Lecture 31: General methods for finding intervals of absolute and relative stability

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1. Root Locus Method

The first and most direct-method is the root locus method . This consists of repeatedly solving the polynomial equation (8.34) for a range of values of $\bar{h}$ in the neighborhood of the origin. Any standard numerical method, such as Newton-Raphson iteration, may be employed for the approximate solution of (8.34). A plot of $|r_s|, s = 1, 2, \ldots, K$ against $\bar{h}$ then allows us to deduce intervals of stability in the neighborhood of the origin.

2. Schur Criteria

We shall call the second method we consider the Schur criterion. In fact, several criteria based on theorem of Schur have been proposed unstably the Wilf stability criterion.

We state the criterion for a general K^{th}degree polynomial, with complex coefficients

$$\phi(r) = C_K r^K + C_{K-1} r^{K-1} + \cdots + C_1 r + C_0$$

where $C_K \neq 0$, $C_0 \neq 0$. The polynomial $\phi(r)$ is said to be a Schur polynomial if its root r_s satisfy $|r_s| < 1, s = 1, 2, \ldots, K$. Define the polynomials

$$\tilde{\phi}(r) = C_0^* r^K + C_1^* r^{K-1} + \cdots + C_{K-1}^* r + C_K^*$$

where C_j^* is the complex conjugate of C_j and

$$\phi_1(r) = \frac{1}{r}\left[\tilde{\phi}(0)\,\phi(r) - \phi(0)\tilde{\phi}(r)\right].$$

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Clearly $\phi_1(r)$ has degree at most $K-1$. Then by a theorem of Schur $\phi\ (r)$ is a Schur polynomial if and only if $\left|\hat{\phi}(0)\right| > \ |\phi\ (0)|$ and $\phi_1\ (r)$ is a Schur polynomial.

Clearly, the interval (α,β) is an interval of absolute stability if, for all $\bar{h}\ \in(\alpha,\beta)$ the (real) stability polynomial $\pi\ (r,\bar{h}\ )$, is a Schur polynomial. Writing $\pi\ (r,\bar{h}\ )$ for $\phi\ (r)$, we can construct $\hat{\pi}(r,\bar{h})$ and $\pi_1(r,\bar{h})$. The first condition $\left|\ \hat{\pi}(0,\bar{h})\right| > \left|\pi\ (0,\bar{h}\ )\right|$ yields our first inequality in $\bar{h}$, while the second condition may be tested by writing $\pi_1(r,\bar{h})$ for $\phi\ (r)$ and repeating the process, there by obtaining a second inequality for $\bar{h}$, and so on. At each stage, the degree of the polynomial under test in reduced by one, so that eventually we merely have to state a criterion for a polynomial of degree one to be a Schur polynomial, and, obviously, this can easily be done.

We cannot use the Schur criterion directly to determine intervals of relative stability is the sense previously defined. However, if we adopt a definition of relative stability which requires that $|r_s| < \exp(\bar{h})\,, s = 1\,,2,\ldots,K$, then it is technically possible to use the Schur criterion. Substituting $r = R\exp(\bar{h})$ into (8.34) gives a polynomial equation in R,

$$\rho\ (R\exp(\bar{h})) - \ \bar{h}\ \sigma\ (R\exp(\bar{h})) = 0$$

If the roots of this equation are R_s, $s = 1, 2, \ldots, K$, then the Schur criterion gives necessary and sufficient conditions for $|R_s| < 1, s = 1, 2, \ldots, K$, that is, for $|r_s|\ < \exp\ (\bar{h})$. $s = 1, 2, \ldots, K$.

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Example: Use the Schur criterion to investigate the weak stability of the method

$$y_{n+2} - y_n = \frac{1}{2} h \left(f_{n+1} + 3f_n\right)$$

Solution: The stability polynomial is

$\pi\left(r,\bar{h}\right) = r^2 - \frac{1}{2}\bar{h}\, r - \left(1 + \frac{3}{2}\bar{h}\right)$. Thus,

$$\hat{\pi}\left(r,\bar{h}\right) = -\left(1 + \frac{3}{2}\bar{h}\right) r^2 - \frac{1}{2}\bar{h}\, r + 1$$

and the condition $\left|\hat{\pi}\left(0,\bar{h}\right)\right| > \left|\pi\left(0,\bar{h}\right)\right|$ is

satisfied if $\left|1 + \frac{3}{2}\bar{h}\right| < 1$, that is, if $\bar{h} \in \left(-\frac{4}{3}, 0\right)$

Now

$$\pi_1\left(r,\bar{h}\right) = \frac{1}{r}\left[r^2 - \frac{1}{2}\bar{h}\, r - \left(1 + \frac{3}{2}\bar{h}\right) + \left(1 + \frac{3}{2}\bar{h}\right)\left\{-\left(1 + \frac{3}{2}\bar{h}\right) r^2 - \frac{1}{2}\bar{h}\, r + 1\right\}\right]$$

$$= -\frac{1}{2}\bar{h}\left(2 + \frac{3}{2}\bar{h}\right)(3r + 1)$$

which has its only root at $-\frac{1}{3}$, and is therefore a Schur polynomial. It follows that the interval of absolute stability is $\left(-\frac{4}{3}, 0\right)$.

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Using the device described above to find, for this example, the interval of relative stability, given by the requirement $|r_s| < \exp(\bar{h})$, $s = 1,2,\ldots,K$, we consider the polynomial

$\phi(R,\bar{h}) = \exp(2\bar{h})R^2 - \frac{1}{2}\bar{h}\exp(\bar{h})R - \left(1+\frac{3}{2}\bar{h}\right)$. Then

$$\tilde{\phi}(R,\bar{h}) = -\left(1+\frac{3}{2}\bar{h}\right)R^2 - \frac{1}{2}\bar{h}\exp(\bar{h})R + \exp(2\bar{h})$$

and we obtain the first condition

$$\left|1+\frac{3}{2}\bar{h}\right| < (\exp(2\bar{h}))$$

After a little manipulation, we find that the first degree polynomial $\phi_1(R,\bar{h})$ is Schur if and only if

$$\left|\left\{\frac{1}{2}\bar{h}\exp(\bar{h})\right\}/\left\{\exp(2\bar{h}) - 1 - \frac{3}{2}\bar{h}\right\}\right| < 1.$$

A full solution of this pair of simultaneous inequalities for $\bar{h}$ involves considerable computation but on expanding the exponentials in powers of $\bar{h}$, it becomes clear that both inequalities are satisfied for all positive $\bar{h}$ and that the second is not satisfied for small negative $\bar{h}$. We conclude that $(0,\infty)$ is an interval of relative stability.