Module 10: Finite Difference Methods for Boundary Value Problems

Lecture 37: Finite Difference Methods

The Lecture Contains:

Linear Boundary Value Problems

Consider the single linear second-order equation

$$L\,y(x) = -y'' + p(x)\,y' + q(x)\,y = r(x) \qquad a < x < b \tag{10.1}$$

subject to

$$y(a) = \alpha, \quad y(b) = \beta \tag{10.2}$$

A unique solution exists if $p(x)$, $q(x)$ and $r(x)$ are continuous on $[a,b]$ and $q(x)$ is positive there. But since these functions are continuous on a closed bounded interval, there must exist positive constants P^*, Q^* and Q_* such that

$$|p(x)| \le P^*\,,\ 0 < Q_* \le q(x) \le Q^* \qquad a \le x \le b$$

We shall now study finite difference methods for computing approximations to the solution of the BVP (10.1)-(10.2).

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Discretization

On the interval $[a,b]$, we place a uniform mesh, say

$$x_j = a + j\,h,\;\; j = 0,1,\dots,J+1,\;\; h = \frac{b-a}{J+1}$$

To approximate $y(x)$ on this mesh, we define a mesh function $\{u_j\}$ as the solution of a system of finite difference equations which are in some sense an approximation to the above problem. An obvious such difference formulation is

$$L_h\,u_j \equiv -\left(\frac{u_{j+1} - 2\,u_j + u_{j-1}}{h^2}\right) + p(x_j)\left(\frac{u_{j+1} - u_{j-1}}{2\,h}\right) + q(x_j)\,u_j = r\,(x_j) \qquad 1 \le j \le J \qquad (10.3)$$

$$u_0 = \alpha$$

$$u_{J+1} = \beta$$

We now proceed to show that the linear system (10.3) can be solved by a simple algorithm if h is sufficeintly small, and then we estimate the accuracy of this numerical solution.

Multiplying (10.3) by $\frac{h^2}{2}$, we have

$$\frac{h^2}{2}\,L_h\,u_j \equiv a_j\,u_{j-1} + b_j\,u_j + c_j\,u_{j+1} = \frac{h^2}{2}\,r(x_j) \qquad 1 \le j \le J \qquad (10.4)$$

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where

$$a_j \equiv -\frac{1}{2}\left[1+\frac{h}{2}\,p\left(x_j\right)\right]$$

$$b_j \equiv \left[1+\frac{h^2}{2}\,q\left(x_j\right)\right] \qquad 1 \le j \le J$$

$$c_j \equiv -\frac{1}{2}\left[1-\frac{h}{2}\,p\left(x_j\right)\right]$$

or in vector notation ,

$$A\,u = r \tag{10.5}$$

where we have introduced the J-dimensional vectors

$$u \equiv \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_J \end{pmatrix}, \quad r \equiv \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_J \end{pmatrix} \equiv \frac{h^2}{2}\begin{pmatrix} r(x_1) \\ r(x_2) \\ \vdots \\ r(x_J) \end{pmatrix} - \begin{pmatrix} a_1\,\alpha \\ 0 \\ \vdots \\ 0 \\ c_J\,\beta \end{pmatrix}$$

and the J-order matrix

$$A = \begin{pmatrix} b_1 & c_1 & 0 & \cdots & \cdots & 0 \\ a_2 & b_2 & c_2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & a_{J-1} & b_{J-1} & c_{J-1} \\ 0 & \cdots & \cdots & 0 & a_J & b_J \end{pmatrix}$$

This is called a tri diagonal matrix.

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Solution of Algebraic System(LU-Factorization)

We assume that A is non-singular and can be factored into the product

$A = L\,U$

where

$$L = \begin{pmatrix} \beta_1 & 0 & & & 0 \\ a_2 & \beta_2 & & & \\ 0 & & & & 0 \\ \vdots & & & & \\ 0 & \dots & 0 & a_J & \beta_J \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & r_1 & 0 & 0 \\ 0 & 1 & r_2 & 0 \\ & & & r_{J-1} \\ 0 & & 0 & 1 \end{pmatrix}$$

It follows that β_j and r_j must satisfy

$$\beta_1 = b_1 \qquad r_1 = \frac{c_1}{\beta_1}$$

$$\beta_j = b_j - a_j\, r_{j-1} \quad j = 2,3,\dots,J\,; \tag{10.6}$$

$$r_j = \frac{c_j}{\beta_j},\quad j = 2,3,\dots,J-1.$$

The given system can now be replaced by an equivalent pair of systems

$$L\,z = r,\quad Uu = z \tag{10.7}$$

But since L and U are triangular, the solution of these systems are easily obtained

$$z_1 = {}^{r_1}/_{\beta_1} \qquad z_j = \left(r_j - a_j\, z_{j-1}\right)/\beta_j, j = 2,3,\dots,J$$

and

$$u_J = z_J \qquad u_j = z_j - r_j\, u_{j+1} \quad j = J-1,\ J-2,\dots 1$$

The factorization procedure can fail if some β_j vanishes. This difficulty can be avoided by means of the following:

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