Module 9: Stiff-Initial Value Systems

Lecture 33: First order linear systems with constant coefficient

The Lecture Contains:

We briefly discuss linear systems with constant coefficients and illustrate with an example,how to find the general solution of such a system. This is followed by a discussion on solving these initial value systems, say, by an Euler's method which finally leads to the notion of stiffness.

Previous Next

The first order system $y' = f(t, y)$, where y and f are m-dimensional vectors, is said to be linear if $f(t, y) = A(t)y + \phi(t)$, where $A(t)$ is an $m \times m$ matrix and $\phi(t)$ an m-dimensional vector; if in addition, $A(t) = A$, a constant matrix, the system is said to be linear with constant coefficients. We require the general solution of such a system

$$y' = Ay + \phi(t) \qquad (9.1)$$

Let $\hat{y}(x)$ be the general solution of the corresponding homogeneous system

$$y' = Ay \qquad (9.2)$$

If $\psi(t)$is any particular solution of (9.1), then $y(t) = \hat{y}(t) + \psi(t)$ is the general solution of (9.1).

A set of m linearly independent solutions $\hat{y}_j(t), j = 1,2, \dots \dots, m$of (9.2), is said to form a fundamental system of (9.2), and the most general solution of (9.2) may be written as a linear combination of the members of the fundamental system. It is easily seen that $\hat{y}_j(t) = e^{\lambda_j t} C_j$, where C_j is an m-dimensional vector, is a solution of (9.2) if $\lambda_j C_j = A C_j$, that is, if λ_j is an eigen value of A and C_j is the corresponding eigen vector. We consider only the case where A possesses m distinct possibly complex, eigen values $\lambda_j, j = 1,2, \dots, m$.The corresponding eigen vectors $C_j, j = 1,2, \dots, m$ are then linearly, independent, and it follows that the solutions $\hat{y}_j(t) = e^{\lambda_j t} C_j, j = 1,2, \dots, m$ form a fundamental system of (9.2). The most general solution of (9.1) is then

$$y(t) = \sum_{j=1}^{m} K_j e^{\lambda_j t} C_j + \psi(t)$$

Previous Next

Consider the example given by

$y' = A\,y;\; y(0) = [1,0,-1]^T$

where $A = \begin{bmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{bmatrix}$

The eigen values of A are the roots of the equation det $(A - \lambda I) = 0$ and are found to be $\lambda_1 = -2, \lambda_2 = -40 + 40\,i\,, \lambda_3 = -40 - 40\,i$, and are distinct. The corresponding eigen vectors are

$C_1 = [1,1,0]^T, C_2 = [1,-1,-2\,i\,]^T$and $C_3 = [1,-1,2\,i\,]^T$. The general solution of $y' = A\,y$ is

$y(t) = K_1 + e^{\lambda_1 t}\,C_1 + K_2\,e^{\lambda_2 t}\,C_2 + K_3\,e^{\lambda_3 t}\,C_3$.

For this problem, $\psi(t)$ is identically zero, and the given initial vector $[1,0,-1]^T$ can be expressed as the following linear combination of C_1, C_2 and C_3:

$[1,0,-1]^T = \frac{1}{2}[1,1,0\,]^T + \frac{1}{4}(1 - i\,)[1,-1,-2\,i\,]^T + \frac{1}{4}(1 + i\,)[1,-1,2\,i\,]^T$.

We thus choose $K_1 = \frac{1}{2}, K_2 = \frac{1}{4}(1 - i\,)$ and $K_3 = \frac{1}{4}(1 + i\,)$, giving the solution

$$y(t) = \frac{1}{2}\,e^{-2t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{4}(1 - i\,)\,e^{(-40 + 40\,i\,)t}\begin{bmatrix} 1 \\ -1 \\ -2i \end{bmatrix} + \frac{1}{4}(1 + i)\,e^{(-40\,-\,40\,i\,)t}\begin{bmatrix} 1 \\ -1 \\ 2i \end{bmatrix}$$

or

$u(t) = \frac{1}{2}\,e^{-2t} + \frac{1}{2}\,e^{-40\,t}(\cos 40\,t + \sin 40\,t)$

$v(t) = \frac{1}{2}\,e^{-2t} - \frac{1}{2}\,e^{-40\,t}(\cos 40\,t + \sin 40\,t)$

$w(t) = -\,e^{-40t}(\cos 40\,t - \sin 40\,t)$

where $y\,(t) = [u(t),\; u(t),\; w(t)]^T$.

Previous Next

If we now attempt to solve this problem by Euler's method with $h = 0.04$ in the range $0.1 \leq t \leq 1.0$with $y(0.1)$ given by the exact solution. We find that for the given problem in the range $0.1 \leq t \leq 1.0$, the choice of $h = 0.04$ causes $\bar{h}$ to lie outside the region of absolute stability, which is the circle R with center -1, radius 1, and it follows that for $\bar{h}(= h\,\lambda)$ to lie within R for all three values of λ, we must satisfy $h < 0.025$. Note that the eigen values responsible for this severe restriction in $h$are $-40 \pm 40\,i$; that is, the very eigen values whose contributions to the theoretical solution are negligible in the range $0.1 \leq t \leq 1.0$

On the other hand, consider the IVP

$$y' = A\,y \qquad y(0.1) = \left[\frac{1}{2}\,e^{-0.2}, \frac{1}{2}e^{-0.2}, 0\right]^T$$

where $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

whose theoretical solution $y(t) = \left[\frac{1}{2}\,e^{-2t}, \frac{1}{2}e^{-2t}, 0\right]^T$

is, in the range $0.1 \leq t \leq 1.0$, virtually indistinguishable from that of the previous problem, is integrated perfectly satisfactorily by Euler's rule with step length 0.04. The Eigen values of the system for this problem are $-2, -2, 0$ and for absolute stability we require only $h < 1.0$.

Previous Next

We know that the $m \times m$ linear systems

$$y' = A\,y + \phi(t)$$

where the matrix A has distinct eigen values λ_j and corresponding eigen vector $C_j, j = 1,2,\ldots,m$ has a general solution of the form

$$y(t) = \sum_{j=1}^{m} K_j\, e^{\lambda_j t}\, C_j + \psi(t)$$

Let us assume that $\operatorname{Re}\lambda_j < 0,\ j = 1,2,\ldots,m$ then the term $\sum_{j=1}^{m} K_j\, e^{\lambda_j t}\, C_j \to 0$ as $t \to \infty$ we therefore call this term the transient solution, and call the remaining term $\psi(t)$ the steady state solution. Let λ_μ and λ_u be two eigen values of A such that

$$|\operatorname{Re}\lambda_\mu| \geq |\operatorname{Re}\lambda_j| \geq |\operatorname{Re}\lambda_u|,\ j = 1,2,\ldots,m$$

If our aim is to find numerically the steady state solution $\psi(t)$, then we must pursue the numerical solution until the slowest decaying exponential in the transient solution, namely $e^{\lambda_u t}$ is negligible. Thus, the smaller $|\operatorname{Re}\lambda_u|$, the longer will be the range of integration. On the other hand, the presence of eigen values of A far out to the left in the complex plane will force us to use excessively small step lengths in order that $\bar{h}$ will lie within the range of absolute stability of the method. The further out such eigen values lie, the more severe is the restriction on step length. A rough measure of this difficulty is the magnitude of $|\operatorname{Re}\lambda_\mu|$. If $|\operatorname{Re}\lambda_\mu| \gg |\operatorname{Re}\lambda_u|$, we are forced into the highly undesirable computational situation of having to integrate numerically over a long range, using a step length which is everywhere excessively small relative to the interval, this is the problem of stiffness. We can make the following somewhat heuristic definition.

Previous Next