Module 8: Linear Multistep Methods

Lecture 32: Some more methods for Absolute & Relative Stability

The Lecture Contains:

3. Routh-Hurwitz Criterion:

An alternative to Schur criterion consists of applying a transformation which maps the interior of the unit circle into the left hand half plane, and then appealing to the well-known Routh-Hurwitz criterion, which gives necessary and sufficient condition for the roots of a polynomial to have negative real parts. The appropriate transformation is $r = (1+Z)/(1-Z)$; this maps the circle $|r| = 1$ into the imaginary axis $R_e\, Z = 0$, the interior of the circle into the half plane $R_e Z < 0$, and the point $r = 1$ into $Z = 0$. Under this transformation, the stability polynomial becomes

$$\rho\,((1+Z)/(1-Z)) - \bar{h}\,\sigma\,((1+Z)/(1-Z)) = 0.$$

On multiplying through by $(1-Z)^K$, this becomes a polynomial equation of degree K, which we write

$$a_0 Z^K + a_1\, Z^{k-1} + \cdots + a_K = 0 \qquad (8.37)$$

where, we assume without loss of generality that $a_0 > 0$. The necessary and sufficient condition for the roots of (8.37) to lie in the half plane $R_e Z < 0$, that is, for the roots of $\pi\left(r, \bar{h}\right) = 0$ to lie within the circle $|r| < 1$, is that all the leading principal minors of Q be positive, where Q is the $K \times K$ matrix defined by

$$Q = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & a_{2K-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2K-2} \\ 0 & a_1 & a_3 & \cdots & a_{2K-3} \\ 0 & a_0 & a_2 & \cdots & a_{2K-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_K \end{bmatrix}$$

and where $a_j = 0$ if $j > K$. It can be shown that this condition implies $a_j > 0, j = 0, 1, \ldots, K$. Thus the positivity of the coefficients in (8.37) is a necessary but not sufficient condition for absolute stability. For $K = 2, 3, 4$ the necessary and sufficient condition for absolute stability given by this criterion are as follows,

$K = 2:\ a_0 > 0,\ a_1 > 0,\ a_2 > 0.$

K = 3 : $a_0 > 0,\ a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_1 a_2 - a_3 a_0 > 0$

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Example: Use the Routh-Hurwitz criterion to investigate stability of $y_{n+2} - y_n = \frac{1}{2} h \left(f_{n+1} + 3f_n\right)$.

Solution: The stability polynomial is

$$\pi\left(r, \bar{h}\right) = r^2 - \frac{1}{2}\bar{h}\, r - \left(1 + \frac{3}{2}\bar{h}\right)$$

giving on transformation

$$(1-Z)^2 \left[\left(\frac{1+Z}{1-Z}\right)^2 - \frac{1}{2}\bar{h}\left(\frac{1+Z}{1-Z}\right) - \left(1 + \frac{3}{2}\bar{h}\right)\right] = a_0\, Z^2 + a_1\, Z + a_2$$

where $a_0 = -\bar{h}$, $a_1 = 4 + 3\bar{h}$, $a_2 = -2\bar{h}$.

The Routh-Hurwitz criterion is clearly satisfied if and only if $\bar{h} \in \left(-\frac{4}{3}, 0\right)$, which is the required interval of absolute stability. If we investigate relative stability as given by the requirement $|r_s| < \exp(\bar{h})$, $s = 1, 2, \ldots, K$, we obtain

$$(1-Z)^2 \left[\left(\frac{1+Z}{1-Z}\right)^2 \exp(2\bar{h}) - \frac{1}{2}\bar{h}\left(\frac{1+Z}{1-Z}\right)\exp(\bar{h}) - \left(1 + \frac{3}{2}\bar{h}\right)\right] = a_0\, Z^2 + a_1 Z + a_2$$

where, for relative stability

$$a_0 = \exp\left(2\bar{h}\right) + \frac{1}{2}\bar{h}\exp\left(\bar{h}\right) - 1 - \frac{3}{2}\bar{h} > 0$$

$$a_1 = 2\exp\left(2\bar{h}\right) + 2 + 3\bar{h} > 0$$

$$a_2 = \exp\left(2\bar{h}\right) - \frac{1}{2}\bar{h}\, \exp\left(\bar{h}\right) - 1 - \frac{3}{2}\bar{h} > 0$$

we find once again that $(0, \infty)$ is an interval of relative stability.

4. Boundary Locus method: It requires neither the computation of roots of the polynomial nor the solving of simultaneous inequalities. The roots r_s of the stability polynomial are, in general, complex numbers; for the moment let us regard $\bar{h}$ as complex. Then instead of defining an interval of absolute stability to be an interval of the real $\bar{h}$ line such that the roots of $\pi\left(r,\bar{h}\right)=0$ lie within the unit circle whenever $\bar{h}$ lies in the interior of the interval, we define a region of absolute stability to be a region of the complex $\bar{h}$ – plane such that the roots of $\pi\left(r,\bar{h}\right)=0$ lie within the unit circle whenever $\bar{h}$ lies in the interior of the region. Let us call the region R and its boundary ∂R. Since the roots of the roots of $\pi\left(r,\bar{h}\right)=0$ are continuous functions of $\bar{h}$, $\bar{h}$ will lie on ∂R when all of the roots of $\pi\left(r,\bar{h}\right)=0$ lie on the boundary of the unit circle, i.e., when $\pi\left(\exp(i\theta),\bar{h}\right)=\rho\left(\exp(i\theta)\right)-\bar{h}\,\sigma\left(\exp(i\theta)\right)=0$. It follows that the of ∂R is given by $\bar{h}\,(\theta)=\rho(\exp(i\theta))/\sigma\left(\exp\,(i\theta)\right)$. For real $\bar{h}$, the end points of the interval of absolute stability will be given by the points at which ∂R cuts the real axis.

Example: Let us illustrate the method for $y_{n+2}-y_n=\frac{1}{2}\,h\,(f_{n+1}+3\,f_n)$.

For this method, $\rho\,(r)=r^2-1$, $\sigma\,(r)=\frac{1}{2}\,(r+3)$

$$\bar{h}\,(0)=\rho(\exp\,(i\theta))/\sigma\,\rho(\exp\,(i\theta))=2\,[\exp(2\,i\,\theta)-1]/[\exp(i\,\theta)+3]$$

$$=\;[3(\cos 2\theta-1)+\;i\,(3\sin 2\theta+2\sin\theta)](\,5+3\cos\theta)$$

This is the locus of ∂R, and it crosses the real axis where $\sin\theta=0$ or $3\cos\theta=-1$, i.e. $\theta=0,\quad\pi,\quad\pi\pm\cos^{-1}\left(\frac{1}{3}\right)$. At $\theta=0,\pi\quad\bar{h}\,(\theta)=0$, while at $\theta=\pi\pm\cos^{-1}\left(\frac{1}{3}\right),\quad\bar{h}\,(\theta)=-\frac{4}{3}$. The end points of the interval of absolute stability are thus $-\frac{4}{3}$ and 0.

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Problems

1. Show that the operator associated with the difference method

$$y_{n+4} - y_n = \frac{4}{3}h\left(2f_{n+3} - f_{n+2} + 2f_{n+1}\right)$$

is of order 4 and its error constant is $\frac{7}{90}$.

2. Determine the constants α and β in such a way that the operator associated with

$$Y_{n+3} - Y_n + \alpha(Y_{n+2} - Y_{n+1}) = h\beta(f_{n+2} + f_{n+1})$$

is of order 4, and determine the error constant. Verify that the resulting method is unstable.

3. Find the most accurate implicit linear two-step method. Find its principle part of the local truncation error.

4. Show that the order of the linear multistep method

$$Y_{n+2} - Y_{n+1} = \frac{h}{12}\left(4f_{n+2} + \delta f_{n+1} - f_n\right)$$

is zero. By finding the exact solution of the difference equation which arise when this method is applied to the initial value problem

$$Y' = 1, \qquad y(0) = 0$$

and demonstrate that the method is indeed divergent.

5. Show that the order of the linear multistep method

$$Y_{n+2} + (b-1)\,Y_{n+1} - b\,Y_n = \frac{h}{4}\left[(b+3)\,f_{n+2} + (3b+1)\,f_n\right]$$

is 2 if $b \neq -1$ and is 3 if $b = -1$. Show that the method is zero-unstable if $b = -1$. Illustrate the resulting divergence of the method with $b = -1$ by applying it to the initial value problem

$$Y' = Y \qquad Y(0) = 1$$

and solving exactly the resulting difference equation when the starting values are $Y_0 = 1,\ \ Y_1 = 1$.

6. Find the range of α for which the linear multistep method

$$Y_{n+3} + \alpha(Y_{n+2} - Y_{n+1}) - Y_n = \frac{(3+\alpha)h}{2}\left[f_{n+2} + f_{n+1}\right]$$

is zero-stable. Show that there exists a value of α for which the method has order 4 but that if the method is to be zero-stable, its order cannot exceed 2.

7. If $\rho(\xi) = \xi^4 - 1$, find a $\sigma(\xi)$ of degree four such that the method has maximum order. What is that order and what is the error constant?

8. If $\sigma(\xi) = \xi^2$. Find $\rho(\xi)$ such that $\rho(\xi)$ is of second degree and the order is two.

9. Find the region of absolute stability of the method given by

$$y_{n+1} - y_n = h\, f_{n+1}$$

10. Find the interval of absolute stability for the two-step Adams-Bashforth method given by

$$y_{n+2} - y_{n+1} = \frac{h}{2}\,(3\, f_{n+1} - f_n)$$