Lecture 15: Convergence of General One-Step Methods

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Necessary and Sufficient condition for Convergence

We shall now study the necessary and sufficient condition for convergence of a general one-step method which is given in the following theorem:

Theorem: If $\Phi(y,t,h)$ is continuous in $y,t,h.$ for $0 \le t \le b, 0 \le h \le h_0$ and all y, and if it satisfies a Lipschitz condition in y in that region, a necessary and sufficient condition for convergence is that

$$\Phi(y(t),t,0) = f(y(t),t) \qquad (5.4)$$

Remark: The equation (5.4) is called the condition of consistency. Since, by suitable choice of initial conditions, $y(t)$ can take on any value for a given t, the equation (5.4) will hold for any y in the form

$$\Phi(y,t,0) = f(y,t)$$

Proof: Let $\Phi(y,t,0) = g(y,t)$

Since g satisfies the conditions of existence and uniqueness of the solution, the IVP

$$z' = g(z,t)$$

$$z_0 = y_0 \qquad (5.5)$$

has a unique differentiable solution. We shall show that the numerical solution given by (5.1) converges to $z(t)$, and hence $f = g$ is a necessary and sufficient condition.

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The numerical solution satisfies

$$y_{n+1} = y_n + h\Phi(y_n, t_n, h) \tag{5.6}$$

By the mean value theorem,

$z^i(t_{n+1}) = z^i(t_n) + hg^i\left(z(t_n + \theta^i h), t_n + \theta^i h\right)$ for $0 < \theta^i < 1$

Subtracting this from (5.6) and setting

$\varepsilon_n = y_n - z(t_n)$, we get

$$\varepsilon^i{}_{n+1} = \varepsilon^i{}_n + h\left[\Phi^i(y_n, t_n, h) - g^i\left(z(t_n + \theta^i h), t_n + \theta^i h\right)\right]$$

$$= \varepsilon^i{}_n + h\left[\phi^i(y_n, t_n, h) - \phi^i(z(t_n), t_n, h) + \phi^i(z(t_n), t_n, h) - \phi^i(z(t_n), t_n, 0) + \phi^i(z(t_n), t_n, 0) - g^i\left(Z(t_n + \theta^i h), t_n + \theta^i h\right)\right] \tag{5.7}$$

Now, if we assume that ϕ satisfies a Lipschitz condition in t and h, as will happen in practice, we get the following bounds:

$$\| \phi(y_n, t_n, h) - \phi(z(t_n), t_n, h) \| \leq L \| y_n - z(t_n) \| = L \| \varepsilon_n \|$$

$$\| \phi(z(t_n), t_n, h) - \phi(z(t_n), t_n, o) \| \leq L_1 h$$

and

$$\left|\left\{\phi^i\left((z(t_n), t_n, 0) - g^i(z(t_n + \theta^i h), t_n + \theta^i h\right)\right\}\right|$$
$$= \left|\left\{g^i(z(t_n), t_n) - g^i(z(t_n + \theta^i h), t_n + \theta^i h)\right\}\right| \leq L\left|z'(t_n + \xi\,\theta^i h)\right|\theta^i h + L_3\theta^i h$$

(by using Lipschitz condition and mean value theorem)

$\leq L_2 h$

Hence the norm of the last two expressions on the R.H.S. of (5.7) can be bounded by $L_2 h$. Substituting these in (5.7), we get

$$\|\varepsilon_{n+1}\| \le \|\varepsilon_n\| + hL\,\|\,\varepsilon_n\,\| + h^2(L_1+L_2) = (1+hL)\|\varepsilon_n\| + h^2(L_1+L_2) \quad (5.8)$$

This is a difference equation of the type given in Lemma in Module 2 from which we have

$$\|\varepsilon_N\| \le (L_1+L_2)h\frac{e^{Lb}-1}{L} + e^{Lb}\|\varepsilon_0\| \qquad (Nh=b)$$

This converges to zero as h and $\|\varepsilon_0\| \to 0$, so the numerical solution converges to the solution of (5.5). Sufficiency of the condition $g(y,t)=f(y,t)$ follows immediately.

If, on the other hand, we have convergence, then $z(t)$, the solution of (5.5), is identical to $y(t)$, the solution of $y'(t)=f(y(t),t)$.

Suppose also that f and g differ at some point (y_a,t_a). If we consider the initial value problem starting from (y_a,t_a), we have

$$y'(t_a) = f(y(t_a),t_a) \neq g(y(t_a),t_a) = g(z(t_a),\,t_a) = z'(t_a)$$

leading to a contradiction. Hence the theorem.

Application of the above theorem:

Illustrative Example:

Let us now apply the above theorem to illustrate the convergence of Classical Runge-Kutta method applied to the system of first order equations $y' = f(y)$.

We are given that f satisfies a Lipschitz condition. Thus, $K_1(y) = hf(y)$satisfies

$$\|K_1(y) - K_1(y^*)\| \leq hL\|y - y^*\|$$

$K_2(y) = hf\left(y + \frac{1}{2}K_1(y)\right)$ satisfies

$$\|K_2(y) - K_2(y^*)\| \leq hL\left\|y - y^* + \frac{1}{2}k_1(y) - \frac{1}{2}K_1(y^*)\right\| \leq hL\left(1 + \frac{1}{2}hL\right)\| y - y^* \|$$

$K_3(y) = hf\left(y + \frac{1}{2}K_2(y)\right)$ satisfies

$$\|K_3(y) - K_3(y^*)\| \leq hL\left\|y - y^* + \frac{1}{2}K_2(y) - \frac{1}{2}K_2(y^*)\right\| \leq hL\left[1 + \frac{1}{2}hL + \frac{1}{4}(hL)^2\right]\|y - y^*\|$$

and $K_4(y) = hf(y + K_3(y))$ satisfies

$$\begin{aligned}\|K_4(y) - K_4(y^*)\| &\leq hL\|y - y^* + K_3(y^*) - K_3(y^*)\| \\ &\leq hL\left[1 + hL + \frac{1}{2}(hL)^2 + \frac{1}{4}(hL)^3\right]\|y - y^*\|\end{aligned}$$

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Therefore,

$\phi(y,t,h) = \frac{1}{6h}(K_1 + 2\,K_2 + 2K_3 + K_4)$ satisfies

$$\|\,\phi(y,t,h)\| - \phi(y^*,t,h)\,\|$$
$$\leq \frac{L}{6}\left[1 + 2 + hL + 2 + hL + \frac{1}{2}(hL)^2 + 1 + hL + \frac{1}{3}(hL)^2 + \frac{1}{4}(hL)^3\right]\|y - y^*\|$$
$$\leq L\left[1 + \frac{1}{2}(hL) + \frac{1}{6}(hL)^2 + \frac{1}{24}(hL)^3\right]\|y - y^*\|$$

Hence ϕ satisfies a Lipschitz condition in y. It can also be seen to be continuous in h. Thus we can conclude that the classical fourth order Runge-Kutta method converges for a system of equations.

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