Module 2: Single Step Methods

Lecture 5: Convergence of Euler's Method

The Lecture Contains:

This lecture starts with broad definitions of local truncation error, round–off error and convergence of a difference method (more precise definitions to follow in subsequent lectures). We also discuss convergence of Euler's method and derive a bound (a-priori) for the error.

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In general, the following question arises:

Does the Euler method converge as $h \to 0$?

The answer is given in the following theorem. Before we state the theorem, following definitions are in order:

Definition: The local truncation error (or local error) associated with a given difference method is that quantity which fails to satisfy the exact solution of the difference equation.

Definition : The round-off error associated with a given method is that quantity which must be added to a finite representation of a computed number in order to make it the exact representation of that number.

Definition : We roughly define a method as convergent for a problem if, as more grid (mesh) points are taken, the numerical solution converges to the true solution in the absence of round-off errors.

We shall make these definitions more precise when specific classes of methods are discussed later.

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Theorem: If $f(t, y)$ satisfies a Lipschitz condition in y and is continuous in t for $0 \le t \le b$ and a ll y, if the sequence $\{y_i\}, i = 1, \dots, n$ is defined by (2.1) and if $y_0 \to y(0)$, then $y_n \to y(t)$ as $n \to \infty$ uniformly in t, where $y(t)$ is the solution of the IVP.

$$y' = f(t, y),\ y(0) = y_0.$$

Remark: We will call y_0 the starting value to distinguish it from the initial value $y(0)$. In practice, we can only expect the starting value used in numerical computations to approach the initial value as the mesh size h decreases and as we use more precision in our computation. In this theorem, we are assuming that (2.1) is solved without round-off errors.

The proof of the above theorem is given below. It consists of deriving a bound for the error

$$e_n = y_n - y(t_n)$$

and showing that this bound can be made arbitrarily small. If a bound for the error depends only on the knowledge of the problem but not on its solution $y(t)$, it is called an a priori bound. If, on the other hand, a knowledge of the properties of the solution is required, its error bound is called an a posteriori bound.

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Proof: To get an a priori bound, let us write

$$y(t_{n+1}) = y(t_n) + h\, f\big(t_n, y(t_n)\big) - d_n \qquad \textbf{(2.2)}$$

where d_n is called the local truncation error. It is the amount by which the solution fails to satisfy the difference method. Subtracting (2.2) from (2.1), we get

$$e_{n+1} = e_n + h[f\,(t_n, y_n) - f(t_n, y(t_n)] + d_n \qquad \textbf{(2.3)}$$

Let us write

$$f(t_n, y_n) - f\big(t_n, y(t_n)\big) = e_n L_n \qquad \textbf{(2.4)}$$

Therefore,

$$e_{n+1} = e_n(1 + h\, L_n) + d_n$$

This is a difference equation for e_n. The error e_0 is known, so it can be solved if we know L_n and d_n. We have a bound of the Lipschitz constant L for $|L_n|$. Suppose we also have $D \geq |d_n|$. Then we have

$$|e_{n+1}| \leq |e_n|(1 + hL) + D \qquad \textbf{(2.5)}$$

To proceed further, we need the following lemma.

Lemma: If $|e_n|$ satisfies (2.5) and $0 \le n\,h \le b$, then

$$|e_n| \le D\frac{(1+hL)^n - 1}{h\,L} + (1+hL)^n\,|e_0| \le \frac{D}{hL}\left(e^{L\,b} - 1\right) + e^{Lb}|e_0| \quad \textbf{(2.6)}$$

Proof of the Lemma: The first inequality of (2.6) follows by induction. It is trivially true for $n = 0$. Assuming that is true for n, we have from (2.5)

$$\begin{aligned}
|e_{n+1}| &\le D\frac{(1+hL)^n - 1}{hL}(1+hL) + D + (1+hL)^{n+1}\,|e_0| \\
&= D\frac{(1+hL)^{n+1} - (1+hL) + hL}{hL} + (1+hL)^{n+1}|e_0| \\
&= D\frac{(1+hL)^{n+1} - 1}{hL} + (1+hL)^{n+1}\,|e_0|
\end{aligned}$$

Hence (2.6) is true for $n + 1$, and thus for all n.

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The second inequality in (2.6) follows from the fact that $nh \le b$, and for $hL \ge 0, (1+hL) \le e^{Lh}$, so that $(1+hL)^n \le e^{Lnh} \le e^{Lb}$, proving the lemma.

To continue the proof of the theorem, we need to investigate D, the bound on the local truncation error.

From (2.2), we have

$$-d_n = y(t_{n+1}) - y(t_n) - h\, f(t_n, y(t_n))$$

By the Mean value theorem, we get for $0 \le \theta \le 1$,

$$|d_n| = \left|h\, f(t_n + \theta\, h, y(t_n + \theta\, h)) - h\, f(t_n, y(t_n))\right|$$

$$\le h\left|f(t_n + \theta h, y(t_n)) - f(t_n, y(t_n))\right| + h\left|f(t_n + \theta h, y(t_n + \theta h)) - f(t_n + \theta h, y(t_n))\right| \quad \textbf{(2.7)}$$

$$\le h|f(t_n + \theta h, y(t_n)) - f(t_n, y(t_n))| + hL|y(t_n + \theta h) - y(t_n)|$$

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The last term can be treated by the Mean value theorem to get a bound

$L\theta h^2 |y'(g)| \leq h^2 LZ$

where $Z = \max|y'(t)|$, which exists because of the continuity of y and f in a closed region. The treatment of the first term in (2.7) depends on our hypothesis. If we are prepared to assume that $f(t, y)$ also satisfies a Lipschitz condition in t (as will happen in practice), we can bound the first term in (2.7) by $K\,\theta\,h^2$, where K is the Lipschitz constant for f as a function of t. Consequently,

$|d_n| \leq\ h^2\ (K + LZ) = D$

and so from (2.6), we get

$$|e_n| \leq h\frac{K+LZ}{L}\left(e^{Lb} - 1\right) + e^{Lb}|e_o| \qquad \textbf{(2.8)}$$

Thus the numerical solution converges as $h \to 0$, if $|e_0| \to 0$.

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