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Module 3: Higher order Single Step Methods

Lecture 8: Higher Order Methods

Higher Order Methods

The Euler method has an error that behaves like $O(h)$ as $h \to 0$. For this reason, we call it a first order method. If we have a method in which the error behaves like $O(h^p)$ for some $p > 0,$ we would call it a p^{th} order method.

Question: Why higher order methods?

A simple answer at this stage is that if h is small, then h^2 is even smaller, so more accuracy can be achieved with higher order methods for small h, and as h goes to zero, a higher order method will converge faster. Additional features of higher order method will become apparent later.

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The Taylor Series Method

The Euler's method can be viewed as an approximation by the first two terms of a Taylor's series. If we calculate the higher order derivatives of y, we can write

$$y_{n+1} = y_n + h\, y_n' + \frac{h^2}{2}\, y_n'' + \cdots + \frac{h^p}{p!}\, y_n^{(p)} \qquad (3.1)$$

We have

$$y_n' = f(t_n, y_n),$$

and so we can evaluate higher order derivatives as follows:

Writing f for $f\,(t,y)$, f_t for $\frac{\partial f}{\partial t}$, f_y for $\frac{\partial f}{\partial y}$ etc., we have the next two derivatives as

$$y'' = f_t + f f_y \qquad (3.2)$$

$$y''' = f_{tt} + 2\, f f_{ty} + f_t\, f_y + f f_y^2 + f^2 f_{yy}$$

It is immediately clear that this is not a practical method unless the function $f\,(t,y)$ is simple enough that many of these partial derivatives vanish, but it is theoretically possible to develop as many methods like (3.1) as necessary and evaluate these derivatives for substitution in (3.1)

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Richardson Extrapolation to h = 0

We observe that the Euler method gives an error of the form $h\,\delta(b) + O(h^2)$, where $\delta(b)$ depends on the differential equation only if round–off errors are ignored. To halve the error produced by the Euler method, it is necessary to halve the step size if the $O(h^2)$ term can be neglected. Let us consider two integrations of a differential equation, one using step size h and one using step size $h/2$. If the answers are $y_h(b)$ and $y_{h/2}(b)$ respectively, we can write

$$y_h(b) = y(b) + h\,\delta(b) + O(h^2) \qquad (3.3)$$

$$y_{\frac{h}{2}}(b) = y(b) + \frac{h}{2}\,\delta(b) + O(h^2)$$

Eliminating $\delta(b)$ from (3.3), we get

$$y(b) = 2\,y_{h/2}(b) - y_h(b) + O(h^2) \qquad (3.4)$$

This can be used as a better numerical approximation for it is more accurate than the Euler formula, and that it gains accuracy more rapidly as h is decreased because the error is $O(h^2)$rather than $O(h)$. This process of deriving a higher order method from a lower order method is also called the deferred approach to the limit.

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The Mid Point Method

Let us look at the approximation of the previous section in more detail as it is applied to a single step of si ze h. We get

$$y_h\,(t_n + h) = y_n + \,h\,f(t_n, y_n)$$

with a step size of h, and

$$q_1 = y_{\frac{h}{2}}\left(t_n + \frac{h}{2}\right) = \; y_n + \frac{h}{2}\,f\,(t_n, y_n)$$

$$y_{\frac{h}{2}}\,(t_n + \,h) = q_1 + \frac{h}{2}\,f\left(t_n + \frac{h}{2}, q_1\right)$$

with two steps of size $\frac{h}{2}$. Therefore,

$$y(t_n + h) = 2\,y_{\frac{h}{2}}(t_n + \,h) - \,y_h\,(t_n + \,h) + 0(h^2)$$

$$= 2\,q_1 + h\,f\left(t_n + \frac{h}{2}, q_1\right) - y_n - h\,f(t_n, y_n) + 0(h^2)$$

The actual calculations involved are

$$q_1 = y_n + \frac{h}{2} f\,(t_n, y_n)$$

$$y_{n+1} = y_n + h\,f\left(t_n + \frac{h}{2}, q_1\right) \qquad (3.5)$$

Thus, the form of the method is similar to Euler's method in that the value at the end of the step is obtained by adding something to the value at the beginning of the step. It is called the mid-point method.

Note: It is easy to verify the local truncation error of (3.5) is $O(h^3)$.

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Trapezoidal Method

The trapezoidal method is obtained by using some other approximations to approximate the terms in Taylor's series.

For example, we have

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}\, y''\,(t_n) + O(h^3) \qquad (3.6)$$

Also,

$$\frac{h}{2}[y'(t_n) + y'(t_{n+1})] = hy'(t_n) + \frac{h^2}{2}\, y''\,(t_n) + O(h^3) \qquad (3.7)$$

and so we can write (3.6) as

$$y(t_{n+1}) - y(t_n) = \frac{h}{2}\left[f\left(t_n, y(t_n)\right) + f\left(t_{n+1}, y(t_{n+1})\right)\right] + O(h^3) \qquad (3.8)$$

If the $O(h^3)$ term is neglected and (3.8) is used to approximate the value of $y(t_{n+1})$,

we have the trapezoidal method

$$y_{n+1} = y_n + \frac{h}{2}\,[f\,(t_n, y_n) + f(t_{n+1}, y_{n+1})] \qquad (3.9)$$

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Note: Since y_{n+1} is not known, the RHS of (3.9) cannot be evaluated. Two approaches are possible. One is to attempt to solve the nonlinear equation (3.9) for y_{n+1}. This is called an implicit method. Alternatively, one can estimate y_{n+1} by another method, such as the Euler method, and get an explicit method. Then we get the formula

$$\tilde{y}_{n+1} = y_n + h\, f(t_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, \tilde{y}_{n+1})\right] \qquad (3.10)$$

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