Module 3: Higher order Single Step Methods

Lecture 10: Error bounds for Runge-Kutta methods

The Lecture Contains:

Previous Next

Let us first define the local truncation error at t_{n+1} of the general explicit one-step method defined by

$$y_{n+1} - y_n = h\,\phi\left(t_{n,}\ y\ (t_n), h\right) \qquad (3.31)$$

Definition: The local truncation error at t_{n+1} of the one step method (3.31) is defined to be T_{n+1} where $T_{n+1} = y(t_{n+1}) - y(t_n) - h\,\phi\left(t_{n,}\ y(t_n), h\right)$ (3.32)

and $y(t)$ is the true solution of the initial value problem.

If we assume that no previous errors have been made, viz. $y_n = y(t_n)$, then from (3.32) and (3.11) , it follows that

$$T_{n+1} = y(t_{n+1}) - y_{n+1}$$

and the truncation error defined by (3.32) is local.

The global truncation error of the one step method (3.11), denoted by e_{n+1}, and is defined by $e_{n+1} = y(t_{n+1}) - y_{n+1}$where it is no longer assumed that no previous truncation errors have been made.

Definition: The local truncation error for the non-linear method (3.11) of order p is

$$T_{n+1} = \psi(t_{n,}\ y(t_n))h^{p+1} + 0(h^{p+2}) \qquad (3.33)$$

where the function $\psi\left(t, y(t)\right)$ is called the principal error function , and $\psi(t_{n,}\ y(t_n))\ h^{p+1}$ is called the principal local truncation error . In other words, one can also define that the method (3.11) is of order p, if its local truncation error is of $O(h^{p+1})$.

Previous Next

Module 3: Higher order Single Step Methods

Lecture 10: Error bounds for Runge-Kutta methods

Example: Let us consider the general two-stage Runge-Kutta method obtained by getting $R = 2$ in (3.12). Then, by (3.17), (3.23) and (3.32), the local truncation error is

$$T_{n+1} = h\,[f + \tfrac{1}{2}hF + \tfrac{1}{6}h^2(F\,f_y + G)] - h\,[(c_1 + c_2)f + h\,c_2\,a_2\,F + \tfrac{1}{2}h^2\,c_2\,a_2{}^2\,G] + O\,(h^4) \quad (3.34)$$

If the order is two, then (3.24) must hold, and we obtain

$$T_{n+1} = h^3\left[\frac{1}{6}F\,f_y + \left(\frac{1}{6} - \frac{1}{4}a_2\right)G\right] + O(h^4) \quad (3.35)$$

Thus the principal error function for the general second order Runge-Kutta method is given by

$$\psi(t, y) = \frac{1}{6}\,F\,f_y + \left(\frac{1}{6} - \frac{1}{4}a_2\right)G \quad (3.36)$$

Previous Next

Lotkins Bounds

We can find a bound for $\psi(t,y)$ if we assume that the following bounds for f and its partial derivatives hold for $t \in [a,b], y \in (-\infty,\infty)$:

$$|f(t,y)| < Q$$

$$\left|\frac{\partial^{i+j} f}{\partial t^i \partial y^j}\right| < \frac{P^{i+j}}{Q^{j-1}}, \qquad i+j \le p \tag{3.37}$$

Where P and Q are positive constants and p is the order of the method. These bounds are due to Lotkin and are called Lotkin's bounds. Here in this example, we have

$$|f_y| < P$$

$$|F| = |f_t + ff_y| < 2PQ$$

$$|G| = |f_{tt} + 2ff_{ty} + f^2 f_{yy}| < 4P^2Q$$

Hence from (3.36), we have

$$|\psi(t,y)| < \left(\frac{1}{3} + \left|\frac{2}{3} - a_2\right|\right)P^2Q \tag{3.38}$$

and we obtain the following bound for the principal local truncation error:

$$|\psi(t_n, y(t_n))h^3| < \left(\frac{1}{3} + \left|\frac{2}{3} - a_2\right|\right)h^3P^2Q \tag{3.39}$$

Previous Next

Remark: For all one-step explicit methods, the bound for the principal local truncation error is also a bound for the whole local truncation error. In view of this, we may write

$$|T_{n+1}| < \left(\frac{1}{3} + \left|\frac{2}{3} - a_2\right|\right) h^3 P^2 Q \qquad (3.40)$$

as the bound (Lotkin) for the local truncation error for a two-stage Runge-Kutta method.

In the case of general three stage and four stage Runge-Kutta methods, the bound for the local truncation error is very complicated. For the classical fourth order Runge-Kutta method (3.30) the bound for the local truncation error (using Lotkin's bounds) is given by

$$|T_{n+1}| < \frac{73}{720} h^5 P^4 Q \qquad (3.41)$$

For the general explicit one-step method (3.11), the bound for the global truncation error is an order of magnitude greater than the bound for the local truncation error. If the local truncation error T_{n+1} defined by (3.32) satisfies

$$|T_{n+1}| \le K\, h^{p+1} \qquad (3.42)$$

where K is a constant, then the global truncation error

$$e_n = y(t_n) - y_n$$

satisfies the inequality

$$|e_n| \le \frac{h^p K}{L}\left[\exp\left(L(t_n - a)\right) - 1\right] \qquad (3.43)$$

where $L > 0$ is the Lipschitz constant of $f(t,y)$ with respect to y.

Previous Next

Error estimates for Runge-Kutta Methods:

We have seen that the bounds for the local truncation error, as discussed earlier, are very complicated and of little use in practice for deciding the appropriate step size control policy. What is needed, in place of a bound, is a readily computable estimate of the local truncation error. The most commonly used estimate arises from an application of the Richardson extrapolation. Under the usual localizing assumption that no previous errors have been made, we can write

$$y(t_{n+1}) - y_{n+1} = \psi(t_n, y(t_n))\, h^{p+1} + O(h^{p+2}) \qquad (3.44)$$

where p is the order of the Runge-Kutta method. Now let us compute

y^*_{n+1}, a second approximation to $y(t_{n+1})$, obtained by applying the same method at $t = t_{n-1}$ but with step size 2h. With the same localizing assumption, it follows that

$$y(t_{n+1}) - y^*_{n+1} = \psi(t_{n-1}, y(t_{n-1}))(2h)^{p+1} + O(h^{p+2})$$

$$= \psi(t_n, y(t_n))(2h)^{p+1} + O\,(h^{p+2}) \qquad (3.45)$$

on expanding $\psi(t_{n-1}, y(t_{n-1}))$ about $(t_n, y(t_n))$. Subtracting (3.44) from (3.45),

We get

$$y_{n+1} - y^*_{n+1} = (2^{p+1} - 1)\, \psi(t_n, y(t_n))\, h^{p+1} + O\,(h^{p+2})$$

Thus, the principal local truncation error, which is taken as an estimate for the local truncation error, may be written as

$$\psi(t_n, y(t_n))h^{p+1} = (y_{n+1} - y^*_{n+1})/(2^{p+1} - 1) \qquad (3.46)$$

This estimate is quite adequate for step size control policy.

Previous Next