Module 10: Finite Difference Methods for Boundary Value Problems

Lecture 41: Special Boundary Value Problems

The Lecture Contains:

Perhaps the simplest boundary value problem can be represented by the conditions

$$y'' = f(x,y), \quad y(a) = A, \quad y(b) = B \qquad (10.15)$$

where $b > a$ and A and B are given constants. It is theoretically always possible to reduce the solution of a boundary value problem to the solution of a sequence of initial value problems.

Let $y(x,\alpha)$ denote the solution of the initial value problem resulting from the above problem by replacing the condition for $y(b)$ by the condition $y'(a) = \alpha$, where α is a parameter.

The above boundary value problem is then equivalent to solving the (in general non linear) equation $y(b,\alpha) = B$ for α. This can be effected by one of the standard methods such as Newton's method.

Each evaluation of the function $y(x,\alpha)$ requires the solution of an initial value problem. The above 'shooting' technique nevertheless may represent a feasible procedure. However, if the systems of differential equations are involved or if the initial value problems show signs of instability, other direct procedures may be preferable.

Even for the simple boundary value problem considered above it may happen that there are infinitely many solutions-as in the problem

$y'' + \pi^2 y = 0, \quad y(0) = 0, \quad y(1) = 0$, for which $y(x) = c \sin \pi x$ is a solution for arbitrary c – or that there is no solution, as in the problem

$$y'' + \pi^2 y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

We now discuss a direct method based on implicit difference approximations for solving a class of non-linear boundary value problems of the second order.

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Definition: A boundary value problem will be said to be of class M if it is of the form

$$y'' = f(x,y)$$

$$y(a) = A \qquad (10.16)$$

$$y(b) = B$$

where $-\infty < a < b < \infty$, A and B are arbitrary constants, and the function $f(x,y)$, in addition to satisfying the conditions of the existence and uniqueness, is such that $f_y(x,y)$ is continuous and satisfies

$$f_y(x,y) \geq 0 \qquad a \leq x \leq b \qquad -\infty < y < \infty \qquad (10.17)$$

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Existence of a unique solution:

The main motivation for concentrating on problems of class M is the following theorem:

Theorem: A boundary value problem of class M has a unique solution.

Proof: Let $y(x,\alpha)$ denote the solution of the initial value problem

$$y'' = f(x,y)$$

$$y(a) = A$$

$$y'(a) = \alpha.$$

By virtue of a standard theorem in the theory of differential equation (theorem 7.5 of chapter 1 in Coddington and Leuinson), $y(x,\alpha)$ is a continuous function of x and α, and $y_\alpha(x,\alpha)$ also exists and is continuous for $x \in [a,b]$ and all values of α.

In order to show that the equation for α, $y\,(b,\alpha) = B$, has exactly one solution, we shall prove that

$$y_\alpha(b,\alpha) \geq b - a \qquad (10.18)$$

The desired result then follows from the fact that a monotone function defined for all values of α whose derivative is bounded away from zero assumes every value exactly once.

In order to establish (10.18), we differentiate the identity

$$y''(x,\alpha) = f\big(x, y(x,\alpha)\big)$$

with respect to α. We get

$$y''_\alpha\,(x,\alpha) = f_y\big(x, y(x,\alpha)\big)\, y_\alpha(x,\alpha)$$

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Now writing $\eta(x) = y_\alpha(x,\alpha)$, we obtain

$$\eta''(x) = f_y\big(x, y(x,\alpha)\big)\,\eta(x) \qquad a \le x \le b \tag{10.19}$$

From the definition of $y(x,\alpha)$, we have

$$\eta(a) = y_\alpha(a,\alpha) = 0 \qquad \begin{bmatrix} \because y(a,\alpha) = A \\ \text{and } y_\alpha(a,\alpha) = 0 \end{bmatrix}$$

and

$$\eta'(a) = y'_\alpha(a,\alpha) = 1 \qquad \begin{bmatrix} \because y'(a,\alpha) = \alpha \\ \therefore y'_\alpha(a,\alpha) = 1 \end{bmatrix} \tag{10.20}$$

We shall show that

$\eta(x) \ge \xi - a$ for $a \le x \le b$.

Assume that

$\eta(\xi) < \xi - a$ for some $\xi \in (a,b)$.

Since $\eta(x) > 0$ for small positive values of $x - a$, we may assume without loss of generality that

$$\eta(x) > 0 \text{ for } a < x \le \xi \tag{10.21}$$

By the mean value theorem,

$\eta(\xi) = \eta(a) + (\xi - a)\,\eta'(\xi_1)$ for some $\xi_1 \in (a,\xi)$

or $\eta(\xi) = (\xi - a)\,\eta'(\xi_1)$ for $\xi_1 \in (a,\xi)$ $(\because \eta(a) = 0)$

it follows that

$\eta'(\xi_1) < 1$ (because of the assumption that $\eta(\xi) < \xi - \alpha$ for $\xi \in (a,b)$) .

Applying the mean value theorem to the function $\eta'(x)$, we have

$\eta'(\xi_1) - \eta'(a) = (\xi_1 - a)\,\eta''(\xi_2)$ for some $\xi_2 \in (a,\xi_1)$. In view of $\eta'(a) = 1$, we then have $\eta''(\xi_2) < 0$ (because $\eta'(\xi_1) - \eta'(a) < 0$ and $\xi_1 - a > 0$). This contradicts the differential equation (10.19) in view of $f_y \ge 0$ and (10.21). It thus follows that $\eta(x) \ge x - a$. The desired relation (10.18) is the special ease $x = b$.

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