NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 9

Āryabhaṭīya of Āryabhaṭa - Part 3

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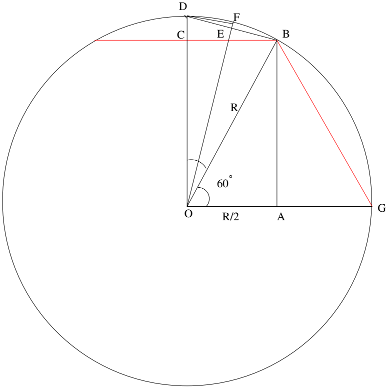
Outline

Āryabhaṭīya of Āryabhaṭa – Part 3

- Finding tabular Rsines
  - Geometrical approach (summary)
  - Analytic approach (that has no parallel)
- Problems related to gnomonic shadow
  - Application in astronomy
  - Finding the height of a lamp-post
- *Bhujā-koṭi-karṇa-nyāya* & *Jyā-śara-saṃvarga-nyāya*
  - The hawk-rat problem
  - The bamboo and the lotus problem
  - The fish-cane problem
- Dealing with arithmetic progressions
- Finding sum of a series, Sum of sums, and so on

Finding tabular sines: Geometrical approach (contd.)



- In the triangle CBD, $BC = R \sin 30^\circ$ and $CD = OD - OC = R$ vers30° are known. Hence, $BD =$ chord 30°

$$BD = \sqrt{(R\sin 30)^2 + (R \text{ vers} 30)^2},$$

is known. $R \sin 15^\circ = \frac{1}{2}BD = 890.$

- At this stage, we need to note that

$$R\sin\theta \rightsquigarrow R\cos\theta \rightsquigarrow R \text{ vers } \theta$$

$$R\sin\theta \;\&\; R \text{ vers } \theta \rightsquigarrow R\sin\frac{\theta}{2}$$

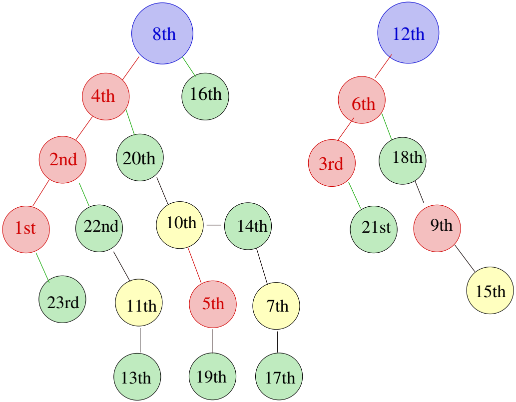
- Now considering the triangle ODE,

$$\begin{aligned} OE &= \sqrt{OD^2 - DE^2} \\ &= \sqrt{R^2 - (R\sin 15)^2} \end{aligned}$$

gives $R \sin 75^\circ$.

Finding tabular sines: Geometrical approach (contd.)

- Most of the Indian astronomers have presented their sine tables by dividing the quadrant (90°) into 24 parts.
- By the principle outlined above, it can be easily shown that all the 24 Rsines can be obtained provided the 24th, 12th and 8th Rsines are known.



- The circumference of the circle was taken by Āryabhaṭa to be 21600 units.
- From that using the approximation for π given by him, we get $R = 24th$ Rsine ≈ 3438.
- Once this it known, it is noteworthy that in the proposed scheme of constructing the table, all that is required is extraction of square root, for which Āryabhaṭa had clearly evolved an efficient algorithm.

Finding tabular sines: Analytic approach

Āryabhaṭa's recursion relation for the construction of tabular Rsines

प्रथमात् चापज्यार्धात् यैरूनं खण्डितं द्वितीयार्धम्।
तत्प्रथमज्यार्धांशैः तैस्तैरूनानि शेषाणि॥

- This is one of the most terse verses in *Āryabhaṭīya* and its content may be expressed as:

$$R\sin(i+1)\theta - R\sin i\theta = R\sin i\theta - R\sin(i-1)\theta - \frac{R\sin i\theta}{R\sin\theta}.$$

- In fact, the values of the 24 *Rsines* themselves are explicitly noted in another verse.
- The exact recursion relation for the Rsine-differences is:

$$R\sin(i+1)\theta - R\sin i\theta = R\sin i\theta - R\sin(i-1)\theta - R\sin i\theta\ \ 2(1-\cos\theta).$$

- Approximation used by Āryabhaṭa is $2(1-\cos\theta) = \frac{1}{225}$.
- While, $2(1-\cos\theta) = 0.0042822$, $\frac{1}{225} = 0.00444444$.

Comment on Āryabhaṭa's Method (Delambre)

Commenting upon the method of Āryabhaṭa in his monumental work Delambre[1] observes:

> "The method is curious: it indicates a method of calculating the table of sines by means of their second differences. . . . The differential process has not up to now been employed except by Briggs, who himself did not know that the constant factor was the square of the chord . . . Here then is a method which the Indians possessed and which is found neither amongst the Greeks nor amongst the Arabs."[2]

[1] ". . . an astronomer of wisdom and fortitude, able to review 130 years of astronomical observations, assess their inadequacies, and extract their value." – Prix prize citation 1789.

[2] Delambre, *Historie de l'Astronomie Ancienne*, t 1, Paris 1817, p.457; cited from B. Datta and A. N. Singh, Hindu Trigonometry, IJHS 18, 1983, p.77.

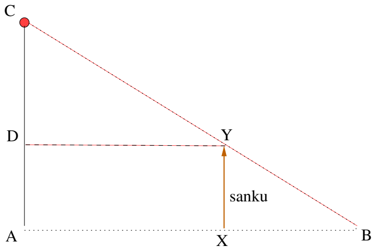
Problems related to gnomonic shadow

शङ्कुगुणं शङ्कुभुजाविवरं शङ्कुभुजयोर्विशेषहृतम्।
यल्लब्धं सा छाया ज्ञेया शङ्कोः स्वमूलाद्धि॥ १५ ॥

The [height of the] gnomon multiplied by the distance between the gnomon and the lamp-post [the latter being referred to as *bhujā*[3]] is to be divided by the difference between the lamp-post and the gnomon. The quotient (thus obtained) should be known as the length of the shadow measured from the foot of the gnomon.[4]

Triangles XYB and DCY are similar. Hence,

$$
\begin{aligned}
XB &= \frac{XY \times DY}{DC} \\
&= \frac{XY \times AX}{AC - XY}.
\end{aligned}
$$



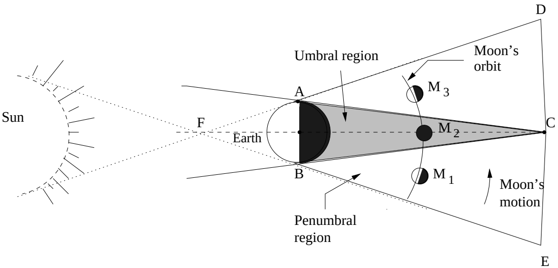
Such problems are very common in designing temple, wherein sunlight has to fall on shrine.

[3] Bhāskara's comm: भुजाशब्देन प्रदीपोच्छ्रायः उच्यते। What kind of lamps?

[4] This rule occurs also in *BrSpSi*, xii.53; *GSS*, *SiŚe*; *Lil.* *GK*, *II*.

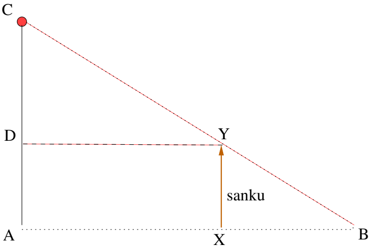
Problems related to gnomonic shadow

Application in astronomy: Case of a lunar eclipse



Mapping the above figure with the one below,

XY – semi-diameter of the earth (known)
AC – semi-diameter of the sun (known)
AX – distance of the sun (given)
XB – earth's shadow(?)



Triangles XYB and DCY are similar. Hence,

$$XB = \frac{XY \times DY}{DC} = \frac{XY \times AX}{AC - XY}.$$

Problems related to gnomonic shadow

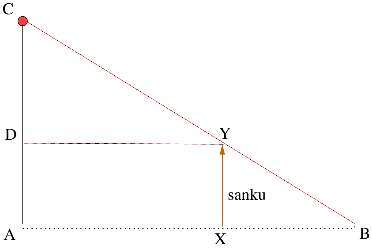
Obtaining the distance of the post from gnomon, when height of the post is known

छाया षोडश दृष्टा द्वासप्तत्युच्छ्रितस्य दीपस्य।
मूलं कियता शङ्कोः द्वादशकस्य त्वया वाच्यम्॥

The shadow [of the gnomon] cast by a lamp-post of height 72 was noted to be 16. You have to tell me how far is the base of the lamp-post from that of the gnomon of height 12.

Triangles XYB and DCY are similar. Hence,

$$\frac{XB}{DY} = \frac{XY}{DC}$$



- Since the heights of the the gnomon (XY) and lamp-post are given, DC is known.
- The length of the shadow (XB) is also given.
- Hence $DY = AX$ is known. (Ans. 80)

Bhāskara in his commentary presents several such examples, that are of practical relevance (estimating distances, etc.)

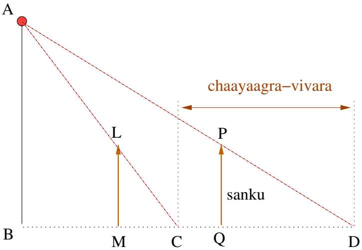
Problems related to gnomonic shadow

Rule for finding the height of a lamp-post

छायागुणितं छायाग्रविवरम् ऊनेन भाजितं कोटी।
शङ्कुगुणा कोटी सा छायाभक्ता भुजा भवति॥ १६ ॥

The distance between the tips of the shadows [of the two gnomons] is multiplied by the [larger or shorter] shadow and divided by the larger shadow diminished by the shorter one. The result is the upright (i.e., the distance of the tip of the larger or shorter shadow from the foot of the lamp-post). The upright multiplied by [the height of] the gnomon and divided by the (larger or shorter) shadow gives the base (i.e., the height of the lamp-post).

- By considering two pairs of similar triangles ABC and LMC, and ABD and PQD, Āryabhaṭa presents a rule by which the height of the lamp-post can be calculated.
- Bhāskara presents an interesting discussion on the propreity of the application of this rule to find the distance of separation between the sun and the earth.



Problems related to gnomonic shadow

Rule for finding the height of a lamp-post

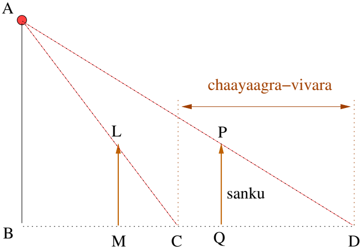
छायागुणितं छायाग्रविवरम् ऊनेन भाजितं कोटी।
शङ्कुगुणा कोटी सा छायाभक्ता भुजा भवति॥ १६ ॥

Triangles ABC and LMC are similar. Hence,

$$\frac{AB}{LM} = \frac{BC}{MC} \qquad (1)$$

Similarly triangles ABD and PQD are similar.

$$\frac{AB}{PQ} = \frac{BD}{QD} \qquad (2)$$



Since the *saṅkus* employed are same, $PQ = LM$. Therefore

$$\frac{BD}{QD} = \frac{BC}{MC} = \frac{CD}{QD - MC} \qquad \text{(using componendo-dividendo)} \qquad (3)$$

Using (3), (1) and (2), we get the expression for *koṭis* and hence the *bhujā*

$$BD = \frac{CD \times QD}{QD - MC}; \quad BC = \frac{CD \times MC}{QD - MC}; \quad \text{and} \quad AB = \frac{BD \times PQ}{QD} = \frac{BC \times LM}{MC}.$$

Problems related to gnomonic shadow

Impropriety in employing the technique to find the distance of the sun

विषुवदहनि गगनतल(मध्य)वर्तिनि सवितरि, समदक्षिणोत्तरछायाग्रान्तराल-
योजनैः, छायाविशेषेण शङ्कुना च केचित् विवस्वदवनितलान्तराल-
योजनान्यानयन्ति, तदयुक्तम्। अत्र प्रदीपच्छायाद्वयकर्मालापावतारोऽपि
नोपपद्यते। कुतः?
यस्मादाह 'भूरविविवरम् विभजेत्' (गोल ३९) इति।भूः शङ्कुः,
रवियोजनकर्णः शङ्कुभुजाविवरं सकलजगदेकप्रदीपो भगवान् भास्करः
स्वयमेव प्रदीपोच्छ्रायः ...

On the equinoctial day when the sun (savitr) is [in the middle] of the vault of the sky (i.e., on prime meridian), some people compute the *yojanas* that make up the distance between the sun and the surface of the earth ... This is improper. Even to resort to mentioning the method with a light and the shadows of two gnomons, does not stand to reasoning. Why? The [diameter of the] earth is the gnomon, the true distance (*karṇa*) in *yojanas* to the sun is the distance of separation between the gnomon and the base, and the sole illuminator of the entire world, the Lord Sun is itself the height of the light.

Bhujā-koṭi-karṇa-nyāya & *Jyā-śara-saṃvarga-nyāya*

Theorem on the square of hypotenuse & Theorem on the square of chords

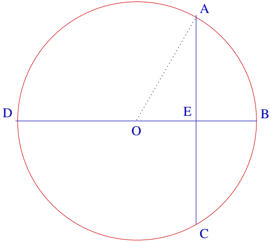
- Āryabhaṭa has presented both the theorems in a single *āryā*:

  यश्चैव भुजावर्गः कोटीवर्गश्च कर्णवर्गः सः ।
  वृत्ते शरसंवर्गः अर्धज्यावर्गः स खलु धनुषोः ॥

- The words *varga* and *samvarga* refer to square and product. Similarly, *dhanus* and *śara* refer to arc and arrow respectively.
- Using modern notations the *nyāyas* may be expressed as:

$$\begin{aligned} bhuj\bar{a}^2 + ko\d{t}\bar{\imath}^2 &= kar\d{n}a^2 \\ \text{side}^2 + \text{upright}^2 &= \text{hypotenuse}^2 \\ OE^2 + EA^2 &= OA^2 \end{aligned}$$

$$\begin{aligned} \text{product of } \acute{s}aras &= \text{Rsine}^2 \\ DE \times EB &= AE^2 \end{aligned}$$



Interesting examples (Hawk-rat problem)

अत्रैव श्येनमूषकोद्देशान् व्यावर्णयन्ति। तद्यथा –
अर्धज्या भुजा, अर्धज्यामण्डलकेन्द्रान्तरालं कोटिः, तद्वर्गयोगमूलं कर्णः
मण्डलव्यासार्धम्। तत्तु प्रदर्श्यते – इयमर्धज्या श्येनस्थानोच्छ्रायः ...

अष्टादशकोच्छ्राये श्येनः स्तम्भे ह्याखुः।
आवासन्निष्कान्तस्त्वेकाशीत्या भयात्छ्येयेनात्॥
गच्छन्नानालयदृष्टिः क्रूरेण निपातितस्ततो मार्गे।
कियता प्राप्नोति बिलं श्येनगतिर्वा तदा वाच्यम्॥

A falcon is [seated] on a pole whose height is eighteen. And the rat that has departed from his residence is at a distance of eighty one. Due to the fear of the falcon starts running [towards the hole at the base of the pole. As he is moving with his residence in his eyes, he is killed on the way by the cruel [falcon]. One should then state, by covering what distance would the rat reach the hole, and what distance is covered by the falcon?

The answers given by Bhāskara are: $38\frac{1}{2}$ & $42\frac{1}{2}$.

Interesting examples

The bamboo problem (*Ans: 10 - from top, 6 - from ground*)

षोडशहस्तो वंशः पवनेन निपातितः स्वमूलात्।
अष्टौ गत्वा पतितः कस्मिन् भग्नो मरुत्वतो वाच्यः॥

A bamboo of sixteen *hastas* was made to fall by the wind. It fell such that its tip hit the ground at eight [*hastas*] from its root. Where was it broken by the possesser (Lord) of the wind, is to be said.

The lotus problem (*Ans: 40 - lotus, 32 - water*)

कमलं जलात् प्रदृश्यं विकसितमष्टाङ्गुलं निवातेन।
नीतं मज्जति हस्ते शीघ्रं कमलाम्भसी वाच्ये॥

A full-bloomed lotus is noted to be of height eight *aṅgulas* above [the surface of] the water. Carried by the wind, it sinks in one hasta. Quickly, [the height of] the lotus and water are to be told.

Interesting examples (Fish-crane problem)

मत्स्यबकोद्देशकेष्वपि एवमेव आयतचतुरश्रक्षेत्रस्य एको बाहुः अर्धज्या, बाहुद्वयं महाशरः शेषं मूषकोद्देशवत् कर्म . . .

षड्द्वादशिका वापी तस्यां पूर्वोत्तरे स्थितो मत्स्यः ।
वायव्यकोणेस्याद् बकः स्थितस्तद्भयात् तूर्णम्॥
भित्वा वापीं मत्स्यः कर्णेन गतो दिशं ततो याम्याम्।
पार्श्वेनागत्य हतः बकेन वाच्यं तयोर्यातम्॥

A dimension of the tank is six and twelve. In its north-east [corner] lies the fish.— In the north-west corner stands a crane. The fish, by fear of that crane, quickly cuting through the tank diagonally went towards the southern direction. However, it was killed by the crane by travelling along the sides. The path traced by them should be stated.

The path traced by the crane or fish is $10 units$.

Dealing with arithmetic progressions

इष्टं व्येकं दलितं सपूर्वम् उत्तरगुणं समुखं मध्यम्।
इष्टगुणितम् इष्टधनं तु, अथवा आद्यन्तं पदार्धहतम्॥

The given number of terms is dimished by one, and then divided by two. This is increased by the number of the preceding terms (if any), and then multiplied by the common difference, and then increased by the first term of the (whole) series: the result is the arithmetic mean (of the given number of terms). This multiplied by the given number of terms is the sum of the given terms. Alternatively, by multiplying the sum of the first and last terms (of the series or partial series which is to be summed up) by half the number of terms.

It has been pointed out by Bhāskara that many formulae have been set out in this verse separately (*muktaka*). They have to be obtained by appropriate combination of words.

अत्र बहूनि सूत्राणि मुक्तकव्यवस्थितानि। तेषां यथायोगं सम्बन्धः।

Dealing with arithmetic progressions

Formulae to compute the mean value (*madhyadhana*) and sum (*gaccha*) of n terms

इष्टं व्येकं दलितं सपूर्वम् उत्तरगुणं समुखं मध्यम्।
इष्टगुणितम् इष्टधनं तु, अथवा आद्यन्तं पदार्धहतम्॥

- Formula 1:

  इष्टं व्येकं दलितं उत्तरगुणं समुखं – इति मध्यमधनानयनार्थम् सूत्रम्।
  The desired number of terms decreased by one, halved, multiplied by the common difference, and increased by the first term – is the formula for computing the mean value.

- Formula 2:

  मध्यमम् इष्टगुणितं इष्टधनम् – इति गच्छधनानयनार्थम्।
  The mean value multiplied by the desired number . . . – is [the formula] for computing the sum of n terms.

- Considering an arithmetic series of the form $a + (a + d) + (a + 2d) + \ldots$, the two formulae given may be expressed as:

$$\text{Mean} = a + \frac{n-1}{2}d; \qquad \text{Sum} = \left(a + \frac{n-1}{2}d\right)n$$

Thanks!

THANK YOU

More of *Āryabhaṭīya* in the next lecture!