Module 4: Systems of Equations and Equations of Order Greater Than One

Lecture 13: Direct Methods For Higher Order Equations

The Lecture Contains:

- Taylor's Series Methods
- Runge-Kutta Methods
- Problems

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Rather than expand a higher order equation into a larger system, we have direct methods to solve them. Here we will outline the extension of some of the one-step methods to higher order equations such as (4.1).

Taylor's Series Methods

The Taylor's series method can be extended in the obvious way as follows: We are given $y_0, y_0^{(1)}, \ldots, y_0^{(p-1)}$. Using (4.1) we can calculate $y_0^{(p)}$, and by differentiation of (4.1) a number of times we can also

calculate $y_0^{(q)}$, $q = p+1, \ldots, r$. We can then write

$$y_1 = y_0 + h\, y_0^{(1)} + \frac{h^2}{2}\, y_0^{(2)} + \cdots + \frac{h^r}{r!}\, y_0^{(r)}$$

$$y_1^{(1)} = y_0^{(1)} + h\, y_0^{(2)} + \frac{h^2}{2}\, y_0^{(3)} + \cdots + \frac{h^{r-1}}{(r-1)!}\, y_0^{(r)}$$

$$y_1^{(p-1)} = y_0^{(p-1)} + h\, y_0^{(p)} + \cdots + \frac{h^{r-p+1}}{(r-p+1)!}\, y_0^{(r)}$$

so that the values of $y, y^{(1)}, \ldots, y^{(p-1)}$ can be approximated at t_1.

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Runge-Kutta Methods:

Let us consider the second order equation

$$y'' = f(y, y').$$

we know $y_n,\ y'_n$. A general two-stage explicit Runge-Kutta method has the form

$$K_1 = \frac{h^2}{2} f(y_n, y'_n)$$

$$K_2 = \frac{h^2}{2} f\left(y_n + \alpha_1 h\, y'_n + \alpha_2 K_1, y'_n + \frac{\alpha_3 K_1}{h}\right)$$

$$y_{n+1} = y_n + h\, y'_n + r_1 K_1 + r_2 K_2 \qquad (4.6)$$

$$y'_{n+1} = y'_n + \frac{\delta_1}{h} K_1 + \frac{\delta_2}{h} K_2$$

Expanding K_2, we get

$$K_2 = \frac{h^2}{2}\left[f(y_n, y'_n) + (\alpha_1 h\, y' + \alpha_2 K_1)\frac{\partial f}{\partial y} + \alpha_1^2 \frac{h^2 (y')^2}{2} \frac{\partial^2 f}{\partial y^2} + \frac{\alpha_3 K_1}{h} \frac{\partial f}{\partial y'} + \frac{\alpha_3^2 K_1^2}{2h^2} \frac{\partial^2 f}{\partial y'^2} + \alpha_1 \alpha_3 K_1 \frac{\partial^2 f}{\partial y\, \partial y'}\, y'\right] + 0(h^5) \qquad (4.7)$$

where everything is evaluated at $y_n,\ y'_n$. We also have

$$y''_n = f,$$

$$y'''_n = \left(\frac{\partial f}{\partial y}\right) y' + \left(\frac{\partial f}{\partial y'}\right) y''$$

$$y_n^{(4)} = \frac{\partial^2 f}{\partial y^2} (y')^2 + 2 \frac{\partial^2 f}{\partial y\, \partial y'} y' y'' + \frac{\partial f}{\partial y} y'' + \frac{\partial^2 f}{\partial y'^2} (y'')^2 + \frac{\partial f}{\partial y'} y'''$$

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Substituting (4.7) into (4.6), we get

$$y_{n+1} = y_n + h\,y'_n + (r_1 + r_2)\frac{h^2}{2}y''_n + r_2\frac{h^3}{2}\left[\alpha_1\frac{\partial f}{\partial y}\,y' + \frac{\alpha_3}{2}\,\frac{\partial f}{\partial y'}\,y''\right] + r_2\frac{h^4}{4}\left[\alpha_2\frac{\partial f}{\partial y}\,y'' + \alpha_1^2\,\frac{\partial^2 f}{\partial y^2}\,(y')^2 + \alpha_1\alpha_3\,\frac{\partial^2 f}{\partial y\,\partial y'}\,y'y'' + \frac{{\alpha_3}^2}{4}\,\frac{\partial^2 f}{\partial y'^2}\,(y'')^2\right] + 0(h^5) \tag{4.8}$$

If $r_1 + r_2 = 1$ and $\alpha_1 r_2 = \frac{\alpha_3 r_2}{2} = \frac{1}{3}$, (4.8) will agree with the first four terms of the Taylor's series expansion of $y(t_{n+1})$. The term in h^4will not match the fifth term in the Taylor's series for any choice of α_i and r_i. Similarly, if $\delta_1 + \delta_2 = 2$ and $\alpha_1\delta_2 = \frac{\alpha_3\delta_2}{2} = 1$, then y'_{n+1}agrees with $y'_n + h\,y''_n + \frac{h^2}{2}\,y'''_n$ to order h^2. Thus, one solution is

$$r_1 = r_2 = \frac{1}{2}$$

$$\alpha_1 = \alpha_2 = \frac{2}{3}, \quad \alpha_3 = \frac{4}{3} \tag{4.9}$$

$$\delta_1 = \frac{1}{2}$$

$$\delta_2 = \frac{3}{2}$$

The local truncation error is thus $0(h^4)$ in y and $0(h^3)$ in y'.

If the equations are of a restricted type, then these methods can be more practical. The system

$y'' = f(y)$

with first derivatives absent frequently occurs in celestial mechanics. In this case $y'' = f(y),\ y^{(3)} = f_y\, y',\ y^{(4)} = f_y\, y'' + f_{yy}(y')^2$ and $f_{y'} = 0$.

Consequently, (4.8) is replaced by

$$y_{n+1} = y_n + h\, y_n' + \frac{h^2}{2}\, y_n''\,(r_1 + r_2) + \frac{h^3}{2} r_2 \alpha_1\, f_y\, y' + \frac{h^4}{4} r_2 \left[\alpha_2\, f_y\, y'' + \alpha_1^2\, f_{yy}(y')^2\right] + 0(h^5) \quad (4.10)$$

while y_{n+1}' is given by

$$y_{n+1}' = y_n' + \frac{h}{2}\, y_n''(\delta_1 + \delta_2) + \frac{h^2}{2}\, \delta_2 \alpha_1\, fy\, y' + \frac{h^3}{4}\, \delta_2\left[\alpha_2\, f_y\, y'' + \alpha_1^2\, f_{yy}(y')^2\right] + 0(h^4) \qquad (4.11)$$

The first four terms of both (4.10) and (4.11) can be made to agree with the Taylor's series terms by choosing

$\delta_2\, \alpha_2 = \frac{2}{3}\, \delta_2\, \alpha_1^2 = \frac{2}{3}$

$\delta_1 + \delta_2 = 2\delta_2\, \alpha_1 = 1$

$\gamma_2\, \alpha_1 = \frac{1}{3}\gamma_1 + \gamma_2 = 1$

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which gives the Nystrom Formula for special second order equations:

$$K_1 = \frac{h^2}{2} f(y_n)$$

$$K_2 = \frac{h^2}{2} f\left(y_n + 2h\frac{y'_n}{3} + \frac{4K_1}{9}\right)$$

$$y_{n+1} = y_n + h\,y'_n + \frac{1}{2}(K_1 + K_2)$$

$$y'_{n+1} = y'_n + \frac{K_1 + 3K_2}{2h}$$

This has a local truncation error of $O(h^4)$ in both $y$and y', giving a global error of $O(h^3)$ is each component.

Problems

1. Consider methods of the form

   $$y'_{n+1} = y'_n + \alpha_1 h f_n + \alpha_2 h f_{n+1}$$

   $$y_{n+1} = y_n + h y'_n + \beta_1 h^2 f_n + \beta_2 h^2 f_{n+1}$$

   for the second order equation

   $$y'' = f(y, y', t)$$

   f_{n+1} is taken to mean $f\left(y_{n+1}, y'_{n+1}, t_{n+1}\right)$.

   What is the highest order that can be achieved globally? Is this order changed if $\frac{\partial f}{\partial y'} = 0$?

2. Verify that the method

   $$y_{n+1} = y_n + \frac{1}{2} h[f(y_n) + f(y_{n+1})] + \frac{1}{12} h^2 [f'(y_n) - f'(y_{n+1})]$$

   is of fourth order for a system of equations of the first order.

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