Lecture 21: The local error of the formulas based on integration

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Adams-Bashforth & Adams-Moulton Methods

We have seen that the exact solution of the differential equation $y' = f(t, y)$ satisfies the identity

$$y(t+K) - y(t) = \int_t^{t+K} f(t, y)\, dt$$

for any two points t and $t+K$ in the interval $[a, b]$. In the methods discussed earlier, we tried to replace the function $f(t, y(t))$, which is unknown, by an interpolating polynomial having the values $f_n = f(t_n, y_n)$ on a set of points t_n. The Newton backward difference formula was used to find such an interpolating polynomial. We assume here that the interpolating points and $f(t, y)$ has a continuous $(q+1)$ the derivative. Using the notations that $t_p = t_0 + ph$,$f_p = f(t_p, y_p)$ and

$$\nabla^{q+1} f_p = \nabla^q f_p - \nabla^q f_{p-1}$$

where ∇ is the backward difference operator with $\nabla^0 f_p = f_p,$ we have the interpolating polynomial, for the unknown function $f(t, y)$, with the remainder term as

$$f(t) = f_p + \frac{t-t_p}{h} \nabla f_p + \frac{(t-t_p)(t-t_{p-1})}{2!}$$

$$\frac{\nabla^2 f_p}{h^2} + \ldots + \frac{(t-t_p)(t-t_{p-1}) \ldots (t-t_{p-q+!})}{q!}$$

$$\frac{\nabla^q f_p}{h^q} + \frac{(t-t_p)(t-t_{p-1}) \ldots (t-t_{p-q})}{(q+1)!} f^{(q+1)}(\xi)$$

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where $\overset{(q+1)}{f(\xi)}$ is the $(q+1)^{th}$ derivative of f evaluated at some point in an interval containing t, t_{p-q} and t_p. On setting $S = \frac{t-t_p}{h}$, we have

$$f(t) = \binom{-S}{0} f_p - \binom{-S}{1} \nabla f_p + \cdots (-1)^q \binom{-S}{q} \nabla^q f_p + (-1)^{q+1} h^{q+1} \binom{-S}{q+1} . f^{(q+1)}(\xi)$$

or

$$f(t) = \sum_{m=0}^{q} (-1)^m \binom{-S}{m} \nabla^m f_p + (-1)^{q+1} h^{q+1} \binom{-S}{q+1} f^{(q+1)}(\xi)$$

The interpolating points corresponding to $t + K$ and t in the Adams-Bashforth formula are t_{p+1} and t_p and integrating $y'(t)$ between these limits, we have

$$y(t_{p+1}) - y(t_p) = \int_{t_p}^{t_{p+1}} \sum_{m=0}^{q} (-1)^m \binom{-S}{m} \nabla^m y'(t_p) dt + \int_{t_p}^{t_{p+1}} (-1)^{q+1} h^{q+1} \binom{-S}{q+1} f^{(q+1)}(\xi)\, dt$$

$$= h \sum_{m=0}^{q} \gamma_m \nabla^m y'(t_p) + R_q^{AB} \qquad (7.24)$$

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where $\gamma_m = \frac{1}{h}\int_{t_p}^{t_{p+1}} (-1)^m \binom{-S}{m}\, dt$

and

$$R_q^{AB} = (-1)^{q+1} h^{q+1} \int_{t_p}^{t_{p+1}} \binom{-S}{q+1} y_{(\xi)}^{(q+2)}\, dt$$

The Adams-Bashforth formula is obtained by neglecting the term R_q^{AB} in (7.24) and is given as

$$y_{p+1} - y_p = h \sum_{m=0}^{q} \gamma_m \nabla^m f_p$$

we now make use of the following two facts about R_q^{AB}:

a. $\binom{-S}{q+1}$ is of constant sign in the interval $t_p \le t \le t_{p+1}$ and

b. $y_{(\xi)}^{(q+2)}$ is a continuous function of t. We are thus in a position to apply the second mean value theorem of the integral calculus with the result that

$$R_q^{AB} = (-1)^{q+1} h^{q+1} y_{(\xi_1)}^{(q+2)} \int_{t_p}^{t_{p+1}} \binom{-S}{q+1} dt$$

where ξ_1 is one of the values of ξ corresponding to values of t in (t_p, t_{p+1}), $t_{p-q} < \xi_1 < t_{p+1}$. By the definition of γ_{q+1}, this may be written as

$$R_q^{AB} = h^{q+2} y_{(\xi_1)}^{(q+2)} \gamma_{q+1} \qquad (7.25)$$

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This is the desired expression for the remainder in the Adams-Bashforth formula.

In a completely analogous manner, we find for the Adams-Moulton formula

$$y(t_p) - y(t_{p-1}) = \int_{t_{p-1}}^{t_p} \sum_{m=0}^{q} (-1)^m \binom{-S}{m} \nabla^m y'(t_p).dt + \int_{t_{p-1}}^{t_p} (-1)^{q+1} h^{q+1} \binom{-S}{q+1} y_{(\xi)}^{(q+2)} dt$$

$$= h \sum_{m=0}^{q} \gamma_m^* \nabla^m y'(t_p) + R_q^{AM} \qquad (7.26)$$

where

$$\gamma_m^* = \frac{1}{h} \int_{t_{p-1}}^{t_p} (-1)^m \binom{-S}{m} dt$$

and

$$R_q^{AM} = \int_{t_{p-1}}^{t_p} (-1)^{q+1} h^{q+1} \binom{-S}{q+1} y_{(\xi)}^{(q+2)}.dt$$

The Adams-Moulton formula is obtained by neglecting the remainder term R_q^{AM} in (7.26). Again, we use the following two facts about R_q^{AM}:

a. $\binom{-S}{q+1}$ is of constant sign in the interval $t_{p-1} \le t \le t_p$ and

b. $y_{(\xi)}^{(q+2)}$ is a continuous function of t.

Thus applying second mean value theorem of integral calculus, we have

$$R_q^{AM} = h^{q+1} (-1)^{q+1} y_{(\xi)}^{(q+2)} \int_{t_{p-1}}^{t_p} \binom{-S}{q+1} dt$$

for some ξ satisfying $t_{p-q} < \xi < t_p$. By definition of γ_{q+1}^*, we have

$$R_q^{AM} = h^{q+2} y_{(\xi)}^{(q+2)} r_{q+1}^* \qquad (7.27)$$

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