Module 8: Linear Multistep Methods

Lecture 30: Absolute Stability and Relative Stability

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Stability Polynomial

Consider the general linear multistep method

$$\sum_{j=0}^{K} \alpha_j \, y_{n+j} = h \sum_{j=0}^{K} \beta_j \, f_{n+j} \qquad (8.26)$$

which we assume to be consistent and zero–stable. The theoretical solution $y(t)$ of the initial value problem satisfies

$$\sum_{j=0}^{K} \alpha_j \, y\left(t_{n+j}\right) = h \sum_{j=0}^{K} \beta_j \, f\left(t_{n+j}, y(t_{n+j})\right) + T_{n+K} \qquad (8.27)$$

where $T_{n+K} = L\,[y(t_n): h]$, the local truncation error. If we denote by $\{\breve{y}_n\}$ the solution of (8.26) when a round-off error R_{n+K} is committed at the n^{th} application of the method, then

$$\sum_{j=0}^{K} \alpha_j \, \breve{y}_{n+j} = h \sum_{j=0}^{K} \beta_j \, f\left(t_{n+j}, \breve{y}_{n+j}\right) + R_{n+K} \qquad (8.28)$$

on subtracting (8.28) from (8.27) and defining the global error $\breve{e}_n$ by $\breve{e}_n = y\left(t_n\right) - \breve{y}_n$ we find

$$\sum_{j=0}^{K} \alpha_j \, \breve{e}_{n+j} = h \sum_{j=0}^{K} \beta_j \left[f\left(t_{n+j}, y(t_{n+j})\right) - f\left(t_{n+j}, \breve{y}_{n+j}\right)\right] + \phi_{n+K} \qquad (8.29)$$

where $\phi_{n+K} = T_{n+K} - R_{n+K}$

If we assume that the partial derivatives $\dfrac{\partial f}{\partial y}$ exists for all $t \in [a, b]$, then by the mean value theorem, there exists a number ξ_{n+j} lying in the open interval whose end points are $y\left(t_{n+j}\right)$ and $\breve{y}_{n+j}$, such that

$$f\left(t_{n+j}, y(t_{n+j})\right) - f\left(t_{n+j}, \breve{y}_{n+j}\right)$$

$$= \left[y\left(t_{n+j}\right) - \breve{y}_{n+j}\right] \frac{\partial f}{\partial y}\left(t_{n+j}, \xi_{n+j}\right)$$

$$= \breve{e}_{n+j} \, \frac{\partial f}{\partial y} \left(t_{n+j}, \xi_{n+j}\right)$$

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Hence

$$\sum_{j=0}^{K} \alpha_j \ \breve{e}_{n+j} = h \ \sum_{j=0}^{K} \beta_j \frac{\partial f\left(t_{n+j}, \xi_{n+j}\right)}{\partial y} \ \breve{e}_{n+j} + \phi_{n+K} \qquad (8.30)$$

We now make two important simplifying assumptions:

$\frac{\partial f}{\partial y} = \lambda$, Constant

$\phi_n = \phi$, Constant

The equation for $\breve{e}_n$ now reduces to the linearized error equation

$$\sum_{j=0}^{K} \left(\alpha_j - h \, \lambda \, \beta_j\right) \ \breve{e}_{n+j} = \phi \qquad (8.31)$$

The general solution of this equation is

$$\breve{e}_n = \sum_{s=1}^{K} d_s \ r_s^n - \phi / h \, \lambda \sum_{j=0}^{K} \beta_j \qquad (8.32)$$

where the d_s are arbitrary constants and the r_s are the roots, assumed distinct, of the polynomial equation

$$\sum_{j=0}^{K} \left(\alpha_j - h \, \lambda \, \beta_j\right) \ r^j = 0 \qquad (8.33)$$

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This equation may be written in terms of first and second characteristic polynomial as

$$\lambda(r,\bar{h}) \equiv \rho(r) - \bar{h}\,\sigma(r) = 0 \qquad (8.34)$$

where

$\bar{h} = h\,\lambda$

The polynomial $\pi(r,\bar{h})$ is frequently referred to as the characteristic polynomial of the method. However, we shall call $\pi(r,\bar{h})$ the stability polynomial of the method defined by ρ and σ.

Definition: The linear multistep method (8.26) is said to be absolutely stable for a given $\bar{h}$ if, for that $\bar{h}$, all the roots r_s of $\pi(r,\bar{h})$ satisfy $|r_s| < 1,\ \ s = 1, 2, \ldots, K$, and to be absolutely unstable for that $\bar{h}$otherwise. An interval (α,β) of the real line is said to be an interval of absolute stability if the method is absolutely stable for all $\bar{h} \in (\alpha,\beta)$. If the method is absolutely unstable for all $\bar{h}$ it is said to have no interval of absolute stability.

Definition: The linear multistep method (8.26) is said to be relatively stable for a given $\bar{h}$ if for that $\bar{h}$, the roots of $\pi(r,\bar{h})$ satisfy $|r_s| < |r_1|,\ \ s = 2, 3, \ldots, K$ and to be relatively unstable otherwise. An interval (α,β) of the real line is said to be an interval of relative stability if the method is relatively stable for all $\bar{h} \in (\alpha,\beta)$.

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$O(\bar{h})$ approximation to roots of Stability Polynomial:

From

$$\pi\left(r,\bar{h}\right) = \rho\,(r) - \bar{h}\,\sigma\,(r) = 0$$

We see that when $\bar{h} = 0$ the roots r_s coincide with the zeros ξ_s of the first characteristic polynomial $\rho\,(\xi)$, which, by zero-stability, all lie in or the unit circle. Consistency and zero stability imply that $\rho\,(\xi)$has a simple zero at +1; we have labeled this zero ξ_1. Let r_1 be the root of (8.34) which tends to ξ_1 as $\bar{h} \to 0$. We shall now show that

$$r_1 = \exp\left(\bar{h}\right) + 0\left(\bar{h}^{p+1}\right) \text{as } \bar{h} \to 0$$

where p is the order of the linear multistep method. By definition of order, $L\,\{y\,(t),h] = 0\,(h^{p+1})$ for any sufficiently differentiable function $y(t)$.

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Choosing $y(t) = \exp(\lambda t)$, we obtain

$$L[\exp(\lambda t);\ h] = \sum_{j=0}^{K} \left\{ \alpha_j \exp[\lambda (t_{n+jh})] - h\beta_j \lambda \exp[\lambda (t_{n+jh})] \right\} = 0(\bar{h}^{p+1}) \quad (8.35)$$

or $\exp(\lambda t_n) \sum_{j=0}^{K} \left\{ \alpha_j [\exp(\bar{h})]^j - \bar{h}\beta_j [\exp(\bar{h})]^j \right\} = 0(\bar{h}^{p+1})$

on dividing by $\exp(\lambda t_n)$, we obtain

$$\pi(\exp(\bar{h}), \bar{h}) \equiv \rho(\exp(\bar{h})) - \bar{h}\,\sigma(\exp(\bar{h})) = 0(\bar{h}^{p+1})$$

Since the roots of (8.34) are $r_s, s = 1, 2, \ldots, K$, we may write

$$\pi(r, \bar{h}) \equiv \rho(r) - \bar{h}\,\sigma(r) \equiv (\alpha_K - \bar{h}\beta_K)(r - r_1)(r - r_2)(r - r_K)$$

and set $r = \exp(\bar{h})$ to obtain

$$(\exp(\bar{h}) - r_1)(\exp(\bar{h}) - r_2) \ldots (\exp(\bar{h}) - r_k) = 0(\bar{h}^{p+1}) \quad (8.36)$$

As $\bar{h} \to 0, \exp(\bar{h}) \to 1$ and $r_s \to \xi_s, s = 1, 2, \ldots, K$.

Thus the first factor on the left hand side of (8.36) tends to zero as $\bar{h} \to 0$, no other factor may do so, since, by zero stability. $\xi_1\ (= +1)$ is a simple zero of $\rho(\xi)$. Hence we may conclude that exp $(\bar{h}) - r_1 = 0(\bar{h}^{p+1})$ establishing the result.

Remark: It now follows that for sufficiently small $\bar{h}; r_1 > 1$ whenever $\bar{h} > 0$. Thus every consistent zero stable method is absolutely unstable for small positive $\bar{h}$.

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Method not absolutely stable but relatively stable :

Example: Consider, for example, Simpson's rule which, being an optimal method, turns out to have no interval of absolute stability. A method is said to be an optimal method, if its order is $K+2$, where K is the step number of the method. The Simpson's rule is

$$y_{n+2} - y_n = \frac{h}{3}[f_{n+2} + 4f_{n+1} + f_n].$$

For this method, we have

$$\rho(r) = r^2 - 1$$

$$\sigma(r) = \frac{1}{3}(r^2 + h\,r + 1)$$

And the stability polynomial is

$$\left(1 - \frac{1}{3}\bar{h}\right) r^2 - \frac{h}{3}\bar{h}\;r - \left(1 + \frac{1}{3}\bar{h}\right) = 0$$

It is easily established that the roots r_1 and r_2 of this equation are real for all values of $\bar{h}$. Using $0(\bar{h})$approximation of the roots, we can take

$r_1 = 1 + \bar{h} + 0(\bar{h}^2)$. The spurious root of $\rho(\xi) = 0$ is $\xi_2 = -1$ and so we write,

$r_2 = -1 + \gamma\,\bar{h} + 0(\bar{h}^2)$. Substituting this value in the stability polynomial gives $r = \frac{1}{3}$. For sufficiently small $\bar{h}$, we can ignore $O(\bar{h}^2)$ and have $r_1 = 1 + \bar{h}$ and $r_2 = -1 + \frac{1}{3}\bar{h}$.

This gives that the method is not absolutely stable but is relatively stable for $\bar{h} > 0$.

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