Course Name	Numerical Solution to ODE
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The Lecture Contains:

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Many problems in engineering and science can be formulated in terms of differential equations.

An equation involving a relation between an unknown function and one or more of its derivatives is called a differential equation. These differential equations basically fall into two classes, ordinary and partial, depending upon the number of independent variables present in the differential equation; one for ordinary and more than one for partial differential equation.

We shall confine our attention here only to the numerical solution of ordinary differential equations (ODE's).

The general form of an ODE can be written as

$$L\,(y) = r\,(t) \qquad (1.1)$$

Where L is a differential operator and $r(t)$ is a given function of the independent variable t. The order of the differential equation is the order of its highest derivative. If the dependent variable $y\,(t)$ and its derivatives occur in the first degree and are not multiplying each other, the equation is said to be linear, otherwise it is non linear. The general n^{th} order non linear differential equation can be written as

$$F\left(t, y, y', \ldots\ldots\ldots, y^{(n-1)}, y^{(n)}\right) = 0 \qquad (1.2)$$

This may be written in the form

$$y^{(n)}(t) = f\left(t, y, y', \ldots\ldots\ldots, y^{(n-1)}\right) \qquad (1.3)$$

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which is called a canonical representation of (1.2). In such a form, the highest order derivative is expended in terms of lower order derivatives and the independent variable. It is known that the general solution of the n^{th} order ODE contains n independent arbitrary constants. In order to determine a particular solution or to describe a specific physical situation, supplementary conditions to determine the arbitrary constants in the general solution are prescribed. If these supplementary conditions are prescribed at one point, then these conditions are called initial conditions and if the conditions are prescribed at more than one point, then these are called boundary conditions.

The differential equation together with the initial conditions is called an Initial Value Problem (IVP) and the differential equation together with the boundary conditions is called a Boundary Value Problem (BVP).

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Applications: Some Model Problems Involving Differential Equations

Let us give here some typical examples of these problems that arise in applications:

Example 1: Mechanical Vibrations

Let $y(t)$ denote the displacement at time t of a block of mass m that is connected to a spring of stiffness k, a damper of resistance c of an oscillator $f(t)$. If the system is released from position y_0 with a velocity y'_0, then its subsequent motion satisfies

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = f(t), \qquad t > 0$$
$$y(0) = y_{0,}$$
$$\frac{d}{dt}y(0) = y'_0$$

This is an example of an initial value problem for a second order linear ODE.

Example 2: Ecology (Lotka-Volterra equations)

Consider a population of predators and prey living in an ecological niche. The predators survive by eating the prey and the prey exist on independent source of food. Let $P(t)$ and $p(t)$, respectively, denote the population of predators and prey as time t. Given the initial populations P_0 and p_0 of predators and prey, their subsequent populations satisfy the Lotka-Volterra equations:

$$\frac{dp}{dt} = p(a - \alpha P), \qquad t > 0, \qquad p(0) = p_0$$
$$\frac{dP}{dt} = P(-c + rp), \qquad t > 0, \qquad P(0) = P_0$$

where a, c, α and r are positive constant corresponding to the prey's natural growth rate, the predator's natural death rate, the prey's death rate upon coming into contact with predators, and the predator's growth rate upon coming into contact with prey.

This example involves an IVP for a system of the first order non linear ODEs.

Example 3: Buckling of columns (Euler-Bernoulli Equations)

The lateral displacement $y(t)$ at position t of a clamped-hinged bar of length l that is subjected to a load P may be approximated by the Euler-Bernoulli equations

$$\frac{d^4y}{dt^4} + \lambda \frac{d^2y}{dt^2} = 0 , \qquad 0 < t < l$$

In the boundary conditions

$$y(0) = \frac{d}{dt} y(0) = 0$$
$$y(l) = \frac{d^2}{dt^2} y(l) = 0$$

The parameter $\lambda = P/EI$, where EI is the flexural rigidity of the bar.

This is an example of a Boundary Value Problem (BVP) for a fourth order linear ODE.

Note: Observe that $y(t) = 0$ is a solution of this problem in all value of λ and l. A more interesting problem is to determine y and those values of λ and l for which non-trivial $(y(t) \neq 0)$ solutions exist. As such, this BVP is also a differential eigen value problem. Non- trivial solutions $y(t)$ are called eigen functions and their corresponding values of λ are eigen values.

Example 4: Pendulum Oscillations

The position ($x(t), y(t)$) at time t of a particle of mass m oscillating on a pendulum of length l is

$$m\frac{d^2x}{dt^2} = -T\sin\theta = -\frac{T}{l}x, \qquad t>0$$
$$m\frac{d^2y}{dt^2} = mg - T\cos\theta = mg - \frac{T}{l}y, \quad t>0$$

where g is the acceleration due to gravity, $T(t)$ is the tension in the string and $\theta(t)$ is the angle of the pendulum relative to the vertical at time t. These equations are however insufficient to guarantee that the particle stays on the string. To ensure that this is so, we must supplement the ODEs by the algebraic constraint.

$$x^2 + y^2 = l^2$$

The two second order differential equations and the constraint comprise a system of three differential algebraic equations (DAE's) for the unknowns $x(t), y(t)$ and $T(t)$.

Initial conditions specify $x(0)$, $\frac{d}{dt}x(0)$, $y(0)$ and $\frac{d}{dt}y(0)$, but not $T(0)$.

Note: Of course, for this problem, it is easy to eliminate the constraint by introducing the change of variables

$$x = l\sin\theta\,, \quad y = l\cos\theta$$

This would reduce the DAEs to the second order ODE

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

IVP's will comprise our initial study and followed by a brief discussion of BVP's.

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