Lecture 26: Linear Multistep Methods (Contd)

The Lecture Contains:

A Constructive Example:

Construct an implicit linear multistep (two-step) method of maximal order, containing one free parameter, and find its order.

Solution: Here $K = 2$; $\alpha_2 = +1$ by hypothesis. Let $\alpha_0 = a$ be the free parameter. There remain four undetermined coefficients $\alpha_1, \beta_0, \beta_1, \text{and } \beta_2$ and we can thus set $C_0 = a + \alpha_1 + 1 = 0$

$$C_0 = a + \alpha_1 + 1 = 0$$

$$C_1 = \alpha_1 + 2 - (\beta_0 + \beta_1 + \beta_2) = 0$$

$$C_2 = \frac{1}{2!}(\alpha_1 + 4) - (\beta_1 + 2\beta_2) = 0$$

$$C_3 = \frac{1}{3!}(\alpha_1 + 8) - \frac{1}{2!}(\beta_1 + 4\beta_2) = 0.$$

Solving, we get

$$\alpha_1 = -1 - a, \beta_0 = -\frac{1}{12}(1 + 5a), \beta_1 = \frac{2}{3}(1 - a), \beta_2 = -\frac{1}{12}(5 + a)$$

and the method is

$$y_{n+2} - (1 + a)\, y_{n+1} + a\, y_n = \frac{h}{12}[(5 + a)\, f_{n+2} + 8\,(1 - a) f_{n+1} - (1 + 5a) f_n]^{(*)}$$

Moreover,

$$C_4 = \frac{1}{4!}(\alpha_1 + 16) - \frac{1}{3!}(\beta_1 + 8\beta_2)$$

$$= -\frac{1}{4!}(1 + a)$$

$$C_5 = \frac{1}{5!}(\alpha_1 + 32) - \frac{1}{4!}(\beta_1 + 16\beta_2) = -\frac{1}{3!\,5!}(17 + 13a)$$

If $a \neq -1$, then $c_4 \neq 0$ and the method $(*)$ is of order 3.

If $a = -1$, then $C_4 = 0, C_5 \neq 0$ and the method $(*)$ which is now Simpson's rule, is of order 4. Note that when $a = 0, (*)$ is the two step Adam-Moultan method, while if $a = -5$, it is an explicit method.

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Associated Difference Polynomials; Necessary & Sufficient Condition for Order

The order p and the error coefficient C_{p+1} can be expressed in a more convenient way as follows:

We define the polynomials

$$\rho(\xi) = \alpha_K \xi^K + \alpha_{K-1} \xi^{K-1} + \cdots + \alpha_0$$

and $\sigma(\xi) = \beta_K \xi^K + \beta_{K-1} \xi^{K-1} + \cdots + \beta_0$

The difference operator associated with (8.1) is given by

$$L[y(t);h] = \alpha_K y(t+Kh) + \alpha_{K-1} y(t+(K-1)h) + \cdots + \alpha_0 y(t) - h\{\beta_K y'(t+Kh) + \beta_{K-1} y'(t+(K-1)h) + \ldots + \beta_0 y'(t)\}$$

Consider the function $y(t) = e^{\lambda t}$, we have

$$L[y(t);h] = \alpha_K e^{\lambda(t+Kh)} + \alpha_{K-1} e^{\lambda(t+(K-1)h)} + \ldots + \alpha_0 e^{\lambda t} - \lambda h\{\beta_K e^{\lambda(t+(Kh))} + \cdots + \beta_0 e^{\lambda t}\}$$

$$= e^{\lambda t}[\{\alpha_K e^{\lambda hK} + \alpha_{K-1} e^{\lambda h(K-1)} + \cdots + \alpha_0\} - \lambda h\{\beta_K e^{\lambda hK} + \beta_{K-1} e^{\lambda h(K-1)} + \cdots + \beta_0\}]$$

$$= e^{\lambda t}\left[\rho\left(e^{\lambda h}\right) - \lambda h\, \sigma\left(e^{\lambda h}\right)\right] \qquad (8.5)$$

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If the method is of order p, then we have

$$L\,[y(t);h] = C_{p+1}\, h^{(p+1)}\, y^{(p+1)}(t) + 0(h^{p+2})$$

$$= C_{p+1}\,(\lambda h)^{p+1}\, e^{\lambda t} + \; 0\,(h^{p+2})$$

$$= C_{p+1}\,(\lambda h)^{p+1}\, e^{\lambda t} + \; 0\,(h^{p+2}) \qquad (8.6)$$

Therefore, form (8.5) and (8.6),

$$\rho\left(e^{\lambda h}\right) - \; \lambda\, h\, \sigma\left(e^{\lambda h}\right) = \; C_{p+1}\,(\lambda h)^{p+1} + \; 0(h^{p+2}) \qquad (8.7)$$

Now writing $\lambda h = \log\,(1+Z)$ and noting that $\lambda h = Z + 0\,(Z^2)$, we have then

$$\rho\,(1+Z) - \log(1+Z)\;\sigma\,(1+Z) = \; C_{p+1}\, Z^{p+1} + 0(Z^{p+2}) \qquad (8.8)$$

Consequently, Equation (8.8) is a necessary and sufficient condition for a method to have order p.

If the polynomial $\sigma\,(\xi)$ is given, Equation (8.8) shows how a unique polynomial $\sigma\,(\xi)$ of degree K can be found such that the method has order $\geq K$. This is illustrated by the following example.

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Illustrative Examples

Example 1: If $\sigma(\xi) = \frac{3}{2}\xi - \frac{1}{2}$ and $K = 2$, we get

$$\rho(1+Z) = \log(1+Z)\,\sigma(1+Z) + C_{p+1} Z^{p+1}$$

$$= \log(1+Z)\left\{\frac{3}{2}(1+Z) - \frac{1}{2}\right\} + 0(Z^3) \quad (\because \text{ order is } 2)$$

$$= \left(Z - \frac{Z^2}{2}\right)\left(\frac{3}{2}Z + 1\right) + 0(Z^3)$$

$$= \frac{3}{2}Z^2 - \frac{Z^2}{2} + Z$$

$$= Z^2 + Z$$

$$= (1+Z)^2 - (1+Z)$$

Hence $\rho(\xi) = \xi^2 - \xi$, which gives the second order Adams-Bashforth method

$$y_{n+2} - y_{n+1} = h\left\{\frac{3}{2}y'_{n+1} - \frac{1}{2}y_n\right\}$$

or $y_{n+2} - y_{n+1} = \frac{3}{2}\,h y'_{n+1} - \frac{h}{2}y'_n$

If, on the other hand, $\rho(\xi)$ is given, there exists a $\sigma(\xi)$ of degree K such that the method is of order $\geq K+1$. We find this by dividing (8.8) by log $(1+Z)$ to get $(\text{with } p = K+1)$

$$\sigma(1+Z) = \frac{Z}{\log(1+Z)}\cdot\frac{\rho(1+Z)}{Z} - C_{K+2}Z^{K+1} - 0(Z^{K+2}) \quad (\because \log(1+Z) = Z + 0(Z^2)).$$

Since $\frac{Z}{\log(1+Z)}$ is analytic at $Z = 0$, $\frac{\rho(1+Z)}{Z}$ must be analytic at $Z = 0$, if $K \geq 0$, that is,

$$\rho(1) = \alpha_K + \alpha_{K-1} + \cdots + \alpha_0 = 0$$

Example 2: If $\rho(\xi) = -\xi^2 + \xi$ and $K = 2$ we get

$$\sigma(1+Z) = -\frac{(1+Z)^2 + (1+Z)}{\log(1+Z)} + 0(Z^3)$$

$$= \frac{-Z^2 - Z}{Z - \frac{Z^2}{2} + \frac{Z^3}{3}} + 0(Z^3)$$

$$= -(1+Z)\left(1 - \frac{Z}{2} + \frac{Z^2}{3}\right)^{-1} + 0(Z^3)$$

$$= -(1+Z)\left(1 + \frac{Z}{2} - \frac{Z^2}{3} + \frac{Z^2}{4}\right) + 0(Z^3)$$

$$= -1 - Z - \frac{Z}{2} - \frac{Z^2}{2} + \frac{Z^2}{3} - \frac{Z^2}{4}$$

$$= -1 - \frac{3Z}{2} - \frac{5}{12} Z^2$$

$$= -\frac{5}{12}(1+Z)^2 - \frac{2}{3}(1+Z) + \frac{1}{12}$$

Hence, $\sigma(\xi) = -\frac{5}{12}\xi^2 - \frac{2}{3}\xi + \frac{1}{12}$, which gives the third order Adams-Moulton method:

$$-y_{n+2} + y_{n+1} = h\left\{-\frac{5}{12}y'_{n+2} - \frac{2}{3}y'_{n+1} + \frac{1}{12}y'_n\right\}$$

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