NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 29

Magic Squares - Part 2

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Outline

- The *turaga-gati* method of generating magic squares
- Properties of 4×4 pan-diagonal magic squares
- Generating magic squares based on the above properties
- Constructing an 8×8 square (method due to Ṭhakkura Pherū)
- Nārāyaṇa's algorithm for constructing *Samagarbha* squares
- Illustrative examples
- Nārāyaṇa's folding method for 8×8 magic squares
- Nārāyaṇa's folding method for odd squares
- Modified algorithm for getting pan-diagonal squares
- References

Obtaining 4 × 4 PD squares: Horse-move method

चतुरङ्गतुरगगत्या द्वौ द्वौ श्रेढीसमुद्भवावङ्कौ।
न्यस्य क्रमोत्क्रमेण च कोष्ठैक्य-एकान्तरेण च॥ १० ॥
सव्यासव्यतुरङ्गमरीत्या कोष्ठान् प्रपूरयेदङ्कैः।
समगर्भे षोडशगृहभद्रे प्रोक्तो विधिश्चायम्॥ ११ ॥
तिर्यक्कोष्ठगतानां ऊर्ध्वस्थानां च कर्णगानां च।
अङ्कानां संयोगः पृथङ्मितो जायते तुल्यः॥ १२ ॥
इह समगर्भानामप्यन्येषां उद्भवश्चतुर्भद्रात्।

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

चतुरङ्गतुरगगत्या → like the movement of horse in chess
द्वौ द्वौ → choose pairs of numbers [from the sequence]
कोष्ठैक्य → in two adjacent cells
एकान्तरेण च → and at an interval of one cell
सव्यासव्यतुरङ्गमरीत्या → by the method of the horse moving to the left and right
षोडशगृहभद्रे → in a magic square with 16 cells
समगर्भानामप्यन्येषां → other magic squares of order $4m$

Obtaining 4 $\times$ 4 PD squares: Horse-move method

A few illustrative examples

Example 1: (1,2) same a previous example; 3 placed in *koṣṭhaikya*

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

- It may be noted here that for the first *yamalayugalas*, *turaga* motion is 'south-east' ($\searrow$).
- For the next it is 'south-west' ($\swarrow$)
- For the next it is 'north-west' ($\nwarrow$)
- For the next it is 'north-east' ($\nearrow$)

Example 2: (1,2,3,4) same a previous example; position of 5 swapped with 9.

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

- It may be noted here that for the first *yamalayugalas*, *turaga* motion is 'south-east' ($\searrow$).
- For the next it is 'north-west' ($\nwarrow$)
- For the next it is 'south-west' ($\swarrow$)
- For the next it is 'north-east' ($\nearrow$)

Note: Main-diagonals remain unchanged; Off-diagonal elements get swapped.

Obtaining 4×4 PD squares: Horse-move method

A few illustrative examples

Magic squares corresponding to previous examples:

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

Magic squares obtained by swapping the positions of 3, (5,9).

1	14	7	12
8	11	2	13
10	5	16	3
15	4	9	6

1	14	11	8
12	7	2	13
6	9	16	3
15	4	5	10

Note: It is seen that these squares are obtained by simply swapping the elements of II and IV columns, as expected.

Properties of 4×4 pan-diagonal magic squares

Property 1: Let M be a pan-diagonal 4×4 magic square with entries $1, 2, \ldots, 16$, which is mapped on to the torus by identifying opposite edges of the square. Then the entries of any 2x2 sub-square formed by consecutive rows and columns on the torus add up to 34.

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

$$1 + 12 + 15 + 6 = 1 + 12 + 14 + 7 = 34$$

Property 2: Let M be a 4×4 pan-diagonal magic square with entries $1, 2, \ldots, 16$, which is mapped on to the torus. Then, the sum of an entry on M with another which is two squares away from it along a diagonal (in the torus) is always 17.

$$1 + 16 = 6 + 11 = 15 + 2 = 4 + 13 = 14 + 3 = 9 + 8 = 17$$

Properties of 4×4 pan-diagonal magic squares

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

The "neighbours" of an element of a 4×4 pan-diagonal magic square (which is mapped on to the torus as before) are the elements which are next to it along any row or column. For example, $3, 5, 2$ and 9 are the "neighbours" of 16 in the magic square below.

Property 3: Let M be a 4×4 pan-diagonal magic square with entries $1, 2, \ldots, 16$, which is mapped on to the torus. Then the neighbours of the entry 16 have to be the entries $2, 3, 5$ and 9 in some order.

- We can use the above properties, and very easily construct 4×4 pan-diagonal magic squares starting with 1, placed in any desired cell.
- Let us work out some examples.

Samagarbha magic squares

Constructing an 8×8 square following the method given by Ṭhakkura Pherū

- This seems to be an old method for construction of *samagarbha* or $4n \times 4n$ magic square from a 4×4 magic square which is also described by Ṭhakkura Pherū and Nārāyaṇa. Consider 4×4 PD square:

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

- With this we construct an 8×8 as follows:

- Conceive the 8×8 to be made up of four 4×4 magic squares.
- The four *yamala-yugalāṅkas* are to be placed using *turagagati* as shown.

1				5			
		2				6	
4				8			
		3				7	
9				13			
		10				14	
12				16			
		11				15	

Samagarbha magic squares

- In the previous step we moved horizontally to the right and filled the cells.
- Now, we start with the quarter where we left and move horizontally to the left.
- The four *yamala-yugalāṅkas* are to be placed using *turagagati* as earlier.

1	32			5	28		
		2	31			6	27
4	29			8	25		
		3	30			7	26
9	24			13	20		
		10	23			14	19
12	21			16	17		
		11	22			15	18

Finally we arrive at the pan-diagonal 8×8 magic square

- Again we start with the quarter where we left and move horizontally to the right.
- One of the properties of an 8×8 pan-diagonal magic square seems to be that the sum of four alternating cells along any diagonal adds to half the magic sum.

1	32	61	36	5	28	57	40
62	35	2	31	58	39	6	27
4	29	64	33	8	25	60	37
63	34	3	30	59	38	7	26
9	24	53	44	13	20	49	48
54	43	10	23	50	47	14	19
12	21	56	41	16	17	52	45
55	42	11	22	51	46	15	18

Algorithm for constructing *Samagarbha* Magic Squares

The Folding Method or *Samputavidhi* given by Nārāyaṇa

समगर्भे द्वे कार्ये छादकसंज्ञं तयोर्भवेदेकम्।
छाद्याभिधानमन्यत् करसम्पुटवच्च सम्पुटो ज्ञेयः॥
इष्टादीष्टचयाङ्का भद्रमिता मूलपङ्क्तिसंज्ञाद्या।

- Consider two $n \times n$ squares where $n = 4m$.
- Of the two, one is called the coverer ($ch\bar{a}daka$)
- The other is called the covered ($ch\bar{a}dya$)
- The folding here is just like folding the palms.
- The first sequence known as $m\bar{u}lapa\dot{n}kti$ [has],
- any desired number as the first term ($i\d{s}\d{t}\bar{a}di$) and so too the common difference ($caya$) [and], the number of terms in the sequence is limited by the order of the magic square

तद्वदभीप्सितमुखचयपङ्क्तिश्च[1] अन्या पराख्या स्यात्॥

Similarly, (another) sequence having desired number as the first item and also as the common difference is known as the *parāpaṅkti*.

Given below are a few examples of *mūlapaṅkti* and *parāpaṅkti*.

mūlapaṅkti				*caya*	*parāpaṅkti*				*caya*
1	2	3	4	1	0	1	2	3	1
2	4	6	8	2	1	2	3	4	1
0	3	6	9	3	2	3	4	5	1
3	6	9	12	3	4	6	8	10	2
4	8	12	16	4	0	3	6	9	3

[1] The *vigraha* is: मुखञ्च चयश्च मुखचयौ। अभीप्सितौ मुखचयौ अभीप्सितमुखचयौ। तौ यस्याः पङ्क्तेः सा अभीप्सितमुखचयपङ्क्तिः।

Algorithm for constructing *Samagarbha* Magic Squares

The Folding Method or *Samputavidhi* given by Nārāyaṇa

मूलाख्यपङ्क्तियोगोनितं फलं परसमाससंभक्तम्।
लब्धहतापरपङ्क्तिः गुणजाख्या सा भवेत् पङ्क्तिः॥

- The result obtained by decreasing the sum of the *mūlapaṅti* [from the desired magic sum],
- when divided by the sum of the *parapaṅti* [is the *guṇa*].
- The elements of the *parapaṅti* multiplied by that *guṇa* obtained is known as the *guṇapaṅti*.

Example 1: Suppose the desired sum $\boxed{S = 40}$

- *Mūla-paṅti* – | 1 | 2 | 3 | 4 |. Its sum $s_m = 10$
- *Parā-paṅti* – | 0 | 1 | 2 | 3 |. Its sum $s_p = 6$. Now,

$$\frac{S - s_m}{s_p} = \frac{40 - 10}{6} = 5$$

- Using this we obtain *Guṇa-paṅti* – | 0 | 5 | 10 | 15 |.

Algorithm for constructing *Samagarbha* Magic Squares

The Folding Method or *Samputavidhi* given by Nārāyaṇa

मूलगुणाख्ये पङ्क्ती ये ते भद्रार्धतस्तु परिवृत्ते।

- The two sequences that are known as *mūlapaṅti* and *guṇapaṅti*,
- are to be arranged [in the cells] in a cyclic manner—clockwise or anti-clockwise[2]—from the centre of the magic square.

- *Mūla-paṅti* – | 1 | 2 | 3 | 4 |.
- *Guṇa-paṅti* – | 0 | 5 | 10 | 15 |.

2	3	2	3	clockwise
1	4	1	4	
3	2	3	2	anti-clockwise
4	1	4	1	

clockwise		anti-clockwise	
10	15	5	0
5	0	10	15
10	15	5	0
5	0	10	15

[2] specified so in the next few lines of the verses.

Algorithm for constructing *Samagarbha* Magic Squares

The Folding Method or *Samputavidhi* given by Nārāyaṇa

ऊर्ध्वस्थितैः तदङ्कैः छादकसञ्छाद्ययोः पृथग्यानि॥
तिर्यक्कोष्ठान्याद्ये अन्यतरस्मिन्नूर्ध्वगानि कोष्ठानि।
भद्रस्यार्धं क्रमगैः उत्क्रमगैः पूरयेदर्धम्।

- The horizontal blocks in the first and the vertical ones in the other
- Half of the magic square *bhadra* is to be filled in order
- And the other half in the reverse order

2	3	2	3	clockwise
1	4	1	4	
3	2	3	2	anti-clockwise
4	1	4	1	

clockwise		anti-clockwise	
10	15	5	0
5	0	10	15
10	15	5	0
5	0	10	15

Algorithm for constructing *Samagarbha* Magic Squares

The Folding Method or *Sampuṭavidhi* given by Nārāyaṇa

भद्राणामिह सम्पुटविधिः उक्तो नृहरितनयेन॥

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

$+$ (folded)

10	15	5	0
5	0	10	15
10	15	5	0
5	0	10	15

$=$

2	8	17	13
16	14	1	9
3	7	18	12
19	11	4	6

छाद्य-square छादक-square भद्र-square

Notable features:

- Apart from the rows, columns and the principal diagonals, the broken diagonals too add up to the magic sum (pan-diagonal).
- The 16 distinct quadruplets that can be considered also add up to the magic sum.

A few examples of 4 $\times$ 4 Magic Squares

Example 2: Suppose the desired sum $\boxed{S = 120}$

- *Mūla-paṅti* – $\boxed{2 \quad 4 \quad 6 \quad 8}$. Its sum $s_m = 20$
- *Parā-paṅti* – $\boxed{1 \quad 2 \quad 3 \quad 4}$. Its sum $s_p = 10$. Now,

$$\frac{S - s_m}{s_p} = \frac{120 - 20}{10} = 10$$

- Using this we obtain *Guṇa-paṅti* – $\boxed{10 \quad 20 \quad 30 \quad 40}$.

4	6	4	6
2	8	2	8
6	4	6	4
8	2	8	2

$+$ (folded)

30	40	20	10
20	10	30	40
30	40	20	10
20	10	30	40

=

14	26	44	36
42	38	12	28
16	24	46	34
48	32	18	22

छाद्य-square

छादक-square

भद्र-square

A few examples of 4×4 Magic Squares

Example 3: Suppose the desired sum $\boxed{S = 120}$ (with a different *mūla* and *parā-paṅti*).

- *Mūla-paṅti* – $\boxed{3 \quad 6 \quad 9 \quad 12}$. Its sum $s_m = 30$
- *Parā-paṅti* – $\boxed{0 \quad 3 \quad 6 \quad 9}$. Its sum $s_p = 18$. Now,

$$\frac{S - s_m}{s_p} = \frac{120 - 30}{18} = 5$$

- Using this we obtain *Guṇa-paṅti* – $\boxed{0 \quad 15 \quad 30 \quad 45}$.

6	9	6	9
3	12	3	12
9	6	9	6
12	3	12	3

$+$ (folded)

30	45	15	0
15	0	30	45
30	45	15	0
15	0	13	45

$=$

6	24	51	39
48	42	3	27
9	21	54	36
57	33	12	18

छाद्य-square

छादक-square

भद्र-square

A few examples of 4×4 Magic Squares

Example 4: Suppose the desired sum $\boxed{S = 128}$.

- *Mūla-paṅti* – | 3 | 6 | 9 | 12 |. Its sum $s_m = 30$
- *Parā-paṅti* – | 2 | 3 | 4 | 5 |. Its sum $s_p = 14$. Now,

$$\frac{S - s_m}{s_p} = \frac{128 - 30}{14} = 7$$

- Using this we obtain *Guṇa-paṅti* – | 14 | 21 | 28 | 35 |.

6	9	6	9
3	12	3	12
9	6	9	6
12	3	12	3

$+$ (folded)

28	35	21	14
21	14	28	35
28	35	21	14
21	14	28	35

=

20	30	41	37
38	40	17	33
23	27	44	34
47	31	26	24

छाद्य-square छादक-square भद्र-square

Nārāyaṇa's folding method for 8×8 magic square

Nārāyaṇa's Example: *Sum* = 260

Mūlapaṅkti: 1, 2, 3, 4, 5, 6, 7, 8, *Parapaṅkti*: 0.1.2.3.4.5.6, 7

Guṇa = $\frac{[260-(1+2+\ldots+8)]}{[0+1+2+\ldots+7]} = 8$ and *Guṇapaṅkti*: 0, 8, 16, 24, 32, 40, 48, 56

4	5	4	5	4	5	4	5
3	6	3	6	3	6	3	6
2	7	2	7	2	7	2	7
1	8	1	8	1	8	1	8
5	4	5	4	5	4	5	4
6	3	6	3	6	3	6	3
7	2	7	2	7	2	7	2
8	1	8	1	8	1	8	1

+

24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0

$\rightsquigarrow$

60	53	44	37	4	13	20	29
3	14	19	30	59	54	43	38
58	55	42	39	2	15	18	31
1	16	17	32	57	56	41	40
61	52	45	36	5	12	21	28
6	11	22	27	62	51	46	35
63	50	47	34	7	10	23	26
8	9	24	25	64	49	48	33

Nārāyaṇa's folding method for odd squares

पङ्क्ती मूलगुणाख्ये स्तः प्राग्वत्साध्ये तदादिमम्।
आदिमायामूर्ध्वपङ्क्तौ मध्यमे कोष्ठके लिखेत्॥
तदधः क्रमं पङ्क्त्याङ्कान् शिष्टाङ्कान् ऊर्ध्वतः क्रमात्।
द्वितीयाद्यास्तु तद्वच्च द्वितीयाद्यांश्च संलिखेत्॥
छाद्यच्छादकयोः प्राग्वद्विधिः संपुटने भवेत्।

Two sequences referred to as the *mūlapaṅkti* and the *guṇapaṅkti* are to be determined as earlier. The first number should be written in the middle cell of the top row and below this the numbers of the sequence in order. The rest of the numbers are to be entered in order from above. The first number of the second sequence is to be written in the same way [in the middle cell of the top row]; the second etc. numbers are also to be written in the same way. The rule of combining the covered and the coverer is also the same as before.

Nārāyaṇa's folding method for odd squares

Example : 5 $\times$ 5 square adding to 65

Mūlapaṅkti: $1, 2, 3, 4, 5$; *Parapaṅkti*: $0, 1, 2, 3, 4$

$$Guṇa = \frac{[65 - (1 + 2 + 3 + 4 + 5)]}{[0 + 1 + 2 + 3 + 4]} = 5;$$

Guṇapaṅkti: $0, 5, 10, 15, 20$

4	5	1	2	3
5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2

+

15	20	0	5	10
20	0	5	10	15
0	5	10	15	20
5	10	15	20	0
10	15	20	0	5

=

14	10	1	22	18
20	11	7	3	24
21	17	13	9	5
2	23	19	15	6
8	4	25	16	12

Nārāyaṇa's method happens to be an instance of combining two Mutually Orthogonal Latin Squares. However, it does not yield a pan-diagonal magic square as the diagonal elements of the squares are not all different.

Modification of Nārāyaṇa's *sampuṭa* for odd squares

Example : 5 × 5 square adding to 65

We may modify the above prescription and construct pan-diagonal magic squares for all orders $n \leq 5$ as follows.

Mūlapaṅkti: $1, 2, 3, 4, 5$; *Parapaṅkti*: $0, 1, 2, 3, 4$

$$Guṇa = \frac{[65 - (1 + 2 + 3 + 4 + 5)]}{[0 + 1 + 2 + 3 + 4]} = 5; \quad Guṇapaṅkti: 0, 5, 10, 15, 20$$

2	4	1	3	5
3	5	2	4	1
4	1	3	5	2
5	2	4	1	3
1	3	5	2	4

+

5	15	0	10	20
10	20	5	15	0
15	0	10	20	5
20	5	15	0	10
0	10	20	5	15

=

22	14	1	18	10
3	20	7	24	11
9	21	13	5	17
15	2	19	6	23
16	8	25	12	4

- The resulting square is clearly pan-diagonal.
- It may be noted that, in the *chādaka* all that was done is to start arranging the sequence based on *turagagati*, and not from diagonally below in the next column.

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Thanks!

THANK YOU