Module 7: Multistep Methods

Lecture 18: Multistep Methods

The Lecture Contains:

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Introduction: The methods discussed so far have required knowledge of the differential equations and initial values only. Consequently, given an approximation to the value of $y(t)$ at $t = n$, say, y_n, they have provided a technique for computing $y_{n+1} \cong y(t_{n+1})$. They could therefore be called one-step methods because they only required the value at one mesh point to compute the value at the next. Once the values at a number of points, say for $t \leq t_n$, have been computed, they can be used to obtain the solution at $t = t_{n+1}$. Thus the methods which make use of the information about the dependent variable at $t \leq t_n$, to compute the value of the dependent variable at $t = t_{n+1}$ are called multistep methods.

Consequently, a K-step multistep method is a method which uses the information about the dependent variable at K different mesh points $t_n, t_{n-1}, t_{n-2}, \ldots\ldots, t_{n-k+1}$ to compute the value of the dependent variable at $t = t_{n+1}$.

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Methods based on numerical integration:

An exact solution of the differential equation $y' = f(t, y)$ by definition satisfies the identity

$$y(t+K) - y(t) = \int_t^{t+K} f\,(t, y(t))\,dt$$

for any two points t and $t+K$ in the interval $[a, b]$. The methods now to be discussed are based on replacing the function $f\,(t,\ y(t))$, which is unknown, by an interpolating polynomial having the values $f_n = f\,(t_n, y_n)$ on a set of points t_n where y_n has already been computed or is just about to be computed, evaluating the integral and accepting its value as the increment of the approximate values $y_n$between t and $t+K$. We shall assume that the interpolating points are $t_p, t_{p-1}, \ldots\ldots, t_{p-q}$. To approximate $f\,(t, y)$ by an interpolating polynomial through the above values, we use the Newton backward difference formula. If $f\,(t, y) = f\,(t)$ has a continuous $(q+1)$ the derivative, $t_p = t_0 + ph, f_p = f\,(t_p)$ and backward differences are given by

$$\nabla^{q+1} f_p = \nabla^q f_p - \nabla^q f_{p-1}$$

where $\nabla^0 f_p = f_p$, then

$$f\,(t) = f_p + \frac{t - t_p}{h} \nabla f_p + \frac{(t - t_p)\,(t - t_{p-1})}{2!}$$

$$\frac{\nabla^2 f_p}{h^2} + \cdots \frac{(t - t_p)\,(t - t_{p-1}) \ldots (t - t_{p-q+1})}{q!}$$

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$$\frac{\nabla^q f_p}{h^q} + \frac{(t-t_p)(t-t_{p-1})\dots(t-t_{p-q})}{(q+1)!} f^{(q+1)}(\xi) \qquad (7.1)$$

where $f^{(q+1)}(\xi)$ is the $(q+1)^{th}$ derivative of f evaluated at some point is an interval containing t, t_{p-q}, and t_p.

If we set

$$S = \frac{t-t_p}{h}$$

(7.1) becomes

$$f(t) = \binom{-S}{0} f_p - \binom{-S}{1} \nabla f_p + \dots + (-1)^q \binom{-S}{q} \nabla^q f_p + (-1)^{q+1} h^{q+1} \binom{-S}{q+1} f^{(q+1)}(\xi) \quad (7.2)$$

where

$$\binom{S}{q} = \frac{s(s-1)\dots\dots(s-q+1)}{q!}$$

and $\binom{S}{0} = 1$

Alternatively, (7.2) can be written as

$$f(t) = \sum_{m=0}^{q} (-1)^m \binom{-S}{m} \nabla^m f_p + (-1)^{q+1} h^{q+1} \binom{-S}{q+1} f^{(q+1)}(\xi)$$

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The positive integer q is arbitrary in principle, but in practice rarely exceeds 6, say. Several classes of methods can now be distinguished according to the position of t and $t+K$ relative to the interpolating points, see the following Table.

Method	t	$t+K$
Adams-Bashforth	t_p	t_{p+1}
Adams-Moulton	t_{p-1}	t_p
Nystrom	t_{p-1}	t_{p+1}
Milne-Simpson	t_{p-2}	t_p

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i) The Adams-Bashforth method:

Here we have

$$y_{p+1} - y_p = \int_{t_p}^{t_{p+1}} f(t)\,dt$$

$$= h \sum_{m=0}^{q} \gamma_m \nabla^m f_p \qquad (7.3)$$

where the constants

$$\gamma_m = (-1)^m \frac{1}{h} \int_{t_p}^{t_{p+1}} \binom{-S}{m}\,dt$$

$$= (-1)^m \int_0^1 \binom{-S}{m}\,dS \qquad (7.4)$$

are independent of f and will be calculated numerically below. If the values $y_p, y_{p-1}, \dots \dots .., y_{p-q}$ are known, the corresponding values $f_n = f(t_n, y_n)$ can be calculated and the differences $\nabla^m f_p$ are easily formed. The expression on the right of (7.3) is thus known, and $y_{p+1} - y_p$ and hence y_{p+1} can be calculated. The index p is then increased by 1, and the same formula is used to calculate y_{p+2}, etc. The method breaks down if some of the values $y_p, y_{p-1}, \dots \dots .., y_{p-q}$ are not known. Such is the case at the beginning of the computation, where the initial condition furnishes only one of the required $q+1$values. In such cases the missing values have to be secured by an independent method.

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In order to find recurrence relation, numerical values, and other useful properties of the coefficients γ_m we use the method of generating functions. Such we call the function $G\,(t)$ which has the γ_m as coefficients in its Maclaurin expansion. Using the general form of the binomial theorem, we find

$$G\,(t_1) = \sum_{m=0}^{\infty} \gamma_m\, t_1^m = \sum_{m=0}^{\infty} (-\,t_1)^m \int_0^1 \binom{-S}{m} dS$$

$$= \int_0^1 \sum_{m=0}^{\infty} (-\,t_1)^m \binom{-S}{m}\, dS$$

$$= \int_0^1 (1-\,t_1)^{-S}\, dS$$

The integral is easily evaluated by writing $(1-\,t_1)^{-S} = e^{-S\,\log(1-t_1)}$. We thus find

$$G\,(t_1) = -\frac{t_1}{(1-t_1)\log\,(1-t_1)} \qquad (7.5)$$

which can be written as

$$-\frac{\log\,(1-t_1)}{t_1}\,G\,(t_1) = \frac{1}{1-t_1},$$

Using the well-known expansions

$$\frac{1}{1-t_1} = 1 + t_1 + t_1^2 + \cdots$$

$$-\frac{\log\,(1-t_1)}{t_1} = 1 + \frac{1}{2}\,t_1 + \frac{1}{3}\,t_1^2 + \cdots \qquad (7.6)$$

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we obtain the following identity between power series:

$$\left(1+\frac{1}{2}\,t_1+\frac{1}{3}\,t_1^2+\cdots\right)(\gamma_0+\gamma_1\,t_1+\gamma_2\,t_1^2+\cdots)=1+t_1+t_1^2+\cdots$$

By comparing the coefficients of corresponding powers of t, we find the relation

$$\gamma_m+\frac{1}{2}\,\gamma_{m-1}+\frac{1}{3}\,\gamma_{m-2}+\cdots.+\frac{1}{m+1}\,\gamma_0=1,\quad m=0,1,2,....$$

which makes it possible to calculate the γ_m recursively.

m	0	1	2	3	4	5	6
γ_m	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{251}{720}$	$\frac{95}{288}$	$\frac{19087}{60480}$

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