Lecture 4: The Euler Method

The Lecture Contains:

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We shall now describe methods for solving a scalar IVP

$$y' = f(t, y) \qquad t \in [0, b]$$

$$y(0) = y_0$$

Most of the methods that follow can be easily extended to vector systems. Since it is assumed that the given IVP is not amenable to analytical solution, we approximate its solution at a set of discrete points, called the mesh (or grid) points. We subdivide the internal $[0, b]$ into a finite number of equally spaced N subintervals as

$$0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots t_N = b$$

where $t_j = t_0 + jh, j = 1,2, \dots, N$

and h is called the mesh (or step) size. Since the solution at $t = t_0$ is known (initial condition), we need to approximate the solution at the grid points $t = t_j$ for $j = 1,2, \dots, N$.

A method which involves the knowledge of the solution only at the previous point $t = t_n$ in order to find the solution at the current point $t = t_{n+1}$, is called a single step method.

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The Euler Method

It is instructive to examine the simplest method, the Euler method, for solving the first-order scalar IVP given by

$$y' = f(t, y) \qquad t \in [0, b]$$

$$y(0) = y_0$$

(This method is also called the Euler-Cauchy, forward Euler, or explicit Euler method).

In this method, the value of the dependent variable at the current point is calculated by straight line extrapolation from the previous point. Since the initial data is known, we can evaluate

$$y'(0) = f\left(0, y(0)\right)$$

and from this, we can calculate an approximation to $y(h)$ by using the first two terms of a Taylor' series

$$y(h) \simeq y_0 + h\, y'(0)$$

We let $t_1 = h$, and define our approximation to $y(t_1)$ as y_1. Thus

$$y_1 = y_0 + h\, f(t_0, y_0)$$

Similarly,

$$y_2 = y_1 + h\, f(t_1, y_1)$$

and, in general, for $n = 0, 1, 2, \ldots ..$

$$y_{n+1} = y_n + h\, f(t_n, y_n), \text{ where } t_n = n\, h \qquad (2.1)$$

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Module 2: Single Step Methods

Lecture 4: The Euler Method

Euler's Method (Analytical Interpretations)

1. If we approximate the derivative appearing in the differential equation at the point (t_n, y_n) by a forward difference, we obtain

   $$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n)$$

   Solving for y_{n+1} yields the formula for the Euler's method.

2. Integrating the identity

   $$y'(t) = f(t, y(t))$$

   between the limits t and $t + k$, we obtain

   $$y(t + k) - y(t) = \int_t^{t+k} f(x, y(x))\, dx$$

   In particular, if $t = t_n$ and $k = h$, we get

   $$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(x, y(x))\, dx$$

   Approximating the integral by a crude rule for numerical integration (length of the interval times the value of integrand at left end point) and identifying $y(t_n)$ with y_n, we obtain the Euler's method.

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3. We finally may assume the possibility of expanding the solution in a Taylor series around the point t_n:

$$y(t_n + h) = y(t_n) + h\, f\big(t_n, y(t_n)\big) + \frac{h^2}{2}\, y''(t_n) + \cdots$$

Truncating the series after the linear term in h yields the Euler's method.

Remark: Each of these interpretations points the way to a class of generalizations of Euler's method and it is interesting to note that the generalization indicated by (i) (numerical differentiation), which seems to be the most straight forward, has proved to be the least fruitful of the three.

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An Analytical Example

If $f(t, y)$ is a sufficiently simple function, it may be possible to solve the recurrence relation for y_n as a function of n and h. Such an explicit solution is rarely of practical interest, because it can usually only be found in cases where the differential equation itself can be solved in closed form. However, it can be helpful in the study of theoretical properties of the method under consideration.

Let us find the explicit form of the Euler approximation..

Euler approximation to the solution of the IVP

$$y' = y, \quad y(0) = 1.$$

Here we have $f(t, y) = y$ and hence

$$y_{n+1} = y_n + h\, y_n = (1+h)\, y_n, \qquad n = 0,1, \ldots$$

In view of $y_0 = 1$, we thus find

$$y_1 = 1 + h$$

$$y_2 = (1+h)\, y_1 = (1+h)^2$$

and generally

$$y_n = (1+h)^n, \quad n = 1,2, \ldots \ldots$$

Since $n = \frac{t_n}{h}$, the value approximating the solution at the point $t_n = t$ is thus given by

$$y_n = (1+h)^{t/h} = \left[(1+h)^{1/h}\right]^t$$

As $h \to 0$, this tends to e^t. We thus have shown that by decreasing the mesh size, the exact solution of the IVP can be approximated arbitrarily well in the special example under consideration.

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