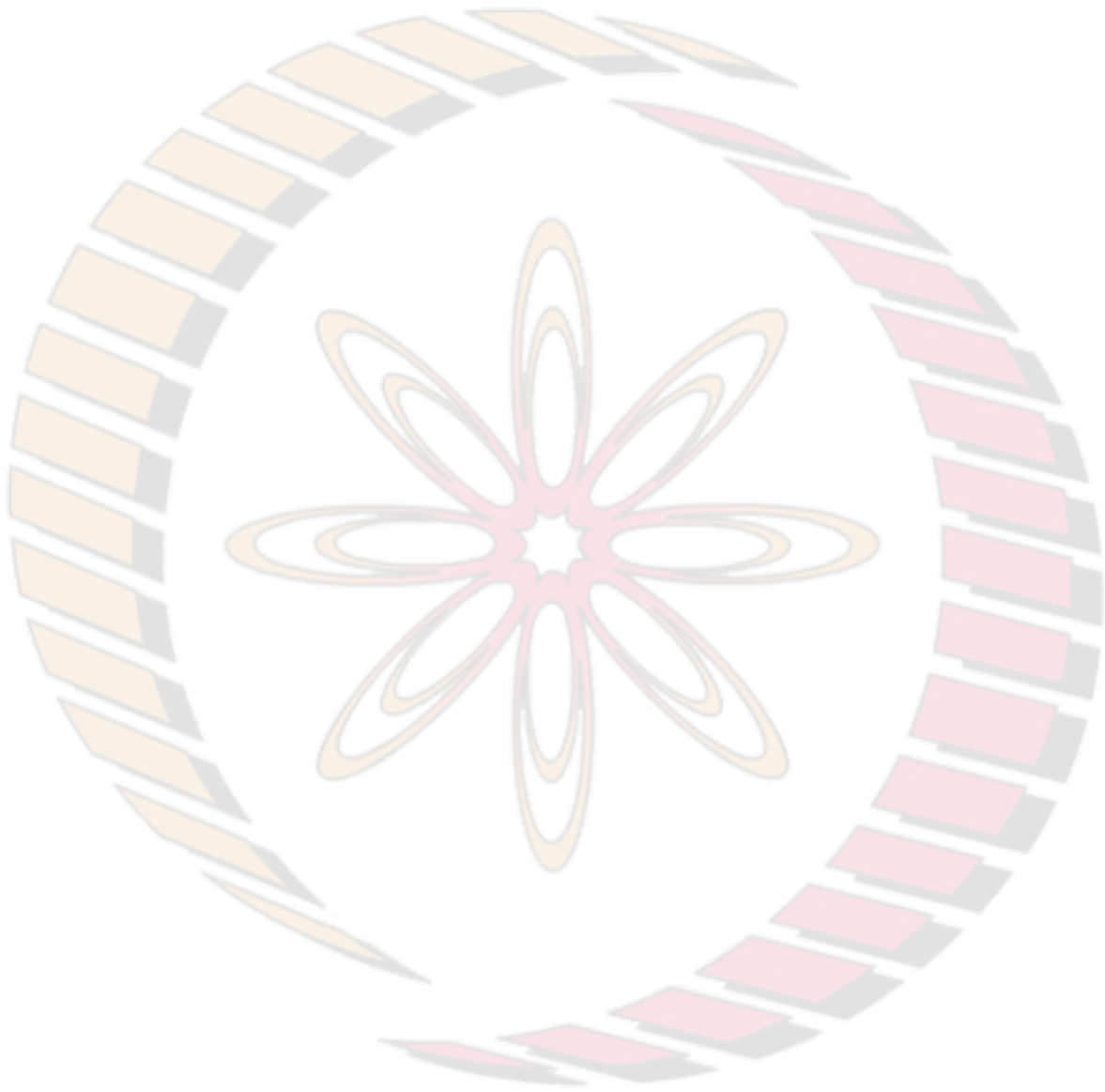


NPTEL Lecture Notes on
Introduction to Algebraic Topology
Part-II

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Jan. 2021

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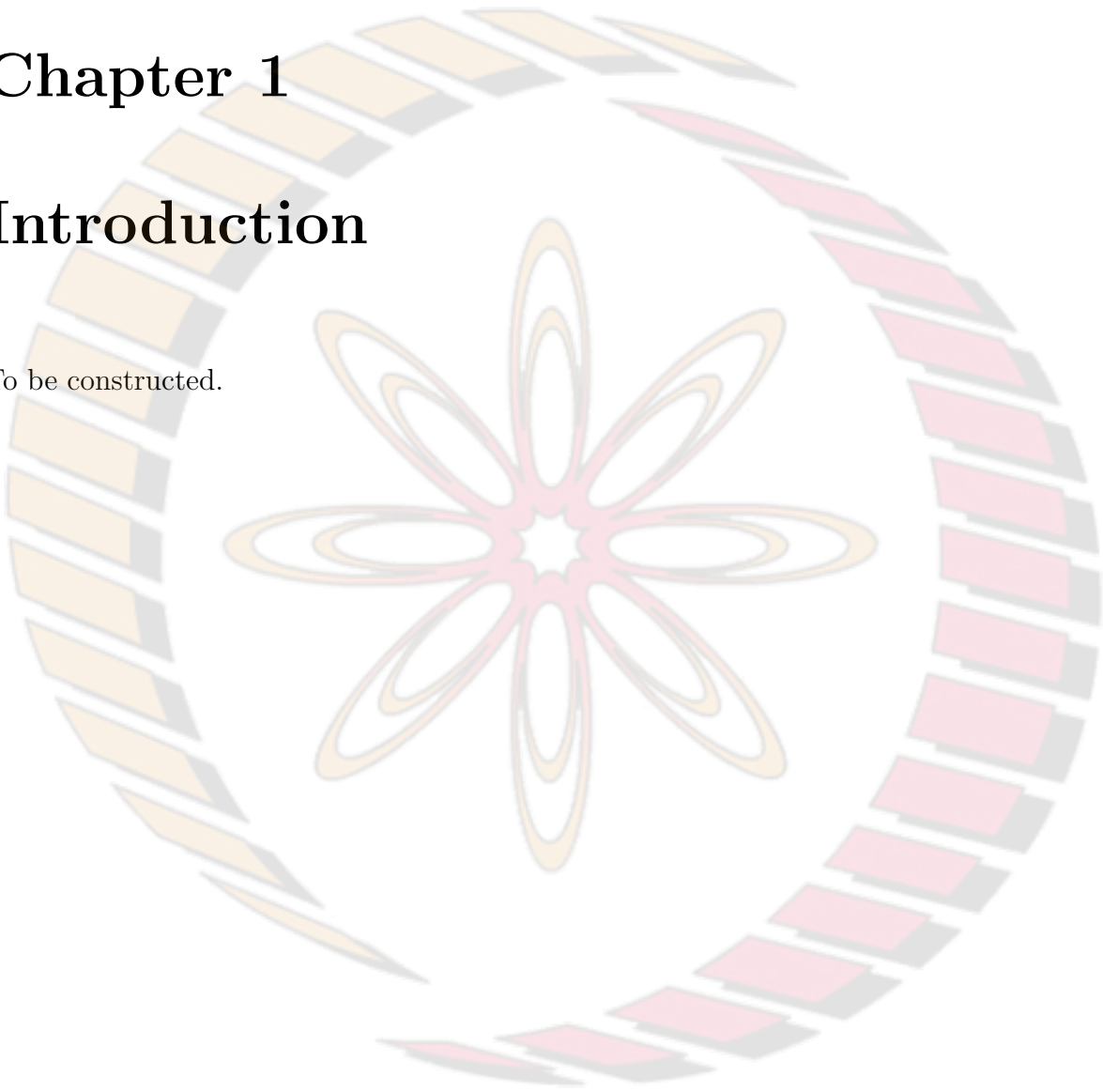
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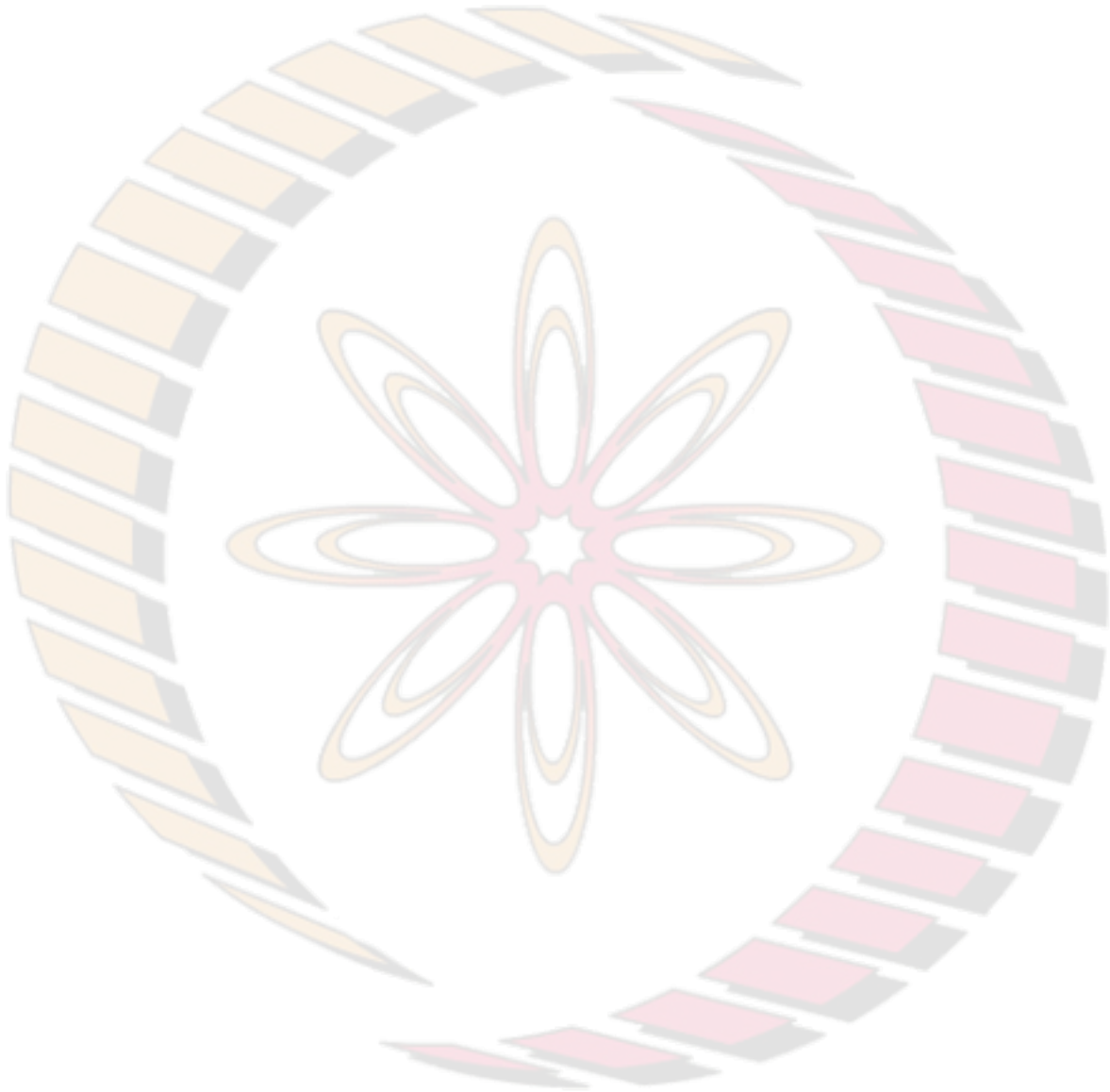
Chapter 1

Introduction

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Chapter 2

Cell Complexes

In this chapter, we shall first introduce the most important class of topological spaces in algebraic topology, viz., the CW-complexes. We shall study some fundamental point-set topological properties of these spaces, relate it to the simplicial complexes that you have studied in Part-I. In subsequent chapters, we shall study the homology of CW-complexes also. Some of these results you have met in Part-I and just for refreshing your memory, we shall recall them here.

2.1 Attaching cells

Definition 2.1.1 Let $k \geq 1$ be an integer and for each index $\alpha \in \Lambda$, let D_α^k denote a copy of the closed unit ball $\mathbb{D}^k$ in $\mathbb{R}^k$. Given two spaces X and Y , we say X is obtained by attaching k -cells to Y if there exists a family of maps $f_\alpha : \mathbb{S}^{k-1} \rightarrow Y$, $\alpha \in \Lambda$, such that X is the quotient space of the disjoint union

$$Y \sqcup_{\alpha \in \Lambda} D_\alpha^k \tag{2.1}$$

by the relation $x \sim f_\alpha(x)$ for each $x \in \partial D_\alpha^k$, and for each α .

Let us denote the quotient map by

$$q : Y \sqcup_{\alpha \in \Lambda} D_\alpha^k \rightarrow X.$$

It is worth recalling that the quotient topology on X is defined by the following rule:

$$A \subset X \text{ is open iff } q^{-1}(A) \text{ is open in } Y \sqcup_{\alpha \in \Lambda} D_\alpha^k.$$

The maps $\{f_\alpha\}$ are called *attaching maps for the cells*. Let us denote the quotient map restricted to a cell D_α^k by ϕ_α . Observe that $\phi_\alpha|_{\partial D_\alpha^k} = f_\alpha$ and ϕ_α is injective in the

interior of D_α^k . Thus ϕ_α defines a homeomorphism of the interior of D_α^k onto the image. The image $\phi_\alpha(\text{int}(D_\alpha^k))$ is called an *open cell* in X . The maps ϕ_α are called *characteristic maps of the cells*. Observe that the image of each D_α^k is a compact subspace of X . We call them the *closed k -cells* in (X, Y) and denote them by e_α^k .

Remark 2.1.2 Note that q is injective on Y . Moreover, $A \subset Y$ is closed in Y iff it is closed in X . Therefore, we identify Y with a closed subset of X .

With the additional assumption that Y is a Hausdorff space, we can say something more.

Lemma 2.1.3 Let X, Y , etc., be as in the definition above and Y be a Hausdorff space. Then

- (a) Each e_α^k is a closed subset of X .
- (b) Each characteristic map $\phi_\alpha : D_\alpha^k \rightarrow e_\alpha^k$ is a quotient map.
- (c) A subset A of X is closed in X iff $A \cap Y$ is closed in Y and $A \cap e_\alpha^k$ is closed in e_α^k for each $\alpha \in \Lambda$.

Proof: (a) Since Y is Hausdorff and $\mathbb{S}^{k-1}$ is compact, we first note that for each α ,

$$e_\alpha^k := f_\alpha(\mathbb{S}^{k-1})$$

is a closed subset of Y . Fix a β . Now to see that e_β^k is closed in X , appeal to the quotient topology: e_β^k is closed, iff $q^{-1}(e_\beta^k)$ is closed in $Y \sqcup_\alpha D_\alpha^k$. Now

$$q^{-1}(e_\beta^k) \cap Y = e_\beta^k; \quad q^{-1}(e_\beta^k) \cap D_\beta^k = D_\beta^k; \quad q^{-1}(e_\beta^k) \cap D_\alpha^k = f_\alpha^{-1}(e_\beta^k), \quad \alpha \neq \beta.$$

Since each of these three sets is a closed subset of the respective space in the decomposition (2.1), we are done.

(b) Let $A \subset e_\alpha^k$ be such that $\phi_\alpha^{-1}(A)$ is closed in D_α^k . To show that A is closed in e_α^k is the same as saying that it is closed in X . Once again, this is true if $q^{-1}(A)$ is closed in $Y \sqcup_\beta D_\beta^k$. By hypothesis, it follows that $f_\alpha^{-1}(A)$ is closed in $\mathbb{S}^{k-1}$ and hence $A \cap e_\alpha^k$ is closed. From this the claim follows.

(c) This is now a direct consequence of (b) and the quotient topology. ♠

Remark 2.1.4

(a) Observe that in the definition above, the family $\{f_\alpha\}$ may be empty also. Of course this is the most uninteresting case. However, we should include this case for technical reasons.

(b) If the family has just one member f , it may be noted that X can be identified with the mapping cone of f . In this case, it is customary to denote the resulting space by

$$Y \cup_f e^k. \quad (2.2)$$

(c) In general, i.e., without the assumption of Hausdorffness on Y , e_α^k need not be a closed subset of X . However, Y is always a closed subset of X .

(d) Also, in general, e_α^k is not homeomorphic to $\mathbb{D}^k$. However, its interior is homeomorphic to $\text{int}(\mathbb{D}^k)$.

(e) In Figure 2.1 below, $k = 1$ and $\Lambda = \{1, 2, 3\}$. Observe that f_1 and f_2 are injective and f_3 is not injective.

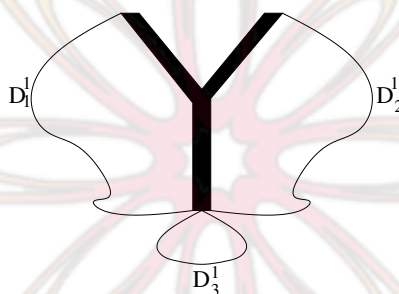


Figure 2.1. Attaching maps need not be injective.

The quotient topology retains a large number of topological properties and hence we can expect a certain Euclidean behavior in the spaces obtained by attaching cells. In order to put this to some good use, we need to prepare ourselves a bit. The proof of following lemma is obvious (see Figure 2.2).

Lemma 2.1.5 Consider a subset $A \subset \mathbb{S}^{n-1}$ and let $0 < \epsilon < 1$. Let us put

$$N_\epsilon(A) = \{x \in \mathbb{D}^n : \|x\| > \epsilon \text{ \& } \frac{x}{\|x\|} \in A\}.$$

Then

- (i) $N_\epsilon(A) \cap \mathbb{S}^{n-1} = A$;
- (ii) $N_\epsilon(A)$ is an open subset of $\mathbb{D}^n$ iff A is an open subset of $\mathbb{S}^{n-1}$.
- (iii) $(x, t) \mapsto (1 - t)x + \frac{tx}{\|x\|}$ defines a SDR of $N_\epsilon(A)$ onto A .

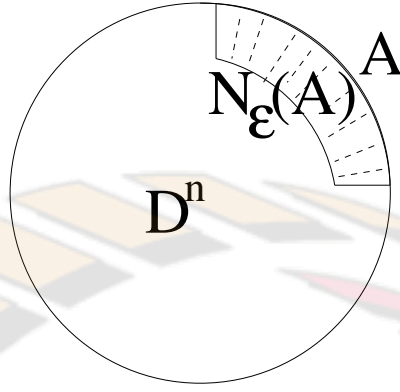


Figure 2.2. Extending neighbourhoods

Proposition 2.1.6 Let X be a space obtained by attaching a family of k -cells $\{e_\alpha^k : \alpha \in \Lambda\}$ to Y with attaching maps $\{f_\alpha : \alpha \in \Lambda\}$ and let $q : Y \sqcup_{\alpha \in \Lambda} D_\alpha^k \rightarrow X$ be the quotient map. Let $\epsilon : \Lambda \rightarrow (0, 1)$ be any set function. For any subset A of Y define

$$N_\epsilon(A) := q(A \sqcup \{N_{\epsilon(\alpha)}(f_\alpha^{-1}(A)) : \alpha \in \Lambda\}).$$

Then

- (i) $N_\epsilon(A) \cap Y = A$;
- (ii) $N_\epsilon(A)$ is open in X iff A is open in Y ;
- (iii) A is a SDR of $N_\epsilon(A)$.

Proof: We simply appeal to Lemma 2.1.5: Parts (i) and (ii) follow directly from the corresponding parts of this lemma. To obtain (iii), let

$$h_\alpha : N_{\epsilon(\alpha)}(f_\alpha^{-1}(A)) \times \mathbb{I} \rightarrow N_{\epsilon(\alpha)}(f_\alpha^{-1}(A))$$

be the strong deformation as in Lemma 2.1.5. On the disjoint union, $(A \sqcup \{N_{\epsilon(\alpha)}f_\alpha^{-1}(A), \alpha \in \Lambda\}) \times \mathbb{I}$, let h be defined by the rule:

$$h|_{A \times \mathbb{I}} = Id; \quad h|_{N_{\epsilon(\alpha)}f_\alpha^{-1}(A) \times \mathbb{I}} = h_\alpha; \quad \alpha \in \Lambda.$$

Then h factors down to define a deformation $\hat{h} : N_\epsilon \times \mathbb{I} \rightarrow N_\epsilon$ onto A . ♠

Corollary 2.1.7 Let Y be a T_2 -space. Then X is also a T_2 -space. If Y is T_3 or T_4 so is X .

Proof: Exercise.

Definition 2.1.8 A relative CW-complex (or a relative cell complex) (X, A) consists of a topological space X and a closed subspace A which is Hausdorff, together with a sequence of closed subspaces $\{X^{(n)}\}, n \geq 0$ such that

- (i) $A \subset X^{(0)}$ and $X^{(0)} \setminus A$ is a discrete space;
- (ii) $X^{(k)}$ is obtained by attaching k -cells to $X^{(k-1)}$ for all $k \geq 1$;
- (iii) $X = \cup_k X^{(k)}$;
- (iv) The topology on X is the weak topology with respect to the family $\{X^{(k)}\}$, i.e., a subset $F \subset X$ is closed in X iff $F \cap X^{(k)}$ is closed in $X^{(k)}$ for every $k \geq 0$.

Remark 2.1.9

- (a) An interesting case is when $A = \emptyset$. Then we say X is a *CW-complex*. It follows that $X^{(0)}$ is a discrete space (possibly empty). Thus, in building up a CW-complex, we start off with a discrete topological space, then attach 1-cells to this space, then attach 2-cells to the space obtained, and so on, to obtain the CW-complex X . We call elements of $X^{(0)} \setminus A$ the 0-cells of (X, A) . Note that a 0-cell is both an open 0-cell as well as a closed 0-cell.
- (b) It may happen that $X = X^{(k)}$ for some k . In that case, condition (iv) is automatically satisfied. (Keep using Lemma 2.1.3.) Further if $X^{(k-1)} \neq X$, then we say (X, A) is k -dimensional. We also write $\dim(X, A) = k$.
- (c) The terminology that we introduced in Definition 2.1.1, such as *open cells*, *characteristic maps*, etc., holds good here also. Caution: open cells in a CW-complex need not be open subsets!
- (d) Note that we allow the case when $X^{(0)} = \emptyset$. Of course, it then follows that $X^{(k)} = \emptyset$ for all k and $X = \emptyset = A$.

2.2 Relative CW-complexes

Definition 2.2.1 By a subcomplex (Y, B) of a relative CW-complex (X, A) , we mean a relative CW-complex where, $Y \subset X, B \subset A$ and each cell in Y is also a cell in X with precisely the same attaching map.

Remark 2.2.2

- (a) If (Y, B) is a sub-pair of a relative CW-complex (X, A) and Y is the union of B and some closed cells in X then (Y, B) is a subcomplex of (X, A) .
- (b) If Y is a subcomplex of X then (X, Y) is a relative CW-complex. More generally, if

(Y, B) is a subcomplex of (X, A) then $(X, Y \cup A)$ is a relative CW-complex.

(c) For all k , $X^{(k)}$ is a subcomplex of any CW-complex X . It is called the k^{th} -skeleton of X . For a relative CW-complex (X, A) , observe that $(X^{(k)}, A)$ is a subcomplex.

Example 2.2.3

- (i) Any discrete space can be thought of as a CW-complex with only 0-cells.
- (ii) The n -dimensional unit sphere $\mathbb{S}^n$ is a CW-complex with a single 0-cell and a single n -cell. The attaching map of the n -cell is the constant map. This follows from the fact that the quotient space $\mathbb{D}^n/\mathbb{S}^{n-1}$ is homeomorphic to $\mathbb{S}^n$. Observe that with these CW-structures, even though $\mathbb{S}^{n-1}$ is a subspace of $\mathbb{S}^n$ via the equatorial inclusion, it is not a subcomplex. (See Figure 2.3(a).)

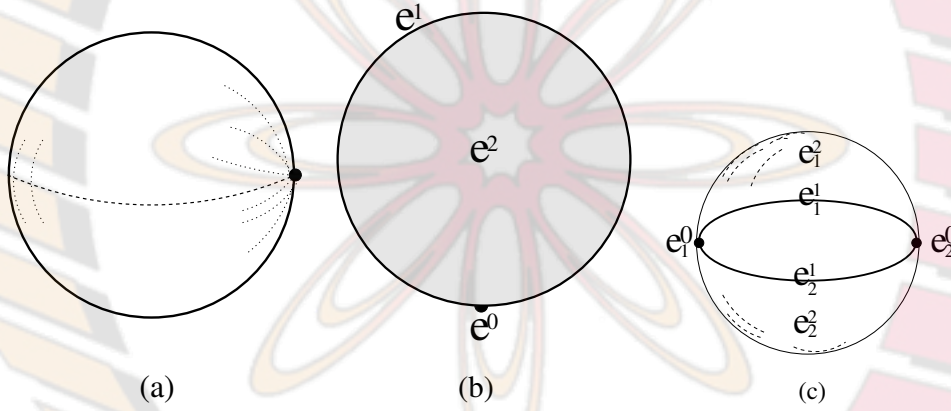


Figure 2.3. Standard examples of CW-complexes

- (iii) Fix some CW-structure on $\mathbb{S}^n$, for instance given as in Figure 2.3(a). Then $\mathbb{D}^{n+1}$ itself can be viewed as a CW-complex obtained by attaching one $(n+1)$ -cell to $\mathbb{S}^n$ via the identity map $\mathbb{S}^n \rightarrow \mathbb{S}^n$. It follows that $\mathbb{S}^n$ is then a subcomplex of $\mathbb{D}^{n+1}$. The case $n = 1$ is illustrated in Figure 2.3(b).
- (iv) We can have different CW-structures on the same topological space. For instance, take $\mathbb{S}^n$, $n \geq 1$ and consider the usual equatorial inclusions

$$\mathbb{S}^0 \subset \mathbb{S}^1 \subset \dots \subset \mathbb{S}^{n-1} \subset \mathbb{S}^n.$$

Then each $\mathbb{S}^k$, $k \geq 1$ can be obtained from $\mathbb{S}^{k-1}$ by attaching two k -cells, viz., the upper and lower hemispheres. (For instance, $\mathbb{S}^1$ has two 0-cells and two 1-cells. See Figure 2.3(c).) This cell decomposition of $\mathbb{S}^n$ is more useful than the

earlier simpler one given in (2). It immediately gives us a cell structure for the real projective space, which we shall discuss now.

- (v) Recall that the real projective space $\mathbb{P}^n$ is the quotient space of $\mathbb{S}^n$ under the anti-podal action $x \sim -x$. By the definition of quotient topology, if $q : \mathbb{S}^n \rightarrow \mathbb{P}^n$ denotes the quotient map then a subset U of $\mathbb{P}^n$ is open iff its inverse image $q^{-1}(U)$ is open in $\mathbb{S}^n$. We first observe that the quotient map q is both an open mapping as well as a closed mapping. This follows easily from the fact that for any subset $F \subset \mathbb{S}^n$, $F \cup (-F)$ is open (closed) if F is open (closed, respectively). From this, many of the topological properties of $\mathbb{S}^n$ pass onto the quotient space $\mathbb{P}^n$. For instance, using the openness of q we can easily conclude that $\mathbb{P}^n$ is II-countable. Indeed given any base $\mathcal{B}$ for the topology of $\mathbb{S}^n$, it follows that $\{q(U) : U \in \mathcal{B}\}$ is a base for $\mathbb{P}^n$. Since $\mathbb{S}^n$ is compact, it follows that $\mathbb{P}^n$ is also compact.

The CW-structure of $\mathbb{S}^n$ as described in (iv) is compatible with this action in the sense that the action preserves each skeleton and merely permutes the various cells. In such a situation, the quotient space acquires a natural CW-structure: We begin with $X^{(0)}$ as a single 0-cell for $\mathbb{P}^n$, which is the image of $\mathbb{S}^0$ under the quotient map. Inductively, having defined $X^{(k-1)}$ whose underlying space happens to be $\mathbb{P}^{k-1}$, we attach a single k -cell to $X^{(k-1)}$ which could be either the upper or the lower hemisphere of $\mathbb{S}^{k-1}$ to get $X^{(k)}$. For definiteness, let us choose the upper-hemisphere of $\mathbb{S}^k$ and take the attaching map to be q restricted to $\mathbb{S}^{k-1}$. The space so obtained is indeed equal to $\mathbb{P}^k$. Thus, $\mathbb{P}^n$ has a CW-structure with one cell for each dimension $0 \leq k \leq n$. Indeed, this is really the first non trivial example of a CW-complex.

- (vi) The infinite sphere $\mathbb{S}^\infty$ which is the union of all spheres

$$\mathbb{S}^0 \subset \mathbb{S}^1 \subset \dots \subset \mathbb{S}^n \subset \dots$$

is a CW-complex with two cells in each dimension. It is infinite dimensional. Indeed, let $\mathbb{R}^\infty$ denote the infinite sum of copies of $\mathbb{R}$ as a vector space. Elements of $\mathbb{R}^\infty$ can be written as infinite sequences of real numbers in which all but finitely many entries are zero. The standard Euclidean inner product extends to $\mathbb{R}^\infty$ and makes it into a Hilbert space. However, the topology that we give it is the *weak topology* coinduced from the family

$$\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \dots,$$

viz., a subset $F \subset \mathbb{R}^\infty$ is closed iff $F \cap \mathbb{R}^n$ is closed in $\mathbb{R}^n$ for every n . Note that the weak topology is finer than the metric topology and so all vector space operations are continuous. The unit sphere $\mathbb{S}^\infty$ inherits this weak topology.

The cell structure on $\mathbb{S}^\infty$ is compatible with the antipodal action and hence we get an induced cell structure on the infinite real projective space $\mathbb{P}^\infty$, with exactly one cell in each dimension $0 \leq k < \infty$.

- (vii) The case of complex projective spaces is not much different. Recall that $\mathbb{CP}^n$ is defined as the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ by the relation

$$(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n), \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Under the standard inclusions

$$\mathbb{C} \subset \mathbb{C}^2 \subset \dots \subset \mathbb{C}^n \subset \dots$$

the above relations are compatible and hence we can use the same notation q for all the quotient maps $q : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$.

As before, restricted to the unit sphere $\mathbb{S}^{2n+1}$, q is surjective and hence we can view $\mathbb{CP}^n$ as the quotient space of $\mathbb{S}^{2n+1}$ modulo the same relation as above wherein λ is now restricted to being of unit length. Clearly, $\mathbb{CP}^0$ is a singleton. Next, the quotient map $q : \mathbb{S}^3 \rightarrow \mathbb{CP}^1$ sends the circle $z_1 = 0$ to a single point which is our $\mathbb{CP}^0$. The subspace

$$e^2 = \{(z_0, r) : |z_0|^2 + r^2 = 1, r \geq 0\}$$

is clearly homeomorphic to $\mathbb{D}^2$ with its boundary being the circle $z_1 = 0$. Note that $q(e_2) = \mathbb{CP}^1$ and q restricted to the interior of the 2-cell e^2 is injective. Hence $\mathbb{CP}^1$ is nothing but a 2-disc with its boundary collapsed to a single point and hence is homeomorphic to $\mathbb{S}^2$. Indeed the map $q : \mathbb{S}^3 \rightarrow \mathbb{S}^2$ is the familiar Hopf fibration. Inductively, having established that $\mathbb{CP}^k$ is obtained by attaching a $2k$ -cell to $\mathbb{CP}^{2k-1}$ via the quotient map $\mathbb{S}^{2k-1} \rightarrow \mathbb{CP}^{k-1}$, we see that the subspace $e_{2k+2} \subset \mathbb{S}^{2k+3}$ given by

$$e^{2k+2} = \{(z_0, \dots, z_{k+1}, r) : \sum_{i=0}^{k+1} |z_i|^2 + r^2 = 1, r \geq 0\}$$

is homeomorphic to $\mathbb{D}^{2k+2}$ with its boundary being the sphere given by $z_{k+2} = 0$. One merely checks that $q(e^{2k+2}) = \mathbb{CP}^{k+1}$ and restricted to the interior of the cell

e^{2k+2} is injective. Therefore, $\mathbb{CP}^{2k+2}$ is obtained by attaching the $(2k+2)$ -cell e^{2k+2} to $\mathbb{CP}^k$ via the map $q : \mathbb{S}^{2k+1} \rightarrow \mathbb{CP}^k$.

Thus the infinite complex projective space $\mathbb{CP}^\infty$ has a CW-structure one cell for each even dimension.

2.3 Topological Properties

We shall now study some of the topological properties of a CW-complex. Proofs of corresponding properties for relative CW-complexes, with appropriate modifications, will be left to the reader as exercises.

Proposition 2.3.1 Let X be a CW-complex.

- (a) Each skeleton $X^{(k)}$ is a closed subset of X .
- (b) Each closed cell is also a closed subset of X .
- (c) A subset S of X is closed iff S intersects each closed cell e^k of X in a closed subset of e^k .
- (d) The topology on X is compactly generated.

Proof: (a) By remark 2.1.2, each $X^{(k)}$ is closed in $X^{(k+1)}$ and hence each $X^{(k)}$ is closed in $X^{(r)}$ for $r > k$. This means $X^{(k)} \cap X^{(r)}$ is closed in $X^{(r)}$ for every r . Hence by part (iv) of Definition 2.1.8, $X^{(k)}$ is closed in X .

(b) $X^{(0)}$ being a discrete space is Hausdorff. By Corollary 2.1.7, applied inductively, it follows that each skeleton $X^{(k)}$ is a Hausdorff space. Now if e^k is a closed cell, then it is a compact subset of $X^{(k)}$ and hence is a closed subset of $X^{(k)}$. Since each $X^{(k)}$ is closed in X , we are done.

(c) By (b), if S is closed in X then $S \cap e$ is closed in X for each closed cell e . Conversely, suppose $S \cap e$ is closed in X for every closed cell e . Inductively, we shall prove that $S \cap X^{(k)}$ is closed in $X^{(k)}$, for every k . Again for $k = 0$, there is nothing to prove since $X^{(0)}$ is discrete. Having proved this for some k , we appeal to Lemma 2.1.3 to conclude that $S \cap X^{(k+1)}$ is closed in $X^{(k+1)}$.

(d) Recall that a topology is *compactly generated* if it is coherent with the family of its compact subsets, i.e., a subset S of X is closed iff $S \cap K$ is closed in K for every compact subset K of X . In this situation, since each closed cell is compact, $S \cap e$ is closed for each cell e and hence we are through by (c). ♠

Just as in Corollary 2.1.7, it is not difficult to prove the following:

Theorem 2.3.2 *Let (X, A) be a relative CW-complex. If A is T_2, T_3 or T_4 so is X .*

Lemma 2.3.3 *Every CW-complex is the disjoint union of its open cells.*

Proof: By definition, we know that

$$X = \coprod_{k \geq 0} (X^{(k)} \setminus X^{(k-1)}).$$

Here $X^{(-1)} = \emptyset$ and $X^{(0)}$ is the union of 0-cells which are, by convention open cells. And for $k \geq 1$ each $X^{(k)} \setminus X^{(k-1)}$ is the disjoint union of open k -cells. ♠

Lemma 2.3.4 *Let S be a subset of X such that S contains at most one point from each open cell of X . Then S is a closed discrete subset of X .*

Proof: To show that S is a closed set, clearly $S \cap X^{(0)}$ is closed in $X^{(0)}$. Assume that $S \cap X^{(k)}$ is closed in $X^{(k)}$. Then for any closed $(k+1)$ -cell e , $e \cap S$ is either equal to $e \cap S \cap X^{(k)}$ or has one extra point coming from the interior of e . And in either case this is a closed subset of $X^{(k+1)}$. Therefore, $S \cap X^{(k+1)}$ is closed in $X^{(k+1)}$. Again by condition (iv) of Definition 2.1.8, we are through. Now take any subset A of S . It also satisfies the same condition as S and therefore is a closed subset. This means every subset of S is closed. Therefore S is discrete. ♠

Theorem 2.3.5 *Every compact subset K of a CW-complex X is contained in the union of finitely many open cells.*

Proof: If not, K will intersect infinitely many open cells which, we know are disjoint. Selecting one point in each such intersection, we will get a subset S of K which is, by Lemma 2.3.4, a discrete subset of the compact set K . But S is infinite?! ♠

Corollary 2.3.6 *The closure of every cell in X meets only finitely many open cells.*

Remark 2.3.7 This property of a CW-complex is known as ‘closure finiteness’ and it explains the presence of the letter ‘C’ in this mysterious nomenclature, ‘CW-complex’. Historically, this property was an additional part of the definition itself. We have just proved that we don’t need to put this extra condition in the definition itself. By the way, the other letter ‘W’ in the name corresponds to the ‘weak topology’ on X . So, it might have been appropriate to call these spaces W -complexes. However, we shall stick to the present-day fashion of calling them *cell complexes* or the old name CW-complexes.

2.4 Paracompactness of CW-complexes

One of the salient features of the coherent topology that we take on a CW-complex is that it provides a practical method to construct continuous functions as well as a method to verify whether a given function is continuous or not. We shall have many instances of this. Let us begin with an illustration by showing the existence of partition of unity on CW-complexes. We assume that the reader is familiar with the concept of partition of unity and related results. However, let us recall a few the definitions and conventions.

Definition 2.4.1 Let X be a topological space. By a *partition of unity on X* , we mean a family $\{\theta_\alpha : \alpha \in \Lambda\}$ of continuous function $\theta_\alpha : X \rightarrow \mathbb{I}$ such that

- (i) for each $x \in X$, there exists a neighborhood N_x of x in X , such that only finitely many of θ_α are non zero on N_x ; (this property is called ‘locally finiteness’);
- (ii) $\sum_\alpha \theta_\alpha(x) = 1$, for all $x \in X$.

Definition 2.4.2 Given an open covering $\mathcal{U} = \{U_\beta\}$ of X , the family $\{\theta_\alpha\}$ is said to be subordinate to $\mathcal{U}$, if for each α there exists $\beta = \beta(\alpha)$ such that the support of θ_α :

$$\text{supp } \theta_\alpha := \text{cl}(\{x \in X : \theta_\alpha(x) \neq 0\})$$

is contained in U_β .

It is a standard result that over any subspace of a Euclidean space, there is always a partition of unity subordinate to a given open cover. Similar to Proposition 2.1.6, the key result that we need is the following relative version for partition of unity so that functions can be ‘patched up’ using the patching up process in a CW-complex. Let us make a definition.

Definition 2.4.3 Let $Y \subset X$ and $\mathcal{U}$ be an open covering for X . Suppose $\{\eta_\alpha : \alpha \in \Lambda\}$ is a partition of unity on Y subordinate to the cover $\mathcal{U}|_Y = \{U \cap Y : U \in \mathcal{U}\}$. Let $\psi_\alpha : \alpha \in \Lambda'\}$ be a partition of unity on X , subordinate to $\mathcal{U}$ such that

- (i) $\Lambda \subset \Lambda'$;
- (ii) For each $\alpha \in \Lambda$, $\psi_\alpha|_Y = \eta_\alpha$ and
- (iii) there is a neighbourhood V of Y in X , such that for each $\alpha \in \Lambda' \setminus \Lambda$, $\text{Supp } \psi_\alpha \cap V = \emptyset$.

We then call $\{\psi_\alpha : \alpha \in \Lambda'\}$ an extension of $\{\eta_\alpha : \alpha \in \Lambda\}$.

Remark 2.4.4 Thus, an ‘extended partition of unity’ consists of extensions of all the old functions to the larger space, together with some extra functions to cover up the rest

of the larger space. It is important that we make the technical assumption that these extra functions, do not enter into a neighbourhood of the smaller space.

Now the only pre-knowledge that we use here is that for every open covering $\mathcal{V}$ of $\mathbb{D}^n$, there is a partition of unity on $\mathbb{D}^n$ subordinate to the covering $\mathcal{V}$. We shall leave this to the reader as an assignment to learn this from somewhere, if not done already.

Lemma 2.4.5 Let $\mathcal{U}$ be an open covering of $\mathbb{D}^n$, $\{\eta_\alpha\}$ be a partition of unity on $\mathbb{S}^{n-1}$ subordinate to the restricted covering $\mathcal{U}|_{\mathbb{S}^{n-1}} = \{U \cap \mathbb{S}^{n-1} : U \in \mathcal{U}\}$. Then there is an extension of $\{\eta_\alpha\}$ to a partition of unity on $\mathbb{D}^n$ subordinate to $\mathcal{U}$.

Proof: Since $\mathbb{S}^{n-1}$ is compact and since the partition of unity is locally finite, it follows that only finitely many η_α are non zero. For each such α , let the support of η_α be contained in $U_\alpha \cap \mathbb{S}^{n-1}$, say. By Tietze's extension theorem there exist extensions $\eta'_\alpha : \mathbb{D}^n \rightarrow \mathbb{I}$ of η_α with support contained in U_α . (For, we can, if necessary multiply by a continuous function $g : \mathbb{D}^n \rightarrow \mathbb{I}$ which is 0 on U_α^c and 1 on the support of η_α .) For some $0 < \epsilon < 1$, put

$$V := V_\epsilon = \left\{ x \in \mathbb{D}^n : \sum_{\alpha} \eta'_\alpha > 1 - \epsilon \right\}$$

Then V_ϵ is a neighbourhood of $\mathbb{S}^{n-1}$ in $\mathbb{D}^n$. Let F be the closed set where $\sum_{\alpha} \eta'_\alpha = 0$. Choose a finite subcover $\{U_1, \dots, U_k\}$ from $\mathcal{U}$ for F and let $\{\phi_1, \dots, \phi_r\}$ be a partition on unity on $\mathbb{D}^n$ subordinate to $\{U_1, \dots, U_k, \mathbb{D}^n \setminus F\}$. Finally, let $\phi : \mathbb{D}^n \rightarrow \mathbb{I}$ be a continuous function such that $\phi(F) = 1$ and such that $\text{Supp } \phi \subset V_\epsilon^c$. If we set up

$$\Phi = \sum_{\alpha} \eta'_\alpha + \phi \left(\sum_{i=1}^r \phi_i \right),$$

then Φ does not vanish on $\mathbb{D}^n$ at all. Therefore, we can put

$$\hat{\eta}_\alpha = \eta'_\alpha / \Phi : \quad \lambda_i = \phi \phi_i / \Phi.$$

Since $\eta_\alpha = \eta'_\alpha$ on $\mathbb{S}^{k-1}$, $\Phi \equiv 1$ on $\mathbb{S}^{n-1}$ and hence $\hat{\eta}_\alpha = \eta'_\alpha = \eta_\alpha$ on $\mathbb{S}^{k-1}$. It follows that $\{\hat{\eta}_\alpha\} \cup \{\lambda_1, \dots, \lambda_r\}$ is a partition of unity on $\mathbb{D}^n$, subordinate to $\mathcal{U}$ and an extension of $\{\eta_\alpha\}$. ♠

Remark 2.4.6 It may be noted that if the set $F = \emptyset$ then bringing the extra functions λ_i 's is not necessary.

Proposition 2.4.7 Let X be obtained from Y by attaching k -cells $\{e_j^k : j \in J\}$. Let $\mathcal{U}$ be an open covering of X and $\{\eta_\alpha : \alpha \in \Lambda\}$ be a partition of unity on Y subordinate to $\mathcal{U}|_Y$. Then there is an extension of $\{\eta_\alpha\}$ to a partition of unity on X subordinate to $\mathcal{U}$.

Proof: For each $j \in J$, let $f_j : \mathbb{S}^{k-1} \rightarrow Y$ be the attaching map and $\phi_j : \mathbb{D}^k \rightarrow X$ be the characteristic map. Consider $\{\eta_\alpha \circ f_j\}$ which is a partition of unity on $\mathbb{S}^{k-1}$ subordinate to the cover $f_j^{-1}(\mathcal{U})$. Apply the above lemma to obtain an extension $\{\psi_\alpha : \alpha \in \Lambda \sqcup \Lambda_j\}$ on $\mathbb{D}^k$, subordinate to the cover $\phi^{-1}(\mathcal{U})$. For each j , we also have a neighbourhood V_j of $\mathbb{S}^{k-1}$ in $\mathbb{D}^k$ such that $\text{supp}(\psi_\alpha) \subset V_j^c$. Clearly, each of these maps factors down to a continuous function on the closed cell e_j^k which can be extended continuously by 0 on the cells $e_{j'}^k, j' \neq j$ to functions $\phi_\alpha : X \rightarrow [0, 1]$ in an obvious way using the properties (ii) and (iii) in the definition 2.4.3. Take $\Lambda' = \Lambda \sqcup (\sqcup_j \Lambda_j)$. Putting $V = Y \cup (\cup_j \phi(V_j))$, we get a neighbourhood of Y such that $\text{supp} \phi_\alpha \subset V^c$ for all $\alpha \notin \Lambda$.

The family $\{\phi_\alpha : \alpha \in \Lambda'\}$ is the required partition of unity which is an extension of $\{\eta_\alpha, \alpha \in \Lambda\}$. ♠

Theorem 2.4.8 Let X be a CW complex and $\mathcal{U}$ be an open covering for X . Then there exists a partition on X which is subordinate to $\mathcal{U}$.

Proof: Take $\Lambda_0 = X^{(0)}$ and define $p_{0,x} : X^{(0)} \rightarrow \mathbb{I}$ by the formula

$$p_{0,x}(y) = \begin{cases} 1 & \text{if } y = x \\ 0, & \text{otherwise.} \end{cases}$$

Since $X^{(0)}$ is discrete, this is clearly a partition of unity on $X^{(0)}$. Inductively, by the previous proposition, we get

(i) A sequence of indexing sets

$$\Lambda_0 \subset \Lambda_1 \subset \dots$$

(ii) partitions of unity $\{p_{n,\alpha} : \alpha \in \Lambda_n\}$ on $X^{(n)}$ which are extensions of $\{p_{n-1,\alpha}\}$

(iii) neighbourhoods V_n of $X^{(n)}$ in $X^{(n+1)}$ such that $\text{Supp}(p_{n+1,\alpha}) \subset V_n^c$ for all $\alpha \notin \Lambda_n$.

Now, take $\Lambda = \cup_n \Lambda_n$ and for each $\alpha \in \Lambda$, define $p_\alpha : X \rightarrow \mathbb{I}$ by the property $p_\alpha|_{X^{(n)}} = p_{n,\alpha}|_{X^{(n)}}$ for n sufficiently large. Clearly, each $p_{n,\alpha}$ is well defined and continuous.

Given any $\alpha \in \Lambda$, say $\alpha \in \Lambda_n \setminus \Lambda_{n-1}$, then $p_{n,\alpha}$ is first chosen on $X^{(n)}$ with its support inside $X^{(n)} \cap U_\alpha$ and iteratedly gets extended with support contained all the time in U_α . Therefore the family p_α is subordinate to $\mathcal{U}$.

Given any $x \in X$, say $x \in X^{(n)}$. Since the family $\{p_{n,\alpha} : \alpha \in \Lambda_n\}$ is a partition of unity restricted to $X^{(n)}$, we get a neighbourhood W_x of x in $X^{(n)}$ such that W_x intersects supports of only finitely many of $\{p_{n,\alpha} : \alpha \in \Lambda_n\}$. Now as in Proposition 2.1.6 we can extend this neighbourhood to a neighbourhood W_x^{n+1} inside V_{n+1} in $X^{(n+1)}$ and so on to get a nested sequence

$$W_x \subset W_x^{(n+1)} \subset \dots$$

Clearly then $\hat{W}_x = \cup_{k \geq n+1} W_x^{(k)}$ is a neighbourhood of x in X which will not meet support of any of $p_\alpha : \alpha \notin \Lambda_n$. This verifies that the family $\{p_\alpha : \alpha \in \Lambda\}$ is locally finite. ♠

Corollary 2.4.9 Every CW-complex is paracompact.

Exercise 2.4.10

- (i) Show that the compactly generated topology on $\mathbb{R}^\infty$ is finer than the metric topology, by giving an example of a subset $F \subset \mathbb{R}^\infty$ which is closed in the compactly generated topology but not closed in the metric topology.
- (ii) Let X be a paracompact space and $\mathcal{U}$ be an open cover for X . Show that there exists a family of continuous functions $\theta_j : X \rightarrow \mathbb{I}$ subordinate to $\mathcal{U}$ and such that for every $x \in X$, $\max_j \theta_j(x) = 1$.

2.5 Product of Cell Complexes

We shall now study finite Cartesian product of CW-complexes. Experience shows that Cartesian product neither misbehaves totally nor behaves all too well with respect to several topological/geometric structures. CW-structure is one such instance of this. In this section, we shall study how to give CW-structure to product spaces and obtain conditions under which this construction yields the standard product topology.

Example 2.5.1 Let us begin with an example. Let us fix a CW-structure on $\mathbb{S}^1$ with one 0-cell and one 1-cell as in the previous example (see Figure 2.3(a)). Let us denote them by σ_0, σ_1 . Similarly, let us fix a CW-structure on the unit interval $\mathbb{I}$ with two 0-cells and one 1-cell. We would like to obtain a CW-structure on the cylinder $\mathbb{S}^1 \times \mathbb{I}$. For the 0-skeleton we take the obvious candidate, viz., $\{(\sigma_0, 0), (\sigma_0, 1)\}$, being the product space of the 0-skeleton of the two spaces. What should be the 1-skeleton. We observe that the product of a 1-cell and a 0-cell is again a 1-cell and there are three different possibilities

of obtaining a 1-cell from this process, viz., $\sigma_1 \times \{0\}, \sigma_1 \times \{1\}, \sigma_0 \times \mathbb{I}$. Finally the product of a 1-cell and a 1-cell yields a 2-cell, viz., $\sigma_1 \times \mathbb{I}$. (See the Figure 2.4.) This process can actually be generalized completely, at least when both the CW-complexes are finite (hence compact).

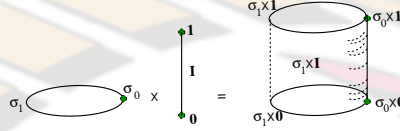


Figure 2.4. Product of CW-complexes

Definition 2.5.2 We fix once for all a set of homeomorphisms $h_{m,n} : \mathbb{D}^{n+m} \longrightarrow \mathbb{D}^n \times \mathbb{D}^m$. Given two CW-complexes X, Y , we define the product CW-complex $X \times_w Y$ as follows: The underlying set is the product set $X \times Y$. For each cell σ in X and a cell τ in Y with characteristic maps ϕ and ψ , respectively, we take a product cell $\sigma \times \tau$ in $X \times Y$ with its characteristic map $(\phi \times \psi) \circ h_{m,n} : \mathbb{D}^{m+n} \longrightarrow X \times Y$. In particular, observe that the 0-skeleton $(X \times_w Y)^{(0)} = X^{(0)} \times Y^{(0)}$. Inductively define the k -skeleton $(X \times_w Y)^{(k)}$ of $(X \times_w Y)$ to be the space obtained from the $(k-1)$ -skeleton by attaching all possible product cells $\sigma \times \tau$ where σ, τ range over cells of X, Y , respectively with the condition $\dim \sigma + \dim \tau = k$. Finally give the weak topology on $X \times_w Y = \bigcup_{k \geq 0} (X \times_w Y)^{(k)}$, viz., a subset S is closed iff $S \cap (X \times_w Y)^{(k)}$ is closed in $(X \times_w Y)^{(k)}$.

Example 2.5.3 Let us consider the cell structure for $\mathbb{S}^1 \times \mathbb{S}^1$. To begin with we must fix some cell structure on $\mathbb{S}^1$ explicitly. We consider $\mathbb{S}^1$ as the standard subspace of $\mathbb{C}$. Take the 0-cell to be the point 1 and the characteristic map of the 1-cell to be $\phi : \mathbb{I} \rightarrow \mathbb{C}$ where $\phi(t) = e^{2\pi it}$.

Now we look at the product CW-structure on the subspace $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^2$. There is just one 0-cell, viz., the point $(1, 1)$, whereas there are two 1-cells given by $\phi_1(t) = (\phi(t), 1)$ and $\phi_2(t) = (1, \phi(t))$. Finally there is just one 2-cell given by $\phi \times \phi : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{C} \times \mathbb{C}$ which fills up the subspace $\mathbb{S}^1 \times \mathbb{S}^1$. It is interesting to note that the attaching map of this 2-cell traces the first circle $\mathbb{S}^1 \times 1$ then the second circle $1 \times \mathbb{S}^1$ then again the first circle but in the opposite direction and finally the second circle, again in the opposite direction. Therefore in the fundamental group this loop represents the element $xyx^{-1}y^{-1} \in \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$.

Remark 2.5.4 Returning to the study of $X \times Y$ in general, it is not difficult to see that the images of the product cells cover $X \times Y$ as σ ranges over all cells of X and τ

ranges over all cells of Y and hence the underlying set of $X \times_w Y$ is $X \times Y$. Our major concern now is to compare the weak topology with the product topology. Note that for each 0-cell x_0 of X , the $x_0 \times Y$ is a subcomplex of $X \times_w Y$. Similarly, for each 0 cell y_0 of Y , $X \times y_0$ is a subcomplex $X \times_w Y$. A product cell can be thought of as obtained by attaching $\mathbb{D}^m \times \mathbb{D}^n$ with the characteristic map as the product of the two characteristic maps, instead of going to the round disc $\mathbb{D}^{m+n}$. (This is what we have done in the above example.) In particular, all the closed cells of $X \times_w Y$ are closed and compact subsets of the product space $X \times Y$. As it turns out, there is a much closer relation between the product topology on $X \times Y$ and the weak topology on $X \times_w Y$ though, in general, they may not be the same. To understand this, we begin with recalling one of the fundamental properties of co-induced topology. Not being familiar with it is going to cause problems for us at some stage or other.

Theorem 2.5.5 *Let $(X_i, \mathcal{T}_i)$ be a family of topological spaces and $f_i : X_i \rightarrow X$ be any given set theoretic functions. Then there exists a unique maximal topology $\mathcal{T}$ on X such that each f_i is continuous. This topology is called the topology on X coinduced by the collection $\{f_i\}$. A subset A of X is open (closed) in $\mathcal{T}$ iff $f_i^{-1}(A)$ is open (closed) in $\mathcal{T}_i$ for every i .*

Theorem 2.5.6 *$X = \sqcup_i X_i$ where each X_i is a topological space. Let $\mathcal{T}$ be the topology on X , coinduced by the inclusion maps $\eta_i : X_i \rightarrow X$.*

- (a) *A subset $A \subset X$ is open (closed) in $\mathcal{T}$ iff $A \cap X_i$ is open (closed) in X_i for each i .*
- (b) *Let $q : \sqcup_i X_i \rightarrow X$ be the quotient function $q(x) = \eta_i(x)$ for $x \in X_i$. Then the coinduced topology on X is nothing but the quotient topology from the disjoint union topology on $\sqcup_i X_i$.*
- (c) *Given $A \subset V \subset X$, V is a neighbourhood of A in $(X, \mathcal{T})$ iff $X_i \cap V$ is a neighbourhood of $X_i \cap A$ in X_i , for all i .*

Proof: (a) is an immediate consequence of the last statement of the previous theorem.

(b) is an immediate consequence of (a).

(c) The ‘only if’ part is obvious. Conversely, select open subsets U_i of X_i such that $X_i \cap A \subset U_i \subset X_i \cap V$ for each i . Then $U := q(\sqcup_i U_i)$ is open in X because of (b). Clearly, $A \subset U \subset V$. ♠

Theorem 2.5.7 *Let X be a set and $X = \sqcup_i X_i$, where each X_i is given a topology $\mathcal{T}_i$. Assume that the following compatibility conditions are satisfied:*

(CC1) For each pair (i, j) , the two topologies on $X_i \cap X_j$ induced from $(X_i, \mathcal{T}_i)$ and $(X_j, \mathcal{T}_j)$ are the same.

(CC2) For each pair (i, j) , $X_i \cap X_j$ is an open (respectively, closed) subspace of both X_i and X_j . Let $(X, \mathcal{T})$ be the coinduced topology. Then the subspace topology $\mathcal{T}_{X_i}$ for each X_i coincides with $\mathcal{T}_i$ for all i . Moreover, each X_i is open (or closed) in X , according to the condition (CC2).

Proof: (Note that for an affirmative answer condition (CC1) is a must. So, at this stage we are apologetic about (CC2) only.)

We need to check $\mathcal{T}_i = \mathcal{T}|_{X_i}$, for each i . Of course, by the definition of $\mathcal{T}$ it is clear that $\mathcal{T}|_{X_i}$ is contained in $\mathcal{T}_i$. For the converse, suppose $A \subset X_i$ is open in X_i . In the first case when X_i is open in X , it follows that A is open in X and we are through. In the second case when X_i is closed in X , we have $X_i \setminus A$ is closed in X_i and hence is closed in X . Therefore $X \setminus (X_i \setminus A)$ is open in X . But then $A = X_i \cap (X \setminus (X_i \setminus A))$. Therefore $A \in \mathcal{T}|_{X_i}$. This proves that $\mathcal{T}$ is the coherent topology. The last assertion in the theorem is obvious. ♠

2.6 Compactly Generated Topology, k -spaces

We shall assume that all spaces in this section are Hausdorff space.

Lemma 2.6.1 Given any topological space X , let X_w denote X with the topology coinduced from the collection of compact subsets of X , i.e., a subset S of X is closed in X_w iff $S \cap K$ is closed in K for each compact subset K of X . Then

- (a) The identity map $Id : X_w \longrightarrow X$ is continuous.
- (b) Id is a homeomorphism iff X is compactly generated.
- (c) Id defines a bijection of compact sets in the two topologies.

Proof: (a), (b) are straightforward, being just a rewording of the definition. In (c), again if K is compact in X_w , by continuity of Id , K is compact in X . To see the converse, let K be a compact set in X . Let $\mathcal{F}$ be a family of subsets of K closed in X_w with finite intersection property. By the very definition, X_w , for each $F \in \mathcal{F}$, we have $F = F \cap K$ is a closed subset of K . Since K is compact, it follows that $\bigcap \{F : F \in \mathcal{F}\} \neq \emptyset$. ♠

Lemma 2.6.2 The identity map $Id : (X \times Y)_w \longrightarrow X \times Y$ is a homeomorphism if one of the following holds:

- (a) X and Y are compact.
- (b) X is compactly generated and Y is locally compact.
- (c) X is compactly generated and Y is compact and Hausdorff.

Proof: (a) Since X and Y are compact, so is $X \times Y$. For any compact space, the compactly generated topology coincides with the given topology.

(b) Let U be an open subset of $(X \times Y)_w$ and $(x_0, y_0) \in U$. Let K be a compact neighbourhood of y_0 in Y . Then $U \cap \{x_0\} \times K$ is open in $\{x_0\} \times K$ and hence there exists a compact neighbourhood K' of y_0 such that $\{x_0\} \times K' \subset U$. Let $V = \{x \in X : \{x\} \times K' \subset U\}$. Let L be a compact subset of X , such that $x_0 \in L$. Then $U \cap L \times K'$ is a neighbourhood of $x_0 \times K'$ in $L \times K'$. Hence by Wallman's theorem, there is an open subset W of L such that $x_0 \in W$ and $W \times K' \subset U$. This means $W \subset V$. Therefore, $V \cap L$ is a neighbourhood of x_0 in L . By theorem 2.5.6 (c), it follows that V is a neighbourhood of x_0 in X . Since $V \times K' \subset U$ we have proved that U is a neighbourhood of (x_0, y_0) in $X \times Y$. Therefore, U is open in $X \times Y$.

(c) A compact Hausdorff space is locally compact and hence (c) follows from (b). ♠

Theorem 2.6.3 *Let X, Y be any two CW-complexes. We have:*

- (a) $(X \times_w Y) = (X \times Y)_w$.
- (b) *The identity map $Id : X \times_w Y \rightarrow X \times Y$ is continuous i.e., the CW-topology is finer than the product topology.*
- (c) *Id is a homeomorphism iff the product topology is compactly generated.*
- (d) *The identity map $X \times_w Y \rightarrow X \times Y$ defines a bijection of compact subsets in the two topologies.*

Proof: (a) The family $\mathcal{F}$ of product cells $\{\sigma \times \tau\}$ as σ, τ runs over all cells of X and Y , respectively, is a cover of $X \times Y$ consisting of compact subsets. Therefore the coinduced topology of $X \times_w Y$ from this family is finer than the weak topology on $(X \times Y)_w$. The equality follows from the observation that every compact subset of $X \times Y$ is covered by finitely many members of $\mathcal{F}$. Details are left to the reader.

Other conclusions now follow from Lemma 2.6.1. ♠

Definition 2.6.4 A CW-complex, X is said to be locally countable, if every open cell meets only countably many closed cells.

Lemma 2.6.5 In a locally countable CW-complex X , every point x is contained in a countable subcomplex which is a neighbourhood of x in X .

Proof: Let x belong to the interior of e^m , and K_0 be any finite subcomplex containing e^m . Now for each closed cell e^k which intersects $\text{int } e^m$, there is a finite subcomplex that contains the closure of e^k . Take the union of all these finite subcomplexes to get a countable subcomplex K_1 . Carry on with this procedure to define subcomplexes,

$$K_0 \subset K_1 \subset \dots$$

and take $K = \bigcup_n K_n$. Then clearly $x \in K$ and K is a countable subcomplex.

To see that K is a neighbourhood of x , start with a neighbourhood U_0 of x in K_0 . Let U_1 be an extension of U_0 to the next skeleton in X as in Proposition 2.1.6. It is clear that $U_1 \subset K_1$. Repeating this argument, we get a neighbourhood U_i of x in X which is extension U_{i-1} and each $U_i \subset K_i$. Therefore, $U = \bigcup_i U_i$ will be a neighbourhood of x in X and $U \subset K$. ♠

We can now give a number of instances when the product topology coincides with the weak topology for the product complex $X \times_w Y$.

Theorem 2.6.6 *Let X, Y be any two CW-complexes. The product topology coincides with the CW-topology on $X \times Y$ in the following instances.*

- (1) X, Y are finite.
- (2) If either X or Y is finite.
- (3) If X or Y is locally compact (or equivalently, locally finite).
- (4) If both X and Y have countably many cells.
- (5) If X and Y are both locally countable.

Proof: (1), (2), (3) directly follow from (a), (b), (c) of Lemma 2.6.2. To prove (4), let W be an open subset of $X \times_w Y$ and $(x_0, y_0) \in W$ be an arbitrary point. It is enough to produce neighbourhoods U of x_0 in X and V of y_0 in Y such that $U \times V \subset W$. Write $X = \bigcup_{n \geq 0} K_n, Y = \bigcup_{n \geq 0} L_n$ as an increasing union of finite subcomplexes such that $x_0 \in K_0, y_0 \in L_0$. First thing to note is that the CW-topology on X (similarly on Y) is equal to the coinduced topology by the family $\{K_n\}$ (by the family $\{L_n\}$, respectively.)

In particular, $W \cap (K_n \times L_n)$ is a neighbourhood of (x_0, y_0) in $K_n \times L_n$. So, we can choose neighbourhoods U_n, V_n of x_0, y_0 in K_n, L_n , respectively, such that $U_n \cap K_{n-1} = U_{n-1}, V_n \cap L_{n-1} = V_{n-1}$, and $U_n \times V_n \subset W \cap (K_n \times L_n)$. Put $U = \bigcup_n U_n, V = \bigcup_n V_n$. Then it follows from theorem 2.5.6(c) that U and V are neighbourhoods of x_0 and y_0 in X, Y respectively. Clearly, $U \times V \subset W$ and we are through.

Finally to prove (5), Given $(x_0, y_0) \in X \times_w Y$, by the previous lemma, there are countable subcomplexes $K \subset X, L \subset Y$ such that $x_0 \in K, y_0 \in L$ and K and L are

neighbourhoods of x_0 and y_0 , respectively, in X and Y . Fix $x_0 \in A \subset K, y_0 \in B \subset Y$ such that $A \subset X$ and $B \subset Y$ are open.

Now, let W be open in $X \times_w Y$ and $(x_0, y_0) \in W$. Then $W \cap K \times_w L$ is open in $K \times_w L = K \times L$ from (4). Therefore, there are open sets $U \in K$ and $V \in L$ such that $U \times V \subset W$. But then $A \cap U$ is open in U and therefore open in X . Similarly, $B \cap V$ is open in B and hence open in Y . Clearly, $(x_0, y_0) \in (A \cap U) \times (B \cap V) \subset W$. ♠

Example 2.6.7 (Dowker [?]). This example tells us that we may not be able to generalize the results in the above theorem any further. Let $\mathbb{N}$ denote the set of natural numbers and $\mathcal{L}$ denote the set of all functions $\phi : \mathbb{N} \rightarrow \mathbb{N}$. Consider the real vector spaces $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}^{\mathcal{L}}$ (direct sums) with the topology co-induced by finite dimensional vector subspaces. Let $\mathbf{v}_n, \mathbf{u}_\phi$ denote the standard basis elements in $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}^{\mathcal{L}}$, respectively. Define

$$X = \{r\mathbf{v}_n : n \in \mathbb{N}, 0 \leq r \leq 1\} \subset \mathbb{R}^{\mathbb{N}}; \quad Y = \{r\mathbf{u}_\phi : \phi \in \mathcal{L}, 0 \leq r \leq 1\} \subset \mathbb{R}^{\mathcal{L}}.$$

Then both X and Y are 1-dimensional CW-complexes which are nothing but the 1-point-union of edges indexed by $\mathbb{N}$ and $\mathcal{L}$, respectively. We claim that the product CW-topology on $X \times Y$ is strictly finer than the product topology, i.e., considered as a subspace of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathcal{L}}$. Define

$$P = \left\{ \left(\frac{\mathbf{v}_n}{\phi(n)}, \frac{\mathbf{u}_\phi}{\phi(n)} \right) : n \in \mathbb{N}, \phi \in \mathcal{L} \right\}.$$

Observe that P consists of precisely one element from the interior of each 2-cell in $X \times Y$ and hence is a discrete closed subset of $(X \times Y)_w$. (See Lemma 2.3.4.) However, we shall see that the origin 0 of $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathcal{L}}$ is in the closure of P in the product topology. So, let U, V be neighbourhoods of the origin in $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}^{\mathcal{L}}$, respectively. Then for each n and each ϕ there exist $r_n, s_\phi \in (0, 1]$ such that

$$\{\lambda \mathbf{v}_n : 0 \leq \lambda \leq r_n\} \subset U; \quad \{\lambda \mathbf{u}_\phi : 0 \leq \lambda \leq s_\phi\} \subset V.$$

Consider $\psi : \mathbb{N} \rightarrow \mathbb{N}$ given by

$$\psi(n) = \max \left\{ n, \left\lfloor \frac{1}{r_n} \right\rfloor \right\} + 1.$$

Clearly $\psi(n) \rightarrow \infty$ as $n \rightarrow \infty$ and hence we can choose m such that $\psi(m) > \frac{1}{s_\psi}$. It follows that $(\frac{1}{\psi(m)}\mathbf{v}_m, \frac{1}{\psi(m)}\mathbf{u}_\psi) \in P \cap (U \times V)$. This proves that P is not closed in the product topology.

2.7 Homotopical Aspects

We shall now study the homotopical aspects of cell complexes, which is central to our study. This section will contain only the basics of this aspect on which the entire edifice of algebraic topology will be built.

Theorem 2.7.1 *Let X be a CW-complex. A function $f : X \times \mathbb{I} \rightarrow Y$ is continuous iff the restricted functions $f|_{X^{(k)} \times \mathbb{I}} : X^{(k)} \times \mathbb{I} \rightarrow Y$ are all continuous for $k \geq 0$.*

Proof: Since $\mathbb{I}$ is compact, $Id : (X \times \mathbb{I})_w \rightarrow X \times \mathbb{I}$ is a homeomorphism. Therefore f is continuous iff $f|_{(X \times \mathbb{I})^{(k)}} : (X \times \mathbb{I})^{(k)} \rightarrow Y$ is continuous for each k . Observe that for each k , $(X \times_w \mathbb{I})^{(k)} \subset X^{(k)} \times \mathbb{I} \subset (X \times_w \mathbb{I})^{(k+1)}$ are closed subsets. The claim follows. ♠

The following lemma is central to the homotopical aspects of CW-complexes.

Lemma 2.7.2 *Let U be an open (or a closed) subset of X where (X, A) is a relative CW-complex. Put $U_{-1} = U \cap A$; and $U_n = U \cap (X, A)^{(n)}$, $n \geq 0$. Suppose U_{n-1} is a SDR of U_n for each n . Then U_{-1} is SDR of U .*

Proof: Let $F_n : U_n \times \mathbb{I} \rightarrow U_n$ be a homotopy such that $F_n(x, t) = x$ for all $x \in U_{n-1}$, $t \in \mathbb{I}$ and $F_n(x, 0) = x$ for all $x \in U_n$. Put $f_n(x) = F_n(x, 1)$. Then clearly, $f_n : U_n \rightarrow U_{n-1}$ is a SDR for each n . Therefore taking the composites, viz., $g_n = f_0 \circ f_1 \circ \cdots \circ f_n$, we get a SDR $g_n : U_n \rightarrow U_{-1}$. Observe that $g_{n+1}|_{U_n} = g_n$ for all n . Take $g(x) = g_n(x)$ whenever $x \in U_n$. Then $g : U \rightarrow U_{-1}$ is well-defined, continuous and a retraction. However, to show that it is a SDR needs a little more effort.

Let us define $G_n : U_n \times \mathbb{I} \rightarrow U_n$ inductively as follows:

$$G_0(x, t) = \begin{cases} x, & 0 \leq t \leq \frac{1}{2}; \\ F_0(x, 2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

For $n \geq 1$, having defined G_{n-1} , now define

$$G_n(x, t) = \begin{cases} x, & 0 \leq t \leq \frac{1}{n+2}; \\ F_n(x, (n+1)[(n+2)t - 1]), & \frac{1}{n+2} \leq t \leq \frac{1}{n+1}; \\ G_{n-1}(f_n(x), t), & \frac{1}{n+1} \leq t \leq 1. \end{cases}$$

Inductively, we now verify that each G_n is a SDR of U_n into U_{-1} . Moreover, $G_n|_{U_{n-1} \times \mathbb{I}} = G_{n-1}$. Therefore, there is a well defined map $G : U \times \mathbb{I} \rightarrow U$ given by $G(x, t) = G_n(x, t)$

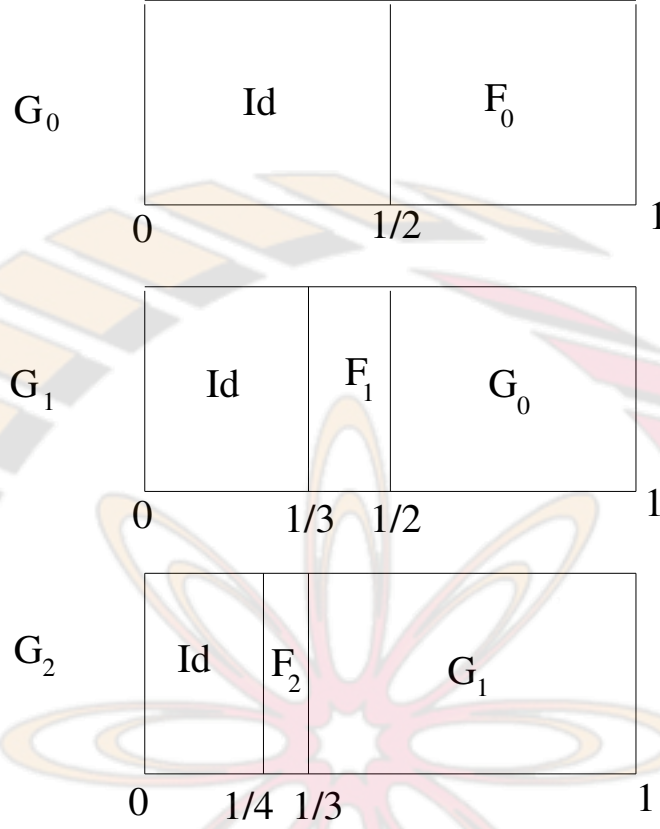


Figure 2.5. Construction of Iterated Homotopy

whenever $x \in U_n$. If V is an open subset of U then $G^{-1}(V) \cap (X, A)^{(n)} = G^{-1}(V) \cap U_n = G_n^{-1}(V \cap (X, A)^{(n)})$ and hence is open in U_n for each n . This means that $G^{-1}(V)$ is open in U . Therefore, G is continuous. It is easily verified that G is a SDR of U into U_{-1} . ♠

Remark 2.7.3 The above proof actually can be modified slightly to show that each U_n is a SDR of U .

Theorem 2.7.4 *Every CW-complex is locally contractible.*

Proof: Given $x \in V \subset X$, where V is an open set, we shall construct a neighbourhood $U \subset V$ of x which is contractible. Using the previous lemma, this construction is done inductively. To begin with, there is a unique open cell e^k in X to which x belongs. First choose a contractible neighbourhood U_k of x in e^k so that the closure $\bar{U}_k$ is contained in $V \cap e^k$. For $n \geq k$, having constructed a neighbourhood U_n so that the closure $\bar{U}_n \subset X^{(n)} \cap V$ is compact and such that U_{n-1} is a SDR of U_n , the inductive step is carried out as follows. Let Λ be the indexing set of all $(n+1)$ -cells. For each $\alpha \in \Lambda$,

it follows that $\phi_\alpha^{-1}(\bar{U}_n)$ is a compact subset of $\mathbb{S}^n$ contained in the open set $\phi_\alpha^{-1}(V)$. Therefore (by Wallman's theorem) there exists $0 < \epsilon(\alpha) < 1$ such that

$$\overline{N_{\epsilon(\alpha)}(\phi_\alpha^{-1}(\bar{U}_n))} \subset \phi_\alpha^{-1}(V).$$

(See, Remark 2.1.5.) With this choice of $\epsilon : \Lambda \rightarrow (0, 1)$, let $U_{n+1} = N_\epsilon(U_n)$ as defined in Proposition 2.1.6. Now we take $U = \cup_n U_n$. From Lemma 2.7.2, it follows that U is a contractible neighbourhood of x in V . ♠

Lemma 2.7.5 Let X be obtained from Y by attaching k -cells, then $X \times 0 \cup Y \times \mathbb{I}$ is a strong deformation retract of $X \times \mathbb{I}$.

Proof: Let $\{f_\alpha : \alpha \in \Lambda\}$ be the family of attaching maps. One of the standard result that we have proved in Part-I is that there is a strong deformation retraction

$$h : \mathbb{D}^k \times \mathbb{I} \rightarrow \mathbb{D}^k \times 0 \cup \mathbb{S}^{k-1} \times \mathbb{I}$$

Let

$$H : \sqcup_\alpha D_\alpha^k \times \mathbb{I} \rightarrow \sqcup_\alpha (\mathbb{D}_\alpha^k \times 0 \cup \mathbb{S}_\alpha^{k-1} \times \mathbb{I})$$

be just the disjoint union of copies of h .

Let $f : \sqcup_\alpha \mathbb{S}_\alpha^{k-1} \times \mathbb{I} \rightarrow Y \times \mathbb{I}$ be given by

$$f(x, t) = (f_\alpha(x), t), \quad x \in \mathbb{S}_\alpha^{k-1}.$$

It follows that $X \times \mathbb{I}$ is the quotient of

$$Y \times \mathbb{I} \sqcup_\alpha D_\alpha^k \times \mathbb{I}$$

by the one single identification

$$(x, t) \sim (f(x), t).$$

Now the deformation retract H factors down to define a deformation retract

$$X \times \mathbb{I} \rightarrow X \times 0 \cup Y \times \mathbb{I}.$$

Another landmark result that we have proved in Part-I is the following. We quote it here for the ready reference:

Proposition 2.7.6 Let A be any subspace of X . Then (X, A) has HEP with respect to every space, i.e., $A \hookrightarrow X$ is a cofibration iff A is a closed subspace of X and the subspace $Z = A \times \mathbb{I} \cup X \times 0$ is a retract of $X \times \mathbb{I}$.

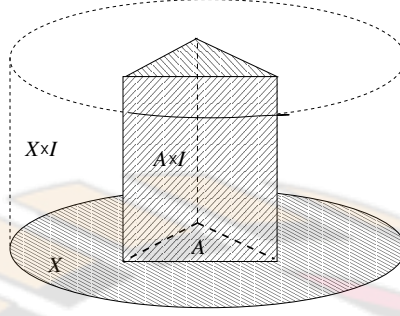


Figure 2.6. A retract and homotopy extension property

Theorem 2.7.7 *If (X, A) is a relative CW-complex, then $A \hookrightarrow X$ is a cofibration.*

Proof: By Proposition 2.7.6, it is enough to see that $X \times 0 \cup A \times \mathbb{I}$ is a retract of $X \times \mathbb{I}$. Applying the above lemma successively to various skeletons of X , we get retractions $r_w : X^{(k)} \times \mathbb{I} \rightarrow X^{(k)} \times 0 \cup X^{(k-1)} \times \mathbb{I}$. Now define $r : X \times \mathbb{I} \rightarrow X \times 0 \cup A \times \mathbb{I}$ by the formula $r|_{X^{(k)} \times \mathbb{I}} = r_0 \circ r_1 \circ \cdots \circ r_w$. Verify that r is well defined continuous and is identity on $X \times 0 \cup A \times \mathbb{I}$. ♠

Exercise 2.7.8

- (i) Show that a CW-complex X is locally compact iff each closed cell meets finitely many other closed cells. CW-complexes satisfying this condition are called locally finite. (Compare corollary 2.3.6.)
- (ii) If X and Y are CW-complexes then show that

$$[[\mathbb{S}^n, X \times_w Y]] \cong [[\mathbb{S}^n, X \times Y]] \cong [[\mathbb{S}^n, X]] \times [[\mathbb{S}^n, Y]], \text{ for all } n \geq 1.$$

2.8 Cellular Maps

In mathematical studies, the study of objects always goes hand in hand with the study of appropriate maps between them. In this section, we introduce the cellular maps from one CW-complex to another, thereby completing the introduction to the category of CW-complexes. The most important property of cellular maps stems from the fact that every continuous map can be approximated by a cellular map. This result goes under the name ‘cellular approximation theorem’ and is a part of the tool-kit in the study of algebraic topology. We shall indicate only a few applications here.

Definition 2.8.1 Let $f : (X, A) \rightarrow (Y, B)$ be a continuous function of CW-pairs. We say f is cellular, if $f((X, A)^{(q)}) \subset (Y, B)^{(q)}$ for all $q \geq 0$.

Remark 2.8.2 Thus a cellular map takes each q -skeleton of the domain inside the q -skeleton of the codomain. It is easy to check that the composite of cellular maps is again cellular and hence CW-complexes and cellular maps form a category as mentioned.

Theorem 2.8.3 (Cellular approximation theorem) *Let $f : (X, A) \rightarrow (Y, B)$ be any continuous map of CW-pairs. Suppose (X', A) is a subcomplex on which f is cellular. Then there exists a cellular map $g : (X, A) \rightarrow (Y, B)$ such that $g = f$ on (X', A) and $g \simeq f$ (rel X').*

We shall give a proof of this based on smooth approximations and Sard's theorem which you may have learnt in your calculus course (see [?] Theorem 1.7.2 and 2.2.1). In a later section, we shall indicate how to modify this proof so that we can use simplicial approximation theorem in place of calculus.¹ The basic idea is in the following lemma:

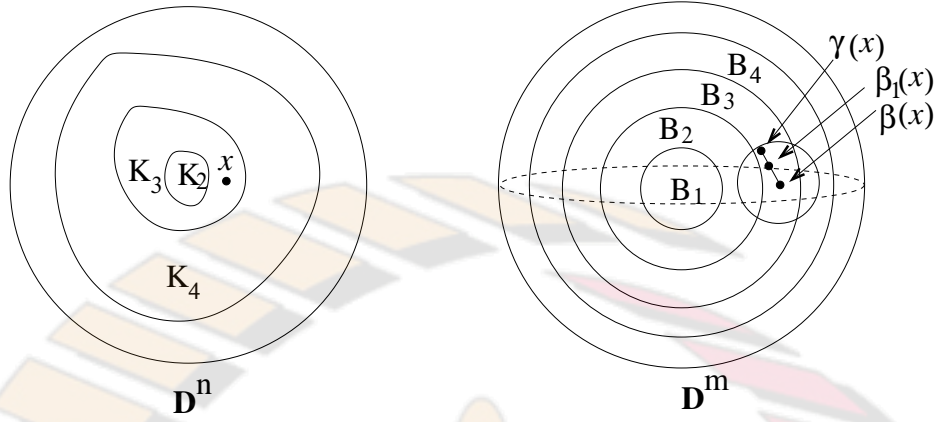
Lemma 2.8.4 Let $\alpha : (\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow (Y, B)$ be a continuous map, where Y is got by attaching a single cell e^m to B . If $m > n$, then there is a homotopy $H : \mathbb{D}^n \times \mathbb{I} \rightarrow Y$ such that $H(x, 0) = \alpha(x)$, $x \in \mathbb{D}^n$, $H(x, t) = \alpha(x)$, $x \in \mathbb{S}^{n-1}$, $0 \leq t \leq 1$, and $H(x, 1) \in B$, for all $x \in \mathbb{D}^n$.

Proof: Suppose the image of α misses a point $z \in \text{int } e^m$. If $r : Y \setminus \{z\} \rightarrow B$ is a strong deformation retraction (which exists), then α is homotopic to $r \circ \alpha$ and H can be taken to be such a homotopy. Therefore, it is enough to prove that there is a homotopy H of α to a map α_1 relative to $\mathbb{S}^{n-1}$ such that image of α_1 misses a point in the interior of e^m . Let $\phi : \mathbb{D}^m \rightarrow Y$ be the characteristic map of e^m . Choose $0 < r < 1/5$ and put

$$B_j = jr\mathbb{D}^m, \quad C_j = \phi(B_j), \quad K_j = \alpha^{-1}(C_j), \quad j = 1, 2, 3, 4.$$

We then have compact subsets $C_1 \subset \cdots \subset C_4 \subset \text{int } e^m$ and $K_1 \subset \cdots \subset K_4 \subset \text{int } \mathbb{D}^n$. Let $\beta = \phi^{-1} \circ \alpha : K_4 \rightarrow \mathbb{D}^m$. Let $\eta : \mathbb{D}^n \rightarrow \mathbb{I}$ be a smooth map such that $\eta \equiv 1$ on K_2 and $\equiv 0$ outside $\text{int } K_3$. Let $\gamma : \text{int } K_4 \rightarrow \mathbb{D}^m$ be a smooth r -approximation to β . Consider the homotopy $t \mapsto (1 - t\eta)\beta + t\eta\gamma$ from β to $\beta_1 = (1 - \eta)\beta + \eta\gamma$. On K_2 , β_1 is equal to γ and outside K_3 , it is equal to β . Hence β_1 can be extended over all of K_4 by β . Indeed, the same holds for the entire homotopy as well and hence we get a homotopy of β with the map β_1 on K_4 . (See Figure 2.7.)

¹A teacher/student who does not like to use smooth approximation theorem may post-pone the proof of this theorem till Section 2.9.

Figure 2.7. Homotoping away from a point of B_1

We claim that $\beta_1(K_4)$ does not contain some points in $r\mathbb{D}^m$. For, first of all, any point x outside K_2 is mapped by β outside of the ball $2r\mathbb{D}^m$. Since γ is a r -approximation to β , $\gamma(x)$ is in a ball of radius r with centre $\beta(x)$ which misses the ball $r\mathbb{D}^m$. But then $\beta_1(x)$ belongs to the line segment $[\beta(x), \gamma(x)]$ and hence cannot be in $r\mathbb{D}^m$. Therefore, $\beta_1(K_4) \cap r\mathbb{D}^m = \beta_1(K_2)$. Now β_1 is a smooth map on K_2 and since $n < m$, by Sard's theorem $\beta_1(K_2)$ is of measure zero. Therefore, $r\mathbb{D}^m \setminus \beta_1(\mathbb{D}^n) = r\mathbb{D}^m \setminus \beta_1(K_2) \neq \emptyset$.

Composing back with ϕ we obtain a homotopy $G : K_4 \times \mathbb{I} \rightarrow Y$ of α with a map $\alpha' = \phi \circ \beta_1$ which is constant outside K_3 and hence can be extended by α outside K_4 . This however does not affect its image inside $\phi(r\mathbb{D}^m)$, i.e., $\alpha'(\mathbb{D}^n)$ misses a point of $\text{int } e^m$.

♠

Proof of Theorem 2.8.3: For simplicity put $X^{(k)} = (X, X')^{(k)}$; $Y^{(k)} = (Y, B)^{(k)}$. We shall construct inductively a family $H_n : X \times \mathbb{I} \rightarrow (Y, B)$ such that

- (i) $H_0(x, 0) = f(x), x \in X$;
- (ii) $H_n(x, 0) = H_{n-1}(x, 1), x \in X, n \geq 1$;
- (iii) $H_n(x, t) = H_{n-1}(x, 1)$ for all $x \in X^{(n-1)}, 0 \leq t \leq 1$;
- (iv) $H_n(X^{(n)} \times 1) \subset Y^{(n)}$.

Then as in the proof of Lemma 2.7.2, we can piece them together to get a homotopy H as required, viz.,

$$H(x, t) = \begin{cases} H_{n-1}(x, (n+1)(nt - n - 1)), & \frac{n-1}{n} \leq t \leq \frac{n}{n+1}; \\ H_n(x, 1), & x \in X^{(n)}. \end{cases}$$

Note that $X^{(-1)} = (X', A)$ and we take H_0 to be identically f on it. If $x \in X^{(0)} \setminus X^{(-1)}$, choose a path ω_x from $f(x)$ to some point in $Y^{(0)}$ and define $H_0(x, t) = \omega_x(t)$. This gives

us a map $X \times 0 \cup X^{(-1)} \times \mathbb{I} \rightarrow Y$. Since $X^{(-1)} \subset X$ is a cofibration, this map extends to $H_0 : X \times \mathbb{I} \rightarrow Y$ as required. Inductively suppose we have defined H_{n-1} with properties as specified. For each n -cell e^n in $X^{(n)}$, we need to define homotopies $h_n : e^n \times \mathbb{I} \rightarrow Y$ such that $h_n|_{\partial e^n \times \mathbb{I}} = H_{n-1}|_{\partial e^n \times \mathbb{I}}$, $h_n(x, 0) = H_{n-1}(x, 0)$ and $h_n(e^n \times 1) \subset Y^{(n)}$. For once we have done this for each n -cell in $X^{(n)}$, we can put all h_n together, use the cofibration property of $X^{(n)} \subset X$ and get the map H_n as desired.

Now since $H_{n-1}(e^n \times 1)$ is a compact set it meets the interior of only finitely many cells in Y . If all these cells are of dimension $\leq n + 1$, there is nothing to prove. So, suppose e^m is a cell, $m > n$ and $H_{n-1}(e^n \times 1)$ meets the interior of e^m .

Composing with the characteristic map of the n -cell e^n in X and taking $Z = Y \setminus \text{int } e^m$ makes this problem precisely the same as in Lemma 2.8.4. Therefore we can find a homotopy of $H_{n-1}(-, 1)$ so as to avoid the interior of e^m . The required homotopy h_n is then the composite of all these finitely many homotopies.

Putting all these h_n together and using the cofibration property as mentioned, this completes the definition of H_n and thereby the proof of Theorem 2.8.3. ♠

Remark 2.8.5 Since composite of cellular maps is cellular, there is a category **CW** of CW-complexes and cellular maps. This category is of prime importance from the point of view of algebraic topology. We shall meet CW-complexes all over this book.

2.9 Higher connectivity and Homotopy Groups

??? Following the important Lemma 2.8.4 we now make a definition.

Definition 2.9.1 Let $n \geq 0$ be an integer. A topological pair (X, A) is said to be n -connected, if for each $0 \leq k \leq n$, every map $f : (\mathbb{D}^k, \mathbb{S}^{k-1}) \rightarrow (X, A)$ is homotopic relative to $\mathbb{S}^{k-1}$ to a map $g : \mathbb{D}^k \rightarrow A$. (Here by convention, for $n = 0$, $\mathbb{D}^0$ is a singleton space and $\mathbb{S}^{-1} = \emptyset$.) In particular, if $A = \{x_0\}$, where x_0 is the base point then we say X is n -connected.

Remark 2.9.2 Note that a based space is n -connected iff it is path connected and every map $\alpha : \mathbb{S}^k \rightarrow X$ is null-homotopic for every $1 \leq k \leq n$. (See Part-I notes.)

Theorem 2.9.3 The pair $(\mathbb{D}^{n+1}, \mathbb{S}^n)$ and the space $\mathbb{S}^{n+1}$ are n -connected, for all $n \geq 0$.

Proof: Each $\mathbb{S}^{k-1}$ is a CW-complex with a single 0-cell and a single $(k-1)$ -cell. Each $\mathbb{D}^k$ is a CW-complex of dimension k obtained by attaching a single n -cell to the CW-complex $\mathbb{S}^{k-1}$. The theorem follows immediately from the Lemma 2.8.4. ♠

The CW-approximation theorem has many applications. Here is just a sample.

Theorem 2.9.4 *Let X be a connected CW-complex. Then the inclusion induced homomorphism $\iota_{\#} : \pi_1(X^{(2)}) \rightarrow \pi_1(X)$ is an isomorphism.*

Proof: We choose one of the vertices $\star$ as the base point for X . Given a map $\alpha : (\mathbb{S}^1, 1) \rightarrow (X, \star)$, by the cellular approximation theorem, there is a homotopy $(\mathbb{S}^1, 1) \times \mathbb{I} \rightarrow (X, \star)$ relative to 1 of α to a map $\beta : (\mathbb{S}^1, 1) \rightarrow (X^{(1)}, \star)$. This clearly implies that $\iota_{\#}([\beta]) = [\alpha]$ and hence the surjectivity of $\iota_{\#}$. The injectivity is proved similarly, by starting with a map $f : (\mathbb{D}^2, \mathbb{S}^1, 1) \rightarrow (X, X^{(1)}, \star)$. ♠

Exercise 2.9.5

- (i) Show that a CW-complex X is connected iff its 1-skeleton $X^{(1)}$ is connected.
- (ii) Let X be a locally finite CW-complex which is connected. Show that X has countably many cells.

More exercise?

NPTTEL

Chapter 3

Categories and Functors

This chapter is a quick introduction to the language of *Categories and Functors*. No need to master this section in one go. You may keep coming back to it now and then as and when the need arises. However, we hope that at the end of the course, you have acquired some mastery over this ‘language’.

All sciences are essentially the study of patterns. This is more so in the case of mathematics. The fundamental concepts such as number and space combine to produce enumerable patterns that we come across in day-to-day life. All mathematical concepts involve a certain family of objects which fit into a pattern and the study involves relations among these objects.

Let us consider some examples. In topology we study the so-called topological properties of spaces. The objects that we consider here are topological spaces. The relation between two topological spaces is studied by considering continuous functions from one to the other. A fundamental property of continuous functions is that the composite of two continuous functions if defined, is again continuous. Also, for every space X , there is at least one continuous function, viz., the identity map $Id_X : X \longrightarrow X$. Similar statements are also true for groups and homomorphisms, viz., the objects that we study here are groups and the relations are group homomorphisms; composite of two homomorphisms, if defined, is a homomorphism and there is always the identity homomorphism $Id_G : G \longrightarrow G$ for any group G . We can go on listing similar properties in other contexts such as vector spaces and linear maps between them, modules over a ring and linear maps between them and so on.

Category theory is the language that sums up this aspect of mathematics in a technically precise way. Not so surprisingly, it can prove theorems too.

This chapter is divided into four sections. In 3.1, you will see the definition of category and a lot of examples. In 3.2 relation between two categories will be discussed. In 3.4-3.5, you see some examples other standard notions in mathematics converted into categorical language.

3.1 Categories-Definitions and Examples

Definition 3.1.1 A category $\mathcal{C}$ consists of a family $Obj(\mathcal{C})$ of *objects* with the following properties:

- (i) To each ordered pair (A, B) of objects with $A, B \in Obj(\mathcal{C})$, there is a set denoted by $M(A, B)$; $M(A, B), M(A', B')$ are disjoint unless $A = A'$ and $B = B'$ in $\mathcal{C}$. The elements of $M(A, B)$ are called *morphisms* with *domain* A and *codomain* B .
- (ii) For each triple (A, B, C) of objects, there is a binary operation

$$\circ : M(A, B) \times M(B, C) \longrightarrow M(A, C) \quad (f, g) \mapsto g \circ f;$$

these are, collectively associative in the following sense: for $f \in M(A, B), g \in M(B, C), h \in M(C, D)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- (iii) For each object A in $\mathcal{C}$, there exists a (unique) $Id_A \in M(A, A)$ such that for any $f \in M(A, B)$ and $g \in M(C, A)$ we have,

$$f \circ Id_A = f; \quad Id_A \circ g = g.$$

A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ is called an *equivalence* or *invertible* if there exists a morphism $g : B \longrightarrow A$ such that $f \circ g = Id_B$ and $g \circ f = Id_A$. Then g is said to be an inverse to f . If there exists an equivalence $f : A \longrightarrow B$ then we say A and B are equivalent objects in $\mathcal{C}$.

Remark 3.1.2

- (i) Often when more than one category is involved in the discussion, we write $M_{\mathcal{C}}(X, Y), \text{Mor}(X, Y)$ or $\text{Hom}_{\mathcal{C}}(X, Y)$ to denote the set of all morphisms from X to Y , where X and Y are considered as objects in a particular category $\mathcal{C}$.
- (ii) It follows easily that the identity morphism Id_A is unique in $M(A, A)$ for all A . Note that $M(A, B)$ is a set, may or may not be empty, whereas objects of $\mathcal{C}$ need not be sets.

Neither is the collection of objects a set, in general. You need not bother about these purely ‘logical’ problems at this stage.

(iii) It also follows that if f is invertible, then its inverse is unique and hence we can write it as f^{-1} .

(iv) The equivalence of any two objects defines an equivalence relation on the family of all objects of a category. The central theme in any categorical study is to determine this set.

(v) The condition of disjointness of $M(A, B)$ and $M(A', B')$ unless $A = A', B = B'$ is important especially in the categorical language. We are aware of the common practice of denoting a restriction of a function $f : X \rightarrow Y$ to a subset $A \subset X$ by the same notation, and often we too will follow this ‘convenient’ practice. For example the functions $\sin : \mathbb{R} \rightarrow [-1, 1], \sin : \mathbb{R} \rightarrow \mathbb{R}, \sin : \mathbb{C} \rightarrow \mathbb{C}$ are all denoted by $\sin$, (which is a $\sin$ according to the modern practice). The justification is that we are cutting down on clumsy notations which may obscure the mathematical ideas that we want to present. On the other hand, one of the basic objects of the language of category theory is to provide us rigor without being too verbose.

Definition 3.1.3 Let $\mathcal{C}$ be a category. By a *subcategory* $\mathcal{D}$ of $\mathcal{C}$, we mean a category $\mathcal{D}$ each of whose objects are objects of $\mathcal{C}$ and for each pair of objects $A, B \in \mathcal{D}$ we have the set of all morphisms from A to B is a subset of the set of all $\mathcal{C}$ -morphisms from A to B , i.e.,

$$M_{\mathcal{D}}(A, B) \subset M_{\mathcal{C}}(A, B) \quad (3.1)$$

and the binary operations in $\mathcal{D}$ are just the restrictions of the same in $\mathcal{C}$. If equality holds in (3.1), for all possible pairs of objects in $\mathcal{D}$, then we say $\mathcal{D}$ is a *full subcategory* of $\mathcal{C}$.

Example 3.1.4

1. **The category of all sets and functions, $\mathbf{Ens}$** There is the category $\mathbf{Ens}$ whose objects are all sets and whose morphisms are all functions, with the usual law of composition. This is the ‘mother’ (see Example 3.1.6) of many categories that we deal with usually. That is one reason for expressing $f \in M(X, Y)$ by $f : X \rightarrow Y$. The short form $\mathbf{Ens}$ is taken from the French word *ensembles*. You will see that many interesting examples of categories are subcategory of $\mathbf{ENS}$ or related to it through a slightly different manner. The axioms for categories are extracts of the fundamental concepts involved in $\mathbf{ENS}$.

2. **The singleton category $\mathbf{P}$** Consider a category with a single object denoted by $\star$ and with the morphism set $M(\star, \star)$ consisting of the single element, viz., the identity morphism. In some sense this category is the smallest of all categories which are ‘non empty’.
3. **The opposite category** Given any category $\mathcal{C}$, there is another category called the *opposite* of $\mathcal{C}$ and denoted by $\mathcal{C}^{\text{op}}$. Its objects are the same as that of $\mathcal{C}$. The morphisms are defined by $M_{\mathcal{C}^{\text{op}}}(A, B) := M_{\mathcal{C}}(B, A)$ and the composition law is also the same. Note that $(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}$.
4. **The category of topological spaces, $\mathbf{Top}$** As discussed in Remark ?? and in the beginning of this section, there is a category whose objects are topological spaces, for each pair X, Y of spaces, $M(X, Y)$ is the set of all continuous functions from X to Y and the composition is the usual composition of functions as the binary operations. This is called the category of topological spaces and is usually denoted by **Top**. An equivalence in this category is called a homeomorphism. The equivalence class of objects in this category are called homeomorphism types.
5. **The homotopy category of topological spaces, $\mathbf{T}$** Consider the category in which objects are all topological spaces and $M(X, Y)$ is the set of all homotopy class of maps from X to Y . This category is called the *homotopy category* which we shall denote by $\mathcal{T}$. (Notice that the objects in this category are the same as objects in **Top**. In some literature, this category is denoted by **Top** instead of the one we have introduced in Example 1 above.) More generally, given any topological category, we may retain the objects as they are but change the morphisms to homotopy class of maps to obtain the associated homotopy category.
6. **The category of groups, $\mathbf{Gr}$** Likewise, we have the category of all groups and group homomorphisms which is denoted by **Gr**. Here the composition of two homomorphisms is defined in the usual way. An equivalence in this category is an isomorphism. The equivalence classes are isomorphism classes of groups.
7. **The category of abelian groups, $\mathbf{Ab}$** Consider the category **Ab** whose objects are abelian groups and $M(G, H)$ is taken to be all homomorphisms from G to H . Of course we compose two such homomorphisms in the usual way. In some sense this category is a subcategory of **Gr** which we shall define soon.

8. **The category of open sets on X , $\mathcal{U}_X$** Given a topological space X consider the category whose objects are open sets $U \subset X$. For open sets U, V , if $U \subset V$ then take $M(U, V) = \{\iota\}$, the singleton set consisting of the inclusion map $U \hookrightarrow V$; otherwise take $M(U, V) = \emptyset$. In particular, note that $M(U, U) = \{Id_U\}$. We shall denote this category by $\mathcal{U}_X$.
9. **The fundamental groupoid** Let X be a fixed topological space. Let us try to define a category $\mathcal{P}_X$ whose objects are points of X and for any two points $x, y \in X$, let $M(x, y)$ be the set of all paths in X from x to y . We can concatenate a path from x to y with a path from y to z . However, this binary operation fails to satisfy the associativity condition. (Also the identity condition.) So, we rectify this by taking $M(x, y)$ to be the set of all path homotopy classes of paths from x to y . Since concatenation of path homotopic paths are path homotopic, $\mathcal{P}_X$ becomes a category. For each $x \in X$ the class of the constant path at x forms the identity element. Observe that $M(x, y)$ is non empty iff x and y are in the same path component of X . Every morphism in this category is an equivalence, since for each path ω the inverse path ω^{-1} is a homotopy inverse. (See part (iii) of Lemma ?? again.) More generally, any category in which every morphism is an equivalence is called a *groupoid*. The set $M(x, x)$ forms a group which is nothing but the fundamental group $\pi_1(X, x)$ of X at x .
10. **The semi-group** By a semigroup M we mean a set M with an associative binary operation and a two-sided identity. In any category $\mathcal{C}$, for any object A in $\mathcal{C}$, $M(A, A)$ is a semigroup in two different ways: either by taking the operation $f \star g$ as $f \circ g$ or $g \circ f$ (the two semigroups being the opposite of one another). Conversely, given a semigroup M , there are essentially two different ways of thinking M as $M_{\mathcal{C}}(A, A)$: Define the category $\mathcal{L} = \mathcal{L}_M$ with a single object $A = M$ and $M_{\mathcal{L}}(A, A) = \{L_x : x \in M\}$ where $L_x : M \rightarrow M$ denotes the left-multiplication by x . Likewise we can define $\mathcal{R}_M$ using right-multiplication in M . We can almost recover the semigroup M from $\mathcal{L}_M$ or $\mathcal{R}_M$ except for the fact that we cannot tell whether it is the original semigroup or the opposite one. The reason is $\mathcal{L}_M$ is the opposite of $\mathcal{R}_M$.
- A semigroup is called a *monoid* if it is cancellative, i.e., $xy = xz \implies y = z$. This does not necessarily mean that the morphisms in $\mathcal{L}_M$ are invertible. In any case, notice that each morphism in $M(A, A)$ is invertible iff the semigroup is actually a

group.

11. **The poset as a category** Let $(X, \leq)$ be a partially ordered set. Let us define a category associated to this. Take elements of X as objects of this category. For any two $x, y \in X$ take $M(x, y)$ to be a singleton set if $x \leq y$, and $= \emptyset$ otherwise. The binary operations are defined in an obvious way, due to the transitivity condition. For each x , the unique element in $M(x, x)$ plays the role of two-sided identity. These steps can be reversed. Starting with a category whose family of objects is a set and for each pair of objects (A, B) , the set of morphisms $M(A, B)$ is either empty or a singleton and if $M(A, B) \neq \emptyset, A \neq B$ then $M(B, A) = \emptyset$, we obtain a partial order on the set of objects.

The above three examples illustrate the fact that there are many interesting categories in which morphisms are not necessarily functions. We shall see a little later an explicit example in which objects are not sets.

12. **The smooth category $\mathbf{Diff}$** Let the objects be smooth manifolds and morphisms be all smooth functions. This category is called the smooth (differential) category denoted by **Diff**.
13. **The category of vector spaces** Let $\mathbf{k}$ be a field. Consider the family of all (finite dimensional) vector spaces over $\mathbf{k}$ with $M(V, U)$ being the set of all linear maps from V to U . This forms a category called the category of all (finite dimensional) vector spaces over $\mathbf{k}$. We shall denote it by **Vect_k** (**FVect_k**, respectively).
14. **The category of modules** More generally, if R is a commutative ring, we can take the objects to be all modules over R with $M(A, B) = \text{Hom}_R(A, B)$ to be the set of all R -linear homomorphisms from A to B , we get a category called the category of R -modules denoted by **R-mod**.

Example 3.1.5 We have already indicated one such example of a subcategory of the category of all groups, viz., the category of all abelian groups. Indeed, **Ab** is a full subcategory of **Gr**. On the other hand, **Diff** is a subcategory of **Top**, which is not a full subcategory.

Example 3.1.6 List all categories occurring in above the examples which are subcategories of **Ens**. You will find that most of the categories that we come across are subcategories of **Ens**. This explains why we called **Ens** as the ‘mother’ in Example 3.1.4.1.

3.2 Functors

We shall now consider the concept of relations between two categories, which is more general than the concept of subcategory.

Definition 3.2.1 Let $\mathcal{C}, \mathcal{D}$ be any two categories. By a covariant (respectively, contravariant) functor $\mathcal{F} : \mathcal{C} \rightsquigarrow \mathcal{D}$ we mean

(i) as association denoted by $\mathcal{F}$ itself

$$\mathcal{F} : \text{Obj } \mathcal{C} \longrightarrow \text{Obj } \mathcal{D}; \quad A \mapsto \mathcal{F}(A), \text{ and}$$

(ii) to each pair of objects A, B in $\mathcal{C}$ a function, again denoted by

$$\mathcal{F} : M_{\mathcal{C}}(A, B) \longrightarrow M_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(B)); f \mapsto \mathcal{F}(f)$$

$$(\text{respectively } M_{\mathcal{C}}(A, B) \longrightarrow M_{\mathcal{D}}(\mathcal{F}(B), \mathcal{F}(A)); f \mapsto \mathcal{F}(f))$$

satisfying the following properties:

(a) $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$; (respectively $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$);

(b) $\mathcal{F}(Id_A) = Id_{\mathcal{F}(A)}$;

for all objects A, B, C of $\mathcal{C}$ and all morphisms $g \in M_{\mathcal{C}}(A, B); f \in M_{\mathcal{C}}(B, C)$.

Remark 3.2.2

1. The difference between covariance and contravariance is simply in the fact that covariance preserves the direction of the arrow whereas contravariance reverses it. However, in practice, it turns out that contravariance has more mathematical structure in it whereas covariance is more geometrical and easy to understand.
2. If $\mathcal{F}_1 : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ and $\mathcal{F}_2 : \mathcal{C}_2 \longrightarrow \mathcal{C}_3$ are covariant functors then there is an obvious way to define the composite functor $\mathcal{F}_2 \circ \mathcal{F}_1$ which is again a covariant functor from $\mathcal{C}_1$ to $\mathcal{C}_3$. Similarly for contravariant functors. Of course, a composite of two contravariant functors will be covariant.
3. Suppose $\mathcal{F}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$. If two objects A, B are equivalent in $\mathcal{C}$ then $\mathcal{F}(A)$ and $\mathcal{F}(B)$ are equivalent in $\mathcal{D}$. (Verify this.) This is one of the most effective ways functors are exploited, viz., if we know that $\mathcal{F}(A)$ and $\mathcal{F}(B)$ are inequivalent for some functor then we know that A and B are inequivalent. We will have several illustrations of this. For a quick one see the second example below.

Example 3.2.3

1. One of the most important class of functors are the so called *forgetful* functors from a category whose objects may also be considered as objects of a larger category. For instance, from $\mathbf{Ab} \rightsquigarrow \mathbf{Ens}$ we have the forgetful functor which associates to each abelian group its underlying set and to each homomorphism $f : G \rightarrow H$ of abelian groups, the corresponding function. The study of inter-relations between a category and its subcategory is a common feature in all mathematics and that is how the forgetful functors are important. The most widely studied forgetful functor in topology is the one from $\mathbf{Diff}$ to $\mathbf{Top}$ with other categories interspersed between them.
2. Consider the category $\mathbf{Top}$. The set of connected components $\text{conn}(X)$ of a topological space defines a covariant functor from $\mathbf{Top}$ to $\mathbf{Ens}$. Let us illustrate the importance of such functors in this simple example. Suppose now that we are given two topological spaces X and Y with the underlying sets having different cardinality, i.e., $\#(c(X)) \neq \#(C(Y))$. Then from Remark 3 above, it follows that X and Y cannot be equivalent in $\mathbf{Top}$.
3. Let $\mathcal{C}$ be any category and $\hat{\mathcal{C}}$ denote the category whose single object is $\mathcal{C}$ itself and the set of morphisms $M_{\hat{\mathcal{C}}}(\mathcal{C}, \mathcal{C}) = \{Id\}$, the singleton consisting of the identity functor. This is a simple example of a category whose objects may not be sets.
4. Let $\mathcal{C}$ be any category and $A \in \text{Obj } \mathcal{C}$. Then there is a covariant and a contravariant functor, $\text{Hom}(A, -), \text{Hom}(-, A) : \mathcal{C} \rightsquigarrow \mathbf{Ens}$ defined as follows:

$$\text{Hom}(A, -)(B) = M(A, B); \quad \text{Hom}(-, A)(B) = M(B, A)$$

and for any $f \in M(B, C)$, $f_* := \text{Hom}(A, f) : M(A, B) \rightarrow M(A, C)$ and $f^* := \text{Hom}(f, A) : M(C, A) \rightarrow M(B, A)$ are given as follows: For any $g \in M(A, B), h \in M(C, A)$ take

$$f_*(g) = f \circ g; \quad f^*(h) = h \circ f.$$

These are called representable functors. They are very important, since understanding these functors is equivalent to understanding the category $\mathcal{C}$ itself.

5. Given a surjective group homomorphism $f : G \rightarrow H$, the first isomorphism theorem says that the quotient group $G/\text{Ker } f$ is canonically isomorphic to H . Likewise given a finite dimensional vector space V , you are told that its double-dual V^{**}

$$\begin{array}{ccc}
\mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) \\
\eta_A \downarrow & & \downarrow \eta_B \\
\mathcal{G}(A) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(B)
\end{array}$$

Figure 3.1. Naturality

is canonically isomorphic to V . Perhaps, you were not told the true meaning of the adverb ‘canonically’ in these situations. We shall explain this using category theory.

Definition 3.2.4 Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightsquigarrow \mathcal{D}$ be any two covariant functors. By a *natural transformation of functors* $\eta : \mathcal{F} \rightsquigarrow \mathcal{G}$, we mean the following: For each $A \in \text{Obj}\mathcal{C}$, there is a morphism $\eta_A \in M_{\mathcal{D}}(\mathcal{F}(A), \mathcal{G}(A))$ such that for each $f \in M_{\mathcal{C}}(A, B)$, the following diagram is commutative.

In addition, if each η_A is an equivalence, then η is called an *equivalence of functors*. When we have such an equivalence, we say $\mathcal{F}, \mathcal{G}$ are naturally equivalent or naturally isomorphic functors.

The naturality is of utmost importance and cannot be brushed away as merely a rhetoric of pure mathematicians. We end up this section with one more important notion which generalizes the notion of equivalence.

Definition 3.2.5 Let $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \rightarrow \mathcal{C}$ be functors. We say $\mathcal{F}$ is a *left-adjoint of $\mathcal{G}$* (and equivalently, $\mathcal{G}$ is a *right-adjoint of $\mathcal{F}$*) if there is a natural isomorphism

$$\eta : \text{Hom}_{\mathcal{C}}(-, \mathcal{G}(-)) \approx \text{Hom}_{\mathcal{D}}(\mathcal{F}(-), -).$$

This just means that for every pair (C, D) with C in $\mathcal{C}$ and D in $\mathcal{D}$, there is an isomorphism

$$\eta_{C,D} : \text{Hom}_{\mathcal{C}}(C, \mathcal{G}(D)) \rightarrow \text{Hom}_{\mathcal{D}}(\mathcal{F}(C), D)$$

such that for every pair of morphisms $\phi : C \rightarrow C'$ and $\psi : D \rightarrow D'$ in the respective categories, the diagram

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(C', \mathcal{G}(D)) & \xrightarrow{\eta_{C',D}} & \text{Hom}_{\mathcal{D}}(\mathcal{F}(C'), D) \\
\text{Hom}_{\mathcal{C}}(\phi, \mathcal{G}(\psi)) \downarrow & & \downarrow \text{Hom}_{\mathcal{C}}(\mathcal{F}(\phi), \psi) \\
\text{Hom}_{\mathcal{C}}(C, \mathcal{G}(D')) & \xrightarrow{\eta_{C,D'}} & \text{Hom}_{\mathcal{D}}(\mathcal{F}(C), D')
\end{array}$$

commutes.

Remark 3.2.6 We shall leave it to the reader to verify that any two left adjoints of a functor $\mathcal{G}$ are naturally equivalent. Similarly, any two right adjoints of a functor $\mathcal{F}$ are naturally equivalent.

Example 3.2.7

1. Consider the category $\mathbf{Vect}_{\mathbf{k}}$. Let V^* denote the ‘dual’ space of all linear maps from $V \rightarrow \mathbf{k}$. Then the assignment

$$V \rightsquigarrow V^*$$

defines a contravariant functor $\mathbf{Vect}_{\mathbf{k}} \rightarrow \mathbf{Vect}_{\mathbf{k}}$ where for a linear map $f : V \rightarrow W$ we take $f^* : W^* \rightarrow V^*$ defined by the formula

$$f^*(\phi) = \phi \circ f.$$

Take $\mathcal{C} = \mathbf{Vect}_{\mathbf{k}} = \mathcal{D}$ and $\mathcal{F}$ to be the identity functor and $\mathcal{G}$ to be the double dual: $T : V \rightsquigarrow V^{**}$. Define a natural transformation $\eta : \mathcal{F} \rightsquigarrow \mathcal{G}$ as follows: given a vector space V over $\mathbf{k}$, we have to first define a linear map $\eta_V : V \rightarrow V^{**}$. This is done by taking

$$\eta_V(v)(\phi) = \phi(v), \quad v \in V; \phi \in V^*.$$

It follows that $f^{**} : V^{**} \rightarrow W^{**}$ has the property

$$f^{**}(\psi) = \psi \circ f^*, \quad \psi \in V^{**}.$$

It is straightforward to verify that the following diagram is commutative.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \eta_V \downarrow & & \downarrow \eta_W \\ V^{**} & \xrightarrow{f^{**}} & W^{**} \end{array}$$

Thus η is a natural transformation of functors.

Observe that there is a subcategory $\mathbf{FVect}_{\mathbf{k}}$ of $\mathbf{Vect}_{\mathbf{k}}$ whose objects are finite dimensional vector spaces over k . If V is finite dimensional then so is V^{**} . Thus, η restricts to a natural transformation of functors Id and T on $\mathbf{FVect}_{\mathbf{k}}$. Further, we know that η_V is an isomorphism whenever V is finite dimensional. Therefore η

is an equivalence of the two functors on $\mathbf{FVect}_k$. This is the explanation that we promised of the word ‘canonical’ in the Example 3.2.3.5.

In the next example, we explain the word ‘canonical’ occurring in the first isomorphism theorem in group theory.

2. Consider the category $\mathcal{C}$ whose objects are surjective homomorphisms $f : G \rightarrow G'$ of groups and whose morphisms are pairs (α, β) of homomorphisms which make up a commutative diagram:

$$\begin{array}{ccc} G_1 & \xrightarrow{f_1} & G'_1 \\ \alpha \downarrow & & \downarrow \beta \\ G_2 & \xrightarrow{f_2} & G'_2 \end{array}$$

Note that because of the commutativity of the diagram, $\alpha(\text{Ker } f_1) \subset \text{Ker } f_2$. Therefore α induces a homomorphism $\bar{\alpha} : G_1/\text{Ker } f_1 \rightarrow G_2/\text{Ker } f_2$.

On the category $\mathcal{C}$, we consider two functors: $\mathcal{Q}$ and $\mathcal{I}$ defined as follows:

$$\mathcal{Q}(f) = G/\text{Ker } f, \quad \mathcal{Q}(\alpha, \beta) = \bar{\alpha}$$

$$\mathcal{I}(f) = \text{Image of } f; \quad \mathcal{I}(\alpha, \beta) = \beta.$$

The first isomorphism theorem yields an isomorphism: $\phi_f : G/\text{Ker } f \rightarrow G'$ such that the following diagrams are commutative:

$$\begin{array}{ccc} G_1/\text{Ker } f_1 & \xrightarrow{\phi_{f_1}} & G'_1 \\ \bar{\alpha} \downarrow & & \downarrow \beta \\ G_2/\text{Ker } f_2 & \xrightarrow{\phi_{f_2}} & G'_2 \end{array}$$

Thus ϕ defines a natural equivalence of the two functors. This is the meaning of the word ‘canonical’ occurring in the statement of the first isomorphism theorem.

Example 3.2.8 Here are a few examples of adjoint functors. We shall meet more of them as we proceed deeper into algebraic topology.

1. Let $\mathcal{F} : \mathbf{Ens} \rightarrow \mathbf{Ab}$ be the functor that assigns to each set S the free abelian group over it: $\mathcal{F}(S) = F(S)$. A right adjoint $G : \mathbf{Ab} \rightarrow \mathbf{Ens}$ is the forgetful functor which assigns the underlying set to a given abelian group. Verify this and imitate this to obtain many other examples.

2. Let B be an A -algebra, where A is a commutative ring. Let $S : \mathbf{B}\text{-mod} \rightarrow \mathbf{A}\text{-mod}$ be the forgetful functor that associates to each B -module, the underlying A -module. Then the functor $M \rightsquigarrow M \otimes_A B$ is a left-adjoint to S and the functor $N \rightsquigarrow \text{Hom}_A(B, N)$ is a right adjoint to S . Verify these statements.

3.3 Initial and Terminal Objects, Universal Properties

Definition 3.3.1 Let A be an object in a category $\mathcal{C}$. We say A is an *initial object* in $\mathcal{C}$ if for every object $B \in \mathcal{C}$, $M(A, B)$ consists of a single element. Likewise A is called a *terminal object* in $\mathcal{C}$ if for every $B \in \mathcal{C}$, $M(B, A)$ is a singleton. The object A is called a *zero object* if it is both an initial object and a terminal object.

Example 3.3.2 In the category **Ens**, the emptyset $\emptyset$ is an initial object but not a terminal object; also any singleton set $\{p\}$ is a terminal object but not an initial object. Categories such as **Gr**, **Ab**, **Vect_k**, **R-mod**, etc., all have zero objects.

Remark 3.3.3 In any category $\mathcal{C}$, it is easy to see that any two initial (terminal) objects (if they exist) are equivalent. Likewise, it is not difficult to see that any two zero objects in a category $\mathcal{C}$ are equivalent (isomorphic). Moreover, if there are zero objects in $\mathcal{C}$ then for every $A, B \in \mathcal{C}$, there is a special morphism $0_{A,B}$ which is the composite of the unique morphisms:

$$A \rightarrow 0 \rightarrow B.$$

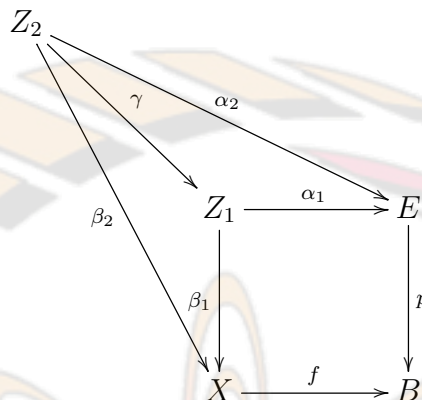
These morphisms have the additional property that for any morphisms $f : D \rightarrow A, g : B \rightarrow E$ in $\mathcal{C}$, we have,

$$0_{A,B} \circ f = 0_{D,B}; \quad g \circ 0_{A,B} = 0_{A,E}.$$

Example 3.3.4 Recall the definition of pull-back of coverings from Part-I. Here we shall describe the same thing in any category $\mathcal{C}$. Let $f \in M(X, B), p \in M(E, B)$ be any two morphisms in $\mathcal{C}$. There is a category $\mathcal{C}(f, p)$ whose objects are commutative diagrams

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & E \\ \beta \downarrow & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

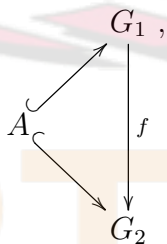
where all objects and morphisms are from $\mathcal{C}$. A morphism in (p, f) is a commutative diagram:



A terminal object in this category is called the pullback of p under f and is denoted by $f^*p : f^*E \rightarrow X$. The uniqueness of the pullback according to this definition is obvious. Lemma ?? relates this definition with the constructive definition ?. This construction is not available in a general category and even if available may not serve to exhibit the pullback.

What we have seen in part-I is that it is available in the category of topological spaces. We have also seen that many topological properties of p also hold for $f^*(p)$.

Example 3.3.5 Let $\mathcal{C}$ be an algebraic category such as **Gr**, **Ab**, **Vect_k**, **R – Mod**. Let A be any set and consider the category $\mathcal{C}(A)$ whose objects are pairs (A, G) where G is an object in $\mathcal{C}$ and A is a subset of G . Morphisms in this category are commutative triangles



where f is a morphism in $\mathcal{C}$. An initial object in this category is called a free object in $\mathcal{C}$ with A as a basis. Check that in the cases sighted above, this is nothing but the free group, the free abelian group, the vector space, and the free R module respectively. Only the proof of existence in each case is different.

3.4 Direct Limits and Inverse Limits

In example 3.1.4.11 we have seen how to view a poset as a small category. A *directed set* $(J, \leq)$ is a poset such that for any two elements $i, j \in J$, there is $k \in J$ such that $i \leq k$ and $j \leq k$. It is not difficult to reformulate this condition in terms of category theory: given $i, j \in J$, there is $k \in J$ such that $M(i, k)$ and $M(j, k)$ are non empty.

We define a *directed system* in a category $\mathcal{C}$ to be a covariant functor $\mathcal{F} : (J, \leq) \rightarrow \mathcal{C}$ where $(J, \leq)$ is a directed set viewed as a category. We shall denote the objects $\mathcal{F}(j) := F_j, j \in J$ and $\mathcal{F}(M(i, j)) = \{f_{ij}\}$ whenever $i < j$. Associated to a directed system $\mathcal{F}$ one defines another category $\bar{\mathcal{F}}$ as follows: the objects are pairs $(A, \{\alpha_j\}_{j \in J})$, where A is an object in $\mathcal{C}$ and $\alpha_j : F_j \rightarrow A$ are morphisms in $\mathcal{C}$ such that $\alpha_i = \alpha_j \circ f_{ij}$ whenever f_{ij} exist. A morphism $\tau : (A, \{\alpha_j\}_{j \in J}) \rightarrow (B, \{\beta_j\}_{j \in J})$ in $\bar{\mathcal{F}}$ is a morphism $\tau : A \rightarrow B$ in $\mathcal{C}$, such that $\tau \circ \alpha_j = \beta_j, j \in J$.

$$\begin{array}{ccc}
 F_i & \xrightarrow{\alpha_i} & A \\
 \downarrow f_{ij} & & \uparrow \alpha_j \\
 F_j & \xrightarrow{\alpha_j} & A
 \end{array}$$

(a)

$$\begin{array}{ccc}
 & & A \\
 & \nearrow \alpha_j & \downarrow \tau \\
 F_j & & B \\
 & \searrow \beta_j &
 \end{array}$$

(b)

(3.2)

By the *direct limit* of the directed system $\mathcal{F}$ we mean an initial object in the category $\bar{\mathcal{F}}$. Note that if $\mathcal{C}$ has initial objects then the category $\bar{\mathcal{F}}$ is non empty; however, there is no guarantee that it will have initial objects. Of course, if an initial object exists, we have seen that it is unique up to equivalence. Thus the same holds for the direct limit also. We denote this object by $\varinjlim F_j$.

Thus the direct limit of a directed system $\{F_i\}$ in $\mathcal{C}$ is an object A in $\mathcal{C}$ together with a collection of morphisms $\alpha_j : F_j \rightarrow A$ satisfying the compatibility condition (3.2(a)) such that for every other compatible family of morphisms $\beta_j : F_j \rightarrow B$ there is a *unique* morphism $\tau : A \rightarrow B$ making up a commutative diagram as in (3.2(b)).

The simplest thing is to see that in the category **Ens** every directed system has a direct limit: Take B to be the quotient of the disjoint union $\coprod_{j \in J} F_j$ by the relation $x \sim f_{ij}(x)$ for all $x \in F_i$ whenever f_{ij} makes sense and $\beta_i : F_i \rightarrow B$ as the composite of the inclusion followed by the quotient map. Similar construction works in many other algebraic categories such as **Ab**, **Vect_k**, **Gr**, **R-mod**, etc. In **Top** also, this works; we

have to take the obvious disjoint union topology on the disjoint union and the quotient topology on the quotient. A typical special case of this is when F_i are subsets of a given set F and f_{ij} are inclusion maps whenever $F_i \subset F_j$. The direct limit of this system is nothing but $F' := \cup_i F_i$ with the coinduced topology: $A \subset F'$ is open iff $A \cap F_i$ is open for every i .

An important observation which often helps in computations is the following. Let $(J, \leq)$ be a sub-poset of a directed set $(I, \leq)$. We say J is ‘final’ in I if to each $i \in I$ there exists some $j \in J$ such that $i \leq j$. In this situation, given a directed system $\{F_i\}_{i \in I}$, we may consider the subsystem $\{F_j\}_{j \in J}$. If one of the two direct limits exists then the other exists and the two are equal:

$$\operatorname{dlim}_{i \in I} F_i = \operatorname{dlim}_{j \in J} F_j.$$

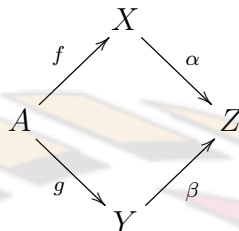
Verification of this statement is straight forward and left to the reader as an exercise.

If we replace ‘covariant’ by ‘contravariant’ and ‘initial’ by ‘terminal’ in the above passages, we obtain the notion of inverse systems and inverse limits and corresponding conclusions. However, about the existence of inverse limits, one needs to be careful. In the category of **Ens**, one may take the subset of $\prod_i F_i$ consisting of those elements (x_i) such that $f_{ij}(x_j) = x_i$ whenever $i \leq j$. Once again, the same construction goes through in many other familiar categories as well. For more details, the reader may refer to [?].

Exercise 3.4.1

- (i) List all possible category-pairs where one is a ‘sub’ of the other, occurring in the above set of Examples 3.1.4.
- (ii) List all the pairs of categories where one is a full subcategory of the other, occurring in the above set of Examples 3.1.4.
- (iii) In any category $\mathcal{C}$ and for any object A in $\mathcal{C}$, let $\operatorname{Aut} A$ denote the set of all invertible morphisms in $M_{\mathcal{C}}(A, A)$. Verify that this forms a group under the composition law coming from $\mathcal{C}$. Now, let $\alpha : A \rightarrow B$ be an invertible morphism in $\mathcal{C}$. Define $\hat{\alpha} : \operatorname{Aut} A \rightarrow \operatorname{Aut} B$ by $\hat{\alpha}(f) = \alpha \circ f \circ \alpha^{-1}$. Show that $\hat{\alpha}$ is an isomorphism of groups.
- (iv) Let $\mathcal{C}$ be any category, $f \in M_{\mathcal{C}}(A, X), g \in M_{\mathcal{C}}(A, Y)$ be any two morphisms. There

is a category $(f, g)\mathcal{C}$ whose objects are commutative diagrams of morphisms in $\mathcal{C}$:



An initial object in this category is called the *pushout* of (g, f) and is denoted by $X_f \cup_g Y$. Show that pushouts exist in the categories **Ens**, **Ab**. It exists in the category **Gr**. But it may be better to read Section ?? before trying to prove this.

(v) Show that the direct limit of the following system

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \dots$$

is isomorphic to $\mathbb{Z}[1/2]$, the group of rational numbers of the form $\frac{m}{2^n}$, $m, n \in \mathbb{Z}$.

NPTTEL

Chapter 4

Homology Groups

You may be familiar with terms such as ‘exact forms’ and ‘closed forms’ from your calculus course. Also in Complex Analysis, you might have come across with the term ‘homologous’. We are going to put various ideas behind these terms and put them on a firm footing here, from the modern point of view of algebraic topology.

Homology and cohomology theories have their roots in complex analysis and the integral calculus of several variables, especially in Stokes’ theorem. However, it was Poincaré who formalized the idea of homology theory in his seminal paper ‘Analysis Situs’ and it took many more years for the arrival of cohomology theory and finally both of them were properly established in 1952 by Eilenberg and Steenrod [?].

In order to bring out the full force of these results, as well as save time, it is good to temporarily isolate these two concepts and study each of them in a more abstract set-up. Naturally, there is a lot of algebra to be worked out. Among these, cohomology is more natural and some of you may have come across with it while studying differential forms. On the other hand, being more geometric, homology is easier to comprehend. So, we shall deal with homology in this chapter. For lack of time cohomology theory, which needs more algebraic preparation is beyond the scope of this course.

In Sections 4.1-4.4, we are going to develop basic ‘homological algebra’ needed. Next four sections, 4.5-4.8 will be devoted to the most important homology theory, viz., singular homology. In sections 4.9-to 4.11, we shall discuss some other homologies such as simplicial homology, CW-homology etc., and their relation with singular homology. In sections 4.12- 4.13, we shall give some applications of homology in deriving certain celebrated theorems. In the last two sections we shall take care of certain proofs that we have post-poned.

4.1 Basic Homological Algebra

In this section, we shall begin with a minimum introduction to ‘homological algebra’ necessary to understand the homology groups. The most important result here is the Snake lemma. The important technique is the so called ‘diagram chasing’ which will be introduced in small doses to begin with. More and more homological algebra will be introduced as and when required. Throughout this chapter, R will denote a general commutative ring with a unit, though often you may assume that this ring is nothing but the ring of integers $\mathbb{Z}$.

Definition 4.1.1 Consider a direct sum

$$C_\bullet := C_* := \bigoplus_{n \in \mathbb{Z}} C_n$$

of R -modules. Often we call C_* , a graded R -module with its n^{th} graded component C_n . Members of C_n are also called homogeneous elements of C_* of degree n . Both these notations for a graded module are common in the literature and hence we would like you to get familiar with both of them right from the beginning.

A R -module homomorphism $f : C_* \rightarrow C'_*$ is called a graded homomorphism if there exists d such that $f(C_r) \subset C'_{r+d}$ for all r . We then call d the (homogeneous) degree of f . We shall denote $f|_{C_r}$ by f_r if necessary—and often we may simply write f itself for f_r provided there is no confusion.

By a *chain complex* (C_*, ∂) of R -modules, we mean a graded R -module C_* , together with an endomorphism $\partial := \partial_* : C_* \rightarrow C_*$ of degree -1 with the property $\partial \circ \partial = 0$. Observe that ∂ consists of a sequence $\{\partial_n : C_n \rightarrow C_{n-1}\}$ of R -module homomorphisms such that $\partial_n \circ \partial_{n-1} = 0$ for all n . The endomorphism ∂ is called the *differential* or the boundary map of the chain complex. Often we shall not mention the map ∂ at all and merely say C_* is a chain complex.

If C_* and C'_* are two chain complexes then by a chain map $f = f_* : C_* \rightarrow C'_*$ we mean a graded module homomorphism of degree 0 that commutes with the corresponding differentials, i.e., a sequence $f_n : C_n \rightarrow C'_n$ of R -module homomorphisms such that $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$ for all n . It is straightforward to check that there is a category of chain complexes of R -modules and chain maps. We shall denote this category by $\mathbf{Ch}_R$ with a further simplification $\mathbf{Ch} := \mathbf{Ch}_{\mathbb{Z}}$.

There is an obvious way to define the direct sum of a family of chain complexes $\{(C^\alpha_\bullet, \partial^\alpha)\}_{\alpha \in \Lambda}$: the graded module is taken to be the direct sum $\bigoplus_\alpha C^\alpha_\bullet$ and the boundary

map $\partial = \oplus \partial^\alpha$. For instance, if $(C^1, \partial^1), (C^2, \partial^2)$ are two chain complexes then their direct sum (C, ∂) is defined by

$$C_n = C_n^1 \oplus C_n^2, \quad \forall n; \quad \partial(c^1 \oplus c^2) = \partial^1(c^1) \oplus \partial^2(c^2).$$

It is straightforward to check that $\partial \circ \partial = 0$.

Definition 4.1.2 A sequence of R -modules

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

is said to be *exact* at M if $\text{Ker } \beta = \text{Im } \alpha$. A sequence

$$\cdots \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_{n+1} \longrightarrow \cdots$$

is said to be exact if it is exact at each of the modules M_n . By a short exact sequence we mean an exact sequence of the form

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0. \quad (4.1)$$

It is called a *split exact* sequence, if there exists a R -linear map $s : M'' \rightarrow M$ such that $\beta \circ s = \text{Id}_{M''}$.

This is the same as saying that there is a R -linear map $t : M \rightarrow M'$ such that $t \circ \alpha = I_{M'}$. In that case, M is the direct sum $\alpha(M') \oplus s(M'')$.

Definition 4.1.3 A sequence of chain complexes and chain maps

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0 \quad (4.2)$$

is said to be exact if for each n the corresponding sequence of modules

$$0 \longrightarrow C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} C''_n \longrightarrow 0 \quad (4.3)$$

is exact. (4.2) is said to split if (4.3) is split exact for each n .

Remark 4.1.4 There is a category of short exact sequences of chain complexes of R -modules, defined in an obvious way. These will be used to ‘split-up’ and study longer sequences of modules. We shall now introduce the homology groups to measure the deviation of a chain complex from being exact.

Definition 4.1.5 Given a chain complex C_* , define the *homology group* of C_* to be the graded R -module

$$H_*(C_*) := \bigoplus_{n \in \mathbb{Z}} H_n(C_*)$$

by taking

$$H_n(C_*) := \text{Ker } \partial_n / \text{Im } \partial_{n+1}, \quad \forall n \in \mathbb{Z}.$$

Observe that, if $f : C_* \rightarrow C'_*$ is a chain map then

$$f(\text{Ker } \partial_n) \subset \text{Ker } \partial'_n; \quad f(\text{Im } \partial_{n+1}) \subset \text{Im } \partial'_{n+1}, \quad \forall n,$$

and hence by II-isomomorphism theorem, f induces a graded homomorphism $H_*(f) : H_*(C_*) \rightarrow H_*(C'_*)$ in the obvious way. In addition, this has the *naturality* property, viz., $H_*(Id) = Id$ and if g is another chain map such that $f \circ g$ is defined, then $H_*(f \circ g) = H_*(f) \circ H_*(g)$. Thus, H_* is a *covariant functor* from the category of chain complexes to the category of graded modules.

Given a chain map between two chain complexes which are themselves exact, we need to dig a little deeper as to how the nearby components of the chain map influence each other. This will prepare us with a powerful technique known as ‘*diagram chasing*’. This is going to be an essential tool in the study of homological algebra in general and snake lemma in particular. We shall now state and prove a number of very useful lemmas:

Lemma 4.1.6 Consider the following commutative diagram of R -modules and R -linear maps in which the two rows are exact. If f_1 and f_3 are isomorphisms then so is f_2 .

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & M_3 & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\ 0 & \longrightarrow & N_1 & \xrightarrow{\beta_1} & N_2 & \xrightarrow{\beta_2} & N_3 & \longrightarrow & 0 \end{array}$$

Proof: Suppose $m_2 \in M_2$ is such that $f_2(m_2) = 0$. This implies $f_3(\alpha_2(m_2)) = \beta_2(f_2(m_2)) = 0$ and since f_3 is injective, we get $\alpha_2(m_2) = 0$. By the exactness at M_2 it follows that $m_2 = \alpha_1(m_1)$ for some $m_1 \in M_1$. But then $\beta_1(f_1(m_1)) = f_2(\alpha_1(m_1)) = 0$ and hence $f_1(m_1) \in \text{Ker } \beta_1 = (0)$. Therefore $f_1(m_1) = 0$ since f_1 injective, we get $m_1 = 0$ and hence $m_2 = 0$. This proves f_2 is injective.

Now let us prove that f_2 is surjective. Given $n_2 \in N_2$ consider $\beta_2(n_2) \in N_3$. Since f_3 is surjective, we get some $m_3 \in M_3$ such that $f_3(m_3) = \beta_2(n_2)$. By the surjectivity

of α_2 we get $m_2 \in M_2$ such that $\alpha_2(m_2) = m_3$. Now look at $f_2(m_2)$. We know that $\beta_2(f_2(m_2)) = f_3(\alpha_2(m_2)) = \beta_2(m_3) = 0$. By the exactness at N_2 , there exists $n_1 \in N_1$ such that $\beta_1(n_1) = f_2(m_2)$. By the surjectivity of f_1 , there is $m_1 \in M_1$ such that $f_1(m_1) = n_1$. But then $f_2(\alpha_1(m_1)) = \beta_2(f_1(m_1)) = \beta_2(n_1) = f_2(m_2)$. This just means that $f_2(\alpha_1(m_1) + m_2) = 0$. ♠

We shall leave the proof of the following two results as assignments.

Lemma 4.1.7 (Four lemma) Consider the following commutative diagram of R modules and R -linear maps in which the two rows are exact. Suppose that f_1 is surjective and f_4 is injective. Then

- (i) f_2 is injective $\implies f_3$ is injective.
- (ii) f_3 is surjective $\implies f_2$ is surjective.

$$\begin{array}{ccccccc}
 M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & M_3 & \xrightarrow{\alpha_3} & M_4 \\
 \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\
 N_1 & \xrightarrow{\beta_1} & N_2 & \xrightarrow{\beta_2} & N_3 & \xrightarrow{\beta_3} & N_4
 \end{array}$$

Proof: Assignment.

Lemma 4.1.8 (Five lemma) In the following diagram of R -modules, the two rows are given to be exact. If f_1, f_2, f_4 and f_5 are isomorphisms then f_3 is also an isomorphism:

$$\begin{array}{ccccccccc}
 M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\
 f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & f_4 \downarrow & & f_5 \downarrow \\
 M'_1 & \longrightarrow & M'_2 & \longrightarrow & M'_3 & \longrightarrow & M'_4 & \longrightarrow & M'_5
 \end{array}$$

Proof: Assignment.

4.2 Snake Lemma

The following result is one of the simplest thing that tells how homology behaes under direct sums. The proof of this is also left to the reader as an assign ment.

Theorem 4.2.1 *The homology of a direct sum of chain complexes is isomorphic to the direct sum of the homology of chain complexes.*

Remark 4.2.2 The above property may be named ‘additivity of the homology’. However, we need to work in a more general set-up of (4.2) wherein the module C in the middle may not be the direct sum of the two end-modules. The question is: *do we have $H_*(C)$ isomorphic to the direct sum of $H_*(C')$ and $H_*(C'')$; or at least, do we have a corresponding short exact sequence of the homology modules?* That is the kind of additivity we will be interested in. However, the answer, in general, is NO. So, we are forced to refer to the property of homology groups as in Theorem 4.2.1 ‘weak additivity’. The best thing that we can say about this additivity of homology viz., Theorem 4.2.5 below is one of the very basic result is homological algebra and the following lemma prepares us for it.

Lemma 4.2.3 (Snake lemma) Given a commutative diagram of R -module homomorphisms: where the two horizontal sequences are exact, there exists a R -module homo-

$$\begin{array}{ccccccc}
 M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow & 0 \\
 f' \downarrow & & \downarrow f & & \downarrow f'' & & \\
 0 \longrightarrow & N' & \xrightarrow{\alpha'} & N & \xrightarrow{\beta'} & N'' &
 \end{array}$$

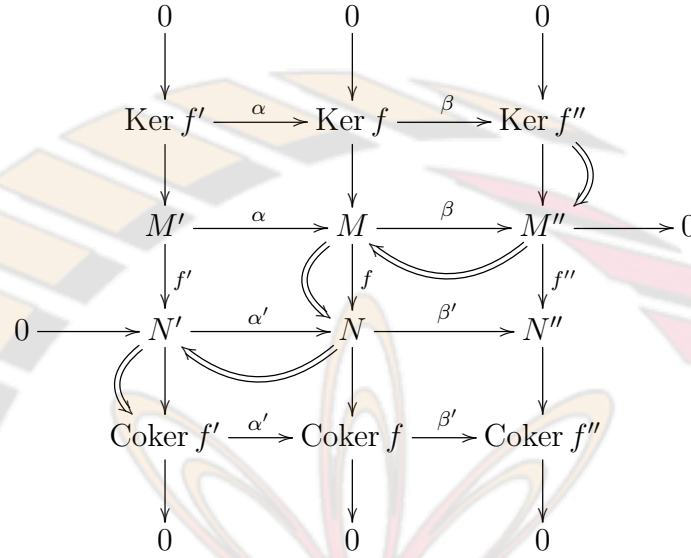
Figure 4.1. Snake lemma

morphism $\delta : \text{Ker } f'' \longrightarrow \text{Coker } f'$, called the *connecting homomorphism* such that the sequence

$$\text{Ker } f' \xrightarrow{\alpha} \text{Ker } f \xrightarrow{\beta} \text{Ker } f'' \xrightarrow{\delta} \text{Coker } f' \xrightarrow{\alpha'} \text{Coker } f \longrightarrow \text{Coker } f''$$

is exact. Moreover, the connecting homomorphism δ has the naturality properties, so that the above assignment of a ‘snake’ to the corresponding ‘six-term’ exact sequence of modules defines a covariant functor.

Proof:



Defining the connecting homomorphism

The homomorphism δ is defined as follows: Given $x'' \in \text{Ker } f''$, pick $x \in M$ such that $\beta(x) = x''$. Then it follows that $\beta' \circ f(x) = 0$ and hence $f(x) \in \text{Im } \alpha'$. Pick $y' \in N'$ such that $\alpha'(y') = f(x)$. Take $\delta(x'') = y' + \text{Im } f'$, the class represented by y' in $\text{Coker } f'$. Observe that, if $x'' = 0$ then we could have picked up anything in $\text{Im } \alpha$ for x . But then $f(x) = \alpha' \circ f'(x')$ for some $x' \in M'$. This would imply that $\delta(x'') = 0$ in $\text{Coker } f'$. This in fact shows that δ is well-defined. That it is a R -module homomorphism and fits the bill in the lemma can be verified in a straightforward way in a similar manner.

We shall verify, say the exactness at $\text{Ker } f''$ and leave the rest of it as an exercise to the reader.

First, we have to show that $\delta \circ \beta = 0$. So let $x'' = \beta x$ where $x \in \text{Ker } f$. In defining δ , we can use this particular x . But then $f(x) = 0$ and hence we are forced to pick up $0 \in N'$ such that $\alpha'(0) = f(x)$. Therefore, $\delta(x'') = 0$. Conversely, let $x'' \in \text{Ker } f''$ be such that $\delta(x'') = 0$. This means that we have $x \in M$ such that $f(x) = \alpha'(y')$ and $y' \in \text{Im } f'$. Say, $y' = f'(x')$ for some $x' \in M'$. Then take $x_1 = x - \alpha(x')$. It follows that $f(x_1) = 0$ and $\beta(x_1) = x''$. Therefore $x'' \in \beta(\text{Ker } f)$. This proves the exactness at $\text{Ker } f''$.

We shall call a commutative diagram of R -modules as in Figure 4.1, with the two horizontal sequences being exact, a ‘snake’. A morphism from one snake to another snake is yet another commutative diagram of R -module homomorphisms as shown in the next

figure.

$$\begin{array}{ccccccc}
 & & M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \longrightarrow 0 \\
 & \swarrow \phi' & \downarrow f' & & \swarrow \phi & \downarrow f & \swarrow \phi'' \\
 P' & \xrightarrow{\tau} & P & \xrightarrow{\sigma} & P'' & \longrightarrow & 0 \\
 \downarrow g' & & \downarrow g & & \downarrow g'' & & \\
 0 \longrightarrow & N' & \xrightarrow{\alpha'} & N & \xrightarrow{\beta'} & N'' \\
 \downarrow \psi' & & \downarrow \psi & & \downarrow \psi'' & & \\
 0 \longrightarrow & Q' & \xrightarrow{\tau'} & Q & \xrightarrow{\sigma'} & Q''
 \end{array}$$

Clearly we then have a diagram of R -module homomorphisms wherein the two horizontal sequences are exact.

$$\begin{array}{ccccccccc}
 \text{Ker } f' & \xrightarrow{\alpha} & \text{Ker } f & \xrightarrow{\beta} & \text{Ker } f'' & \xrightarrow{\delta} & \text{Coker } f' & \xrightarrow{\alpha'} & \text{Coker } f & \xrightarrow{\beta'} & \text{Coker } f'' \\
 \downarrow \phi' & & \downarrow \phi & & \downarrow \phi'' & & \downarrow \psi' & & \downarrow \psi & & \downarrow \psi'' \\
 \text{Ker } g' & \xrightarrow{\tau} & \text{Ker } g & \xrightarrow{\sigma} & \text{Ker } g'' & \xrightarrow{\delta} & \text{Coker } g' & \xrightarrow{\tau'} & \text{Coker } g & \xrightarrow{\sigma'} & \text{Coker } g''
 \end{array}$$

It is straightforward to check that this assignment defines a functor from the category of ‘snakes’ to category of ‘six-terms’. ♠

Remark 4.2.4 The student is advised to go through the definition of connecting homomorphism in the above lemma carefully and memorise it. For, despite all the theories that we are going to develop, often while dealing with the connecting homomorphism, quoting theorems and lemmas does not help—we need to go through this construction of δ itself.

Theorem 4.2.5 *Given a short exact sequence of chain complexes*

$$0 \longrightarrow C'_* \xrightarrow{\alpha} C_* \xrightarrow{\beta} C''_* \longrightarrow 0$$

there is a functorial long exact sequence of homology groups

$$\longrightarrow H_n(C'_*) \xrightarrow{H_n(\alpha)} H_n(C_*) \xrightarrow{H_n(\beta)} H_n(C''_*) \xrightarrow{\delta_n} H_{n-1}(C'_*) \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(C_*) \longrightarrow \quad (4.4)$$

Proof: The chain maps α and β , respectively, induce chain maps of quotient modules:

$$\bar{\alpha} : C'_*/\text{Im } \partial' \longrightarrow C_*/\text{Im } \partial; \quad \bar{\beta} : C_*/\text{Im } \partial \longrightarrow C''_*/\text{Im } \partial''.$$

Also upon restriction to $\text{Ker } \partial'$ and $\text{Ker } \partial$, respectively, they define chain maps:

$$\alpha' : \text{Ker } \partial' \longrightarrow \text{Ker } \partial; \quad \beta' : \text{Ker } \partial \longrightarrow \text{Ker } \partial''.$$

For each n , we then have a snake as follows.

$$\begin{array}{ccccccc}
 C'_n/\text{Im } \partial'_{n+1} & \xrightarrow{\bar{\alpha}_n} & C_n/\text{Im } \partial_{n+1} & \xrightarrow{\bar{\beta}_n} & C''_n/\text{Im } \partial''_{n+1} & \longrightarrow & 0 \\
 \downarrow \partial'_n & & \downarrow \partial_n & & \downarrow \partial''_{n+1} & & \\
 0 \longrightarrow & \text{Ker } \partial'_{n-1} & \xrightarrow{\alpha'_{n-1}} & \text{Ker } \partial_{n-1} & \xrightarrow{\beta'_{n-1}} & \text{Ker } \partial''_{n-1} &
 \end{array}$$

Now notice that the kernel of $[\partial'_n : C'_n/\text{Im } \partial'_{n+1} \rightarrow \text{Ker } \partial'_{n-1}]$ is isomorphic to $\text{Ker } \partial'_n/\text{Im } \partial_{n+1} = H_n(C')$. Likewise the cokernel of this homomorphism is isomorphic to $H_{n-1}(C')$. The same is true of ∂_n and ∂''_n . Therefore the associated ‘six-term’ exact sequence is nothing but

$$H_n(C'_*) \xrightarrow{H_n(\alpha)} H_n(C_*) \xrightarrow{H_n(\beta)} H_n(C''_*) \xrightarrow{\delta_n} H_{n-1}(C'_*) \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(C_*) \xrightarrow{H_{n-1}(\beta)} H_{n-1}(C''_*).$$

Since this is true for all n we obtain (4.4). ♠

4.3 Euler Characteristic

We shall now initiate a discussion on a famous invariant known as Euler-characteristic.

Definition 4.3.1 By an *additive function* on a category of R -modules we mean an integer valued set theoretic function ℓ on the isomorphism classes of R -modules such that whenever we have a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of modules, $\ell(M) = \ell(M') + \ell(M'')$. By a simple induction, one can easily verify that if

$$0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

is exact, then $\sum_i (-1)^i \ell(M_i) = 0$ for any additive function (Exercise). On the category of finitely generated R modules over a ring R , which is a principal ideal domain (PID), the rank function is one of the most important additive functions.

Definition 4.3.2 Let ℓ be an additive function on some category $\mathcal{F}$ of R -modules. A chain complex $C_\bullet$ of R -modules is said to be of *finite type with respect to ℓ* if all C_n are objects in $\mathcal{F}$ and $\ell(C_n) = 0$ for almost all n . In case, R is PID, and ℓ is the rank function, this is the same as saying that all the C_n are of finite rank and most of them have rank equal to zero. We then merely refer to $C_\bullet$ to be of *finite type*.

Remark 4.3.3 Thus, for example, (when $R = \mathbb{Z}$), a finite type chain complex of abelian groups need not be of *finite type with respect to* some additive function other than the rank function.

Theorem 4.3.4 *Let C_* be a chain complex of R -modules of finite type with respect to an additive function ℓ . Then*

$$\sum_n (-1)^n \ell(C_n) = \sum_n (-1)^n \ell(H_n(C_*)).$$

Proof: For each n , we have the short exact sequence

$$0 \longrightarrow \text{Ker } \partial_n \longrightarrow C_n \longrightarrow \text{Im } \partial_n \longrightarrow 0$$

and hence $\ell(C_n) = \ell(\text{Ker } \partial_n) + \ell(\text{Im } \partial_n)$. On the other hand, by definition, $H_n(C_*) = \text{Ker } \partial_n / \text{Im } \partial_{n+1}$. Hence, $\ell(\text{Ker } \partial_n) = \ell(H_n(C_*)) + \ell(\text{Im } \partial_{n+1})$. Substituting these on the left hand side of the equation and observing that

$$\sum_n (-1)^n \ell(\text{Im } \partial_n) = - \sum_n (-1)^n \ell(\text{Im } \partial_{n+1}),$$

establishes the required equality. ♠

Definition 4.3.5 Let R be a PID and M graded module of finite type. (This implies that the rank of M_n is finite for all n and is zero except for finitely many of n .) We define the *Euler characteristic*

$$\chi(M) = \sum_i (-1)^i \text{rk}(M_i),$$

where rk denotes the rank function.

The following theorem is an immediate consequence of Theorem 4.3.4.

Theorem 4.3.6 *Let C_* be a finite type chain complex of R -modules over a PID R . Then*

$$\chi(H(C_*)) = \chi(C_*) \quad (4.5)$$

Remark 4.3.7 The above theorem tells us that the Euler characteristic of a finitely generated chain complex is the same as the Euler characteristic of its homology groups. In our context, R is either the ring of integers or a field. (So, it should not cause you any worry even if you are not familiar with PIDs at this stage.) The notion of the Euler

characteristic plays a very important role in the development of algebraic topology and differential geometry on the whole. It manifests itself in a variety of ways from the simple formula $v - e + f = 2$ of Euler to the Atiyah–Singer index theorem for elliptic differential operators. (See [?] for several equivalent differential topological definitions of Euler characteristic.)

4.4 Chain Homotopy

We shall now introduce the concept of ‘chain-homotopy’ which is the algebraic analogue of homotopy.

Definition 4.4.1 Two chain maps $f, g : C_* \rightarrow C'_*$ of degree 0, are said to be *chain homotopic* if there exists a graded homomorphism $D_* : C_* \rightarrow C'_*$ of degree 1 such that $D \circ \partial + \partial' \circ D = f - g$. It is easily verified (Exercise) that *chain homotopy* is an equivalence relation. The idea behind this definition will be clear when we consider the topological situation, from which it has emerged. The important property of chain homotopy that we are interested in is:

Lemma 4.4.2 Chain homotopic maps induce the same homomorphism of homology groups.

Proof: Exercise. ♠

Remark 4.4.3 By definition a *co-chain complex* is a graded R -module with an endomorphism δ of degree 1 such that $\delta^2 = 0$. The Hom functor converts a chain complex into a cochain complex, viz., given a chain complex

$$\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots$$

denoted by (C_*, ∂) , the following is a co-chain complex:

$$\cdots \xrightarrow{\delta^*} \text{Hom}(C_{n-1}, R) \xrightarrow{\delta^*} \text{Hom}(C_n, R) \xrightarrow{\delta^*} \text{Hom}(C_{n+1}, R) \longrightarrow \cdots$$

denoted by C^* or (C^*, δ^*) where $\delta^*(f) := f \circ \partial$ for $f \in \text{Hom}(C_*, R)$. Cohomology of a chain complex (C_*, ∂) is defined to be $H_*(C^*)$ denoted by $H^*(C_*; R)$. There exists a canonical R -module homomorphism

$$h : H^*(C_*; R) \longrightarrow \text{Hom}(H_*(C_*; R), R)$$

defined as follows: let $\alpha \in H^n(C_*; R)$, i.e., $\alpha \in \text{Ker } \delta^* / \text{Im } \delta^*$. By definition of δ^* , $\alpha \in \text{Ker } \delta^*$ means that $\alpha \circ \partial : C_{n+1} \rightarrow R$ is the 0-map. Now consider the restriction $\text{Res}(\alpha)$ of α to $\text{Ker } \delta^*$. Clearly this gives rise to a welldefined element in $\text{Hom}(H_*(C_*, R), R)$. One can verify that h is an isomorphism if $R = k$ is a field for the simple reason that every linear subspace is a direct summand. The discussion in the general case will be taken in a later chapter.

Exercise 4.4.4

- (i) Let R be a PID. Recall that if M is a finitely generated module over R , and $\text{Tor } M$ denotes the submodule of all torsion elements in M , then the quotient module $M/\text{Tor } M$ is a free module of finite rank which is equal to the rank of M itself. For any endomorphism of $f : M \rightarrow M$, there is an induced endomorphism $\bar{f} : M/\text{Tor } M \rightarrow M/\text{Tor } M$ of the free R -module. Fixing a basis for this module, we can then consider the matrix of $\bar{f}$. We define $\text{tr}(f)$ to be the trace of this matrix, which is independent of the choice of the matrix and hence can be called the trace of f .

(a) Let

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \end{array}$$

be a commutative diagram of finitely generated R -modules and homomorphisms. Show that $\text{tr}(f) + \text{tr}(h) = \text{tr}(g)$.

- (b) Let $C_\bullet$ be a graded module of finite type over R . For any chain endomorphism $f : C_\bullet \rightarrow C_\bullet$ define the *Lefschetz number*

$$L(f) = \sum_n (-1)^n \text{tr}(f_n). \quad (4.6)$$

Further assume that $C_\bullet$ is a chain complex and f is a chain map. If $f_* : H_*(C_\bullet) \rightarrow H_*(C_\bullet)$ is the induced homomorphism on the homology, then show that $L(f_*) = L(f)$. We shall meet this again in Section 4.6.

- (ii) Prove the Five lemma and the Four lemma directly by diagram chasing, i.e., without using the snake lemma.

4.5 Singular Homology Groups

Of all the various kinds of homology theories, the singular homology is the most important. We shall directly construct the singular chain complex of a topological space, discuss basic properties of it such as functoriality, dimension axiom, additivity, excision and homotopy invariance, some of these without proof. The proofs themselves will be given in a separate section. At this stage, it may be a better idea to learn how to use these results than learning the proofs.

We shall also compute the singular homology groups in the simplest cases, viz., for $\mathbb{S}^{n-1}$ and $(\mathbb{D}^n, \mathbb{S}^{n-1})$.

The Construction

Definition 4.5.1 Let X be any topological space and $n \geq 0$ be an integer. By a *singular n -simplex* in X , we mean a continuous map $\sigma : |\Delta_n| \rightarrow X$. By a *singular n -chain* in X , we mean a finite sum $\sum_i n_i \sigma_i$, where $n_i \in \mathbb{Z}$ and σ_i are singular n -simplices in X . We can add any two chains in an obvious way to obtain another chain by simply following the rule

$$n\sigma + m\sigma = (n + m)\sigma \quad (4.7)$$

Then the collection of all singular n -chains in X becomes a free abelian group, with the empty sum playing the role of the zero-element. The collection of all singular n -simplices is then a basis for this group. We denote this group by $S_n(X)$.

If $f : X \rightarrow Y$ is a map of topological spaces, then $\sigma \mapsto f \circ \sigma$ extends to define a graded group homomorphism $f_* : S_*(X) \rightarrow S_*(Y)$. Since $(g \circ f) \circ \sigma = g \circ (f \circ \sigma)$, it follows that the association $X \rightsquigarrow S_*(X)$ defines a functor.

Now if A is a subspace of X and σ is a singular simplex in A then it can be considered as a singular simplex in X via the inclusion map $A \hookrightarrow X$. In this way, $S_n(A)$ becomes a subgroup of $S_n(X)$. We denote the quotient group $S_n(X)/S_n(A)$ by $S_n(X, A)$. Note that $S_n(X, A)$ is a free abelian group with the collection of all singular n -simplexes σ in X whose image is not contained in A as a basis. If $A = \emptyset$, then clearly $S_n(X, A) = S_n(X)$.

We shall now make the graded group $S_*(X, A) = \bigoplus_n S_n(X, A)$ into a chain complex.

Definition 4.5.2 For each integer $r \geq 0$, the *face map* or the *face operator* $F^r : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ is defined by

$$F^r(\mathbf{e}_s) = \begin{cases} \mathbf{e}_s, & \text{if } s < r, \\ \mathbf{e}_{s+1}, & \text{if } s \geq r \end{cases}$$

and extended linearly.

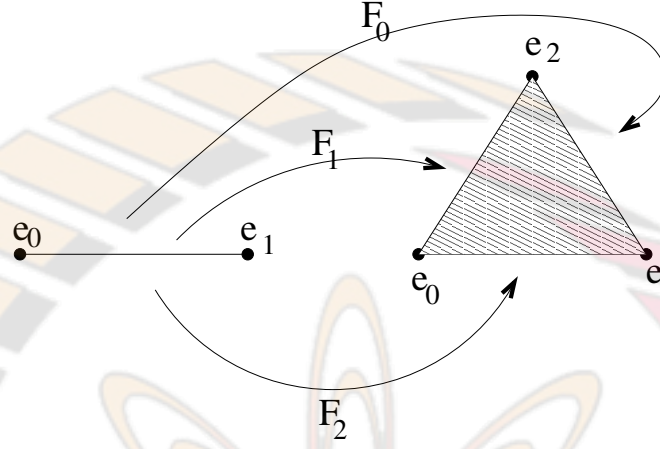


Figure 4.2. The face operators

Remark 4.5.3 Note that for each $n \geq r$, F^r carries Δ_{n-1} onto the $(n-1)$ -face of Δ_n opposite to the vertex e_r in an order preserving manner. So, in order to save on cumbersome notations, we shall denote each F^r restricted to any $\Delta_n, n \geq r$ also by F^r , the exact meaning being understood from the context. Figure 4.2 depicts the face operators F^0, F^1, F^2 restricted to Δ_1 .

Definition 4.5.4 For $n \geq 1$ and for any singular n -simplex σ in X , we define

$$\partial_n(\sigma) = \sum_{r=0}^n (-1)^r \sigma \circ F^r$$

and extend it linearly to obtain homomorphisms $\partial_n : S_n(X, A) \rightarrow S_{n-1}(X, A)$. We define $S_n(X, A) = (0)$ for $n \leq -1$ and $\partial_n = 0$ for $n \leq 0$.

Proposition 4.5.5 $(S(X, A), \partial) = \oplus_n (S_n(X, A), \partial_n)$ is a chain complex and the association $(X, A) \rightsquigarrow (S(X, A), \partial)$ is a functor.

We need to show that $\partial \circ \partial = 0$. For this we need the following lemma, the proof of which is a straightforward exercise.

Lemma 4.5.6 $F^r \circ F^s = F^s \circ F^{r-1}$ if $r > s$.

Proof: Let $k \leq s - 1$. Then both sides fix the vector $\mathbf{e}_k$. So, now let $s \leq k$. Then $F^s(\mathbf{e}_k) = \mathbf{e}_{k+1}$. Therefore

$$F^r \circ F^s(\mathbf{e}_k) = \begin{cases} \mathbf{e}_{k+1} & \text{if } k < r - 1 \\ \mathbf{e}_{k+2} & \text{if } r - 1 \leq k. \end{cases} = F^s \circ F^{r-1}(\mathbf{e}_k).$$

To prove the proposition, we shall first show that $\partial^2(\sigma) = 0$ for any singular n -simplex, $n \geq 2$:

$$\begin{aligned} \partial^2(\sigma) &= \partial(\sum_r (-1)^r \sigma \circ F^r) \\ &= \sum (-1)^{r+s} \sigma \circ F^r \circ F^s \\ &= \sum_{r \leq s} (-1)^{r+s} \sigma \circ F^r \circ F^s + \sum_{r > s} (-1)^{r+s} \sigma \circ F^r \circ F^s \\ &= \sum_{r \leq s} (-1)^{r+s} \sigma \circ F^r \circ F^s + \sum_{r > s} (-1)^{r+s} \sigma \circ F^s \circ F^{r-1} \quad (\diamond) \\ &= \sum_{r \geq s} (-1)^{r+s} \sigma \circ F^r \circ F^s - \sum_{i \geq j} (-1)^{i+j} \sigma \circ F^i \circ F^j \quad (\heartsuit) \\ &= 0 \end{aligned}$$

($\diamond$ = by the lemma above; $\heartsuit$ = by putting $s = i, r - 1 = j$ in the second sum).

Given a map $f : (X, A) \rightarrow (Y, B)$ the fact that f_* defines a chain map follows from the fact that in defining ∂ we take composition with the face operators F^r on the right, whereas, the composition with f is taken on the left and we have $f \circ (\sigma \circ F^r) = (f \circ \sigma) \circ F^r$. This is the essential step in the proof of the functorial property; the rest of the details are left to the reader. ♠

Definition 4.5.7 The *relative singular homology groups* of a topological pair (X, A) are the homology groups $H_*(S_*(X, A))$ and are denoted by $H_*(X, A)$. Taking $A = \phi$, we get the *homology groups* $H_*(X)$.

Remark 4.5.8 Given any commutative ring R , we could have formed the free module $S_n(X; R)$ of all the singular n -chains in X . Exactly as before, we would have got a chain complex $S_*(X; R)$, subchain complex $S_*(A; R)$ and the quotient complex $S_*(X, A; R)$, etc. The corresponding homology groups $H_*(X; R)$, etc., are then called the homology groups X with coefficients in R . Clearly they form a graded R -module by themselves. It is customary to drop the notation R when $R = \mathbb{Z}$.

4.6 Basic Properties of Singular Homology

It is easily seen that $\{S_*(X, A), \partial_*\}$ is a functor from the category of topological pairs to the category of chain complexes; thus $H_*(X, A)$ and $H_*(X)$ are also functors. Given

$f : (X, A) \rightarrow (Y, B)$, we shall denote by f_* , the map induced by f on $S_*(X, A)$ and by f_* the map induced by f on $H_*(X, A)$. The functoriality is summed-up in the following facts:

- (i) Given continuous functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we have, $(g \circ f)_* = g_* \circ f_*$;
- (ii) If Id denotes the identity map on any space, then $(Id)_*$ is again the identity map on the homology.

One of the most fundamental topological properties of $H_*(X, A)$ is the homotopy invariance.

Theorem 4.6.1 (Homotopy invariance) *Let $f, g : (X, A) \rightarrow (Y, B)$ be homotopic to each other. Then $f_* = g_*$ on $H_*(X, A)$.*

Proof: Let $G : (X, A) \times \mathbb{I} \rightarrow (Y, B)$ be a homotopy between f and g (i.e., $G : X \times \mathbb{I} \rightarrow Y$ is a map such that $G(A \times \mathbb{I}) \subset B$, $G(x, 0) = f(x)$, $G(x, 1) = g(x)$, $\forall x \in X$). Consider the inclusion maps $\eta_t : (X, A) \rightarrow (X, A) \times \mathbb{I} = (X \times \mathbb{I}, A \times \mathbb{I})$ given by

$$\eta_t(x) = (x, t), \quad t \in \mathbb{I}.$$

Then, $G \circ \eta_0 = f$ and $G \circ \eta_1 = g$. Passing onto homology, this implies that $G_* \circ (\eta_0)_* = G_* \circ (\eta_1)_*$. Therefore it suffices to prove that $(\eta_0)_* = (\eta_1)_* : H_*(X, A) \rightarrow H_*((X, A) \times \mathbb{I})$. For this purpose, we construct a chain homotopy $h : S_*(X, A) \rightarrow S_*(X \times \mathbb{I}, A \times \mathbb{I})$ between $(\eta_0)_*$ and $(\eta_1)_*$ (at the chain group level). This chain homotopy is called *prism operator*. For future reference we shall state this as a separate lemma:

Lemma 4.6.2 There exist functorial chain homotopies

$$h : S_q(X, A) \rightarrow S_{q+1}((X, A) \times \mathbb{I}, \quad q \geq 0,$$

between the two inclusion induced maps $\eta_0, \eta_1 : S_*(X, A) \rightarrow S_*((X, A) \times \mathbb{I})$.

We shall postpone the proof of this lemma to the last section, and take for granted the Theorem 4.6.1 as proved.

As an immediate consequence, we obtain:

Corollary 4.6.3 Homotopy-equivalent topological spaces have isomorphic homology groups; in particular, a contractible space has the homology groups of a point-space.

Example 4.6.4

(i) **Homology of a point space**

Let n be any positive integer. There is a unique singular n -simplex $\sigma : |\Delta_n| \rightarrow \{\star\}$ and hence $S_n(\star) \approx \mathbb{Z}$ for each $n \geq 0$. Observe that $\partial : S_{2n+1}(\star) \rightarrow S_{2n}(\star)$ is the zero-map because Δ_{2n+1} has an even number of $(2n)$ -faces and the terms in $\partial(\sigma)$ cancel in pairs. However, similar reasoning tells us that $\partial : S_{2n}(\star) \rightarrow S_{2n-1}(\star)$ is an isomorphism. Thus we have a chain complex

$$\cdots \quad \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

Thus it follows that $H_0(\star) \simeq \mathbb{Z}$ and $H_i(\star) = (0)$ for all $i > 0$.

(ii) **Homology of path components**

Let $\{X_j : j \in J\}$ be the set of path connected components of X so that $X = \sqcup_{j \in J} X_j$. Since $|\Delta_n|$ is a path connected space, it follows that any singular n -simplex in X is contained completely in one of the path components X_j . Therefore, it follows that the entire chain complex $S(X)$ is the direct sum of the subchain complexes $S(X_j)$. Hence the homology is also a direct sum of $H_*(X_j), j \in J$. Thus in practice, we can often assume that the space itself is path connected, while dealing with its homology groups.

(iii) **Homology long exact sequence of the pair (X, A)**

By definition of $S(X, A)$, we have an exact sequence of chain complexes

$$0 \rightarrow S(A) \rightarrow S(X) \rightarrow S(X, A) \rightarrow 0.$$

By Theorem 4.2.5, this yields the long exact sequence of homology groups

$$\left. \begin{aligned} \cdots \rightarrow H_i(A) \rightarrow H_i(X) \rightarrow H_i(X, A) \xrightarrow{\delta} H_{i-1}(A) \rightarrow \cdots \\ \cdots \rightarrow H_1(X, A) \xrightarrow{\delta} H_0(A) \rightarrow H_0(X) \rightarrow H_0(X, A), \end{aligned} \right\} \quad (4.8)$$

which is functorial. This sequence comes handy in many computations as we shall see soon.

(iv) **Reduced homology and H_0 of a path connected space**

Let us compute $H_0(X)$ for any path connected space X . Note that $S_0(X)$ is a free abelian group on the underlying set of the space X . For any singular 1-simplex σ in X , we have $\partial_1(\sigma) = \sigma(\mathbf{e}_1) - \sigma(\mathbf{e}_0)$. We define $\varepsilon : S_0(X) \rightarrow \mathbb{Z}$ by $\varepsilon(\sum n_i x_i) = \sum n_i$. Then ε is a surjective homomorphism. Since $\varepsilon \circ \partial_1(\sigma) = 0$ for all 1-simplexes σ , it

follows that $\epsilon \circ \partial_1 = 0$. Finally to show that $\text{Ker } \epsilon \subset \text{Im } \partial_1$, consider an element $\sum_i n_i x_i$ with $\sum_i n_i = 0$. Choose any point $z_0 \in X$. We can then join z_0 to each x_i by a path σ_i which can be treated as a singular 1-simplex. Then

$$\partial_1\left(\sum_i n_i \sigma_i\right) = \sum_i n_i x_i - \left(\sum_i n_i\right) z_0 = \sum_i n_i x_i.$$

This proves that $\partial_1(S_1(X)) = \text{Ker } \epsilon$. Hence

$$H_0(X) = S_0(X)/\partial_1(S_1(X)) = S_0(X)/\text{Ker } \epsilon \approx \mathbb{Z}.$$

The homomorphism $\epsilon : S_0(X) \rightarrow \mathbb{Z}$ is called the *augmentation* homomorphism. We define the extended chain complex $\tilde{S}(X)$ by

$$\tilde{S}_n(X) = \begin{cases} S_n(X), & \text{if } n \geq 0, \\ \mathbb{Z}, & \text{if } n = -1, \\ (0), & \text{if } n < -1; \end{cases} \quad \text{and} \quad \tilde{\partial}_n = \begin{cases} \partial_n, & n \geq 1, \\ \epsilon, & n = 0, \\ 0, & n < 0. \end{cases}$$

The homology groups of this complex are denoted by $\tilde{H}_*(X)$ and are called *reduced homology groups*. Clearly, $\tilde{H}_i(X) = (0)$ for $i < 0$. Also $\tilde{H}_i(X) \approx H_i(X)$ for $i > 1$. However in dimension zero, we have $\tilde{H}_0(X) \oplus \mathbb{Z} \approx H_0(X)$. In particular, for a path connected space $\tilde{H}_0(X)$ is (0) .

Remark 4.6.5

(a) Recall that a contractible space has the homotopy type of a point space and hence by homotopy invariance, it follows that every contractible space has homology isomorphic to that of a point space. Now the only non zero term in the homology of a point space is H_0 which is infinite cyclic. The motivation for introducing the augmentation and thereby the extended chain complex may be attributed to a desire to have homology groups completely vanishing for contractible spaces.

(b) Note that for any non empty subspace $A \subset X$, we have $\tilde{H}_*(X, A) = H_*(X, A)$ and (4.8) is valid if we replace H_i by $\tilde{H}_i$ everywhere. You have to do some checking only at the last three groups.

4.7 Excision

At this stage, we still do not have any powerful tool to compute the homology. We shall now discuss one of the most important tools, viz., *Mayer-Vietoris* in the computation of singular homology.

Given subsets X_1 and X_2 of a topological space X , let us denote the inclusion maps of X_i into X by $\eta_i, i = 1, 2$. They induce inclusion maps of the chain complexes which we shall denote again by $\eta_i : S_*(X_i) \rightarrow S_*(X), i = 1, 2$. Now consider the chain map

$$(\eta_1, -\eta_2) : S_*(X_1) \oplus S_*(X_2) \rightarrow S_*(X).$$

What is the kernel of this map? Remember that $S_*(X)$ is a free module. Thus, if $\alpha_i \in S_*(X_i), i = 1, 2$ are such that $\alpha_1 - \alpha_2 = 0$, then it follows that $\alpha_1 = \alpha_2 \in S_*(X_1 \cap X_2)$. Therefore the kernel of the above chain map is $S_*(X_1 \cap X_2)$. Clearly, the image of this chain map is the submodule generated by $S_*(X_1)$ and $S_*(X_2)$ in $S_*(X)$ which we shall denote by $S_*(X_1) + S_*(X_2)$ so that we have a short exact sequence of chain complexes

$$0 \rightarrow S_*(X_1 \cap X_2) \rightarrow S_*(X_1) \oplus S_*(X_2) \rightarrow S_*(X_1) + S_*(X_2) \rightarrow 0 \quad (4.9)$$

This will then give a long exact sequence of homology modules from which it would be possible to get a lot of information on $H_*(S_*(X_1) + S_*(X_2))$ from $H_*(X_1), H_*(X_2)$ and $H_*(X_1 \cap X_2)$. The crucial question is the following:

Question: *Can we replace the modules $H_*(S_*(X_1) + S_*(X_2))$ with $H_*(X)$; if not always, at least under some suitable conditions?*

Toward an affirmative answer to this question, we proceed, beginning with the following technical lemma.

Lemma 4.7.1 Let $X = A \cup B$. Then the following statements are equivalent.

- (a) $S_*(A) + S_*(B) \rightarrow S_*(X)$ induces isomorphisms in homology.
- (b) $[S_*(A) + S_*(B)]/S_*(B) \rightarrow S_*(X)/S_*(B)$ induces isomorphisms in homology.
- (c) $[S_*(A) + S_*(B)]/S_*(A) \rightarrow S_*(X)/S_*(A)$ induces isomorphisms in homology.
- (d) $S_*(A)/S_*(A \cap B) \rightarrow S_*(X)/S_*(B)$ induces isomorphisms in homology.
- (e) $S_*(B)/S_*(A \cap B) \rightarrow S_*(X)/S_*(A)$ induces isomorphisms in homology.

Proof: The equivalence of (a) and (b) is a consequence of the Five lemma applied to the ladder of homology long exact sequences given by the ladder of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_*(B) & \longrightarrow & S_*(A) + S_*(B) & \longrightarrow & [S_*(A) + S_*(B)]/S_*(B) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S_*(B) & \longrightarrow & S_*(X) & \longrightarrow & S_*(X)/S_*(B) \longrightarrow 0. \end{array}$$

The equivalence of (b) and (d) follows from the commutative diagram below in which the horizontal arrow is the isomorphism given by the Nöether's isomorphism theorem.

$$\begin{array}{ccc}
 S_*(A)/S_*(A \cap B) & \xrightarrow{\quad\quad\quad} & [S_*(A) + S_*(B)]/S_*(B) \\
 & \searrow \quad \swarrow & \\
 & S_*(X)/S_*(B) &
 \end{array}$$

Interchanging A, B gives the equivalence of (a), (c) and (e). ♠

Definition 4.7.2 Let X be a topological space, A, B be two subspaces such that $X = A \cup B$. We then say that the inclusion map $(A, A \cap B) \hookrightarrow (X, B)$ an excision map. A pair $\{A, B\}$ of subspaces of a space X is called an *excisive couple* for singular homology, if the inclusion map

$$S_*(A) + S_*(B) \hookrightarrow S_*(A \cup B)$$

induces isomorphisms in homology or if any of the other four equivalent conditions in the above lemma is satisfied.

Condition (d) of this lemma immediately gives the following theorem.

Theorem 4.7.3 (Homology excision) *The inclusion map $(A, A \cap B) \hookrightarrow (A \cup B, B)$ of an excisive couple $\{A, B\}$ induces an isomorphism of homology groups $H_*(A, A \cap B) \rightarrow H_*(A \cup B, B)$.*

Note that whether a pair $\{A, B\}$ is excisive or not, the inclusion map $(A, A \cap B) \hookrightarrow (A \cup B, B)$ is an excision map. This terminology refers to the set-theoretic fact, viz., $A \cup B \setminus B = A \setminus A \cap B$. Not all pairs are excisive couples. The basic examples of excisive couples are provided by the following theorem.

Theorem 4.7.4 *If $X = X_1 \cup X_2 = \text{int}_X(X_1) \cup \text{int}_X(X_2)$ then, $\{X_1, X_2\}$ is an excisive couple for the singular homology.*

We shall postpone the proof of this theorem to a later section. The reader is advised to skip the proof of this theorem in the first reading, and come back to it at leisure. The knowledge of the proof is not all that essential in understanding most of the material that follows. However, any serious student of algebraic topology has to read at least one proof of this theorem and at least once. For the present, the following remark suffices: Given a singular n -simplex σ in X , one subdivides Δ_n in such a way that, the image of each simplex of this subdivision is contained in X_1 or X_2 . One thinks of the original

singular simplex σ as an appropriate sum of these little pieces. Strictly speaking though, they are different chains and most of the effort is to show that they represent the same element in the homology groups. Of course one has to do all these in a canonical fashion.

Example 4.7.5 Here are a few examples of excisive couples which occur in practice.

(a) $\{\mathbb{D}^n, \mathbb{R}^n \setminus \{0\}\}$ is an excisive couple and hence the inclusion map $(\mathbb{D}^n, \mathbb{D}^n \setminus \{0\}) \hookrightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ will induce isomorphisms in all singular homology groups.

(b) Let N denote the north pole on $\mathbb{S}^n$ and $\bar{U}$ denote the closed upper hemisphere. Then $\{\mathbb{S}^n \setminus \{N\}, \bar{U}\}$ is an excisive couple, which gives isomorphisms of $H_*(\bar{U}, \bar{U} \setminus \{N\}) \rightarrow H_*(\mathbb{S}^n, \mathbb{S}^n \setminus \{N\})$.

In practical situations, often X_1, X_2 are closed subspaces and hence the above theorem cannot be applied immediately. Then the theorem below comes to the rescue. We need to prepare ourselves with yet another important result.

Lemma 4.7.6 Let $\hookrightarrow X$ be a closed inclusion and $\hookrightarrow A$ a cofibration. Then the quotient map $q : X \rightarrow X/A$ induces isomorphism in singular homology, $q_* : H_*(X, A) \rightarrow H_*(X/A; \star)$.

Proof: $A \hookrightarrow X$ is a cofibration, implies that $CA \hookrightarrow X \cup CA$ is a cofibration, where CA denotes the cone over A . (See Exercise ???.?) By Theorem ??, it follows that the quotient map $q : X \cup CA \rightarrow X \cup CA/CA$ is a homotopy equivalence. Consider the following diagram

$$\begin{array}{ccccc}
 & H_*(X, A) & \xrightarrow{q_*} & H_*(X/A, \star) & \\
 & \swarrow \alpha & & \downarrow h_* & \\
 H_*(X_1, X_1 \cap X_2) & \xrightarrow{\eta} & H_*(X \cup CA, CA) & \xrightarrow{p'_*} & H_*(X \cup CA/CA, \star) \\
 & \nwarrow \beta & \searrow p_* & & \\
 & H_*(X \cup CA, \star) & & &
 \end{array}$$

in which we want to prove that q_* is an isomorphism, We have:

- (i) h_* is an isomorphism because h is a homeomorphism;
- (ii) $X_1 = X \cup A \times [0, 1/2]$, $X_2 = CA$, so that the pair $(X_1, X_1 \cap X_2)$ deformation retracts onto the pair (X, A) and hence we have the inclusion induced isomorphism α .
- (iii) The pair $\{X_1, X_2\}$ satisfies the condition in the excision theorem above and hence is an excisive couple which implies that η is an isomorphism.
- (iv) CA is contractible. Therefore, from the homology exact sequence of the pair

$(X \cup CA, CA)$, β is an isomorphism.

(v) p is a homotopy equivalence and therefore p_* is an isomorphism.

It follows that p'_* is an isomorphism and therefore q_* is an isomorphism. ♠

Theorem 4.7.7 *Let $X = A \cup B$ where A, B are closed subspaces. If $A \hookrightarrow X$ or $B \hookrightarrow X$ is a cofibration, then $\{A, B\}$ is an excisive couple.*

Proof: We shall prove condition (e) of Lemma 4.7.1. This is the same as showing that $H_*(B, B \cap A) \rightarrow H_*(X, A)$ is an isomorphism. Note that $A \hookrightarrow A \cup B$ is a cofibration implies $B \cap A \hookrightarrow B$ is a cofibration. We now consider the quotient map $q : X \rightarrow X/A$. Since B is a closed subspace of X , it follows that the restriction $q' : B \rightarrow X/A = B/B \cap A$ is also a quotient map. From the above lemma, both q, q' induce isomorphism in the relative homology and hence we are done. ♠

Corollary 4.7.8 A pair $\{X_1, X_2\}$ of closed subspaces of a space is an excisive couple for singular homology under the following conditions:

- (a) $(X_i, X_1 \cap X_2)$ is a relative CW-complex for $i = 1$ or $i = 2$.
- (b) $X_i = |K_i|$ where K_i are simplicial subcomplexes of a simplicial complex K with $|K| = X$.

Proof: For (a), use Theorems 2.7.7 and 4.7.7. Statement (b) is a special case of (a). ♠

4.8 Mayer-Vietoris

We shall now concentrate on putting the excision theorem to good use, by first deriving an important ready-to-use result:

If $\{X_1, X_2\}$ is an excisive couple then, in the long exact sequence of homology groups given by (4.9), we can replace the groups $H_*(S_*(X_1) + S_*(X_2))$ by $H_*(X_1 \cup X_2)$ to obtain the **Mayer–Vietoris** sequence:

$$\left. \begin{aligned} &\rightarrow H_{i+1}(X_1 \cup X_2) \xrightarrow{\partial} H_i(X_1 \cap X_2) \xrightarrow{i_*} H_i(X_1) \oplus H_i(X_2) \xrightarrow{j_*} H_i(X_1 \cup X_2) \rightarrow \cdots \\ &\rightarrow H_1(X_1 \cup X_2) \xrightarrow{\partial} H_0(X_1 \cap X_2) \rightarrow H_0(X_1) \oplus H_0(X_2) \rightarrow H_0(X_1 \cup X_2) \end{aligned} \right\} (4.10)$$

with

$$i_*(z) = (i_{1*}z, -i_{2*}z); \quad j_*(z_1, z_2) = j_{1*}z_1 + j_{2*}z_2 \quad (4.11)$$

where,

$$i_1 : X_1 \cap X_2 \hookrightarrow X_1, \quad i_2 : X_1 \cap X_2 \hookrightarrow X_2$$

$$j_1 : X_1 \hookrightarrow X_1 \cup X_2, \quad j_2 : X_2 \hookrightarrow X_1 \cup X_2$$

are inclusion maps. The Mayer–Vietoris sequence is one of the major ready-to-use-tools in computing homology.

As a simple consequence of Mayer–Vietoris, we shall prove:

Theorem 4.8.1 (Homology suspension theorem) *Let X be a topological space. Then there is a canonical isomorphism*

$$S : \tilde{H}_n(X) \rightarrow H_{n+1}(SX), \quad n \geq 0,$$

where SX denotes the suspension of X ($SX = \mathbb{S}^0 \star X$).

Proof: Note that the suspension SX is always path connected and hence $H_*(SX) \approx \tilde{H}_*(SX)$ in positive dimensions. So, we shall establish isomorphisms $S : \tilde{H}_n(X) \rightarrow \tilde{H}_{n+1}(SX), n \geq 0$. We have:

$$\mathbb{S}^0 \star X = \{-1, 1\} \star X$$

is the union of two cones over X , viz, $\{N\} \star X$ and $\{S\} \star X$ identified along their common base X . Also for any cone $\{v\} \star X$, we have $\{v\} \star X \setminus \{v\}$ is homeomorphic to $(0, 1] \times X$, which can be deformed to $1 \times X$. It follows that both $\mathbb{S}^0 \star X \setminus \{S\} = A_{-1}$ and $\mathbb{S}^0 \star X \setminus \{N\} = A_1$ are contractible and $X \hookrightarrow A_{-1} \cap A_1$ is a deformation retract. Since A_1 and A_{-1} are both open in $\mathbb{S}^0 \star X$, we have a Mayer–Vietoris sequence

$$\tilde{H}_{i+1}(A_1) \oplus \tilde{H}_{i+1}(A_{-1}) \rightarrow \tilde{H}_{i+1}(\mathbb{S}^0 \star X) \xrightarrow{\partial} \tilde{H}_i(A_1 \cap A_{-1}) \rightarrow \tilde{H}_i(A_1) \oplus \tilde{H}_i(A_{-1})$$

which yields an exact sequence

$$0 \rightarrow \tilde{H}_{i+1}(SX) \xrightarrow{\partial} \tilde{H}_i(A_1 \cap A_{-1}) \rightarrow 0$$

for each i and hence the connecting homomorphism ∂_i is an isomorphism. Being a deformation retract, $\eta : X \hookrightarrow A_1 \cap A_{-1}$ also induces isomorphisms in homology. The composite

$$S : \tilde{H}_i(X) \xrightarrow{\eta_*} \tilde{H}_i(A_1 \cap A_{-1}) \xrightarrow{\partial_i^{-1}} \tilde{H}_{i+1}(SX)$$

is therefore, an isomorphism. ♠

Remark 4.8.2

1. Whenever a pair of closed subspaces $\{X_1, X_2\}$ is given, introducing intermediate pairs of open sets $\{U_1, U_2\}$ such that X_i are deformation retract of U_i is a typical way excision Theorem 4.7.4 is used in practice. This is the role played by subsets $A_{\pm}$ in the proof of the above theorem. Alternatively, we could have used Theorem 4.7.7 as follows: Note that the inclusion $X \hookrightarrow CX$ is a cofibration the inclusion $\{N\} * X \hookrightarrow SX$ is a cofibration. Therefore the pair $\{\{N\} * X, \{S\} * X\}$ is an excisive couple.
2. Here is an explicit description of the homology suspension homomorphism S . For any singular n -simplex σ in X , let $[v]\sigma$ denote the $n+1$ simplex in the cone $\{v\} \star X$ which sends e_0 to v and e_s to $\sigma(e_{s-1})$ for $s \geq 1$. Let $S(\sigma) = [N]\sigma - [S]\sigma$. Then S can be extended linearly to a homomorphism $S : S_n(X) \rightarrow S_{n+1}(SX)$. It is a matter of straightforward verification that S is a chain map and hence gives a homomorphism of the homologies. Indeed, directly going through the definition of ∂ we see that $\partial S c = c$ for any cycle c in X and therefore, S is equal to $\partial^{-1} \circ \eta_*$.

Example 4.8.3 Computation of $H_*(\mathbb{S}^n)$ We know that $\mathbb{S}^n$ is homeomorphic to the suspension of $\mathbb{S}^{n-1}$. Therefore, starting with the homology of $\mathbb{S}^0$, one can directly apply the homology suspension theorem above inductively, and compute the homology of all spheres. At each step the missing information is about $\tilde{H}_0(\mathbb{S}^n), n \geq 1$ which clearly vanishes since $\mathbb{S}^n, n \geq 1$ are path connected. Therefore

$$\tilde{H}_k(\mathbb{S}^n) = \begin{cases} 0, & k \neq n; \\ \mathbb{Z}, & k = n. \end{cases} \quad (4.12)$$

We would like to know explicitly representative cycles of elements of these simplest homology groups $H_n(\mathbb{S}^n) \approx \mathbb{Z}$. For this we shall use the S-triangulation of the spheres introduced in Example ??, Part-I. We need to introduce some elaborate notation here. Let us begin with denoting the two point space $\mathbb{S}^0$ by $\{u_0, v_0\}$. However, when we are taking suspension of this space, we need another copy of $\mathbb{S}^0$ which we shall denote by $\{u_1, v_1\}$. Thus the S-triangulation of $\mathbb{S}^1$ consists of four edges $\{u_0, u_1\}, \{v_0, u_1\}, \{u_0, v_1\}, \{v_0, v_1\}$. Inductively, we shall use $\{u_n, v_n\}$ to denote the n^{th} -copy of $\mathbb{S}^0$ and then the n -simplexes of the S-triangulation of $\mathbb{S}^n$ will be of the form $\sigma \cup \{u_n\}$ or $\sigma \cup \{v_n\}$ where σ runs over all the $(n-1)$ -simplexes of $\mathbb{S}^{n-1}$.

Now if σ is any singular k -simplex in $\mathbb{S}^{n-1}$, we shall denote by $[\sigma, u_n]$ the $(k+1)$ -simplex in $\mathbb{S}^n$ defined by

$$[\sigma, u_n](t\alpha + (1-t)\mathbf{e}_{k+1}) = t\sigma(\alpha) + (1-t)u_n, \quad \alpha \in |\Delta_k|, \quad 0 \leq t \leq 1.$$

Similarly, the singular $(k+1)$ -simplex $[\sigma, v_n]$ is defined. Now if $c = \sum_i n_i \sigma_i$ is any singular k -chain in $\mathbb{S}^{n-1}$, we define $S(c) = \sum_i n_i [\sigma_i, v_n] - \sum_i n_i [\sigma_i, u_n]$. Finally, we define a n -chain g_n in $\mathbb{S}^n$ inductively by the formula,

$$g_0 = v_0 - u_0; \quad g_n = S(g_{n-1}), \quad n \geq 1.$$

We claim that g_n is a n -cycle which represents a generator of $\tilde{H}_n(\mathbb{S}^n)$. For $n = 0$, we have seen this in example 4.6.4.(iv). For $n \geq 1$, observe that $\partial(g_n) = g_{n-1}$ where ∂ is the connecting homomorphism of the Mayer–Vietoris sequence as in the previous theorem. We need to merely go through the initial steps in the proof of the snake lemma in defining this connecting homomorphism. This implies $\eta_*^{-1}\partial(g_n) = (g_{n-1})$ and since $\eta_*^{-1}\partial$ is an isomorphism, we are through.

Example 4.8.4 Homology of the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$

Since $\mathbb{D}^n$ is contractible, its reduced homology vanishes. On the other hand, from the previous example we know $H_*(\mathbb{S}^{n-1})$. Piecing them together via the long exact sequence

$$\cdots \longrightarrow H_i(\mathbb{D}^n) \longrightarrow H_i(\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow H_i(\mathbb{S}^{n-1}) \longrightarrow \cdots$$

we get

$$H_i(\mathbb{D}^n, \mathbb{S}^{n-1}) = \begin{cases} 0, & i \neq n; \\ \mathbb{Z}, & i = n. \end{cases} \quad (4.13)$$

Example 4.8.5 Homology of the torus

As another illustration of application of Mayer–Vietoris sequence, we shall compute the singular homology groups of $\mathbb{S}^1 \times \mathbb{S}^1$. Take $U_{\pm} = \mathbb{S}^1 \setminus \{\mp 1\}$. Then $\mathbb{S}^1 \times U_{\pm}$ are open in $\mathbb{S}^1 \times \mathbb{S}^1$, and

$$\mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{S}^1 \times U_+ \cup \mathbb{S}^1 \times U_-.$$

Therefore we can apply Mayer–Vietoris (4.10) to get an exact sequence. Since $\mathbb{S}^1 \times \mathbb{S}^0 \hookrightarrow (\mathbb{S}^1 \times U_+) \cap (\mathbb{S}^1 \times U_-)$ is a SDR, we can replace $H_*((\mathbb{S}^1 \times U_+) \cap (\mathbb{S}^1 \times U_-))$ by $H_*(\mathbb{S}^1 \times \mathbb{S}^0)$. Therefore we have,

$$\begin{aligned}
0 &\longrightarrow H_2(\mathbb{S}^1 \times \mathbb{S}^1) \xrightarrow{\partial} H_1(\mathbb{S}^1 \times \mathbb{S}^0) \xrightarrow{i_*} \\
&H_1(\mathbb{S}^1 \times U_+) \oplus H_1(\mathbb{S}^1 \times U_-) \xrightarrow{j_*} H_1(\mathbb{S}^1 \times \mathbb{S}^1) \xrightarrow{\partial} \\
&H_0(\mathbb{S}^1 \times \mathbb{S}^0) \xrightarrow{i_*} H_0(\mathbb{S}^1 \times U_+) \oplus H_0(\mathbb{S}^1 \times U_-) \xrightarrow{j_*} H_0(\mathbb{S}^1 \times \mathbb{S}^1).
\end{aligned}$$

The first term in the above sequence actually corresponds to $H_2(\mathbb{S}^1 \times U_+) \oplus H_2(\mathbb{S}^1 \times U_-)$. Since $U_{\pm}$ are contractible, we have $H_*(\mathbb{S}^1 \times U_{\pm}) \approx H_*(\mathbb{S}^1)$. We can therefore use (4.12). In particular, the first entry is 0 as we have indicated above. For the same reason, the third, fourth, sixth and seventh terms are isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. All the higher homology terms, which we have not written, vanish. Since $\mathbb{S}^1 \times \mathbb{S}^1$ is connected the last term is $\mathbb{Z}$. Clearly j_* is surjective on H_0 level and hence its kernel is isomorphic to $\mathbb{Z}$. Therefore, the kernel of i_* is also infinite cyclic and hence $\partial : H_1(\mathbb{S}^1 \times \mathbb{S}^1) \rightarrow H_0(\mathbb{S}^1 \times \mathbb{S}^0)$ has an infinite cyclic image. From (4.11) it follows that $j_*(1, 0)$ is non zero and $j_*(1, -1) = 0$. Therefore both the image and kernel of $j_* : H_1(\mathbb{S}^1 \times U_+) \oplus H_1(\mathbb{S}^1 \times U_-) \rightarrow H_1(\mathbb{S}^1 \times \mathbb{S}^1)$ are infinite cyclic. Therefore, it follows that $H_1(\mathbb{S}^1 \times \mathbb{S}^1) \approx \text{Im } j_* \oplus \text{Im } \partial = \mathbb{Z} \oplus \mathbb{Z}$. It also follows that the kernel of $i_* : H_1(\mathbb{S}^1 \times \mathbb{S}^0) \rightarrow H_1(\mathbb{S}^1 \times \mathbb{S}^1)$ is infinite cyclic and hence $H_2(\mathbb{S}^1 \times \mathbb{S}^1) \approx \mathbb{Z}$. And $H_i(\mathbb{S}^1 \times \mathbb{S}^1) = 0$ for $i > 2$. In conclusion, we have,

$$H_i(\mathbb{S}^1 \times \mathbb{S}^1) = \begin{cases} \mathbb{Z}, & i = 0, 2; \\ \mathbb{Z} \oplus \mathbb{Z}, & i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.8.6

1. The example is a typical way Mayer–Vietoris can be employed in specific situations. Later, we shall have more elegant proof of the fact proved in the above example. The reader can now try the exercise below to get more familiar with Mayer–Vietoris.
2. Incidentally, the discussion in Example 4.8.3 provides a proof of the fact that in Theorem 4.8.1, the isomorphism $H_i(X) \rightarrow H_{i+1}(SX)$ is actually obtained by ‘suspending’ each cycle and is therefore canonical.
3. The functoriality, the homotopy invariance and the excision property of the singular homology are so important that they have been raised to the status of ‘axioms’.

for homology'. Most often, in deriving a certain result concerning singular homology, we need not appeal to the actual construction of singular homology but only to these axioms. Indeed, along with one more property called 'dimension axiom' discussed in Example 4.6.4.(i), Eilenberg and Steenrod ([?]) proved that all homology theories on the category of compact polyhedrons coincide. At that time genuine examples of 'homology theories' which may not satisfy the dimension axiom (such as K-theory) were not known. (Of course, one can artificially 'shift' the dimension even for the singular homology by merely defining $H_i^\dagger(X) = H_{i+k}(X)$ for a fixed k .) However, notice that property discussed in Example 4.6.4.(ii), viz., being a direct sum of homologies of path components, is not a consequence of these axioms.

4. With all this, at this stage, we still do not have any 'good way' of computing singular homology. Even in the simple case of a map $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$, though the homology modules are known, we do not know how to compute $f_* : H_n(\mathbb{S}^n) \rightarrow H_n(\mathbb{S}^n)$. In the next section, we shall introduce a number of alternative homology modules as tools which help us in this task.

Exercise 4.8.7

- (i) Compute the homology of $\mathbb{S}^p \times \mathbb{S}^q$ using Mayer–Vietoris.
- (ii) Compute $H_*(\vee_k \mathbb{S}^{n_k})$ where $\vee_k \mathbb{S}^{n_k}$ denotes the bouquet of spheres of varying dimensions n_k . In particular, compute $H_*(\vee_k \mathbb{S}^n)$ where each $n_k = n$.
- (iii) Use the generator of $H_n(\mathbb{S}^n; \mathbb{Z})$ as given in Example 4.8.3 to compute the induced homomorphism $(\alpha_n)_* : H_n(\mathbb{S}^n) \rightarrow H_n(\mathbb{S}^n)$, where $\alpha_n : \mathbb{S}^n \rightarrow \mathbb{S}^n$ is the antipodal map. [Since this group is isomorphic to $\mathbb{Z}$ and any homomorphism $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ is given by multiplication by an integer, given any continuous map $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$, we define $\deg f$ to be the integer m such that $f_*(c) = mc$ on $H_n(\mathbb{S}^n)$. We call this the *degree* of f .

Now, computing $(\alpha_n)_*$ simply means that we have to determine $\deg \alpha_n$.]

- (iv) Compute the degree of maps $\mathbb{S}^1 \rightarrow \mathbb{S}^1$ given by $z \mapsto z^n$ and $z \mapsto \bar{z}$ and compare with Exercise ???. What do you conclude?
- (v) Compute the degree of any reflection in a plane $r : \mathbb{S}^n \rightarrow \mathbb{S}^n$ say,

$$r : (x_0, x_1, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n).$$

More generally compute the degree of $f_k : \mathbb{S}^n \rightarrow \mathbb{S}^n$ given by

$$f_k(x_0, x_1, \dots, x_n) = (-x_0, \dots, -x_k, x_{k+1}, \dots, x_n).$$

- (vi) (Try this exercise only if you know the definition of degree of a smooth map between two compact, oriented, n -manifolds.) There is a notion of geometric degree of a map $f : M \rightarrow N$ where M and N are orientable manifolds, which you may have come across in differential topology. The two notions coincide: Start with a connected compact orientable manifold M without boundary and of dimension m . Assume that it is triangulable and fix a triangulation. Then it is possible to choose orientations on n -simplexes of M in such a way that whenever two of them meet along a $(n-1)$ -face, they induce opposite orientation on the face. If you take the sum c of all these oriented simplexes, it follows that $c \in S_m(X)$ and $\partial(c) = 0$. Moreover, $\bar{c} \in H_m(M)$ generates the infinite cyclic group $H_m(M)$. Thus if M, N are two such manifolds, then for any smooth map $f : M \rightarrow N$, we can talk about $\deg f$ in two different ways. The two notions coincide.

Prove this for any smooth map $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$.

(vii) **Homology exact sequence of a triple**

Given topological spaces $B \subset A \subset X$, show that there is a long exact sequence of homology groups

$$\cdots \rightarrow H_n(A, B) \rightarrow H_n(X, B) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A, B) \rightarrow \cdots$$

which is functorial in the triples (X, A, B) .

- (viii) Suppose $q : X \rightarrow Y$ is a quotient map $A \hookrightarrow X$ is a cofibration and $q : (X, A) \rightarrow (Y, q(A))$ is a relative homeomorphism, i.e., $q : X \setminus A \rightarrow Y \setminus q(A)$ is a homeomorphism. Then show that q induces isomorphisms in homology $H_*(X, A) \rightarrow H_*(Y, q(A))$.

4.9 Construction of Some Other Homology Groups

In this section we discuss variants of singular homology. These include

- (A) smooth singular homology for smooth manifolds,
- (B) simplicial and singular simplicial homology for simplicial complexes,

(C) CW-homology and cellular CW-homology for CW-complexes.

We shall state how each of them is related with the singular homology of the underlying topological space. The proofs are all postponed to the last section. Each of these homology groups enhances our knowledge of singular homology and helps in computation in special cases.

Smooth Singular Homology of Smooth Manifolds

For this paragraph, we presume that the reader is familiar with the basic theory of smooth manifolds. Consider the category of $\mathcal{C}^1$ -manifolds. We must expand this category a little bit by allowing objects such as $M \times \mathbb{I}$ where M is a manifold with or without boundary. Perhaps the easiest way to do this is to consider subspaces of Euclidean spaces with the induced smooth structure on them. By Whitney's embedding theorem, it follows that this will take care of all smooth manifolds as well as objects such as $M \times \mathbb{I}$. In particular, note that the standard n -simplexes with the induced smooth structure on them are objects in this category. Obviously, morphisms in this category are smooth functions on these objects, i.e., those which are restrictions of some smooth functions on some open neighbourhood of the object in the Euclidean space.

Let M be a smooth object. We denote by $S_n^{sm}(M)$ the free abelian subgroup of $S_n(M)$ generated by smooth maps $\sigma : \Delta_n \rightarrow M$. It is easily verified that $S^{sm}(M) = \bigoplus_{n \geq 0} S_n^{sm}(M)$ is a chain subcomplex of $S(M)$. The homology of this chain complex is called the *smooth singular homology of M* and is denoted by $H_*^{sm}(M)$.

It is easily checked that the assignment $M \rightsquigarrow S^{sm}(M)$ is a covariant functor. The smooth singular homology functor satisfies the standard properties similar to the singular homology. It satisfies the homotopy invariance as in Theorem 4.6.1. We shall prove this later along with the proof of Theorem 4.6.1. In particular it would follow that if M is a smoothly contractible smooth object then its smooth singular homology is the same as that of a point space. Moreover, it is also easy to see the following.

Corollary 4.9.1 $H_*^{sm}(*) = H_*(*)$.

Clearly there is a homology long exact sequence of a pair of smooth objects (X, A) and the inclusion map $\eta : S^{sm} \rightarrow S$ induces a ladder of homomorphisms of the two

exact sequences.

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & H_n^{sm}(A) & \longrightarrow & H_n^{sm}(X) & \longrightarrow & H_n^{sm}(X, A) \xrightarrow{\partial} H_{n-1}^{sm}(A) \longrightarrow \cdots \\
 & & \downarrow \eta_* & & \downarrow \eta_* & & \downarrow \eta_* \\
 \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots
 \end{array} \quad (4.14)$$

Note that the discussion preceding Lemma 4.7.1 and the conclusion of the lemma 4.7.1 holds verbatim for S^{sm} of smooth manifolds. In a similar fashion to that of Definition 4.7.2, one can then define excisive couples for sm-homology. What is somewhat non trivial is that the analogue of Theorem 4.7.4 is also true:

Theorem 4.9.2 *If $X = X_1 \cup X_2 = \text{int}_X(X_1) \cup \text{int}_X(X_2)$ then, $\{X_1, X_2\}$ is an excisive couple for the smooth singular homology.*

We shall prove this later along with the proof of Theorem 4.6.1. Taking this for granted, you are tempted to repeat verbatim the homology suspension theorem, etc., as in the case of singular homology, but hold on: we need to explain what we mean by the suspension in the smooth category... Instead, it is time to have:

Theorem 4.9.3 *The inclusion map $\eta : S^{sm} \rightarrow S$ is a functorial chain equivalence and hence defines a natural equivalence of the two homology modules.*

Here again, you will have to wait for the proof of this, this time until the sixth chapter.

4.10 The Simplicial Homology and the Singular Simplicial Homology

The singular chain complex of a topological space is too huge to carry out certain types of computations of homology groups. At least in the case of polyhedron $|K|$, we can remedy this situation by introducing a subcomplex of $S(|K|)$ which yields the same homology groups as $S(|K|)$. We have seen earlier, via the simplicial approximation theorem, that for a polyhedron, the set of simplicial maps is capable of capturing the topological features of the space, at least up to homotopy type, in a certain loose sense. This being so, it should be possible to have similar treatment while dealing with homology groups of a polyhedron. To be a little more precise, we can think of taking only the simplicial maps $\sigma : |\Delta_n| \rightarrow |K|$, instead of taking all continuous functions, while forming the

singular homology groups. If this turns out to be ‘meaningful’, then it would have the same kind of advantages over the singular chain complex, as a simplicial map over a continuous map, which we have witnessed earlier. For instance, the chain groups become combinatorial objects and will be extremely handy compared to the ordinary singular chain groups. This is what we would like to study now.

Definition 4.10.1 Let (K, L) be a simplicial pair. Let $\mathcal{S}_n(K)$ be the subgroup of $S_n(|K|)$ generated by all simplicial maps $\lambda : |\Delta_n| \rightarrow |K|$. Let $\mathcal{S}_n(K, L) = \mathcal{S}_n(K)/\mathcal{S}_n(L)$. Note that $\mathcal{S}_n(K, L)$ is a free abelian group and can be regarded as a subgroup of $S_n(|K|, |L|) = S_n(|K|)/S_n(|L|)$ with a basis consisting of those simplicial maps $\lambda : \Delta_n \rightarrow K$ such that $\lambda(\Delta_n)$ is not contained in L . If ∂ denotes the boundary map of the singular chain complex $S_*(|K|)$, then clearly $\partial(\mathcal{S}_n(K)) \subset \mathcal{S}_{n-1}(K)$. Therefore, $\mathcal{S}_*(K) = \bigoplus_{n \geq 0} \mathcal{S}_n(K)$ and $\mathcal{S}_*(K, L) = \bigoplus_{n \geq 0} \mathcal{S}_n(K, L)$ form subchain complexes of $S_*(|K|)$ and $S_*(|K|, |L|)$, respectively.

Example 4.10.2 $H_0(\mathcal{S}_*(K))$ of a connected complex K

Recall that if K is connected (i.e., $|K|$ is connected), then any two vertices u, v in K can be joined by a sequence of edges (see Exercises ??). This then yields a 1-chain σ such that $\partial(\sigma) = v - u$. Proceeding as in Example 4.6.4.4, we conclude that $H_0(\mathcal{S}_*(K)) \approx \mathbb{Z}$.

Definition 4.10.3 A simplicial map $\lambda : \Delta_n \rightarrow K$ can be displayed by simply writing down the images of the vertices $\mathbf{e}_i$ of Δ_n under λ in that order. Thus an ordered $(n+1)$ -tuple of vertices of K defines a simplicial map iff all those vertices belong to a simplex of K . Just in order to distinguish them from mere $(n+1)$ -tuples and think of them as simplicial simplexes, we shall display them inside square brackets as:

$$[v_0, v_1, \dots, v_n]$$

where $v_i \in F$ for a k -simplex F in K . (Note that here the vertices v_i need not be distinct.)

In particular, consider now the case when $K = \Delta_n$ is the standard n -simplex. Then the face maps $F^i : \Delta_{n-1} \rightarrow \Delta_n$ are simplicial and can be represented in the form

$$F^i = [\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_n].$$

We introduce some convenient notation here:

$$F^i := [\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_n] =: [\mathbf{e}_0, \dots, \widehat{\mathbf{e}}_i, \dots, \mathbf{e}_n].$$

Here the hat $\widehat{}$ indicates that the corresponding entry is deleted from the sequence. In this notation, it follows that

$$\partial[v_0, \dots, v_n] = \sum_i (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n] \quad (4.15)$$

Example 4.10.4 The chain complex $\mathcal{S}(\Delta_n)$

Clearly, there are precisely $(n+1)^{q+1}$ singular simplicial q -simplices in Δ_n because they are in one-to-one correspondence with sequences of length $(q+1)$ in the letters $\{\mathbf{e}_0, \dots, \mathbf{e}_n\}$. Thus the chain complex $\mathcal{S}(\Delta_n)$ looks like

$$\dots \mathbb{Z}^{(n+1)^{q+1}} \longrightarrow \mathbb{Z}^{(n+1)^q} \longrightarrow \dots \longrightarrow \mathbb{Z}^{(n+1)^2} \longrightarrow \mathbb{Z}^{(n+1)}.$$

Given a singular q -simplex σ in Δ_n and any vertex $v \in \Delta_n$, we shall use the notation $v\sigma$ to denote the singular $(n+1)$ -simplex defined by

$$v\sigma(\mathbf{e}_i) = \begin{cases} v, & i = 0; \\ \sigma(v_{i-1}), & i \geq 1. \end{cases}$$

We can then extend this notation when the n -simplex σ is replaced by any n -chain in Δ_n :

$$v\sigma = v \left(\sum_i n_i \sigma_i \right) = \sum_i n_i v\sigma_i.$$

It follows that

$$\partial(v\sigma) = \sigma - v\partial\sigma. \quad (4.16)$$

This allows us a simple way to construct a chain homotopy of the $Id : \mathcal{S}(\Delta_n) \rightarrow \mathcal{S}(\Delta_n)$ with the chain map $\zeta : \mathcal{S}(\Delta_n) \rightarrow \mathcal{S}(\Delta_n)$ defined as follows. Fix any vertex v_0 of Δ_n . Take $\zeta_0(v) = v_0$, and for all vertices v in Δ_n and take $\zeta_k \equiv 0$ for $k > 0$. Check that ζ is indeed a chain map. Now for each $k \geq 0$, take $P : \mathcal{S}_k(\Delta_n) \rightarrow \mathcal{S}_{k+1}(\Delta_n)$ to be the map

$$P(\sigma) = v_0\sigma.$$

From (4.16), it easily follows that

$$P \circ \partial + \partial \circ P = Id - \zeta.$$

The immediate consequence of this is that on homology the two induced homomorphisms are the same $Id = Id_* = \zeta_* : H(\mathcal{S}(\Delta_n)) \rightarrow H(\mathcal{S}(\Delta_n))$. Since it is easily seen that $\zeta_* = 0$

on H_q for $q > 0$ we conclude that $H_q(\mathcal{S}(\Delta_n)) = 0$ for $q > 0$. Also since any simplex is connected we know $H_0(\mathcal{S}(\Delta_n)) \approx \mathbb{Z}$.

Let Σ_{n+1} denote the permutation group on $(n+1)$ -letters. Then to each $\alpha \in \Sigma_{n+1}$, we have a simplicial homeomorphism $G_\alpha : |\Delta_n| \rightarrow |\Delta_n|$. This defines a right-action of Σ_{n+1} on $\mathcal{S}_n(\Delta_n)$ as follows: First define the right action of α on the basis elements of $\mathcal{S}_n(\Delta_n)$ by the formula,

$$\lambda^\alpha = \lambda \circ G_\alpha$$

where $\lambda : |\Delta_n| \rightarrow |\Delta_n|$ is a simplicial map. Finally define

$$\left(\sum_j n_j \lambda_j \right)^\alpha = \sum_j n_j (\lambda \circ G_\alpha).$$

Definition 4.10.5 A simplicial map $\lambda : |\Delta_n| \rightarrow |K|$ is called a *degenerate n -simplex* if the dimension of the simplex $\lambda(\Delta_n)$ is strictly less than n . Let $\mathcal{S}_n^0(K)$ denote the subgroup generated by degenerate simplicial maps λ , and elements of the form

$$\lambda - \text{sgn}(\alpha)\lambda^\alpha,$$

where λ runs over all simplicial maps $\lambda : \Delta_n \rightarrow K$ and α runs over Σ_{n+1} .

Lemma 4.10.6 $\partial(\mathcal{S}_n^0(K)) \subset \mathcal{S}_{n-1}^0(K)$.

Proof: We shall first check that the boundary of a degenerate simplex is in $\mathcal{S}_{n-1}^0(K)$. So, let $\lambda : \Delta_n \rightarrow K$ be a simplicial map with $\dim \lambda(\Delta_n) = m < n$.

First suppose $m < n - 1$. This means that λ maps either three vertices to the same vertex or two pairs of vertices to the same vertex. Since $\partial(\lambda)$ is a sum of simplicial maps each obtained by deleting only one vertex of Δ_n , it follows that $\partial(\lambda)$ is a sum of degenerate $(n-1)$ -simplexes.

Next, consider the case when $m = n - 1$, say, λ maps i^{th} and j^{th} vertices to the same vertex, for some $0 \leq i < j \leq n$. In this case, $\partial(\lambda)$ consists of a sum of a number of degenerate $(n-1)$ -simplices and the two terms

$$\begin{aligned} & (-1)^i [v_0, v_1, \dots, v_{i-1}, \hat{v}_i, v_{i+1}, \dots, v_j, \dots, v_n] \\ & (-1)^j [v_0, v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_{j-1}, \hat{v}_j, v_{j+1}, \dots, v_n]. \end{aligned} \tag{4.17}$$

But observe that since $v_i = v_j$, $[v_0, v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_{j-1}, \hat{v}_j, v_{j+1}, \dots, v_n]$ is obtained from $[v_0, v_1, \dots, v_{i-1}, \hat{v}_i, v_{i+1}, \dots, v_j, \dots, v_n]$ by performing $i + j - 1$ transpositions. Therefore, the sum in (4.17) is of the form $\tau - \text{sgn}(\alpha)\tau^\alpha$. This proves that $\partial(\lambda) \in \mathcal{S}_{n-1}^0(K)$.

To show that the boundary of elements of the form $\lambda - (\text{sgn}\alpha)\lambda^\alpha$ is in $\mathcal{S}_{n-1}^0(K)$, it suffices to do so under the assumption that α is a transposition. But then $\partial(\lambda + \lambda^\alpha)$ is again a sum of such expressions. ♠

Definition 4.10.7 Let $C_n(K) = \mathcal{S}_n(K)/\mathcal{S}_n^0(K)$. It follows that $\partial : \mathcal{S}_n(K) \rightarrow \mathcal{S}_{n-1}(K)$ induces a homomorphism on these quotient groups, which we shall again denote by $\partial : C_n(K) \rightarrow C_{n-1}(K)$. Clearly $\partial^2 = 0$ and hence $\{C_*(K), \partial\}$ is a chain complex. This chain complex is called the *simplicial chain complex* of K . The homology groups $H_*(C_*(K))$ are called the *simplicial homology groups* of K .

Example 4.10.8 Let us take a closure look at the simplicial chain complex $C(\Delta_n)$ of the n -simplex Δ_n . To begin with, there is no confusion about 0-chains; they are merely linear combinations of vertices of Δ_n and so $C_0(K) = \mathbb{Z}^{n+1}$. Now let us look at 1-chains. We know $\mathcal{S}_1(\Delta_n) = \mathbb{Z}^{(n+1)^2}$. But any simplex given by a non-injective map is a degenerate simplex and so goes into $\mathcal{S}_1^0(\Delta_n)$. Therefore, we need to take only pairs $[u, v]$ with $u \neq v$. But then the action of the permutation group takes $[u, v]$ to $[v, u]$ and hence chains of the form $[u, v] + [v, u]$ belong to $\mathcal{S}_1^0(\Delta_n)$. This just means that we need to count an edge of Δ_n just once, in whichever order we may choose. Since there are precisely $\binom{n+1}{2}$ edges in Δ_n , it follows that $C_1(\Delta_n) = \mathbb{Z}^{\binom{n+1}{2}}$. The boundary operator is clearly given by

$$\partial[u, v] = [v] - [u].$$

Likewise it follows that $C_q(\Delta_n) = \mathbb{Z}^{\binom{n+1}{q+1}}$ and $\partial : C_q(\Delta_n) \rightarrow C_{q-1}(\Delta_n)$ is given by

$$\partial[u_0, \dots, u_q] = \sum_i (-1)^i [u_0, \dots, \widehat{u_i}, \dots, u_q]$$

Thus the entire chain complex $C(\Delta_n)$ looks like

$$0 \cdots 0 \longrightarrow \mathbb{Z}^{\binom{n+1}{n+1}} \longrightarrow \mathbb{Z}^{\binom{n+1}{n}} \longrightarrow \cdots \longrightarrow \mathbb{Z}^{\binom{n+1}{2}} \longrightarrow \mathbb{Z}^{n+1}.$$

Write down a full description of this for the case $n = 3$ along with the boundary operators. Can you show that this chain complex is exact?

Remark 4.10.9 Note that $C_n(K)$ is purely an algebraic object, which can be described as the free abelian group on the n -simplices of K . A priori, this has very little say on the topology on $|K|$. Obviously, the boundary maps themselves depend on the incidence relations within K and hence the homology groups depend ‘merely’ on the combinatorial nature of K . Thus, the following results come as a surprise.

Theorem 4.10.10 (Simplicial versus singular) *The inclusion map*

$$\iota : \mathcal{S}_*(K, L) \rightarrow S_*(|K|, |L|)$$

is a chain homotopy equivalence.

Theorem 4.10.11 (Singular-simplicial versus simplicial) *The quotient map*

$$\varphi : \mathcal{S}_*(K, L) \rightarrow C_*(K, L)$$

is chain homotopy equivalence.

As an easy consequence, we have:

Theorem 4.10.12 *For pairs of simplicial complexes (K, L) , there are canonical isomorphisms of the singular homology groups $H(S_*(|K|, |L|))$, the singular-simplicial homology groups $H(\mathcal{S}_*(K, L))$ and the simplicial homology groups $H(C_*(K, L))$.*

Remark 4.10.13

(1) We are going to postpone the proofs of these theorems also. The homology groups of $C_*(K, L)$ will be referred to as the *simplicial homology of the simplicial pair (K, L)* .

(2) *A priori*, this depends upon the actual simplicial complex. However, once we have proved the above theorem, since singular homology is a homeomorphism invariant, it follows that so is simplicial homology. Thus, while computing the homology groups, we are free to choose suitable triangulations of a polyhedron.

(3) We note that, even though $C_*(K, L)$ is defined as a quotient complex of a free chain complex, it is also a free complex. One begins with a choice of an arbitrary total order on the vertices of K . Then to each n -simplex in K , one can choose the non degenerate singular n -simplex given by the order preserving map. This set will form a free basis for $C_n(K)$. In case K is a finite complex, each $C_n(K)$ will be of finite rank also. Moreover, the boundary homomorphisms are given by incidence matrices, i.e., matrices whose entries are 0 or ± 1 , the data being completely determined by the face relations and the chosen orientations on each simplex. Thus, simplicial homology is quite simple from the computational point of view.

(3) We also note that, $\mathcal{S}_*(K, L)$ plays an intermediary role in connecting $C_*(K)$ and $S_*(|K|)$.

(4) Suppose K_1 and K_2 are subcomplexes of a simplicial complex K such that $K =$

$K_1 \cup K_2$. Since any simplex (singular or not) of K is a simplex in K_1 or in K_2 , it follows that

$$\mathcal{S}(K_1) + \mathcal{S}(K_2) = \mathcal{S}(K); \quad C(K_1) + C(K_2) = C(K).$$

This just means that (K_1, K_2) is an excisive couple for the singular simplicial homology as well as simplicial homology and hence we have the Mayer–Vietoris type exact sequence for both these homologies, which comes very handy in computing the homology of the union from that of the pieces.

An easy but a striking consequence of equivalence of singular and simplicial homology is the following:

Corollary 4.10.14 Let (K, L) be a polyhedral pair. If $\dim K \leq n$, then $H_k(|K|, |L|) = 0$, $\forall k > n$ and $H_n(|K|, |L|)$ is a free abelian group. If K is a compact polyhedron, then $H_*(|K|, |L|)$ is finitely generated.

Proof: Easy. ♠

Remark 4.10.15 Euler characteristic revisited

Recall that in Exercise ?? ??, we have defined

$$\chi(K) := (-1)^i f_i(K)$$

as the Euler characteristic of the simplicial complex K . For the simplicial chain complex $C.(K)$, we have another definition of $\chi(C.(K))$ in 4.3.5. Since each $C_i(K)$ is a free abelian group over the set of i -simplices of K , it immediately follows that

$$\chi(K) = \chi(C.(K)).$$

Since we have also proved that $\chi(C.(K)) = \chi(H(C.(K)))$ and since $H(C.(K))$ is canonically isomorphic to $H_*(|K|)$, we conclude that the Euler characteristic of K is a topological invariant. Indeed, for any topological space X with finitely generated singular homology, we define

$$\beta_i(X) := \text{rank } H_i(X; \mathbb{Z})$$

and call it the i^{th} -Betti number of X . We then have,

$$\chi(X) = \chi(H_*(X)) = \sum_i (-1)^i \beta_i(X).$$

Because of the topological invariance of the homology groups, Betti numbers can be computed using any triangulation of a space and the corresponding simplicial homology.

Exercise 4.10.16

- (i) Let K be a finite simplicial complex. Describe the simplicial chain complex of cK , the cone over K in terms of the simplicial chain complex of K . Deduce that $C_*(cK)$ is exact. [Hint: Fix a total order on the vertices of K . Extend this order on the vertices of cK by taking the apex of cK to be the least element.]
- (ii) Write down explicitly the chain complex $C_*(\Delta_n)$. Using the above exercise, compute $H_*(\mathbb{S}^{n-1})$. (Remember that $|\Delta_n|$ is homeomorphic to $\mathbb{S}^{n-1}$.)
- (iii) Define the subdivision chain map $Sd : C_*(K) \rightarrow C_*(sd K)$, inductively, as follows. On C_0 , it is defined to be linear extension of the inclusion map. Suppose we have defined it on C_{n-1} . For any n -simplex σ in K , define $Sd(\sigma) = \beta(\sigma)Sd(\partial(\sigma))$ where $\beta(\sigma)$ denotes the barycentre of σ .
 - (a) Verify that Sd is a chain map and show that it is functorial.
 - (b) If $\tau : sd K \rightarrow K$ is a simplicial approximation to Id , then show that, $C(\tau) \circ Sd = Id$.
 - (c) Show that $\tau_* : H_*(|sd K|) \rightarrow H_*(|K|)$ is the identity isomorphism.
 - (d) Show that Sd induces the identity isomorphism on homology.

4.11 CW-Homology and Cellular Singular Homology

We begin with:

Theorem 4.11.1 *Let X be obtained by attaching n -cells $\{e^n_j\}$ to Y via the attaching maps $\alpha_j : \mathbb{S}^{n-1} \rightarrow Y$. Let $\alpha : \bigoplus_{j \in J} \mathbb{Z} \rightarrow H_{n-1}(Y)$ be the homomorphism which when restricted to the j^{th} summand is equal to $(\alpha_j)_* : \mathbb{Z} = H_{n-1}(\mathbb{S}^{n-1}) \rightarrow H_{n-1}(Y)$. Then*

(a)

$$H_k(X, Y) = \begin{cases} 0, & k \neq n, \\ \bigoplus_{j \in J} \mathbb{Z}, & k = n. \end{cases}$$

(b) *The inclusion $\iota : Y \rightarrow X$ induces isomorphisms $H_k(Y) \rightarrow H_k(X)$ for all $k \neq n-1, n$.*

(c) *Moreover, $\iota_* : H_{n-1}(Y) \rightarrow H_{n-1}(X)$ is a surjection with its kernel equal to $\text{Im } \alpha$*

(d) *and $\iota_* : H_n(Y) \rightarrow H_n(X)$ is injective with $H_n(X) \approx \iota_*(H_n(Y)) \oplus \text{Ker } \alpha$.*

Proof: (a) Choose a point $p_j \in \text{int } e_j^n$ for each j and put $U = X \setminus \{p_j : j \in J\}$. Then observe that Y is a deformation retract of U . Therefore from the long homology exact sequence of the triple (X, U, Y) , it follows that the inclusion map $\eta_1 : (X, Y) \rightarrow (X, U)$ induces an isomorphism. Put $V = \cup_j \text{int } e_j^n$. Since U is open in X , it follows that $\{U, V\}$ is an excisive couple. Therefore, the inclusion map $\eta_2 : (V, V \cap U) \rightarrow (X, U)$ induces isomorphism of homology groups. Now the pair $(V, V \cap U)$ is the disjoint union of the pairs $(\text{int } e_j^n, \text{int } e_j^n \setminus \{p_j\})$. Hence the homology is the direct sum of the homology of these pairs. Now appeal to Example (4.8.4).

Now (b) (c) and (d) are easy consequences of the long homology exact sequence of the pair (X, Y) . ♠

Remark 4.11.2 We have made a very good beginning in this lemma of understanding the homology of a CW-complex. We shall now launch a systematic study of the homology of a CW-complex. Our theme is ‘from parts to the whole’ and the main tools are long homology exact sequences and excision.

Lemma 4.11.3 Let X be a CW-complex. Then for each integer $n \geq 0$, we have

- (a) $H_k(X^{(n)}, X^{(n-1)})$ vanishes for $k \neq n$ and is isomorphic to the free abelian group of rank equal to the number of n -cells in X .
- (b) $H_k(X^{(n)}) = (0), k > n$.
- (c) The inclusion map $\eta_n : X^{(n)} \rightarrow X$ induces isomorphism in homology for $k < n$ and surjection for $k = n$.

Proof: (a) This is a direct consequence of Lemma 4.11.1 (a).

(b) Since $X^{(0)}$ is a discrete space, we know $H_k(X^{(0)}) = (0)$ for all $k > 0$. Inductively let us assume that the statement (b) holds for $n - 1$. Let $k > n$. In the long exact sequence of $(X^{(n)}, X^{(n-1)})$:

$$H_k(X^{(n-1)}) \rightarrow H_k(X^{(n)}) \rightarrow H_k(X^{(n)}, X^{(n-1)})$$

the first term is zero by induction and the third term is zero by (a). Hence the middle term is also zero.

(c) First, we assume that X is finite dimensional say $\dim X = m$. If $n \geq m$, then $X^{(n)} = X$ and so there is nothing to prove. So we may as well assume $n < m$. Then for each $i \geq 0$, and $k < n$, we have, the inclusion induced map

$$H_k(X^{(n+i)}) \rightarrow H_k(X^{(n+i+1)})$$

is an isomorphism. (Use the homology exact sequence and (a). Composite of finitely many such isomorphisms yields the isomorphism

$$H_k(X^{(n)}) \longrightarrow H_k(X^{(m)}) = H_k(X).$$

For $k = n$, this fails only at the very first instance, viz., $H_k(X^{(n)}) \longrightarrow H_k(X^{(n+1)})$. However, even here, the map is surjective, since the next group in the sequence is $H_n(X^{(n+1)}, X^{(n)}) = (0)$.

Now suppose X is infinite dimensional. Suppose the inclusion map $\eta_n : X^{(n)} \longrightarrow X$ induces a non-injective homomorphism $H_k(X^{(n)}) \longrightarrow H_k(X)$ for some $k < n$. Let c be a k -cycle in $X^{(n)}$ which is a boundary in X say $c = \partial y$. Since y is a finite combination of $(k+1)$ -singular simplices, its support is compact. Therefore y is a $(k+1)$ -chain in a finite skeleton $X^{(m)}$ of X . This means that $H_k(X^{(n)}) \longrightarrow H_k(X^{(m)})$ is not injective, which is a contradiction to what we have proved above, in the finite dimensional case. This proves injectivity. The proof of surjectivity is similar and easier. ♠

Remark 4.11.4 Motivated by the construction of the chain complex $C_*(K)$ for a simplicial complex K and the results obtained above, we now make the following definition of a chain complex associated to a CW-complex.

Definition 4.11.5 Let X be a CW-complex. For each n , let

$$i_n : H_n(X^{(n)}) \longrightarrow H_n(X^{(n+1)}) \text{ and } j_n : H_n(X^{(n)}) \longrightarrow H_n(X^{(n)}, X^{(n-1)})$$

denote the inclusion induced homomorphisms. Put

$$C_n^{CW}(X) := H_n(X^{(n)}, X^{(n-1)}); \quad d_n := j_{n-1} \circ \partial_n, n \geq 1$$

where $\partial_n : H_n(X^{(n)}, X^{(n-1)}) \longrightarrow H_{n-1}(X^{(n-1)})$ is the connecting homomorphism in the long exact sequence of the pair $(X^{(n)}, X^{(n-1)})$.

The following lemma shows that $d_n \circ d_{n+1} = 0$ for all n and hence $C_*^{CW}(X)$ is a chain complex. This is called the *cellular chain complex associated to X* . The homology of this chain complex will be called the *cellular homology of X* .

Lemma 4.11.6 We have, $d_n \circ d_{n+1} = 0$.

Proof: In the commutative diagram, (Figure 4.3) the two vertical sequences are respectively part of the exact sequences of the pairs $(X^{(n+1)}, X^{(n)})$ and $(X^{(n-1)}, X^{(n-2)})$, whereas the horizontal sequence is that of $(X^{(n)}, X^{(n-1)})$. In particular, we have $\partial_n \circ j_n = 0$ and hence $d_n \circ d_{n+1} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1} = 0$. ♠

$$\begin{array}{ccccccc}
& H_{n+1}(X^{(n+1)}, X^{(n)}) & & & 0 & & \\
& \downarrow \partial_{n+1} & \searrow d_{n+1} & & \downarrow & & \\
0 \longrightarrow & H_n(X^{(n)}) & \xrightarrow{j_n} & H_n(X^{(n)}, X^{(n-1)}) & \xrightarrow{\partial_n} & H_{n-1}(X^{(n-1)}) & \\
& \downarrow i_n & & \searrow d_n & & \downarrow j_{n-1} & \\
& H_n(X^{(n+1)}) & & & & H_{n-1}(X^{(n-1)}, X^{(n-2)}) & \\
& \downarrow & & & & & \\
& 0 & & & & &
\end{array}$$

Figure 4.3. Construction of cellular homology

Theorem 4.11.7 $H_n^{CW}(X) \approx H_n(X)$.

Proof: Once again we refer to Figure 4.3. Observe that the appearance of the three zero groups is justified by Lemma 4.11.3. Since j_{n-1} is injective, it follows that $\text{Ker } d_n = \text{Ker } \partial_n = \text{Im } j_n \approx H_n(X^{(n)})$ since j_n is injective. Therefore, $H_n^{CW}(X) = \text{Ker } d_n / \text{Im } d_{n+1} \approx H_n(X^{(n)}) / \text{Im } \partial_{n+1} \approx H_n(X^{(n+1)}) \approx H_n(X)$. ♠

Example 4.11.8 Homology of $\mathbb{CP}^n$

An immediate consequence of this theorem is that if a CW-complex X does not have any cells of dimension k , then $H_k(X) = (0)$. This fact fits very neatly in computing the homology of complex projective spaces. Recall that $\mathbb{CP}^n$ has cell decomposition consisting of one cell of dimension $2k$ for $0 \leq k \leq n$. Therefore the cellular chain complex $C^{CW}(\mathbb{CP}^n)$ looks like

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \dots \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}$$

with 0 and $\mathbb{Z}$ occurring alternatively. Therefore

$$H_i(\mathbb{CP}^n) \approx \begin{cases} \mathbb{Z}, & i = 2k, \quad 0 \leq k \leq n; \\ (0), & \text{otherwise.} \end{cases}$$

Remark 4.11.9 In order to exploit the cellular homology further, we should try to understand the boundary homomorphisms d_n of this chain complex. Here is a description of d_n in terms of the attaching maps of the CW-structure. Let $\phi_\alpha : (\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow (X^{(n)}, X^{(n-1)})$ denote the collection of characteristic maps of n -cells in X and let $f_\alpha : \mathbb{S}^{n-1} \rightarrow X^{(n-1)}$ denote the corresponding attaching maps. We know that $C_n = H_n(X^{(n)}, X^{(n-1)})$ is freely generated over the classes $[\phi_\alpha]$. Therefore, in order to determine d_n we have only to find expressions for $d_n([\phi_\alpha])$ in $H_{n-1}(X^{(n-1)}, X^{(n-2)})$. So, let us denote the collection of characteristic maps of $(n-1)$ -cells by ψ_β . Observe that the quotient space $X^{(n-1)}/X^{(n-2)}$ is homeomorphic to the bouquet of spheres $\vee_\beta \mathbb{S}_\beta^{n-1}$. Indeed the indexing of these spheres is such that the $(n-1)$ -cell ψ_β is mapped onto the sphere $\mathbb{S}_\beta^{n-1}$ under the quotient map $q_{n-1} : X^{(n-1)} \rightarrow X^{(n-1)}/X^{(n-2)}$. Also q_{n-1} induces an isomorphism of C_{n-1} and $H_{n-1}(\vee_\beta \mathbb{S}_\beta^{n-1})$. Let $p_\beta : \vee_\beta \mathbb{S}_\beta^{n-1} \rightarrow \mathbb{S}_\beta^{n-1}$ be the projection onto the β^{th} component. Put $d_n[\phi_\alpha] = \sum_\beta n_{\alpha,\beta}[\psi_\beta]$. Note that $\partial_n[\phi_\alpha] = [f_\alpha]$. It follows that $n_{\alpha,\beta}$ is nothing but the degree of the map $p_\beta \circ q_{n-1} \circ f_\alpha : \mathbb{S}^{n-1} \rightarrow \mathbb{S}_\beta^{n-1}$.

Example 4.11.10 Homology groups of lens spaces

Recall that we have constructed a CW-structure for each lens space in Example ???. You are welcome to use them together with the above remark to compute the corresponding CW-chain complex of the lens spaces and thereby, their homology groups. Here, we shall give yet another (simpler) CW-structure to each lens space $X = L_p(q_1, \dots, q_r)$ ($p \geq 3$) with exactly one cell for each dimension $k, 0 \leq k \leq 2r+1$. The chain complex will look like

$$0 \cdots 0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \cdots \quad . \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}. \quad (4.18)$$

We shall first describe a CW-structure on $\mathbb{S}^{2s+1}$ inductively, so that the action of $\mathbb{Z}_p$ on it will be cellular and then take the quotient structure. So, let $\zeta = e^{2\pi i/p}$. We declare all points ζ^j , $0 \leq j < p$, as 0-cells and the arcs in $\mathbb{S}^1$ from ζ^k to ζ^{k+1} as the 1-cells. For 2-cells, e_k^2 , we take the half-spheres in the $\mathbb{R}$ -linear span of $\mathbb{S}^1 \times 0$ and $(0, \zeta^k)$ containing the point $(0, \zeta^k)$ and with boundary $\mathbb{S}^1 \times 0$. Note that each of these discs have $\mathbb{S}^1 \times 0$ as their boundary which also constitutes their points of intersection. Therefore, the union of any two of them bounds a 3-disc in $\mathbb{S}^3$. For the 3-cells, we take these 3-discs e_k^3 bounded by e_{k-1}^2 and e_k^2 , $0 \leq k \leq p-1$. Verification that this describes a cell structure on $\mathbb{S}^3$ and that the cell structure is $\mathbb{Z}_p$ -action invariant is straightforward.

Now, inductively, suppose we have described the cell-structure on $\mathbb{S}^{2s-1}$. Put $\tau_k = (0, \dots, 0, \zeta^k) \in \mathbb{C}^{s+1}$ and let e_k^{2s} be the $2s$ -dimensional disc which is contained in the

intersection of $\mathbb{S}^{2s}$ with the $\mathbb{R}$ -linear span of $\mathbb{R}^{2s}$ and τ_k , and which contains the point τ_k and has boundary $\mathbb{S}^{2s-1}$. And for $(2s+1)$ -discs, we take the portions of $\mathbb{S}^{2s+1}$ bounded by e_{k-1}^{2s}, e_k^{2s} . Once again, the verification that this extends the cell structure on $\mathbb{S}^{2s-1}$ to a cell structure of $\mathbb{S}^{2s+1}$ and that it is invariant under the $\mathbb{Z}_p$ -action is straightforward. Since the action is clearly transitive on the set of r -cells, it follows that the quotient space has a cell structure with precisely one cell in each dimension, $0 \leq s \leq 2s+1$.

Thus, in particular, it follows that the CW-chain complex of this CW-structure on X has the property $C_k = \mathbb{Z}$, $0 \leq k \leq 2r+1$ and (0) otherwise. It remains to describe $\partial_k : C_k \rightarrow C_{k-1}$. Clearly $\partial_1 = 0$ since the 1-cell is attached to a single vertex. The boundary of any of the oriented 2-cells in $\mathbb{S}^{2r+1}$ is the oriented circle $\mathbb{S}^1$ which is clearly wrapped onto the 2-cell p -times. That is to say that the quotient map $q : \mathbb{S}^{2r+1} \rightarrow X$ restricted to $\mathbb{S}^1$ is of degree p . Therefore it follows that ∂_2 is the multiplication by p . On the other hand, each oriented 3-cell e_k^3 in $\mathbb{S}^3 \subset \mathbb{S}^{2r+1}$ is bounded by precisely two oriented discs e_{k-1}^2, e_{k+1}^2 and hence $\partial(e_k^3) = e_k^2 - e_{k-1}^2$. Since the quotient map restricted to the interior of each of these 2-cells is orientation preserving, it follows that $\partial_3 = 0$. The description of $\partial_{2k}, \partial_{2k+1}, k \geq 2$ is identical to those of ∂_2, ∂_3 , respectively, thereby establishing (4.18).

We conclude that

$$H_i(L_p(q_1, \dots, q_r); \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, 2r+1; \\ \mathbb{Z}_p, & i = 2k-1, 1 \leq k \leq r; \\ (0), & \text{otherwise.} \end{cases} \quad (4.19)$$

Finally, we now come to yet another homology for CW-complexes which lies ‘between’ CW-homology and singular homology and which we call *cellular singular homology*. The case is similar to the simplicial homology and singular simplicial homology of a simplicial complex.

Definition 4.11.11 Let (X, A) be a CW-complex. Let $C_n^{cell}(X)$ be the free abelian group generated by the set of all continuous maps $\sigma : \Delta_n \rightarrow X$ which are cellular, in the standard cell structure of Δ_n . Clearly, this forms a subgroup of $S_n(X)$ and one can easily verify that $\partial(C_n^{cell}) \subset C_{n-1}^{cell}$. In other words, $C_\bullet^{cell}(X) = \bigoplus_n C_n^{cell}(X)$ forms a subchain complex of $S_\bullet(X)$.

Theorem 4.11.12 *The inclusion map $C_\bullet^{cell}(X) \subset S_\bullet(X)$ is a chain equivalence.*

We shall postpone the proof of this theorem to the end of the chapter. However, notice that the CW-chain complex itself can be thought of as a subcomplex of this complex

via the characteristic maps. Moreover, by the naturality, it easily extends to the case of relative CW-complexes as well. This theorem will be of theoretical importance to us. (see, e.g., in the construction of Leray–Serre spectral sequence in Chapter ??.)

Exercise 4.11.13

- (i) Use cellular homology to compute $H_*(\mathbb{S}^p \times \mathbb{S}^q)$.
- (ii) For a finite CW-complex X , show that $\chi(X) = \sum_i (-1)^i f_i(X)$, where $f_i(X)$ is the number of i -cells in X .
- (iii) If X and Y are finite CW-complexes, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.
- (iv) Suppose X is obtained by attaching a n -cell to a space Y , $n \geq 2$, via an attaching map $\phi : \mathbb{S}^{n-1} \rightarrow Y$. Let $\alpha \in H_{n-1}(Y)$ denote the image of a generator of $H_{n-1}(\mathbb{S}^{n-1}; \mathbb{Z}) \approx \mathbb{Z}$. Show that

$$H_q(X) \approx \begin{cases} H_q(Y), & q \neq n-1, n; \\ H_{n-1}(Y)/(\alpha), & q = n-1; \\ H_n(Y), & q = n \text{ and } \alpha \text{ is of infinite order.} \\ H_n(Y) \oplus \mathbb{Z}, & q = n \text{ and } \alpha \text{ is of finite order.} \end{cases}$$

Examine the case when $n = 1$. Write down the corresponding statement and prove it.

4.12 Some Applications of Homology

In this section, we shall give some of the popular applications of homology theory. Many of these results can be arrived at by different ad hoc methods. We begin with Brouwer's fixed point theorem, and then go on to improve this to get Lefschetz's fixed point theorem. We then prove the hairy ball theorem, Jordan-Brouwer's separation theorem and Brouwer's theorem on invariance of domain. The section will end with a brief discussion of embeddings of spheres in spheres and the construction of Alexander's horned sphere, which is an embedded 2-sphere in $\mathbb{S}^3$ such that one of the components of the complement is a 3-disc, whereas the other component is not simply connected.

Lemma 4.12.1 Let A be a retract of X . Then $H_*(A)$ is a retract of $H_*(X)$, i.e., $H_*(A)$ is a direct summand of $H_*(X)$.

Proof: Let $r : X \rightarrow A$ be a retraction. Then $r_* : H_*(X) \rightarrow H_*(A)$ is a retraction, by functoriality, viz., $r_* \circ \eta_* = (r \circ \eta)_* = (Id_A)_*$. Here $\eta : A \rightarrow X$ denotes the inclusion map. ♠

Corollary 4.12.2 For any $n \geq 0$, $\mathbb{S}^n$ is not a retract of $\mathbb{D}^{n+1}$.

Proof: We have seen that, $H_n(\mathbb{S}^n)$ is ‘non trivial’ and hence cannot be a subgroup of the ‘trivial’ group $H_n(\mathbb{D}^{n+1})$. ♠

Remark 4.12.3 Recall that we have proved the above result in a different way using simplicial approximation and the Sperner lemma. We have also seen that the above result is actually equivalent to Brouwer’s fixed point theorem. (See Theorem ??.) The present proof is obviously shorter though it uses the big machine of homology. Our next application is a generalization of this result on any compact polyhedron. Of course, now we shall have some hypothesis on the map itself. Recall that, given an endomorphism of a finite type graded module $\tau : C \rightarrow C$, we define the Lefschetz number $L(\tau)$ to be the alternate sum of the traces (see Exercise 4.4.4)

Also recall that, if τ_* denotes the homomorphism induced on the homology groups, then $L(\tau) = L(\tau_*)$. For any continuous map $f : X \rightarrow X$ on a space X that has finitely generated homology groups, we now define $L(f) = L(f_*)$. The well-known Lefschetz fixed point theorem reads as follows:

Theorem 4.12.4 (Lefschetz fixed point theorem) *Let X be a compact polyhedron, $f : X \rightarrow X$ be a continuous map. If $L(f) \neq 0$, then f must have a fixed point.*

Remark 4.12.5 Notice that, since X is a compact polyhedron, its homology is finitely generated and so $L(f) = L(f_*)$ makes sense. Secondly, if $X = \mathbb{D}^n$ then f is homotopic to the identity map of $\mathbb{D}^n$. Since $L(f)$ is a homotopy invariant, it follows that $L(f) = L(Id)$. But $L(Id)$ is nothing but the Euler characteristic of the space $\mathbb{D}^n$. Therefore, $L(Id) = 1$ since $\mathbb{D}^n$ is contractible. Thus the requirement of the theorem is satisfied. So, in conclusion, we can say that f has a fixed point. This shows that Lefschetz fixed point theorem is a generalization of Brouwer’s fixed point theorem. Indeed, we have just derived a stronger version of BFT, viz., any self map of a compact contractible space has a fixed point.

Proof: Given a compact polyhedron X and a continuous self-map $f : X \rightarrow X$ with no fixed points, we shall show that $L(f) = 0$.

Fixing some linear metric on X , we can find $\epsilon > 0$ such that, $d(x, f(x)) > \epsilon$, $\forall x \in X$. (Why?)

Let K be a simplicial complex that triangulates X and such that

$$\text{mesh } K = \max\{\text{diam } F : F \in K\} < \epsilon/3.$$

[For instance we can start with any simplicial complex that triangulates X and then take K to be the (sufficiently often) iterated barycentric subdivision of it to make its mesh as small as we please (Lemma ??).

Let $\phi : L = sd^k K \rightarrow K$ be a simplicial approximation to f .

Then $|\phi|$ is homotopic to f and hence the two maps have the same Lefschetz number.

Now, let ρ be any simplex of $L = sd^k K$, such that, $|\rho| \subset |\sigma|$ for some simplex σ of K . We claim that

$$(|\phi|(|\rho|)) \cap |\sigma| = \emptyset. \quad (4.20)$$

For otherwise, say $x \in |\rho|$, and $|\phi|(x) \in |\sigma|$. Then, both $x, |\phi|(x)$ belong to $|\sigma|$ and hence $d(x, |\phi|(x)) \leq \text{diam.}(|\sigma|) < \epsilon/3$. On the other hand, $d(|\phi|(x), f(x)) < \epsilon/3$ and hence $d(x, f(x)) < \epsilon$, which is a contradiction.

Our task has been complicated by the fact that, while obtaining the simplicial approximation ϕ , we had to subdivide only the domain, and so ϕ almost never has the same domain and range. This difficulty is overcome at least in two different ways:

Method I: We can use the CW-chain complex associated with K . Since ϕ is a simplicial map $L \rightarrow K$, it follows that $|\phi|$ is a cellular map of the CW-complex K . Let $\alpha_n : H_n(K^{(n)}, K^{(n-1)}) \rightarrow H_n(K^{(n)}, K^{(n-1)})$ be the homomorphism induced by $|\phi|$. If $\tau \in K$ is an oriented n -simplex, it follows from (4.20), that $|\phi|(|\tau|) \cap |\tau| = \emptyset$. Therefore, the coefficient of τ in $\alpha_n(\tau)$ will be zero. It follows that the trace of the matrix representing α_n is zero. Therefore $L(|\phi|) = 0$.

Method II Consider the subdivision chain map $Sd : C(K) \rightarrow C(sd K)$, as defined in Exercise 4.10.16.(iii). We have seen that Sd induces identity isomorphism on homology. By repeated application of this we know that $Sd^k = Sd \circ \cdots \circ Sd$ (k -copies) induces identity isomorphism on homology. Therefore, $L(f) = L(|\phi|) = L(|Sd^k| \circ |\phi|)$. We can compute the Lefschetz number of $|Sd^k \circ \phi| : |Sd^k K| \rightarrow |Sd^k K|$ at the chain level using the chain complex $C(sd^k K)$ with the basis elements from the simplexes of $sd^k K$. Let $\lambda : C_n(Sd^k K) \rightarrow C_n(Sd^k K)$ be the chain map induced by $Sd^k \circ \phi$. From (4.20) it follows that for any n simplex ρ of $Sd^k K$, in the expression for $\lambda(\rho)$, the term ρ does not occur

at all. Hence, the matrix of λ on $C_n(Sd^k K)$ has all its diagonal entries 0. Since this is true for all n , we conclude that, $L(f) = L(Sd^k \circ \phi) = 0$. ♠

Remark 4.12.6 We are now in a position to prove the famous hairy ball theorem. Loosely speaking, this theorem says that one cannot comb one's hair without parting at least at one point. The precise statement, however, is the following:

Theorem 4.12.7 (Hairy ball theorem) *There is no continuous map $f : \mathbb{S}^{2n} \rightarrow \mathbb{S}^{2n}$ such that, for each $x \in \mathbb{S}^{2n}$, $f(x)$ is orthogonal to x .*

Proof: We shall actually prove that for any continuous map $f : \mathbb{S}^{2n} \rightarrow \mathbb{S}^{2n}$ there is a point $x \in \mathbb{S}^{2n}$ such that $f(x) = x$ or $f(x) = -x$. Supposing on the contrary, it follows that the line joining x and $f(x)$ is well defined and does not pass through the origin. Therefore, the map

$$H(x, t) := \frac{(1-t)f(x) + tx}{\|(1-t)f(x) + tx\|}$$

is well defined and continuous. Clearly H is a homotopy from f to the identity map Id of $\mathbb{S}^{2n}$. Hence, f induces identity homomorphism on the homology. In particular, it follows that, $L(f) = \chi(\mathbb{S}^{2n}) = 2$. Hence, by LFT, it follows that, f must have a fixed point contradicting our assumption. ♠

Remark 4.12.8 The above theorem has its origin in differential topology, wherein it is stated as follows: *There is no nowhere vanishing tangent vector field on $\mathbb{S}^{2n}$.* Observe that all odd dimensional spheres possess such vector fields. For example: $(x_1, x_2, \dots, x_{2n}) \rightarrow (x_1, -x_2, \dots, x_{2n-1}, x_{2n})$ is one such.

Let us now give another application of LFT. Recall that for $n \geq 1$, $H_n(\mathbb{S}^n) \approx \mathbb{Z}$. Therefore for any continuous map $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ the homomorphism $f_* : H_n(\mathbb{S}^n) \rightarrow H_n(\mathbb{S}^n)$ is given by multiplication by an integer. This integer is called the degree of f . (See Exercise 4.8.7.)

Also $H_0(\mathbb{S}^n) \approx \mathbb{Z}$ and $f_* : H_0(\mathbb{S}^n) \rightarrow H_0(\mathbb{S}^n)$ is always the identity homomorphism. Since all other homology groups vanish it follows that

$$L(f) = 1 + (-1)^n \deg f.$$

As a special case, consider the antipodal map $\alpha : \mathbb{S}^n \rightarrow \mathbb{S}^n$ given by $x \mapsto -x$. Since this has no fixed points, it follows that $L(\alpha) = 0$. Therefore we have,

Theorem 4.12.9 *The degree of the antipodal map on $\mathbb{S}^n$ is equal to $(-1)^{n+1}$ for all $n \geq 1$.*

Remark 4.12.10 More generally, if $f : \mathbb{S}^{2n} \rightarrow \mathbb{S}^{2n}$ is a homeomorphism, then f_* is an isomorphism and hence $\deg f = \pm 1$ according as f preserves or reverses orientation. If in addition, f has no fixed points then $L(f) = 0$ and hence $\deg f = -1$, which means that f is orientation reversing. In particular, this implies that

Corollary 4.12.11 Let G be a group of odd order acting on $\mathbb{S}^{2n}$ through homeomorphisms. Then for each $g \in G$, there exists $v \in \mathbb{S}^{2n}$ such that $gv = v$.

4.13 Jordan-Brouwer

We end this section with two more celebrated results. The 1-dimensional version of the first result is known as the Jordan curve theorem. The proof that Camille Jordan gave in 1905 was not accepted by many mathematicians.¹ Since then, several authors have given various proofs of this result and the degree of accuracy of these proofs also varies. The general result was proved by L. E. J. Brouwer in 1911, more or less as presented below.

Theorem 4.13.1 (Jordan–Brouwer separation theorem) *An $(n - 1)$ -sphere embedded in an n -sphere separates it into exactly two components. These two components have their common boundary as the embedded $(n - 1)$ -sphere.*

Theorem 4.13.2 (Brouwer’s invariance of domain) *Let U and V be any two subspaces of $\mathbb{R}^n$ homeomorphic to each other. If U is open in $\mathbb{R}^n$ then so is V .*

We need the following lemmas:

Lemma 4.13.3 Let A be a subset of $\mathbb{S}^n$, homeomorphic to $\mathbb{I}^k$ for some $0 \leq k \leq n$. Then the reduced homology groups of $\mathbb{S}^n \setminus A$ all vanish.

Lemma 4.13.4 Let B be a subset of $\mathbb{S}^n$ homeomorphic to $\mathbb{S}^k$, for some $0 \leq k \leq n - 1$. Then the reduced homology groups of $\mathbb{S}^n \setminus B$ are all zero except that, $\tilde{H}_{n-k-1}(\mathbb{S}^n \setminus B) = \mathbb{Z}$.

¹See [?] for an elegant proof.

Granting these two lemmas for a while, let us proceed to prove the Theorems:

Proof of Theorem 4.13.1: Let B be a subset of $\mathbb{S}^n$ homeomorphic to $\mathbb{S}^{n-1}$. From the above lemma, we have $\tilde{H}_0(\mathbb{S}^n \setminus B) = \mathbb{Z}$ and this implies that, $\mathbb{S}^n \setminus B$ has precisely two components. Let us call them U and V . Note that, both U and V are open subsets of $\mathbb{S}^n$. By definition, the boundary set $\partial(U) := \bar{U} \setminus \text{int}(U) = \bar{U} \setminus U$. Since V is open and disjoint from U , no point of V is in the closure of U . Thus it follows that, $\partial(U) \subset B$. By symmetry, it remains to show that, every point b of B is a closure point of U . This is the same as showing that, any arbitrary small neighbourhood N of b intersects $\bar{U}$. Inside N , we can find a neighborhood N_1 of b in B such that $B = N_1 \cup N_2$ and both N_i are homeomorphic to $\mathbb{I}^{n-1}$. Hence by Lemma 4.13.3, it follows that $\mathbb{S}^n \setminus N_2$ is path connected. We choose points $p \in U$ and $q \in V$ and join them by a path ω lying in $\mathbb{S}^n \setminus N_2$. Since U and V are different components of $\mathbb{S}^n \setminus B$, it follows that, ω intersects N_1 . If we now follow the path ω from $p \in U$ till we hit N_1 , the point we get in N_1 is definitely in the closure of U , hence $N \cap \bar{U} \neq \emptyset$. Hence $b \in \bar{U}$. This proves that, $B \subset \partial(U)$. ♠

Proof of Theorem 4.13.2: By taking one-point compactifications we can assume that both U and V are subspaces of $\mathbb{S}^n$. Let $\phi : U \rightarrow V$ be a homeomorphism and let U be open in $\mathbb{S}^n$. Let $x \in U$ and $y = \phi(x)$. We should produce an open subset of $\mathbb{S}^n$ containing y and contained in V . Let A be a neighborhood of x in $\mathbb{S}^n$ homeomorphic to a disc and contained in U . Then its boundary $\partial(A)$ is homeomorphic to $\mathbb{S}^{n-1}$, and so is $B = \phi(\partial(A))$. Hence by Theorem 4.13.1, $\mathbb{S}^n \setminus B$ has precisely two connected components. Clearly, $A \setminus \partial A$ is connected and hence $\phi(A \setminus \partial A) = \phi(A) \setminus B$ is connected. By Lemma 4.13.3, $\mathbb{S}^n \setminus \phi(A)$ is connected. On the other hand, $\mathbb{S}^n \setminus B = (\mathbb{S}^n \setminus \phi(A)) \amalg (\phi(A) \setminus B)$. Hence, these two sets must be the components of $\mathbb{S}^n \setminus B$. In particular, it follows that, $\phi(A) \setminus B$ is an open subset of $\mathbb{S}^n$. Also it contains the point y and is a subset of V . Thus we have succeeded in producing an open neighbourhood of x contained in V . This proves that V is open. ♠

It remains to prove the two lemmas. Since the proof of Lemma 4.13.4 is easier, we shall present that one first, though we need to use Lemma 4.13.3 for it.

Proof of Lemma 4.13.4: We shall use Mayer–Vietoris sequence and induction on k . For $k = 0$, $\mathbb{S}^0$ has two points and we know that $\mathbb{S}^n \setminus B$ is homotopy equivalent to $\mathbb{S}^{n-1}$ and we have already computed the homology groups of $\mathbb{S}^{n-1}$ (see (4.12)). So, the statement is true for $k = 0$.

Assume the statement of the lemma for $k - 1$. Write $B = A_1 \cup A_2$ so that both A_i are homeomorphic to $\mathbb{I}^k$ and $A_1 \cap A_2$ is homeomorphic to $\mathbb{S}^{k-1}$. Both $\mathbb{S}^n \setminus A_i$ are open in

$\mathbb{S}^n$. Hence, we can apply the Mayer–Vietoris sequence:

$$\dots \tilde{H}_{q+1}(\mathbb{S}^n \setminus A_1) \oplus \tilde{H}_{q+1}(\mathbb{S}^n \setminus A_2) \rightarrow \tilde{H}_{q+1}(\mathbb{S}^n \setminus (A_1 \cap A_2)) \rightarrow$$

$$\tilde{H}_q(\mathbb{S}^n \setminus B) \rightarrow \tilde{H}_q(\mathbb{S}^n \setminus A_1) \oplus \tilde{H}_q(\mathbb{S}^n \setminus A_2) \rightarrow \dots$$

Now by Lemma 4.13.3, we have $\tilde{H}(\mathbb{S}^n \setminus A_i) = 0$. Thus the middle arrow is an isomorphism. Now by the inductive assumption the result follows. ♠

Proof of Lemma 4.13.3: Here also, we use induction on k . If $k = 0$, then A is a single point and we know that, $\mathbb{S}^n \setminus A$ is contractible and hence, its reduced homology groups vanish. Assume the result for $k - 1$. Also suppose $\tilde{H}_q(\mathbb{S}^n \setminus A) \neq 0$ for some q and let c be a cycle representing a non zero element $[c] \in \tilde{H}_q(\mathbb{S}^n \setminus A)$. Let $h : \mathbb{I}^k \rightarrow A$ be a homeomorphism. Write $A = A_1 \cup A_2$, where $A_1 = h(\mathbb{I}^{k-1} \times [0, 1/2])$ and $A_2 = h(\mathbb{I}^{k-1} \times [1/2, 1])$. Apply Mayer–Vietoris sequence to the open sets $\mathbb{S}^n \setminus A_1$, $\mathbb{S}^n \setminus A_2$, to obtain the exact sequence:

$$\tilde{H}_{q+1}(\mathbb{S}^n \setminus A_1 \cap A_2) \rightarrow \tilde{H}_q(\mathbb{S}^n \setminus A) \rightarrow \tilde{H}_q(\mathbb{S}^n \setminus A_1) \oplus \tilde{H}_q(\mathbb{S}^n \setminus A_2) \rightarrow \tilde{H}_q(\mathbb{S}^n \setminus A_1 \cap A_2).$$

Since $A_1 \cap A_2$ is homeomorphic to $\mathbb{I}^{k-1}$, the two end-groups in the above sequence are zero by induction hypothesis. Therefore the middle arrow represents an isomorphism. Recall that if $\eta_i : \mathbb{S}^n \setminus A \rightarrow \mathbb{S}^n \setminus A_i$ are the inclusion maps, then the middle arrow is nothing but $((\eta_1)_*, (\eta_2)_*)$. It follows that, $(\eta_i)_*[c]$ is non zero at least for one of $i = 1, 2$, say $i = 1$.

We now iterate this construction on A_1 in place of A and so on, to obtain a sequence of closed sets:

$$A \supset A_1 \supset \dots \supset A_r \supset A_{r+1} \dots$$

such that $\cap_r A_r := T$ is homeomorphic to $\mathbb{I}^{k-1}$ and the image of $[c]$ in $\tilde{H}_q(\mathbb{S}^n \setminus A_r)$ is non zero. But by induction hypothesis $\tilde{H}_q(\mathbb{S}^n \setminus T) = 0$. Thus there exists a $(q + 1)$ -chain τ in $\mathbb{S}^n \setminus T$ such that, $\partial(\tau) = c$. Now every chain is supported on a compact set and every compact subset of $\mathbb{S}^n \setminus T$ is contained in $\mathbb{S}^n \setminus A_r$, for some r . This means that the image of $[c]$ is zero in $\tilde{H}_q(\mathbb{S}^n \setminus A_r)$, which is a contradiction. ♠

Exercise 4.13.5 Let X be a compact polyhedron.

- (i) If $f : X \rightarrow X$ is null homotopic then show that f has a fixed point.

- (ii) Suppose X has a topological group structure, such that the path connected component of e is not $\{e\}$. Then show that $\chi(X) = 0$.
- (iii) Show that $\mathbb{S}^{2n}$ cannot have any topological group structure.
- (iv) Let G be a path connected topological group which acts continuously on X . Suppose $\chi(X) \neq 0$. Show that for each $g \in G$, there exists $x \in X$ such that $gx = x$.

4.14 Relation between π_1 and H_1

We shall assume that our topological space X is path connected. Let $x_0 \in X$ be the base point. Given an element of $\pi_1(X, x_0)$ represented by a loop $\omega : \mathbb{I} \rightarrow X$, we can view ω as a 1-cycle in X , by thinking of $\mathbb{I}$ as the geometric realization of the standard 1-simplex Δ_1 . This 1-cycle then represents an element (ω) in $H_1(X)$. The aim of this section is to see that, this simple minded assignment

$$[\omega] \mapsto (\omega)$$

called *Hurewicz homomorphism* can be used to give a precise relation between $\pi_1(X, x_0)$ and $H_1(X)$.

Lemma 4.14.1 Let $\omega_i : \mathbb{I} \rightarrow X$ be any two paths so that $\omega_1 * \omega_2$ is defined. Then as a singular 1-chain, $\omega_1 * \omega_2 - \omega_1 - \omega_2$ is null-homologous.

Proof: The proof is obvious from Figure 4.4 which defines a singular 2-simplex $\sigma : \Delta_2 \rightarrow X$ such that

$$\partial\sigma = \omega_1 * \omega_2 - \omega_1 - \omega_2.$$



Figure 4.4. Chain equivalence under subdivision

Indeed, with the convention that $\sum_i t_i = 1, 0 \leq t_0, t_1, t_2 \leq 1$, we have:

$$\sigma(t_0 \mathbf{e}_0 + t_1 \mathbf{e}_1 + t_2 \mathbf{e}_2) = \begin{cases} \omega_1 \left(\frac{2-2t_0}{1+t_2} \right), & t_0 \geq t_1; \\ \omega_2 \left(\frac{2t_1+t_2-1}{1+t_2} \right), & t_0 \leq t_1. \end{cases}$$

The lemma follows. ♠

Theorem 4.14.2 *The assignment $[\omega] \mapsto (\omega)$ defines a functorial surjection*

$$\varphi : \pi_1(X, x_0) \rightarrow H_1(X); \quad \varphi[\omega] = (\omega).$$

whose kernel is precisely the commutator subgroup of $\pi_1(X, x_0)$.

(Thus φ defines $H_1(X)$ as the abelianization of $\pi_1(X, x_0)$.)

Proof: The first thing to do is to show that φ is well-defined. (See Figure 4.5.) Let $H : \mathbb{I} \times \mathbb{I} \rightarrow X$ be a path homotopy of ω_1 with ω_2 . Consider the following triangulation of $\mathbb{I} \times \mathbb{I}$. The vertex set of the simplicial complex K consists of points, $v_0 = (0, 0), v_1 = (1, 0), v_2 = (1, 1)$ and $v_3 = (0, 1)$.

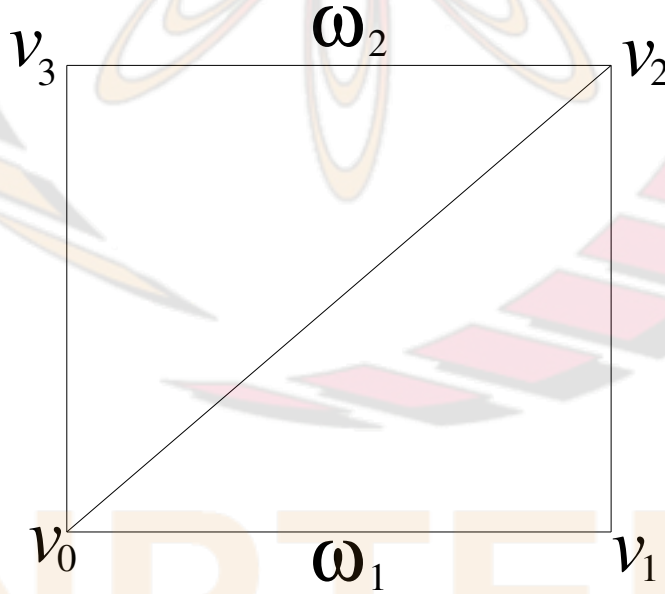


Figure 4.5. Hurewicz map is well-defined.

The edges are $\{v_0, v_1\}, \{v_1, v_2\}, \{v_0, v_2\}, \{v_0, v_3\}$ and $\{v_2, v_3\}$. The 2-simplexes in $|K|$ are $\{v_0, v_1, v_2\}$ and $\{v_0, v_2, v_3\}$. Let $\alpha, \beta : \Delta_2 \rightarrow K$ be defined by $\alpha = [v_0, v_1, v_2], \beta =$

$[v_0, v_2, v_3]$. Consider the 2-chain in K given by $\tau = H \circ \alpha + H \circ \beta$. We claim that $\partial(\tau) = \omega_1 - \omega_2$ and that will show that $(\omega_2) = (\omega_1)$ in $H_1(X)$.

First observe that $H/0 \times \mathbb{I}$ and $H/1 \times \mathbb{I}$ are constant maps at x_0 . Thus $H \circ [v_0, v_3] = H \circ [v_1, v_2]$. Now

$$\partial(\tau) = H \circ ([v_1, v_2] - [v_0, v_2] + [v_0, v_1]) + H \circ ([v_2, v_3] - [v_0, v_3] + [v_0, v_2])$$

which is easily seen to be equal to

$$H \circ [v_0, v_1] + H \circ [v_2, v_3] = \omega_1 - \omega_2.$$

Therefore φ is well-defined.

Now the Lemma 4.14.1 immediately yields that φ is a group homomorphism.

The functoriality of φ is left to the student as a straightforward exercise.

We shall denote $\pi_1(X, x_0)$ simply by π_1 and $H_1(X)$ by H_1 , for the rest of the proof.

Let $\psi : \pi_1 \rightarrow \pi_1^{ab}$ denote the canonical quotient homomorphism of π_1 onto the abelianization of π_1 . Since H_1 is an abelian group, it follows that, $\varphi : \pi_1 \rightarrow H_1$ factors through the abelianization to define a homomorphism

$$\begin{array}{ccc} & \pi_1 & \\ \psi \swarrow & & \searrow \varphi \\ \pi_1^{ab} & \xrightarrow{\varphi'} & H_1 \end{array}$$

such that $\varphi' \circ \psi = \varphi$. It remains to show that, φ' is an isomorphism. We shall construct an explicit inverse to φ' .

For each $x \in X$, choose an arbitrary path ω_x from x_0 to x in X except that, ω_{x_0} should be chosen to be the constant loop at x_0 . To each singular 1-simplex σ in X , define $\theta(\sigma)$ to be the element in π_1^{ab} represented by the loop $\omega_a * \sigma * (\omega_b)^{-1}$, where $a = \sigma(0)$ and $b = \sigma(1)$. Then θ extends to a homomorphism $\theta : S_1(X) \rightarrow \pi_1^{ab}$. We will show that $\theta \circ \partial(S_2(X)) = (0)$. Then it would follow that θ defines a homomorphism, denoted by $\tilde{\theta} : H_1(X) \rightarrow \pi_1^{ab}$. This homomorphism is then easily seen to be the inverse of φ' .

For any singular 2-simplex $\gamma : \Delta_2 \rightarrow X$, we shall prove that $\theta(\partial(\gamma)) = 0$. Let $\gamma(e_0) = a$, $\gamma(e_1) = b$, and $\gamma(e_2) = c$. Let $F^i : \Delta_1 \rightarrow \Delta_2$ be the face maps $F^i : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ as given in 4.5.2. Then

$$\theta(\partial(\gamma)) = \psi(\omega_a * (\gamma \circ F^2) * (\gamma \circ F^0) * (\gamma \circ F^1) * (\omega_a)^{-1})$$

Since $(\gamma \circ F^2) * (\gamma \circ F^0) * (\gamma \circ F^1)$ is a null homotopic loop, it follows that the term on the RHS above is zero. Thus $\tilde{\theta}$ is well defined.

Given $[\tau] \in \pi_1(X, x_0)$, $\varphi([\tau])$ is represented by the 1-cycle τ itself. Therefore, $\tilde{\theta}(\varphi[\tau]) = \psi([\tau])$ in $\pi_1^{ab}(X)$.

On the other hand, given any element $\lambda \in H_1$ represented by a cycle $\sigma = \sum_i \lambda_i$, Lemma 4.14.1 shows that $\varphi(\sum_i \theta(\lambda_i))$ represents the same element λ , since all the extra edges introduced along paths ω'_x s cancel out in pairs. This means that $\varphi' \circ \tilde{\theta}(\lambda) = \lambda$ and so, $\tilde{\theta} = (\varphi')^{-1}$. ♠

Example 4.14.3 As a simple application, we can now compute the CW-chain complex of $\mathbb{S}^1 \times \mathbb{S}^1$. Recall (see Example 2.5.3) that treating $\mathbb{S}^1$ as a CW-complex with a single 0-cell and a single 1-cell, the product CW-complex will then have a single 0-cell and two 1-cells and a single 2-cell. It follows that $C_0 = \mathbb{Z}$, $C_1 = \mathbb{Z} \oplus \mathbb{Z}$, $C_2 = \mathbb{Z}$ and $C_q = 0$ for $q > 2$. It is also clear that $\partial_1 : C_1 \rightarrow C_0$ is the zero map. Now $\partial_2 : C_2 \rightarrow C_1$ is given by the degree of two maps obtained by composing the attaching map $\mathbb{S}^1 = \partial(\mathbb{I} \times \mathbb{I}) \rightarrow \mathbb{S}^1 \vee \mathbb{S}^1$ with the two projections $\mathbb{S}^1 \vee \mathbb{S}^1 \rightarrow \mathbb{S}^1$. We have seen that the attaching map which is actually the boundary of the product of the two characteristic maps of the 1-cells, represents the element $xyx^{-1}y^{-1}$ in $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$. Passing onto the homology via the Hurewicz map this represents the trivial element in $H_1(\mathbb{S}^1 \vee \mathbb{S}^1)$. It follows that the two degrees are zero and hence $\partial_2 = 0$.

4.15 All Postponed Proofs

Homotopy Invariance Theorem for $S_\cdot$ and $S_\cdot^{sm}$

Here we shall present a proof of Theorem 4.6.1. [Those of you who have studied Poincaré's lemma in differential topology may notice some similarity in the proof there (see [?] p. 116) and the proof of this theorem given below.]

We shall first concentrate on singular homology. As seen before, we need to construct the prism operators $h : S_q(X, A) \rightarrow S_{q+1}((X, A) \times \mathbb{I})$ (4.6.2), which define a chain homotopy between $(\eta_0)_*$ and $(\eta_1)_*$ (at the chain group level). This h will be functorial, in the sense that if $\alpha : (X, A) \rightarrow (Y, B)$ is any map, then we have a commutative diagram

$$\begin{array}{ccc} S_\cdot(X, A) & \xrightarrow{h} & S_\cdot((X, A) \times \mathbb{I}) \\ \alpha_* \downarrow & & \downarrow (\alpha \times id)_* \\ S_\cdot(Y, B) & \xrightarrow{h} & S_\cdot((Y, B) \times \mathbb{I}). \end{array}$$

But then, if σ is a singular n -simplex, we would have

$$h(\sigma) = h \circ \sigma_*(\xi_n) = (\sigma \times id)_* h(\xi_n)$$

where $\xi_n : |\Delta_n| \rightarrow |\Delta_n|$ is the identity map. Thus it suffices to define $h(\xi_n)$ for each n . We demand that h satisfies the equation

$$(\eta_1)_*(\sigma) - (\eta_0)_*(\sigma) = h \circ \partial(\sigma) + \partial \circ h(\sigma) \quad (4.21)$$

for all singular n simplexes σ . By functoriality, this will be the case, if we manage to have,

$$(\eta_1)_*(\xi_n) - (\eta_0)_*(\xi_n) = h \circ \partial(\xi_n) + \partial \circ h(\xi_n). \quad (4.22)$$

For we can simply apply $(\sigma \times Id)_*$ to both sides of (4.22) to get (4.21).

We now need to introduce some notation. In $\Delta_n \times \mathbb{I}$, let

$$\begin{aligned} b^i &= (\mathbf{e}_i, 0), \quad c^i = (\mathbf{e}_i, 1) \text{ and} \\ \gamma^n &= (\beta(\Delta_n), 1/2) \quad (\text{where } \beta(\Delta_n) \text{ is the barycentre of } \Delta_n). \end{aligned}$$

(See Example ?? and Figure ??.) For any simplicial complex K , if $\sigma : \Delta_n \rightarrow K$ is a simplicial map with $\sigma(\mathbf{e}_i) = v_i$, then we shall denote σ by $[v_0, \dots, v_n]$. If $x \in |s|$, where $|s| \supset \sigma(\Delta_n)$, then $x\sigma := x[v_0, \dots, v_n]$ denotes the linear singular $(n+1)$ -simplex defined by

$$\left. \begin{aligned} (x\sigma)(\mathbf{e}_0) &= x \quad \text{and} \\ (x\sigma)(\mathbf{e}_i) &= v_{i-1}, \quad i \geq 1 \end{aligned} \right\} \quad (4.23)$$

and extended linearly. If $\rho = \sum_j n_j \sigma_j$ is a singular n -chain with $\text{supp } \rho \subset |s|$ then for any $x \in |s|$, it makes sense to talk of $x\rho = \sum_j n_j x\sigma_j$, which is a singular $(n+1)$ -chain.

Now the definition of h is completed by induction on n : Define

$$h(\xi_0) = \gamma^0([c^0] - [b^0]).$$

Then h is defined on $S_0(X, A)$ for all (X, A) . Note that if $h(\xi_r)$ is defined for all $r \leq n-1$, then h is defined on $S_r(X, A)$ for all $r \leq n-1$ and for all (X, A) . So inductively define

$$h(\xi_n) = \gamma^n([c^0, \dots, c^n] - [b^0, \dots, b^n] - h \circ \partial(\xi_n)).$$

By induction again we have,

$$\begin{aligned} \partial \circ h(\partial \xi_n) &= (\eta_1)_*(\partial \xi_n) - (\eta_0)_*(\partial \xi_n) - h \circ \partial^2 \xi_n \\ &= \partial((\eta_1)_*(\xi_n) - (\eta_0)_*(\xi_n)) \\ &= \partial([c^0, \dots, c^n] - [b^0, \dots, b^n]). \end{aligned}$$

Hence

$$\begin{aligned}\partial \circ h(\xi_n) &= [c^0, \dots, c^n] - [b^0, \dots, b^n] - h \circ \partial(\xi_n) \\ &\quad - \gamma^n(\partial[c^0, \dots, c^n], -\partial[b^0, \dots, b^n] - \partial \circ h \circ \partial(\xi_n)) \\ &= (\eta_1)_*(\xi_n) - (\eta_0)_*(\xi_n) - h \circ \partial(\xi_n)\end{aligned}$$

as required.

Now consider the case of S^{sm} on the smooth category. If we replace the word ‘map’ by ‘smooth map’ everywhere in the above proof, the entire thing goes through without any trouble, until we come to the notation (4.23). However, we need this only for the case when K is the ‘prism’ simplicial complex underlying $\Delta_n \times \mathbb{I}$ as constructed in Example ???. All simplexes here will be automatically smooth and the inductive formula for h takes values inside smooth chains. This gives the proof of the homotopy invariance of smooth singular homology, H^{sm} . ♠

Excision Theorems

Here we shall present proofs of Theorems 4.7.4 and 4.9.2.

We begin with a definition:

Definition 4.15.1 Subdivision chain map We define, respectively, a functorial chain map Sd and a chain homotopy D of Sd with the identity map

$$Sd : S(X) \rightarrow S(X); \quad D : S(X) \rightarrow S(X)$$

of Sd inductively as follows:

- (1) If τ is a 0-chain, define $Sd(\tau) = \tau$ and $D(\tau) = 0$.
- (2) Having defined Sd and D on $(n-1)$ -chains, we shall now define them on n -chains. First, let ξ_n be the identity singular n -simplex on $|\Delta_n|$. Take

$$Sd(\xi_n) = \beta(\Delta_n)(Sd(\partial(\xi_n))); \quad D(\xi_n) = \beta(\Delta_n)Sd(\partial(\xi_n) - \xi_n - D\partial(\xi_n)).$$

Now for any singular n -simplex σ , take

$$Sd(\sigma) = \sigma_*(Sd(\xi_n)); \quad D(\sigma) = \sigma_*(D(\xi_n)).$$

(Here σ_* denotes the homomorphisms induced by σ on the chain complexes.) Then extend them linearly over all the n -chains. Notice that, once we define these maps on the ‘universal’ singular simplexes ξ_n , then the rest of the definition is forced on us by the functoriality. (Just like in the proof of homotopy invariance, this is a typical example of what one generally does in such situations and worth noting down.)

To see that, Sd is a chain map, we prove $\partial \circ Sd(\tau) = Sd(\partial(\tau))$ by induction on the (homogeneous) degree n of the chain τ . If $n = 0$, there is nothing to prove. Having proved this for $n - 1$, we note that it is enough to consider the case $\tau = \xi_n$. Then we have.

$$\begin{aligned}\partial \circ Sd(\xi_n) &= Sd(\partial(\xi_n)) - \beta(\Delta_n)(\partial(Sd(\partial(\xi_n)))) \\ &= Sd(\partial(\xi_n)) - \beta(\Delta_n)(Sd \circ \partial(\partial(\xi_n))) \\ &= Sd(\partial(\xi_n)),\end{aligned}$$

since $\partial \circ \partial = 0$. Now to show that $\partial \circ D + D \circ \partial = Sd - Id$, we again induct on the degree of τ . At degree 0, there is no statement at all. At degree 1, this follows from the fact that $D_0 = 0$. Suppose we have proved the statement for $(n - 2)$ -chains. As a consequence we have,

$$\begin{aligned}\partial \circ D(\partial(\xi_n)) &= (\partial \circ D + D \circ \partial)(\partial(\xi_n)) \\ &= (Sd - Id)(\partial(\xi_n)) \\ &= \partial(Sd(\xi_n)) - \partial(\xi_n) \\ &= \partial(Sd(\xi_n) - \xi_n)\end{aligned}$$

Thus, $\partial(Sd(\xi_n) - \xi_n - D(\partial(\xi_n))) = 0$. Now we have,

$$\begin{aligned}(D \circ \partial + \partial \circ D)(\xi_n) &= D \circ \partial(\xi_n) + \partial(\beta(\Delta_n)(Sd(\xi_n) - \xi_n - D \circ \partial(\xi_n))) \\ &= D \circ \partial(\xi_n) + Sd(\xi_n) - \xi_n - D \circ \partial(\xi_n) \\ &\quad - \beta(\Delta_n)(\partial(Sd(\xi_n) - \xi_n - D(\partial(\xi_n)))) \\ &= Sd(\xi_n) - \xi_n,\end{aligned}$$

as required.

For the sake of future use, we make two observations which we state as theorems.

Theorem 4.15.2 *The subdivision chain map and the chain homotopy D have the property that*

- (a) *for any smooth object M , $Sd(S^{\text{sm}}(M)) \subset S^{\text{sm}}(M)$, $D(S^{\text{sm}}(M)) \subset S^{\text{sm}}(M)$ and define a chain map and chain homotopy of the subcomplex S^{sm} and*
- (b) *for any simplicial complex K ,*

$$Sd(\mathcal{S}(K)) \subset \mathcal{S}(sd K), D(\mathcal{S}(K)) \subset \mathcal{S}(sd K)$$

and respectively define a chain map and a chain homotopy at the subcomplex levels also. Moreover,

(c) this further induces chain map and chain homotopy at the quotient complexes

$$Sd : C(K) \longrightarrow C(sd K); \quad D : C(K) \longrightarrow C(sd K).$$

$$\begin{array}{ccc} S.(|K|) & \xrightarrow{Sd} & S.(|K|) = S.(|sd K|) \\ \uparrow & & \uparrow \\ \mathcal{S}(K) & \xrightarrow{Sd} & \mathcal{S}(sd K) \\ \downarrow & & \downarrow \\ C(K) & \xrightarrow{Sd} & C(sd K) \end{array}$$

Theorem 4.15.3 *If $\tau : sd K \longrightarrow K$ is a simplicial approximation to identity map then on the chain complex $C(K)$ we have $\tau_* \circ Sd = Id$.*

The proofs of the above theorems are straightforward.

Coming to the proof of excision for the singular homology:

Let $\phi : H(S.(X_1) + S.(X_2)) \rightarrow H(X)$ be the map induced by the inclusion. Now given a singular n -simplex σ in X , we get two open sets $(\sigma)^{-1}(\text{int } X_1)$ and $(\sigma)^{-1}(\text{int } X_2)$ which cover $|\Delta_n|$. Clearly, there exists k such that $sd^k \Delta_n$ is finer than this covering. If $Sd^k = Sd \circ Sd \dots \circ Sd$ (k -copies), then it follows that the singular n -chain $Sd^k(\sigma)$ belongs to the subgroup $S.(X_1) + S.(X_2)$. Thus if τ is a singular n -chain, then there exists a k , for which $Sd^k(\tau) \in S.(X_1) + S.(X_2)$. Note that, $Sd^k(\tau) \in S.(X_1) + S.(X_2)$ implies its boundary is also there. Further, note that both Sd and D map the chain subgroup $S.(X_1) + S.(X_2)$ to itself and so Sd induces identity homomorphism on the homology of $S.(X_1) + S.(X_2)$. In particular, for any cycle τ in $S.(X_1) + S.(X_2)$, $Sd^k(\tau)$ and τ represent the same element in $H.(X)$. Now, given a n -cycle τ , for some k , we have $Sd^k(\tau) \in S.(X_1) + S.(X_2)$ and hence the surjectivity of ϕ follows. To prove the injectivity, let τ_1 and τ_2 be any two n -cycles in $S.(X_1) + S.(X_2)$ and let α be a $(n+1)$ -chain in $S.(X)$, such that $\partial(\alpha) = \tau_1 - \tau_2$. Then for large k , $Sd^k(\alpha) \in S.(X_1) + S.(X_2)$ and we have $\partial Sd^k(\alpha) = Sd^k(\tau_1) - Sd^k(\tau_2)$. On the other hand, from our earlier observation, $Sd^k(\tau)$ and τ represent the same element in $H(S.(X_1) + S.(X_2))$. This proves the injectivity of ϕ , thereby completing the proof of the excision Theorem 4.7.4.

Now, in view of Theorem 4.15.2(a) the above proof works verbatim for the smooth singular chain complex as well, thereby proving Theorem 4.9.2. ♠

Proof of Singular-Simplicial versus Singular Theorem

We shall directly prove that induced map $\iota_* : H(\mathcal{S}(K)) \rightarrow H(|K|)$ is an isomorphism in homology. Since both chain complexes are free, it would then follow from general algebraic considerations that ι is a chain equivalence. (See Theorem ??.)

We first note that it is enough to prove this when K is finite. For arbitrary K , we can then take direct limit over finite subcomplexes of K .

Let K be a finite complex. We induct on the number r of simplexes in K . If $r = 1$ then the statement is obviously true. Assume the result for all simplicial complexes with number of simplexes $< r$ and $r > 1$. Let s be a simplex in K of maximal dimension. Let L be the subcomplex of K consisting of all simplexes in K other than s . Then by induction hypothesis $\iota_* : H(\mathcal{S}(L)) \rightarrow H(\mathcal{S}(|L|))$ is an isomorphism, which restricts to an isomorphism $\iota_* : H(\mathcal{S}(\dot{s})) \rightarrow H(\mathcal{S}(|\dot{s}|))$. Here $\dot{s}$ denotes the boundary simplicial complex of the simplex s . We also know that both $H(\mathcal{S}(s))$ and $H(\mathcal{S}(|s|))$ vanish in positive dimensions (see Examples 4.10.4) and hence $\iota_* : H(\mathcal{S}(s)) \rightarrow H(\mathcal{S}(|s|))$ is an isomorphism. Since any pair $\{K_1, K_2\}$ of subcomplexes of a simplicial complex is an excisive couple for both singular and singular simplicial complexes we get the two corresponding Mayer–Vietoris sequences. Denoting temporarily $H(\mathcal{S}(K))$ by $\hat{H}(K)$ we then have a commutative diagram:

$$\begin{array}{ccccccccc}
 \hat{H}_n(\dot{s}) & \longrightarrow & \hat{H}_n(s) \oplus \hat{H}_n(L) & \longrightarrow & \hat{H}_n(K) & \longrightarrow & \hat{H}_{n-1}(\dot{s}) & \longrightarrow & \hat{H}_{n-1}(s) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_n(|\dot{s}|) & \longrightarrow & H_n(|s|) \oplus H_n(|L|) & \longrightarrow & H_n(|K|) & \longrightarrow & H_{n-1}(|\dot{s}|) & \longrightarrow & H_{n-1}(|s|)
 \end{array}$$

in which the two rows are Mayer–Vietoris exact sequences and the vertical arrows are all induced by ι . By the Five lemma, it follows that $\iota_* : \hat{H}_n(K) \rightarrow H_n(|K|)$ is also an isomorphism for all n . This completes the proof of Theorem 4.10.10. ♠

4.16 Equivalence of Singular-Simplicial and Simplicial Homologies

Here we shall present a proof Theorem 4.10.11.

Fix a total order on the set of vertices of K . Then for each n , each n -simplex σ in K can be displayed uniquely as a strictly monotonically increasing sequence $(v_{i_0}, \dots, v_{i_n})$ and hence defines a unique element of $\mathcal{S}_n(K)$. This assignment extended linearly defines

a splitting $\alpha_n : C_n(K) \rightarrow \mathcal{S}_n(K)$ of the quotient map $\varphi : \mathcal{S}_n(K) \rightarrow C_n(K)$. It can be easily checked that the $\alpha = \{\alpha_n\}$ is a chain map $C(K) \rightarrow \mathcal{S}(K)$ such that $\varphi \circ \alpha = Id_{C(K)}$. It remains to define a chain homotopy $h : \alpha \circ \varphi \approx Id_{\mathcal{S}(K)}$.

Observe that both φ and α preserve subcomplexes of K , viz, for any subcomplex $L \subset K$ we have $\varphi(\mathcal{S}(L)) \subset C(L)$ and $\alpha(C(L)) \subset \mathcal{S}(L)$. The chain homotopy that we are going to construct will also have this property. Hence we can easily pass onto relative chain complexes as well.

We have to define $h : \mathcal{S}_n(K) \rightarrow \mathcal{S}_{n+1}(K)$ so that

$$\partial \circ h + h \circ \partial = \alpha \circ \varphi - Id. \quad (4.24)$$

The construction of h is carried out by induction on n . We note that $\alpha_0 : C_0(K) \rightarrow \mathcal{S}_0(K)$ is the identity map. Therefore we can start with $h(\sigma) = 0$ for all 0-simplices. Now suppose we have defined $h : \mathcal{S}_{n-1}(K) \rightarrow \mathcal{S}_n(K)$ so as to satisfy

$$h \circ \partial + \partial \circ h = \alpha \circ \varphi - Id \quad (4.25)$$

and such that for each $(n-1)$ singular simplex σ , the support of $h(\sigma)$ is contained in the support of σ . Now let τ be any singular n -simplex in K and $s = \text{supp } \tau \in K$. We need to find a $(n+1)$ -chain $h(\tau)$ supported in s and such that

$$\partial \circ h(\tau) = \alpha \circ \varphi(\tau) - \tau - h \circ \partial(\tau).$$

Obviously, it is necessary that the chain on the RHS of the above equation must be a cycle, i.e.,

$$0 = \partial(\alpha \circ \varphi(\tau) - \tau - h \circ \partial(\tau)) = \alpha \circ \varphi(\partial\tau) - \partial(\tau) - \partial \circ h(\partial\tau).$$

This follows from the inductive step (4.25). It is also clear that support of this cycle is contained in s . Since the homology of $H_n(\mathcal{S}(\Delta_q)) = 0, n > 0$ (see Example 4.10.4), it follows that this cycle is a boundary and hence there is a chain $h(\tau) \in \mathcal{S}_{n+1}(s) \subset \mathcal{S}_{n+1}(K)$ satisfying our requirement. This completes our hunt for a choice of $h(\tau)$ and thereby completes the construction of the chain homotopy h . ♠

Equivalence of CW-homology and Cellular Singular Homology

Here we shall present a proof of Theorem 4.11.12. The basic idea here is the one we have met in the proof of the homotopy invariance.

Lemma 4.16.1 (Retraction operator) Let C be a subcomplex of $S(X, A)$ (freely) generated by some singular simplexes. Assume that to each singular simplex $\sigma : \Delta_q \rightarrow X$ there exists a *singular prism* (i.e., a continuous map) $P\sigma : \mathbb{I} \times \Delta_q \rightarrow X$ with the following properties:

- (a) $P\sigma(0, z) = \sigma(z)$;
- (b) $P\sigma|_{1 \times \Delta_q} \in C$;
- (c) $P(\sigma \circ F^i) = P\sigma \circ (1 \times F^i)$, for each face operator F^i ;
- (d) if $\sigma \in C$, then $P\sigma(t, z) = \sigma(z)$; and
- (e) if $\sigma \in S(A)$ then $P\sigma(\mathbb{I} \times \Delta_q) \subset A$.

Then there is a chain deformation retraction $\tau : (S(X), S(A)) \rightarrow (C, S(A) \cap C)$. In particular, the inclusion $C \rightarrow S(X)$ induces isomorphism in the homology.

Proof: Taking $\tau(\sigma) = P\sigma|_{1 \times \Delta_q}$, and extending linearly over $S(X, A)$, we obtain the map $\tau : S(X, A) \rightarrow C$. It follows that τ is a chain retraction (use (c)). Now a chain homotopy D from Id to τ is defined by taking $D(\sigma) = h(\sigma)$, where h is the prism operator defined in Lemma 4.6.2. ♠

Now the proof of Theorem 4.11.12 is completed by appealing to the cellular approximation theorem. Given any simplex $\sigma : \Delta_n \rightarrow X$, all we do is choose a cellular approximation to it and a homotopy of the original map with the approximation, which gives the prism $P\sigma$. Of course, if σ is already a cellular map then we take this homotopy itself to be the identity, viz., $P\sigma(z, t) = \sigma(z)$. ♠

Remark 4.16.2 The above lemma plays a crucial role in the proof of the Hurewicz theorem (see Chapter 10.)

4.17 Miscellaneous Exercises to Chapter 4

1. (a) Suppose $f_\alpha : A_\alpha \rightarrow B_\alpha$ is a directed system of homomorphisms of abelian groups with its direct limit $f : A \rightarrow B$. Show that $\{\text{Ker } f_\alpha\}$ and $\{\text{Im } f_\alpha\}$ form directed systems and

$$\text{Ker } f = \varinjlim_\alpha \text{Ker } f_\alpha; \quad \text{Im } f = \varinjlim_\alpha \text{Im } f_\alpha.$$

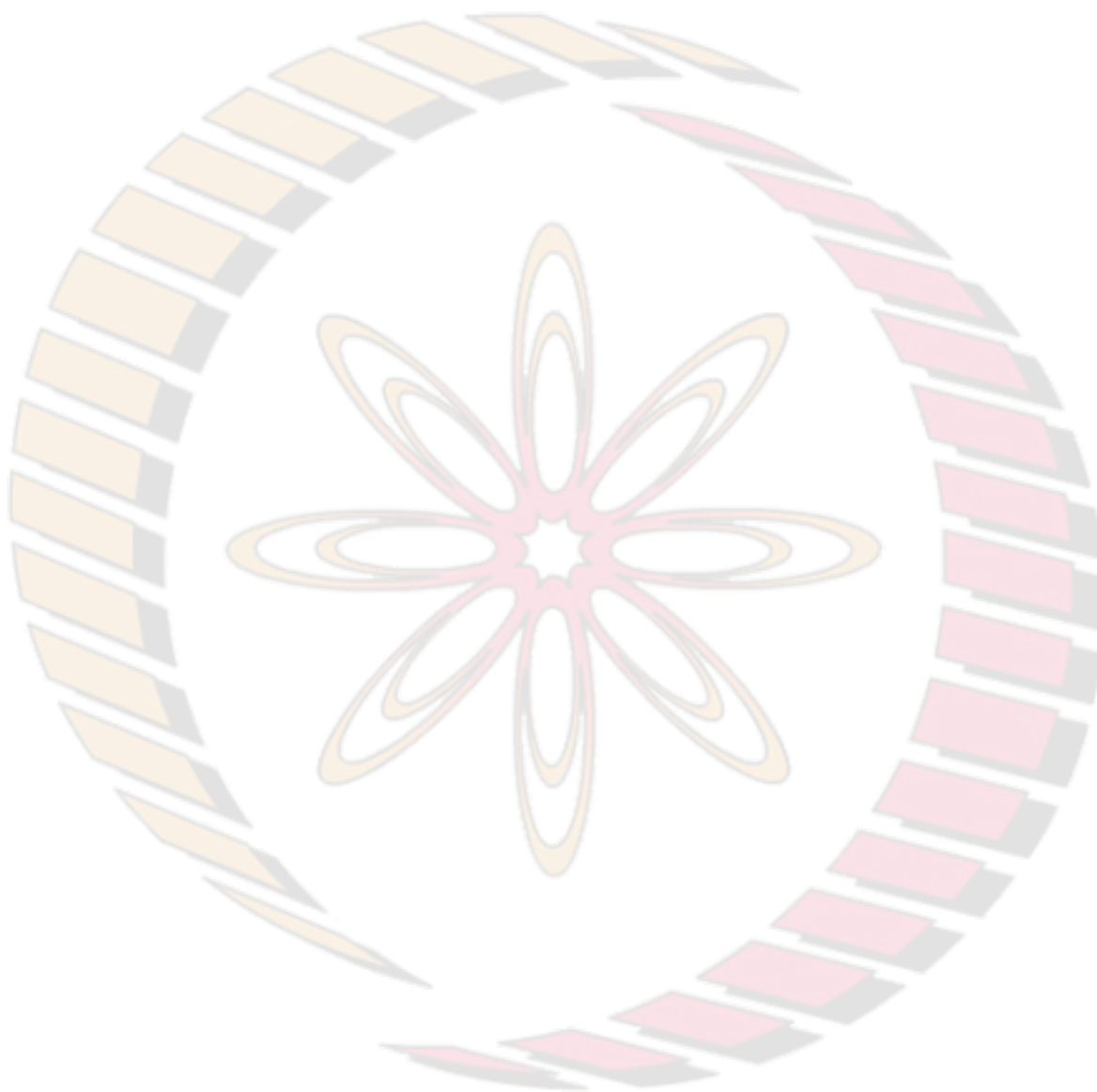
- (b) Show that direct limit of a family of long exact sequences of abelian groups is a long exact sequence.
- (c) Show that the singular chain complex of a topological space is the direct limit of the singular chain complexes of its compact subsets.

- (d) Show that homology commutes with direct limits, i.e., given a directed system $(C.)_\alpha$ of chain complexes

$$H(\lim_\alpha (C.)_\alpha) = \lim_\alpha H((C.)_\alpha).$$

- (e) Conclude from the above two exercises that the singular homology of a space is the direct limit of the singular homology of its compact subsets.

2. Given an open covering $\mathcal{U} = \{U_i\}$ of a space X , we define the chain complex $S_*^\mathcal{U}$ of X to be the subchain complex generated by those singular simplices σ whose image is contained in one of the members of $\mathcal{U}$. Show that the inclusion map $S_*^\mathcal{U}(X) \rightarrow S_*(X)$ induces isomorphism on homology. The chain complexes $S_*^\mathcal{U}$ are called *complexes of small chains*. [Hint: Use iterated barycentric subdivisions as in the proof of the excision theorem.]
3. Let X be the union of two closed sets X_1, X_2 such that $\text{cl}(X_1 \setminus X_2) \cap \text{cl}(X_2 \setminus X_1) = \emptyset$. Show that $\{X_1, X_2\}$ is an excisive couple. Put $A = X_1 \cap X_2$. Deduce that $H(X, A) = H(X_1, A) \oplus H(X_2, A)$. Generalize this to the case when X is the union of finitely many closed sets.
4. Show that if K_1, K_2 are two triangulations of a compact space then $\dim K_1 = \dim K_2$.
5. Compute the integral homology groups of the double comb space given in Exercise ??? and show that it is trivial. (See Exercise ?? in ??.)
6. The graph is called *bipartite* if the vertex set V can be written as a disjoint union $V = V_1 \sqcup V_2$ so that there are no edges within V_1 or V_2 but every vertex of V_1 is joined to every vertex of V_2 . If $\#(V_i) = r_i$, then this graph is denoted by K_{r_1, r_2} . A finite graph on n vertices in which every pair of vertices is joined is called a complete graph and is denoted by K_n . A graph whose underlying topological space can be embedded in $\mathbb{R}^2$ (or in $\mathbb{S}^2$) is called a planar graph.
Show that K_5 and $K_{3,3}$ are non planar. [Hint: Use the Jordan curve theorem.] (In fact a classical theorem due to Kuratowski states that a finite graph is planar iff it contains no subgraphs homeomorphic to K_5 or $K_{3,3}$.)
7. Consider the subspace Y of $\mathbb{R}^2 = \mathbb{C}$ which is the union of the three line segments $[0, 1], [0, \omega], [0, \omega^2]$, where ω is a primitive cube root of 1. Prove or disprove that there is an embedding (piecewise linear) of $Y \times Y$ inside $\mathbb{R}^3$.



NPTEL

Chapter 5

Topology of Manifolds

Manifolds are central objects of study in topology. Earlier we introduced CW-complexes and simplicial complexes. These are in some sense a slight generalization of manifolds and help us in understanding manifolds better.

In Sections 5.1-5.4 we shall study topological and homotopic aspects of manifolds in general. In Section 5.5, we shall classify 1-dimensional manifolds. Section 5.6-5.8 will shall discuss to triangulability of manifolds in general, and, of surfaces in particular. The last four sections will be devoted to the classification of compact surfaces.

5.1 Definitions and Examples

Definition 5.1.1 Let $n \in \mathbb{N}$. Let $X \neq \emptyset$ be a topological space. By a (n -dimensional) *chart* for X we mean a pair (U, ψ) consisting of an open neighbourhood U of x and a homeomorphism $\psi : U \rightarrow \mathbb{R}^n$ onto an open subset of $\mathbb{R}^n$. By an *atlas* $\{(U_j, \psi_j)\}$ for X , we mean a collection of charts for X such that $X = \cup_j U_j$. If there is an atlas for X , we say X is *locally Euclidean*.

A chart (U, ψ) is called a *chart at* $x_0 \in X$ if $\psi(x_0) = 0$. Writing $\psi = (\psi_1, \dots, \psi_n)$, these n component functions ψ_i are called *local coordinate functions* for X at x .

Let $n \geq 1$ be an integer and X be a topological space. We say X is a *topological manifold of dimension n* if X is :

- (i) locally Euclidean, i.e., there is an atlas consisting of n -dimensional charts,
- (ii) a Hausdorff space and
- (iii) II-countable, i.e., it has a countable base for its topology.

Any countable discrete space is called a *0-dimensional manifold*.

Remark 5.1.2

1. We would like to include the empty space also as a topological manifold. However, there is no good way of assigning a dimension to it. Some authors prefer it to be of dimension -1 , and some others $-\infty$. Indeed, the best way would be to treat it as a manifold of any dimension as and when required. In what follows a manifold is always assumed to be nonempty unless it obviously follows from the context that a particular one is empty.
2. Observe that once a chart (U, ψ) exists around a point $x_0 \in X$, then we can choose a chart (V, ϕ) at x_0 such that $\phi(V) = \mathbb{R}^n$. For, by composing with a translation, we can assume that $\psi(x_0) = 0$ and then we can choose $r > 0$ such that the open ball $B_r(0) \subset \psi(U)$ and put $V = \psi^{-1}(B_r(0))$, and $\phi = f \circ \psi$ where $f : B_r(0) \rightarrow \mathbb{R}^n$ is the homeomorphism (actually a diffeomorphism) given by $\mathbf{x} \mapsto \frac{\mathbf{x}}{r^2 - \|\mathbf{x}\|^2}$.
3. For an atlas, it is necessary to assume that the integer n is the same for all the charts. Of course, if X is connected, it is a consequence of the *topological invariance of domain* (see Theorem 4.13.2) that the integer n is the same for all charts.
4. For a topological space that is locally Euclidean, the II-countability condition is equivalent to many others, such as metrizable or paracompactness. We find II-countability the most suitable for our purpose. The Urysohn's metrization theorem which is a standard result in point set topology tells you that every II-countable locally compact Hausdorff space is metrizable. Also, a metric space is always paracompact. Since this is not a standard result, we shall directly prove that a manifold is paracompact.

Example 5.1.3

- (i) Clearly, differential manifolds inside Euclidean spaces that you may have studied in your calculus course are topological manifolds in the above sense.
- (ii) The boundary of the unit square $\mathbb{I} \times \mathbb{I}$ as a subspace of $\mathbb{R}^2$ is a topological 1-manifold. You may have learnt that this is not a smooth submanifold of $\mathbb{R}^2$ because of those corner points.

- (iii) Let X be the union of the two axes in $\mathbb{R}^2$. If U is any connected neighbourhood of $(0,0)$ in X then $U \setminus \{(0,0)\}$ has four components. It follows that X cannot have any chart covering $(0,0)$ and hence fails to be a topological manifold.
- (iv) **Line with double origin:** Let X be the set of all real numbers together with one extra point that we shall denote by $\tilde{0}$. We shall make X into a topological space as follows: Let $\mathcal{T}$ be the collection of all subsets A of X of the form $A = B \cup C$ where
- (a) B is either empty or an open subset of $\mathbb{R}$ in the usual topology and
 - (b) C is either empty or is such that $(C \cap \mathbb{R}) \cup \{0\}$ is a neighbourhood of 0 in $\mathbb{R}$.
- We leave it to you to verify that $\mathcal{T}$ forms a topology on X in which $\mathbb{R}$ is a subspace. Since $\tilde{0}$ also has neighbourhoods that are homeomorphic to an interval, it follows that X has an atlas. It is easily seen that X has a countable base also. But however, observe that X fails to be a Hausdorff space, since neighbourhoods of 0 and $\tilde{0}$ cannot be disjoint. This space can also be thought of as the quotient space obtained by taking two copies of $\mathbb{R}$ and identifying every non zero real number in one copy with the corresponding number in the other copy.
- (v) Likewise, one can also give examples of spaces that are Hausdorff and locally Euclidean but are not II-countable. The typical example is the so-called *long line*: Consider an uncountable set Ω_0 which is well ordered in such a way that every element has only countably many predecessors. Put $X = \Omega_0 \times [0, 1)$. Define a total order $\ll$ on X as follows:

$$(\alpha, t) \ll (\beta, s) \quad \text{if } \alpha < \beta \text{ or } \alpha = \beta \text{ and } t < s.$$

With the order topology induced by this order, X is a connected, Hausdorff space, having a smooth structure, locally diffeomorphic to $\mathbb{R}$. But X does not have a countable base. It is also not path connected. (For more details, See [?].)

- (vi) Another type of non-example is obtained by taking the disjoint union of manifolds of different dimensions. Thus, the subspace of $\mathbb{R}^2$, consisting of the x -axis together with the point $(0, 1)$, is not a manifold.

Let us now consider some examples of manifolds that do not occur naturally as subspaces of any Euclidean space.

- (vii) **The real projective spaces** The foremost one is the n -dimensional real projective space $\mathbb{P}^n$. This is the quotient space of the unit sphere $\mathbb{S}^n$ by the antipodal action,

viz., each element x of $\mathbb{S}^n$ is identified with its antipode $-x$. For more details see Part-I notes wherein we have proved that $\mathbb{P}^n$ is compact.

Given $x \in \mathbb{S}^n$ consider V to be the set of all points in $\mathbb{S}^n$, that are at a distance less than $\sqrt{2}$ from x . Then check that $U = q(V)$ is a neighbourhood of $[x]$ in $\mathbb{P}^n$ and q itself restricts to a homeomorphism from V to U . Since V is anyway homeomorphic to an open subset of $\mathbb{R}^n$, this proves the existence of an n -dimensional atlas for $\mathbb{P}^n$.

To see that $\mathbb{P}^n$ is Hausdorff, let $[x] \neq [y] \in \mathbb{P}^n$ be two points. Clearly, in $\mathbb{S}^n$, we can choose $\epsilon > 0$ such that $B_\epsilon(\pm x) \cap B_\epsilon(\pm y) = \emptyset$. It then follows that $q(B_\epsilon(x))$ and $q(B_\epsilon(y))$ are disjoint neighbourhoods of $[x]$ and $[y]$ in $\mathbb{P}^n$.

- (viii) One can merely take a connected topological manifold X , and take a connected covering space $p : \tilde{X} \rightarrow X$. Since p is a local homeomorphism, it follows that $\tilde{X}$ is locally euclidean. It is not very hard to prove that $\tilde{X}$ is Hausdorff (see Exercise ??(i)). However, why should $\tilde{X}$ be II-countable or metrizable? If we know that the cardinality of the fibre is countable (which is necessary if $\tilde{X}$ has to be II-countable), then of course it easily follows that $\tilde{X}$ is II-countable: We can take a countable base $\{V_i\}$ for X consisting of evenly covered open sets and then take the collection of all open sets in $\tilde{X}$ homeomorphic to V_i 's under p . Thus we are led to the question: Is the fundamental group of any topological manifold countable? The answer is yes (see Corollary 5.4.2) and hence we conclude that $\tilde{X}$ is a manifold of the same dimension as X . Moreover, it is not hard to see that additional structures such as a C^r -structure or a complex analytic structure, etc., on X can also be pulled back onto $\tilde{X}$.

Let us introduce the notation

$$\mathbf{H}^n = \{(x_1, \dots, x_n) : x_n \geq 0\}$$

for the closed upper half space in $\mathbb{R}^n$.

Definition 5.1.4 A topological space X is called a *manifold with boundary* if it is a II-countable, Hausdorff space, such that each point x of X has an open neighbourhood U_x and a homeomorphism $\phi : U_x \rightarrow \mathbf{H}^n$ onto an open subset of $\mathbf{H}^n$.

Denote by $\text{int } X$, the set of all those points in x having a neighbourhood U_x homeomorphic to an open subset of

$$\text{int } \mathbf{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

Clearly this forms an open subset of X and is a topological n -manifold in the old sense. Can you see why $\text{int } X$ is non empty if X is non empty? The complement of $\text{int } X$ in X is denoted by ∂X and is called the *boundary* of X . Clearly it is a closed subset of X .

Remark 5.1.5

1. It may happen that ∂X is empty which means precisely that X is a manifold. The points of ∂X are characterized by the following property. There is a neighbourhood U_x of x and a homeomorphism $\phi : U_x \rightarrow \mathbf{H}^n$ such that the n^{th} -coordinate of $\phi(x)$ vanishes, i.e., $\phi_n(x) = 0$. This is again a simple consequence of the topological invariance of domain (Theorem 4.13.2).
2. It follows that $\hat{U} = \phi^{-1}(\mathbb{R}^{n-1} \times 0)$ is a neighbourhood of x in ∂X if we take ϕ as above. Also, then ϕ itself restricts to a homeomorphism $\phi : \hat{U} \rightarrow (\mathbb{R}^{n-1} \times \{0\}) \cap \phi(U)$. As a consequence, it follows that ∂X , if non empty, is itself a topological $(n-1)$ -dimensional manifold (without boundary).
3. We shall most often use the word ‘manifold’ to mean a manifold without boundary. Often the results that we state for them are valid for manifolds with boundary as well, though we cannot take them for granted. Indeed, whenever, special attention is needed for manifolds with boundary, we shall take care to mention them.

5.2 Paracompactness

One of the most important tools in the study of manifolds is partition of unity. If you have studied this earlier for differential manifolds, you may skip reading the proofs of Lemma 5.2.1 and Theorem 5.2.2 given below.

Lemma 5.2.1 Let X be a topological manifold. Then there exists a nested sequence of open subsets $\{W_i\}$ in X such that

- (i) $\overline{W_i}$ is compact for each i ;
- (ii) $\overline{W_i} \subset W_{i+1}$, for each i ;
- (iii) $X = \cup_i W_i$.

Proof: Using II-countability, it follows that there is a countable family of homeomorphisms $\phi_i : U_i \rightarrow \mathbb{R}^n$, where U_i are open in X and $X = \cup_i \phi_i^{-1}(B_1(0))$. Let

$$B_r(\mathbf{x}) := \{y \in \mathbb{R}^n : \|y - x\| < r\}.$$

Put $V_i = \phi_i^{-1}(B_1(0))$. Then each V_i is an open subset of X . The closure of V_i is compact being homeomorphic to the closure of the open ball $B_1(0)$.

Put $W_1 = V_1$. Inductively having defined W_k , satisfying (i) and (ii), we can select finitely many members of $\{V_i\}$, that cover $\overline{W}_k$. Let W_{k+1} be the union of these members and V_{k+1} . Check the property (iii). ♠

Theorem 5.2.2 Partition of unity on abstract manifolds: *Let X be any subspace of a manifold Y and $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open covering of X . Then there exists a countable family $\{\theta_j\}$ of continuous real valued functions on Y with compact support such that*

- (i) $0 \leq \theta_j(x) \leq 1$, for all j and $x \in Y$;
- (ii) for each $x \in X$ there exists a neighbourhood N_x of x in X , such that only finitely many of θ_j are nonzero on N_x ;
- (iii) for each j , $(\text{supp } \theta_j) \cap X \subset U_{\alpha_j}$ for some α_j ; and
- (iv) $\sum_j \theta_j(x) = 1$, for all $x \in X$.

Proof: Taking $U = \cup_\alpha U_\alpha$, we shall actually prove the result for U in place of X . Therefore, we may as well assume that X itself is open in Y .

By the previous lemma, X is the increasing union of a countable family of open sets $\{K_i\}_{i \geq 1}$ with the closure of each K_i being compact. We set $K_0 = \emptyset$.

We shall first construct a countable family $\{B_{ij}\}$ of open sets in X with homeomorphisms $\psi_{ij} : B_{ij} \rightarrow \mathbb{D}^n$ such that if we put $\frac{1}{2}B_{ij} := \psi_{ij}^{-1}(B_{1/2}(0))$, then $\{\frac{1}{2}B_{ij}\}$ is a covering of X , and is a locally finite open refinement of $\{U_\alpha\}$.

Inductively, suppose $\{B_{ij}\}$ have been constructed for $i \leq k$ so that $\{\frac{1}{2}B_{ij}\}$ covers $\overline{K}_k$. For each point $x \in \overline{K}_{k+1} \setminus K_k$, we can choose a neighbourhood W_x contained in some member of $\{U_\alpha\}$ and not intersecting $\overline{K}_{k-1}$. We can further assume that there is a homeomorphism $\psi_x : W_x \rightarrow B_1(0)^n$ such that $\psi_x(x) = 0$. Since $\{\psi_x^{-1}(B_{1/2}(0))\}$ is an open cover of the compact space $\overline{K}_{k+1} \setminus K_k$, there exist finitely many $x_1, \dots, x_r$ such that

$$\overline{K}_{k+1} \setminus K_k \subset \cup_{1 \leq i \leq r} \psi_{x_i}^{-1}(B_{1/2}(0)).$$

Put $B_{(k+1),j} := \psi_{x_j}^{-1}(B_1(0))$. It then follows that

$$\overline{K}_{k+1} \subset \cup_{ij} \{B_{ij} : i \leq k+1\}.$$

Inductively, the construction of the family $\{B_{ij}\}$ is over. Clearly, it is an open refinement of the family $\{U_\alpha\}$ and covers X . To see that the family $\{B_{ij}\}$ is *locally finite*, given

$x \in X$, suppose $x \in K_k$. Then, take $N_x = K_k$. Clearly, N_x does not meet any of the B_{ij} for $i \geq k+2$ and the family $\{B_{ij} : i \leq k+2\}$ is finite.

Now, let $\beta : B_1(0)^n \rightarrow \mathbb{I}$ be the function

$$\beta(x) := 2 \min \left\{ d(x, \mathbb{S}^{n-1}), \frac{1}{2} \right\}.$$

Put $\eta_{i,j} = \beta \circ \psi_{i,j}$, where $\psi_{i,j} : B_{i,j} \rightarrow B_1(0)$ are the homeomorphisms chosen earlier. Extend $\eta_{i,j}$ by zero over all of Y . Now re-index this family by single integers and call it $\{\eta_j\}$. It is easily verified that this family satisfies properties (i), (ii) and (iii). Now define $\eta = \sum_i \eta_i$ which makes sense and is continuous. Moreover, $\eta(x) \geq 1$ for all $x \in X$ since each x belongs to some $\frac{1}{2}B_j$ and then $\eta_j(x) = 1$. Take $\theta_j := \eta_j/\eta$ and verify that this family of functions is as required. ♠

Remark 5.2.3 If we begin with a ‘smooth’ manifold Y , the maps $\{\theta_j\}$ can be chosen to be smooth. The only difference is replace above β by a smooth ‘bump function with similar properties (see [?]).

As an immediate consequence of existence of partition of unity, we shall now obtain the collar neighbourhood theorem.

Definition 5.2.4 Let X be a manifold with boundary $\partial X \neq \emptyset$. By a collar of ∂X in X we mean an open subset U of X with a homeomorphism $\varphi : U \rightarrow \partial X \times [0, \infty)$ such that for all $x \in \partial X$, $\varphi(x) = (x, 0)$.

Notice that there is nothing special in the choice of the interval $[0, \infty)$ in the above definition. We can as well take $[0, \epsilon)$ for any $\epsilon > 0$.

Theorem 5.2.5 *In every manifold X , ∂X has a collar neighbourhood. Moreover for any collar neighbourhood U of ∂X , $X \setminus U$ is homeomorphic to X .*

Proof: Let Y be the space obtained by ‘attaching an external collar to X ’, viz., Y is the quotient of the disjoint union of X and $\partial X \times [-1, 0]$ by identifying $x \in \partial X$ with $(x, 0) \in \partial X \times [-1, 0]$. Observe that Y is also a n -manifold with its boundary homeomorphic to $\partial X \times \{-1\}$. The idea is to define a homeomorphism $f : X \rightarrow Y$ and then take $U = f^{-1}(\partial X \times [-1, 0])$.

Begin with a (countable) partition of unity $\{\theta_i\}$ on ∂X so that $\text{supp } \theta_i$ is contained in a coordinate open set U_i of ∂X together with a homeomorphism ϕ_i from $U_i \times [0, 1)$ onto an open subset V_i of X . Put $\eta_0 = 0$, $\eta_k = \sum_{i=1}^k \theta_i$; and

$$Z_k := \{(x, t) : x \in U_k, -\eta_{k-1}(x) \leq t \leq 1\}; \quad Z'_k = \{(x, t) : x \in U_k, -\eta_k(x) \leq t \leq 1\}.$$

Let $\alpha_k : Z_k \rightarrow Z'_k$ be the homeomorphism which linearly stretches the segment $[-\eta_{k-1}(x), 1]$ homeomorphically onto the segment $[-\eta_k(x), 1]$, for each x .

Put $Y_k = X \cup \{(x, t) : -\eta_k(x) \leq t \leq 0\}$ and let $\beta_k : Z'_k \rightarrow Y_k$ be the embeddings given by

$$\beta_k(x, t) = \phi_k(x, t), \quad t \geq 0; \text{ and } \beta_k(x, t) = (x, t), \quad t \leq 0.$$

We define homeomorphisms $f_k : Y_{k-1} \rightarrow Y_k$ as follows:

$$f_k(z) = \begin{cases} z, & z \in X \setminus \phi_k(U_k \times [0, 1)); \\ (x, t), & z = (x, t), \quad x \notin U_k; \\ \beta_k \circ \alpha_k(x, t), & x \in U_k. \end{cases}$$

Finally put $f = \cdots \circ f_k \circ f_{k-1} \circ \cdots \circ f_1$.

First of all note that on the complement of $V = \cup_i \phi_i(U_i \times [0, 1))$, f is identity. On V itself, f makes sense, since given any point $x \in \partial X$, there are only finitely many i for which $x \in U_i$ and $f_k(x, t) = (x, t)$ if $x \notin U_k$. Indeed in a neighbourhood of x , all f_k are identity except those k for which $\theta_k(x) \neq 0$. For this reason, f is also a proper mapping. Since $\sum_k \theta_k(x) = 1$, it follows that f is surjective. Since each f_k is an embedding f is injective. Therefore f is a homeomorphism. ♠

Remark 5.2.6 Naturally, we now have many other consequences of the existence of countable partition of unity such as Urysohn's lemma, the Tietze's extension theorem, etc. For example, if $\{\theta_i\}$ is a countable partition of unity on X then $f : X \rightarrow [0, \infty)$ defined by $f(x) = \sum_k k\theta_k(x)$ is a proper mapping.

Exercise 5.2.7

- (i) Show that a manifold is connected iff it is path connected iff its interior is connected.
- (ii) (This exercise is intended as some kind of an explanation for our obsession with Hausdorffness and locally Euclidean spaces.) Consider $\mathbb{R}$ with the cofinite topology. Certainly it is not Hausdorff. But is it locally Euclidean? More generally, take the affine space $\mathbb{C}^n$ with the Zariski topology. Is it locally Euclidean? Is it contractible?

5.3 Embeddings in Euclidean Spaces

The following result tells us that after all, we could have just stuck to the study of subsets of Euclidean spaces for studying manifolds. As we shall see, this single result has several implications on topological, homotopical and homological properties of a manifold.

Theorem 5.3.1 *Every n -manifold is homeomorphic to a closed subset of $\mathbb{R}^{2n+1}$.*

Theorem 5.3.2 *For every topological n -manifold X , the set of embeddings of X in $\mathbb{I}^{2n+1}$ is dense in the space $(\mathbb{I}^{2n+1})^X$ of all continuous maps (with topology of uniform convergence on compact sets, which is the same as the compact-open-topology).*

Remark 5.3.3 The proofs of these theorems are somewhat lengthy and hard. The smooth version of Theorem 5.3.1 goes under the name *easy Whitney embedding theorems* which you may read from many books such as [?]. However, for the topological case, there are not many references available. You are welcome to see this in the excellent old book by Hurewicz and Wallman ([?], Theorem V-3). Or you may choose to read a nice proof of the embedding Theorem 5.3.1 from [?]. Here, we shall be satisfied with an easy proof of the following weaker version:

Theorem 5.3.4 *Every compact manifold is homeomorphic to a closed subset of some Euclidean space.*

Proof: Cover X by finitely many open sets $\{U_i\}_{1 \leq i \leq k}$ such that there is a homeomorphism $f_i : U_i \rightarrow \mathbb{R}^n$. Let $\eta : \mathbb{R}^n \rightarrow \mathbb{S}^n$ be the inverse of the stereographic projection and $g_i : X \rightarrow \mathbb{S}^n$ be the extension of $\eta \circ f_i$ which sends $X \setminus U_i$ to the north pole. Put $g = g_1 \times g_2 \cdots \times g_k : X \rightarrow (\mathbb{S}^n)^k \subset \mathbb{R}^{nk+k}$. Verify that g is a one-one mapping. Since X is compact, and $\mathbb{R}^{nk+k}$ is Hausdorff, g is a homeomorphism onto a closed subset. ♠

5.4 Homotopical aspects

In order to derive some homotopical and homological properties of a manifold, we need the following:

Theorem 5.4.1 *Every locally contractible closed subset X of $\mathbb{R}^N$ is a retract of some neighbourhood of X in $\mathbb{R}^N$.*

Proof: For each $i = 1, 2, \dots, N$ and for some $t \in \mathbb{R}$, let $L_{i,t}$ denote the hyperplane

$$L_{i,t} := \{\mathbf{x} \in \mathbb{R}^N : x_i = t\}$$

Let $P_{1/2^{k+1}}$ be the collection of all closed N -cubes in $\mathbb{R}^N$ with their faces in

$$\{L_{i,r} : r = m/2^{k-1}, m \in \mathbb{Z}\}$$

Let W_1 be the union of all cubes in P_1 which do not meet X . Inductively, let W_k be the union of all those cubes in $P_{1/2^{k-1}}$ which do not meet X and which are not contained in W_{k-1} . Then

$$\mathbb{R}^N \setminus X := W = \bigcup_{k=1}^{\infty} W_k$$

and W has a CW-structure in which all these cubes of various sizes form the N -dimensional cells. (Do not confuse W_k with the k -skeleton of W .) Note that each W_k is a closed subset of W .

We shall construct a subcomplex V of W and a function $r : X \cup V \rightarrow X$ such that $r(x) = x$, $x \in X$. We shall then show that $X \cup V$ contains a neighbourhood of X and r is continuous on this neighbourhood, which will complete the proof of the lemma.

First, the constructions of V and r will be done simultaneously and inductively. Later, we shall check the continuity of r so defined.

Of course, we start with $r(x) = x$, $x \in X$. Take the 0-skeleton of V to be $W^{(0)}$, i.e., all vertices of W . Given $\sigma \in V^{(0)}$, choose any point $r(\sigma) \in X$ such that

$$d(\sigma, r(\sigma)) < 2d(\sigma, X).$$

(Here $d(A, B)$ denotes the infimum of the Euclidean distances $d(a, b)$ where $a \in A, b \in B$.) This completes the constructions of $V^{(0)}$ and $r : X \cup V^{(0)} \rightarrow X$.

Having defined $V^{(k-1)}$ and $r : X \cup V^{(k-1)} \rightarrow X$, let us take σ to be any k -cell of W such that $\partial\sigma \subset V^{(k-1)}$. If $r : \partial\sigma \rightarrow X$ has a continuous extension then let us put this σ inside $V^{(k)}$; not otherwise. Having put this σ inside $V^{(k)}$ take $r : \sigma \rightarrow X$ to be any extension which satisfies the property

$$d(\sigma, r(\sigma)) < 2d(\sigma, r'(\sigma)) \tag{5.1}$$

for all extensions $r' : \sigma \rightarrow X$ of $r : \partial\sigma \rightarrow X$. This completes the constructions of $V^{(k)}$ and $r : X \cup V^{(k)} \rightarrow X$. By induction the constructions of V and r are complete.

Note that by the very choice, r is continuous on V . So we have to verify continuity at points $x \in X$ only.

Given $x \in X$ and $\epsilon > 0$, put $\epsilon_{2N} = \epsilon$ and choose inductively $\epsilon_N > \dots > \epsilon_0$ such that

$$3\epsilon_i < \epsilon_{i+1}, i = 0, \dots, N-1$$

so that if $B_i := X \cap B_{\epsilon_i}(x)$ then the inclusion map $B_i \subset B_{i+1}$ is null homotopic. (This is where local contractibility is used.) Let $U_t = B_t(x)$, where $0 < t < \epsilon_0/4$.

We first claim that if σ is any cell of W contained in U_t then $\sigma \in V$ and $d(x, r(\sigma)) < \epsilon$. Suppose a is a 0-cell and $a \in U_t$. Then

$$d(x, r(a)) \leq d(x, a) + d(a, r(a)) < t + 2d(a, x) < 3t < \epsilon_0.$$

Now suppose for all $(k-1)$ -cells τ of V contained in U_t we have $d(x, r(\tau)) < \epsilon_{2k-2}$. Let σ be a k -cell of W contained in U_t . Then $r(\partial\sigma) \subset B_{\epsilon_{2k-2}}$ and the inclusion $B_{\epsilon_{2k-2}} \subset B_{\epsilon_{2k-1}}$ is null homotopic. So, there exist extensions of $r : \partial(\sigma) \rightarrow B_i$ inside B_{i+1} . From (5.1), it follows that $d(\sigma, r(\sigma)) < 2\epsilon_{2k-1}$ and therefore,

$$d(x, r(\sigma)) \leq d(x, \sigma) + d(\sigma, r(\sigma)) < t + 2\epsilon_{2k-1} < \epsilon_{2k}.$$

Therefore, by induction, we have, $r(U_t \cap V) \subset U_\epsilon$.

Note that this however, does not complete the proof of continuity of r at x ; nor does this prove that V contains a neighbourhood of $x \in X$. For we have not proved $U_t \subset V$ since U_t will necessarily have many points belonging to cells σ in V where σ itself is not completely in U_t .

So, we claim that there exist $s = s(t) > 0$ such that if any cell σ of W intersects U_s then the entire cell σ is contained in U_t . Combined with the earlier claim this will then complete both the proofs.

If this is not the case, there exists a sequence $n_i \rightarrow \infty$ and cells σ_i in W such that $\sigma_i \cap U_{1/n_i} \neq \emptyset$ and $\sigma_i \cap U_t^c \neq \emptyset$. In particular, this implies that the diameter of σ_i is bigger than $t/2$ for all large n_i . If m is an integer such that $t/2 > 1/m$, this means that all σ_i are inside W_m which has a positive distance from x , say, δ . If i is such that $1/n_i < \delta$, this will contradict the choice that $\sigma_i \cap U_{1/n_i} \neq \emptyset$. This completes the proof of the theorem.

♠

Corollary 5.4.2 The fundamental group, the homology and cohomology groups of a compact manifold are finitely generated.

Proof: Embed a given compact manifold X in some Euclidean space $\mathbb{R}^N$ and take a neighbourhood U of X and a retraction $r : U \rightarrow X$. Choose k such that

$$\frac{1}{2^k} < \frac{d(X, \mathbb{R}^N \setminus U)}{3}.$$

Let Y be the union of all those boxes of P_k which meet X . Then Y is a finite CW-complex $X \subset Y \subset U$. The retraction r restricts to a retraction $r : Y \rightarrow X$. Now $r \circ i = Id_X$ implies that $r_\# \circ i_\# = Id_\#$ on the fundamental groups, $r_* \circ i_* = Id_*$ on homology (and $i^* \circ r^* = Id^*$ on cohomology) groups. Since Y has fundamental group, homology (cohomology) modules finitely generated, the same must hold for X as well. ♠

Corollary 5.4.3 The fundamental group and the homology modules of a manifold are countable.

Proof: By Lemma 5.2.1, we can write $X = \cup_i W_i = \cup_i \bar{W}_i$ where each $\bar{W}_i$ is compact. Therefore $\pi_1(X) = \varinjlim \pi_1(\bar{W}_i)$ and each $\pi_1(\bar{W}_i)$ is countable being finitely generated. Therefore $\pi_1(X)$ is countable. The same argument holds for homology modules. ♠

Exercise 5.4.4

- (i) Give examples of Hausdorff, $\mathbb{I}$ -countable spaces that are not manifolds.
- (ii) Can you think of some manifolds, other than projective spaces, that do not **naturally** occur as subspaces of Euclidean spaces?
- (iii) Let X be a manifold, $U \subset X$ be a connected open subset. Given $p, q \in U$ show that there exists a homeomorphism $f : X \rightarrow X$ such that $f(x) = x$ for all $x \in X \setminus U$ and $f(p) = q$. (It is in this strong sense that the group of homeomorphisms of X acts ‘transitively’ on X for any connected manifold. Compare Exercise (iv) below. [Hint: See Exercise ??.]
- (iv) Let X be a connected manifold of dimension $n \geq 2$ and k any positive integer. Given any two k -subsets $\{a_1, \dots, a_k\}, \{b_1, \dots, b_k\} \subset X$, show that there exists a homeomorphism $f : X \rightarrow X$ such that $f(a_i) = b_i, i = 1, 2, \dots, k$. What is the best that we can say here if $\dim X = 1$?
- (v) Given a connected n -manifold and any finite subset $F \subset X$, show that there is an n -cell $E \subset X$ such that F is contained in the interior of E .
- (vi) Prove (without using embeddability results) that every locally Euclidean, $\mathbb{I}$ -countable, Hausdorff space is **paracompact** or read it from some book, say, e.g. [?].
- (vii) Show that every subspace of a manifold is paracompact.

5.5 Classification of 1-dimensional manifolds

In every classification problem, we must first of all have plenty of examples which are ‘likely’ to represent all possible types of objects that we want to classify. Only after that, we can make a probable list of representatives which are mutually of different type. The

final step is to show that every object that we wanted to classify belongs to (precisely) one of the types mentioned in the list.

In order to classify all manifolds, clearly, it suffices to consider only connected ones. For, any manifold is locally connected and hence its connected components are open as well as closed. Therefore any manifold is the disjoint union of its connected components, even as a topological space.

What are the examples of 1-dimensional connected manifolds that we have?

We observe that any two closed intervals are homeomorphic, via an affine linear map. This homeomorphism can then be used to get homeomorphisms between any two finite open intervals or between any two half open intervals as well. Moreover, $x \mapsto \tan x$ defines a homeomorphism of the interval $(-\pi/2, \pi/2) \rightarrow \mathbb{R}$. Thus, as far as subsets of $\mathbb{R}$ are concerned, we have three different classes of connected 1-dimensional manifolds:

- (i) open intervals (ii) half open intervals (iii) closed intervals.

As soon as we go to subspaces of $\mathbb{R}^2$, we get ‘other’ types: circles, ellipses, parabolas and many more ‘smooth’ curves, boundary of a convex polygon and so on. If we have one-to-one parameterization of any of these curves then clearly they will be homeomorphic to an interval. This is the case with a parabola for instance. One can also see easily that any two circles are homeomorphic to each other. Indeed, placing a small circle inside an ellipse and then projecting radially from the centre of the circle produces a homeomorphism of the circle with the ellipse. Write down an explicit formula by yourself:

Do we get any other types of 1-dimensional manifolds, if we look inside higher dimensional Euclidean spaces? The answer is: **NO**.

Theorem 5.5.1 *Let X be a connected 1-dimensional, (Hausdorff and II-countable) abstract smooth manifold. Then X is homeomorphic to one of the following:*

- (i) $(0, 1)$; (ii) $[0, 1)$; (iii) $[0, 1]$ (iv) $\mathbb{S}^1$.

In particular, if $\partial X = \emptyset$, it follows that X is homeomorphic to either $(0, 1)$ or $\mathbb{S}^1$. Granting this, let us complete the proof for the case $\partial X \neq \emptyset$.

It follows that $\text{int } X$ cannot be homeomorphic to $\mathbb{S}^1$. For then, it will be compact subset of X and hence a closed subset of X . But all points in ∂X are in the closure of $\text{int } X$.

So, we may assume without loss of generality, $(0, 1) = \text{int } X \subset X$. By the very definition, if $x \in \partial X$, there is homeomorphism $\psi : [0, \epsilon) \rightarrow U$, where U is an open set

in X , such that $\psi(0) = x$. Clearly, then $\psi(0, \epsilon) = U \setminus \{x\} = (a, b)$ is an open interval in $(0, 1)$ and $\psi(x)$ is in its closure. If $0 < a < b < 1$, then it follows that $\{\psi(x), a\}$ or $\{\psi(x), b\}$ will be a pair of distinct points of X violating Hausdorffness of X . Therefore, it follows that either $x = a = 0$ or $x = b = 1$ which just implies that $[0, 1) \subset X$ or $(0, 1] \subset X$, say $[0, 1) \subset X$. Of course, X may be equal to $[0, 1)$. Then we are done.

So, consider the case $X \setminus [0, 1) \neq \emptyset$. Let y be another boundary point of X . As before, it follows that $y = 1$. Therefore $[0, 1] \subset X$. Since $[0, 1]$ is compact, it is closed in X . It follows that there cannot be any more boundary points in X . So, $X = [0, 1]$. ♠

Case when $\partial X = \emptyset$

We shall now proceed toward the proof of the theorem 5.5.1 in case $\partial X = \emptyset$.

Our starting point is the following simple observation:

Lemma 5.5.2 Let X be any n -manifold. Then X is the quotient of a countable disjoint union of copies of the open disc $\mathbb{D}^n$.

Proof: By II-countability, there exists an atlas $\{U_i, \phi_i\} : i \in \mathbb{N}$ for X with values in $\mathbb{D}^n$. Take

$$M = \sqcup_i \mathbb{D}_i^n$$

as the disjoint union of countable copies of $\mathbb{D}^n$, and $q : M \rightarrow X$ be the map given by $q|_{\mathbb{D}_i^n} = \phi_i^{-1}$. Then clearly, q is a surjective open mapping and hence a quotient map onto X . ♠

Next, by deleting those open sets which are covered by the union of previous members of the atlas, we can assume that the atlas is ‘minimal’ in a certain sense. This can be achieved systematically as follows. Label the members of the atlas $\{(U_i, \phi_i)\}$. Let n_1 be the least number for which U_{n_1} Put $W_2 = U_{n_0} \cup U_{n_1}$. Inductively, having defined U_{n_k} and W_k , let n_{k+1} be the least number such that $U_{n_{k+1}}$ is not contained in W_k . Put $W_{k+1} = W_k \cup U_{n_{k+1}}$. If the family is finite, of course this process will stop at some stage with $W_m = X$. In any case, we get an atlas $\{U_{n_k}\}$ (finite or infinite) which is minimal in the sense that no U_{n_k} is contained in W_{k-1} .

In what follows we shall assume that $\{(U_i, \phi)\}$ is a countable and minimal atlas for X . Put $\psi_i = \phi_i^{-1} : (-1, 1) \rightarrow X$. We may think of X as the quotient of a disjoint union of countably many copies of $(-1, 1)$ via the map $q : \sqcup_i \mathbb{D}_i^1 \rightarrow X$ where $q|_{\text{disc}_i^1} = \psi_i$. Therefore, to understand X we need to unravel what kind of identifications are possible among these disjoint copies of the intervals.

Put

$$U_{ij} := \psi_i^{-1}(\psi_j(\mathbb{D}_j^1)) = \phi_i(\psi_j(\mathbb{D}_j^1)). \quad (5.2)$$

Each of them is an open subset of $\mathbb{D}_i^1$. To begin with, we need to know how they look.

Example 5.5.3 Suppose the atlas for X consists of just two charts, $M = \mathbb{D}_1^1 \sqcup \mathbb{D}_2^1$. Consider the following cases:

(A) $U_{12} = (3/4, 1)$ and $U_{21} = (0, 1/4)$. Here consider the following two subcases:

(i) $\psi_1^{-1} \circ \psi_2(t) = t + 3/4$.

(ii) $\psi_1^{-1} \circ \psi_2(t) = 1 - t$.

In the first case, we can take $\alpha : M \rightarrow (0, 7/4)$ defined by $\alpha|_{\mathbb{D}_1^1}(t) = t$ and otherwise $\alpha(t) = t + 3/4$. It is easily checked that α factors down to define a homeomorphism $\hat{\alpha} : X \rightarrow (0, 7/4)$.

Case (ii) cannot occur! For, then the images of $\psi_1(3/4)$ and $\psi_2(1/4)$ are two distinct points in X which cannot be separated by open sets.

(B) $U_{1,2} = (0, 1/4) \cup (3/4, 1)$ and $U_{2,1} = (0, 1/4) \cup (3/4, 1)$ Here suppose

$$\psi_1^{-1} \circ \psi_2(t) = \begin{cases} t + 3/4, & 0 < t < 1/4; \\ t - 3/4, & 3/4 < t < 1. \end{cases}$$

Define $\beta : M \rightarrow \mathbb{S}^1$ by the formula:

$$\beta(t) = \begin{cases} e^{\pi i(3t/2)}, & t \in \mathbb{D}_1^1; \\ e^{\pi i((3t+2)/2)} & t \in \mathbb{D}_2^1. \end{cases}$$

One checks that β factors down to define a homeomorphism $\hat{\beta} : X \rightarrow \mathbb{S}^1$. All other possibilities can be reduced to the above case by changing the indexing or to the case A(ii) which cannot occur for the same reason.



Figure 5.1. gluing one or two intervals

We can now generalize these facts a little bit, in the following two lemmas.

Lemma 5.5.4 Let $a < b < c$ and $d < e < f$ be real numbers, M be the disjoint union of the two intervals $M = (a, c) \sqcup (d, f)$. Let $\alpha : (b, c) \rightarrow (d, e)$ be an order preserving homeomorphism. Then the identification space

$$M_\alpha = \frac{M}{t \sim \alpha(t), t \in (b, c)}$$

is homeomorphic to $(0, 1)$

Proof: Note that α , first of all, extends to a homeomorphism from $\alpha : [b, c]$ to $[d, e]$ sending b to d and c to e . Choose a homeomorphisms $f_1 : [a, c] \rightarrow [0, 2/3]$ and $f_2 : [e, f] \rightarrow [2/3, 1]$ such that

$$f_1(a) = 0, f_1(b) = 1/3, f_1(c) = 2/3, f_2(e) = 2/3, f_2(f) = 1.$$

Define $h : M \rightarrow (0, 1)$ as follows:

$$h(t) = \begin{cases} f_1(t), & t \in (a, c); \\ f_1\alpha^{-1}(t), & t \in (d, e); \\ f_2(t), & t \in [e, f). \end{cases}$$

It is straight forward to verify that h is well defined and is a continuous bijection. There are different ways to see that h is a homeomorphism. Here in one.

Note that the quotient map $q : M \rightarrow M_\alpha$ is an open map. Therefore, M_α is locally compact. Also, M_α can be thought of as the quotient of $(a, c] \sqcup [d, f)$ by the same relation extended to include $c \sim \alpha(c) = e$ and $b \sim \alpha(b) = d$.

Next, it follows that h is a proper map. For $h^{-1}[1/n, 1 - 1/n]$ is the image of $[f_1^{-1}(1/n), c] \sqcup [d, f_2^{-1}(1 - 1/n)]$ and hence is compact. From this, it follows that h is a homemorphism. ♠

Lemma 5.5.5 Given $-1 < a < b < 1$ and $-1 < c < d < 1$, let $\alpha : (b, 1) \rightarrow (-1, c)$ and $\beta : (-1, a) \rightarrow (d, 1)$ be order preserving homeomorphisms. Let X be the quotient of the disjoint union of two copies of $(-1, 1)$ by the identification $t \sim \alpha(t), b < t < 1$ and $t \sim \beta(t), -1 < t < a$. Then X is homeomorphic to $\mathbb{S}^1$.

Proof: Without loss of generality, we may assume that $a = -1/2 = c, b = 1/2 = d$. On the first copy of $(-1, 1)$, define $f(t) = e^{(t+1)5\pi i/8}$ and on the second copy define $g : (-1, 1) \rightarrow \mathbb{S}^1$ by

$$g(t) = \begin{cases} f(\alpha^{-1}(t)), & -1 < t < -1/2; \\ e^{(5t+13)\pi i/8}, & -1/2 \leq t \leq 1/2; \\ f(\beta^{-1}(t)), & 1/2 < t < 1. \end{cases}$$

Verify that $f \sqcup g$ factors down to define a map $h : X \rightarrow \mathbb{S}^1$ which is a bijection. It is easily checked that X is compact. Therefore h is a homeomorphism.

Having taken care of actual homeomorphisms used in the identifications, we can now concentrate on other topological aspects of U_{ij} . Since U_{ij} is an open subset of $(-1, 1)$, it is a countable union of open subintervals of $(-1, 1)$. Our first claim in this direction is:

Lemma 5.5.6 Let X be a Hausdorff space $\psi_1, \psi_2 : \mathbb{D}^1 \rightarrow X$ be homeomorphisms onto open sets U_1, U_2 of X , respectively. Assume that $\psi_2(\mathbb{D}^1) \not\subset \psi_1(\mathbb{D}^1)$.

(i) Then no component of $\psi_1^{-1}(\psi_2(\mathbb{D}^1))$ will be an open interval of the form (a, b) for some $-1 < a < b < 1$; in particular, $U_1 \cap U_2$ has at most two components.

(ii) Suppose further that $\psi_1(\mathbb{D}^1) \not\subset \psi_2(\mathbb{D}^1)$. Then after changing ψ_i by a reflection in the origin, if necessary, one of the components of $\psi_1^{-1}(\psi_2(\mathbb{D}^1))$ is of the form $(a, -1)$, and

$$\psi_2^{-1} \circ \psi_1 : (a, 1) \rightarrow (-1, d)$$

defines an increasing function which is a homeomorphism onto an interval of the form $(-1, d)$.

Proof: (i) Note that components of $U_1 \cap U_2$ are all homeomorphic to open intervals. The emphasis here is that they will **not** be the image of some ‘middle’ portion of $\mathbb{D}^1$ under ψ_i .

Assume on the contrary. Let $\psi_2^{-1}\psi_1(a, b) = (c, d) \subset \mathbb{D}_j^1$. Since $U_j \not\subset U_i$, it follows that $\{c, d\} \neq \{-1, 1\}$. Say $-1 < c < 1$. It follows that $\psi_2(c)$ and $\psi_1(a)$ are distinct points in X which cannot be separated by open sets contradicting the Hausdorffness of X . It follows that the components of $\psi_1^{-1}(\psi_2(\mathbb{D}^1))$ can be of the form either $(-1, b)$ or $(a, 1)$ or both. In particular, there are at most two possibilities. (ii) Pre-composing ψ_1 with a reflection, if necessary, we may assume that one of the components of $\psi_1^{-1}(\psi_2(\mathbb{D}^1))$ is of the form $(a, 1)$. With the extra hypothesis of (ii), we can do the same thing with $\psi_2^{-1}(\psi_1(\mathbb{D}^1))$ and conclude that $\psi_2^{-1}(\psi_1(a, 1)) = (-1, c)$ for some $-1 < c < 1$. If $\psi_2^{-1} \circ \psi_1$ is decreasing, this would imply that $\psi_1(a)$ and $\psi_2(c)$ (which are, any way, distinct points of X) cannot be separated by open sets in X , contradicting the Hausdorffness of X . ♠

Lemma 5.5.7 Let X be a connected 1-dimensional manifold having an atlas consisting of only two members, U_1, U_2 , such that $U_1 \not\subset U_2$. Then

(i) $U_1 \cap U_2$ is non empty and has at most two components.

(ii) If $U_1 \cap U_2$ has only one component, then X is homeomorphic to an interval.

(iii) If $U_1 \cap U_2$ has two components, then X is homeomorphic to $\mathbb{S}^1$.

Proof: That $U_1 \cap U_2 \neq \emptyset$ follows from connectedness of X .

Now the claim (i) follows from the previous lemma.

(ii) This follows from part(ii) of the above lemma and lemma 5.5.4.

(iii) As in the previous case,

$$\psi_2^{-1} \circ \psi_1 : (a, 1) \rightarrow (-1, d)$$

which takes care of one component of $U_1 \cap U_2$. It follows that the other component of $\psi_1^{-1}(U_1 \cap U_2)$ is of the form $(-1, b)$ and the other component of $\psi_2^{-1}(U_1 \cap U_2)$ is of the form $(c, 1)$. And for the same reason that X is Hausdorff, as before, the homeomorphism $\psi_2^{-1} \circ \psi_1$ must be increasing. Therefore we are in the situation of lemma 5.5.5. ♠

Continuing with the proof of the theorem:5.5.1:

Let $\{U_\alpha\}$ be an atlas for X . By II-countability, there exists a countable sub-cover for $\{U_j\}$ for X . Indeed, we have:

Step I There exists a countable family $\{U_j\}$:

- (i) Each U_j is homeomorphic to an interval;
- (ii) $U_k \not\subset \cup_{j \leq k-1} U_j =: W_{k-1}$.
- (iii) $W_{k-1} \cap U_k \neq \emptyset, k \geq 1$.
- (iv) If $W_k \neq X$, then it is homeomorphic to an open interval.

To construct such a family, we start off with any one member from the countable family $\{U_j\}$, call it $U_1 =: W_1$. Having picked up $U_0, U_1, \dots, U_{k-1}$ so as to satisfy the above requirement, we check whether $W_{k-1} := \cup_{j \leq k-1} U_j$ is the whole space X . If so, we stop. Otherwise, it means there are members in $\{U_j\}$ not contained in W_{k-1} . If none of them intersect W_{k-1} , it would mean that X is disconnected. Therefore there exists a member which we label U_k not contained in W_{k-1} but intersecting it. Depending on whether the number of components of $W_{k-1} \cap U_k$, is equal to one or two, it follows that W_k is homeomorphic to an interval or a circle. In the latter case, W_k is both closed and open in X and therefore $W_k = X$.

It remains to analyse the case when the sequence $\{W_k\}$ is infinite.

Fix a homeomorphism $f_1 : W_1 \rightarrow (-1, 1)$. From the lemma 5.5.5, it follows that W_2 is extended either on the left side or on the right side or both. Accordingly, we can extend f_1 to a homeomorphism $f_2 : W_2 \rightarrow (-2, 1)$ or $(-1, 2)$ or $(-2, 2)$. Suppose we have constructed the homeomorphism $f_k : W_k \rightarrow (i_k, j_k)$ which is an extension of $f_{k-1} : W_{k-1} \rightarrow (i_{k-1}, j_{k-1})$, where $i_k \leq i_{k-1}$ and $j_{k-1} \leq j_k$ are all integers with the property $j_k - i_k = j_{k-1} - i_{k-1} \geq 1$. We then extend f_k to a homeomorphism $f_{k+1} :$

$W_{k+1} \rightarrow (i_{k+1}, j_{k+1})$. Put $a = \lim_k i_k$ and $b = \lim_k j_k$. It follows that $a = -\infty$ or $b = \infty$. We define $f : X \rightarrow (a, b)$ by $f(x) = f_k(x)$ whenever $x \in W_k$. It follows that f is a homeomorphism. We can now compose f with a homeomorphism $(a, b) \rightarrow (0, 1)$ and conclude that X is homeomorphic to the open interval $(0, 1)$. ♠

Remark 5.5.8 The case when X has boundary, viz., $\partial X \neq \emptyset$, is settled easily. Apply the above theorem to $\text{int } X$. It follows that $\text{int } X$ must be homeomorphic to $(0, 1)$. Therefore X can have at most two boundary points and is homeomorphic to $[0, 1]$, $[0, 1)$ or $(0, 1]$.

5.6 Triangulation of Manifolds

In this section, we shall begin a discussion on triangulation of manifolds. The three fundamental classical questions here are:

- (A) Can every topological (smooth) manifold be triangulated?
- (B) Given two triangulations K_1, K_2 of a topological (smooth) manifold, is K_1 combinatorially equivalent to K_2 ?
- (C) Does every triangulated manifold carry a smooth structure? From the classification of 1-dimensional manifold, it follows easily, that every 1-dimensional manifold is triangulable. In the next section, we shall present a proof of the result that all (compact) 2-dimensional manifolds are triangulable (a result due to Rado, proved in 1924). The triangulability of all 3-dimensional manifolds is a deep result due to Edwin Moise [?], who also proved that any two triangulations K_1, K_2 of the same 3-manifold are combinatorially equivalent, i.e., there are subdivisions of K'_i of K_i such that K'_1 is isomorphic to K'_2 .

A theorem of Cairns (1935) says that every smooth manifold is triangulable. (See [?] for a proof and more.)

And there are triangulable manifolds which do not admit any smooth structure. The first example came in 1960 due to Kervaire in dimension 10 which was soon improved to dimension 8 by Eells and Kuiper. Siebenmann has constructed an example of a 5-manifold which cannot be triangulated. With these classical results, we may safely say that the above three questions have been answered satisfactorily.

A triangulation of a manifold has necessarily more combinatorial structure. We make a beginning of the study of these properties. Besides theoretical importance, it provides a very effective tool in the study of topological properties of manifolds. We shall then introduce the notion of ‘combinatorial triangulation’ and use it to sketch a proof of

Poincaré duality as envisaged by Poincaré himself. More technical versions and rigorous proofs of Poincaré duality will be discussed in a subsequent chapter. Finally, we shall describe a result due to Munkres about a certain local homological condition. This result has found applications in the algebra of face rings.

We begin with a general lemma about ‘local’ homology of a simplicial complex. In what follows, we have suppressed the coefficient group in the homology groups. One can take it to be any field or the group of integers according to one’s requirements.

Lemma 5.6.1 Let $x \in \text{int } F$, i.e., a point in the interior of $|F|$ for some simplex $F \in K$. Then

$$H_i(|K|, |K| \setminus \{x\}) \approx \tilde{H}_{i-\dim F-1}(Lk(F)).$$

Proof: For any simplex $F \in K$, the open star, $st(F)$ is an open subset of $|K| = X$ and hence by Theorem 4.7.4, $\{st(F), X \setminus \{x\}\}$ is an excisive couple in $X = |st(F) \cup X \setminus \{x\}|$. By Theorem 4.7.3,

$$H_i(|st(F)|, |st(F)| \setminus \{x\}) \approx H_i(X, X \setminus \{x\}).$$

Recall that $|st(F)| \setminus \{x\}$ contains $|Lk(F) * \mathcal{B}(F)|$ as a deformation retract (see Corollary ??). Therefore

$$H_i(|st(F)|, |Lk(F) * \mathcal{B}(F)|) \approx H_i(|st(F)|, |st(F)| \setminus \{x\}).$$

Since $|st(F)|$ is contractible, we have

$$\begin{aligned} H_i(|st(F)|, |Lk(F) * \mathcal{B}(F)|) &\stackrel{\partial}{\approx} \tilde{H}_{i-1}(|Lk(F) * \mathcal{B}(F)|) \\ &\approx \tilde{H}_{i-1-\dim F}(Lk(F)) \end{aligned}$$

(by the homology suspension Theorem 4.8.1). ♠

Definition 5.6.2 Let K be a finite simplicial complex. We say K is *pure* (of dim n) if each simplex in K is a face of a n -simplex.

For a topological manifold X without boundary, given any point $x \in X$ we can choose a neighbourhood U of x homeomorphic to $\mathbb{R}^n$. By excision, it follows that

$$H_i(X, X \setminus \{x\}) = H_i(U, U \setminus \{x\}) \approx H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}).$$

Combined with the above lemma, we immediately have the following.

Theorem 5.6.3 *Let X be a connected, compact topological manifold (without boundary) and K be a simplicial complex such that $|K| = X$. Then the following holds:*

- (i) *For all non empty faces F of K , we have, $\widetilde{H}_i(Lk(F)) = (0)$ for $i < \dim Lk(F)$ and $\approx \mathbb{Z}$ for $i = \dim Lk(F)$, i.e., $Lk(F)$ is a homology sphere of dimension equal to $\dim Lk(F)$.*
- (ii) *K is pure of dimension n .*
- (iii) *Every $(n-1)$ -simplex of K occurs as the face of exactly two n -simplices.*
- (iv) *Given any two n -simplexes σ and τ in K , there is a chain of n -simplices connecting σ and τ , i.e., there exist n -simplices $s_1, \dots, s_k$ in K such that $s_i \cap s_{i+1}$ is an $(n-1)$ -face for $i = 1, \dots, k-1$ and $s_1 = \sigma, s_k = \tau$.*

Proof: We shall first prove (ii). If F is a maximal simplex then $\langle F \rangle$ is an open set in $|K| = X$ and hence for any $x \in \langle F \rangle$, $\{X \setminus \{x\}, |F|\}$ is an excisive couple. Also $\partial|F|$ is a SDR of $|F| \setminus \{x\}$ and therefore we have,

$$H_i(X, X \setminus \{x\}) \approx H_i(|F|, |F| \setminus \{x\}) \approx H_{i-1}(|F| \setminus \{x\}) \approx H_{i-1}(\partial|F|).$$

This means $\partial|F|$ is a homology $(n-1)$ -sphere, which means $\dim F = n$.

- (i) is immediate from the Lemma 5.6.1, since $\dim Lk(F) = n - \dim F - 1$.
- (iii) If F is any $(n-1)$ simplex, then from (i) we get

$$\mathbb{Z} \approx H_n(X, X \setminus \{x\}) \approx \tilde{H}_0(Lk(F)).$$

Since $Lk(F)$ is just a set of vertices one for each n -simplex of which F is a face, we are done.

(iv) Clearly, on the set of all n -simplices in K , having a chain of n -simplices connecting them is an equivalence relation. We need to show that there is just one equivalence class. Assuming on the contrary, let A be the subcomplex spanned by all n -simplices in one of the equivalence classes and B the subcomplex spanned by the rest of the n -simplices. Then clearly $|A|, |B|$ are closed subspaces of X . Since X is connected, it follows that $A \cap B \neq \emptyset$. Let F be a maximal simplex in $A \cap B$. It follows that $\dim F < n-1$. This implies $\dim Lk_K(F) > 0$. By part (i), we conclude that $\tilde{H}_0(Lk_K(F)) = (0)$ and hence $Lk_K(F)$ is connected. Clearly $Lk_K(F) \cap A \neq \emptyset \neq Lk_K(F) \cap B$. It follows that there is an edge $e \in Lk_K(F)$ with one vertex $u \in A \setminus B$ and another vertex $v \in B \setminus A$. If s is a n -simplex such that $F \cup e \subset s$, then either $s \in A$ or $s \in B$, which implies $v \in A$ or $u \in B$, which is a contradiction. ♠

Definition 5.6.4 A simplicial complex which satisfies the properties (ii), (iii) and (iv) of the above theorem is called a *pseudo-manifold* of dimension n . If we weaken the condition (iii) by replacing the phrase ‘exactly two’ by ‘at most two’ we get the definition of a pseudo-manifold with boundary.

As an immediate corollary of this definition we get the following important result:

Theorem 5.6.5 *Given any connected, n -dimensional triangulated pseudo-manifold K with boundary, there exists a triangulated convex polyhedron P in $\mathbb{R}^n$ and affine linear isomorphisms $\phi_i : F_i \rightarrow F'_i$ where $F_1, \dots, F_k, F'_1, \dots, F'_k$ are some (all, if $\partial|K| = \emptyset$) distinct facets of P such that K is the quotient of P by the identifications*

$$x \sim \phi_i(x), \quad x \in F_i, \quad i = 1, 2, \dots, k. \quad (5.3)$$

Proof: Label the n -simplexes of K so that for $i \geq 2$, any n -simplex σ_i has at least one $(n-1)$ -face (i.e., a facet) common with some $\sigma_j, j < i$. [This is possible because of condition (iv) in Theorem 5.6.3.] Let K_j be the subcomplex spanned by $\cup_{i \leq j} \sigma_i$. Inductively, we shall construct triangulated convex polyhedrons $P_1 \subset \dots \subset P_j \subset P_{j+1} \subset \dots \subset \mathbb{R}^n$ and surjective simplicial maps $\Theta_j : P_j \rightarrow K_j$ such that for each j

- (a) Θ_j is a bijection on each simplex and defines a homeomorphism restricted to the interior of P_j ;
- (b) $\Theta_j|_{P_{j-1}} = \Theta_{j-1}$.

For $j = 1$, we can take P_1 to be any geometric n -simplex in $\mathbb{R}^n$ and $\Theta_1 : P_1 \rightarrow K_1$ to be a bijection of vertices. Suppose we have arrived at the stage j . Now σ_{j+1} shares one $(n-1)$ -face F with one of the n -simplexes in K_j say, τ , i.e., $\sigma_{j+1} = F \cup \{u\}$ and $\tau = F \cup \{u'\}$. Let $F' \subset P_j$ be such that $\Theta_j(F') = F$. It follows that $F' \subset \partial P_j$. Consider the convex region bounded by the hyperplanes spanned by F' and other facets of P_j intersecting F . If v' is any point in the interior of this convex region, and τ_{j+1} is the convex hull of v' and F' , we put $P_{j+1} = P_j \cup \tau_{j+1}$. It is easy to check that P_{j+1} is again a triangulated convex polyhedron. We define Θ_{j+1} to be equal to Θ_j on P_j and $\Theta_{j+1}(v') = v$. Clearly Θ extends to a simplicial map which surjects onto P_{j+1} . Injectivity of Θ_{j+1} in the interior of P_{j+1} is easily verified. By induction, this completes the construction of $P_k = P$ and $\Theta_k = \Theta$. The only points where Θ is not injective are in ∂P . There again, on each facet, Θ is injective. It may happen that two distinct faces are mapped on to the same facet in K . Label them in pairs as F_i, F'_i . Put $\psi_j = \Theta|_{F_i}, \psi'_j = \Theta|_{F'_i}$. Define $f_i = (\psi'_i)^{-1} \circ \psi_i$. It follows that K is then isomorphic as a simplicial complex to the quotient space of P by the relations (5.3), as required. ♠

Remark 5.6.6

(a) The method of proof in the above theorem is the beginning of a technique known as ‘cut-and-paste-technique’, in low-dimensional topology. We have illustrated this a little bit right in Section 1.1. In the next section, we shall use this technique to classify surfaces.

(b) A partial converse of the above theorem is also true with some minor modifications. First of all, you will have to take the second barycentric subdivision, viz., given a triangulated convex polyhedron P and an equivalence relation on the boundary faces as above, the quotient of the second barycentric subdivision of P is a pseudo-manifold. We leave the details to the reader as an exercise. This result raises the question as to when the quotient is actually a manifold. Indeed, it is not even true that condition (i) of Theorem 5.6.3 holds for such a quotient space in general. However, in dimension ≤ 2 , there is no need to put any extra condition and the result is true. In dimension 3, there is a nice criterion in terms of the Euler characteristic due to Poincaré. It uses two important facts, viz.,

- (i) A closed surface S is homeomorphic to $\mathbb{S}^2$ iff $\chi(S) = 2$. (See section 5.7)
- (ii) If L is a compact odd dimensional manifold with non empty boundary, then $2\chi(L) = \chi(\partial L)$ (see Remark ?? (b) below). Let us assume these and now state and see how to prove the above result of Poincaré.

Theorem 5.6.7 *Let K be a pseudo-manifold of dimension 3 (without boundary) obtained by identification of pairs of facets of a convex polyhedron P in $\mathbb{R}^3$. If $\chi(X) = 0$ then $X := |K|$ is a closed 3-manifold.*

Proof: We need to prove the local Euclideaness of X . For points which are images of points of interior of P , there is no problem. Similarly, those points which are in the image of an interior of a facet also, there is no problem, since two 3-simplexes will come together at such a facet and the neighbourhood of the point in the quotient space is the union of two half discs along their base and hence is a full disc. Next we consider a point in the interior of an edge e . The topology of the neighbourhood of such a point depends on the topology of the link of e because $St(e) = Lk(e) \star e$ and hence $|St(e)|$ is homeomorphic to the iterated cone $C(C(|Lk(e)|))$. Clearly, the $Lk(e)$ is 1-dimensional pseudo-manifold. We claim that it must be connected. For if not, we can write $Lk(e) = A \sqcup B$ as a disjoint union of two subcomplexes. We can then partition the set of identifications on the boundary of P into two sets and obtain a quotient space X' in which the points lying over the interior of the edge e will be partitioned to define two different edges e_1, e_2 and

X is obtained by a further identification of e_1 with e_2 . Since such extra identifications are not allowed, this is a contradiction. Therefore $Lk(e)$ is a circle and hence $St(e)$ is a 3-disc.

It remains to check the local Euclideanness at the image v_i of vertices of ∂P . By passing to a subdivision of K , we may assume that $St(v_i)$ are disjoint in $|K| = X$. We need to show that each $Lk(v_i)$ is a topological 2-sphere, which is the same as showing that $\chi(Lk(v_i)) = 2$ for all i . Let $S = \cup_i St(v_i)$, and let L be the closure of $K \setminus S$ so that $K = L \cup S$ and $L \cap S = \sqcup Lk(v_i)$. Then L is a topological 3-manifold with boundary $\partial L = L \cap S$. We know from ?? (b) that $\chi(S \cap L) = \chi(\partial L) = 2\chi(L)$. Also,

$$0 = \chi(X) = \chi(K) = \chi(S) + \chi(L) - \chi(S \cap L) = k - \frac{1}{2}\chi(S \cap L).$$

This means that $\sum_{i=1}^k \chi(Lk(v_i)) = 2k$. Notice that each $Lk(v_i)$ is a surface and hence $\chi(Lk(v_i)) \leq 2$ with equality holding iff $Lk(v_i) = 2$. It follows that each $Lk(v_i)$ is a 2-sphere for each i . ♠

Definition 5.6.8 Let X be a topological n -manifold. We say a triangulation $f : |K| \rightarrow X$ is combinatorial, if the link of every k -simplex in K is combinatorially equivalent to the boundary complex of the standard $(n - k)$ -simplex.

Remark 5.6.9 Obviously, the requirement in the above definition is more stringent than the conclusion in the above theorem which holds for any triangulation of a manifold. J. H. C. Whitehead (1940) strengthened the theorem of Cairns by proving that every smooth manifold has a combinatorial triangulation ([?]). Kirby and Siebenmann and independently Lashof and Rosenberg came up with a unique obstruction class in $H^4(M, \mathbb{Z}_2)$ for any given topological manifold M of dimension ≥ 5 to admit a combinatorial triangulation. And soon, Siebenmann actually showed that for each $n \geq 5$ there are closed manifolds M of dimension n , which have no combinatorial triangulations.

Remark 5.6.10 We shall end this section with a digressional note if only to indicate how a very important result due to Munkres can be derived effortlessly from Lemma 5.6.1. In the following theorem statement (b) is Reisner's condition for the face ring of a simplicial complex K to be *Cohen–Macaulay*. These results were used in a non trivial way by Stanley in the solution of the upper bound conjecture.

Theorem 5.6.11 (Munkres) *With $X = |K|$ where K is a finite simplicial complex of dimension n , the following two statements are equivalent:*

- (a) $\tilde{H}_i(X) = \tilde{H}_i(X, X \setminus \{x\}) = (0)$ for $i < n$ and for every $x \in X$.
- (b) $\tilde{H}_i(Lk(F)) = (0)$ for $i < \dim Lk(F)$ and for every face $F \in K$.

Observe that since the empty set is also a face in K , according to our convention, and $Lk_K(\emptyset) = K$, condition (b) implies that $\tilde{H}_i(K) = (0)$ for $i < n$.

We shall first prove two lemmas.

Lemma 5.6.12 Either of the conditions (a) and (b) implies that K is pure.

Proof: Suppose (a) holds. Let F be a maximal simplex in K and x be a point in the open simplex $\langle F \rangle$. Then

$$\begin{aligned} H_i(X, X \setminus \{x\}) &\approx H_i(|St(F)|, |St(F)| \setminus \{x\}) \\ &= H_i(|F|, |F| \setminus \{x\}) \\ &\approx H_i(|F|, |\mathcal{B}(F)|) \approx H_i(B_1(0)^k, \mathbb{S}^{k-1}) \end{aligned}$$

where $k = \dim F$. By (a), $k \geq n$ and hence $k = n$. Now suppose (b) holds. Suppose K is not pure. Let L_1 be the collection of all n -simplices and their faces. Let L_2 be the collection of all simplices F which are not a face of any n -simplex, and all the faces of such simplices. Then L_1 and L_2 are subcomplexes of K , $K = L_1 \cup L_2$ and $L_1 \neq \emptyset \neq L_2$. Since X is connected $L_1 \cap L_2 \neq \emptyset$. Let σ be a maximal simplex in $L_1 \cap L_2$. Then $\sigma \neq \emptyset$ and is a proper face of a simplex F in L_2 and hence

$$\dim \sigma < \dim F < n.$$

Also σ is a proper face of a n -simplex $F' \in L_1$. Hence $Lk(\sigma)$ intersects both $L_1 \setminus \sigma$ and $L_2 \setminus \sigma$. Since σ is maximal in $L_1 \cap L_2$, it follows that $Lk(\sigma)$ is disconnected. Hence $\tilde{H}_0(Lk(\sigma)) \neq (0)$. But, $\dim \sigma \leq n - 2$ and $\sigma \subset s' \in L_1$ and so $\dim(Lk(\sigma)) \geq 1$. This is contradicting (b). ♠

Remark 5.6.13 Observe that we have used only the second part of condition (a) and condition (b), only for non empty faces of K in proving the purity. It is worth watching our steps in this light, in the proof of Theorem 5.6.11 that follows.

Proof of the Theorem 5.6.11: By Lemma 5.6.12 either of the conditions implies K is pure. Hence for each $F \in K$, $|St(F)|$ has dimension n . Hence

$$\dim Lk(F) + \dim F + 1 = n.$$

Now,

$$\begin{aligned}
 & \text{condition (a)} \\
 \iff & \tilde{H}_i(X) = (0) \text{ for } i < n \quad \text{and} \\
 & \tilde{H}_{j-\dim F-1}(Lk(F)) = (0) \quad \forall \emptyset \neq F \in K, \quad \text{and } j < n \text{ (by Lemma 5.6.1).} \\
 \iff & \tilde{H}_j(X) = (0) \quad \text{and} \\
 & \tilde{H}_{j-n+\dim Lk(F)}(Lk(F)) = (0) \text{ for } j < n \text{ and for every } F \in K. \\
 \iff & \text{condition (b)}.
 \end{aligned}$$

This completes the proof of Munkres' theorem. ♠

5.7 Existence of Triangulations of Surfaces

In this section, we shall study the classification problem for compact surfaces. Originally, this is attributed to August Möbius (1790–1868) and Camille Jordan (1838–1922) who considered surfaces embedded in the 3-dimensional Euclidean space. Our treatment is somewhat general. We shall first of all show that every compact surface is triangulable and then use this triangulation for getting a classification up to homeomorphism. For a classification of smooth surfaces up to diffeomorphism using Morse theory, see [?]. It turns out that both these classifications coincide, in the sense that every homeomorphism class of a compact surface contains a unique diffeomorphism class.

We remark that the classification of non compact surfaces is a formidable problem and is beyond the scope of this course.

Definition 5.7.1 By a *surface* we shall mean a compact 2-dimensional connected topological manifold (without boundary).

In this section we shall first show that every connected compact 2-dimensional manifold without boundary is *triangulable* and then use this to classify them. The proof of the classification is combinatorial in nature. It turns out that just orientability and Euler characteristic are enough invariants to classify them. The connectivity of the surface is not a logical necessity but only a convenience as we immediately see that we can first write the given compact surface as a union of its connected components.

The proof of the existence of triangulations on a surface that we are going to present now is due to Tibor Rado (1920), and we shall follow the presentation given in Ahlfors and Sario [?], with minor modifications. (These modifications consist chiefly of adding extra explanations to certain claims that are made without providing justifications).

For the sake of concentrating on the new ideas involved here and to keep more involved arguments at a bay, we restrict ourselves to the case when the surface is compact. The same proof works for non compact surfaces as well but, in a few places, we need to workout more point-set topology.

We observe that we need to use the famous Jordan curve theorem(JCT). This is not a problem since we have already proved the celebrated Jordan–Brouwer separation theorem. Indeed, we shall actually need the stronger form of JCT which goes under the name Jordan–Schönflies Theorem(JST). This was proved by Schönflies around 1892. A proof of this can be found in [?].

Attempts have been made to avoid JST. However, they are as complicated as learning JST if not more. More recently, another proof of this appeared in the American Mathematical Monthly (see [?]). According to the author himself, this proof is not much simpler than the earlier ones but somewhat shorter. The novelty in [?] is that it uses graph theory: the entire thing hinges on the Kuratowski's criterion for planarity of graphs.

The point of the above remarks is to indicate to what extent we can keep these results elementary, for example without bringing in heavy machinery of homology theory. However, as we shall see, for the uniqueness part of the classification, we need to bring in homology or the fundamental group.

Theorem 5.7.2 (Jordan–Schönflies Theorem (JST)) *Every Jordan curve γ in $\mathbb{C}$ can be mapped onto the unit circle in $\mathbb{C}$ by a homeomorphism $\mathbb{C} \rightarrow \mathbb{C}$.*

Here, by a *Jordan curve* or a *Jordan loop* we mean the image of S^1 under a homeomorphism. Analogously, we will use the word *Jordan arc* (respectively, *open Jordan arc*) to mean the image of a closed interval (respectively, an open interval) under a homeomorphism. We shall use the term *Jordan path* to mean any three of these.

The bounded component of the complement of a Jordan loop in $\mathbb{C}$ is called a *Jordan region*. The two important immediate consequences of the JST that we are going to use are:

Lemma 5.7.3 A Jordan path in $\mathbb{C}$ has no interior.

Lemma 5.7.4 The closure of each Jordan region is a 2-cell, i.e., homeomorphic to the closed disc.

Remark 5.7.5 These two consequences can also be derived directly from JCT and the Riemann mapping theorem if you like. For Lemma 5.7.3, we can also use the theorem of invariance of domain. So, to that extent, we need not really have to depend upon JST. In any case, using JST, observe that Lemmas 5.7.3 and 5.7.4 follow from the corresponding statements about the unit circle in place of an arbitrary Jordan curve. For the circle itself these statements are obvious. The proof of Lemmas 5.7.6, 5.7.7 and 5.7.8 are left to the reader just to catch her attention to what is going on.

Lemma 5.7.6 Let A be a Jordan region in $\mathbb{C}$ and let γ be a Jordan arc in $\bar{A}$ such that $\gamma \cap \partial A = \partial \gamma$. Then $A \setminus \gamma$ consists of two Jordan regions.

Put a diagram here

Lemma 5.7.7 Let A be a Jordan region and B be a Jordan subregion. Let the Jordan loop $\gamma = \partial B$ be such that $\gamma \cap \partial \bar{A}$ is a singleton set. Then $A \setminus \bar{B}$ is an open 2-cell.

Lemma 5.7.8 Let Γ be a finite connected 1-dimensional CW complex embedded in $\mathbb{C}$. Then $\mathbb{C} \setminus \Gamma$ consists of an unbounded component and a finite number of open 2-cells.

Lemma 5.7.9 Let S be a compact surface. Suppose $J \subset S$ is an open Jordan arc and $\bar{J}$ is its closure in S . Then $\bar{J} \setminus J$ has at most two points.

Proof: This is an immediate consequence of Hausdorffness of S . Let $f : (0, 1) \rightarrow S$ be a parameterization of J . Let $\lim_{t \rightarrow 0} f(t)$ and $\lim_{t \rightarrow 1} f(t)$ may constitute at most two points in $\bar{J} \setminus J$. If there is $z \in \bar{J}$, which not equal to the above limit points, by Hausdorffness of S , there will be some $\epsilon > 0$ such that arbitrary small neighbourhoods of z will intersect $f(\epsilon, 1 - \epsilon)$. But then $z \in f([\epsilon, 1 - \epsilon]) \subset J$. ♠.

5.8 Open Covering of Finite Character

Definition 5.8.1 Let $\{U_i\}$ be a family of open sets in a surface S . We say $\{U_i\}$ is of *finite character* if the following conditions are satisfied:

- (i) The family $\{U_i\}$ is locally finite, i.e., each point of S has a neighbourhood which intersects only finitely many U_i 's.
- (ii) The closure $\bar{U}_i$ of U_i in S is a closed 2-cell; in particular, $J := \partial U_i = \partial \bar{U}_i$ is a Jordan loop in S , for each i .
- (iii) Each J_i meets at most finitely many other J_j 's.
- (iv) $J_i \cap J_j$ has finitely many connected components (which may be either arcs or points) for each $i \neq j$.

Remark 5.8.2 The above definition is made in the general context when the surface S may not be compact, just to illustrate the kind of modifications needed if we do not assume that S is compact. Since we are assuming that S is compact, in application, we can always assume that the family $\{U_i\}$ is finite and so, then (i) and (iii) are obvious. Condition (ii) is always satisfied in the situation that we are interested in, because we start with an atlas for S . Condition (iv) requires us to pay more attention.

We shall first prove:

Lemma 5.8.3 Suppose $\{U_i\}$ is a finite open covering of finite character for a surface S . Put $\Gamma = \cup_i \partial \bar{U}_i$. Then

- (a) Γ is a pseudo-graph, i.e., Γ can be given the structure of a 1-dimensional CW-complex.
- (b) Suppose further that $S \setminus \Gamma$ is a disjoint union of open 2-cells and annuli. Then S is triangulable.

Proof: (a) We induct on the number of members in the cover $\{U_i\}$. Clearly, since each $\partial U_i = J_i$ is a Jordan loop, we can select an arbitrary point on it as a vertex and the complement of this point as the open 1-cell. That gives a CW decomposition for each J_i .

Suppose we have given a CW-decomposition for $\Gamma_n := \cup_{i=1}^n J_i$. Now the intersection of J_{n+1} with Γ_n will consist of finitely many arcs and points (may be empty). These extra points and end points of the arcs give a ‘subdivision’ Γ'_n of Γ_n . Now $J_{n+1} \setminus \Gamma_n$ will consist of finitely many open 1-cells. Together, we get a CW-decomposition for Γ_{n+1} . This completes the proof of (a).

(b) First of all by adding extra vertices, if necessary, make the boundary of each of the component A of $S \setminus \Gamma$ into a simplicial complex. Next, it may happen that one of the component of ∂A may not be a Jordan loop i.e., may not be homeomorphic to a circle. In that case, cut A further by introducing open Jordan arc γ in it with $\partial \gamma$ at two distinct vertices of ∂A . Now the boundary of each of the new component is homeomorphic to a circle. Now, for each component A of $S \setminus \Gamma$, which is an annulus select an extra Jordan arc $\gamma_A \subset A$ which joins two of the vertices lying on different components of ∂A .

If Γ' is taken as the union of Γ with all these extra points and extra Jordan arcs, it follows that Γ' is also a 1-dim CW-complex and all components of $S \setminus \Gamma'$ are open 2-cells.

Finally, select a point $v(G') \in G'$ for each G' and perform the ‘star-construction’. The simplicial complex so obtained is a triangulation of S . ♠

We are now ready to prove:

Proposition 5.8.4 A surface is triangulable if it has an open covering of finite character.

Proof: Let $\{U_i\}$ be a finite open covering of S that is of finite character. By discarding any set U_i contained inside another U_j , we shall assume that no U_i is contained in any other U_j . Clearly the number of U_i is at least two.

From the previous lemma, it suffices to prove that condition (b) is satisfied by the family $\{U_i\}$.

Case 1. Suppose that for some $i \neq j$, we have $J_i \subset U_j$. In this case we will actually show that $J_j \subset U_i$, $S = U_i \cap U_j$ and $S \setminus \Gamma$ consists of three components two of them are open 2-cells and one of them is an open annulus so that condition (b) of the above lemma are satisfied.

Note that J_i being a Jordan loop in U_j $U_j \setminus J_i = D \cup A$ where D is an open cell and A is an annulus.

Let $\psi : \mathbb{D}^2 \rightarrow \bar{U}_i$ be a homeomorphism, and let $x \in \partial\mathbb{D}^2$. Let G be a neighbourhood of $y = \psi(x) \in J_i$ in U_j homeomorphic to an open disc such that $G \setminus J_i$ consists of two connected components D_1, A_1 contained in D and A respectively. By continuity, there exists a δ such that $\psi(B_\delta(x) \cap \mathbb{D}^2)$ is contained in G . By connectivity, $\psi(B_\delta(x) \cap \mathbb{D}^2) \setminus J_i$ is contained either in

- (a) A_1 or in
- (b) D_1 .

In the first case (a), we claim that $J_j \subset U_i$. If not, we can join a point $z \in J_j \setminus U_i$ to a point in $A_1 \cap U_i$ by a Jordan arc γ inside $\bar{A} \setminus J_i$. This means γ contains a point of $\bar{U} \setminus U_i = J_i$ which is absurd.

Next, we claim the second case (b) does not occur. For suppose $\psi(B_\delta(x) \cap \mathbb{D}^2) \setminus J_i \subset D_1$. We claim that this will imply that $U_i \subset D$. This implies $U_i \subset U_j$ contradicting our assumption. Therefore this case does not occur. So, let $z \in U_i \setminus D$. Then we can join z to a point in D , by a Jordan arc which is completely contained in U_i . As before, this arc will contain a point of J_j which is also a point of $\bar{D}$. Since $\bar{D}$ is disjoint from J_j , this is absurd.

Thus we are in the situation in which $J_i \subset U_j$ and $J_j \subset U_i$. Therefore,

$$\overline{U_i \cup U_j} \subseteq \overline{U_i} \cup \overline{U_j} \subseteq U_i \cup J_i \cup U_j \cup J_j \subseteq U_i \cup U_j.$$

Therefore, $U_i \cup U_j$ is closed as required. Clearly, J_j cuts U_i into a 2-cell D' and an annular region A' which is part of U_j . It follows that

$$S \setminus \Gamma = D_1 \cup A' \cup D'.$$

This completes the case I.

Case 2. We can now assume that no J_i is contained in any U_j . This means that for each i, j such that $U_i \cap U_j \neq \emptyset$, we have $J_i \cap J_j \neq \emptyset$. Also, $J_i \cap \overline{U_j}$ consists of finitely many Jordan curves and Jordan arcs each with its boundary on J_j . (Observe that it may happen that $J_i \subset \overline{U_j}$ with $J_i \cap J_j$ being a singleton set.) By iterated application of Lemma 5.8.1, it follows that $U_i \setminus \Gamma$ is the disjoint union of open 2-cells. We need to show that condition (b) of lemma 5.8.3 is satisfied. SO suppose A is a path component of $S \setminus \Gamma$. We claim that A is contained in some U_i . (This will imply that every component of $S \setminus \Gamma$ is an open 2-cell.) If this is not the case, There will be $i \neq j$ such that $A \cap U_i \neq \emptyset$ and $A \cap (U_i \setminus U_j) \neq \emptyset$. Choose points $x \in A \cap U_i$ and $y \in A \cap (U_i \setminus U_j)$. Let $f : [0, 1] \rightarrow A$ be a path such that $f(0) = x$ and $f(1) = y$. Let t_0 be the infimum of the set

$$\{t \in [0, 1] : f(t) \notin U_i\}.$$

Then $f(t_0) \in J_i \subset \Gamma$ which is absurd because $f(t) \in A$.

This completes the proof of the proposition. ♠

Before proceeding with the proof of the existence of a covering with finite character we make a technical definition.

Definition 5.8.5 Let Γ be a set of Jordan paths on a surface S . We say Γ is *discrete* in S if every point of S has a neighbourhood U that meets finitely many arcs or curves in Γ .

Observe that any finite set of Jordan paths is discrete. However, if we intersect them with an open set, this finiteness may be destroyed! Therefore, following lemma is crucial, in the construction of a covering of finite character.

Lemma 5.8.6 Let Γ be discrete in S . Then for any region Ω in S , $\Omega \cap \Gamma$ is discrete in Ω .

Proof: Let $x \in \Omega$. Choose a neighbourhood U of x such that $\overline{U} \subset \Omega$ and U meets only finitely many paths in Γ . If infinitely many paths in $\Gamma \cap \Omega$ meet U then it follows that there is at least one path, say γ , in Γ such that infinitely many components of $\gamma \cap \Omega$ meet

U . Starting at some point on γ and moving along γ in one or the other direction, we can obtain an inter-laced sequence of points on the (Jordan) path γ say $x_1, y_1, x_2, y_2 \dots$ such that $x_i \in U$ and $y_i \notin \Omega$. Passing to a subsequence if necessary, we may assume that this sequence converges to a point z . Since this point is the limit of $\{x_i\}$ it is in $\overline{U}$. Since it is also the limit of $\{y_i\}$, it cannot be in the open set Ω . This contradicts the fact $\overline{U} \subset \Omega$. Hence U meets finitely many arcs of $\Gamma \cap \Omega$. ♠

Lemma 5.8.7 $Y \subset S$ is a finite union of Jordan arcs and is connected, then any two points in Y can be joined by a Jordan arc γ such that $\gamma \subset Y$.

Proof: This can be proved in various ways. For instance, we can induct on the number n of Jordan curves in $Y = \cup_{i=1}^n \gamma_i$. For $n = 1$, there is nothing to prove. For $n = 2$, say, $x \in \gamma_1$ and $y \in \gamma_2$. Start at x and trace γ_2 till you hit a point of z on γ_2 and now trace the curve γ_2 till you hit y . For $n > 2$, start from x and trace γ_1 till you hit a point $z \in \cup_{i=2}^n \gamma_i$ to get a Jordan sub-arc λ_1 of γ_1 . By induction, we can join z to y by a Jordan arc λ_2 inside $\cup_{i=2}^n \gamma_i$. Then $x, y \in \lambda_1 \cup \lambda_2$ we are reduced to the case $n = 2$. ♠

Lemma 5.8.8 Let Γ be a set of Jordan paths in the closed unit disc $\mathbb{D}^2$ in $\mathbb{R}^2$, so that $\Gamma \cap \text{int } \mathbb{D}^2$ is discrete in $\text{int } \mathbb{D}^2$. Given any two points $z_1, z_2 \in A := \mathbb{D}^2 \setminus (\Gamma \cap \partial \mathbb{D}^2)$, there exists a Jordan path γ in $\mathbb{D}^2$, joining z_1, z_2 and intersecting each component of Γ in finitely many arcs and points and lying in $\text{int } \mathbb{D}^2$ except perhaps for its end-points.

Proof: Fix a point $z_1 \in A$. If we can join z_1 to z_2 and also to z_3 inside $\mathbb{D}^2$ in the manner described in the lemma, then it is clear that we can join z_2 to z_3 in the same manner. So, let X be the set of all points of A that can be joined to z_1 in this manner. We have to show that $X = A$. (We use a typical topological argument here: first showing that X is a non empty open and closed subset of A and then appealing to the connectivity of A .)

Let $z \in A$ be any point. Choose $\epsilon > 0$ such that $U = B_\epsilon(z)$, the open ball around z of radius ϵ , intersects only those arcs of Γ to which z belongs. If z is on the boundary of $\mathbb{D}^2$, then this is possible since z is not a point of Γ . (Indeed, in case $z \notin \Gamma$, and hence U does not meet Γ .) If z is in $\text{int } \mathbb{D}^2$ then this is possible since $\Gamma \cap \text{int } \mathbb{D}^2$ is discrete and so there is an open set V around z which intersects only finitely many curves in Γ . Let us denote the union of these curves by Γ' .

Now suppose $z \in X$ and let γ be the Jordan path in $\mathbb{D}^2$ joining z_1 to z . We claim that $U \subset X$. For this, it suffices to show that each point in U can be joined to z by a

Jordan arc τ in U , intersecting $\Gamma \cap U$ in finitely many arcs and points. We can then take a Jordan arc inside $\gamma * \tau^{-1}$ joining z_1 to x .

So, given any $x \in U$, take τ to be the line segment $[x, z]$. If τ has the required property, then well and good! Otherwise, let x_1 be the first point of intersection of $[x, z]$ with $\gamma \cup \Gamma'$ and $\tau' = [x, x_1]$. Then both x, z_1 lie on the connected set $\gamma \cup \Gamma' \cup \tau'$. By the previous lemma, we can now get a Jordan subarc from z_1 to x contained in $\gamma \cup \Gamma' \cup \tau'$. Therefore $U \subset X$. This proves X is open.

The next proposition will complete the proof of the existence of triangulation.

Proposition 5.8.9 For any surface S , there exists an open covering of S of finite character.

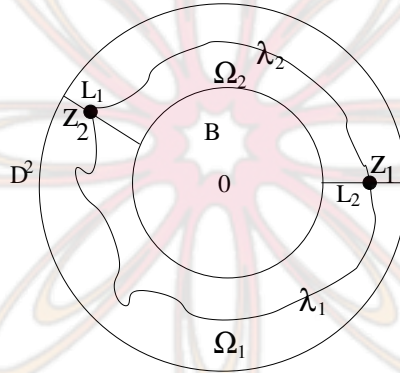


Figure 5.2. Construction of an open covering of finite character

Proof: Let $h_i : \mathbb{D}^2 \rightarrow S$ be homeomorphisms of the closed disc onto closed subsets of S such that the images of the half-discs form a finite open cover of S , i.e., $S = \cup_{i=1}^n h_i(B)$, where B denotes the open disc of radius $1/2$ around 0 . Put $U_i = h_i(B)$, $V_i = \text{int } h_i(\mathbb{D}^2)$. We shall find Jordan regions W_i such that $U_i \subseteq W_i \subset V_i$, $\forall i$ and such that the covering $\{W_i\}$ is of finite character.

Take $W_1 = U_1$. Assume that we have already found W_i , $1 \leq i \leq k-1$, such that $\{W_1, W_2, \dots, W_{k-1}\}$ is of finite character. We should find W_k such that $U_k \subseteq W_k \subset V_k$ and such that $\gamma_k := \partial W_k$ intersects $\gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_{k-1}$ in finitely many arcs and points. (If ∂U_k has this property then we can choose $W_k = U_k$, but this may not be the case). Let $\Gamma = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_{k-1}$. Then, each curve in Γ is a Jordan curve and Γ has finitely many such curves. In particular, Γ is discrete in S and hence $\Gamma \cap V_k$ is also discrete in V_k . Moreover, Γ has no interior. Let us put $\Gamma' = h_k^{-1}(\Gamma)$. Then Γ' has no interior in $\mathbb{D}^2$.

So we can choose points z_1 and z_2 in the interior of the annulus $\mathbb{D}^2 \setminus B$ such that z_i do not lie on Γ' and also they are not on the same line through 0. Let L_i be the radial segments through z_i to the boundary of the annulus. Then the annulus $\mathbb{D}^2 \setminus B$ is cut into two Jordan regions Ω_1 and Ω_2 , respectively, by L_1 and L_2 . (See Figure 5.2.) By Lemma 5.8.6, it follows that $\Omega_i \cap \Gamma'$ is discrete in Ω_i , for $i = 1, 2$, respectively. Join z_1 and z_2 by two Jordan arcs λ_i with their interior lying inside Ω_i , respectively, and intersecting Γ in finitely many arcs and points (Lemma 5.8.8). Then $\gamma'_k := \lambda_1 \cup \lambda_2$ is a Jordan loop lying in the annulus, bounding Jordan region R inside $\mathbb{D}^2$. Take $W_k = h_k(R)$. It follows that $\{W_1, \dots, W_k\}$ is of finite character. This completes the proof of the proposition. ♠

5.9 Classification of Compact Surfaces

Toward the classification of surfaces, we fix a triangulation on a connected compact surface X . We then appeal to Theorem 5.6.5 which provides us a representation of X as a quotient of a convex polygon P of $2n$ sides whose sides have to be identified pairwise by homeomorphisms which are linear on each side. This process can be completely described purely combinatorially as follows:

Let us agree once and for all, that we shall trace the boundary of any convex polygon in the anticlockwise direction. We are free to start from any vertex. We then label the edges by letters **a**, **b**, **c**, etc. As soon as we meet an edge which is being identified with an edge which has already been labeled, we shall use the same letter to label this new edge also. However, we have to take care of another aspect, viz., whether the identification is orientation preserving or orientation reversing. To indicate the orientation reversing we shall use labellings $\mathbf{a}^{-1}$, $\mathbf{b}^{-1}$, $\mathbf{c}^{-1}$, etc. Of course, we stop as soon as we have arrived back where we have started. Since the starting point is arbitrary, what this process yields is that the surface X is completely determined by the cyclic sequence

$$\mathbf{a}_1^{\epsilon_1} \mathbf{a}_2^{\epsilon_2} \dots, \mathbf{a}_n^{\epsilon_n} \quad (5.4)$$

in which each letter $\mathbf{a}_i$ occurs precisely twice. In particular, n is even. For the simplicity of the notation, for an edge $\mathbf{a}^{+1}$ occurring with $+1$ sign, we drop this sign and simply write it as **a**. A sequence such as (5.4) is called a *canonical polygon*. We shall use bold face capitals **A**, **B**, etc., to denote any sequence such as (5.4). For $n \geq 4$, we can represent a canonical polygon by a regular convex polygon $\mathcal{P}$ in $\mathbb{R}^2$ with n sides with

its sides appropriately labeled. Observe that we allow the exceptional case when $n = 2$ also. In this case, we do not get a convex polygon in $\mathbb{R}^2$. However, in this case, we take $\mathcal{P}$ to be the unit disc with its boundary being divided into two edges, the sequence itself being $\mathbf{aa}^{-1}$ or $\mathbf{aa}$. From Exercise ?? (a),(b), it follows that we get $\mathbb{S}^2$ in the first case, and $\mathbb{P}^2$ in the second case. Therefore, in what follows we can concentrate only on the case $n \geq 4$.

If both $\mathbf{a}$ and $\mathbf{a}^{-1}$ occur in this sequence for some edge $\mathbf{a}$, we call that pair of edges $\{\mathbf{a}, \mathbf{a}^{-1}\}$ *type I pair*. Otherwise, the pair is of *type II*. While representing a canonical polygon by a picture, instead of the exponents ± 1 over the letters, we shall use arrows to indicate direction.

Several canonical polygons may define the same surface up to homeomorphism. Our next step is to make a list which should include all possible topological types as well as have no redundancy. Next, we must identify some simple transformations on the canonical polygons which do not change the topological type and then keep applying these transformations so as to bring any given canonical polygon to one in the list. Let us illustrate this with an example.

Example 5.9.1 Consider the surface given by the sequence $\mathbf{abab}^{-1}$. We have seen that this represents the Klein bottle. Mark the diagonal of the square piece of the paper with a (thick) arrow and letter $\mathbf{c}$ and cut the square along this arrow as shown in Figure 5.3. Bring the upper triangular piece down, flip it and identify the two triangles along the edge $\mathbf{a}$, taking care to preserve the orientation of the edge while identifying. The new figure is not a rectangle but we treat it as a rectangle with the sides marked $\mathbf{bbcc}$ which is a simpler canonical polygon which represents connected sum of two copies of $\mathbb{P}^2$.

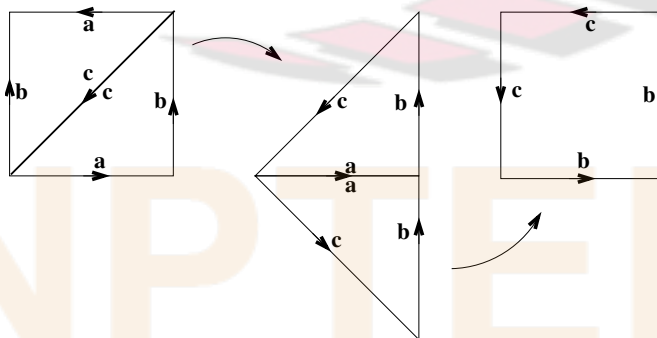


Figure 5.3. Transformations of canonical polygons

Theorem 5.9.2

(A) Every compact connected surface without boundary is homeomorphic to the surface defined by one of the following canonical polygons:

- (i) $\mathbf{aa}^{-1}$.
- (ii) $\mathbf{a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}}$, $g \geq 1$.
- (iii) $\mathbf{a_1a_1 \cdots a_na_n}$, $n \geq 1$.

(B) Moreover, any two distinct members of this list give surfaces which are non homeomorphic.

The canonical polygons listed above are said to be in the *normal form*. The rest of this section will be occupied by the proof of part (A) of the theorem which will be achieved in five steps, and then giving three different proofs of the uniqueness part (B).

5.10 Combinatorial Transformations on Canonical Polygons

Beginning with an arbitrary canonical polygon, we would like to *transform* it to one in the normal form without changing the homeomorphism type of the surface defined by it.

First observe that in (ii) and (iii), all the vertices are identified to a single point. This may not be true for an arbitrary canonical polygon. So, our first aim would be find a reduction process which will ensure that all vertices are identified to a single point. One of the simplest thing to do is to cancel out a pair of edges $\cdots \mathbf{aa}^{-1} \cdots$.

Step 1 Elimination of adjacent edges of type I:

We will now show that if D has at least four edges, then an adjacent pair of edges of type I can be eliminated, until we end up with case (i) or there are no adjacent pairs of edges of type I. Figure 5.4 illustrates this process. And Exercise ??(iii) provides justification. The relation may be coded as:

$$\mathbf{Aaa}^{-1} \sim \mathbf{A}. \quad (5.5)$$

We can assume that the new $\mathcal{P}$ is again convex. By repeated application of this, we eliminate all adjacent pairs of type I.

Step 2 Reduction to the single-vertex-class case:

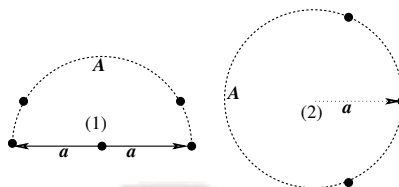


Figure 5.4. Elimination of adjacent edges of type I

We consider the equivalence classes of vertices under the quotient map $\mathcal{P} \rightarrow X$. By step 1, we assume that there are no adjacent edges of type I. Suppose there are at least two equivalence classes of vertices and consider a class $[P]$ with the least number of elements in it. We can pick a vertex P in this class so that the next vertex Q on ∂P is not in this class (see Figure 5.5). Let R be the other vertex adjacent to P . Let a, b denote the edges RP and PQ .

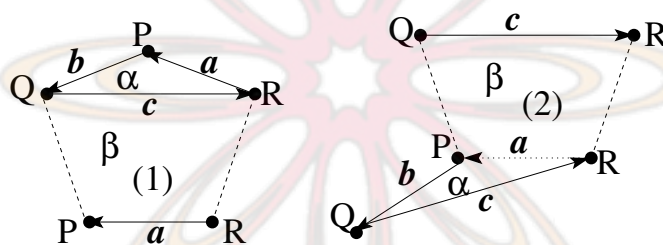


Figure 5.5. Reduction to single vertex class

Make a cut along the line labeled c from Q to R . Glue the two edges labelled a together. In the new polygon, there is one less vertex in the equivalence class of P and one more vertex in the equivalence class of Q . The corresponding relation reads as follows:

$$abAa^{-1}B \sim c^{-1}AbcB. \quad (5.6)$$

If needed, we now carry out Step 1 again. Note that each time Step 1 is performed, the number of edges as well as the number of vertices go down without increasing the number of equivalence classes of vertices. Step 2, on the other hand, keeps all these numbers the same. By repeated applications of these two steps, we keep reducing the number of vertices from the classes with the least number of vertices each time and hence, at some stage one of this class has to disappear. (Indeed, if a class of vertices has just one vertex in it, this implies the two edges incident at this vertex must of the type

I and hence we can perform Step 1 to get rid of them.) This way we keep reducing the number of classes themselves until there is only one class left.

Thus from now on we shall also assume that all vertices are identified to a single point. It is worth noting that none of the reduction operations that we are going to perform from now onward will disturb this property. (In fact, the only way to create another equivalence class of vertices is to perform the reverse operation of Step 1.)

5.11 Final Reduction

Step 3 To make any pair of edges of type II adjacent:

Suppose there is a non-adjacent pair of edges of type II labeled **a**, as depicted in Figure 5.6.

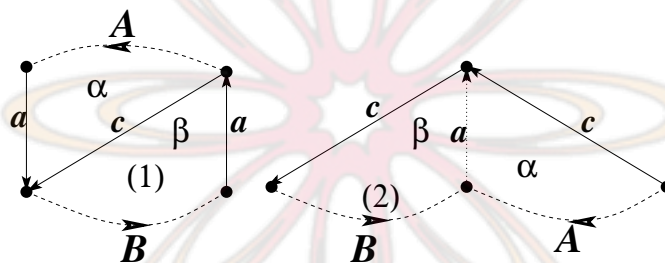


Figure 5.6. Bringing type-II edges together

Cut along the line labeled **c** and paste along **a**. By repeated application of this, we make all pairs of edges of type II adjacent. The corresponding relation may be coded as:

$$\mathbf{aAaB} \sim \mathbf{ccBA}^{-1}. \quad (5.7)$$

Note that such an operation does not disturb some other pair **bb** which is already adjacent. Therefore, repeated application of this will make all the pairs of type II adjacent. At this stage if there is no pair of edges of type I left, we have found the canonical polygon has become $\mathbf{a_1a_1a_2a_2 \cdots a_na_n}$ as in (iii) of Theorem 5.9.2.

Step 4 Handling a pair of edges of type I: Suppose there is at least one pair of edges of type I, say labeled **a**, (which is necessarily not adjacent). Then P has the form as shown in Figure 5.7. where both A, B are non empty.

If no edge in **A** is identified with any edge in **B**, then there will be at least two equivalence classes of vertices, viz., the endpoints of **a**. This contradicts the assumption that there

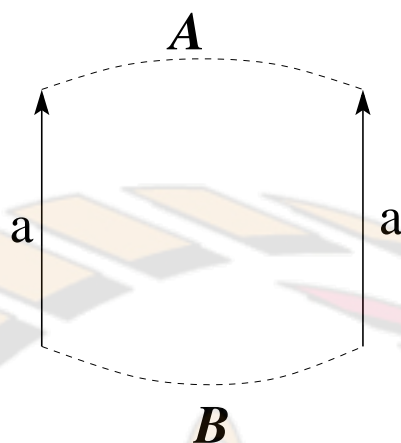


Figure 5.7. Handling the interlocked edges

is only one equivalence class of vertices. Therefore at least one edge in $\mathbf{A}$ is identified with one edge in $\mathbf{B}$. This pair cannot be of type II, since all pairs of type II are adjacent. Therefore the polygon P has the form as depicted in Figure 5.8(1).

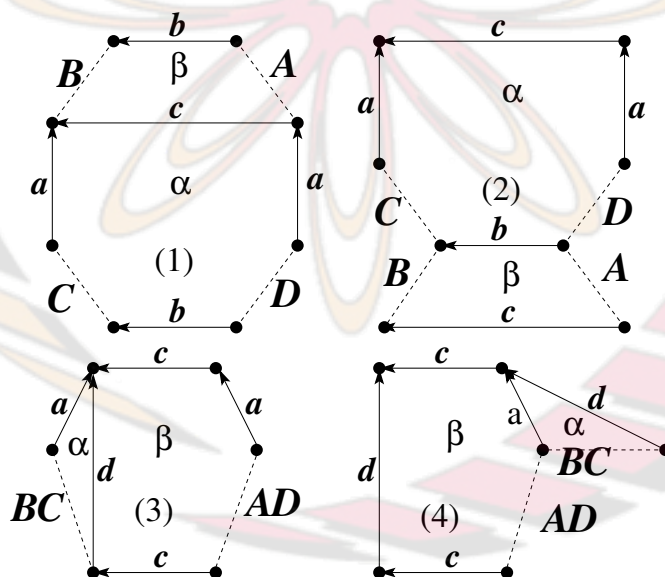


Figure 5.8. Inter-locked edges brought together

We will transform the polygon so that the two interlocked pairs of edges become consecutive.

First cut Figure 5.8(1) along $\mathbf{c}$ and paste together along $\mathbf{b}$ to obtain 5.8(2). The polygon 5.8(3) is nothing but 5.8(2) redrawn so as to be represented by a convex polygon. Cut 5.8(3) along $\mathbf{d}$ and paste together along $\mathbf{a}$ to obtain 5.8(4), as desired. The

corresponding relation may be coded as:

$$\mathbf{aAbBa}^{-1}\mathbf{Cb}^{-1}\mathbf{D} \sim \mathbf{aca}^{-1}\mathbf{CBc}^{-1}\mathbf{AD} \sim \mathbf{dcd}^{-1}\mathbf{c}^{-1}\mathbf{ADCB}. \quad (5.8)$$

By repeated application of this, we get the canonical polygon with edges marked

$$\mathbf{a_1b_1a_1^{-1}b_1^{-1} \cdots a_mb_ma_m^{-1}b_m^{-1}c_1c_1c_2c_2 \cdots c_nc_n}.$$

If either $m = 0$ or $n = 0$ then we are done. It remains to consider the case when both m and n are positive. See Figure 5.9.

Step 5 Handling the case of mixed types:

Combining results of the two cut-and-pastes done in Figure 5.9, we obtain the following relation:

$$\mathbf{Aaba}^{-1}\mathbf{b}^{-1}\mathbf{cc} \sim \mathbf{Axyzzyz}. \quad (5.9)$$

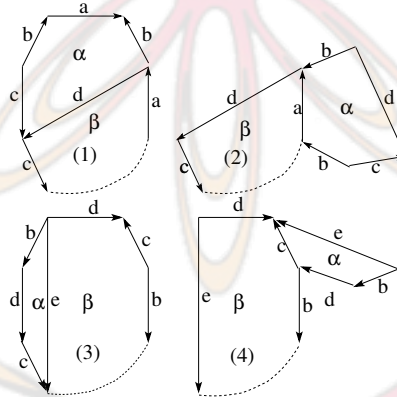


Figure 5.9. Mixed type

We can now apply step 3 successively to make this into $\mathbf{Axxyyzz}$.

By repeated application of this step, all interlocking pairs of type I can be replaced by two pairs of type II, so that the canonical polygon

$$\mathbf{a_1b_1a_1^{-1}b_1^{-1} \cdots a_mb_ma_m^{-1}b_m^{-1}c_1c_1c_2c_2 \cdots c_nc_n}$$

is transformed into $\mathbf{x_1x_1 \cdots x_kx_k}$, where $k = 2m+n$. This completes the proof of Theorem 5.9.2.

5.12 Proof of Part (B)

Clearly the fundamental group of a surface can be easily determined by the CW-structure K given by the canonical polygon. (See Corollary ??.)

The 1-skeleton $K^{(1)}$ of the CW-structure is a bouquet of r circles, the number r being equal to either $2g$ or n for T_g and for P_n , respectively. The convex polygon gives just one 2-cell. The attaching map of this 2-cell represents a certain element t in $\pi_1(K^{(1)})$ which is easily determined by the sequence representing the canonical polygon. The fundamental group $\pi_1(K)$ is the quotient of the free group $\pi_1(K^{(1)})$ by the normal subgroup generated by t . Therefore we have:

Theorem 5.12.1 *The fundamental group of a surface is given by the generators and relation:*

$$\begin{aligned}\pi_1(T_g) &= \langle x_1, y_1, \dots, x_g, y_g \mid \prod_{i=1}^g [x_i, y_i] \rangle; \\ \pi_1(P_n) &= \langle x_1, \dots, x_n \mid x_1^2 \cdots x_n^2 \rangle.\end{aligned}$$

Remark 5.12.2

1. It is not hard to see that each of the groups listed in the theorem belongs to a distinct isomorphism class. Indeed, their abelianizations themselves will be non isomorphic. As a consequence we can conclude that the topological type of a surface is completely determined by the fundamental group which comes as a small surprise. (There is no such surprise in a similar result for compact 1-dimensional manifold though.) This may be attributed to a deeper property of surfaces: viz., barring two exceptional cases of $\mathbb{S}^2$ and $\mathbb{P}^2$, all connected closed surfaces have the universal covering space either $\mathbb{C}$ or the upper half plane, both of which are contractible. (This result is not at all obvious without the use of function theory of one complex variable.)

One can now ask whether such is the case in higher dimensional manifolds as well, viz., suppose we have a n -manifold X covered by a Euclidean space. Is the homeomorphism type of X determined by the fundamental group of X ? This problem goes under the name Borel's conjecture which has been verified in every known case. A complete solution is yet to come. See [?] or

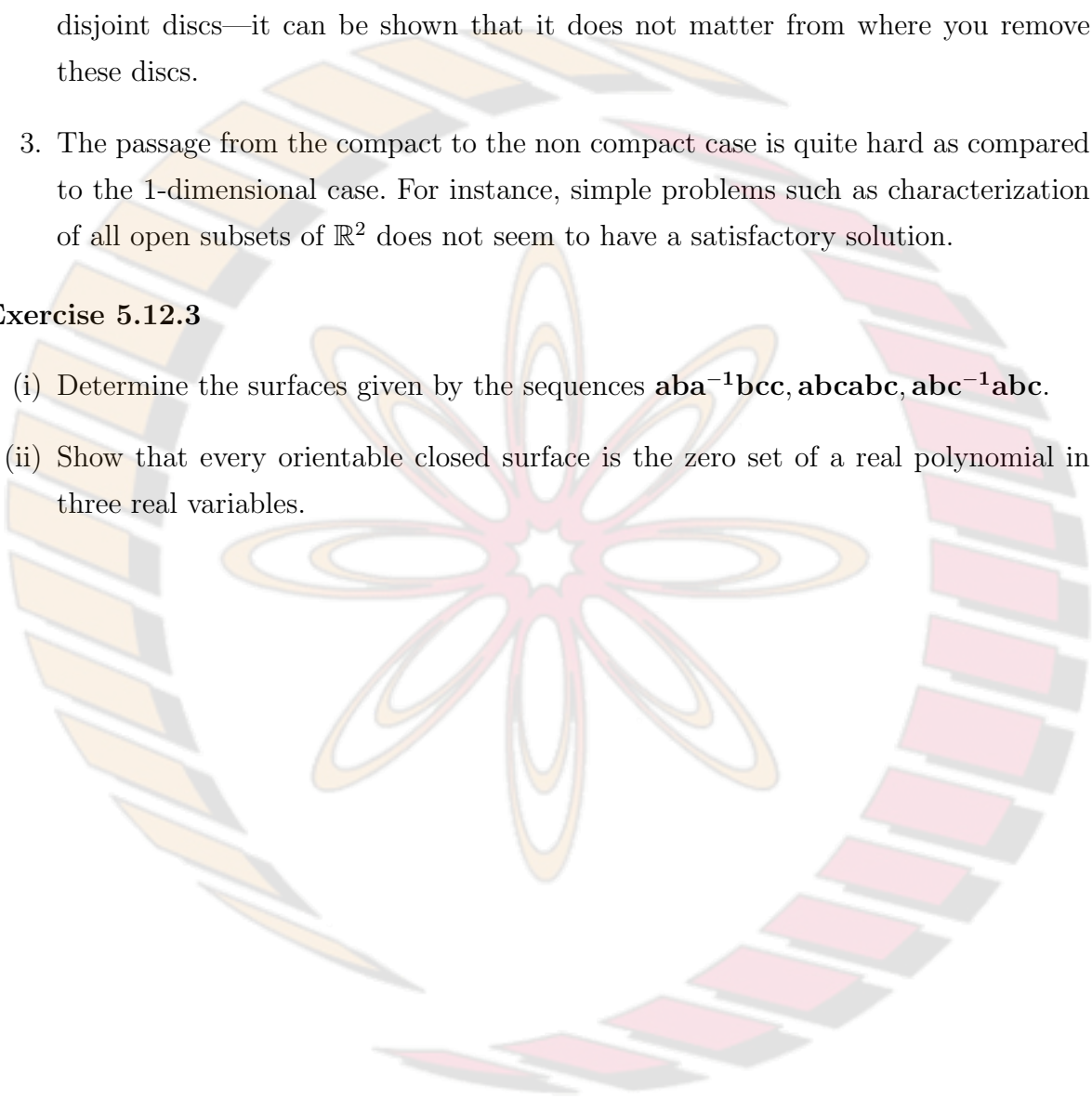
<http://publications.ictp.it/lns/vol.9.html>

for more details.

2. It is not hard to obtain a classification of all connected compact 2-manifolds with boundary. By capping-off all the boundary components we obtain a surface. Thus any compact 2-manifold with boundary is obtainable by removing finitely many disjoint discs—it can be shown that it does not matter from where you remove these discs.
3. The passage from the compact to the non compact case is quite hard as compared to the 1-dimensional case. For instance, simple problems such as characterization of all open subsets of $\mathbb{R}^2$ does not seem to have a satisfactory solution.

Exercise 5.12.3

- (i) Determine the surfaces given by the sequences **$aba^{-1}bcc$** , **$abcabc$** , **$abc^{-1}abc$** .
- (ii) Show that every orientable closed surface is the zero set of a real polynomial in three real variables.

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