Lecture 2: Existence, Uniqueness, and Wellposedness

The Lecture Contains:

Reduction of n^{th} Order IVPs

An n^{th}order initial value problem (IVP) can be expressed as

$$y^{(n)}(t) = f\left(t, y, y', \ldots, y^{(n-1)}\right)$$
$$y^{(v)}(t_0) = Y_0^v, \qquad v = 0, 1, \ldots \ldots \ldots \ldots \ldots, n-1 \tag{1.4}$$

This is equivalent to the following system of n first order IVPs:

Let

$y = y_1$, then

$y' = y'_1 = y_2\,;\ y_1(t_0) = y_0$

$y'' = y'_2 = y_3;\ y_2(t_0) = y_0^1$

$\vdots$

$y^{(n-1)} = y'_{(n-1)} = y_n;\ y_{(n-1)}(t_0) = y_0^{n-2}$

and

$y'_n = f(t, y_1, y_2, \ldots \ldots \ldots \ldots, y_n);\ \ y_n\ (t_0) = y_0^{n-1}$

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Thus, in almost all cases, it will suffice to develop and analyze numerical methods for IVPs that are written as first order vector systems in the explicit form

$$Y'(t) = f(t, Y), \qquad t > 0 \qquad (1.5)$$

where

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \qquad f(t, Y) = \begin{bmatrix} f_1(t, y_1, \dots \dots, y_n) \\ f_2(t, y_1, \dots \dots, y_n) \\ \vdots \\ f_n(t, y_1, \dots \dots, y_n) \end{bmatrix}$$

The initial combination would be specified as

$$Y(0) = Y_0 = \begin{bmatrix} y_0 \\ y_0^1 \\ \vdots \\ y_0^{(n-1)} \end{bmatrix}$$

Before proceeding with the numerical solution of the differential equation, we must ask ourselves about the existence and uniqueness of the solution of the given problem. This is stated by the following theorem:

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Existence & Uniqueness

Theorem: If $y' = f(t, y)$ is a differential equation such that $f(t, y)$ is continuous in the region $0 \leq t \leq b$, and if there exists a constant L such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \qquad (1.6)$$

For all $0 \leq t \leq b$ and all y_1, y_2 (this is called the Lipschitz condition and L is called the Lipschitz constant), then there exists a unique continuously differentiable function $y(t)$ such that

$$y'(t) = f\left(t, y(t)\right)$$

and $y(0) = y_0$, the initial condition.

An example that illustrates the fact that an initial value problem may have no solution is the equation

$$y' = \frac{y}{t}$$

The family of solutions is $y = ct$. Since all solution pass through $y = t = 0$, initial values cannot be given at the singular point $t = 0$.

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Remark: The continuity of $f(t, y)$ in both t and y guarantees the existence of a solution and the Lipschitz condition guarantees uniqueness of the solution and is not needed for existence.

For example, the IVP

$$y' = 3y^{2/3}, \qquad y(0) = 0$$

has two solutions given by

$y(t) = 0$ and $y(t) = t^3$

The function $f(t, y)$ does not satisfy a Lipschitz condition at $t = 0$.

Well-posedness

In addition to existence and uniqueness, we will want to know something about the stability of solutions of the IVP. In particular, we will usually be interested in the sensitivity of the solution to small changes in the data. Perturbations arise naturally in numerical computation due to discretization and round off errors. A formal study of sensitivity would lead us to the following notion of a well posed problem.

Definition: The IVP

$$y' = f(t,y) \quad 0 \le t \le b, \quad y(0) = y_0 \qquad (1.7)$$

is well-posed if there exist positive constants K and $\hat{\varepsilon}$ such that, for any $\varepsilon \le \hat{\varepsilon}$, the perturbed IVP

$$\begin{aligned} z' &= f(t,z) + \delta(t) \\ z(0) &= y_0 + \varepsilon_0 \end{aligned} \qquad (1.8)$$

satisfies

$$|y(t) - z(t)| \le K\varepsilon \qquad (1.9)$$

whenever $|\varepsilon_0| < \varepsilon$ and $|\delta(t)| < \varepsilon$ for $t \in [0,b]$.

Remark: By well-posedness, we mean that small perturbations in the stated problem will lead to small changes in the solutions.

Sufficient Conditions for Wellposedness

We shall now give a sufficient condition for wellposedness of an IVP and this is given in the following theorem:

Theorem: If $f(t, y)$ satisfies a Lipschitz condition on

$\{(t, y) \mid (0 \le t \le b, -\infty < y < \infty)\}$

then $y' = f(t, y)$ is well-posed on $[0, b]$ with respect to all initial data.

Proof: We can show this by considering the perturbed problem

$$Z' = f(t, z) + \delta(t) \qquad (1.10)$$

$Z(0) = y_0 + \varepsilon_0$

where $\delta(t)$ and ε_0 are the small perturbations. Let $\hat{\varepsilon}$ be the norm $\|\varepsilon_0, \delta\|$ defined as $\max[|\varepsilon_0|, \max|\delta(t)|]$.

Let $\varepsilon(t) = z(t) - y(t)$. Here $z(t)$ is the changed solution and $y(t)$ is the true solution. Now subtracting (1.7) from (1.10), we get

$\varepsilon'(t) = f(t, z) - f(t, y) + \delta(t)$

and $|\varepsilon(0)| = |\varepsilon_0| \le \hat{\varepsilon}$.

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Therefore,

$$|\varepsilon'(t)| \leq |f(t,z) - f(t,y)| + |\delta(t)| \leq L|\varepsilon(t)| + \hat{\varepsilon}$$

By integration, we find

$$|\varepsilon(t)| \leq \frac{\hat{\varepsilon}}{L}[(L+1)e^{Lt} - 1]$$

Consequently, the largest change in the solution of the perturbed problem is bounded by

$$\max_{0 \leq t \leq b} |\varepsilon(t)| \leq \|\varepsilon_0, \delta\| \frac{1}{L}\,[(L+1)e^{Lb} - 1] = K\hat{\varepsilon}$$

Where K is independent of $\hat{\varepsilon}$.