Lecture 7: Stability

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The concept of stability is loosely defined as follows: If there exists an $h_0 > 0$ for each differential equation such that a change (perturbation) in the starting value by a fixed amount produces a bounded change in the numerical solution for all $0 \le h \le h_0$,then the method is stable. In other words, let $y_n$be the solution of (2.1) with initial condition y_0 and let z_n be the solution of the same method (2.1) with a perturbed initial condition $y_0 + \delta_0$. Then the method (2.1) is stable if there exists positive constants h_0 and K such that

$$|y_n - Z_n| \le K\delta \quad \forall\, n\, h \le b, \quad h \in (0, h_0)$$

whenever $|\delta_0| \le \delta$.

This definition will be modified when multistep methods are discussed.

Note: We also see that stability is related to a method and well posedness is related to a problem.

The Euler's method for solving

$$y' = f(t, y) \text{ with } y(0) = y_0$$

is given by

$$y_{n+1} = y_n + h\, f\,(t_n, y_n) \qquad (2.16)$$

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A change in one of the computed values from y_n to z_n will cause us to solve

$$z_{n+1} = z_n + h\, f\,(t_n, z_n) \qquad (2.17)$$

instead of (2.16). Subtracting (2.16) from this and setting $e_n = z_n - y_n$, we get

$$|e_N| \leq (1 + h\, L)^{N-n} |e_0|$$

$$\leq\ e^{bL}\ |e_0|$$

which is a bounded multiple of the introduced error $|e_0|$ and is independent of h. Hence the Euler's method is stable.

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Example: Consider solving

$$y' = -1000(y - t^2) + 2t, \qquad t \in [0,1] \qquad (2.18)$$

$$y(0) = 0$$

by Euler's method. Let us choose $h = 0.1$ and 0.01. Let us find the value $y(1)$. We get

Table 1.1

h	N	y(1)
0.1	10	0.9044E16
0.01	100	overflow

Overflow means that the largest number retainable in the computer was exceeded and in this case greater than 10^{38}.

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Question: What goes wrong with this example even though the Euler's method is shown to be stable. This points to the shortcomings of the definition of stability given above. These are

- the definition is applicable in the limit of small step size, and
- the definition allows some growth of the solution for bounded times.

Remark: If the solution of the IVP is stable or asymptotically stable, we cannot tolerate any growth in the computed solution and if the solution of the IVP is unstable, some growth of the perturbations is acceptable. The following concept of absolute stability will provide a more useful tool when the solutions of IVP are not growing in time.

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Absolute Stability

Definition: A method is absolutely stable for a given step size h and a given differential equation if the change due to a perturbation of size δ in one of the computed values y_n is no larger than δ in all subsequent values $y_m, m > n.$

Remark: In contrast to the definition of stability, absolute stability is applied at a specific value of h rather than in the limit as $h \to 0.$ Also the definition of absolute stability depends heavily on the differential equation. In order to reduce this dependence, it is common to apply the concept to the "test equation"

$$y' = \lambda y, \qquad y(0) = 0 \qquad (2.19)$$

where λ is a complex constant.

Definition: The region of absolute stability of a method is that set of all non-negative real values of h and complex values of λ for which a perturbation in a single computed value y_n will produce a change in subsequent values that does not increase from step to step, when applied to the test equation (2.19).

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To examine absolute stability of the Euler's method, we consider the test equation $y' = \lambda y.$ For this, we get

$$y_{n+1} = y_n + \lambda\, h\, y_n = (1 + \lambda h) y_n \qquad (2.20)$$

The true solution of $y' = \lambda y.$ is

$y(t) = Ce^{\lambda t}$, so that by Taylor's series

$$y(t_{n+1}) = y(t_n) e^{\lambda h} \qquad (2.21)$$

Let $y_n = y(t_n) + e_n$, we have from (2.20)

$$y_{n+1} = (1 + \lambda h)\, y_n = (1 + \lambda h)(y(t_n) + e_n)$$

and therefore from (2.20) & (2.21), we have

$$y_{n+1} - y(t_{n+1}) = (1 + \lambda\, h)\, y(t_n) + (1 + \lambda\, h) e_n - y(t_n) e^{\lambda h}$$

Or

$$e_{n+1} = \left(1 + \lambda h - e^{\lambda h}\right) y(t_n) + (1 + \lambda\, h) e_n \qquad (2.22)$$

The first expression on the RHS of (2.22) gives the local truncation error and the second expression is the inherited error.

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Consequently, Euler's method is absolutely stable, when

$$|1+\lambda\,h| \le 1$$

which is a unit circle in the complex λh- plane concerned at $(-1,0)$.

We can see the effect of absolute stability very clearly in the following

Example: Let us consider the same example as given above, viz.

$$y' = -1000(y-t^2) + 2t$$

$$y(0) = 0$$

We solve it by Euler's method using mesh size $h = 10^{-m}, \;\; m = 0,1,\ldots\ldots\ldots,5$. We compute $y(1)$ and the results are tabulated as follows:

Table-2.2

Effect of Absolute Stability

h	N	y(1)
1	1	0
0.1	10	0.9044 E 16
0.01	100	Overflow
0.001	1000	0.99999
0.0001	10000	0.99999
0.00001	100000	0.99999

We can notice that overflow occurs because $\lambda = -1000$for this problem and so with $h = 0.01$, the error is amplified by $1 + \lambda h = -9$ at each step. In 100 steps, the error in the first step is increased by nearly 10^{100}in size. Once we find $|1 + \lambda \; h| < 1$, the computed solutions are meaningful and converge to the true solution.

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Problems

1. Show that Euler's method fails to approximate the solution $y(t) = (\frac{2}{3}t)^{3/2}$ of the initial value problem

   $y' = y^{1/2},$, $y(0) = 0$. Explain.

2. Determine analytically the Euler approximation to the initial value problem

   $$y' = \frac{2}{t}, y(1) = 1.$$

   Find also the exact solution of the problem and determine the magnified error function.

3. How large is the discretization error of the approximation to the solution of the initial value problem

   $$y' = \left(\frac{1}{t} + 1\right)y, y(1) = e$$

   obtained by Euler's method?

4. What step size would you use with Euler's method to integrate $y' = 2t$ from $t = 0$ to $t = 1$ in order to achieve errors (ignoring round off errors) of not more than the following:

   - 0.1
   - 0.01
   - 0.001

5. Consider solving the following initial value problem by Euler's method

   - $y' = 1 + 3t^2 - y + t^3, y(0) = 1$
   - $y' = 2t + 1000(2 + t^2) - 1000y, y(0) = 2$

   What step size would one use to achieve an error (ignore round off errors) less than 0.01?

6. Determine error bounds (a priori, a posteriori, and an error estimate) when solving the following initial value problems over $[0, 1]$ by Euler's method

   - $y' = 2ty, y(0) = 1$
   - $y' = -2ty, y(0) = 1$
   - $y' = t^{\frac{1}{2}}, y(0) = 1$

7. Determine the magnified error function for the numerical solution of the initial value problem

$$y' = -\frac{2}{t}y, \; y(-2) = \frac{1}{e^4}$$

by the Euler's method.