Module 9: Stiff-Initial Value Systems

Lecture 34: Stiffness and Problem of Stiffness

The Lecture Contains:

Previous Next

Definition: The linear systems $y' = Ay + \phi(t)$ is said to be stiff if (i) $\mathrm{Re}\,\lambda_j < 0, j = 1,2,\ldots,m$ and (ii) $\max_{j=1,2,\ldots,m}|\mathrm{Re}\,\lambda_j| \gg \min_{j=1,\ldots,m}|\mathrm{Re}\,\lambda_j|$, where $\lambda_j, j = 1,2,\ldots,m$ are the eigen values of A. The ratio

$$\left[\max_{j=1,2,\ldots,m}|\mathrm{Re}\,\lambda_j|\right] : \left[\min_{j=1,\ldots,m}|\mathrm{Re}\,\lambda_j|\right]$$

is called the stiffness ratio.

Non linear systems $y' = f(t, y)$exhibit stiffness if the eigen values of the Jacobian $\frac{\partial f}{\partial y}$ behave in a similar fashion. The eigen values are no longer constant but depend on the solution, and therefore vary with t. Accordingly we say that the system $y' = f(t, y)$ is stiff in an interval I of t if, for $t \in I$, the eigen values $\lambda_j(t)$ of $\frac{\partial f}{\partial t}$ satisfy (i) and (ii) above.

Note that if the partial derivatives appearing in the Jacobian $\frac{\partial f}{\partial y}$ are continuous and bounded in an appropriate region, then the Lipschitz constant L of the system $y' = f(t, y)$ may be taken to be $L = \left\|\frac{\partial f}{\partial y}\right\|$. For any matrix A, $\|A\| \geq \rho(A)$where $\rho(A)$ is the spectral radius, is defined to be $\max_{j=1,2,\ldots,m}|\lambda_j|$, $\lambda_j = 1,2,\ldots,m$ being the eigen values of A. If $\max_{j=1,2,\ldots,m}|\mathrm{Re}\,\lambda_j| \gg 0$ it follows that $L \gg 0$. Thus stiff systems are occasionally referred to as systems with large Lipschitz constants'.

Previous Next

The problem of stability for stiff systems

We have seen that a basic difficulty (but not the only one) in the numerical solution of stiff systems is the satisfaction of the requirement of absolute stability. Thus, several definitions which call for the method to possess some adequate region of absolute stability, have been proposed.

Definition: (Dahlquist) A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half-plane $\mathrm{Re}\, h\lambda < 0$.

If an A-stable method is applied to a stiff system, then the difficulties described earlier disappear, since, no matter, how large $\max\left|\mathrm{Re}\,\lambda_j\right|$, no stability restriction on h can result. However, A-stability is a severe requirement to ask of a numerical method as the following somewhat depressing theorem of Dahlquist shows:

Theorem: (i) An explicit linear multistep method cannot be A-stable. (ii) The order of an A-stable implicit linear multistep method cannot exceed two. (iii) The second order A-stable implicit linear multistep method with smallest error constant is the Trapezoidal rule.

Previous Next

The restriction on order implied by (i) is a severe one. In view of this, several less demanding stability definitions have been proposed; we present two here:

Definition: (Widlund) A numerical method is said to be $A(\alpha)-\text{stable},\ \alpha \in \left(0, \frac{\pi}{2}\right)$, if its region of absolute stability contains the infinite wedge $W_\alpha = \{h\lambda \mid -\alpha < \pi - \arg h\lambda < \alpha\}$. It is said to be A(0)-stable if it is $A(\alpha)$-stable for some(sufficiently small) $\alpha \in \left(0, \frac{\pi}{2}\right)$

Theorem: (i) An explicit linear multistep method cannot be A(0)-stable. (ii) There is only one A(0)-stable linear K-step method whose order exceeds $K+1$, namely the trapezoidal rule, (iii) For all $\alpha \in \left(0, \frac{\pi}{2}\right)$, there exist $A(\alpha)$-stable linear K-step methods of order p for which $K = p = 3$ and $K = p = 4$.

An alternative slackening of the A-stability requirement is incorporated in the following definitions:

Definition (Gear): A numerical method is said to be stiffly stable if (i) its region of absolute stability contains R_1 and R_2. and (ii) it is accurate for all $h \in R_2$ when applied to the scalar test equation $y' = \lambda y, \lambda$ a complex constant with $Re(h\lambda) < 0$, where

$R_1 = \{h\lambda \mid Re(h\lambda) < -a\}$,

$R_2 = \{h\lambda \mid -a \le Re(h\lambda) \le b, -c \le Im(h\lambda) \le c\}$

and a, b, and c are positive constants.

The motivation for this definition is that those eigen values which represent rapidly decaying terms in the transient solution will correspond to values of $h\lambda$ in R_1; we generally have no interest in representing such terms accurately but only stably-that is, we will be satisfied with any representation of these terms which decays sufficiently rapidly. The remaining eigen values of the system represent terms in the solution which we would like to represent accurately as well as stably; by suitable choice of h, values of $h\lambda$ corresponding h these eigen values can be made to lie in R_2.

Previous Next

A-Stability Versus L-Stability

Severe as the requirement of A-stability is, in one sense it is not severe enough. Consider, for example, the application of the Trapezoidal rule to the scalar test equation $y' = \lambda\ y, \lambda$ a complex constant with $\mathrm{Re}\,\lambda < 0$. We obtain

$$\frac{y_{n+1}}{y_n} = \left(1 + \frac{h\lambda}{2}\right) / \left(1 - \frac{h\lambda}{2}\right)$$

Since the Trapezoidal rule is A-stable, $y_n \to 0$, as $n \to \infty$ for all fixed h.

However,

$$\left|\frac{y_{n+1}}{y_n}\right| = \left|\frac{1+h\frac{\lambda}{2}}{1-h\frac{\lambda}{2}}\right| = \left[\frac{1+h\,\mathrm{Re}\,\lambda+\frac{1}{4}h^2|\lambda|^2}{1-h\,\mathrm{Re}\,\lambda+\frac{1}{4}h^2|\lambda|^2}\right]^{\frac{1}{2}}$$

and if $|\mathrm{Re}\,\lambda| \gg 0$and h is not small then $\left|\frac{y_{n+1}}{y_n}\right|$ will be close to $+1$. Thus $|y_n|$ will decay to zero only very slowly, and it follows that an A-stable method may be unsatisfactory for an excessively stiff system. Contrast the behavior of the method $y_{n+1} - y_n = h\,f_{n+1}$ [the backward Euler method) for which

$$\left|\frac{y_{n+1}}{y_n}\right| = \left|\frac{1}{1-h\lambda}\right| = \left[\frac{1}{1-2h\,\mathrm{Re}\,\lambda+h^2\,|\lambda|^2}\right]^{\frac{1}{2}}$$

and the RHS tends to zero as $\mathrm{Re}\,\lambda \to -\infty$ and we can expect a rapid decay of $|y_n|$ even for moderately large h.

Definition: (Ehle) A one step method is said to be L-stable if it is A-stable and, in addition, when applied to the scalar test equation $y' = \lambda y, \lambda$ a complex constant with $\mathrm{Re}\,\lambda < 0$, it yields $y_{n+1} = R(h\,\lambda)y_n$, where $|R(h\,\lambda)| \to 0$ as $|R(h\,\lambda)| \to 0$. Note that $L-\text{stability} \Rightarrow A-\text{stability} \Rightarrow A(0)-\text{stability}$

Previous Next