Module 7: Multistep Methods

Lecture 19: Multistep Methods (Contd.)

The Lecture Contains:

Adams-Bashforth Method in terms of Ordinates :

Now if we assume that $t_p = t_0 + ph$, w here h is a constant and p is an integer, and write $f_p = f\left(t_p, y_p\right)$. Then, the first backward difference of the function $f\left(t\right)$ at the point $t = t_p$ is defined by

$$\nabla f_p = f_p - f_{p-1}$$

Higher backward differences are defined by

$\nabla^q f_p = \nabla\left(\nabla^{q-1} f_p\right)$, so that for instance

$$\nabla^2 f_p = \nabla\left(f_p - f_{p-1}\right)$$

$$= \nabla f_p - \nabla f_{p-1}$$

$$= f_p - f_{p-1} - f_{p-1} + f_{p-2}$$

$$= f_p - 2 f_{p-1} + f_{p-2}$$

If we also put $\nabla^0 f_p = f_p$. One can easily verify by induction that

$$\nabla^q f_p = \sum_{m=0}^{q} (-1)^m \binom{q}{m} f_{p-m} \qquad q = 0, 1, \ldots \qquad (7.7)$$

where $\binom{q}{m}$ denotes the binomial coefficient

$$\binom{q}{o} = 1$$

$$\binom{q}{m} = \frac{q\,(q-1)\,\ldots.\,(q-m+1)}{1.2.3\,\ldots.\,m}, \qquad m = 1, 2, \ldots.$$

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Expressing the differences in terms of ordinates defined by (7.7) in (7.3) and collecting the coefficients of equal ordinates, the Adams- Bashforth formula appears in the form

$$y_{p+1} - y_p = h \sum_{\rho=0}^{q} \beta_{q\rho} \; f_{p-\rho} \qquad (7.8)$$

where the coefficients $\beta_{q\rho}$ are given by

$$\beta_{q\rho} = (-1)^{\rho} \left\{ \binom{\rho}{\rho} \gamma_\rho + \binom{\rho+1}{\rho} \gamma_{\rho+1} + \cdots \binom{q}{\rho} \gamma_q \right\}, \quad \rho = 0, 1, \ldots, q; q = 0, 1, \ldots$$

It should be noted that the coefficients $\beta_{q\rho}$ depend on q as well as ρ, which makes it more difficult to change the number of differences employed. Some numerical values of the coefficients $\beta_{q\rho}$ are given as under:

ρ	0	1	2	3	4	5
$\beta_{0\rho}$	1					
$2\ \beta_{1\rho}$	3	-1				
$12\ \beta_{2\rho}$	23	-16	5			
$24\ \beta_{3\rho}$	55	-59	37	-9		
$720\ \beta_{4\rho}$	1901	-2774	2616	-1274	251	
$1440\ \beta_{5\rho}$	4227	-7673	9482	-6798	-2627	-425

The numerically large values of the coefficients and the alternating signs are a disadvantage of the method.

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ii) The Adams- Moulton Method

Here we have

$$y_p - y_{p-1} = \int_{t_{p-1}}^{t_p} f\left(t,\ y\left(t\right)\right) dt$$

$$= h \sum_{m=0}^{q} \gamma_m^* \ \nabla^m \ f_p \qquad (7.9)$$

where

$$\gamma_m^* = (-1)^m \ h^{-1} \int_{t_{p-1}}^{t_p} \binom{-S}{m} dt$$

$$= (-1)^m \int_{-1}^{0} \binom{-S}{m} \ dS$$

The coefficients γ_m^* are determined by using the method of generating functions.

$$G^* \left(t_1\right) = \sum_{m=0}^{\infty} \gamma_m^* \ t_1^m = \sum_{m=0}^{\infty} (-t_1)^m \int_{-1}^{0} \binom{-S}{m} \ dS$$

$$= \int_{-1}^{0} \sum_{m=0}^{\infty} \ (-t_1)^m \binom{-S}{m} \ ds$$

$$= \int_{-1}^{0} (1 - t_1)^{-S} \ dS$$

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$$= \left[- \frac{(1-t_1)^{-S}}{\log(1-t_1)} \right]_{-1}^{0}$$

$$= \left[- \frac{1}{\log(1-t_1)} + \frac{(1-t_1)}{\log(1-t_1)} \right]$$

$$= \frac{-t_1}{\log(1-t_1)} \qquad (7.10)$$

Thus, we have

$$- \frac{\log(1-t_1)}{t_1} G^*(t_1) = 1$$

Using the expression

$$- \frac{\log(1-t_1)}{t_1} = 1 + \frac{1}{2} t_1 + \frac{1}{3} t_1^2 + \cdots.$$

we have

$$\left(1 + \frac{1}{2} t_1 + \frac{1}{3} t_1^2 + \cdots \right) (\gamma_0^* + \gamma_1^* t_1 + \gamma_2^* t_1^2 + \ldots) = 1$$

It follows that

$$\gamma_m^* + \frac{1}{2} \gamma_{m-1}^* + \frac{1}{3} \gamma_{m-2}^* + \cdots + \frac{1}{m+1} \gamma_0^* = \begin{cases} 1, & m = 0 \\ 0, & m = 1, 2, 3, \ldots \end{cases}$$

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The numerical values can now be easily found from these recurrence relations and are given as under:

m	0	1	2	3	4	5	6
γ_m^*	1	$\frac{-1}{2}$	$\frac{-1}{12}$	$\frac{-1}{24}$	$\frac{-19}{720}$	$\frac{-3}{160}$	$\frac{-863}{60480}$

We also note the relation

$$\frac{1}{1-t_1} G^*(t_1) = G(t_1)$$

or

$$(1 + t_1 + t_1^2 + \cdots)(\gamma_0^* + \gamma_1^* t_1 + \cdots) = \gamma_0 + \gamma_1 t_1 + \gamma_2 t_1^2 + \ldots.$$

comparing the coefficients of t_1^m, we obtain

$$\gamma_0^* + \gamma_1^* + \cdots + \gamma_m^* = \gamma_m \qquad m = 0, 1, 2, \ldots \qquad (7.11)$$

Formula (7.9) is used like the Adam- Bashforth formula except that now only the values $y_{p-1}, y_{p-2}, \ldots, y_{p-q}$ are known and (7.9) is used to determine y_p. Since y_p occurs as an argument in $f_p = f(t_p, y_p)$ in the right hand term of (7.9), this equation now represents a nontrivial equation for y_p. In general, it will not be possible to solve this equation explicitly. Fortunately, the special from of the equation suggests an iteration procedure which furnishes the solution very rapidly if h is sufficiently small.

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Assuming that from some source an approximation $y_p^{(0)}$ of a solution o f (6.9) has been obtained, we calculate $f_p^{(0)} = f\left(t_p, y_p^{(0)}\right)$

and from the differences $\nabla f_p^{(0)} = f_p^{(0)} - f_{p-1}, \ \nabla^2 f_p^{(0)} = \nabla f_p^{(0)} - \nabla f_{p-1}, \ldots.$

A better approximation $y_p^{(1)}$ is then obtained from

$$y_p^{(1)} = y_{p-1} + h \sum_{m=0}^{q} \gamma_m^* \ \nabla^m f_p^{(0)} \tag{7.12}$$

Calculating

$f_p^{(1)} = f\left(t_p, y_p^{(1)}\right)$ and re-evaluating the differences, a still better value $y_p^{(2)}$ is

$$y_p^{(2)} = y_{p-1} + h \sum_{m=0}^{q} \gamma_m^* \ \nabla^m f_p^{(1)} \tag{7.13}$$

Generally, a sequence $y_p^{(v)}$ $(v = 0, 1, 2, \ldots)$ of approximations is obtained recursively from the relation

$$y_p^{(v+1)} = y_{p-1} + h \sum_{m=0}^{q} \gamma_m^* \ \nabla^m f_p^{(v)} \tag{7.14}$$

where $f_p^{(v)} = f\left(t_p, y_p^{(v)}\right)$. It can be proved that the sequence of numbers $y_p^{(0)}, y_p^{(1)}, y_p^{(2)}, \ldots$ thus defined converges for sufficiently small values of h to a solution y_p of (7.9) and that this solution is unique.

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