

MODULE 5

SOME SPECIAL ABSOLUTELY CONTINUOUS DISTRIBUTIONS

LECTURE 22

Topics

5.2 GAMMA AND RELATED DISTRIBUTIONS

Definition 2.2

An absolutely continuous type random variable X is said to follow a *gamma distribution* with shape parameter $\alpha (> 0)$ and scale parameter $\theta (> 0)$ (written as $X \sim G(\alpha, \theta)$) if its probability density function is given by

$$f_X(x|\alpha, \theta) = \begin{cases} \frac{1}{\theta^\alpha \Gamma(\alpha)} e^{-\frac{x}{\theta}} x^{\alpha-1}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases} . \blacksquare$$

Note that, for $\alpha > 0$ and $\theta > 0$, $f_X(x|\alpha, \theta) \geq 0$, $\forall x \in \mathbb{R}$ and

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x|\alpha, \theta) dx &= \frac{1}{\theta^\alpha \Gamma(\alpha)} \int_0^{\infty} e^{-\frac{x}{\theta}} x^{\alpha-1} dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-t} t^{\alpha-1} dt \\ &= 1. \end{aligned}$$

Theorem 2.1

Suppose that $X \sim G(\alpha, \theta)$, for some $\alpha > 0$ and $\theta > 0$. Define the random variable $Z = \frac{X}{\theta}$. Then $Z \sim G(\alpha, 1)$, i. e, the p.d.f of Z is given by

$$f_Z(z) = \begin{cases} \frac{e^{-z} z^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } z > 0 \\ 0, & \text{otherwise} \end{cases} .$$

Proof. Clearly $S_X = \{x \in \mathbb{R}: f_X(x|\alpha, \theta) > 0\} = (0, \infty)$ and the p.d.f. of X is

$$f_X(x|\alpha, \theta) = \begin{cases} \frac{1}{\theta^\alpha \Gamma(\alpha)} e^{-\frac{x}{\theta}} x^{\alpha-1}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

We have $Z = \frac{X}{\theta} = h(X)$, say. The transformation $h(x) = \frac{x}{\theta}, x \in S_X$, is strictly monotone on S_X with $h'(x) = \frac{1}{\theta}, x \in S_X$. Also $S_Z = h(S_X) = (0, \infty)$, and $h^{-1}(z) = \theta z, z \in S_Z$. Thus $Z = h(X) = \frac{X}{\theta}$ has the p.d.f.

$$\begin{aligned} f_Z(z) &= f_X(h^{-1}(z)) \left| \frac{d}{dz} h^{-1}(z) \right| I_{h(S_X)}(z) \\ &= f_X(\theta z) |\theta| I_{(0, \infty)}(z) \\ &= \begin{cases} \frac{e^{-z} z^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } z > 0 \\ 0, & \text{otherwise} \end{cases}. \blacksquare \end{aligned}$$

Note that if $Z \sim G(\alpha, 1)$, for some $\alpha > 0$, then

$$\begin{aligned} E(Z^r) &= \int_{-\infty}^{\infty} z^r f_Z(z) dz \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-z} z^{\alpha+r-1} dz \\ &= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}, \quad r > -\alpha. \end{aligned}$$

Thus

$$\boxed{Z \sim G(\alpha, 1) \Rightarrow E(Z^r) = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}, \quad r > -\alpha.} \quad (2.8)$$

Clearly, for $r \in \mathbb{N}$,

$$\boxed{E(Z^r) = \alpha(\alpha+1) \cdots (\alpha+r-1)} \quad (2.9)$$

Also if $X \sim G(\alpha, \theta)$ (so that $Z = \frac{X}{\theta} \sim G(\alpha, 1)$) then, for $r > -\alpha$,

$$E(X^r) = E((\theta Z))^r = \theta^r E(Z^r).$$

Using (2.8) we get

$$E(X^r) = \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)} \theta^r.$$

Therefore,

$$\mu'_r = E(X^r) = \alpha(\alpha + 1) \cdots (\alpha + r - 1)\theta^r, \quad r \in \mathbb{N}.$$

$$\text{Mean} = \mu'_1 = E(X) = \alpha\theta,$$

$$\mu'_2 = E(X^2) = \alpha(\alpha + 1)\theta^2,$$

$$\text{Variance} = \mu_2 = E(X^2) - (E(X))^2 = \alpha\theta^2,$$

$$\mu_3 = E((X - \mu'_1)^3) = \mu'_3 - 3\mu'_1\mu'_2 + 2(\mu'_1)^3 = 2\alpha\theta^3,$$

$$\mu_4 = E((X - \mu'_1)^4) = \mu'_4 - 4\mu'_1\mu'_3 + 6(\mu'_1)^2\mu'_2 - 3(\mu'_1)^4 = 3\alpha(\alpha + 2)\theta^4,$$

$$\text{Coefficient of skewness} = \beta_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2}{\sqrt{\alpha}}.$$

and

$$\text{Kurtosis} = \gamma_1 = \frac{\mu_4}{\mu_2^2} = \frac{3(\alpha + 2)}{\alpha} = 3 + \frac{6}{\alpha}.$$

Note that, as $\alpha \rightarrow \infty$, $\beta_1 \rightarrow 0$ and $\gamma_1 \rightarrow 3$. Also $\beta_1 > 0$ and $\gamma_1 > 3$. Thus the gamma distribution is positively skewed and has sharper peaks than the normal distribution. For $\alpha \approx 0$ the distribution is heavily (positively) skewed. For large α ($\alpha \rightarrow \infty$) the gamma distribution very much behaves like the normal distribution.

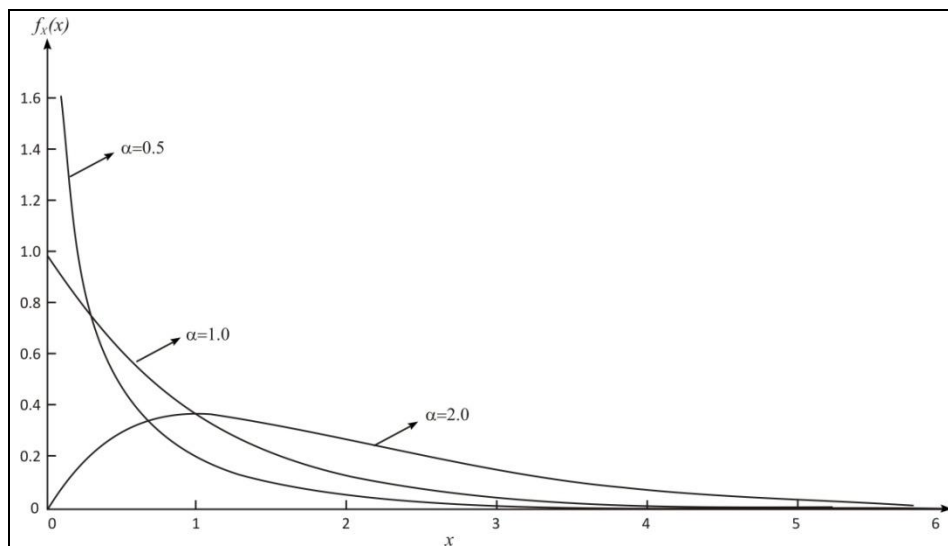


Figure 2.1. Plots of p.d.f.s of $G(\alpha, 1)$ distribution

The m.g.f. of $X \sim G(\alpha, \theta)$ is given by

$$M_X(t) = E(e^{tX}) = E(e^{t\theta Z}),$$

where $Z = \frac{X}{\theta} \sim G(\alpha, 1)$ (see Theorem 2.1). Thus

$$M_X(t) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-(1-t\theta)z} z^{\alpha-1} dz$$

$$\Rightarrow M_X(t) = (1 - t\theta)^{-\alpha}, \quad t < \frac{1}{\theta}. \quad (2.10)$$

The following theorem provides a relationship between gamma probabilities and Poisson probabilities.

Theorem 2.2

For a positive integer n and for real constants $\theta > 0$ and $t > 0$, let $X \sim G(n, \theta)$ and $Y \sim P\left(\frac{t}{\theta}\right)$. Then

$$P(\{X > t\}) = P(\{Y \leq n - 1\})$$

i. e.,

$$\int_t^{\infty} \frac{e^{-\frac{x}{\theta}} x^{n-1}}{\Gamma(n)\theta^n} dx = \sum_{j=0}^{n-1} \frac{e^{-\frac{t}{\theta}} \left(\frac{t}{\theta}\right)^j}{j!}, \quad t > 0, \theta > 0.$$

Proof. Let $Z = \frac{X}{\theta}$, so that, by Theorem 2.1, $Z \sim G(n, 1)$. Then, for $t > 0$,

$$P(\{X > t\}) = P\left(\left\{Z > \frac{t}{\theta}\right\}\right). \quad (2.11)$$

For $x > 0$, we have

$$P(\{Z > y\}) = \int_y^\infty \frac{e^{-x} x^{n-1}}{(n-1)!} dx = I_n(\text{say}), \quad n \in \mathbb{N},$$

with the convention that $0! = 1$.

On integrating by parts we get

$$\begin{aligned} I_n &= \frac{1}{(n-1)!} \left[\{-e^{-x} x^{n-1}\}_y^\infty + (n-1) \int_y^\infty e^{-x} x^{n-2} dx \right] \\ &= \frac{e^{-y} y^{n-1}}{(n-1)!} + \frac{1}{(n-2)!} \int_y^\infty e^{-x} x^{n-2} dx \\ &= \frac{e^{-y} y^{n-1}}{(n-1)!} + I_{n-1}, \quad n \geq 2 \\ &= \frac{e^{-y} y^{n-1}}{(n-1)!} + \frac{e^{-y} y^{n-2}}{(n-2)!} + I_{n-2}, \quad n \geq 3 \\ &\vdots \\ &= \sum_{j=1}^{n-1} \frac{e^{-y} y^j}{j!} + I_1 \\ &= \sum_{j=1}^{n-1} \frac{e^{-y} y^j}{j!} + \int_y^\infty e^{-x} dx \\ &= \sum_{j=0}^{n-1} \frac{e^{-y} y^j}{j!}. \end{aligned}$$

Thus, for $y > 0$,

$$P(\{Z > y\}) = \sum_{j=0}^{n-1} \frac{e^{-y} y^j}{j!}$$

$$\Rightarrow P(\{X > t\}) = P\left(\left\{Z > \frac{t}{\theta}\right\}\right) = \sum_{j=0}^{n-1} \frac{e^{-\frac{t}{\theta}} \left(\frac{t}{\theta}\right)^j}{j!}, t > 0, \quad (\text{using (2.11)}). \blacksquare$$

Example 2.1

For a positive integer n and $\theta > 0$, let $X \sim G(n, \theta)$. Define the random variable

$$Y = \sum_{j=0}^{n-1} \frac{e^{-\frac{X}{\theta}} \left(\frac{X}{\theta}\right)^j}{j!}.$$

Find the probability distribution of random variable Y .

Solution. Note that $P(\{X \geq 0\}) = 1$ and, by Theorem 2.2, the distribution function of $X \sim G(n, \theta)$ is given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - \sum_{j=0}^{n-1} \frac{e^{-\frac{x}{\theta}} \left(\frac{x}{\theta}\right)^j}{j!}, & \text{if } x \geq 0 \end{cases}.$$

Clearly $P(\{Y = 1 - F(X)\}) = P(\{X \geq 0\}) = 1$, and therefore $Y \stackrel{d}{=} 1 - F(X)$. By probability integral transformation (Theorem 1.3) $U \stackrel{\text{def}}{=} F(X) \sim U(0, 1)$. Moreover $U \stackrel{d}{=} 1 - U \sim U(0, 1)$. From the above discussion it follows that $Y \stackrel{d}{=} 1 - U \sim U(0, 1)$. \blacksquare

Definition 2.3

For $\theta > 0$, a $G(1, \theta)$ distribution is called an *exponential distribution* with scale parameter θ (denoted by $\text{Exp}(\theta)$). \blacksquare

The p.d.f. of $T \sim \text{Exp}(\theta)$ is given by

$$f_T(t) = \begin{cases} \frac{1}{\theta} e^{-\frac{t}{\theta}}, & \text{if } t > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Note that if $T \sim \text{Exp}(\theta)$, then

$$\text{Mean} = \mu'_1 = E(T) = \theta,$$

$$\text{Variance} = \mu_2 = \theta^2,$$

$$\mu'_r = E(T^r) = r! \theta^r, \quad r \in \{1, 2, \dots\},$$

$$\text{Coefficient of skewness} = \beta_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = 2,$$

and

$$\text{Kurtosis} = \gamma_1 = \frac{\mu_4}{\mu_2^2} = 9.$$

The m.g.f. of $T \sim \text{Exp}(\theta)$ is given by

$$M_X(t) = (1 - t\theta)^{-1}, \quad t < \frac{1}{\theta},$$

and the d.f. of $T \sim \text{Exp}(\theta)$ is given by

$$F_T(t) = \int_{-\infty}^t f_T(x) dx$$

$$\text{i. e., } F_T(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1 - e^{-\frac{t}{\theta}}, & \text{if } t \geq 0 \end{cases}.$$

Clearly, for every $s > 0$, and $t > 0$,

$$P(\{T > s + t \mid T > s\}) = \frac{P(\{T > s + t\})}{P(\{T > s\})} = e^{-\frac{t}{\theta}} = P(\{T > t\}), \quad (2.12)$$

$$\text{i. e., } P(\{T > s + t\}) = P(\{T > s\})P(\{T > t\}), \quad \forall s > 0, \quad t > 0. \quad (2.13)$$

Let $T \sim \text{Exp}(\theta)$ denote the lifetime of a component. Then the property (2.12) (or equivalently the property (2.13)) about the lifetime T of the component has the following interesting interpretation. Given that the component has survived s units of time the probability that it will survive additional t units of time is the same as the probability that a fresh unit (of age 0) will survive t units of time. In other words the component is not aging with time (i.e., the used component is as good as the new one). This property of a

continuous type random variable is also known as the *lack of memory property* (at each stage the component forgets its age and behaves like a fresh component).

In the following theorem it is shown that the lack of memory property characterizes the exponential among all continuous distributions having mass concentrated on $(0, \infty)$.

Theorem 2.2

Let Y be a random variable of continuous type with $F_Y(0) = 0$, where $F_Y(\cdot)$ is the distribution function of Y . Then Y has the lack of memory property if, and only if, $Y \sim \text{Exp}(\theta)$, for some $\theta > 0$.

Proof. We have seen that if $Y \sim \text{Exp}(\theta)$, for some $\theta > 0$, then

$$P(\{Y > s + t | Y > s\}) = P(\{Y > t\}), \quad \forall s, t > 0,$$

i.e., Y has the lack of memory property.

Conversely suppose that Y has the lack of memory property, i.e.,

$$P(\{Y > s + t\}) = P(\{Y > s\})P(\{Y > t\}), \quad \forall s, t > 0.$$

Let $\bar{F}_Y(t) = 1 - F_Y(t)$, $t \in \mathbb{R}$. Then we have

$$\bar{F}_Y(s + t) = \bar{F}_Y(s)\bar{F}_Y(t), \quad \forall s, t > 0$$

$$\Rightarrow \bar{F}_Y(s_1 + s_2 + \dots + s_m) = \bar{F}_Y(s_1)\bar{F}_Y(s_2) \dots \bar{F}_Y(s_m), \quad \forall s_1, s_2, \dots, s_m > 0 \quad (2.14)$$

$$\begin{aligned} \Rightarrow \bar{F}_Y\left(\frac{m}{n}\right) &= \bar{F}_Y\left(\underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{m \text{ times}}\right) \\ &= [\bar{F}_Y\left(\frac{1}{n}\right)]^m, \quad \forall m, n \in \mathbb{N}, \quad (\text{using (2.14)}) \end{aligned} \quad (2.15)$$

$$\Rightarrow \bar{F}_Y(1) = [\bar{F}_Y\left(\frac{1}{n}\right)]^n, \quad \forall n \in \mathbb{N}. \quad (2.16)$$

Using (2.15) and (2.16), we get

$$\begin{aligned} \bar{F}_Y\left(\frac{m}{n}\right) &= \left[\bar{F}_Y\left(\frac{1}{n}\right)\right]^m \\ &= [\bar{F}_Y(1)]^{\frac{m}{n}}, \quad \forall m, n \in \mathbb{N}. \end{aligned} \quad (2.17)$$

Let $\lambda = \bar{F}_Y(1)$, so that $0 \leq \lambda \leq 1$. Clearly, if $\lambda = 0$, then by (2.16)

$$\begin{aligned}
\bar{F}_Y\left(\frac{1}{n}\right) &= 0, \quad \forall n \in \mathbb{N} \\
\Rightarrow F_Y\left(\frac{1}{n}\right) &= 1, \quad \forall n \in \mathbb{N} \\
\Rightarrow \lim_{n \rightarrow \infty} F_Y\left(\frac{1}{n}\right) &= 1, \\
\Rightarrow F_Y(0) &= 1, \quad (\text{since } F_Y(\cdot) \text{ is continuous}),
\end{aligned}$$

which is not true as $F_Y(0) = 0$.

Therefore, $\lambda = \bar{F}_Y(1) > 0$. Similarly, if $\lambda = \bar{F}_Y(1) = 1$, then using (2.14)

$$\begin{aligned}
\bar{F}_Y(n) &= \bar{F}_Y\left(\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}\right) \\
&= [\bar{F}_Y(1)]^n \\
&= 1, \quad \forall n \in \mathbb{N} \\
\Rightarrow F_Y(n) &= 0, \quad \forall n \in \mathbb{N} \\
\Rightarrow \lim_{n \rightarrow \infty} F_Y(n) &= 0,
\end{aligned}$$

which is not true as $\lim_{n \rightarrow \infty} F_Y(n) = 1$.

Thus we have $\bar{F}_Y(1) = \lambda \in (0,1)$. Then $\lambda = \bar{F}_Y(1) = e^{-\frac{1}{\theta}}$, for some $\theta > 0$. ($\theta^{-1} = -\ln \lambda$). Using (2.17) we have

$$\bar{F}_Y(r) = e^{-\frac{r}{\theta}}, \quad \forall r \in \mathbb{Q} \cap (0, \infty),$$

where \mathbb{Q} denotes the set of rational numbers. Now let $x \in \mathbb{Q}^c \cap (0, \infty)$. Then there exists a sequence $\{r_n : n = 1, 2, \dots\}$ of rational numbers in $\mathbb{Q} \cap (0, \infty)$ such that $\lim_{n \rightarrow \infty} r_n = x$. Therefore

$$\begin{aligned}
\bar{F}_Y(x) &= \bar{F}_Y\left(\lim_{n \rightarrow \infty} r_n\right) \\
&= \lim_{n \rightarrow \infty} \bar{F}_Y(r_n) \quad (\text{since } \bar{F}_Y(x) = 1 - F_Y(x), \text{ is continuous on } \mathbb{R}) \\
&= \lim_{n \rightarrow \infty} e^{-\frac{r_n}{\theta}} \\
&= e^{-\frac{x}{\theta}}.
\end{aligned}$$

Also $F_Y(0) = 0$ implies that $F_Y(x) = 0$ for every $x \leq 0$, i.e., $\bar{F}_Y(x) = 1, \forall x \leq 0$. Therefore

$$\begin{aligned}\bar{F}_Y(x) &= \begin{cases} 1, & \text{if } x \leq 0 \\ e^{-\frac{x}{\theta}}, & \text{if } x > 0 \end{cases} \\ \Rightarrow F_Y(x) &= \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-\frac{x}{\theta}}, & \text{if } x \geq 0 \end{cases} \\ \Rightarrow Y &\sim \text{Exp}(\theta). \blacksquare\end{aligned}$$

Example 2.2

The waiting time for occurrence of an event E (say repair time of a machine) is exponentially distributed with mean of 30 minutes. Find the conditional probability that the waiting time for occurrence of event E is at least 5 hours given that it has not occurred in the first 3 hours.

Solution. Let X be the waiting time (in hours) for the occurrence of event E . Then $X \sim \text{Exp}(\frac{1}{2})$. By the lack of memory property of exponential distribution, the required probability is

$$P(\{X > 5\}|\{X > 3\}) = P(\{X > 2\}) = e^{-4}. \blacksquare$$

Definition 2.4

For a positive integer n , a $G(\frac{n}{2}, 2)$ distribution is called the *chi-squared* distribution with n degrees of freedom (d.f.) (denoted by χ_n^2). \blacksquare

The p.d.f. of $Y \sim \chi_n^2$ is given by

$$f_Y(y) = \begin{cases} \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} e^{-\frac{y}{2}} y^{\frac{n}{2}-1}, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Note that if $Y \sim \chi_n^2$ then

$$\text{Mean} = \mu_1' = E(Y) = n,$$

$$\text{Variance} = \mu_2 = 2n,$$

$$\text{Coefficient of skewness} = \beta_1 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = 2 \sqrt{\frac{2}{n}},$$

and

$$\text{Kurtosis} = \gamma_1 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{12}{n}.$$

Moreover the m.g.f. of $Y \sim \chi_n^2$ is given by

$$M_Y(t) = E(e^{tY}) = (1 - 2t)^{-\frac{n}{2}}, \quad t < \frac{1}{2}.$$

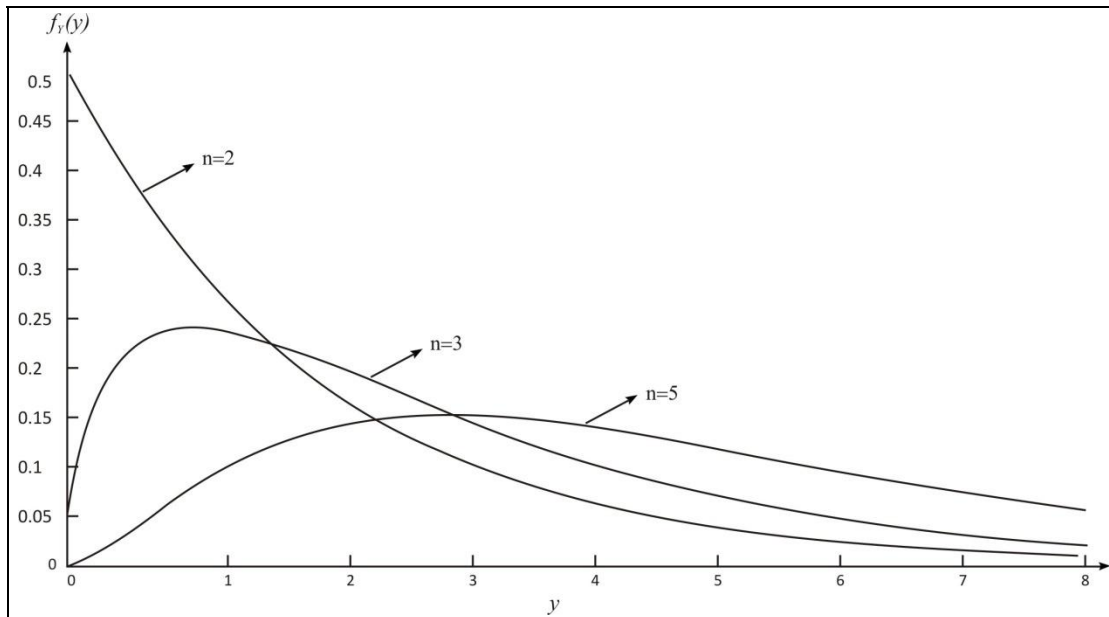


Figure 2.2. Plots of p.d.f.s of χ_n^2 distribution

Recall that $\chi_{n,1-\alpha}^2$, the $(1 - \alpha)$ - th quantile of $Y \sim \chi_n^2$, is given by

$$\begin{aligned} P(Y \leq \chi_{n,1-\alpha}^2) &= 1 - \alpha \\ \Leftrightarrow \int_0^{\chi_{n,1-\alpha}^2} \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} e^{-\frac{y}{2}} y^{\frac{n}{2}-1} dy &= 1 - \alpha. \end{aligned}$$

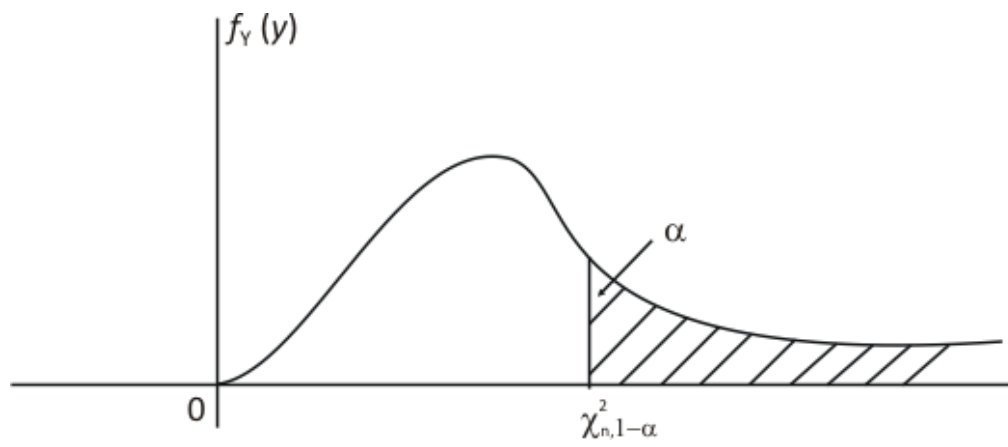


Figure 2.3. $(1 - \alpha)$ -th quantile of $Y \sim \chi_n^2$

Table 2.1. $(1 - \alpha)$ -th quantile of $Y \sim \chi_n^2$ ($P(Y \leq \chi_{n,1-\alpha}^2) = 1 - \alpha$)Values of $\chi_{n,1-\alpha}^2$

n/α	.995	.990	.975	.950	.900	.100	.050	.025	.010	.005
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	31.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169