Module 7: Multistep Methods

Lecture 20: Multistep Methods(Contd.)

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Predictor-Corrector Formula:

A formula which furnishes a first approximation $y_p^{(0)}$ for the iteration procedure described above is called a predictor formula . Formula (7.14) in this connection is called a corrector formula . The above iteration process thus consists in predicting a tentative value $y_p^{(0)}$ of y_p and correcting it (possibly a number of times) by means of the corrector formula. In the interest of minimizing the number of corrections it is clearly desirable to predict $y_p^{(0)}$ as accurately as possible. If the predictor formula is sufficiently accurate, it may make it unnecessary to correct more than once. For these reasons, the Adams-Bashforth formula [(7.3) with p diminished by 1] is recommended as an accurate predictor formula in connection with the Adams- Moulton method.

Expressing differences in terms of ordinates, the Adams-Moulton formula (7.9) can be written in the from

$$y_p - y_{p-1} = h \sum_{\rho=0}^{q} \beta_{q\rho}^{*} f_{p-\rho} \qquad (7.15)$$

where

$$\beta_{q\rho}^{*} = (-1)^{\rho}\left\{\binom{\rho}{\rho}\gamma_\rho^{*} + \binom{\rho+1}{\rho}\gamma_{\rho+1}^{*} + \cdots + \binom{q}{\rho}\gamma_q^{*}\right\}$$

$\rho = 0, 1, \ldots \ldots, q; \quad q = 0, 1, \ldots$

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Some numerical values of β^*_{qp} are given as under:

p	0	1	2	3	4	5
β^*_{0p}	1					
$2\ \beta^*_{1p}$	1	1				
$12\ \beta^*_{2p}$	5	8	-1			
$24\ \beta^*_{3p}$	9	19	-5	1		
$720\ \beta^*_{4p}$	251	646	-264	106	-19	
$1440\ \beta^*_{5p}$	475	1427	-798	482	-173	27

The Adams- Moulton method is given by

$$y_p - y_{p-1} = h \sum_{m=0}^{q} \gamma^*_m \nabla^m f_p$$

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Special Case q=1:

For $q = 1$, we have the formula

$$y_p - y_{p-1} = h \sum_{m=0}^{1} \gamma_m^* \nabla^m f_p$$

$$= h\left[\gamma_0^* \nabla^0 f_p + \gamma_1^* \nabla^1 f_p\right]$$

$$= h\left[\gamma_0^* f_p + \gamma_1^* f_p - \gamma_1^* f_{p-1}\right]$$

$$= h\left[f_p - \frac{1}{2} f_p + \frac{1}{2} f_{p-1}\right]$$

$$= \frac{h}{2}\left[f_p + f_{p-1}\right]$$

which gives the trapezoidal method.

iii) Nystrom's Method :

Here we set

$$y_{p+1} - y_{p-1} = \int_{t_{p-1}}^{t_{p+1}} f(t, y(t))\, dt$$

$$= h \sum_{m=0}^{q} K_m \nabla^m f_p \qquad (7.16)$$

where

$$K_m = (-1)^m \frac{1}{h} \int_{t_{p-1}}^{t_{p+1}} \binom{-S}{m} dt$$

$$= (-1)^m \int_{-1}^{1} \binom{-S}{m} dS \qquad (7.17)$$

This method is explicit and works much like the Adams-Bashforth method, except that now the increment of y_n is calculated over two steps in place of one.

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The generating function of the coefficients K_m is found to be

$$g(t_1) = \sum_{m=0}^{\infty} K_m\, t_1^m$$

$$= \sum_{m=0}^{\infty} (-\,t_1)^m \int_{-1}^{1} \binom{-S}{m}\, dS$$

$$= \int_{-1}^{1} \sum_{m=0}^{\infty} (-\,t_1)^m \binom{-S}{m}\, dS$$

$$= \int_{-1}^{1} (1 - t_1)^{-S}\, dS$$

$$= \left[-\,\frac{(1-t_1)^{-s}}{\log(1-t_1)}\right]_{-1}^{1}$$

$$= \left[-\,\frac{1}{(1-t_1)\,\log(1-t_1)} + \frac{1-t_1}{\log(1-t_1)}\right]$$

$$= \left[\frac{-1+(1-t_1)^2}{(1-t_1)\,\log(1-t_1)}\right]$$

$$= -\,\frac{t_1}{\log(1-t_1)}\,.\,\frac{2-t_1}{1-t_1}$$

using the expansion (7.6) and

$$\frac{2-t_1}{1-t_1} = 1 + \frac{1}{1-t_1} = 2 + t_1 + t_1^2 + \ldots$$

we thus have the identity

$$\left(1 + \frac{1}{2}\,t_1 + \frac{1}{3}\,t_1^2 + \cdots\right)(K_0 + K_1\,t_1 + K_2\,t_1^2 + \cdots)$$

$$= 2 + t_1 + t_1^2 + \cdots$$

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from which we derive the recurrence relations

$$K_m + \frac{1}{2} K_{m-1} + \frac{1}{2} K_{m-2} + \cdots + \frac{1}{m+1} K_0 = \begin{cases} 2; & m = 0 \\ +1; & m = 1, 2, 3, \ldots \end{cases}$$

The numerical values of K_m are given as under:

m	0	1	2	3	4	5	6
K_m	2	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{29}{90}$	$\frac{14}{45}$	$\frac{1139}{3780}$

Special case q=1 of Nystrom's method:

In this case, we have

$$y_{p+1} - y_{p-1} = h \sum_{m=0}^{0} K_m \nabla^m f_p$$

$$= h K_0 f_p$$

$$= h\, 2 f_p$$

or

$$y_{p+1} = y_{p-1} + 2h f_p$$

which gives the mid-point method. Since $K_1 = 0$, it yields the same accuracy as could be expected from taking $q = 1$, which accounts for some of the popularity of the method. The mid-point rule has also been used as a predictor formula for the trapezoidal rule (Adams-Moulton method with q=1). For general q the Nystrom method may be used as a predictor formula for the generalized Milne-Simpson method to be discussed now.

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iv) The Generalized Milne-Simpson Method:

Here we have

$$y_p - y_{p-2} = \int_{t_{p-2}}^{t_p} f(t)\ dt$$

$$= h \sum_{m=0}^{q} K_m^* \ \nabla^m \ t_p \qquad (7.18)$$

where

$$K_m^* = (-1)^m \frac{1}{h} \int_{t_{p-2}}^{t_p} \binom{-S}{m} \ dt$$

$$= (-1)^m \int_{-2}^{0} \binom{-S}{m} \ dS \qquad (7.19)$$

Formula (7.18) resembles the Adams-Moulton formula in being implicit. However, the integration is now over two steps, which may cause weak stability. On the other hand, the method is, for comparable $q \geq 2$, more accurate than any of the methods previously considered. This is true in particular in the case $q \geq 2$, which is known as the Milne method. The reason why it is more accurate will be understood by studying the coefficients K_m^*. Their generating function is given by

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$$K^*(t_1) = \sum_{m=0}^{\infty} K_m^* t_1^m = \sum_{m=0}^{\infty} (-t_1)^m \int_{-2}^{0} \binom{-S}{m} dS$$

$$= \int_{-2}^{0} \sum_{m=0}^{\infty} (-t_1)^m \binom{-S}{m} dS$$

$$= \int_{-2}^{0} (1-t_1)^{-S} dS$$

$$= \left[-\frac{(1-t_1)^{-S}}{\log(1-t_1)}\right]_{-2}^{0}$$

$$= -\frac{1}{\log(1-t_1)} + \frac{(1-t_1)^2}{\log(1-t_1)}$$

$$= -\frac{t_1}{\log(1-t_1)} \cdot (2-t_1)$$

Or

$$-\frac{\log(1-t_1)}{t_1} \cdot K^*(t_1) = 2 - t_1$$

where from it follows that

$$\left(1 + \frac{1}{2}t_1 + \frac{1}{3}t_1^2 + \cdots\right)(K_0^* + K_1^* t + K_2^* t^2 + \cdots) = 2 - t_1$$

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Comparing coefficients, we find

$$K_m^* + \frac{1}{2} K_{m-1}^* + \frac{1}{3} K_{m-2}^* + \cdots + \frac{1}{m+1} K_0^* = \begin{cases} 2; & m = 0 \\ -1; & m = 1 \\ 0; & m = 2, 3, \ldots \end{cases}$$

or which can be written as

$$\sum_{n=0}^{m} \frac{1}{m+1-n} K_n^* = \begin{cases} +2, & m = 0 \\ -1, & m = 1 \\ 0, & m = 2, 3, \end{cases}$$

Numerical values of K_m^* are given as

m	0	1	2	3	4	5	6
K_m^*	2	-2	$\frac{1}{3}$	0	$\frac{-1}{90}$	$\frac{-1}{90}$	$\frac{-37}{3780}$

we note that $K_3^* = 0,$ which shows that the Milne-Simpson formula with $q = 2$ produces an effect which normally would be expected only with $q = 3$. We also note that

$$\frac{1}{1 - t_1} K^*(t_1) = K(t_1)$$

from which there follows the relation

$$K_0^* + K_1^* + K_2^* + \cdots + K_m^* = K_m, \quad m = 0, 1, \ldots \qquad (7.20)$$

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The numerical values given in the above table suggest that (7.18) be better written and used in the form

$$y_p - y_{p-2} = h\left[2\,f_{p-1} + \frac{1}{3}\nabla^2 f_p - \frac{1}{90}\left(\nabla^4 f_p + \nabla^5 f_p\right) + \cdots\right] \qquad (7.21)$$

The Milne formula resulting from $q = 2$ or 3 may be written out in terms of ordinates as follows

$$y_p - y_{p-2} = \frac{1}{3}\,h\left(f_p + 4\,f_{p-1} + f_{p-2}\right) \qquad (7.22)$$

Equations (7.18), (7.21) and (7.22) represent implicit equations for y_p and are usually solved by iteration, starting with a predicted first approximation $y_p^{(0)}$. Any explicit formula can be used in principle to calculate $y_p^{(0)}$. In the interest of economizing the number of corrections, one will choose, if possible, a predictor formula whose accuracy is comparable to that of the corrector formula. Milne suggests the formula

$$y_p - y_{p-4} = \frac{1}{3}\,h\left(8\,f_{p-1} - 4\,f_{p-2} + 8\,f_{p-3}\right) \qquad (7.23)$$

(resulting from integrating the quadratic polynomial interpolating $f(t, y(t))$ at $t_{p-1}, t_{p-2}, t_{p-3}$ between t_{p-4} and t_p) as a predictor formula. Now the iteration procedure may be arranged in the same manner as explained in Adams-Moulton method.

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