Lecture 38: Analysis of Difference System

The Lecture Contains:

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Uniqueness

We shall first show that the difference system obtained above has a unique solution.

Theorem: Let the element of A satisfy

$$|b_j| > |a_j| + |c_j| \qquad\qquad j = 1,2,\ldots,J$$

Then A is non singular and the quantities β_j and r_j are bounded by

$a)\quad |r_j| > 1 \qquad b)\ |b_j| - |a_j| \leq |\beta_j| \leq |b_j| + |a_j|$

Proof: If $\beta_j \neq 0$for $1 \leq j \leq J$, the factorization $A = LU$ is valid. Then $\det A = (\det L)\,(\det U)$

$= \beta_1\,\beta_2 \ldots \beta_J \neq 0$ so that A is nonsingular

From the hypothesis, $|r_1| = \left|\frac{c_1}{b_1}\right| < 1$.

For an inductive proof of (a), assume that $|r_j| < 1$for $i \leq j-1$. But we know that

$$r_j = \frac{e_j}{(b_j - a_j \quad r_{j-1})}$$

and thus $|r_j| \leq \frac{|c_j|}{|b_j| - |a_j|\,|r_{j-1}|} \leq \frac{|c_j|}{|b_j| - |a_j|} < 1$, so part a) follows. Now we use $|r_j| < 1$in $\beta_j = b_j - \ a_j\ r_{j-1}$ and take absolute values to conclude part (b).

Cor. Let $p(x)$ and $q(x)$ satisfy the inequalities $|p(x)| < p^*, \quad 0 < \ Q_* \leq q(x) \leq Q^*$

and the mesh spacing h satisfy

$h \leq \frac{2}{p^*}$. Then the finite difference system has a unique solution.

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Proof: we have

$$|b_j| \geq 1 + \frac{h^2}{2} Q_*$$

But if $h \leq \frac{2}{p^*}$, then

$$|a_j| = \frac{1}{2}\left[1 + \frac{h}{2}\, p(x_j)\right]$$

$$|c_j| = \frac{1}{2}\left[1 - \frac{h}{2}\, p(x_j)\right]$$

and so $|a_j| + |c_j| = 1$.

Thus the difference system is strictly diagonally dominant and the result follows.

Truncation error

To estimate the error in the numerical solution of BVP by finite difference method we first define the local truncation error $T_j[v]$ in L_h, as an approximation to L, for any smooth function $v(x)$, by

$$T_j[v] \equiv L_h\, v(x_j) - L\, v(x_j) \qquad\qquad 1 \leq j \leq J.$$

If $v(x)$ has continuous fourth derivatives on $[a, b]$, then

$$T_j\,[v] = -\left[\frac{v(x_j+h) - 2\,v(x_j) + v\,(x_j-h)}{h^2} - v''(x_j)\right] + p(x_j)\left[\frac{v(x_j+h) - v(x_j-h)}{2\,h} - v'(x_j)\right]$$

$$= -\frac{h^2}{12}\left[v''''\ (\xi_j) - 2\,p(x_j)\,v'''(\eta_j)\right] \qquad 1 \leq j \leq J.$$

Here ξ_j and η_j are values in $[x_{j-1},\ x_{j+1}]$.

Thus we find that L_h is consistent with L i.e. $T_j[v] \to 0$ as $h \to 0$ for all factors $v(x)$ having a Cont. second derivative on $[a, b]$. Further L_h has second order accuracy.

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Error estimates

Theorem: Let $p(x)$ and $q(x)$ satisfy the inequality given earlier and h satisfy $h \leq \frac{2}{P^*}$. Then the numerical solution $\{u_j\}$ and the solution $y(x)$ satisfy with

$$M \equiv \max\left(1, \frac{1}{Q_*}\right)$$

$$|u_j - y(x_j)| \leq M \max_{1 \leq i \leq J} |T_i(y)| \qquad 0 \leq j \leq J+1$$

If $y(x)$ has four cont. derivatives on $[a,b]$, then

$$|u_j - y(x_j)| \leq M \frac{h^2}{12} (M_4 + 2 P^* M_3) \qquad 0 \leq j \leq J+1$$

where

$$M_v \equiv \max_{a \leq x \leq b} \left|\frac{d^v y(x)}{d x^v}\right|; \; v = 3,4.$$

Difference corrections and $h \to 0$ Extrapolation

The difference scheme given above has been shown to yield an approximation to the solution of the BVP to within an error that is $O(h^2)$. We shall briefly examine two ways, in which, with additional calculations, the difference scheme can be made to yield $O(h^4)$ accuracy. These error reduction procedures are Richardson's deferred approach to the limit or as we prefer to call it, extrapolation to zero mesh width, and the method of difference correction.

The theoretical basis for both methods is the same, namely that some function $e(x)$, independent of the mesh spacing h, such that the error has the form

$$[y(x_j) - u_j] = h^2 e(x_j) + O(h^4) \qquad 0 \le j \le J+1 \qquad \textbf{(10.8)}$$

Suppose, we compute $\{E_j\}$, an $O(h^2)$approximation to $\{e(x_j)\}$; then clearly

$$\bar{u}_j \equiv u_j + h^2 E_j$$

is an $O(h^4)$ approximation to $y(x_j)$ on the mesh. This is essentially the difference correction method and there may be various ways in which the E_j can be determined.

For the $h \to 0$ extrapolation, we solve the difference system twice, with the net spacing h and $\frac{h}{2}$. Let the respective solutions of these difference problems be denoted by $\{u_i(h)\}$ and $\left\{u_j\left(\frac{h}{2}\right)\right\}$. For any point x common to both meshes, say $x = jh = 2j\left(\frac{h}{2}\right)$, we have from (10.8)

$$\frac{1}{3}\left\{4\left[y(x) - u_{2j}\left(\frac{h}{2}\right)\right] - \left[y(x) - u_j(h)\right]\right\} = 0(h^4).$$

Thus an $O(h^4)$ approximation to $y(x)$ on the net with spacing h is given by

$$\bar{y}_j \equiv \frac{4}{3} u_{2j}\left(\frac{h}{2}\right) - \frac{1}{3} u_j(h) \qquad 0 \le j \le J+1.$$

A derivation of (10.8) is contained in the proof of the following:

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