Module 8: Linear Multistep Methods

Lecture 25: Linear Multistep Methods

The Lecture Contains:

Introduction:

Most of the methods discussed earlier (particularly for 1st order equation) can be considered as a special case of the formula

$$\alpha_K\, y_{n+K} + \alpha_{K-1}\, y_{n+K-1} + \cdots + \alpha_0\, y_n = h\,\{\beta_K\, f_{n+K} + \beta_{K-1}\, f_{n+K-1} + \cdots + \beta_0\, f_n\},\quad n = 0, 1, 2, \ldots \tag{8.1}$$

where K is a fixed integer,

$f_m = f\left(t_m, y_m\right)\ (m = 0, 1, 2, \ldots)$, and where α_μ and β_μ $(\mu = 0, 1, 2, \ldots, K)$

denote real constants which do not depend on n. We shall always assume that $\alpha_K \neq 0,\ |\alpha_0| + |\beta_0| > 0$. Equation (8.1) is said to define the General linear K-step method. The method is called linear because the values f_m enter linearly in (8.1); it is not assumed that f is a linear function of y.

One way in which we were able to derive the coefficients of Adams method was by requiring that they are exact for polynomials of degree $\leq q$. There are $2K + 2$ unknowns in (8.1). There is an arbitrary normalizing factor so we set $\alpha_0 = -1$, leaving $2K + 1$ unknowns.

Consequently, we expect to be able to choose the α and β so that this method is exact for polynomials of degree upto $2K$. This is possible. However, it has been observed that such methods are never useful for $K > 2$, and only marginally useful when $K = 2$. If we were only concerned with local truncation error and the problem had well-behaved derivatives, we would be tempted to use K-step methods of maximal order $2K$. But it is known that for $K > 2$ such methods cause the small truncation errors committed in one step to be unacceptably amplified in later steps due to instability. However, there are stable K-step methods of order $K + 1$ (the Adams-Moulton method, for example) and order $K + 2$ if K is even.

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The associated difference operator; Order and error constant:

The condition of stability has the purpose of preventing a small initial error in the computation from growing at such a rate that convergence is jeopardized. It is clear, however, that stability alone does not guarantee convergence. A further condition must be added which ensures that the difference equation (8.1) is a good approximation to the differential equation $y' = f(t, y)$.

When considering the problem of measuring the accuracy of a One-step method, we looked at the expression

$y(t+h) - y(t) - h\,\psi\,(t, y\,(t), h\,)$ where $y\,(t)$ is a solution of the given differential equation. The smaller this quantity as a function of h, the higher was the accuracy of the method. Similarly, if (8.1) is to define a good method, we expect the discrepancy between the two sides of (8.1) to be small if h is small and if the values y_m are replaced by $y\,(t_m)$, where $y\,(t)$ is an exact solution of the given differential equation. In order to measure this discrepancy, we associate with (8.1) the difference operator.

$$L\,[y(t);h] = \alpha_K\, y\,(t+K\,h) + \alpha_{K-1}\, y\,(t+(K-1)h) + \cdots + \alpha_0\, y\,(t) - h\,\{\beta_K\, y'(t+Kh) + \beta_{K-1}\, y'(t+(K-1)h) + \ldots + \beta_o\, y'(t)\} \qquad (8.2)$$

This may be regarded as a linear operator that acts on any differentiable function $y\,(t)$. For the time being, however, we shall apply the operator L only to functions which have continuous derivatives of sufficiently high order. We then may expand $L[y\,(t);h]$ in powers of h, and the expansion can be pushed as far as we please. In view of the formulas

$$y\,(t+m\,h) = y\,(t) + m\,h\,y'(t) + \frac{1}{2}m^2\,h^2 y''(t) + \cdots$$

$$h\,y'(t+m\,h) = h\,y'(t) + m\,h^2\,y''\,(t) + \cdots$$

it turns out that

$$L\,[y\,(t);h] = C_0\,y\,(t) + C_1\,h\,y'(t) + \ldots + C_q\,h^q\,y^{(q)}\,(t) + \cdots$$

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where the coefficients $C_q\ (q = 0, 1, \dots)$ are constants which do not depend on the choice of the function $y\ (t)$:

$$C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_K$$

$$C_1 = \alpha_1 + 2\alpha_2 + \cdots + K\,\alpha_K - (\beta_0 + \beta_1 + \beta_2 + \dots + \beta_K)$$

$$C_q = \frac{1}{q!}(\alpha_1 + 2^q\,\alpha_2 + \cdots + K^q\,\alpha_K) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\,\beta_2 + \dots + K^{q-1}\,\beta_K);\ q = 2, 3, \dots$$

A given difference operator of the form (8.2) is said to be of order p if $C_0 = C_1 = \cdots C_p = 0$, but $c_{p+1} \neq 0$. As in the case of one-step methods, the order may be considered as a first crude measure of the accuracy of the method.

From the practical point of view the difference equation (8.1) is completely equivalent to the equation

$$\alpha_K\ y_{n+l+K} + \cdots + \alpha_0\,y_{n+l} = h\,\{\beta_K\,f_{n+l+K} + \cdots + \beta_0\,f_{n+l}\} \qquad (8.3)$$

where l is any fixed (positive or negative) integer. Proceeding as above,

we may associate with (8.3) the difference operator

$$L_l\,[y(t);\,h] = \alpha_K\,y\,(t + (l+K)h) + \cdots + \alpha_0 y\,(t + l\,h) - h\,\{\beta_K y'(t + (l+K)h) + \cdots + \beta_0\,y'(t + l\,h)\} \qquad (8.4)$$

and define its order as the order of the first non-vanishing term in its Taylor expansion in powers of h minus 1. It is an important fact that the order p as well as the constant C_{p+1} do not depend on l . For, expanding (8.4) in powers of h is equivalent to expanding (8.2) in powers of h, where y(t) is replaced by $y\ (t + l\,h)$. We thus find, if L is of the order p,

$$L_l\,[y\,(t);h] = L\,[y\,(t + l\,h);\,h] = C_{p+1}\,h^{p+1}\,y^{(p+1)}\,(t + l\,h) + O\,(h^{p+2})$$

Since $y(t)$ was assumed sufficiently differentiable,

$$y^{(p+1)}\,(t + l\,h) = y^{(p+1)}\,(t) + O\,(h)$$

and thus

$$L_l\,[y\,(t);h] = C_{p+1}\,h^{p+1}\,y^{(p+1)}\,(t) + O\,(h^{p+2})$$

This proves the assertion, even without making use of the assumption that l was an integer.

As an example, consider the mid-point rule [Nystrom's method with $q = 0$], which in the standardized form (8.1) appears as follows:

$$y_{n+2} - y_n = h \,.\, 2\, f_{n+1}$$

The corresponding operator (8.2) is given by

$$L\,[y\,(t); h] = y\;(t+2\,h) - y\,(t) - 2\,h\,y'(t+h)$$

$$= 2\,h\,y' + \frac{(2h)^2}{2}\,y'' + \frac{(2h)^3}{6}\,y''' + \frac{(2h)^4}{24}\,y^4 + \cdots - 2h\,y' - 2h^2\,y' - 2\,\frac{h^3}{2}\,y\,''' - 2\,\frac{h^4}{6}\,y^{(4)} - \ldots$$

We readily find $C_0 = C_1 = C_2 = 0,\; C_3 = \frac{1}{3}, C_4 = \frac{1}{3}$ and thus $p = 2$.

Alternatively, by choosing $l = -1$, we may consider the operator

$$L_{-1}\,[y\,(t); h] = y\;(t+\,h) - y\,(t-h) - 2\,h\,y'(t)$$

$$= 2\,h\,y' + \;2\,\frac{h^3}{6}\,y''' + \;2\,\frac{h^5}{120}\,y^{(5)} + \cdots - 2hy'$$

Again $C_0 = C_1 = C_2 = 0, C_3 = \frac{1}{3}$ and hence $p = 2$ but now $C_4 = 0$. This indicates that the constants $C_{p+2}, C_{p+3}, \ldots$ depend, in general, on l. From the point of view of practical computation, the second method of calculating p and C_{p+1} is clearly preferable, since in the Taylor's expansion only terms of odd order occur.

The error constant of the method defined by (8.1) is

$$C = \frac{C_{p+1}}{\beta_0 + \beta_1 + \ldots + \beta_k}$$

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