The Lecture Contains:

We continue with the details about the derivation of the two–stage implicit Runge-Kutta methods. A brief description of semi-explicit Runge-Kutta methods is also given. Finally, an advantage of implicit Runge-Kutta methods in terms of absolute stability characteristics is illustrated by an example.

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Note: It is of interest to note that this is precisely the number of undetermined coefficients we had at our disposal when we derived the general three-stage explicit Runge-Kutta method. There, however, the form of the expansion for $\phi(t, y, h)$ precluded any possibility of attaining an order greater than three. In the present case, the form of the expansion for $\phi(t, y, h)$ holds out a possibility of attaining order four.

We now have eight equations to be satisfied but have only six coefficients at our disposal. However, if we solve (6.8), (6.9) and the last equations of (6.10) and (6.11), we find

$$C_1 = C_2 = \frac{1}{2}$$

$$a_1 = \frac{1}{2} \pm \frac{\sqrt{3}}{6} \qquad (6.12)$$

$$a_2 = \frac{1}{2} \mp \frac{\sqrt{3}}{6}$$

with superior alternative signs to be taken together. On substituting these values into the rest of equations, we find that all of the remaining equations of (6.8)-(6.11) are satisfied if

$$b_{11} = b_{22} = \frac{1}{4}$$

$$b_{12} = a_1 - \frac{1}{4}$$

$$b_{21} = a_2 - \frac{1}{4}$$

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Module 6: Implicit Runge-Kutta Methods

Lecture 17: Derivation of Implicit Runge-Kutta Methods(Contd.)

It follows from symmetry that the two solutions indicated by the alternative signs in (6.12) lead to the same method. Thus there exists a unique two-stage implicit Runge-Kutta method of order four, defined by

$$y_{n+1} - y_n = \frac{h}{2}(K_1 + K_2)$$

$$K_1 = f\left(t_n + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)h,\ y_n + \frac{1}{4}h\,K_1 + \left(\frac{1}{4} + \frac{\sqrt{3}}{6}\right)h\,K_2\right)$$

$$K_2 = f\left(t_n + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)h,\ y_n + \left(\frac{1}{4} - \frac{\sqrt{3}}{6}\right)h\,K_1 + \frac{1}{4}h\,K_2\right) \qquad (6.13)$$

a method originally proposed by Hammer and Hollingsworth.

Note: The method we have derived above extends to general R. For any $R \geq 2$, there exists an R-stage implicit Runge-Kutta method of order $2R$.

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This considerable increase in attainable order for a given number of stages does not necessarily make implicit Runge-Kutta methods more computationally efficient than explicit ones. In order to apply an R-stage implicit Runge-Kutta method, at each stage it is necessary to solve the system of R implicit non-liner equations which define the K_r. This needs to be done by some iterative process. The discussion on this would be dealt with later in these notes.

To overcome this implicitness, a somewhat less formidable problem arises in the case when $b_{rs} = 0 \text{ for } r < s$, and the resulting methods are called Semi-explicit Runge-Kutta methods. $(\text{for explicit methods } b_{rs} = 0 \text{ in } r \leq s)$.

One can find that R-stage semi-explicit method can attain higher order than an R-stage explicit Runge-Kutta method as demonstrated by the following fourth order three-stage method given by

$$y_{n+1} - y_n = \frac{h}{6}(K_1 + 4K_2 + K_3)$$

$$K_1 = f(t_n, y_n)$$

$$K_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{1}{4}hK_1 + \frac{1}{4}hK_2\right) \qquad (6.14)$$

$$K_3 = f(t_n + h, y_n + hK_2)$$

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Note: The main interest of implicit Runge-Kutta methods lies in their absolute stability characteristics, which are much superior to those of explicit methods.

Example: For example, if we analyze the method given by (6.14) for absolute stability, we obtain

$$\frac{y_{n+1}}{y_n} = \frac{1+\frac{1}{2}\bar{h}+\frac{1}{12}\bar{h}^2}{1-\frac{1}{2}\bar{h}+\frac{1}{12}\bar{h}^2}, \text{ where } \bar{h} = \lambda h \qquad (6.15)$$

This is a fourth order rational approximation (or (2, 2) Pade approximation) to $\exp(\bar{h})$, whereas a fourth order explicit Runge-Kutta method produces a fourth order polynomial approximation to the exponential. It follows from (6.15) that the interval of absolute stability for the method (6.14) is $(-\infty, 0)$.

Remark: One can conclude that implicit Runge-Kutta method can offer substantially improved regions of absolute stability but at such a high computational cost. The demand for such methods arises when weak stability considerations are paramount; such as solving stiff systems of initial value problems to be discussed later.

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Problems

1. Prove that the alternative solutions given in (6.12) both lead to the same method given by (6.13).
2. Show that when $f(t,y) \equiv g(t)$, the implicit method (6.13) reduces to a quadrature formula which is equivalent to the two- point Gauss-Legendre quadrature formula $\int_{-1}^{1} F\,(t)dt = F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)$.
3. In addition to (6.13), Hammer and Hollingsworth proposed the method

   $$y_{n+\frac{1}{2}} = y_n + \frac{h}{3}\left(y'_{n+\frac{1}{2}} + y'_n\right)$$

   $$y_{n+1} = y_n + \frac{h}{4}\left(3y'_{n+\frac{1}{2}} + y'_n\right)$$

   where, $y'_n = f(t_n,\ y_n)$, $\quad y'_{n+\frac{1}{2}} = f\left(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}\right)$..

   Write this method in the form (6.1)-(6.3) and use (6.8)-(6.11) to show that it is of third order.
4. Prove that the semi–explicit method (6.14) has order four and find its interval of absolute stability.
5. Find the order of the Implicit Runge-Kutta method

   $$y_{n+1} = y_n + \frac{1}{6}h\left[4f(t_n, y_n) + 2f(t_{n+1}, y_{n+1}) + hf'(t_n, y_n)\right]$$

   and determine its interval of absolute stability.
6. Find the order of the method

   $$y_{n+1} = y_n + \frac{h}{2}\left[f(y_n) + f(y_{n+1})\right] + \frac{h^2}{12}\left[(f'(y_n) - f'(y_{n+1})\right]$$

   where $y' = f(y)$.

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