Lecture 6: Improvement of the error bound

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A posteriori bound

The error bound (2.8) for the solution demonstrates that the error behaves like $O(h)$ if $\frac{\partial f}{\partial t}$ exists, is continuous, and is bounded. Generally, the function f will be differentiable, and the bounds K, L, and Zcan be calculated. However, the error bound so obtained many not be very good because the largest value of $|y'|$, $\left|\frac{\partial f}{\partial y}\right|$, and $\left|\frac{\partial f}{\partial t}\right|$will have to be chosen. If we have some knowledge of the solution, and assume that its second derivative is continuous and bounded by a known quantity, say C, we can get a better bound.

We first express d_n by using a Taylor's series expansion at t_n with remainder term to get

$$-d_n = y(t_{n+1}) - y(t_n) - h\, f\left(t_{n,}\, y(t_n)\right)$$

$$= \frac{h^2}{2} y''(\xi) \text{ for } \xi \in (t_n, t_{n+1})$$

Therefore,

$$|d_n| \le C\frac{h^2}{2}, \text{and}$$

$$|e_n| \le \frac{h}{2} \cdot \frac{C}{L}\left(e^{Lb} - 1\right) + e^{Lb}|e_0| \qquad (2.9)$$

This is an A posteriori bound because it depends on knowledge of the second derivative of the solution.

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Remark: It is difficult to improve the bound given in (2.9) for the error, but we can instead look for an estimate of the error:

Error Estimate

Suppose that $f(t,y)$ has a second derivative which is continuous and bounded in the region R. Under this hypothesis, the third derivative of $y(x)$ exists, and we may write

$$y(t_{n+1}) = y(t_n) + h\, f\big(t_n, y(t_n)\big) + \frac{h^2}{2} y''(t_n) + \frac{1}{6} h^3\, y'''(\xi) \qquad (2.10)$$

where $t_n < \xi < t_{n+1}$. Subtracting this from the corresponding relation satisfied by the approximate values,

$$y_{n+1} = y_n + h\, f(t_n, y_n)$$

we obtain $(\text{where } y_n = y(t_n) + e_n)$,

$$e_{n+1} = e_n + h\big[f(t_n, y(t_n) + e_n) - f\big(t_n, y(t_n)\big)\big] - \frac{1}{2} h^2\, y''(t_n) - \frac{1}{6} h^3\, y'''(\xi) \quad (2.11)$$

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By Taylor's formula, the expression in the brackets can be written as

$$f_y\left(t_n, y(t_n)\right)\, e_n + \frac{1}{2}\, f_{yy}\left(t_n, y_n^*\right) e_n^2$$

where y_n^* is a value between $y(t_n)$ and y_n.

We now divide (2.11) by h and introduce the quantities

$\bar{e}_n = h^{-1} e_n$, and thus (2.11) can now be written in the form

$$\bar{e}_{n+1} = \bar{e}_n + h\left[f_y\left(t_n, y(t_n)\right)\bar{e}_n - \frac{1}{2} y''(t_n)\right] + h^2 r_n \qquad (2.12)$$

where $|r_n| \leq C$, and C is a constant

Define the function

$g(t) = f_y\left(t, y(t)\right)$, we can look at (2.12) as the result of applying Euler's method to the solution of a new differential equation for a function $e(x)$,

$$e'(t) = g(t)\, e(t) - \frac{1}{2} y''(t) \qquad (2.13)$$

making at each step an additional error not exceeding $h^2 C$. The initial value $\bar{e}_0$ is zero, because $e_0 = 0$.

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Thus, the error estimate has the form

$$e_n = h\delta(t_n) + O(h^2) \qquad (2.14)$$

where $\delta(t)$ is the solution of the IVP

$$\delta'(t) = g(t)\,\delta(t) - \frac{1}{2}y''$$

$$\delta(0) = {}^{e_0}/_{h} \qquad (2.15)$$

where $g(t) = {}^{\partial f}/_{\partial y}$.

The function $\delta(t)$ is called the magnified error function.

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An Illustrative Example:

Let us consider solving the IVP

$$y' = t + y, \qquad t \in [0.1]$$

$$y(0) = 1$$

by Euler's method. Let us also determine the error bounds and an estimate for the error.

Here $f(t, y) = t + y$, and $t_0 = 0, y_0 = 1$.

Therefore, with $h = 0.1$, we have

$$y_1 = y_0 + h\, f(t_0, y_o) = 1.1$$

$$y_2 = y_1 + h\, f(t_1, y_1) = 1.22$$

$$\vdots$$

$$y_{10} = y_9 + h(t_9, y_9) = 3.1874$$

The a priori bound is obtained as

$$|e_n| \leq h\frac{K+LZ}{L}\left(e^{Lb} - 1\right) + e^{Lb}|e_0|$$

The Lipschitz constants $K \text{ and } L$ are given by

$$\left|\frac{\partial f}{\partial y}\right| \leq L \quad \text{and} \quad \left|\frac{\partial f}{\partial t}\right| \leq K$$

Now, $f(t, y) = t + y,$ therefore $K = L = 1.$

Also, since $y(t) = 2\, e^t - t - 1$, we have

$y'(t) = 2e^t - 1$ and hence

$$Z = \max\ |y'(t)| = 2\, e - 1, \quad \text{for } t \in [0,1].$$

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Thus the a priori bound for the error is,

$$|e_n| \leq h.\frac{1+1(2e-1)}{1}\ (e-1) \qquad (\because\ e_0 = 0)$$

$$= h.\,2e(e-1) \simeq h(9.26)$$

or $|e_n| \leq 0.926$

For the a posteriori bound,

$$|e_n| \leq \frac{h}{2}.\frac{C}{L}\left(e^{Lb} - 1\right) + e^{Lb}|e_o|$$

where $C = \max|y''(t)|$. Therefore,

$$|e_n| \leq \frac{h}{2}.\,2e(e-1)$$

$$\simeq h\ (4.63)$$

or $|e_n| \leq 0.463$

Now the error estimate is given by

$e_n \simeq h\ \delta(t),\ \text{where}\ \delta(t)$ is the solution of

$$\delta'(t) = \delta(t) - e^t, \delta(0) = 0$$

Consequently, $\delta(t) = -t\,e^t$, and

$$\delta(1) = -e = -2.71828$$

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Therefore, the estimate for the error is

$e_n \simeq -0.271$

The exact solution of the given IVP is

$y(t) = 2\,e^t - t - 1$ so that

$y(1) = 2\,(e - 1) \simeq 3.43656$

The Euler's method gives $(\text{with } h = 0.1)$

$y_{10} \simeq y(1) = 3.187485$, and the actual error is 0.2491.

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