MODULE 3

FUNCTION OF A RANDOM VARIABLE AND ITS DISTRIBUTION

LECTURE 14

Topics

3.3 EXPECTATION AND MOMENTS OF A RANDOM VARIABLE

Some special kinds of expectations which are frequently used are defined below.

Definition 3.2

Let X be a random variable defined on some probability space.

(i) $\mu_1' = E(X)$, provided it is finite, is called the *mean* of the (distribution of) random variable X;

(ii) For $r \in \{1, 2 \cdots\}$, $\mu_r' = E(X^r)$, provided it is finite, is called the *r-th moment* of the (distribution of) random variable X;

(iii) For $r \in \{1, 2 \cdots\}$, $E(|X|^r)$, provided it is finite, is called the *r-th absolute moment* of the (distribution of) random variable X;

(iv) For $r \in \{1, 2 \cdots\}$, $\mu_r = E\left(\left(X - \mu_1'\right)^r\right)$, provided it is finite, is called the *r-th central moment* of the (distribution of) random variable X;

(v) $\mu_2 = E\left(\left(X - \mu_1'\right)^2\right)$, provided it is finite, is called the *variance* of the (distribution of) random variable X. The variance of a random variable X is denoted by $\text{Var}(X)$. The quantity $\sigma = \sqrt{\mu_2} = \sqrt{E((X-\mu)^2)}$ is called the standard deviation of the (distribution of) random variable X.

(vi) Suppose that the distribution function F_X of a random variable X can be decomposed as

$$F_X(x) = \alpha F_d(x) + (1-\alpha) F_{AC}(x), x \in \mathbb{R}, \ \ \alpha \in [0,1],$$

where F_d is a distribution function of a discrete type random variable (say X_d) and F_{AC} is a distribution function of an absolutely continuous type random variable (say X_{AC}). Then, for a Borel function $h: \mathbb{R} \to \mathbb{R}$, the expectation of $h(X)$ is defined by

$$E\big(h(X)\big) = \alpha E\big(h(X_d)\big) + (1-\alpha)E\big(h(X_{AC})\big)$$

provided $E(h(X_d))$ and $E(h(X_{AC}))$ are finite. ■

Theorem 3.3

Let X be a random variable.

(i) If h_1 and h_2 are Borel functions such that $P(\{h_1(X) \le h_2(X)\}) = 1$, then $E(h_1(X)) \le E(h_2(X))$, provided the involved expectations are finite;

(ii) If, for real constants a and b with $a \le b, P(\{a \le X \le b\}) = 1$, then $a \le E(X) \le b$;

(iii) If $P(\{X \ge 0\}) = 1$ and $E(X) = 0$, then $P(\{X = 0\}) = 1$;

(iv) If $E(|X|)$ is finite, then $|E(X)| \le E(|X|)$;

(v) For real constants a and b, $E(aX + b) = aE(X) + b$, provided the involved expectations are finite;

(vi) If $h_1, \ldots, h_m$ are Borel function then

$$E\left(\sum_{i=1}^{m} h_i(X)\right) = \sum_{i=1}^{m} E\big(h_i(X)\big),$$

provided the involved expectations are finite.

Proof. We will provide the proof for the situation when X is of absolutely continuous type. The proof for the discrete case is analogous and is left as an exercise. Also assertions (iv)-(vi) follow directly from the definition of the expectation of a random variable and using elementary properties of integrals. Therefore we will provide the proofs of only first three assertions.

(i) Define $A = \{x \in \mathbb{R}: h_1(x) \le h_2(x)\},\ S_X^* = S_X \cap A$ and

$$g(x) = \begin{cases} f_X(x), & \text{if } x \in S_X^* \\ 0, & \text{otherwise} \end{cases}.$$

Then $g(x) \ge 0, \forall x \in \mathbb{R}, P(\{X \in A^c\}) = 0, P(\{X \in S_X \cap A^c\}) = 0.$

$$\begin{aligned} P(\{X \in S_X^*\}) &= P(\{X \in S_X \cap A\}) \\ &= P(\{X \in S_X \cap A\}) + P(\{X \in S_X \cap A^c\}) \\ &= P(\{X \in S_X\}) \\ &= 1, \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} g(x)dx &= \int_{-\infty}^{\infty} f_X(x) I_{S_X^*}(x)dx \\ &= P(\{X \in S_X^*\}) \\ &= 1 \end{aligned}$$

and, for any $B \in \mathcal{B}_1$,

$P(\{X \in B\}) = P(\{X \in S_X \cap B\}) \qquad$ (since $P(\{X \in S_X\}) = 1$)

$$= P(\{X \in S_X \cap A \cap B\}) \quad (\text{since } P(\{X \in S_X \cap A^c \cap B\}) = 0)$$
$$= P(\{X \in S_X^* \cap B\})$$
$$= \int_{-\infty}^{\infty} g(x) I_B(x) dx.$$

It follows that g is also a p.d.f. of X with support $S_X^* = S_X \cap A \subseteq A$. The above discussion suggests that, without loss of generality, we may take $S_X \subseteq A = \{x \in \mathbb{R}: h_1(x) \leq h_2(x)\}$ (otherwise replace $f_X(\cdot)$ by $g(\cdot)$ and S_X by S_X^*). Then

$$h_1(x) I_{S_X}(x) f_X(x) \leq h_2(x) I_{S_X}(x) f_X(x), \forall x \in \mathbb{R}$$

$$\Longrightarrow E\big(h_1(X)\big)$$
$$= \int_{-\infty}^{\infty} h_1(x) I_{S_X}(x) f_X(x) dx \leq \int_{-\infty}^{\infty} h_2(x) I_{S_X}(x) f_X(x) dx = E\big(h_2(X)\big).$$

(ii) Since $P(\{a \leq X \leq b\}) = 1$, as in (i), without loss of generality we may assume that $S_X \subseteq [a, b]$. Then

$$a I_{S_X}(x) f_X(x) \leq x I_{S_X}(x) f_X(x) \leq b I_{S_X}(x) f_X(x), \forall x \in \mathbb{R}$$

$$\Rightarrow a = \int_{-\infty}^{\infty} a I_{S_X}(x) f_X(x) dx \leq \int_{-\infty}^{\infty} x\, I_{S_X}(x) f_X(x) dx \leq \int_{-\infty}^{\infty} b\, I_{S_X}(x) f_X(x) dx = b,$$

i.e., $a \leq E(X) \leq b$.

(iii) Since $P(\{X \geq 0\}) = 1$, without loss of generality we may take $S_X \subseteq [0, \infty]$. Then $(-\infty, 0) \subseteq S_X^C = \{x \in \mathbb{R}: f_X(x) = 0\}$ and therefore, for $n \in \{1, 2, \cdots\}$,

$$0 = E(X)$$
$$= \int_{-\infty}^{0} x\, f_X(x) dx + \int_{0}^{\infty} x\, f_X(x) dx$$
$$= \int_{0}^{\infty} x\, f_X(x) dx$$
$$\geq \int_{\frac{1}{n}}^{\infty} x\, f_X(x) dx$$
$$\geq \frac{1}{n} \int_{\frac{1}{n}}^{\infty} f_X(x) dx$$

$$= \frac{1}{n} P\left(\left\{X \geq \frac{1}{n}\right\}\right)$$

$$\Rightarrow P\left(\left\{X \geq \frac{1}{n}\right\}\right) = 0, \quad \forall\, n \in \{1, 2, \cdots\}$$

$$\Rightarrow \lim_{n\to\infty} P\left(\left\{X \geq \frac{1}{n}\right\}\right) = 0$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty}\left\{X \geq \frac{1}{n}\right\}\right) = 0 \qquad \left(\text{since } \left\{x \geq \frac{1}{n}\right\} \uparrow\right)$$

$$\Rightarrow P(\{X > 0\}) = 0$$

$$\Rightarrow P(\{X = 0\}) = P(\{X \geq 0\}) - P(\{X > 0\}) = 1. \blacksquare$$

Corollary 3.1

Let X be random variable with finite first two moments and let $E(X) = \mu$. Then,

(i) $\mathrm{Var}(X) = E(X^2) - \big(E(X)\big)^2$;
(ii) $\mathrm{Var}(X) \geq 0$. Moreover, $\mathrm{Var}(X) = 0$ if, and only if, $P(\{X = \mu\}) = 1$;
(iii) $E(X^2) \geq \big(E(X)\big)^2$ (Cauchy – Schwarz inequality);
(iv) For real constants a and b, $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$.

Proof.

(i) Note that $\mu = E(X)$ is a fixed real number. Therefore, using Theorem 3.3 (v)-(vi), we have

$$\begin{aligned}\mathrm{Var}(X) &= E((X-\mu)^2)\\ &= E(X^2) - 2\,\mu E(X) + \mu^2\\ &= E(X^2) - \mu^2\\ &= E(X^2) - \big(E(X)\big)^2.\end{aligned}$$

(ii) Since $P(\{(X-\mu)^2 \geq 0\}) = P(\Omega) = 1$, using Theorem 3.3 (i), we have $\mathrm{Var}(X) = E((X-\mu)^2) \geq 0$. Also, using theorem 3.3 (iii), if $\mathrm{Var}(X) = E((X-\mu)^2) = 0$ then $P(\{(X-\mu)^2 = 0\}) = 1$, i.e; $P(\{X = \mu\}) = 1$.
Conversely if $P(\{X = \mu\}) = 1$, then $E(X) = \mu$ and $E(X^2) = \mu^2$. Now using (i), we get

$$\mathrm{Var}(X) = E(X^2) - \big(E(X)\big)^2 = 0.$$

(iii) Follows from (i) and (ii).

(iv) Let $Y = aX + b$. Then

$$E(Y) = aE(X) + b \qquad \text{(using Theorem 3.3 (v))}$$

$$Y - E(Y) = a\big(X - E(X)\big)$$

and

$$\begin{aligned}\mathrm{Var}(Y) &= E\left(\big(Y - E(Y)\big)^2\right) \\ &= E\left(a^2\big(X - E(X)\big)^2\right) \\ &= a^2 E\left(\big(X - E(X)\big)^2\right) \\ &= a^2 \mathrm{Var}(X) \cdot \blacksquare\end{aligned}$$

Example 3.5

Let X be a random variable with p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if} - 2 < x < -1 \\ \frac{x}{9}, & \text{if } 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

(i) If $Y_1 = \max(X, 0)$, find the mean and variance of Y_1;

(ii) If $Y_2 = 2X + 3e^{-\max(X,0)} + 4$, find $E(Y_2)$.

Proof. Using Theorem 3.2 (ii) we get, for $r > 0$,

$$\begin{aligned}E(Y_1^r) &= E((\max(X,0))^r) \\ &= \int_{-\infty}^{\infty} (\max(x,0))^r f_X(x)dx \\ &= \int_0^3 \frac{x^{r+1}}{9} dx \\ &= \frac{3^r}{r+2} \cdot\end{aligned}$$

It follows that $E(Y_1) = 1$, $E(Y_1^2) = {}^9\!/_4$ and $\mathrm{Var}(Y_1) = E(Y_1^2) - \big(E(Y_1)\big)^2 = {}^5\!/_4 \cdot$

(iii) We have

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x) dx$$
$$= \int_{-2}^{-1} \frac{x}{2} dx + \int_{0}^{3} \frac{x^2}{9} dx$$
$$= \frac{1}{4}$$

and

$$E\left(e^{-\max(X,0)}\right) = \int_{-\infty}^{\infty} e^{-\max(x,0)} f_X(x) dx$$

$$= \int_{-2}^{-1} \frac{1}{2} dx + \int_{0}^{3} \frac{x}{9} e^{-x} dx$$

$$= \frac{11 - 8\, e^{-3}}{18}.$$

Therefore,

$$E(Y_2) = E\left(2X + 3e^{-\max(X,0)} + 4\right)$$

$$= 2\, E(X) + 3E\left(e^{-\max(X,0)}\right) + 4$$

$$= \frac{19 - 4e^{-3}}{3}. \blacksquare$$

Example 3.6

Let X be random variable with p.m.f.

$$f_X(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & \text{if } x \in \{0, 1, \cdots, n\} \\ 0, & \text{otherwise} \end{cases},$$

where $n \in \{1, 2, \cdots, \}, p \in (0, 1)$ and $q = 1 - p$.

(i) For $r \in \{1, 2, \cdots\}$, find $E\left(X_{(r)}\right)$, where $X_{(r)} = X(X-1)\cdots(X-r+1)$ ($E\left(X_{(r)}\right)$ is called the r-th factorial moment of $X, r = 1,2, \ldots$);

(ii) Find mean and variance of X;

(iii) Let $T = e^X + 2e^{-X} + 6X^2 + 3X + 4$. Find $E(T)$.

Solution.

(i) Fix $r \in \{1, 2, \cdots, n\}$. Then

$$E\left(X_{(r)}\right) = E\left(X(X-1)\cdots(X-r+1)\right)$$

$$= \sum_{x=0}^{n} x\,(x-1)\cdots(x-r+1)\binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=r}^{n} x\,(x-1)\cdots(x-r+1)\frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= n(n-1)\cdots(n-r+1)p^r \sum_{x=r}^{n}\binom{n-r}{x-r} p^{x-r} q^{(n-r)-(x-r)}$$

$$= n(n-1)\cdots(n-r+1)p^r \sum_{x=0}^{n-r}\binom{n-r}{x} p^{x} q^{(n-r)-x}$$

$$= n(n-1)\cdots(n-r+1)p^r (q+p)^{n-r}$$

$$= n(n-1)\cdots(n-r+1)p^r \cdot$$

(ii) Using (i), we get

$E(X) = E\left(X_{(1)}\right) = np$

$E\left(X(X-1)\right) = E\left(X_{(2)}\right) = n(n-1)p^2 \cdot$

Therefore,

$$E(X^2) = E\left(X(X-1)\right) + E(X)$$
$$= n(n-1)p^2 + np$$

and $\mathrm{Var}(X) = E(X^2) - \left(E(X)\right)^2 = npq$.

(iii) For $t \in \mathbb{R}$, we have

$$E(e^{tX}) = \sum_{x=0}^{n} e^{tx}\binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n}\binom{n}{x}(pe^t)^x q^{n-x}$$

$$= (q+pe^t)^n.$$

Therefore ,

$$E(T) = E(e^X + 2e^{-X} + 6X^2 + 3X + 4)$$

$$= E(e^X) + 2\,E(e^{-X}) + 6\,E(X^2) + 3\,E(X) + 4$$

$$= (q + pe)^n + 2e^{-n}(qe + p)^n + 6n(n-1)p^2 + 3np + 4. \blacksquare$$

We are familiar with the Laplace transform of a given real-valued function defined on $\mathbb{R}$. We also know that, under certain conditions, the Laplace transform of a function determines the function almost uniquely. In probability theory the Laplace transform of a p.d.f./p.m.f. of a random variable X plays an important role and is referred to as moment generating function (of probability distribution) of random variable X.

Definition 3.3

Let X be a random variable and let $A = \{t \in \mathbb{R}\colon E(|e^{tX}|) = E(e^{tX}) \text{ is finite}\}$. Define $M_X\colon A \to \mathbb{R}$ by

$$M_X(t) = E(e^{tX}), \;\; t \in A.$$

(i) We call the function $M_X(\cdot)$ the moment generating function (m.g.f.) (of probability distribution) of random variable X;

(ii) We say that the m.g.f. of a random variable X exists if there exists a positive real number a such that $(-a, a) \subseteq A$ (i.e., if $M_X(t) = E(e^{tX})$ is finite in an interval containing 0). $\blacksquare$

Note that $M_X(0) = 1$ and, therefore, $A = \{t \in \mathbb{R}\colon E(e^{tX}) \text{ is finite}\} \neq \phi$. Moreover, using Theorem 3.3 (ii)-(iii), we have $M_X(t) > 0, \forall t \in A$. Also if $M_X(t) = E(e^{tX})$ exists and is finite on an interval $(-a, a), a > 0$, then for any real constants c and d the m.g.f. of $Y = cX + d$ also exists and $M_Y(t) = M_{cX+d}(t) = E\left(e^{t(cX+d)}\right) = e^{td}E(e^{tcX}) = e^{td}M_X(ct), \; t \in \left(\frac{-a}{|c|}, \frac{a}{|c|}\right)$, with the convention that $a/0 = \infty \cdot$

The name moment generating function to the transform M_X is derived from the fact that $M_X(\cdot)$ can be used to generate moments of random variable X, as illustrated in the following theorem.

Theorem 3.4

Let X be a random variable with m.g.f. M_X that is finite on an interval $(-a, a)$, for some $a > 0$ (i.e., m.g.f. of X exists). Then,

(i) for each $r \in \{1, 2, \ldots\}$, $\mu_r' = E(X^r)$ is finite;

(ii) for each $r \in \{1, 2, \dots\}$, $\mu_r' = E(X^r) = M_X^{(r)}(0)$, where $M_X^{(r)}(0) = \left[\frac{d^r t}{dt^r} M_X(t)\right]_{t=0}$, the r-th derivative of $M_X(t)$ at the point 0;

(iii) $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r',\ t \in (-a, a)$.

Proof. We will provide the proof for the case where X is of absolutely continuous type. The proof for the case of discrete type X follows in the similar fashion with integral signs replaced by summation sings.

(i) We have $E(e^{tX}) < \infty, \forall t \in (-a, a)$. Therefore,

$$\int_{-\infty}^{0} e^{tx} f_X(x)dx < \infty, \forall t \in (-a, a) \text{ and } \int_{0}^{\infty} e^{tx} f_X(x)dx < \infty,\ \ \forall t \in (-a, a)$$

$$\Rightarrow \int_{-\infty}^{0} e^{-t|x|} f_X(x)dx < \infty, \forall t \in (-a, a) \text{ and } \int_{0}^{\infty} e^{t|x|} f_X(x)dx < \infty, \forall t \in (-a, a)$$

$$\Rightarrow \int_{-\infty}^{0} e^{-|t||x|} f_X(x)dx < \infty, \forall t \in (-a, a) \text{ and } \int_{0}^{\infty} e^{|t||x|} f_X(x)dx < \infty, \forall t \in (-a, a)$$

$$\Rightarrow \int_{-\infty}^{0} e^{-|tx|} f_X(x)dx < \infty, \forall t \in (-a, a) \text{ and } \int_{0}^{\infty} e^{|tx|} f_X(x)dx < \infty, \forall t \in (-a, a)$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{|tx|} f_X(x)dx < \infty, \forall t \in (-a, a),$$

i.e., $E\left(e^{|tX|}\right) < \infty, \forall t \in (-a, a)$. Fix $r \in \{1, 2, \dots\}$ and $t \in (-a, a) - \{0\}$. Then $\lim_{x \to \infty} \frac{|x|^r}{e^{|tx|}} = 0$ and therefore there exists a positive real number $A_{r,t}$ such that $|x|^r < e^{|tx|}$, whenever $|x| > A_{r,t}$. Thus we have

$$E(|X|^r) = \int_{-\infty}^{\infty} |x|^r f_X(x)dx$$

$$= \int_{|x| \le A_{r,t}} |x|^r f_X(x)dx + \int_{|x| > A_{r,t}} |x|^r f_X(x)dx$$

$$\leq A_{r,t}^r \int_{|x|\leq A_{r,t}} f_X(x)\,dx + \int_{|x|>A_{r,t}} e^{|tx|} f_X(x)\,dx$$

$$\leq\; A_{r,t}^r + \int_{-\infty}^{\infty} e^{|tx|}\, f_X(x)dx$$

$$< \infty.$$

(ii) Fix $r \in \{1, 2, \dots\}$. Then, for $t \in (-a, a)$,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx}\, f_X(x)dx$$

and $$M_X^{(r)}(t) = \frac{d^r}{dt^r} \int_{-\infty}^{\infty} e^{tx}\, f_X(x)dx.$$

Under the assumption that $M_X(t) = E(e^{tX}) < \infty, \forall t \in (-a, a)$, using arguments from advanced calculus, it can be shown that the derivative can be passed through the integral sign. Therefore, for $t \in (-a, a)$,

$$M_X^{(r)}(t) = \int_{-\infty}^{\infty} \frac{d^r}{dt^r} e^{tx} f_X(x)dx$$

$$= \int_{-\infty}^{\infty} x^r\, e^{tx} f_X(x)dx$$

$$\Rightarrow M_X^{(r)}(0) = \int_{-\infty}^{\infty} x^r\, f_X(x)dx = E(X^r).$$

(iii) Fix $r \in \{1, 2, \cdots\}$. Then, for $t \in (-a, a)$,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx}\, f_X(x)dx$$

$$= \int_{-\infty}^{\infty} \left(\sum_{r=0}^{\infty} \frac{t^r x^r}{r!} \right) f_X(x)dx.$$

Under the assumption that $M_X(t) = E(e^{tX}) < \infty, \forall t \in (-a, a)$, using arguments of advanced calculus, it can be shown that the integral sign can be passed through the summation sign, i.e.,

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r \, f_X(x) dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^{'}. \blacksquare$$

Corollary 3.2

Under the notation and assumptions of Theorem 3.4, define $\psi_X: (-a, a) \to \mathbb{R}$ by $\psi_X(t) = \ln M_X(t), t \in (-a, a)$. Then

$$\mu_1^{'} = \psi_X^{(1)}(0) \text{ and } \mu_2 = \text{Var}(X) = \psi_X^{(2)}(0),$$

where $\psi_X^{(r)}(\cdot)$ denotes the r-th $(r \in \{1, 2\})$ derivative of ψ_X.

Proof. We have, for $t \in (-a, a)$,

$$\psi_X^{(1)}(t) = \frac{M_X^{(1)}(t)}{M_X(t)} \text{ and } \psi_X^{(2)}(t) = \frac{M_X(t) M_X^{(2)}(t) - \left(M_X^{(1)}(t)\right)^2}{\left(M_X(t)\right)^2}.$$

Using the facts that $M_X(0) = 1$ and $M_X^{(r)}(0) = E(X^r), r \in \{1, 2\}$, we get

$$\psi_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = E(X),$$

and

$$\psi_X^{(2)}(0) = \frac{M_X(0) M_X^{(2)}(0) - \left(M_X^{(1)}(0)\right)^2}{\left(M_X(0)\right)^2}$$

$$= E(X^2) - \left(E(X)\right)^2$$

$$= \text{Var}(X). \blacksquare$$

Example 3.7

Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!}, & \text{if } x \in \{0, 1, 2, \cdots\} \\ 0, & \text{otherwise} \end{cases},$$

where $\lambda > 0$.

(i) Find the m.g.f. $M_X(t),\ t \in A = \{s \in \mathbb{R}: E(e^{sX}) < \infty\}$, of X. Show that X possesses moments of all orders. Find the mean and variance of X;

(ii) Find $\psi_X(t) = \ln\big(M_X(t)\big), t \in A$. Hence find the mean and variance of X;

(iii) What are the first four terms in the power series expansion of $M_X(\cdot)$ around the point 0?

Solution.

(i) We have

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}, \forall t \in \mathbb{R}.$$

Since $A = \{s \in \mathbb{R}: E(e^{sX}) < \infty\} = \mathbb{R}$, by Theorem 3.4 (i), for every $r \in \{1, 2, \cdots\}, \mu_r' = E(X^r)$ is finite. Clearly

$M_X^{(1)}(t) = \lambda e^t e^{\lambda(e^t-1)}$ and $M_X^{(2)}(t) = \lambda e^t e^{\lambda(e^t-1)}(1+\lambda e^t), t \in \mathbb{R}.$

Therefore,

$$E(X) = M_X^{(1)}(0) = \lambda,$$

$$E(X^2) = M_X^{(2)}(0) = \lambda(1+\lambda),$$

and $\quad \mathrm{Var}(X) = E(X^2) - E\big((X)\big)^2 = \lambda.$

(ii) We have, for $t \in \mathbb{R}$,

$\psi_X(t) = \ln\big(M_X(t)\big) = \lambda(e^t - 1),$

$\Rightarrow \psi_X^{(1)}(t) = \psi_X^{(2)}(t) = \lambda e^t.$

Therefore,

$E(X) = \psi_X^{(1)}(0) = \lambda$ and $\mathrm{Var}(X) = \psi_X^{(2)}(0) = \lambda.$

(iii) We have

$M_X^{(3)}(t) = \lambda e^t e^{\lambda(e^t-1)}(\lambda^2 e^{2t} + 3\lambda e^t + 1), t \in \mathbb{R}$

$\Rightarrow \mu_3' = E(X^3) = M_X^{(3)}(0) = \lambda(\lambda^2 + 3\lambda + 1).$

Since $A = \{s \in \mathbb{R}: E(e^{sX}) < \infty\} = \mathbb{R}$, by Theorem 3.4 (iii) ,we have

$$\begin{aligned} M_X(t) &= 1 + \mu_1' t + \mu_2'\frac{t^2}{2!} + \mu_3'\frac{t^3}{3!} + \cdots \\ &= 1 + \lambda t + \lambda(\lambda+1)\frac{t^2}{2!} + \lambda(\lambda^2+3\lambda+1)\frac{t^3}{3!} + \cdots,\quad t \in \mathbb{R}. \ \blacksquare \end{aligned}$$

Example 3.8

Let X be a random variable with p.d.f.

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

(i) Find the m.g.f. $M_X(t), t \in A = \{s \in \mathbb{R}: E(e^{sX}) < \infty\}$ of X. Show that X posseses moments of all orders. Find the mean and variance of X;

(ii) Find $\psi_X(t) = \ln\big(M_X(t)\big), t \in A$. Hence find the mean and variance of X;

(iii) Expand $M_X(t)$ as a power series around the point 0 and hence find $E(X^r), r \in \{1, 2, \cdots\}$.

Solution.

(i) We have

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}\, e^{-x} dx = \int_0^\infty e^{-(1-t)x}\, dx < \infty, \text{ if } t < 1.$$

Clearly $A = \{s \in \mathbb{R}: E(e^{sX}) < \infty\} = (-\infty, 1) \supset (-1, 1)$ and $M_X(t) = (1-t)^{-1}, t < 1$. By Theorem 3.4 (i), for every $r \in \{1, 2, \cdots\}, \mu_r'$ is finite. Clearly

$$M_X^{(1)}(t) = (1-t)^{-2} \text{ and } M_X^{(2)}(t) = 2(1-t)^{-3}, t < 1,$$
$$E(X) = M_X^{(1)}(0) = 1,$$
$$E(X^2) = M_X^{(2)}(0) = 2,$$

and $$\text{Var}(X) = E(X^2) - \big(E(X)\big)^2 = 1.$$

(ii) We have

$$\psi_X(t) = \ln\big(M_X(t)\big) = -\ln(1-t), \quad t < 1$$
$$\Rightarrow \psi_X^{(1)}(t) = \frac{1}{1-t} \text{ and } \psi_X^{(2)}(t) = \frac{1}{(1-t)^2}, \quad t < 1$$
$$\Rightarrow E(X) = \psi_X^{(1)}(0) = 1 \text{ and } \text{Var}(X) = \psi_X^{(2)}(0) = 1.$$

(iii) We have

$$M_X(t) = (1-t)^{-1} = \sum_{r=0}^{\infty} t^r, \quad t \in (-1, 1).$$

Since $A = \{s \in \mathbb{R}: E(e^{sX}) < \infty\} = (-\infty, 1) \supset (-1, 1)$, using Theorem 3.4 (iii), we conclude that

$\mu_r' =$ coefficient of $\frac{t^r}{r!}$ in the power series expansion of $M_X(t)$ around 0

$= r!$. ■

Example 3.9

Let X be a random variable with p.d.f.

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}\ , -\infty < x < \infty.$$

Show that the m.g.f. of X does not exist.

Solution. From Example 3.4 we know that the expected value of X is not finite. Therefore, using Theorem 3.4 (i), we conclude that the m.g.f. of X does not exist. ■