Lecture 29: Necessary & Sufficient Conditions for Convergence

The Lecture Contains:

This lecture gives necessary and sufficient conditions for convergence of a linear multistep method and proves such a result.

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Theorem:

A stable and consistent linear multistep method is convergent.

Proof: Let the function $f(t,y)$ satisfy the conditions of the existence and uniqueness, and let η be an arbitrary constant. We shall denote by $y(t)$ the solution of the initial value problem $y' = f(t,y),\ y(a) = \eta$. Let $y_n\ (n = 0,1,2,\dots)$ be the solution of the difference equation (8.1), defined by the starting values

$y_\mu = \eta_\mu(h), \quad \mu = 0,1,\dots,K-1$. We set

$$\delta = \delta(h) = \max_{\mu=0,1,\dots,K-1} \left|n_\mu(h) - y(a+\mu h)\right|$$

and assume that

$$\lim_{h\to 0} \delta(h) = 0 \qquad (8.21)$$

We then have to show that for any $t \in [a,b]$

$$\lim_{h\to 0} y_n = y(t)$$

We begin by estimating the quantity $|L[y\ (t_m);h]|$, where L denotes the difference operator defined earlier.

The function $y'(t) = f(t,\ y(t))$ is continuous in the closed interval $[a,b]$. We define for $\varepsilon \geq 0$the quantity

$$\chi(\varepsilon) = \max_{\substack{|t^*-t| \leq \varepsilon \\ t,t^* \epsilon [a,b]}} |y'\ (t^*) - y'(t)|$$

For $\mu = 0,1,2,\dots,K$, we can write

$$y'(t_{m+\mu}) = y'(t_m) + \theta_\mu\, \chi(\mu h)$$

where $|\theta_\mu| \leq 1$. Furthermore, since

$$y(t_{m+\mu}) = y(t_m) + \mu h y'(\xi_\mu)$$

where $t_m < \xi_\mu < t_{m+\mu}$, we have

$$y(t_{m+\mu}) = y(t_m) + \mu h\left[y'(t_m) + \theta'_\mu\, \chi(\mu h)\right]$$

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where $|\theta'_\mu| \le 1$. Hence it follows that

$$L[y(t_m;h)] = (\alpha_0 + \alpha_1 + \cdots + \alpha_K)\ y(t_m) + (\alpha_1 + \alpha_2 + \cdots + K\,\alpha_K)\, y'(t_m)h + \ \theta'(|\alpha_1| + 2|\alpha_2|$$

$$+ \cdots + K\,|\alpha_K|)\ \chi(Kh) - \left(\beta_0 + \beta_1 + \cdots + \beta_k\right)\ y'(t_m)h - \theta\left(|\beta_1| + \cdots + |\beta_K|\right)\chi(kh)h$$

where $|\theta| \le 1,\ \ |\theta'| \le 1$. Since L is assumed to be consistent, we have

$$\alpha_0 + \alpha_1 + \cdots + \alpha_K = 0$$

$$\alpha_1 + 2\alpha_2 + \cdots + K\alpha_K - \beta_0 - \beta_1 - \cdots - \beta_K = 0$$

Hence

$$|L\ [y(t_m);h]| \le K\,\chi(kh)h \qquad (8.22)$$

where

$$K = |\alpha_1| + 2|\alpha_2| + \cdots + K\,|\alpha_K| + |\beta_0| + \cdots + |\beta_K|$$

We now subtract $L[y(t_m);h]$ from the corresponding relation

$$\alpha_K\, y_{m+K} + \cdots + \alpha_0\, y_m - h\{\beta_K\, f_{m+K} + \cdots + \beta_0\, f_m\} = 0$$

satisfied by the values y_m. Writing $\varepsilon_m = y_m - y(t_m)$, $\ \ m = 0, 1, \ldots$ and setting

$$g_m = \begin{cases} [f(t_m, y_m) - f(t_m, y(t_m))]\varepsilon_m^{-1} \ ; \ \varepsilon_m \ne 0 \\ 0 \quad ; \ \varepsilon_m = 0 \end{cases}$$

we get

$$\alpha_K\,\varepsilon_{m+K} + \cdots + \alpha_0\,\varepsilon_m - h\{\beta_K\, g_{m+K}\,\varepsilon_{m+K} + \cdots + \beta_0\, g_m\,\varepsilon_m\} = \theta_m\, K\,\chi(kh)h$$

where $|\theta_m| \le 1$. In view of the Lipschitz condition, $|g_m| \le L$, $\ \ m = 0, 1, 2, \ldots$

The following Lemma is now used in the remaining part of the proof of the theorem.

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Lemma: Let the polynomial $\rho(\xi) = \alpha_K \xi^K + \cdots + \alpha_0$ satisfy the condition of zero-stability, let β^*, β and $\wedge$ be non-negative constants such that

$$\left|\beta_{K,n}\right| + \left|\beta_{K-1,n}\right| + \cdots + \left|\beta_{0,n}\right| \le \beta^*,$$

$$\left|\beta_{K,n}\right| \le \beta\ , \quad |\lambda_n| \le \wedge, n = 0, 1, 2, \ldots, K \qquad (8.23)$$

and let $0 \le h < |\alpha_K|\ \beta^{-1}$. Then every solution of

$$\alpha_K z_{m+K} + \alpha_{K-1} z_{m+K-1} + \cdots + \alpha_0 z_m = h\left\{\beta_{K,m} Z_{m+K} + \beta_{K-1,m} z_{m+K-1} + \cdots + \beta_{0,m} z_m\right\} + \lambda_1$$

for which

$$\left|z_\mu\right| \le Z \qquad \mu = 0, 1, \ldots \ldots \ldots, K-1$$

satisfies

$$L^* = \Gamma^*\beta^*, \quad K^* = \Gamma^*(N \wedge + AZK)$$

where

$$L^* = \Gamma^*\beta^*, \quad K^* = \Gamma^*(N \wedge + AZK)$$

$$A = |\alpha_K| + |\alpha_{K-1}| + \cdots + |\alpha_0|$$

$$\Gamma^* = \frac{\Gamma}{1 - h\ |\alpha_K|^{-1}\beta}$$

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Now, using the above lemma with $Z_m = \varepsilon_m, Z = \delta(h), \wedge = K\,\chi(Kh)h, N = (t_n - a)/h$ and $\beta^* = \beta L$, where $\beta = |\beta_0| + |\beta_1| + \ldots + |\beta_K|$. It follows that

$$|\varepsilon_n| \leq \Gamma^*[A\delta(h) + (t_n - a)K\,\chi(kh)]\exp(t_n - a)L\,\Gamma^*\,\beta \tag{8.24}$$

where

$$A = |\alpha_0| + |\alpha_1| + \cdots + |\alpha_K|$$

$$\Gamma^* = \frac{\Gamma}{1 - h\,|\alpha_K^{-1}\beta_K|\,L} \tag{8.25}$$

Since $y'(t)$ is uniformly continuous on $[a,b]$, $\chi(kh) \to 0$ as $h \to 0$. In view of this fact and of (8.21) the above bound for $|\varepsilon_n|$ tends to zero as $h \to 0$ for every $t_n \in [a,b]$, establishing the proof of the theorem.

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Module 8: Linear Multistep Methods

Lecture 29: Necessary & Sufficient Conditions for Convergence

The quantity used above is given by the following lemma:

Lemma: Let the polynomial $\rho(\xi) = \alpha_K \xi^K + \alpha_{K-1} \xi^{K-1} + \cdots + \alpha_0$ satisfy the condition of stability, and let the coefficients $\gamma_l (l = 0, 1, 2, \ldots)$ be defined by

$$\frac{1}{\alpha_K + \alpha_{K-1} \xi + \cdots + \alpha_0 \xi^K} = \gamma_0 + \gamma_1 \xi + \gamma_2 \xi^2 + \ldots$$

then

$$\Gamma = \mathrm{Sup} \left|\gamma_l\right| < \infty$$

$$l = 0, 1, \ldots$$

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