Module 10: Finite Difference Methods for Boundary Value Problems

Lecture 39: Analytic Expressionof the Error

## The Lecture Contains:

**Theorem:** Let $p(x)$ and $q(x)$ satisfy the said inequalities and $h$ satisfy $h \leq \frac{2}{p^*}$.

In addition, let $p(x), q(x)$ and $r(x)$ be so smooth that $y(x)$, the solution of the given BVP has a continuous sixth derivative on $[a,b]$. Then (10.8) holds with $\{u_j\}$, the solution of the difference system, and $e(x)$, the solution of the BVP

$$L\, e(x) = \theta(x), \qquad a < x < b \qquad (10.9)$$

$$e(a) = e(b) = 0$$

where

$$\theta(x) \equiv -\frac{1}{12}\left[y^{iv}(x) - 2p(x)y'''(x)\right].$$

**Proof:** We first note that the BVP (10.9) has a unique solution. Now define the net function $\{e_j\}$ by

$$e_j \equiv h^{-2}\left[y(x_j) - u_j\right] \qquad 0 \leq j \leq J+1$$

using the given BVP, difference system, and the T.E. we obtain

$$L_h\, e_j = h^{-2}\left[L_h\, y(x_j) - L_h\, u_j\right]$$

$$= h^{-2}\left[L_h\, y(x_j) - Ly\,(x_j)\right] + h^{-2}\left[L\, y(x_j) - L_h\, v_j\right]$$

$$= h^{-2}\, T_j\,[y] \qquad 1 \leq j \leq J.$$

Now the T.E. can be written as

$T_j[y] = h^2\, \theta(x_j) + O(h^4)$(using the continuity of $y^{vi}(x)$).

Thus we have

$$L_h\, e_j = \theta(x_j) + O(h^2) \qquad 1 \leq j \leq J$$

Previous Next

In addition

$$e_0 = e_{J+1} = 0 \qquad \begin{bmatrix} \because y(a) = \alpha, y(b) = \beta \\ u_0 = \alpha, u_{J+1} = \beta \end{bmatrix}$$

Now $L_h\left[e(x_j) - e_j\right] = L_h\, e(x_j) - L\, e(x_j) - O(h^2)$

$= T_j[e] + O(h^2)$

$= O(h^2) \quad 1 \le j \le J$

where we have used the smoothness of $e(x)$ in estimating $T(e)$.

From the stability of $L_h$, we get

$$\left|e(x_j) - e_j\right| \le O(h^2) \qquad (10.10)$$

Then multiplying by $h^2$yields (10.8).

Previous Next

## More general boundary conditions

Let us consider the BVP

$$L\,y(x) \equiv -y'' + p(x)y' + q(x)\,y = r(x) \qquad (10.11)$$

subject to

$$a_0\,y(a) - a_1\,y'(a) = \alpha \qquad a_0 \geq 0, a_1 \geq 0$$

$$b_0\,y(b) + b_1\,y'(b) = \beta \qquad b_0 \geq 0, b_1 \geq 0 \qquad a_0 + b_0 \neq 0 \quad (10.12)$$

Let the difference equation, be

$$L_h\,u_j = r(x_j) \qquad 0 \leq j \leq J+1$$

$$a_0\,u_0 - a_1\,\frac{u_1 - u_{-1}}{2\,h} = \alpha$$

$$b_0\,u_{J+1} + b_1\,\frac{u_{J+2} - u_J}{2\,h} = \beta$$

Again, we can show that for $h < \dfrac{2}{P^*}$, the difference system has a unique solution.

Previous Next