Module 10: Finite Difference Methods for Boundary Value Problems

Lecture 40: Nonlinear second order equations

The Lecture Contains:

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We now apply the difference method to BVP of the form

$$L\,y(x) \equiv -y'' + f(x,y,y') = 0 \qquad a < x < b \qquad (10.13)$$

$$y\,(a) = \alpha, \quad y(b) = \beta$$

we assume that $f(x,y,z)$ has cont. derivatives which satisfy

$$\left|\frac{\partial f}{\partial z}\right| \le P^*, \quad 0 < Q_* \le \frac{\partial f}{\partial y} \le Q^*$$

for some positive constants, P^*, Q^* and Q_*.

The difference approximation to the BVP on the uniform mesh is given by

$$L_h\,u_j \equiv -\left(\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}\right) + f\left(x_j, u_j, \frac{u_{j+1} - u_{j-1}}{2\,h}\right) = 0 \qquad 1 \le j \le J \qquad (10.14)$$

$$u_0 = \alpha$$

$$u_{J+1} = \beta.$$

The resulting equations are nonlinear and we shall employ iterative methods to solve them.

The local truncation error $T_j[v]$ is given by

$$T_j[v] \equiv L_h\,v(x_j) - L\,v(x_j)$$

$$= -\frac{h^2}{12}\left[v^{iv}(\xi_j) - 2\,\frac{\partial f(x_j,\ v(x_j),\ v'(y_j))}{\partial z}\,v'''(\xi_j)\right]$$

where $v(x) \in c^4[a,b]$ and $\xi_j,\ \eta_j,\ \xi_j$ and points in $[x_{j-1}, x_{j+1}]$.

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Error estimates:

Theorem: Let $f(x,y,z)$ have cont. derivatives which satisfy $\left|\frac{\partial f}{\partial z}\right| \leq P^*$, $0 < Q_* \leq \frac{\partial f}{\partial y} \leq Q^*$. Then the numerical solution $\{u_j\}$ of the difference equation (10.14) and the solution $y(x)$ of the BVP satisfy, with $M \equiv \max\left(1, \frac{1}{Q_*}\right)$

$$\left|u_j - u(x_j)\right| \leq M \max_{1 \leq i \leq J} |T_i[y]| \qquad 0 \leq j \leq J+1$$

It $y(x)$ has a cont. fourth derivative on $[a,b]$, then

$$\left|u_j - y(x_j)\right| \leq M\frac{h^2}{12}\left(M_4 + 2P^*M_3\right) \; 0 < j \leq J+1$$

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Iteration scheme:

Multiply (10.14) by $\frac{h^2}{2}$ and add wu_j to each side assuming $w \neq -1$, the result can be written as

$$u_j^{n+1} = (1+w)^{-1}\left[\frac{1}{2}\left(u_{j+1}^{(n)} + u_{j-1}^{(n)}\right) + wu_j^{(n)} - \frac{h^2}{2} f\left(x_j, u_j^{(n)}, \frac{\left(u_{j+1}^{(n)} - u_{j-1}^{(n)}\right)}{2h}\right)\right] \quad 1 \leq j \leq J$$

$$u_0^{(n)} = \alpha \qquad\qquad n = 0, 1, \ldots$$

$$u_{J+1}^{(n)} = \beta \qquad\qquad u_j^{(0)} = \text{arbitrary} \qquad 1 \leq j \leq J.$$

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Convergence:

Let $f(x,y,z)$ have cont. derivatives which satisfy $\left|\frac{\partial f}{\partial z}\right| \leq P^*, \quad 0 < Q_* \leq \frac{\partial f}{\partial y} \leq Q^*$(10.14)

Then the system of difference equations (10.14) have a unique solution for each h such that

$$h < \frac{2}{P^*}$$

This solution is the limit of the iterates $\left\{u_j^{(n)}\right\}$as $n \to \infty$ with any finite w satisfying

$$w \geq \frac{h^2}{2} Q^*.$$

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