NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 35

Trigonometry and Spherical Trigonometry 3

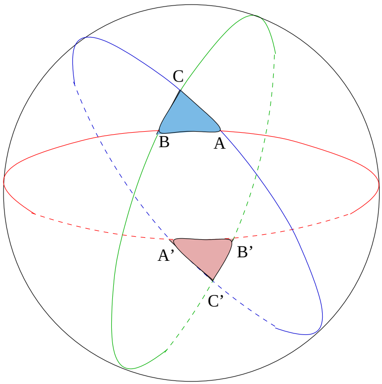
M. S. Sriram
University of Madras, Chennai.

Outline

- Spherical Trigonometry
- Celestial sphere and the co-ordinate systems
- Declination formula
- Ten problems in *Tantrasaṅgraha*: *daśapraśnāḥ*
- A typical example
- Distance between solar and lunar disks

Spherical triangle

A spherical triangle is formed by the intersection of three great circles on the surface of a sphere.



Spherical triangle

Cosine formula for spherical triangles

There are several formulae connecting the sides and angles of a spherical triangle.

If ABC is the spherical triangle, with sides a, b, c, then the law of cosines is given by

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Clearly, there are two companions to the above formula. They are easily obtained by cyclically changing the sides and the angles, and are given by

$$\begin{aligned} \cos b &= \cos c \cos a + \sin c \sin a \cos B \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C. \end{aligned}$$

Sine formula for spherical triangles

The relation between the ratio of the sides to that of the angles of a spherical triangle is given by

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

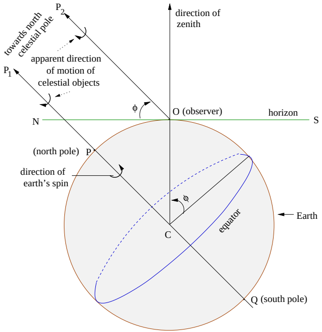
When the sides $a, \ b$ and c are small, it is quite evident that the above formula reduces to

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

which is the sine formula for a plane triangle.

Earth and Observer

All the celestial objects seem to be situated on the surface of a sphere of very large radius, with the observer at the centre. This is the celestial sphere. Though fictitious, the celestial sphere is the basic tool in discussing the motion (both diurnal and relative) of celestial objects.



The horizon and the north celestial pole as seen by the observer on the surface of the Earth.

Celestial sphere

C: Centre of the Earth

O: Observer on the surface of the Earth whose northerly latitude is ϕ.

Tangential plane drawn at the location of the observer, represented by NOS: *Horizon*.

As the Earth rotates about the axis PQ, it appears as if the entire celestial sphere rotates in the opposite direction about P_1.

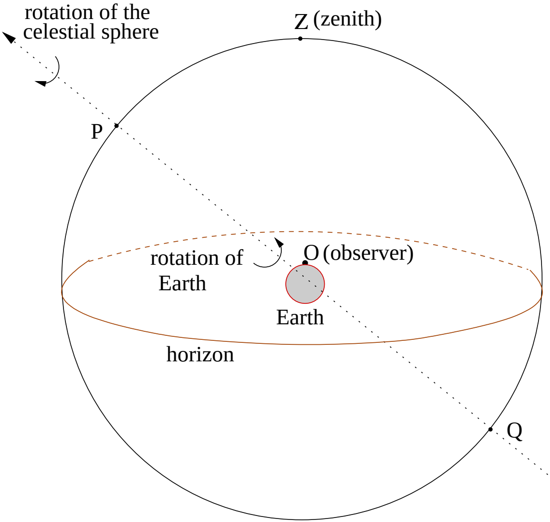
Line OP_2: Parallel to CP_1.

P_1, P_2: Very close

All the celestial bodies seem to be rotating around the axis OP_2 with a period equal to the period of rotation of the Earth (nearly 4 seconds less than 24 hours).

The point P_2: Denoted by P: *North celestial pole*. The celestial sphere for the observer with latitude ϕ is shown next.

Celestial sphere



The celestial sphere for an observer in the northern hemisphere with latitude ϕ.

Coordinate Systems on the Celestial Sphere

Anyone who observes the sky even for short periods of time will have the impression that the objects in it are in continuous motion. This motion consists of two parts. One of them is the apparent motion of all celestial objects, including stars, from east to west, which is actually due to the rotation of the Earth from west to east. This is the diurnal motion. The other is due to the relative motion of any particular celestial object like the Sun, Moon or a planet with respect to the seemingly fixed background of stars.

Just as one uses latitude and longitude (two numbers) to specify any location on the surface of the Earth, so also one employs different coordinate systems to specify the location of celestial objects on the celestial sphere at any instant. We now explain the three commonly employed coordinate systems—namely, the horizontal, the equatorial and the ecliptic.

Three co-ordinate systems for locating an object on the celestial sphere

An object situated at any point on the surface of the celestial sphere, which is a two dimensional surface, can be uniquely specified by two angles. Based on the choice of the fundamental great circle—the horizon, the celestial equator or the ecliptic—we have the following sytems listed in the table.

Coordinate system	Fundamental plane/circle	Poles of circle	Coordinates and notation used
Horizontal	Horizon	Zenith/nadir	Altitude and azimuth $(a,\ A)$
Equatorial	Celestial equator	Celestial poles	Declination and right ascension/hour angle $(\delta,\ \alpha)$ or $(\delta,\ H)$
Ecliptic	Ecliptic	Ecliptic poles	Celestial latitude and longitude $(\beta,\ \lambda)$

The different coordinate systems generally employed to specify the location of a celestial object.

The three co-ordinate systems

Each of these systems has its own advantages and the choice depends upon the problem at hand, somewhat like the choice of coordinate system that is made in order to solve problems in physics. Table below presents the Sanskrit equivalents of the different coordinates and the fundamental reference circles employed for specifying a celestial object.

Coordinates		Reference circles	
Modern name	Skt equivalent.	Modern name	Skt equivalent
Altitude	उत्क्रम	Horizon	क्षितिज
Azimuth	उदग्रा	Prime meridian	दक्षिणोत्तरवृत्त
Hour angle	नत	Prime meridian	दक्षिणोत्तरवृत्त
Declination	क्रान्ति	Celestial equator	विषुवद्वृत्त/घटिकावृत्त
Right Ascension	काल	Celestial equator	विषुवद्वृत्त/घटिकावृत्त
Declination	क्रान्ति	its secondary	तद्विपरीत
Longitude	भोग	Ecliptic	क्रान्तिवृत्त
Latitude	विक्षेप	its secondary	तद्विपरीत

Sanskrit equivalents for different coordinates and the reference circles.

The horizontal (Alt-Azimuth) system

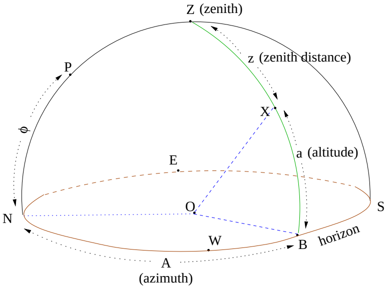
In this system, which is also known as the *alt-azimuth* system, the horizon is taken to be the fundamental reference place.

N, S, E, W: North, South, East, West points: Four Cardinal directions.

Verticle circles: The circles passing through the zenith and perpendicular to the horizon. For any object:

$$\begin{aligned} \text{altitude } (a) &= X\hat{O}B \qquad (\text{range: } 0 - 90^\circ) \\ \text{azimuth}(A) &= N\hat{O}B \qquad (\text{range: } 0 - 360^\circ W). \end{aligned}$$

are the co-ordinates. Some times $z = 90 - a$, instead of a

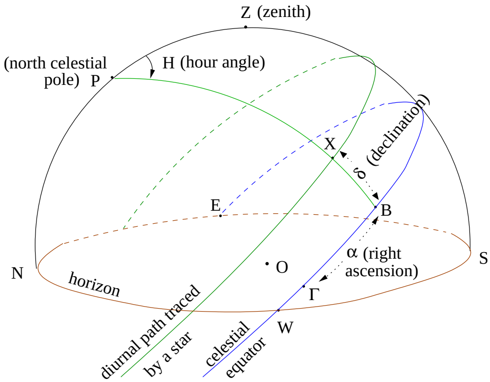


Altitude, azimuth and zenith distance in the horizontal system.

The equatorial system

Here, Celestial equator: Fundamental plane with reference to which the coordinates are specified.

Celestial equator: Great circle whose plane is perpendicular to OP. Inclined to the horizon by an angle equal to the co-latitude $(90 - \phi)$ of the observer.



Declination, hour angle and right ascension in the equatorial system.

The equatorial system

All circles passing through the pole P and perpendicular to the equator are known as *meridian* circles. Consider the meridian passing through the star X and the north celestial pole P, intersecting the equator at B.

Two quantities *declination* and *hour angle* of the star are defined as follows:

$$\begin{aligned} \text{declination } (\delta) &= X\hat{O}B \qquad (\text{range: } 0 - 90^\circ N/S) \\ \text{hour angle } (H) &= Z\hat{P}X \qquad (\text{range: } 0 - 360^\circ/24\,h\ W). \end{aligned}$$

The co-ordinate pairs a, A and δ, H are related. Consider the triangle ZPX in Fig. C.4, where $PX = 90^\circ - \delta$, $PZ = 90^\circ - \phi$, $PZX = A$ and $ZPX = H$ and applying the cosine formula, we have

$$\begin{aligned} \cos(PX) &= \cos(PZ)\cos(ZX) + \sin(PZ)\sin(ZX)\cos(PZX) \\ \text{or} \qquad \sin\delta &= \sin\phi\sin a + \cos\phi\cos a\cos A, \end{aligned}$$

and

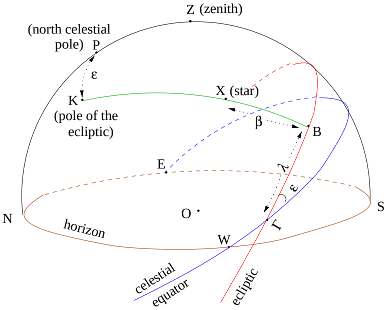
$$\begin{aligned} \cos(ZX) &= \cos(ZP)\cos(PX) + \sin(ZP)\sin(PX)\cos(ZPX) \\ \text{or} \qquad \sin a &= \sin\phi\sin\delta + \cos\phi\cos\delta\cos H. \end{aligned}$$

Sometimes, Right Ascension, α, instead of hour angle H.

The ecliptic system

Ecliptic: Apparent path of the Sun in the background of the stars. Ecliptic system: Ecliptic is the fundamental reference plane. Two angles called the celestial longitude and the celestial latitude, or simply the longitude and the latitude, are used to specify the location of an object on the celestial sphere. These are defined as follows:

$$\begin{aligned} \text{latitude } (\beta) &= X\hat{O}B \qquad \text{(range: } 0 - 90^\circ \text{N/S)} \\ \text{longitude } (\lambda) &= \Gamma\hat{K}X \qquad \text{(range: } 0 - 360^\circ/24\,\text{h East).} \end{aligned}$$



Celestial latitude and longitude in the ecliptic system.

The ecliptic system

Here K is the pole of the ecliptic. β is positive when it is north, and negative when it is south.

Ecliptic inclined to the equator. This inclination, denoted by ε ($\approx 23.5^\circ$), known as the *obliquity of the ecliptic*. The ecliptic and the celestial equator inersect at two points known as the *vernal equinox* and *autumnal equinox*. The Sun's motion on the ecliptic is eastwards. At the vernal equinox Γ, it moves from south to north, or its declination changes sign from $-$ to $+$.

Among the various great circles represented on the celestial sphere, the ecliptic is very important. This is because the Sun moves along the ecliptic, and the inclinations of the orbits of all the planets and the Moon with the ecliptic are small.

Using the formulae of spherical trigonometry, it can be shown that the ecliptic coordinates (β, λ) and the equatorial coordinates (δ, α) are related through the following equations:

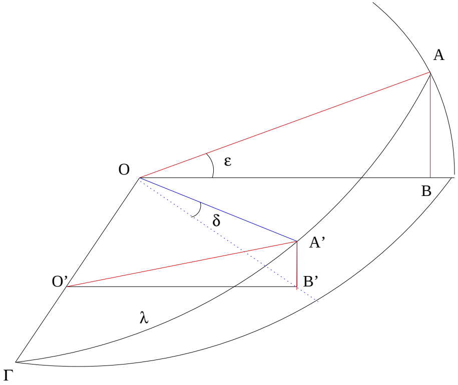
$$
\begin{aligned}
\sin\beta &= \sin\delta\cos\epsilon + \cos\delta\sin\epsilon\sin\alpha \\
\sin\delta &= \sin\beta\cos\epsilon + \cos\beta\sin\epsilon\sin\lambda.
\end{aligned}
$$

Declination formula

When the latitude, $\beta = 0$, the expression for the declination is even simpler:

$$R \sin \delta = \frac{R \sin \varepsilon R \sin \lambda}{R}.$$

This can be derived using planar triangles.



Spherical Trigonometry in modern texts and *Tantrasaṅgraha* and *Yuktibhāṣā*

All the results based on spherical trigonometry are exact in *Tantrasaṅgraha*. In the earlier texts, most of the relations would be correct, except for some small errors at certain places. In *Tantrasaṅgraha* all these errors are removed. Moreover, the treatment of problems related to spherical trigonometry is very systematic. But *Tantrasaṅgraha* gives only the results, but not the explanations. *Yuktibhāṣā* explains all these results systematically, in its astronomy part. In fact the earlier sections of the astronomy part in *Yuktibhāṣā* closely resemble corresponding sections in a modern texts on spherical astronomy. But the derivations and proofs of results are quite different from the modern treatment. The declination type formula play a crucial role in the method of derivation. In the following we give some results from *Tantrasaṅgraha*, briefly indicating the proofs in *Yuktibhāṣā*.

The Ten problems in *Tantrasaṅgraha*

इह शङ्कु-नत-क्रान्ति-दिगग्राऽक्षेषु पञ्चसु ।
द्वयोर्द्वयोरानयनं दशधा स्यात् परैस्त्रिभिः ॥ ६० ॥
सशङ्कवो नतक्रान्तिदिगक्षाः सनतास्तथा ।
अपक्रमदिगग्राक्षा दिगक्षौ क्रान्तिसंयुतौ ॥ ६१ ॥
दिगक्षाविति नीयन्ते द्वन्द्वीभूयेतरैस्त्रिभिः ।

Out of the five quantities *śaṅku*, *nata*, *krānti*, *digagrā* and *akṣa*, any two of them can be determined from the other three and this happens in ten different ways. Pairs from the sequences (i) *śaṅku*, *nata*, *krānti*, *digagrā* and *akṣa*; (ii) *nata*, *krānti*, *digagrā* and *akṣa*; (iii) *krānti*, *digagrā* and *akṣa*; (iv) *digagrā* and *akṣa*; are [formed and] determined with the other three.

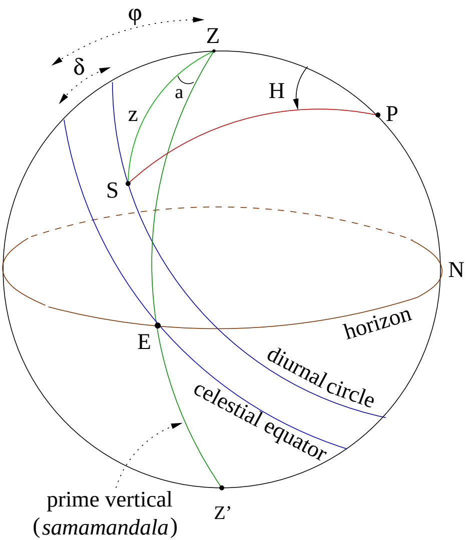
The Ten problems in *Tantrasaṅgraha*

The modern equivalents of the five quantities listed in the above verses and the notation used to represent them are given in the table below.

Sanskrit name	Modern equivalent Rsine of	Notation
śaṅku	zenith distance	$R \sin z$
nata	hour angle	$R \sin H$
krānti	declination	$R \sin \delta$
digagrā	amplitude	$R \sin a$
akṣa	latitude	$R \sin \phi$

The five quantities associated with the problem of the *daśapraśna*.

The Ten problems in *Tantrasaṅgraha*



Celestial sphere with markings of the five quantities, namely, *śaṅku, nata, krānti, digagrā, akṣa* associated with the *daśapraśnāḥ*.

The Ten problems in *Tantrasaṅgraha*

The order in which the ten pairs are selected, as given in verse 61 and the first half of verse 62, is shown in following table:

Set	Pairs formed from this set
$\{z, H, \delta, a, \phi\}$	(z, H), (z, δ), (z, a), (z, ϕ)
$\{H, \delta, a, \phi\}$	(H, δ), (H, a), (H, ϕ)
$\{\delta, a, \phi\}$	(δ, a), (δ, ϕ)
$\{a, \phi\}$	(a, ϕ)

The ten pairs that can be formed out of the five quantities associated with the *daśapraśnāḥ*.

Verses 62–87 in *Tantrasaṅgraha* describe the explicit procedure for the solution of these 'ten problems'. The detailed demonstration of the solution of each of these problems is presented in Jyeṣṭhadeva's *Yuktibhāṣā*.

Zenith distance and hour angle from the declination, amplitude and latitude (Problem 1)

आशाग्रा लम्बकाभ्यस्ता त्रिज्याभक्ता च कोटिका ॥ ६२ ॥
भुजाक्षज्या तयोर्वर्गयोगमूलं श्रुतिर्हरः ।
क्रान्त्यक्षवर्गौ तद्वर्गात् त्यक्त्वा कोट्यौ तयोः पदे ॥ ६३ ॥
कुर्यात् क्रान्त्यक्षयोर्घातं कोट्योर्घातं तथा परम् ।
सौम्ये गोले तयोर्योगात् भेदात् याम्ये तु घातयोः ॥ ६४ ॥

The *āśāgrā* multiplied by the *lambaka* and divided by the *trijyā* is the *koṭi*. The *bhujā* is the *akṣajyā*. The square root of the sum of their squares is the hypotenuse and it is the *hara* [or *hāra*, the divisor, which will be used later].

Then find the square roots of the squares of the *krānti* and the *akṣa* subtracted from it. They form the *koṭis*. Similarly find the products of the *krānti* and the *akṣa* and also their *koṭis*.

The sum and the differences of the products are multiplied by the *trijyā* and divided by the square of the divisor [when the Sun is] in the northern and southern hemispheres respectively.

Zenith distance and hour angle from the declination, amplitude and latitude (Problem 1)

आद्यघातेऽधिके सौम्ये योगभेदद्वयादपि।
त्रिज्याघ्नात् हारवर्गाप्तः शङ्कुरिष्टदिगुद्भवः॥ ६५ ॥
छाया तत्कोटिराशाग्राकोटिघ्ना सा द्युजीवया।
भक्ता नतज्या क्रान्त्यक्षदिगग्राभिर्भवेदिति॥ ६६ ॥

This gives the *śaṅku* that is formed in the desired direction. If the first product is greater than the second one, in the northern hemisphere, then the *śaṅku* is obtained from both the sum and the difference.

Its (the *śaṅku*'s) *koṭi* (compliment) is the *chāyā* (the shadow). When that is multiplied by the *koṭi* (compliment) of the *āśāgrā* and divided by the *dyjyā*, the resultant is the *natajyā*. Thus the *śaṅku* and the *nata* can be obtained from the *krānti*, the *akṣa* and the *āśāgrā*.

Zenith distance and hour angle from the declination, amplitude and latitude (Problem 1)

क्रान्यक्षघाते तत्कोटयोः घातात् याम्येऽधिके सति।
नेष्टः शङ्कुर्भवेत् सौम्ये हाराद्यापक्रमेऽधिके॥ ६७ ॥

In the southern hemisphere, when the product of the *krānti* and the *akṣa* is greater than the product of the *koṭis*, there is no *śaṅku* [i.e. no solution for z with $z < 90°$]. Similarly, in the nothern hemisphere, when the *apakrama* is greater than the divisor, there is no *śaṅku*.

Here, the problem is to obtain the zenith distance (*śaṅku*) and hour angle (*nata*) in terms of declination (*krānti*), latitude (*akṣa*) and amplitude (*āśāgrā*), that is, z and H are to be determined in terms of δ, ϕ and a. It is to be understood that the amplitude in Indian astronomy is always less than 90° and is measured towards either the north or the south from the prime vertical.

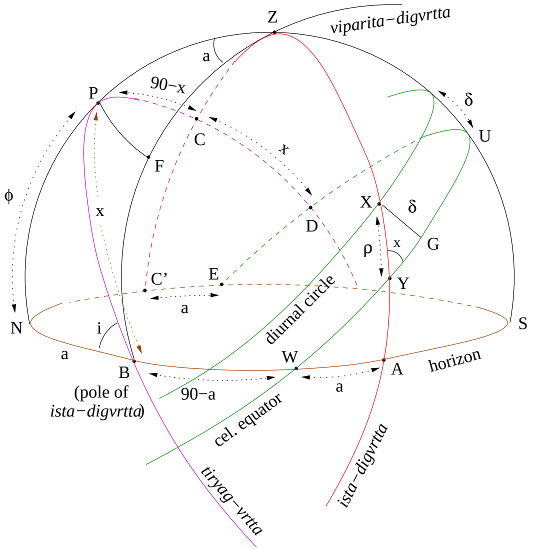
Zenith distance and Hour angle

These verses imply the following expressions for the *śaṅku*

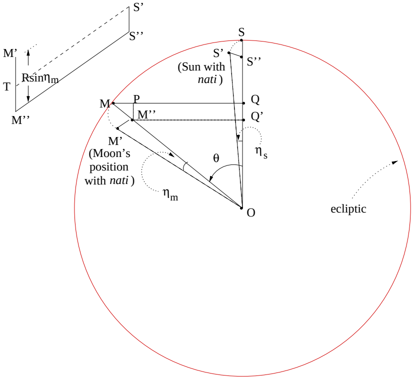
$$\cos z = \frac{(\sin\phi\sin\delta \pm \cos\phi\sin a\,\sqrt{\sin^2\phi + \cos^2\phi\sin^2 a - \sin^2\delta})}{(\sin^2\phi + \cos^2\phi\sin^2 a)}$$

$$\sin H = \frac{\sin z\cos a}{\cos\delta},$$

The important circles and their secondaries considered for the 'ten problems' in *Yuktibhāṣā*



Distance between centers of solar and lunar disks



Distance of separation between the centers of the solar and lunar disk.

$$d^2 = (S'M')^2 = (S''Q')^2 + (M''Q')^2 + (M'M'' \pm S'S'')^2,$$

where the various quantities depend upon the difference in longitudes of the Sun and Moon, their parallaxes and Moon's latitude.

This is as in three dimensional co-ordinate geometry.

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Thanks!

Thank You