Lecture 28: Convergence of Linear Multistep Methods

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Definition:

The linear multistep method defined by (8.1) is called Convergent , if the following statement is true for all functions $f(t,y)$ satisfying the existence and uniqueness conditions and all values of η: If $y(t)$ denotes the solution of the initial value problem

$$y' = f(t,y) \quad . \quad y(a) = \eta$$

then

$$\lim_{h\to 0} y_n = y(t) \qquad (8.9)$$

holds for all $t \in [a,b]$ and all solutions $\{y_n\}$ of the difference equation (8.1) having starting values $y_\mu = \eta_\mu(h)$ satisfying

$$\lim_{h\to 0} \eta_\mu(h) = \eta \qquad \mu = 0,1,\dots,k-1 \qquad (8.10)$$

It should be noted that this definition requires that condition (8.9) be satisfied not only for the sequence $\{y_n\}$ defined with the exact starting values- for these (8.10) is certainly satisfied- but also for all sequences whose starting values tend to the right value as $h \to 0$. This more stringent condition is imposed because in practice it is almost never possible to start a computation with mathematically exact values.

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Some Necessary Conditions

Theorem 1: A necessary condition for convergence of the linear multistep method (8.1) is that the modulus of no root of the associated polynomial $\rho(\xi)$ exceeds one, and that the roots of modulus one be simple.

The condition thus imposed on $\rho(\xi)$ is called the condition of zero-stability.

Proof: If the method is convergent, it is convergent for the initial value problem $y' = 0,\ y(0) = 0$, whose exact solution is $y(t) = 0$. For this problem (8.1) reduces to the difference equation with constant coefficients

$$\alpha_k\, y_{n+k} + \alpha_{k-1}\, y_{n+k-1} + \ldots + \alpha_0\, y_n = 0 \qquad (8.11)$$

If the method is convergent, then by (8.10), for any $t > 0$,

$$\lim_{n\to\infty} y_n = 0 \qquad \left(h = \frac{t}{n}\right) \qquad (8.12)$$

for all solutions $\{y_n\}$ of (8.11) satisfying

$$\lim_{h\to 0}\ \eta_\mu(h) = 0 \qquad \mu = 0,1,\ldots\ldots\ldots\ldots,K-1 \qquad (8.13)$$

where $y_\mu = \eta_\mu(h)$. Let $\xi = r\ e^{i\phi}$ $(r \geq 0, 0 \leq \phi < 2\pi)$ be a root of $\rho(\xi)$. Then, the numbers

$$y_n = h\, r^n \cos n\phi \qquad (8.14)$$

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define a solution of (8.11), and they also satisfy (8.13). If the method is convergent, (8.12) must hold. If $\phi = 0$ or $\phi = \pi$, this immediately implies $r \le 1$. If $\phi \ne 0,\ \phi \ne \pi$, we note that

$$y_n^2 = h^2 r^{2n} \cos^2 n\phi$$

$$y_{n+1} = h\, r^{n+1} \cos(n+1)\phi$$

$$y_{n-1} = h\, r^{n-1} \cos(n-1)\phi$$

and

$$\frac{y_n^2 - y_{n+1}\, y_{n-1}}{\sin^2\phi} = h^2 r^{2n}$$

Since the term on the left tends to zero as $n \to \infty$, $h = \frac{t}{n}$, the term on the right must do the same, which again implies $r \le 1$. This proves the first part of the assertion of the theorem. In order to prove the second part, assume that $\xi = re^{i\phi}$ is a root of $\rho(\xi)$ of multiplicity exceeding 1. Then, again the numbers

$$y_n = h^{1/2}\, n\, r^n \cos n\phi \qquad (8.15)$$

represent a solution of (8.11). They also satisfy (8.13). Hence they must satisfy (8.12) for a convergent method. If $\phi = 0$ or $\phi = \pi$, we have for $h = \frac{t}{n}$ that $|y_n| = t^{1/2}\, n^{1/2}\, r^n$, and it follows immediately that $r < 1$. If $\phi \ne 0,\ \phi \ne \pi$, we can make use of the relation

$$\frac{z_n^2 - z_{n+1}\, z_{n-1}}{\sin^2\phi} = r^{2n}$$

where $Z_n = n^{-1}\, h^{-1/2}\, y_n$. Since $Z_n \to 0$ as $n \to \infty$ in view of (8.12), the term on the left tends to zero as $n \to \infty$, and we conclude that $r < 1$. This proves the theorem.

Theorem 2: A necessary condition for convergence of the linear multistep method defined by (8.11) is that the order of the associated difference operator be at least 1.

The condition that the order $p \geq 1$ is called the condition of consistency. In terms of the constants introduced earlier (in associated difference operator) the condition is equivalent to $C_0 = 0, C_1 = 0$; in terms of the polynomials $\rho(\xi)$ and $\sigma(\xi)$, the condition of consistency is expressed by the relations

$$\rho(1) = 0\,, \qquad \rho'(1) = \sigma(1) \qquad (8.16)$$

Proof: We begin by showing that $C_0 = 0$. If the method is convergent, it is convergent in the initial value problem $y' = 0,\ y(0) = 1$, with the exact solution $y(t) = 1$. The difference equation (8.1) again reduces to

$$\alpha_K\, y_{n+K} + \alpha_{K-1}\, y_{n+K-1} + \ \ldots\ldots\ldots\ldots + \alpha_0\, y_n = 0 \qquad (8.17)$$

Assuming that the method is convergent, the solution $\{y_n\}$ of (8.17) assuming the exact starting values $y_\mu = 1\ (\mu = 0, 1, \ldots\ldots, K-1)$ must satisfy $y_n \to 1$ as $h \to 0, nh = t$. Since in this case y_n does not depend on h, this is the same as saying that $y_n \to 1$ as $n \to \infty$. Letting $n \to \infty$ in (8.17), we obtain $\alpha_K + \alpha_{K-1} + \cdots + \alpha_0 = 0$. This is equivalent to $C_0 = 0$. It follows that $p \geq 0$.

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In order to show that $C_1 = 0$, consider the initial value problem $y' = 1,\ y(0) = 0$. The exact solution is $y(t) = t$. The difference equation (8.1) now reads

$$\alpha_K\, y_{n+K} + \alpha_{K-1}\, y_{n+K-1} + \cdots + \alpha_0\, y_n = h\left(\beta_K + \beta_{K-1} + \cdots + \beta_0\right) \qquad (8.18)$$

For a convergent method every solution of (8.18) satisfying

$$\lim_{h \to 0}\ \eta_\mu(h) = 0 \qquad \mu = 0, 1, \ldots, K-1 \qquad (8.19)$$

where $y_\mu = n_\mu(h)$, must also satisfy

$$\lim_{h \to 0}\ \ y_n = t \qquad (8.20)$$

For a convergent method we may further more assume that

$$K\,\alpha_K + (K-1)\ \alpha_{K-1} + \cdots + \alpha_1 = \rho'(1) \neq 0$$

in view of the previous theorem. Let the sequence $\{y_n\}$ be defined by $y_n = nhk$, where

$$K = \frac{\beta_K + \beta_{K-1} + \cdots + \beta_0}{K\alpha_K + (K-1)\ \ \alpha_{K-1} + \cdots + \alpha_1}$$

This sequence obviously satisfies (8.19) and is easily shown to be a solution of (8.18).

From

$$\lim_{h \to 0} nhk = tk \quad (\because nh = t),$$

we conclude that $K = 1$. This is equivalent to $C_1 = 0$. This completes the proof of the theorem.

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