i = 1, 2, \ldots$ and let $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$, $n = 2, 3, \ldots$. Then

$$S_n^2 = \frac{n}{n-1}(\bar{Y}_n - \bar{X}_n^2),$$

where $Y_1, Y_2, \ldots$ is a sequence of i.i.d. random variables with mean $E(Y_1) = E(X_1^2) = \sigma^2 + \mu^2$. By the WLLN

$$\bar{Y}_n \xrightarrow{p} \sigma^2 + \mu^2, \qquad \text{as } n \to \infty$$

and

$$\bar{X}_n \xrightarrow{p} \mu, \qquad \text{as } n \to \infty.$$

Using the continuity of function $h(x) = x^2,\ x \in \mathbb{R}$, and Theorem 3.1 (i) we have $\overline{X}_n^2 \xrightarrow{p} \mu^2,$ as $n \to \infty$. Since $\frac{n}{n-1} \to 1$, on using Theorem 3.2 (i) and (iii) we get

$$S_n^2 = \frac{n}{n-1}\left(\overline{Y}_n - \overline{X}_n^2\right) \xrightarrow{p} \sigma^2, \qquad \text{as } n \to \infty.$$

Since $f(x) = \sqrt{x},\ x \in (0, \infty)$, is a continuous function, it follows that $S_n \xrightarrow{p} \sigma,$ as $n \to \infty$, and therefore $\frac{\sigma}{S_n} \xrightarrow{p} 1$, as $n \to \infty$. Using the CLT we have

$$Z_n \stackrel{\text{def}}{=} \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1), \qquad \text{as } n \to \infty$$

$$\Rightarrow T_n = \frac{\sigma}{S_n} Z_n \xrightarrow{d} Z \sim N(0,1), \text{ as } n \to \infty, \qquad \text{(using Theorem 3.2 (iv)).}$$

(ii) Let $T_i = \frac{X_i - \mu}{\sigma}, i = 1, \ldots, n$, so that $T_1, T_2, \ldots$ are i.i.d. random variables with mean 0 and variance 1. Moreover $X_i = \mu + \sigma T_i,\ i = 1, 2, \ldots, \bar{X}_n = \mu + \sigma \bar{T}_n,\ \bar{T}_n = \frac{1}{n}\sum_{i=1}^{n} T_i$ and

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

$$= \frac{\sigma^2}{n-1}\sum_{i=1}^{n}(T_i - \bar{T}_n)^2$$

$$= \frac{n}{n-1}\sigma^2\left[\frac{1}{n}\sum_{i=1}^{n} T_i^2 - \bar{T}_n^2\right]$$

$$= \frac{n}{n-1}\sigma^2\left[\frac{1}{n}\sum_{i=1}^{n} Y_i - \bar{T}_n^2\right]$$

$$= \frac{n}{n-1}\sigma^2[\bar{Y}_n - \bar{T}_n^2],$$

where $Y_i = T_i^2, i = 1, 2, \ldots$ and $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i, n = 2, 3, \ldots$. Then $Y_1, Y_2, \ldots$ are i.i.d. random variables with mean $E(Y_1) = E(T_1^2) = 1$ and $\mathrm{Var}(Y_1) = E(T_1^4) - \left(E(T_1^2)\right)^2 = \gamma_1 - 1$. By the CLT

$$U_n \stackrel{\text{def}}{=} \frac{\sqrt{n}(\bar{Y}_n - 1)}{\sqrt{\gamma_1 - 1}} \xrightarrow{d} U \sim N(0,1), \qquad \text{as } n \to \infty$$

and $$V_n = \sqrt{n}\bar{T}_n \xrightarrow{d} V \sim N(0,1), \qquad \text{as } n \to \infty.$$

Also,

$$\sqrt{n}(S_n^2 - \sigma^2) = \frac{n}{n-1}\sigma^2\sqrt{\gamma_1 - 1}\, U_n + \frac{\sqrt{n}}{n-1}\sigma^2 - \frac{\sqrt{n}}{n-1}\,\sigma^2 V_n^2, \;\; n = 2, 3, \ldots.$$

Using continuity of function $h(x) = x^2,\ x \in (0, \infty)$, and Theorem 3.1 (iii) we have $V_n^2 \xrightarrow{d} V^2$, as $n \to \infty$. Since, as $n \to \infty$, $\frac{n}{n-1}\sigma^2\sqrt{\gamma_1 - 1} \to \sigma^2\sqrt{\gamma_1 - 1}$ and $\frac{\sqrt{n}}{n-1}\sigma^2 \to 0$, using Theorem 3.2, we conclude that

$$\sqrt{n}(S_n^2 - \sigma^2) \xrightarrow{d} W \sim N(0, (\gamma_1 - 1)\sigma^4), \qquad \text{as } n \to \infty,$$

where $W = \sigma^2\sqrt{\gamma_1 - 1}U \sim N(0, (\gamma_1 - 1)\sigma^4)$. ■

7.3.1 Normal Approximation to the Student-t Distribution

Corollary 3.1

Let $\{T_n\}_{n\geq 1}$ be a sequence of random variables such that $T_n \sim t_n$, the Student-t distribution with n degrees of freedom. Then $T_n \xrightarrow{d} Z \sim N(0,1)$, as $n \to \infty$.

Proof. Let $Z_1, Z_2, \ldots$ be a sequence of i.i.d. $N(0,1)$ random variables. Let $\bar{Z}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$ and $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2, n = 2, 3, \ldots$. Define

$$V_n = \frac{\sqrt{n}\bar{Z}_n}{S_n}, \qquad n = 2, 3, \ldots.$$

By Corollary 11.1, Module 6, $V_n \stackrel{d}{=} T_{n-1}, n = 2, 3, \ldots$. By Theorem 3.3 (i) we have

$$V_n \xrightarrow{d} Z \sim N(0,1), \text{as } n \to \infty$$

$$\Rightarrow T_{n-1} \xrightarrow{d} Z \sim N(0,1), \text{as } n \to \infty$$

$$\Rightarrow T_n \xrightarrow{d} Z \sim N(0,1), \text{as } n \to \infty. \blacksquare$$

7.4 THE DELTA-METHOD

Generally we have a sequence $\{X_n\}_{n\geq 1}$ of random variables such that, for real constants c and $b > 0, X_n \xrightarrow{p} c$, and $n^b(X_n - c) \xrightarrow{d} X, \text{as } n \to \infty$, where X is some random variable. Then, for any continuous function $g(\cdot)$, we know that $g(X_n) \xrightarrow{p} g(c), \text{as } n \to \infty$. The Delta-method is a tool for providing a non-degenerate limiting distribution to a normalized version of $g(X_n), n = 1, 2, \ldots$.

Theorem 4.1

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that, for some real constants $b > 0$ and c and some random variable $X, n^b(X_n - c) \xrightarrow{d} X, \text{as } n \to \infty$. Let $g: \mathbb{R} \to \mathbb{R}$ be a function that is differentiable at c. Then

$$n^b\left(g(X_n) - g(c)\right) \xrightarrow{d} g^{(1)}(c)X, \text{as } n \to \infty,$$

where $g^{(1)}(c)$ is the derivative of $g(\cdot)$ at the point c.

Proof. Let $\Psi_1: \mathbb{R} \to \mathbb{R}$ be such that $\Psi_1(c) = 0$ and

$$g(x) = g(c) + (x - c)\left(g^{(1)}(c) + \Psi_1(x)\right), x \in \mathbb{R},$$

i.e.,

$$\Psi_1(x) = \begin{cases} \dfrac{g(x) - g(c)}{x - c} - g^{(1)}(c), & \text{if } x \in \mathbb{R} - \{c\} \\ 0, & \text{if } x = c \end{cases}.$$

Then $\lim_{x\to c} \Psi_1(x) = g^{(1)}(c) - g^{(1)}(c) = 0 = \Psi_1(c)$ (i.e., $\Psi_1(\cdot)$ is continuous at c) and

$$n^b\left(g(X_n) - g(c)\right) = g^{(1)}(c)n^b(X_n - c) + \Psi_1(X_n)n^b(X_n - c), n = 1, 2, \ldots.$$

By Theorem 3.2 (iv),

$$X_n = n^{-b}\left(n^b(X_n - c)\right) + c \xrightarrow{d} 0 \times X + c, \text{as } n \to \infty$$

$$\Rightarrow X_n \xrightarrow{p} c, \text{as } n \to \infty$$

$$\Rightarrow \Psi_1(X_n) \xrightarrow{p} \Psi_1(c) = 0, \text{as } n \to \infty \qquad (\text{since } \Psi_1 \text{ is continuous at } c)$$

$$\Rightarrow \Psi_1(X_n)n^b(X_n - c) \xrightarrow{p} 0, \text{ as } n \to \infty \quad \text{(Theorem 3.2 (ii))}$$
$$\Rightarrow n^b\big(g(X_n) - g(c)\big) = g^{(1)}(c)n^b(X_n - c) + \Psi_1(X_n)n^b(X_n - c)$$
$$\xrightarrow{d} g^{(1)}(c)X, \text{as } n \to \infty \qquad \text{(Theorem 3.2).} \blacksquare$$

Remark 4.1

Note that, in the above theorem, if we have $g^{(1)}(c) = 0$ then we conclude that

$$n^b\big(g(X_n) - g(c)\big) \xrightarrow{d} 0, \text{as } n \to \infty$$

i.e., $$n^b\big(g(X_n) - g(c)\big) \xrightarrow{p} 0, \text{as } n \to \infty,$$

and we get a degenerate limiting distribution. Now suppose that $g^{(1)}(c) = 0$ and $g(\cdot)$ is twice differentiable at c with second derivatives at the point c given by $g^{(2)}(c)$. Define $\Psi_2: \mathbb{R} \to \mathbb{R}$ by

$$\Psi_2(x) = \begin{cases} \dfrac{g(x) - g(c)}{(x-c)^2/2} - g^{(2)}(c), & \text{if } x \neq c \\ 0, & \text{if } x = c \end{cases}.$$

The, using L' Hospital rule ($0/0$ form), we have

$$\lim_{x\to c} \Psi_2(x) = \lim_{x\to c} \frac{g^{(1)}(x)}{x - c} - g^{(2)}(c)$$
$$= \lim_{x\to c} \frac{g^{(1)}(x) - g^{(1)}(c)}{x - c} - g^{(2)}(c) \ \ (\text{since } g^{(1)}(c) = 0)$$
$$= g^{(2)}(c) - g^{(2)}(c)$$
$$= 0$$
$$= \Psi_2(c),$$

i.e., $\Psi_2(\cdot)$ is continuous at point c. Consequently, using Theorem 3.2,

$$n^{2b}\big(g(X_n) - g(c)\big) = \frac{g^{(2)}(c)}{2}\big(n^b(X_n - c)\big)^2 + \frac{\big(n^b(X_n - c)\big)^2}{2}\Psi_2(X_n)$$
$$\xrightarrow{d} \frac{g^{(2)}(c)}{2}X^2,$$

since $\Psi_2(X_n) \xrightarrow{p} \Psi_2(c) = 0$ (as Ψ_2 is continuous at c and $X_n \xrightarrow{p} c$, as $n \to \infty$) and $\left(n^b(X_n - c)\right)^2 \xrightarrow{d} X^2$ (as $h(x) = x^2$ is a continuous function on $\mathbb{R}$ and $n^b(X_n - c) \xrightarrow{d} X$, as $n \to \infty$). ■

The following example demonstrates that the conclusion of Theorem 4.1 (The Delta-Method) may not hold if $b = 0$.

Example 4.1

Let $\{Z_n\}_{n\geq 1}$ be a sequence of random variables such that $Z_n \sim N(0,1), n = 1, 2, \ldots$. Then $n^0(Z_n - 0) = Z_n \xrightarrow{d} Z \sim N(0,1)$, as $n \to \infty$. Let $g(x) = x^2$, $x \in \mathbb{R}$. Then

$$n^0\big(g(Z_n) - g(0)\big) = Z_n^2 \xrightarrow{d} Z_1^2 \sim \chi_1^2, \text{as } n \to \infty.$$

However $g^{(1)}(0)Z = 0 \times Z = 0$. ■

Corollary 4.1

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, each having the mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in (0, \infty)$. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$ and let $g: \mathbb{R} \to \mathbb{R}$ be a function that is differentiable at μ. Then

$$\sqrt{n}\big(g(\bar{X}_n) - g(\mu)\big) \xrightarrow{d} W \sim N\left(0, \left(g^{(1)}(\mu)\right)^2 \sigma^2\right), \text{as } n \to \infty,$$

provided $g^{(1)}(\mu) \neq 0$. If $g^{(1)}(\mu) = 0$ then

$$\sqrt{n}\big(g(\bar{X}_n) - g(\mu)\big) \xrightarrow{p} 0, \text{as } n \to \infty \cdot$$

Proof. Let $Z \sim N(0,1)$ and let $V = \sigma Z$. Then by the CLT

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1), \text{as } n \to \infty$$

$$\Rightarrow \sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} \sigma Z = V \sim N(0, \sigma^2), \text{as } n \to \infty$$
$$\Rightarrow \sqrt{n}\big(g(\bar{X}_n) - g(\mu)\big) \xrightarrow{d} g^{(1)}(\mu)V, \text{as } n \to \infty \cdot$$

If $g^{(1)}(\mu) \neq 0$, then $W = g^{(1)}(\mu)V \sim N\big(0, (g^{(1)}(\mu))^2\sigma^2\big)$. However if $g^{(1)}(\mu) = 0$, then the random variable $g^{(1)}(\mu)V$ is degenerate at 0. Hence the result follows. ■

Example 4.2

Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables such that $X_n \sim \chi_n^2, n = 1, 2, ...$ Show that

$$\sqrt{2}\left(\sqrt{X_n} - \sqrt{n}\right) \xrightarrow{d} Z \sim N(0,1), \text{as } n \to \infty.$$

Solution. Let $Y_1, Y_2, ...$ be a sequence of i.i.d. χ_1^2 random variables.Then $E(Y_1) = 1, \text{Var}(Y_1) = 2$ and $X_n \stackrel{d}{=} \sum_{i=1}^n Y_i = n\overline{Y}_n$, $n = 1, 2, ...$ (see Example 7.6 (i), Module 6). By the CLT

$$\frac{\sqrt{n}(\bar{Y}_n - 1)}{\sqrt{2}} \xrightarrow{d} Z \sim N(0,1), \text{as } n \to \infty$$

$$\Rightarrow \sqrt{n}(\bar{Y}_n - 1) \xrightarrow{d} \sqrt{2}Z \sim N(0, 2), \text{as } n \to \infty \cdot$$

Since $g(x) = \sqrt{x}, x \in (0, \infty)$ is differentiable at $x = 1$, using the delta-method, we have

$$\sqrt{n}\left(\sqrt{\bar{Y}_n} - 1\right) \xrightarrow{d} \frac{1}{2} \times \sqrt{2}Z = \frac{Z}{\sqrt{2}} \sim N\left(0, \frac{1}{2}\right), \text{as } n \to \infty.$$

$$\Rightarrow \sqrt{2}\left(\sqrt{X_n} - \sqrt{n}\right) \xrightarrow{d} Z \sim N(0,1), \text{as } n \to \infty. \blacksquare$$