The Lecture Contains:

This lecture includes the definitions of a general one –step method and its consistency, stability and convergence. A sufficient condition for stability of such methods is given. (We also discuss and give necessary and sufficient condition for convergence of a general one-step method in the next lecture). This result is illustrated on the midpoint rule as an example.

Previous Next

The common basis of the single-step methods discussed earlier is that each requires an amount to be added to y_n in order to get y_{n+1}. Formally, we define a general single-step method by:

Definition: A general single-step method for approximating the solution of a differential equation is a method which can be written in the form

$$y_{n+1} = y_n + h\,\phi\,(y_n, t_n, h) \qquad (5.1)$$

where the function $\phi\,(y_n, t_n, h)$ is called the increment function and is determined by f. The increment function ϕ is a function of y_n, t_n and h only.

Convergence for Single–step methods is defined by:

Definition: The single-step method (5.1) is convergent if $y_n \to y\,(t)$ for all $0 \le t \le b$ as $n \to \infty$ and $y_0 \to y(0)$ with

$h = \frac{t}{n}$ for any differential equation $y' = f\,(y)$ which satisfies a Lipschitz condition.

Previous Next

Remark: Convergence assures that the exact solution can be approximated arbitrarily closely by making h smaller and smaller using greater precision. Stability is concerned with the effect of perturbation on the numerical solution.

Definition: A single-step method (5.1) is stable if for each differential equation satisfying a Lipschitz condition, there exist positive constants h_0 and K such that the difference between two different numerical solutions y_n and $\check{y}_n$, each satisfying (5.1) is such that

$$\|y_n - \check{y}_n\| \leq K\|y_0 - \check{y}_0\|$$

for all $0 \leq h \leq h_0$

Remark: Stability is nearly automatic for single–step methods as the following theorem shows:

Theorem: If the increment function $\phi(y, t, h)$ satisfies a Lipschitz Condition

in y, then the method given by (5.1) is stable.

Previous Next

Proof: A single-step method for approximating the solution of the IVP $y' = f(t, y), t \in [0, b]$.

$y(0) = y_0$ is given by

$$y_{m+1} = y_m + h\,\phi\,(y_m, t_m, h) \qquad (5.2)$$

A change in one of the computed values y_m to $\check{y}_m$ will lead us to solve

$$\check{y}_{m+1} = \check{y}_m + h\,\phi\,(\check{y}_m, t_m, h) \qquad (5.3)$$

instead of (5.2).

Subtracting (5.3) from (5.2), we get

$$\|y_{m+1} - \check{y}_{m+1}\| \le (1 + hL)\,\|y_m - \check{y}_m\|$$

Since $\phi(y, t, h)$ satisfies a Lipschitz condition in y, we get

$$\|y_n - \check{y}_n\| \le (1 + hL)^{n-m}\,\|y_m - \check{y}_m\| \le e^{bL}\|y_m - \check{y}_m\|$$

which is a bounded multiple of the introduced error $\|y_m - \check{y}_m\|$and is independent of h. Hence the given method (5.1) is stable provided $\phi(y, t, h)$ satisfies a Lipschitz c ondition in y.

Previous Next

Remark: If, for the differential equation $y' = f(y,t)$, the function $f(y,t)$ is a continuous function of t and satisfies the Lipschitz condition in y in the region $0 \leq t \leq b, -\infty < y < \infty$, then we can see that for all the method discussed earlier, the increment function Φ will also satisfy these conditions for $0 \leq h \leq h_0$. For example, in the case of the mid-point rule, we have

$$\Phi(y,t,h) = f\left(y + \frac{h}{2} f(y,t), t + \frac{h}{2}\right)$$

which is continuous in t and y if f is, and

$$\| \Phi(y,t,h) - \Phi(y_1,t,h) \| = \| f\left(y + \frac{h}{2} f(y,t), t + \frac{h}{2}\right) - f(y_1 + \frac{h}{2} f(y_1,t), t + \frac{h}{2}) \|$$

$$\leq L \| y + \frac{h}{2} f(y,t) - y_1 - \frac{h}{2} f(y_1,t) \|$$

$$\leq L \| y - y_1 \| + L\frac{h}{2} \| f(y,t) - f(y_1,t) \|$$

$$\leq L\left(1 + L\frac{h}{2}\right) \| y - y_1 \|$$

Thus, the increment function Φ satisfies the Lipschitz condition in y for

$0 \leq h \leq h_0$. Also note that Φ is continuous in h if f is continuous in $y$and t.

Previous Next