Module 7: Multistep Methods

Lecture 24: Special 2nd order equations(Contd.)

The Lecture Contains:

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2. Cowell's Method:

Here we have

$$y_p - 2\,y_{p-1} + y_{p-2} = h^2\ \sum_{m=0}^{q}\ \sigma_m^*\ \nabla^m\ f_p \ldots \qquad (7.43)$$

where

$$\sigma_m^* = \frac{(-1)^m}{h^2} \int_{x_{p-1}}^{x_p} (x_p - x)\left[\binom{-S}{m} + \binom{S+2}{m}\right] dx \qquad (7.44)$$

$$= (-1)^m \int_{-1}^{0} (-S)\left[\binom{-S}{m} + \binom{S+2}{m}\right] dS$$

The coefficients σ_m^* are obtained by using the method of generating function;

$$S^*(t) = \sum_{m=0}^{\infty}\ \sigma_m^*\ t^m$$

$$= \sum_{m=0}^{\infty} (-t)^m \int_{-1}^{0} (-S)\left[\binom{-S}{m} + \binom{S+2}{m}\right] dS$$

$$= \int_{-1}^{0} \sum_{m=0}^{\infty} (-t)^m\ (-S)\binom{-S}{m} dS + \int_{-1}^{0} \sum_{m=0}^{\infty} (-t)^m\ (-S)\binom{S+2}{m} dS$$

$$= \int_{-1}^{0} (-S)\,(1-t)^{-S}\ dS + \int_{-1}^{0} (-S)\ (1-t)^{S+2}\ dS$$

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Using integration by parts, we get

$$= \left[\frac{t}{\log(1-t)}\right]^2 \qquad (7.45)$$

or

$$\left[\frac{\log\,(1-t)}{t}\right]^2 \; S^*(t) = 1$$

This gives

$$\left(1 + \frac{2}{3}\,h_2 t + \frac{2}{4}\,h_3\,t^2 + \cdots\right)(\sigma_0^* + \sigma_1^*\,t + \sigma_2^*\,t^2 + \cdots) = 1$$

By comparing coefficients, we get the recurrence solution

$$\sigma_0^* = 1$$

$$\sigma_m^* = -\frac{2}{3}\,h_2\,\sigma_{m-1}^* - \frac{2}{4}\,h_3\,\sigma_{m-2}^* - \cdots - \frac{2}{m+2}\,h_{m+1}\,\sigma_0^*\,; \qquad m = 1,2,\ldots$$

The numerical values of σ_m^* for certain values of m are given in the following table:

m	0	1	2	3	4	5	6
σ_m^*	1	-1	$\frac{1}{12}$	0	$-\frac{1}{240}$	$-\frac{1}{240}$	$-\frac{221}{60480}$

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The relation between σ_m and σ_m^* is obtained as follows:

we know that

$$S(t) = \left[\frac{t}{\log(1-t)}\right]^2 \cdot \frac{1}{1-t}$$

Also

$$S^*(t) = \left[\frac{t}{\log(1-t)}\right]^2$$

Thus , we have

$$S^*(t) = (1-t)\, S(t)$$

from which it follows that for $m = 1, 2, \ldots$

$$\sigma_m^* = \sigma_m - \sigma_{m-1} \qquad (7.46)$$

The Cowell formula is not used for $q = 0$. For $q = 1$, we have

$$y_p - 2y_{p-1} + y_{p-2} = h^2 \sum_{m=0}^{1} \sigma_m^* \nabla^m f_p$$

$$= h^2 \left[\sigma_0^* \nabla^0 f_p + \sigma_1^* \nabla f_p\right]$$

$$= h^2 \left[f_p + (-1)\{f_p - f_{p-1}\}\right]$$

$$= h^2 f_{p-1}$$

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Lecture 24: Special 2nd order equations(Contd.)

which is the same as the Stormer's formula with $q = 0$ or 1. For $q = 2$ and $q = 3$ we obtain the frequently used formula

$$y_p - 2\,y_{p-1} + y_{p-2} = h^2 \left\{f_{p-1} + \frac{1}{12}\,\nabla^2\, f_p\right\}$$

$$= \frac{1}{12}\, h^2 \left\{f_p + 10f_{p-1} + f_{p-2}\right\} \qquad (7.47)$$

The Cowell formulas are implicit for $q \geq 2$; i. e. the unknown value y_p occurs not only on the left but also, as an argument in f_p, on the right. The resulting equation for y_p, which is non-linear except when the differential equation is linear, can be solved by iteration. As a predictor formula, Stormer formula (7.38) is recommended, because it uses the same combination of values of y_n on the left.

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3. Some further Special methods:

Further useful formulas can be derived by doubling the step K, analogously to the doubling of the step in the derivation of the Nystrom and Milne-Simpson methods for first order equations. By taking $K = 2h$ and $x = x_{p-1}$, we obtain the formula

$$y_{p+1} - 2y_{p-1} + y_{p-3} = h^2 \sum_{m=0}^{q} \tau_m \nabla^m f_p \qquad (7.48)$$

where the coefficients τ_m are given by

$$\tau_m = \frac{(-1)^m}{h^2} \int_{x_{p-1}}^{x_{p+1}} (x_{p+1} - x) \left[\binom{-S}{m} + \binom{S+2}{m}\right] dx$$

$$= (-1)^m \int_{-1}^{1} (1-S) \left[\binom{-S}{m} + \binom{S+2}{m}\right] ds \qquad (7.49)$$

Again, the method of generating function is used to obtain the coefficients τ_m as follows:

$$T(t) = \sum_{m=0}^{\infty} \tau_m t^m$$

$$= \sum_{m=0}^{\infty} (-t)^m \int_{-1}^{1} (1-S)\left[\binom{-S}{m} + \binom{S+2}{m}\right] ds$$

$$= \int_{-1}^{1} \sum_{m=0}^{\infty} (-t)^m (1-S) \binom{-S}{m} ds + \int_{-1}^{1} \sum_{m=0}^{\infty} (-t)^m (1-S) \binom{S+2}{m} ds$$

$$= \int_{-1}^{1} (1-S)(1-t)^{-S} ds + \int_{-1}^{1} (1-S).(1-t)^{S+2} ds$$

On integration, one finds that

$$T(t) = \left[\frac{t}{\log(1-t)}\right]^2 \frac{4-4t+t^2}{1-t} \qquad (7.50)$$

or

$$\left[\frac{\log(1-t)}{t}\right]^2 T(t) = \frac{4-4t+t^2}{1-t}$$

This gives

$$\left(1 + \frac{2}{3} h_2 t + \frac{2}{4} h_3 t^2 + \cdots\right)(\tau_0 + \tau_1 t + \tau_2 t^2 + \cdots)$$

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$$= (1 + t + t^2 + \cdots)(4 - 4t + t^2)$$

which leads to the recurrence relation as

$$\tau_0 = 4, \quad \tau_1 = -4$$

$$\tau_m = 1 - \frac{2}{3} h_2 \tau_{m-1} - \frac{2}{4} h_3 \tau_{m-2} - \cdots - \frac{2}{m+2} h_{m+1} \tau_0, \quad m = 2, 3, \ldots$$

Some numerical values of τ_m are given as under:

m	0	1	2	3	4	5	6
τ_m	4	-4	$\frac{4}{3}$	0	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{61}{945}$

Formula (7.48) is explicit. It is not recommended for $q < 2$. For $q = 2$ there results the simple and relatively accurate formula

$$y_{p+1} - 2y_{p-1} + y_{p-3} = \frac{4}{3} h^2 \{f_p + f_{p-1} + f_{p-2}\} \qquad (7.51)$$

Since $\tau_3 = 0$, the same formula results also from picking the value $q = 3$.

Another set of formulas is derived by choosing $K = 2h,\ x = x_{p-2}$ in (7.37).

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we obtain

$$y_p - 2y_{p-2} + y_{p-4} = h^2 \sum_{m=0}^{q} \tau_m^* \Delta^m f_p \qquad (7.52)$$

where

$$\tau_m^* = \frac{(-1)^m}{h^2} \int_{x_{p-2}}^{x_p} (x_p - x) \left[\binom{-S}{m} + \binom{S+4}{m}\right] dx$$

$$= (-1)^m \int_{-2}^{0} (-S)\left[\binom{-S}{m} + \binom{S+4}{m}\right] dS \qquad (7.53)$$

The generating function is given as:

$$T^*(t) = \sum_{m=0}^{\infty} \tau_m^* t^m$$

$$= \sum_{m=0}^{\infty} (-t)^m \int_{-2}^{0} (-S)\left[\binom{-S}{m} + \binom{S+4}{m}\right] dS$$

$$= \int_{-2}^{0} \sum_{m=0}^{\infty} (-t)^m (-S) \binom{-S}{m} dS + \int_{-2}^{0} \sum_{m=0}^{\infty} (-t)^m (-S) \binom{S+4}{m} dS$$

$$= \int_{-2}^{0} (-S)(1-t)^{-S} dS + \int_{-2}^{0} (-S)(1-t)^{S+4} dS$$

on integration, we find

$$T^*(t) = \left[\frac{t}{\log(1-t)}\right]^2 (4 - 4t + t^2) \qquad (7.54)$$

The recurrence relation is obtained as

$$\tau_0^* = 4, \quad \tau_1^* = -8, \quad \tau_2^* = 4 + \frac{4}{3},$$

$$\tau_m^* = -\frac{2}{3} h_2 \tau_{m-1}^* - \frac{2}{4} h_3 \tau_{m-2}^* - \cdots - \frac{2}{m+2} h_{m+1} \tau_0^*, \qquad m = 3, 4, \ldots$$

By comparing (7.50) and (7.54), we also find that

$$\tau_m^* = \tau_{m-}\tau_{m-1}, \qquad m = 1, 2, \ldots$$

The numerical values of τ_m^* are readily found as follows:

m	0	1	2	3	4	5	6
τ_m^*	4	-8	$\frac{16}{3}$	$\frac{-4}{3}$	$\frac{1}{15}$	0	$-\frac{2}{945}$

For $q = 0, 1, 2,$ (7.52) has an irregular appearance, and its use for practical purposes is not recommended. For $q = 3.$, the formula reads

$$y_p - 2y_{p-2} + y_{p-4} = \frac{4}{3} h^2 \{f_{p-1} + f_{p-2} + f_{p-3}\} \qquad (7.55)$$

which is equivalent to (7.51). For $q = 4$ and, since $\tau_5^* = 0,$ also for $q = 5$, (7.52) reduces to

$$y_p - 2\,y_{p-2} + y_{p-4} = \frac{1}{15}\,h^2\left\{f_p + 16\,f_{p-1} + 26\,f_{p-2} + 16\,f_{p-3} + f_{p-4}\right\} \quad (7.56)$$

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Problems

1. Using generating functions, show that

$$K_m = 2\gamma_m - \gamma_{m-1}$$

$$K_m^* = 2\gamma_m^* - \gamma_{m-1}^*, \qquad m = 0, 1, 2, ...$$

2. Solve the initial value problem

$$y' = 1 - xy^2,\ y(0) = 0$$

numerically, using the step $h = 0.1$, by

   a. determining the initial values y_{-1} and y_{+1} by the Runge-Kutta method, and
   b. Continuing the computation by the Adams-Moulton method with $q = 3.$, using suitable Adams-Bashforth formula as predictors.

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