Module 3: Higher order Single Step Methods

Lecture 9: Runge-Kutta Methods

The Lecture Contains:

Derivation of Runge-Kutta methods

The general R-stage Runge-kutta method is defined by

$$y_{n+1} - y_n = h\,\phi\,(t_n, y_n, h) \qquad (3.11)$$

where

$$\phi\,(t, y, h) = \sum_{r=1}^{R} C_r\, K_r \qquad (3.12)$$

with

$$K_1 = f\,(t, y) \qquad (3.13)$$

$$K_r = f\Big(t + h\, a_r,\, y + h\, \sum_{s=1}^{r-1} b_{rs}\, K_s\Big), r = 2, 3, \ldots, R \qquad (3.14)$$

and

$$a_r = \sum_{s=1}^{r-1} b_{rs} \quad r = 2, 3, \ldots, R \qquad (3.15)$$

Note: Here R-stage Runge-kutta method involves R function evaluations per step. Each of the functions $K_r(t, y, h), r = 1, 2, \ldots, R$ may be interpreted as an approximation to the derivative $y'(t)$, and the function $\phi\,(t, y, h)$ as a weighted mean of these approximations. We may also note that consistency demands that $\sum_{r=1}^{R} C_r = 1$ (The notion of consistency would be defined later; it means that the order of the method is at least one). The goal is to determine values for the parameters C_r, a_r, and b_{rs} such that the order of the method is as high as possible.

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In view of tedious calculations involved in deriving Runge–kutta methods of higher order, we shall derive only methods of order upto three and quote same well known methods of order four. We may recall that the Taylor's method of order p can be written in the form (3.11) with

$$\phi(t,y,h) = \phi_T\,(t,y,h) = f\,(t,y) + \frac{h}{2!}\,f'(t,y) + \cdots + \frac{h^{p-1}}{p!}f^{(p-1)}(t,y) \quad (3.16)$$

where

$$f^{(q)}\,(t,y) = \frac{d^q}{dt^q}f\,(t,y), \quad q = 1,2,\ldots,(p-1)$$

The equation (3.16) can also be written as

$$\phi_T\,(t,y,h) = f + \tfrac{1}{2}hF + \frac{h^2}{6}\left(F\,f_y + G\right) + O(h^3) \quad (3.17)$$

where

$$F = f_t + ff_y \quad (3.18)$$

$$G = f_{tt} + 2f\,f_{ty} + f^2\,f_{yy}$$

We also write

$$K_1 = f\,(t,y)$$

$$K_2 = f\,(t + h\,a_2,\; y + h\,a_2\,K_1)$$

$$K_3 = f\,(t + h\,a_3,\; y + h\,(a_3 - b_{32})K_1 + h\,b_{32}\,K_2)$$

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Expanding K_2 as a Taylor series about the point (t, y), we obtain

$$K_2 = f + h\, a_2 \left(f_t + K_1 f_y\right) + \frac{h^2}{2} a_2^2 \left(f_{tt} + 2K_1 f_{ty} + k_1^2 f_{yy}\right) + O(h^3)$$

Substituting for K_1, and using (3.18), we get

$$K_2 = f + h\, a_2\, F + \frac{h^2}{2} a_2^2\, G + O(h^3) \tag{3.19}$$

Expanding K_3 by Taylor series and after substituting for K_1 and K_2, we obtain

$$K_3 = f + h\, a_3\, F + h^2 \left(a_2 b_{32}\, F f_y + \frac{1}{2} a_3^2\, G\right) + O(h^3) \tag{3.20}$$

Substituting the expansion of $K_1, K_2, \text{and } K_3$ in (3.12), we have

$$\phi(t, y, h) = (c_1 + c_2 + c_3) f + h(c_2 a_2 + c_3 a_3) F + \frac{h^2}{2}\left[2\, c_3 a_2 b_{32}\, F f_y + (c_2 a_2^2 + c_3 a_3^2)\, G\right] + O(h^3) \tag{3.21}$$

We now have to match (3.21) with (3.16) to find the parameters. We do this in the following manner:

We first let $R = 1$, so that $c_2 = c_3 = 0$ and (3.21) reduces to

$$\phi(t, y, h) = c_1 f + O(h^3) \tag{3.22}$$

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On setting $c_1 = 1$, (3.22) differs from the expansion (3.16) for ϕ_T by a term of order h. Thus, the resulting method, which is the Euler's method, has order one.

Now let $R = 2$, so that $c_3 = 0$ and (3.21) reduces to

$$\phi(t, y, h) = (c_1 + c_2)\, f + h\, c_2\, a_2\, F + \frac{1}{2}\, h^2\, c_2 a_2^2\, G + O(h^3) \qquad (3.23)$$

Matching this with the expansion (3.16), we have

$$c_1 + c_2 = 1$$

$$c_2\, a_2 = \frac{1}{2} \qquad (3.24)$$

This gives a set of two equations in three unknowns and there exists a one parameter family of solutions. Thus there exists an infinite number of two–stage Runge–kutta methods of order two and none of order more than two. Two particular solutions of (3.24) yield the following well known methods:

i) $c_1 = 0, c_2 = 1,\ a_2 = \frac{1}{2}$.

The resulting method is

$$y_{n+1} - y_n = \frac{h}{2}\, [f\,(t_n, y_n) + f\,(t_n + h, y_n + h\, f\,(t_n, y_n))] \qquad (3.25)$$

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This method is referred to as the modified Euler method.

ii) $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$, $a_2 = 1$.

The resulting method is

$$y_{n+1} - y_n = \frac{h}{2}\left[f\left(t_n, y_n\right) + f\left(t_n + h, y_n + h\, f\left(t_n, y_n\right)\right)\right] \qquad (3.26)$$

which is known as the Improved Euler Method.

For $R = 3$, we can match (3.21) with (3.16) upto and including h^2 term if we satisfy the following set of equations

$$c_1 + c_2 + c_3 = 1$$

$$c_2\, a_2 + c_3\, a_3 = \frac{1}{2} \qquad (3.27)$$

$$c_2\, a_2^2 + c_3\, a_3^2 = \frac{1}{3}$$

$$c_3\, a_2\, b_{32} = \frac{1}{6}$$

These are now four equations in six unknowns and there exists a two–parameter family of solutions. Thus there exist infinite family of three-stage Runge-kutta methods of order three and none of order more than three. Two particular solutions of (3.27) lead to well-known third order Runge–kutta methods

- $c_1 = \frac{1}{4}$. $c_2 = 0$, $c_3 = \frac{3}{4}$, $a_2 = \frac{1}{3}$,

$a_3 = \frac{2}{3}$, $b_{32} = \frac{2}{3}$. The resulting method is:

$$y_{n+1} - y_n = \frac{h}{4}\left(K_1 + 3K_3\right)$$

$$K_1 = f\left(t_n, y_n\right)$$

$$K_2 = f\left(t_n + \frac{h}{3},\ y_n + \frac{h}{3} K_1\right) \qquad (3.28)$$

$$K_3 = f\left(t_n + \frac{2}{3}h, y_n + \frac{2}{3}\, h\, K_2\right)$$

This is known as Heun's third order method.

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- $c_1 = \frac{1}{6}$, $c_2 = \frac{2}{3}$, $c_3 = \frac{1}{6}$, $a_2 = \frac{1}{2}$, $a_3 = 1$, $b_{32} = 2$. The resulting method is

$$y_{n+1} - y_n = \frac{h}{6}\left(K_1 + 4K_2 + K_3\right)$$

$$K_1 = f\left(t_n, y_n\right) \qquad (3.29)$$

$$K_2 = f\left(t_n + \frac{h}{2}, \quad y_n + \frac{h}{2} K_1\right)$$

$$K_3 = f\left(t_n + h, \quad y_n - h K_1 + 2 h K_2\right)$$

This is known as Kutta's third order method.

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Classical Runge-Kutta Method

The derivation of fourth order Runge-kutta methods involve tedious calculations. It transpires that with R=4, the fourth order and no higher can be obtained. The well-known fourth order classical Runge–Kutta method is given by

$$y_{n+1} - y_n = \frac{h}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)$$

$$K_1 = f\left(t_n, y_n\right)$$

$$K_2 = f\left(t_n + \frac{h}{2},\ y_n + \frac{h}{2} K_1\right) \tag{3.30}$$

$$K_3 = f\left(t_n + \frac{h}{2},\ y_n + \frac{h}{2} K_2\right)$$

$$K_4 = f\left(t_n + h,\ y_n + h\ K_3\right)$$

Remark: We have seen that there exists an R-stage method of order R and none of order greater than R for $R = 1, 2, 3, 4$. We have the following result that relates the no. of stages in the Runge-Kutta process with the order of the method:

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Attainable Order of Runge-Kutta Methods

Let $p^*(R)$ be the highest order that can be attained by an R-stage Runge-Kutta method. Then

$p^*(R) = R\,, \;\; R = 1, 2, 3, 4$

$p^*(5) = 4$

$p^*(6) = 5$

$p^*(7) = 6$

$p^*(R) \leq R - 2, R = 8, 9, \ldots.$

It is clear from the above why Runge–Kutta methods of fourth order are most popular.

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