Module 7: Multistep Methods

Lecture 23: Multistep Methods for Special Equations of the Second Order

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Introduction:

In the earlier section, we have discussed multistep methods for a single differential equation, but these methods can also be generalized to systems of differential equations. It is thus possible, for instance, to integrate a differential equation of the second order,

$$y'' = f(x, y, y') \tag{7.32}$$

by reducing it to the system

$$y' = z,$$

$$z' = f(x, y, z) \tag{7.33}$$

and applying one of the methods described earlier. As in the case of one-step methods, this procedure is a perfectly legitimate one. No accuracy is lost, nor is there any unnecessary outlay of computational effort.

The situation is slightly different if the equation to be integrated is of the form

$$y'' = f(x, y) \tag{7.34}$$

i.e., if no derivatives appear in the right hand member of the differential equation. Equations of this type will be called special differential equations, and so will those of the form $y^{(n)} = f(x, y)$ where n is an integer > 2. Special differential equations of the second order, and in particular systems of such equations, occur frequently, e.g., in mechanical problems without dissipation. If one is not particularly interested in the values of the first derivatives, it seems unnatural to introduce them artificially in order to produce systems of first order equations. This section is, however, denoted to the study of the methods which deal directly with such type of equations.

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Some Special Methods:

Let us consider the differential equation

$$y''(x) = f(x, y(x)) \qquad (7.35)$$

By integrating twice, we obtain the formula

$$y(x+K) - y(x) = Ky'(x) + \int_x^{x+K} (x+K-t)\, f(t, y(t))\, dt \qquad (7.36)$$

which also may be regarded as a form of Taylor's formula with a remainder term. This result is not yet satisfactory for our proposes, because we do not wish to use the first derivative $y'(x)$. However, by writing down the same result with K replaced by –K and adding, the first derivative drops out, and the sum of the two integrals, writing $f(t) = f(t, y(t))$, may be transformed as follows:

$$\int_x^{x+K} (x+K-t)\, f(t)\, dt + \int_x^{x-K} (x-K-t)\, f(t)\, dt$$

$$= \int_x^{x+K} (x+K-t)\, \{f(t) + f(2x-t)\}\, dt$$

(This is obtained by putting $t = 2\,x - t$ in the second integral).

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This results in the identity

$$y(x+K) - 2\,y(x) + y\,(x-K) = \int_x^{x+K} (x+K-t)\,\{f(t) + f(2\,x-t)\}\,dt \quad (7.37)$$

which forms the basis of many integration formulas. Replacing $f(t)$ by the interpolating polynomial of degree q, using the points $x_p, \ldots, x_{p-q}$, we can obtain a number of special methods according to our choice of x, and K. Some of them are listed in the following table:

Method	x	x + K
Stormer	x_p	x_{p+1}
Cowell	x_{p-1}	x_p
Special method	x_{p-1}	x_{p+1}
Special method	x_{p-2}	x_p

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1. Stormer's methods:

Here we have

$$y_{p+1} - 2y_p + y_{p-1} = h^2 \sum_{m=0}^{q} \sigma_m \nabla^m f_p \qquad (7.38)$$

where

$$\sigma_m = \frac{(-1)^m}{h^2} \int_{x_p}^{x_{p+1}} (x_{p+1} - x) \left[\binom{-S}{m} + \binom{S}{m}\right] dx \qquad \left(S = \frac{x - x_p}{h}\right) \qquad (7.39)$$

$$= (-1)^m \int_0^1 (1 - S) \left[\binom{-S}{m} + \binom{S}{m}\right] dS$$

We now use the method of generating function to determine the recurrence relation for σ_m. We have

$$S(t) = \sum_{m=0}^{\infty} \sigma_m t^m$$

$$= \sum_{m=0}^{\infty} (-t)^m \int_0^1 (1 - S) \left[\binom{-S}{m} + \binom{S}{m}\right] dS$$

$$= \int_0^1 \sum_{m=0}^{\infty} (-t)^m (1 - S) \left[\binom{-S}{m} + \binom{S}{m}\right] dS$$

$$= \int_0^1 (1 - S) \sum_{m=0}^{\infty} (-t)^m \binom{-S}{m} dS + \int_0^1 (1 - S) \sum_{m=0}^{\infty} (-t)^m \binom{S}{m} dS$$

$$= \int_0^1 (1 - S)\ (1 - t)^{-S}\ dS + \int_0^1 (1 - S)\ (1 - t)^{S}\ dS$$

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Using integration by parts, we get

$$= \left[\frac{t}{\log(1-t)}\right]^2 \cdot \frac{1}{1-t} \qquad (7.40)$$

From

$$\frac{d}{dt}[\log(1-t)]^2 = -\frac{2}{1-t}\log(1-t)$$

$$= 2(1+t+t^2+\cdots)\left(t+\frac{1}{2}t^2+\frac{1}{3}t^3+\cdots\right)$$

$$= 2\,(h_1\,t+h_2\,t^2+h_3\,t^3+\cdots)$$

where $h_m = 1+\frac{1}{2}+\cdots+\frac{1}{m}$ denotes the m^{th} partial sum of the harmonic series.

Thus it follows

$$\left[\frac{\log(1-t)}{t}\right]^2 = 1+\frac{2}{3}h_2\,t+\frac{2}{4}h_3\,t^2+\cdots \qquad (7.41)$$

Multiplying both sides of (7.41) by (7.40), we thus find

$$\left(1+\frac{2}{3}h_2\,t+\frac{2}{4}h_3\,t^2+\cdots\right)(\sigma_0+\sigma_1 t+\sigma_2 t^2+\cdots)$$

$$= 1+t+t^2+\cdots$$

There follows the recurrence relation

$$\sigma_0 = 1,$$

$$\sigma_m = 1-\frac{2}{3}h_2\,\sigma_{m-1}-\frac{2}{4}h_3\,\sigma_{m-2}-\cdots-\frac{2}{m+2}h_{m+1}\,\sigma_0,\ \ m=1,2,\ldots$$

The numerical values of σ_m for a few values of m are given in the following table:

m	0	1	2	3	4	5	6
σ_m	1	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{19}{240}$	$\frac{3}{40}$	$\frac{863}{12096}$

Formula (7.38) is used in much the same manner as the Adams-Bashforth formula. Once the values $y_p,\ldots,y_{p-q}$ are known, y_{p+1} can be calculated explicitly, without iteration.

For $q=0$ and $q=1$, stormer's formula reduces to the simple rule

$$y_{p+1}-2y_p+y_{p-1} = h^2\,f_p \qquad (7.42)$$