Lecture 22: Local Error of Nystrom & Milne-Simpson Methods

The Lecture Contains:

This lecture introduces the local error in the case of Nystrom formula and the generalized Milne-Simpson method and derives such errors for the special cases,viz. the mid-point rule and the Milne-Simpson formula.

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The error in Nystrom's formula is given by

$$y\left(t_{p+1}\right) - y\left(t_{p-1}\right) =$$
$$\int_{t_{p-1}}^{t_{p+1}} \sum_{m=0}^{q} (-1)^m \binom{-S}{m} \nabla^m y'\left(t_p\right) dt + \int_{t_{p-1}}^{t_{p+1}} (-1)^{q+1} \binom{-S}{q+1} h^{q+1} y_{(\xi)}^{(q+2)} \, dt$$

$$= h \sum_{m=0}^{q} K_m \nabla^m y'\left(t_p\right) + R_q^{NY}$$

and similarly in the Generalized Milne-Simpson method, we have the error R_q^{MS} as

$$y\left(t_p\right) - y\left(t_{p-2}\right) = h \sum_{m=0}^{q} K_m^* \nabla^m y'\left(t_p\right) + R_q^{MS}$$

The method used above for getting the expression for the errors in Adams-Bashforth and Adams-Moulton formulas does not work here, since $\binom{-S}{q+1}$ now changes sign in the interval of integration. However, we confine ourselves to the derivation, again by special methods, of error formulas for the mid-point rule and for the special Milne formula. Specifically, we shall show that

$$R_0^{NY} = \frac{1}{3} h^3 y'''(\xi) \qquad t_{p-1} < \xi < t_{p+1} \qquad (7.28)$$

and

$$R_2^{MS} = -\frac{1}{90} h^5 y^{(5)}(\xi) \qquad t_{p-2} < \xi < t_p \qquad (7.29)$$

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In order to prove (7.28), we may assume without loss of generality that $p = 0$ and we then have to show that (7.28) holds with $t_{p\pm 1} = \pm h$

and

$$R_0^{NY} = y\left(t_{p+1}\right) - y\left(t_{p-1}\right) - h\, K_0\, \nabla^0\, y'\left(t_p\right)$$

$$= y\,(h) - y\,(-h) - 2\,h\,y'\,(0) \qquad (7.30)$$

Using Taylor's expansion with remainder, we have

$$y\,(h) = y\,(0) + h\,y'\,(0) + \frac{1}{2}\,h^2\,y''\,(0) + \frac{1}{6}\,h^3\,y'''(\xi_1)$$

where $0 < \xi_1 < h$, and similarly

$$y\,(-h) = y\,(0) - h\,y'\,(0) + \frac{1}{2}\,h^2\,y''\,(0) - \frac{1}{6}\,h^3\,y'''(\xi_2)$$

where $-h < \xi_2 < 0$. Inserting in (7.30), we get

$$R_0^{NY} = \frac{1}{6}\,h^3\,[y'''\,(\xi_1) + y'''\,(\xi_2)]$$

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Since the function $y'''(\xi_1)$ is continuous, it assumes all values between $y'''(\xi_1)$ and $y'''(\xi_2)$ in the interval $(\xi_2,\ \xi_1)$. Therefore, for some $\xi \in (\xi_2,\ \xi_1)$

$$y'''(\xi_1) + y'''(\xi_2) = 2\,y'''(\xi)$$

which establishes the desired result.

In order to prove (7.29), we may assume, again without loss of generality, that $p = 1, t_p = h$. We then have to show that (7.29) holds for some $\xi \in (-h, h)$ and

$$R_2^{MS} = y(t_p) - y(t_{p-2}) - h \sum_{m=0}^{2} K_m^* \nabla^m y'(t_p)$$

$$= y(t_p) - y(t_{p-2}) - h\left[2\,y'(t_p) - 2\,\nabla^1 y'(t_p) + \tfrac{1}{3}\nabla^2 y'(t_p)\right]$$

$$= y(h) - y(-h) - h\left[2\,y'(h) - 2\,y'(h) + 2\,y'(0) + \tfrac{1}{3}\nabla^2 y'(h)\right]$$

or

$$R_2^{MS} = y(h) - y(-h) - h\left[2\,y'(0) + \tfrac{1}{3}\nabla^2 y'(h)\right]$$

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Module 7: Multistep Methods

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It is easily verified that R_2^{MS} can be represented in terms of the definite integral

$$R_2^{MS} = \int_0^h \left[y'(t) + y'(-t) - 2y'(0) - \frac{t^2}{h^2} \nabla^2 y'(h) \right] dt$$

we now consider the function

$$F(x) = \int_0^x \left[y'(t) + y'(-t) - 2y'(0) - \frac{t^2}{h^2} \nabla^2 y'(h) \right] dt$$

$$- \lambda \int_0^x \left[\binom{1 - t/h}{4} + \binom{1 + t/h}{4} \right] dt$$

We have $F(0) = 0,\ F(-x) = -F(x)$, and select λ such that $F(h) = 0$. This is possible in view of

$$\int_0^h \left[\binom{1 - t/h}{4} + \binom{1 + t/h}{4} \right] dt = h \int_{-2}^{0} \binom{-S}{4} dS$$

$$= h\, K_4^* = -\frac{1}{90} h \neq 0$$

with this choice of λ, we have

$$F(\pm h) = R_2^{MS} + \frac{1}{90} \lambda h = 0 \qquad (7.31)$$

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Since $F(\pm h) = F(0) = 0,\ F'(\pm\xi) = 0$ for some $\xi \epsilon (0,h)$. But

$$F'(x) = y'(x) + y'(-x) - 2y'(0) - \frac{x^2}{h^2}\nabla^2 y'(h)$$

$$-\lambda\left[\left(1 - \frac{x}{4}/h\right) + \left(1 + \frac{x}{4}/h\right)\right]$$

vanishes also for $x = 0$ and $x = \pm h$. Thus, in view of the fact that $F'(x)$ has five distinct zeros in the closed interval $(-h,h)$, the fifth derivative $F^{(5)}(x)$ has at least one zero in $(-h,h)$, the fifth derivative $F^{(5)}(x)$ has at least one zero in $(-h,h)$. We easily find

$$F^{(5)}(x) = y^{(5)}(x) + y^{(5)}(-x) - 2\lambda h^{-4}$$

and thus have, for some $\xi_1 \epsilon (-h,h)$

$$\lambda = \frac{1}{2}h^4\left[y^{(5)}(\xi_1) + y^{(5)}(-\xi_1)\right]$$

But, by the continuity of $y^{(5)}(t)$ it follows as before that

$$\lambda = h^4 y^{(5)}(\xi)$$

for some ξ between ξ_1 and $-\xi_1$. Inserting this value in (7.31), (7.29) follows: