Lecture 3: Stability and Asymptotic Stability

The Lecture Contains:

This lecture introduces some basic notions (definitions) of an equilibrium point, stable point and an asymptotically stable point. It also gives very briefly the study of stability of an autonomous system.

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The notion of a well posed problem is related to the more common notion of stability as indicated by the following definition.

Definition: Consider the differential equation $y' = f(t, y)$ and without loss of generality, let the origin $(0,0)$ be an equilibrium point, i.e. $f(0,0) = 0$. Then the origin is:

1. Stable, if a perturbation of the initial condition $|y(0)| < \varepsilon$ grows no larger than ε for subsequent times, i.e. if $|y(t)| < \varepsilon$ for $t > 0$
2. Asymptotically Stable, if it is stable and $|y(0)| < \varepsilon$ implies that
$$\lim_{t\to\infty} |y(t)| = 0$$
grows no larger than ε for subsequent times, i.e. if $|y(t)| < \varepsilon$ for $t > 0$
3. Unstable if it is not stable.

Remark: This definition could also involve perturbations of $f(t, y)$, which are omitted for simplicity.

An autonomous system is one where $f(t, y)$ does not explicitly depend on t, i.e. $f(t, y) = f(y)$.

If $(0,0)$ is an equilibrium point then, in this case, $f(0) = 0$. Expanding the solution in a Taylor's series, we have

$$y'(t) = f(y) = f(0) + f_y(0)\, y(t) + O(y^2)$$

Since $f(0) = 0$, we have

$$y'(t) = f_y(0)\, y(t) + O(y^2)$$

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Letting $\lambda = f_y(0)$, we see that the stability of an autonomous system is related to that of the simple linear IVP

$$y' = \lambda y \qquad t > 0$$

$$y(0) = \varepsilon \qquad (1.11)$$

Its solution is given by

$$y(t) = \varepsilon e^{\lambda t}$$

and hence, (1.11) is

1. stable when $\mathrm{Re}(\lambda) \le 0$,
2. asymptotically stable when $\mathrm{Re}(\lambda) < 0$,and
3. unstable when $\mathrm{Re}(\lambda) > 0$.

Remark: These conclusions remain true for the original non linear autonomous problem when $\mathrm{Re}(\lambda) \neq 0$ and y is small enough for the $O(y^2)$ term to be negligible relative to λy. This cannot happen in the stable case $(\mathrm{Re}(\lambda) = 0)$; hence, it requires a more careful analysis.

The analysis of non-autonomous systems is similar.

Analyzing the stability of autonomous vector systems

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}(\mathbf{t})) \qquad (1.12)$$

is more complicated. Once again, assume that $\mathbf{y} = 0$ is an equilibrium point and expand by Taylor's series

$$y' = f(0) + f_y(0)\,y + 0(\|\mathbf{y}\|^2)$$

$$= f_y\,(o)\,y + 0(\|\mathbf{y}\|^2).$$

We need a brief digression for a few definitions.

Definition: The Jacobian matrix of a vector-valued function $\mathbf{f}(\mathbf{y})$ with respect to $\mathbf{y}$ is the matrix

$$\mathbf{f_y} = \frac{\partial \mathbf{f}}{\partial \mathbf{y}} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix} \qquad (1.13)$$

Definition: The norm of a vector $\mathbf{y}$ is a scalar $\|\mathbf{y}\|$ such that

1. $\|\mathbf{y}\| \geq 0$ and $\|\mathbf{y}\| = 0$ if and only if y=0,
2. $\|\alpha\mathbf{y}\| = |\alpha|\|\mathbf{y}\|$ for any scalar α, and
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

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Let us return to our stability analysis and let $A = \mathbf{f_y}(0)$. Then the stability of the non-linear autonomous system (1.12) is related to that of the linear system

$$\mathbf{Y}' = \mathbf{A}\mathbf{Y}$$

$$\mathbf{Y}(0) = \mathbf{Y_0} \quad , \quad \|\mathbf{Y_0}\| \le \varepsilon \qquad (1.14)$$

The solution of this system is

$$\mathbf{Y}(\mathrm{t}) = \mathrm{e}^{\mathrm{At}}\mathbf{Y}_0 \qquad (1.15)$$

where the matrix exponential is defined by the series

$$\mathrm{e}^{\mathrm{At}} = \mathrm{I} + \mathrm{tA} + \frac{\mathrm{t}^2}{2!}\mathrm{A}^2 + \frac{\mathrm{t}^3}{3!}\mathrm{A}^3 + \;\ldots\ldots\ldots \qquad (1.16)$$

Often, the matrix A can be diagonalized as

$$\mathrm{T}^{-1}\mathrm{AT} = \Lambda = \begin{bmatrix} \lambda_1 & & & \text{O} \\ & \lambda_2 & & \\ & & \ddots & \\ \text{O} & & & \lambda_n \end{bmatrix}$$

where $\lambda_1, \lambda_2, \ldots\ldots\ldots\ldots\lambda_n$ are the eigen values of A and the columns of T are the corresponding eigen vectors. In this case, we may easily verify that the solution of (1.14) is

$$\mathrm{Y}(\mathrm{t}) = \mathrm{T}\ \mathrm{e}^{\Lambda \mathrm{t}}\mathrm{T}^{-1}\mathrm{Y}_0$$

Thus, (1.14) is stable if all of the given values have non-positive real parts, and asymptotically stable if all of the eigen values have negative real parts, and unstable otherwise.

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Unfortunately, not all matrices are diagonalizable. However, A can always be reduced to the Jordan Canonical form

$$T^{-1}AT = \begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_i \end{bmatrix} \qquad (1.17)$$

where each Jordan block has the form

$$\Lambda_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{bmatrix} \qquad (1.18)$$

The dimension of the Jordan block Λ_i corresponds to the multiplicity of the eigen value λ_i. Thus, if λ_i is simple, the block is a scalar. With this, it is relatively easy to show that $Y = 0$ is

1. stable when either $(\lambda_i) < 0$ or $\mathrm{Re}(\lambda_i) = 0$ and λ_i is simple, $i = 1, 2, \ldots, n$.
2. asymptotically stable when $\mathrm{Re}(\lambda_i) < 0, i = 1,2 \ldots, n$, and
3. unstable otherwise.

Problems

1. Suppose $\frac{\partial f}{\partial y}$ is continuous on a closed convex domain D. Show that $f(t, y)$ satisfies a Lipschitz condition on D with

$$L = \max_{(t, y) \in D} \left| \frac{\partial f}{\partial y} \right|$$

2. Are the following IVPs well posed?
   a. $y' = \sqrt{1 - y^2}, y(0) = 0$
   b. $y' = \sqrt{y^2 - 1}, y(0) = 2$