MODULE 6

RANDOM VECTOR AND ITS JOINT DISTRIBUTION

LECTURE 32

Topics

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6.10 DISTRIBUTION OF FUNCTION OF RANDOM VECTORS

Let $\underline{X} = (X_1, \ldots, X_p)$ be a random vector of either discrete type or of absolutely continuous type and let $f_{\underline{X}}(\cdot)$ denote the p.m.f./p.d.f. of $\underline{X}$. Let $g: \mathbb{R}^p \to \mathbb{R}$ be a Borel function. As the following example illustrates, in many situations, it may be of interest to find the probability distribution of $g(\underline{X})$.

Example 10.1

Consider a company that manufactures electric bulbs. The lifetimes of electric bulbs manufactured by the company are random. Past experience with testing on electric bulbs manufactured by the company suggests that the lifetime of a randomly chosen electric bulb manufactured by the company can be described by a random variable X having the p.d.f.

$$f_X(x|\theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}, \theta > 0.$$

However the value of θ (> 0) is not evident from the past experience and therefore θ is unknown. One way to obtain information about unknown θ is to do testing independently and under identical conditions, on a number (say n) of electric bulbs manufactured by the company. Let X_i denote the lifetime of the i-th bulb, $i = 1, \dots, n$. We call $X_1, \dots, X_n$ (which are independent and identically distributed random variables from the distribution $f_X(\cdot\,|\theta), \theta > 0$) the random sample from distribution $f_X(\cdot\,|\theta), \theta > 0$. Clearly the joint p.d.f. of $\underline{X} = (X_1, \dots, X_n)$ is given by

$$f_{\underline{X}}(\underline{x}|\theta) = \prod_{i=1}^{n} f_{X_i}(x_i|\theta) = \begin{cases} \dfrac{1}{\theta^n} e^{-\frac{\sum_{i=1}^{n} x_i}{\theta}}, & \text{if } x_i > 0,\ i = 1, \dots, n \\ 0, & \text{otherwise} \end{cases}.$$

Since $E(X) = \theta$, a natural estimator of θ is the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.To study theoretical properties of the estimator $\bar{X}$ we need probability distribution of $\bar{X}$. ▄

Definition 10.1

(i) A function of one or more random variables that does not depend on any unknown parameter is called a *statistic*.

(ii) Let $X_1, \dots, X_n$ be a collection of independent random variables each having the same p.m.f./p.d.f. f(or distribution function F). We then call $X_1, \dots, X_n$ a *random sample* (of size n) from a distribution having p.m.f./p.d.f f (or distribution function F). In other words a random sample is a collection of independent and identically distributed random variables.

Remark 10.1

(i) Let $\underline{X} \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho), -\infty < \mu_i < \infty, \sigma_i, i = 1, 2, -1 < \rho < 1$. Then the random variable $Y_1 = X_1 + X_2$ is a statistic but the random variable $Y_2 = \frac{X_1 - \mu_1}{\sigma_1}$ is not a statistic unless μ_1 and σ_1 are known parameters.

(ii) Although a statistic does not depend upon any unknown parameters, the distribution of a statistic may very well depend upon unknown parameters. For example, in (i) above, $Y_1 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$.

(iii) If $X_1, \dots, X_n$ is a random sample from a distribution having p.m.f./p.d..f. $f(\cdot)$, then the joint p.m.f./p.d.f. of $\underline{X} = (X_1, \dots, X_n)$ is

$$f_{\underline{X}}(x_1, \dots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

$$= \prod_{i=1}^{n} f(x_i), \quad \underline{x} = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

(iv) Let $X_1, \dots, X_n$ be a random sample from a distribution. Some of the commonly used statistics are

(a) Sample Mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$;

(b) Sample Variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right], n \geq 2$;

(c) $\mathrm{r-th}$ Order Statistics $X_{r:n} = \mathrm{r-th}$ smallest of $(X_1, \dots, X_n), r = 1,2, \dots, n$;

(d) Sample Range $R = X_{n:n} - X_{1:n}$;

(e) Median $M = \begin{cases} X_{\frac{n+1}{2}:n}, & \text{if } n \text{ is odd} \\ \dfrac{X_{\frac{n}{2}:n} + X_{\frac{n}{2}+1:n}}{n}, & \text{if } n \text{ is even} \end{cases}$.

Theorem 10.1

Let $X_1, \dots, X_n$ be a random sample from a distribution having p.m.f./p.d.f. $f(\cdot)$. Then, for any permutation $(\beta_1, \dots, \beta_n)$ of $(1, \dots, n)$,

$$(X_1, \dots, X_n) \stackrel{d}{=} \left(X_{\beta_1}, \dots, X_{\beta_n}\right).$$

Proof. Let $(\beta_1, \dots, \beta_n)$ be a permutation of $(1, \dots, n)$ and let $(\gamma_1, \dots, \gamma_n)$ be the inverse permutation of $(\beta_1, \dots, \beta_n)$. Then, for $\underline{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$f_{X_{\beta_1}, \dots, X_{\beta_n}}(x_1, \dots, x_n) = f_{X_1, \dots, X_n}\left(x_{\gamma_1}, \dots, x_{\gamma_n}\right)$$

$$= \prod_{i=1}^{n} f_{X_i}\left(x_{\gamma_i}\right)$$

$$= \prod_{i=1}^{n} f\left(x_i\right)$$

$$= f_{X_1, \dots, X_n}(x_1, \dots, x_n).$$

It follows that

$$f_{X_{\beta_1}, \dots, X_{\beta_n}}\left(\underline{x}\right) = f_{X_1, \dots, X_n}\left(\underline{x}\right), \quad \forall \underline{x} \in \mathbb{R}^n$$

$$\Rightarrow \left(X_{\beta_1}, \dots, X_{\beta_n}\right) \stackrel{d}{=} (X_1, \dots, X_n). \blacksquare$$

Example10.2

Let $X_1, \dots, X_n$ be a random sample from a given distribution.

(i) If X_1 is of absolutely continuous type then show that $P(\{X_1 < X_2 < \cdots < X_n\}) = P\left(\left\{X_{\beta_1} < X_{\beta_2} < \cdots < X_{\beta_n}\right\}\right) = \frac{1}{n!}$, for any permutation $(\beta_1, \dots, \beta_n)$ of $(1, \dots, n)$;

(ii) If X_1 is of absolutely continuous type then show that $P(\{X_i = X_{r:n}\}) = \frac{1}{n}, i = 1, \dots, n$, where, for $r \in \{1, \dots, n\}, X_{r:n} = r$-th smallest of $\{X_1, \dots, X_n\}$;

(iii) Show that

$$E\left(\frac{X_i}{X_1 + X_2 + \cdots + X_n}\right) = \frac{1}{n}, i = 1,2, \dots, n,$$

provided the expectations are finite;

(iv) Show that

$$E\left(X_i \middle| \sum_{i=1}^{n} X_i = t\right) = \frac{t}{n}, i = 1, \dots, n.$$

Solution. Let S_n denote the set of all permutations of $(1, \dots, n)$. Using Theorem 10.1 we have

$$(X_1, \dots, X_n) \stackrel{d}{=} \left(X_{\beta_1}, \dots, X_{\beta_n}\right), \forall \underline{\beta} = (\beta_1, \dots, \beta_n) \in S_n.$$

$$\Rightarrow E\big(\Psi(X_1, \dots, X_n)\big) = E\left(\Psi\left(X_{\beta_1}, \dots, X_{\beta_n}\right)\right), \forall \underline{\beta} \in S_n \qquad (10.1)$$

(i) On taking

$$\Psi(x_1, \dots, x_n) = \begin{cases} 1, & \text{if } x_1 < x_2 < \cdots < x_n \\ 0, & \text{otherwise} \end{cases}.$$

We conclude that

$$P(\{X_1 < X_2 < \cdots < X_n\}) = P\left(\left\{X_{\beta_1} < X_{\beta_2} < \cdots < X_{\beta_n}\right\}\right), \forall \underline{\beta} \in S_n. \qquad (10.2)$$

Since $P\big(\{X_i = X_j\}\big) = 0$ for $i \neq j$ (as (X_i, X_j) is of absolutely continuous type; see Remark 2.1 (x)), we have

$$\sum_{\underline{\beta} \in S_n} P\left(\left\{X_{\beta_1} < X_{\beta_2} < \cdots < X_{\beta_n}\right\}\right) = 1$$

$$\Rightarrow P(\{X_1 < X_2 < \cdots < X_n\}) = P\left(\left\{X_{\beta_1} < X_{\beta_2} < \cdots < X_{\beta_n}\right\}\right) = \frac{1}{n!}. \text{ (using (10.2))}$$

(ii) Fix $i \in \{1, 2, \dots, n\}$. On taking

$$\Psi(x_1, \cdots, x_n) = \begin{cases} 1, & \text{if } x_i = r-\text{th smallest of } \{x_1, \dots, x_n\} \\ 0, & \text{otherwise} \end{cases},$$

in (10.1) and noting that, for any permutation $\underline{\beta} = (\beta_1, \dots, \beta_n) \in S_n$, r-th smallest of $\left\{X_{\beta_1}, \dots, X_{\beta_n}\right\} = r$-th smallest of $\{X_1, \dots, X_n\} = X_{r:n}$,we have

$$P(\{X_i = X_{r:n}\}) = P\left(\left\{X_{\beta_i} = X_{r:n}\right\}\right) \forall\, \underline{\beta} \in S_n$$

$$\Rightarrow P(\{X_i = X_{r:n}\}) = P(\{X_1 = X_{r:n}\}) i = 1, \dots, n.$$

But

$$\sum_{i=1}^{n} P(\{X_i = X_{r:n}\}) = 1,$$

and therefore

$$P(\{X_i = X_{r:n}\}) = P(\{X_1 = X_{r:n}\}) = \frac{1}{n} \cdot$$

(iii) On taking

$$\Psi(x_1, \dots, x_n) = \frac{x_1}{x_1 + \cdots + x_n}, \quad \underline{x} \in \mathbb{R}^n,$$

in (10.1) we get, for all $\underline{\beta} = (\beta_1, \dots, \beta_n) \in S_n$,

$$E\left(\frac{X_1}{X_1 + \cdots + X_n}\right) = E\left(\frac{X_{\beta_1}}{X_{\beta_1} + \cdots + X_{\beta_n}}\right)$$

$$= E\left(\frac{X_{\beta_1}}{X_1 + \cdots + X_n}\right)\left(\text{since} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} X_{\beta_i}\right)$$

$$\Rightarrow E\left(\frac{X_1}{X_1 + \cdots + X_n}\right) = E\left(\frac{X_i}{X_1 + \cdots + X_n}\right), i = 1, \dots, n. \qquad (10.3)$$

But

$$\sum_{i=1}^{n} E\left(\frac{X_i}{X_1 + \cdots + X_n}\right) = E\left(\sum_{i=1}^{n} \frac{X_i}{X_1 + \cdots + X_n}\right)$$

$$= E\left(\frac{X_1 + \cdots + X_n}{X_1 + \cdots + X_n}\right)$$

$$= 1.$$

Therefore, from (10.3), we get

$$E\left(\frac{X_1}{X_1+\cdots+X_n}\right) = E\left(\frac{X_i}{X_1+\cdots+X_n}\right) = \frac{1}{n}, \quad i = 1, \dots, n.$$

(iv) For fixed t

$$X_1 \left| \left(\sum_{j=1}^{n} X_j = t\right) \stackrel{d}{=} X_{\beta_1} \right| \left(\sum_{j=1}^{n} X_{\beta_j} = t\right) \forall \underline{\beta} \in S_n$$

$$\Rightarrow X_1 \left| \left(\sum_{j=1}^{n} X_j = t\right) \stackrel{d}{=} X_{\beta_1} \right| \left(\sum_{j=1}^{n} X_j = t\right), \quad \forall \underline{\beta} \in S_n \left(\text{since} \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} x_{\beta_j}\right)$$

$$\Rightarrow E\left(X_1 \left| \sum_{j=1}^{n} X_j = t\right.\right) = E\left(X_{\beta_1} \left| \sum_{j=1}^{n} X_j = t\right.\right), \forall \underline{\beta} \in S_n$$

$$\Rightarrow E\left(X_1 \left| \sum_{j=1}^{n} X_j = t\right.\right) = E\left(X_i \left| \sum_{j=1}^{n} X_j = t\right.\right), i = 1, \dots, n. \tag{10.4}$$

But

$$\sum_{i=1}^{n} E\left(X_i \left| \sum_{j=1}^{n} X_j = t\right.\right) = E\left(\sum_{i=1}^{n} X_i \left| \sum_{j=1}^{n} X_j = t\right.\right) = t.$$

Now using (10.4) we get

$$E\left(X_1 \left| \sum_{j=1}^{n} X_j = t\right.\right) = E\left(X_i \left| \sum_{j=1}^{n} X_j = t\right.\right) = \frac{t}{n}, \quad i = 1, \dots, n. \blacksquare$$

In the following subsections we will discuss various techniques to find the distribution of functions of random variables.

6.10.1 Distribution Function Technique

Let $\underline{X} = (X_1, \dots, X_p)$ be a random vector and let $g: \mathbb{R}^p \to \mathbb{R}$ be a Borel function. The distribution of $Y = g(X_1, \dots, X_p)$ can be determined by computing the distribution function

$$F_Y(y) = P\left(\left\{g\left(X_1, \dots, X_p\right) \le y\right\}\right), -\infty < y < \infty.$$

6.10.1.1 Marginal Distribution of Order Statistics of a Random Sample of Absolutely Continuous Type Random Variables

Example 10.1.1

Let $X_1, \dots, X_n$ be a random sample of absolutely continuous type random variables, each having the distribution function $F(\cdot)$ and p.d.f. $f(\cdot)$. Suppose that F is differential everywhere except (possibly) on a countable set C, so that $f(x) = \frac{d}{dx}F(x), \forall x \in C$ and

$$\int_{-\infty}^{\infty} f(x) I_{C^c}(x) dx = 1.$$

Then the joint distribution function of $\underline{X} = (X_1, \dots, X_n)$ is

$$F_{\underline{X}}(x_1, \dots, x_n) = \prod_{i=1}^{n} F(x_i), \quad \underline{x} \in \mathbb{R}^n.$$

We have

$$\begin{aligned} F_{\underline{X}}\left(\underline{x}\right) &= \prod_{i=1}^{n} F(x_i) \\ &= \prod_{i=1}^{n} \int_{-\infty}^{x_i} f(t_i) dt_i \\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \prod_{i=1}^{n} f(t_i)\, dt_n \cdots dt_1 \\ &= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{\underline{X}}\left(\underline{t}\right) dt_n \cdots dt_1. \end{aligned}$$

It follows that $\underline{X}$ is of absolutely continuous type with joint p.d.f. $f_{\underline{X}}(\cdot)$. Therefore, for $i \ne j$,

$$P\left(\left\{X_i = X_j\right\}\right) = 0.$$

Define

$$X_{r:n} = r - \text{th smallest of } \{X_1, \dots, X_n\}, \quad r = 1, \dots, n,$$

so that

$$P(\{X_{1:n} < X_{2:n} < \cdots < X_{n:n}\}) = 1.$$

First let us derive the distribution of $X_{r:n}, r = 1, \cdots, n$. Note that, for $x \in \mathbb{R}$,

$$X_{r:n} \le x \Leftrightarrow \text{at least } r \text{ of } \{X_1, \dots, X_n\} \text{ are } \le x.$$

Therefore

$$\begin{aligned} F_{X_{r:n}}(x) &= P(\{X_{r:n} \le x\}) \\ &= P(\{\text{at least } r \text{ of } X_1, \dots, X_n \text{ are} \le x\}) \\ &= \sum_{i=r}^{n} P\left(\{i \text{ of } X_1, \dots, X_n \text{ are} \le x\}\right), \quad x \in \mathbb{R}\,. \end{aligned}$$

Fix $x \in \mathbb{R}$, and consider a sequence of n trials where at the i-th trial we observe X_i and consider the trial having resulted in a success if $X_i \le x$ and it having resulted in a failure if $X_i > x, i = 1, \dots, n$. Since $X_1, \dots, X_n$ are independent and the probability of success in the i-th trial is $P(\{X_i \le x\}) = F(x)$ (same for all the trials), the above sequence of trials may be considered as a sequence of independent Bernoulli trials with probability of success in each trial as $F(x)$. Therefore

$$\begin{aligned} P(\{i \text{ of } X_1, \dots, X_n \le x\}) &= P(\{i \text{ successes in } n \text{ trials}\}) \\ &= \binom{n}{i} \left(F(x)\right)^i \left(1 - F(x)\right)^{n-i}, \end{aligned}$$

and consequently

$$F_{X_{r:n}}(x) = \sum_{i=r}^{n} \binom{n}{i} \left(F(x)\right)^i \left(1 - F(x)\right)^{n-i}, \quad x \in \mathbb{R}.$$

Recall that for $s \in \{0, 1, \dots, n\}$ and $p \in (0, 1)$ (see Theorem 3.1, Module 5)

$$\sum_{j=s}^{n} \binom{n}{j} p^j (1-p)^{n-j} = \frac{1}{B(s, n-s+1)} \int_0^p t^{s-1} (1-t)^{n-s} dt.$$

Therefore,

$$F_{X_{r:n}}(x) = \frac{1}{B(r, n-r+1)} \int_0^{F(x)} t^{r-1} (1-t)^{n-r} dt, \quad x \in \mathbb{R}.$$

Let

$$f_{X_{r:n}}(x) = \frac{1}{B(r, n-r+1)} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x), \; x \in \mathbb{R}, \qquad (10.1.1)$$

so that

$$\frac{d}{dx} F_{X_{r:n}}(x) = f_{X_{r:n}}(x), \; \forall x \notin C^c,$$

and

$$\int_{-\infty}^{\infty} f_{X_{r:n}}(x) I_{C^c}(x) dx = \int_{-\infty}^{\infty} f_{X_{r:n}}(x) I_{C^c}(x) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{B(r, n-r+1)} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x)\, I_{C^c}(x) dx$$

$$= \frac{1}{B(r, n-r+1)} \int_0^1 t^{r-1} (1-t)^{n-r} dt$$

$$= 1.$$

Using Remark 4.2 (vii), Module 2, it follows that the random variable $X_{r:n}$ is of absolutely continuous type with p.d.f. given by (10.1.1). A simple heuristic argument for expression (10.1.1) is as follows. Interpret $f_{X_{r:n}}(x)\Delta x$ as the probability that $X_{r:n}$ lies in an infinitesimal interval $[x, x+\Delta x]$. Realizing that the probability of more than one $X_i's$ falling in the infinitesimal interval $[x, x+\Delta x]$ may be negligible, $f_{X_{r:n}}(x)\Delta x$ may be interpreted as probability that one of the $X_i's$ falls in the infinitesimal interval $[x, x+\Delta x]$, $(r-1)X_i's$ fall in the interval $(-\infty, x]$ and $(n-r)X_i's$ fall in the interval $(x+\Delta x, \infty) \simeq (x, \infty)$. Since $X_1, \ldots, X_n$ are independent and the probabilities of an observation falling in intervals $[x, x+\Delta x]$, $(-\infty, x]$ and (x, ∞) are $f(x)\Delta x, F(x)$ and $1-F(x)$ respectively, $f_{X_{r:n}}(x)\Delta x$ is given by the multinomial probability

$$f_{X_{r:n}}(x)\Delta x \equiv \frac{n!}{1!\,(r-1)!\,(n-r)!} (f(x)\Delta x)^1 \big(F(x)\big)^{r-1} \big(1-F(x)\big)^{n-r},$$

i.e.,

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), -\infty < x < \infty.$$

Now we will derive the joint distribution of $(X_{r:n}, X_{s:n})$, where r and s are fixed positive integers satisfying $1 \le r < s \le n$. For $-\infty < x < y < \infty$,

$$F_{X_{r:n},X_{s:n}}(x,y)$$

$$= P(\{X_{r:n} \le x, X_{s:n} \le y\})$$

$$= P(\{\text{at least } r \text{ of } \{X_1,\dots,X_n\} \text{ are} \le x \text{ and at least } s \text{ of } X_1,\dots,X_n \text{ are} \le y\})$$

$$= \sum_{\substack{i=0 \\ r\le i\le n \\ s\le i+j\le n}}^{n} \sum_{j=0}^{n} P(\{i \text{ of } \{X_1,\dots,X_n\} \text{ are in } (-\infty,x] \text{ and } j \text{ of } \{X_1,\dots,X_n\} \text{ are in } (x,y]\}).$$

Since $X_1,\dots,X_n$ are independent and probabilities of an observation falling in intervals $(-\infty,x]$, $(x,y]$ and (y,∞) are $F(x), F(y)-F(x)$ and $1-F(y)$ respectively, using property of multinomial distribution, we have, for $r \le i \le n, s \le i+j \le n$ and $-\infty < x < y < \infty$

$$P(\{i \text{ of } \{X_1,\dots,X_n\} \text{ are in } (-\infty,x] \text{ and } j \text{ of } \{X_1,\dots,X_n\} \text{ are in } (x,y]\})$$

$$= \frac{n!}{i!\,j!\,(n-i-j)!}[F(x)]^i[F(y)-F(x)]^j[1-F(y)]^{n-i-j}\ .$$

Therefore, for $-\infty < x < y < \infty$,

$$F_{X_{r:n},X_{s:n}}(x,y) = \sum_{\substack{i=0 \\ r\le i\le n \\ s\le i+j\le n}}^{n} \sum_{j=0}^{n} \frac{n!}{i!\,j!\,(n-i-j)!}[F(x)]^i[F(y)-F(x)]^j[1-F(y)]^{n-i-j}$$

$$= \sum_{i=r}^{n} \sum_{j=\max(0,s-i)}^{n-i} \frac{n!}{i!\,j!\,(n-i-j)!}[F(x)]^i[F(y)-F(x)]^j[1-F(y)]^{n-i-j}$$

$$= \sum_{i=r}^{s-1} \sum_{j=s-i}^{n-i} \frac{n!}{i!\ j!\,(n-i-j)!}[F(x)]^i[F(y)-F(x)]^j[1-F(y)]^{n-i-j}$$

$$+\sum_{i=s}^{n}\sum_{j=0}^{n-i}\frac{n!}{i!\;j!\,(n-i-j)!}[F(x)]^{i}[F(y)-F(x)]^{j}[1-F(y)]^{n-i-j}$$

$$=\sum_{i=r}^{s-1}\binom{n}{i}[F(x)]^{i}\left\{\sum_{j=s-i}^{n-i}\binom{n-i}{j}[F(y)-F(x)]^{j}[1-F(y)]^{n-i-j}\right\}$$

$$+\sum_{i=s}^{n}\binom{n}{i}[F(x)]^{i}\left\{\sum_{j=0}^{n-i}\binom{n-i}{j}[F(y)-F(x)]^{j}[1-F(y)]^{n-i-j}\right\}$$

$$=\sum_{i=r}^{s-1}\binom{n}{i}[F(x)]^{i}[1-F(x)]^{n-i}\left\{\sum_{j=s-i}^{n-i}\binom{n-i}{j}\left(\frac{F(y)-F(x)}{1-F(x)}\right)^{j}\left(1-\frac{F(y)-F(x)}{1-F(x)}\right)^{n-i-j}\right\}$$

$$+\sum_{i=s}^{n}\binom{n}{i}[F(x)]^{i}[1-F(x)]^{n-i}$$

$$=\sum_{i=r}^{s-1}\binom{n}{i}[F(x)]^{i}[1-F(x)]^{n-i}\left\{\int_{0}^{\frac{F(y)-F(x)}{1-F(x)}}\frac{1}{B(s-i,n-s+1)}t^{s-i-1}(1-t)^{n-s}dt\right\}$$

$$+\frac{1}{B(s,n-s+1)}\int_{0}^{F(x)}t^{s-1}\,(1-t)^{n-s}dt.\quad\text{(using Theorem 3.1, Module 5)}$$

Thus, for $-\infty<x<y<\infty, x\notin C, y\notin C$

$$\frac{\partial}{\partial y}F_{X_{r:n},X_{s:n}}(x,y)=\sum_{i=r}^{s-1}\binom{n}{i}[F(x)]^{i}[1-F(x)]^{n-i}\frac{(n-i)!}{(s-i-1)!\,(n-s)!}\left(\frac{F(y)-F(x)}{1-F(x)}\right)^{s-i-1}$$

$$\times\left[1-\frac{F(y)-F(x)}{1-F(x)}\right]^{n-s}\frac{f(y)}{[1-F(x)]}$$

$$=\frac{n!\left(1-F(y)\right)^{n-s}}{(s-1)!\,(n-s)!}f(y)\sum_{i=r}^{s-1}\binom{s-1}{i}[F(x)]^{i}[F(y)-F(x)]^{s-i-1}$$

$$=\frac{n!}{(s-1)!\,(n-s)!}\left(F(y)\right)^{s-1}\left(1-F(y)\right)^{n-s}f(y)$$

$$\times\sum_{i=r}^{s-1}\binom{s-1}{i}\left[\frac{F(x)}{F(y)}\right]^{i}\left[1-\frac{F(x)}{F(y)}\right]^{s-i-1}$$

$$= \frac{n!}{(s-1)!\,(n-s)!}\big(F(y)\big)^{s-1}\big(1-F(y)\big)^{n-s} f(y)$$

$$\times \frac{1}{B(r, s-r)} \int_0^{\frac{F(x)}{F(y)}} t^{r-1}(1-t)^{s-r-1}\, dt$$

$$\Rightarrow f_{X_{r:n},X_{s:n}}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X_{r:n},X_{s:n}}$$

$$= \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}\big(F(y)\big)^{s-1}\big(1-F(y)\big)^{n-s} f(y)$$

$$\times \left[\frac{F(x)}{F(y)}\right]^{r-1} \left[1-\frac{F(x)}{F(y)}\right]^{s-r-1} \frac{f(x)}{F(y)}$$

$$= \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}\big(F(x)\big)^{r-1}\big(F(y)-F(x)\big)^{s-r-1}$$

$$\times [1-F(y)]^{n-s} f(x) f(y), \qquad -\infty < x < y < \infty, \qquad x, y \in C^c.$$

Also, for $-\infty < y < x < \infty, x, y \in C^c$, and $1 \le r < s \le n$

$$\{X_{s:n} \le y\} \subseteq \{X_{r:n} \le x\}$$

and therefore

$$F_{X_{r:n},X_{s:n}}(x,y) = F_{X_{r:n}}(x)$$

$$\Rightarrow \frac{\partial^2}{\partial x \partial y} F_{X_{r:n},X_{s:n}}(x,y) = 0, -\infty < x < y < \infty, \qquad x, y \in C^c..$$

Let

$$f_{r,s}(x,y) = \begin{cases} \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(x) f(y), & \text{if} -\infty < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

(10.1.2)

so that

$$\frac{\partial^2}{\partial x \partial y} F_{X_{r:n},X_{s:n}}(x,y) = f_{r,s}(x,y) \forall (x,y) \in \mathbb{R}^2 - (C \times C) = D \text{ (say)}.$$

Clearly D^c is countable. It is easy to verify that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{r,s}(x,y)\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{r,s}(x,y)\,I_D(x,y)\,dxdy = 1.$$

Using Remark 2.1 (xiii), it follows that the random vector $(X_{r:n}, X_{s:n})$ is of absolutely continuous type with joint p.d.f. given by (10.1.2). One can give the following heuristic argument for the expression (10.1.2). For $-\infty < x < y < \infty$,

$f_{r,s}(x,y)\Delta x\Delta y =$ probability that $(r-1)\ X_i^{'}s$ fall in $(-\infty, x]$, one X_i falls in $(x, x+\Delta x]$, $(s-r-1)\ X_i^{'}s$ fall in $(x+\Delta x, y]$ $(\approx (x,y])$, one X_i falls in $(y, y+\Delta y]$ and $(n-s)\ X_i$ s fall in $(y+\Delta y, \infty)$ $\left(\approx (y,\infty)\right)$. Using the property of multinomial distribution, we have

$f_{r,s}(x,y)\Delta x\Delta y$

$$= \frac{n!}{(r-1)!\,1!\,(s-r-1)!\,1!\,(n-s)!}[F(x)]^{r-1}[f(x)\Delta x]^1[F(y)-F(x)]^{s-r-1}[f(y)\Delta y]^1[1-F(y)]^{n-s}$$

i.e.,

$$f_{r,s}(x,y) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}$$

$$\times [1-F(y)]^{n-s} f(x) f(y), -\infty < x < y < \infty. \blacksquare$$

Example 10.1.2

Let $X_1, \dots, X_n$ be a random sample from a discrete distribution having support S, distribution function $F(\cdot)$ and p.m.f. $f(\cdot)$. Define $X_{1:n} = \min\{X_1, \dots, X_n\}$ and $X_{n:n} = \max\{X_1, \dots, X_n\}$. Find the p.m.f.s of $X_{1:n}$ and $X_{n:n}$.

Solution. For $x \in \mathbb{R}$, the distribution function of $X_{1:n}$ is

$$\begin{aligned} F_{X_{1:n}}(x) &= P(\{X_{1:n} \le x\}) \\ &= 1 - P(\{X_{1:n} > x\}) \\ &= 1 - P(\{X_i > x, i = 1, \dots, n\}) \\ &= 1 - \prod_{i=1}^{n} P(\{X_i > x\}) \\ &= 1 - \prod_{i=1}^{n}[1 - F(x)] \end{aligned}$$

$$= 1 - [1 - F(x)]^n.$$

Note that

$$D_{X_{1:n}} = \{x \in \mathbb{R}: F_{X_{1:n}}(\cdot) \text{ is discontinuous at } x\}$$

$$= \{x \in \mathbb{R}: F(\cdot) \text{ is discontinuous at } x\}$$

$$= S.$$

Thus $X_{1:n}$ is a discrete type random variable with support S and p.m.f.

$$f_{X_{1:n}}(x) = \begin{cases} F_{X_{1:n}}(x) - F_{X_{1:n}}(x-), & \text{if } x \in S \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} [1 - F(x-)]^n - [1 - F(x)]^n, & \text{if } x \in S \\ 0, & \text{otherwise} \end{cases}.$$

Also the distribution function of $X_{n:n}$ is given by

$$F_{X_{1:n}}(x) = P(\{X_{n:n} \le x\})$$

$$= P(\{X_i \le x, i = 1, \dots, n\})$$

$$= \prod_{i=1}^{n} P(\{X_i \le x\})$$

$$= \prod_{i=1}^{n} F(x)$$

$$= [F(x)]^n, x \in \mathbb{R}.$$

Since $F_{X_{n:n}}(\cdot)$ is continuous at x if, and only if, $F(\cdot)$ is continuous at x, the random variable $X_{n:n}$ is of discrete type with support S and p.m.f.

$$f_{X_{n:n}}(x) = \begin{cases} F_{X_{n:n}}(x) - F_{X_{n:n}}(x-), & \text{if } x \in S \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} [F(x)]^n - [F(x-)]^n, & \text{if } x \in S \\ 0, & \text{otherwise} \end{cases}. \blacksquare$$

Example 10.1.3

Let X_1, X_2 be a random sample from $U(0,1)$ distribution. Find the distribution function of $Y = X_1 + X_2$. Hence find the p.d.f. of Y.

Solution. The joint p.d.f. of (X_1, X_2) is given by

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$$

$$= \begin{cases} 1, & \text{if } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

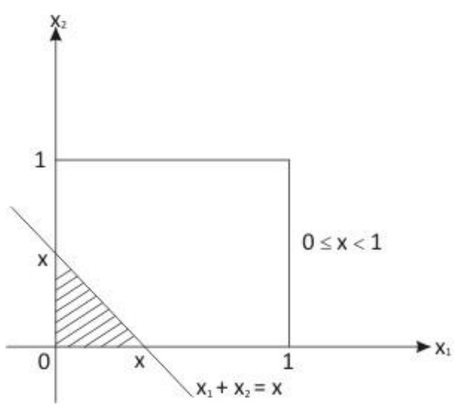
Therefore the distribution function of Y is given by

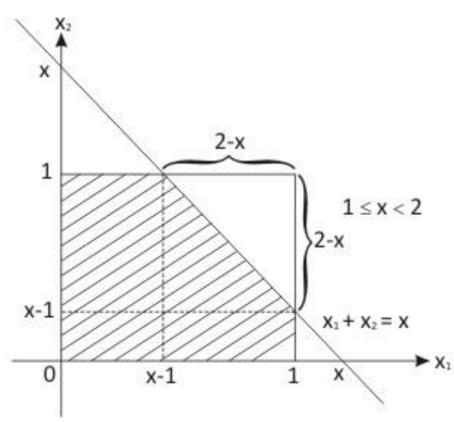
$$F_Y(x) = P(\{X_1 + X_2 \le x\})$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\, I_{(-\infty,x]}(x_1+x_2)\,dx_1 dx_2$$

$$= \underset{x_1+x_2 \le x}{\int_0^1 \int_0^1} dx_1 dx_2$$

$$= \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{2} \times x \times x, & \text{if } 0 \le x < 1 \\ 1 - \frac{1}{2}(2-x)\times(2-x), & \text{if } 1 \le x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}.$$





$$\Rightarrow F_Y(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x^2}{2}, & \text{if } 0 \le x < 1 \\ \frac{4x - x^2 - 2}{2}, & \text{if } 1 \le x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}.$$

Clearly $F_Y(\cdot)$ is differentiable everywhere except on a finite set $C \subseteq \{0,1,2\}$. Let

$$g(x) = \begin{cases} x, & \text{if } 0 < x < 1 \\ 2-x, & \text{if } 1 < x < 2, \\ 0, & \text{otherwise} \end{cases}$$

so that

$$\frac{d}{dx} F_Y(x) = g(x), \forall x \in \mathbb{R} - C$$

and

$$\int_{-\infty}^{\infty} g(x) dx = 1.$$

It follows that Y is of absolutely continuous type with a p.d.f.

$$g(x) = \begin{cases} x, & \text{if } 0 < x < 1 \\ 2-x, & \text{if } 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}. \blacksquare$$

Example 10.1.4

Let X_1, X_2 be a random sample from a distribution having p.d.f.

$$f(x) = \begin{cases} 2x, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find the distribution function of $Y = X_1 + X_2$. Hence find the p.d.f. of Y.

Solution. The joint p.d.f. of (X_1, X_2) is given by

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$$

$$= \begin{cases} 4\, x_1 x_2, & \text{if } 0 < x_1 < 1, 0 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}.$$

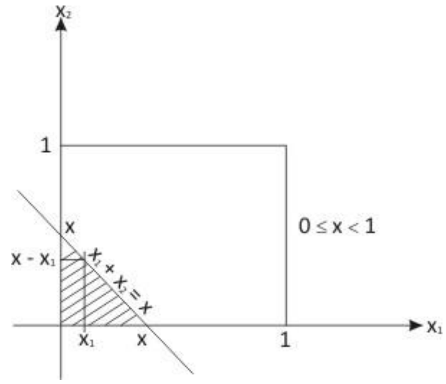
The distribution function of Y is given by

$$F_Y(x) = P(\{X_1 + X_2 \le x\}) = \underset{x_1+x_2 \le x}{\int_0^1 \int_0^1} 4x_1 x_2 dx_1 dx_2.$$

Clearly, for $x < 0, F_Y(x) = 0$ and, for $x \geq 2, F_Y(x) = 1$.

For $0 \le x < 1$

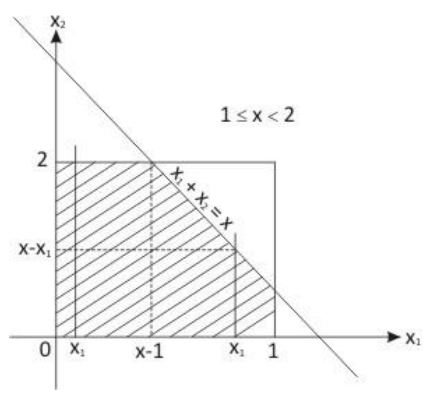
$$F_Y(x) = \int_0^x \int_0^{x-x_1} 4x_1 x_2 \, dx_2 \, dx_1 = \frac{x^4}{6}.$$



For $1 \leq x < 2$

$$F_Y(x) = \int_0^{x-1} \int_0^1 4x_1 x_2 \, dx_2 \, dx_1 + \int_{x-1}^1 \int_0^{x-x_1} 4x_1 x_2 \, dx_2 \, dx_1$$

$$= (x-1)^2 + \frac{(4x-3)-(x+3)(x-1)^3}{6}.$$



Therefore,

$$F_Y(x) = \begin{cases} 0, & \text{if } x < 0 \\ \dfrac{x^4}{6}, & \text{if } 0 \leq x < 1 \\ (x-1)^2 + \dfrac{(4x-3)-(x+3)(x-1)^3}{6}, & \text{if } 1 \leq x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}.$$

Clearly $F_Y(\cdot)$ is differentiable everywhere except on a finite set $C \subseteq \{0, 1, 2\}$. Let

$$g(x) = \begin{cases} \frac{2}{3}x^3, & \text{if } 0 < x < 1 \\ 2(x-1) + \frac{2}{3}(1-(x+2)(x-1)^2), & \text{if } 1 < x < 2 \\ 0, & \text{otherwise} \end{cases},$$

so that

$$\frac{d}{dx}F_Y(x) = g(x), \forall x \in \mathbb{R} - C$$

and

$$\int_{-\infty}^{\infty} g(x)dx = 1.$$

It follows that Y is of absolutely continuous type with a p.d.f.

$$g(x) = \begin{cases} \frac{2}{3}x^3, & \text{if } 0 < x < 1 \\ 2(x-1) + \frac{2}{3}(1-(x+2)(x-1)^2), & \text{if } 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}. \blacksquare$$

Example 10.1.5

Let X_1, X_2, X_3 be a random sample and let $X_1 \sim N(0,1)$. Find the distribution function of $Y = X_1^2 + X_2^2 + X_3^2$. Hence find the p.d.f. of Y.

Solution. The joint p.d.f. of $\underline{X} = (X_1, X_2, X_3)$ is

$$f_{\underline{X}}(x_1, x_2, x_3) = \prod_{i=1}^{3} f_{X_i}(x_i)$$

$$= \prod_{i=1}^{3} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)}, \quad -\infty < x_i < \infty, i = 1, 2, 3.$$

Therefore the distribution function of $Y = X_1^2 + X_2^2 + X_3^2$ is

$$F_Y(y) = \underset{x_1^2 + x_2^2 + x_3^2 \le y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}} \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)} dx_1 dx_2 dx_3.$$

On making the spherical coordinates transformation

$$x_1 = r\sin\theta_1 \sin\theta_2,$$
$$x_2 = r\sin\theta_1 \cos\theta_2,$$
$$x_3 = r\cos\theta_1 ,$$

so that $r > 0, 0 < \theta_1 \leq \pi, 0 < \theta_2 \leq 2\pi$ and the Jacobian of the transformation is $J = r^2 \sin\theta_1$, we get for $y > 0$

$$F_Y(y) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_0^{\sqrt{y}} \int_0^{\pi} \int_0^{2\pi} e^{-\frac{r^2}{2}} r^2 \sin\theta_1 d\theta_2 d\theta_1 dr$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{y}} e^{-\frac{r^2}{2}} r^2 dr$$

$$= \frac{1}{2^{\frac{3}{2}} \Gamma(\frac{3}{2})} \int_0^{y} e^{-\frac{t}{2}} t^{\frac{3}{2}-1} dt$$

Therefore

$$F_Y(y) = \begin{cases} 0, & \text{if } y \leq 0 \\ \dfrac{1}{2^{\frac{3}{2}} \Gamma(\frac{3}{2})} \displaystyle\int_0^{y} e^{-\frac{t}{2}} t^{\frac{3}{2}-1} dt, & \text{if } y > 0 \end{cases}.$$

Clearly $F_Y(\cdot)$ is the distribution function of χ_3^2 distribution having the p.d.f.

$$f_Y(y) = \begin{cases} \dfrac{e^{-\frac{y}{2}} y^{\frac{3}{2}-1}}{2^{\frac{3}{2}} \Gamma(\frac{3}{2})}, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases},$$

Thus $Y \sim \chi_3^2$ (also see Example 7.6 (ii)). ■

In many situations finding distribution function

$$F_Y(y) = P\left(\left\{g\left(X_1, \dots, X_p\right) \leq y\right\}\right), -\infty < y < \infty,$$

of random variable $Y = g\left(X_1, \dots, X_p\right)$ may be difficult or quite tedious. For example, consider a random sample $X_1, \dots, X_n$ $(n \geq 4)$ from $N(0, 1)$ distribution and suppose that the distribution function of $Y = \sum_{i=1}^{n} X_i^2$ is desired. Clearly, for $y > 0$

$$F_Y(y) = \underset{x_1^2+\cdots+x_n^2 \le y}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}} \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^{n} x_i^2} dx_1 dx_2 \cdots dx_n.$$

On making the spherical coordinates transformation

$$\begin{aligned}
x_1 &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \sin\theta_{n-1} \\
x_2 &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \cos\theta_{n-1} \\
x_3 &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-3} \cos\theta_{n-2} \\
&\vdots \\
x_{n-1} &= r \sin\theta_1 \cos\theta_2 \\
x_n &= r \cos\theta_1 ,
\end{aligned}$$

So that $r > 0, \sum_{i=1}^{n} x_i^2 = r^2, 0 < \theta_i \le \pi, i = 1, \dots, n-2, 0 < \theta_{n-1} \le 2\pi$ and the Jacobian of transformation is $J = r^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}$, we get for $y > 0$

$$F_Y(y) = \int_0^{\sqrt{y}} \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} \frac{1}{(2\pi)^{n/2}} e^{-\frac{r^2}{2}} r^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}\, d\theta_{n-1} d\theta_{n-2} \cdots d\theta_1 dr.$$

Clearly evaluating the above integral may be tedious. This points towards desirability, if possible, of other methods of determining the distributions of functions of random variable. We will see that other techniques are available and, in a given situation, often one technique is more elegant than the others.