NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 22

$\boxed{\textit{Līlāvatī}\text{ of Bhāskarācārya 3}}$

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Outline

- Regular polygons inscribed in a circle
- Expression for a chord in a circle
- Excavations and contents of solids
- Shadow problems: Advanced problems
- Importance of rule of proportions
- Combinations: Advanced problems

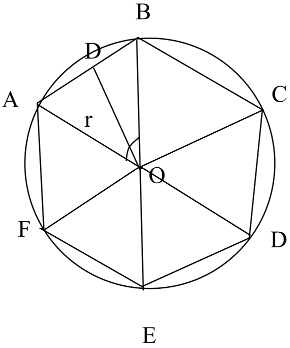
Sides of Polygons inscribed in a circle

Sides of Polygons inscribed in a circle in Verse 209-211 :

त्रिद्व्यङ्काग्निनभश्चन्द्रैः त्रिबाणाष्टयुगाष्टभिः।
वेदाग्निपञ्चखाश्वैश्च खखाभ्राभ्ररसैः क्रमात्॥ २०९ ॥
शैलर्तुनखबाणैश्च द्विद्विनन्देषुसागरैः।
त्रिवेददशवेदैश्च वृत्तव्यासे समाहते॥ २१० ॥
खखखाभ्रार्कसंभक्ते लभ्यन्ते क्रमशो भुजाः।
वृत्ततत्र्यश्रपूर्वाणां नवान्तानां पृथक् पृथक्॥ २११ ॥

"By 103923, 84853, 70534, 60000, 52067, 45922 and 41043 multiply the diameter of a circle, and divide the respective products by 120000; the quotients are severally, in their order, the sides of polygons from the triangle to nonagon (inscribed) within the circle."

Side of a regular polygon with n sides



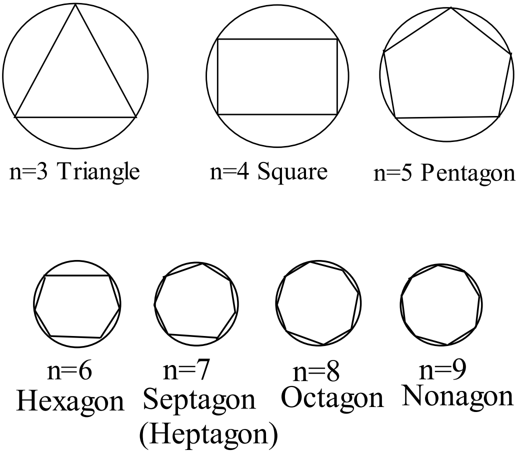
Side of a regular polygon with n sides

It is easily seen that interior angle $A\hat{O}B$ subtended at the centre O by side AB is $A\hat{O}B = \dfrac{2\pi}{n}$. Hence, Half angle $A\hat{O}D = \frac{\pi}{n}$.

$$\therefore \text{ AB } = \text{ Side of a n-sided polygon= 2AD } = 2r\sin\left(\frac{\pi}{n}\right).$$

Side of a regular polygon with *n* sides

$\therefore$ Side of a regular n -sided polygon inscribed in a circle is $=$ (Diameter) $\times$ $sin\left(\frac{\pi}{n}\right)$.



n-sided regular polygons inscribed in a circle

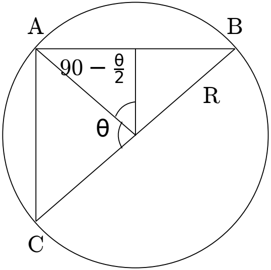
Value of sides for different *n*

Bhāskara gives the values of the sides for $n = 3, 4, 5, 6, 7, 8, 9$.

	n=3	n=4	n=5	n=6	n=7	n=8	n=9
$\frac{\text{Side}}{\text{Diameter}} =$	$\frac{103923}{120000}$	$\frac{84853}{120000}$	$\frac{70534}{120000}$	$\frac{60000}{120000}$	$\frac{52067}{120000}$	$\frac{45922}{120000}$	$\frac{41043}{120000}$
Bhāskara's Value =	0.8660254	0.7071083	0.5877834	0.5000	0.4337916	0.3826383	0.341925
	$\downarrow$	$\downarrow$	$\downarrow$				
Modern value	0.866025	0.7071067	0.5877853	0.500	0.4338879	0.3826383	0.3420201

Expression for the chord in a circle

Expression for the chord in a circle in Verse 213:



Chord of a circle

चापोननिघ्नपरिधिः प्रथमाह्वयः स्यात्
पञ्चाहतः परिधिवर्गचतुर्थभागः।
आद्योनितेन खलु तेन भजेच्चतुर्घ्न-
व्यासाहतं प्रथममाप्तमिह ज्यका स्यात्॥ २१३॥

Chord in a circle

"The circumference less the arc being multiplied by the arc, the product is termed first. From the quarter of the square of the circumference multiplied by five, subtract that first product, and by the remainder divide the first product taken into four times the diameter. The quotient will be the chord."

$$\text{Given : Chord AB} = \frac{4 \times d \times [\text{Circumference - Arc AB }][\text{Arc AB }]}{\frac{5}{4}\text{Circumference}^2 - (\text{Circumference - Arc AB})(\text{Arc AB})}$$

$$\text{Now, Arc AB + Arc CA} = \frac{\text{Circumference}}{2} = \frac{C}{2}$$

.

Chord in a circle

$$\therefore \text{ Arc AB } = \left(\frac{C}{2} - \text{Arc CA}\right)$$

$$\text{Circumference - Arc AB } = C - \left(\frac{C}{2} - \text{Arc CA}\right) = \frac{C}{2} + \text{Arc CA}.$$

$$\begin{aligned}\therefore \text{ (Circumference - Arc AB)(Arc AB)} &= \left(\frac{C}{2} - \text{Arc CA}\right)\left(\frac{C}{2} + \text{Arc CA}\right) \\ &= \left(\frac{C}{2}\right)^2 - (\text{Arc CA})^2\end{aligned}$$

$$\therefore \text{ Chord AB } = \frac{8r\left[\left(\frac{1}{2}C^2\right) - (\text{Arc CA})^2\right]}{\frac{5}{4}C^2 - \left\{\left(\frac{1}{2}C^2\right)^2 - (\text{Arc CA})^2\right\}}.$$

$$\text{Arc CA } = R\theta. \qquad \frac{C}{2} = \pi R.$$

$$\therefore \text{ Chord AB} = 8R\frac{(\pi^2 - \theta^2)}{(4\pi^2 + \theta^2)} = 2R\frac{(4\pi^2 - 4\theta^2)}{(4\pi^2 + \theta^2)}$$

Chord in a circle

$$\text{Now, Chord AB } = 2R\sin\left(\frac{\pi-\theta}{2}\right) = 2R\cos\left(\frac{\theta}{2}\right)$$

So this amounts to :

$$\cos\left(\frac{\theta}{2}\right) = \frac{\left(4\pi^2-4\theta^2\right)}{\left(4\pi^2+\theta^2\right)} = \frac{\left(\pi^2-\theta^2\right)}{\left(\pi^2+\theta^2/4\right)}$$

$$\text{or } \cos\theta = \frac{\left(\pi^2-4\theta^2\right)}{\left(\pi^2+\theta^2\right)}$$

$$\therefore\ \sin\theta = \cos\left(\frac{\pi}{2}-\theta\right) = \frac{\pi^2-4\left(\frac{\pi}{2}-\theta\right)^2}{\pi^2+\left(\frac{\pi}{2}-\theta\right)^2} = \frac{4(\pi-\theta)\theta}{\frac{5}{4}\pi^2-(\pi-\theta)\theta}$$

This is the same as the remarkable expression for $\sin\theta$ given by Bhāskara - I in 7th century CE.

Excavation and contents of solids

Chapter 7. Excavation and contents of solids.

Find average length, average breadth and average depth.

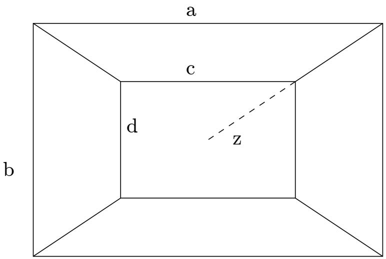
Volume $=$ Average length $\times$ Average breadth $\times$ Average depth.

The rules for finding the Volume of objects which do not have constant cross-section are given in Verse 211, which follows:

मुखज-तलज-तद्युतिज-क्षेत्रफलैक्यं हृतं षड्भिः।
क्षेत्रफलं सममेवं वेधगुणं घनफलं स्पष्टम्॥
समखातफलत्र्यंशः सूचीखाते फलं भवति ॥ २२१॥

The Volume of a Frustrum or tank

"The aggregate of the areas at the top and bottom and of that resulting from the sum (of the sides of the summit and base), being divided by six, the quotient is the mean area; that multiplied by the depth in the neat content (exact Volume). A third part of the content of the regular equal solid is the content of the actual one."



Volume of the tank (frustrum)

Volume of a tank

Suppose a tank is excavated with sides a and b at the top and c and d at the bottom and the depth is z.

$$\text{Volume of the tank } = \frac{1}{6}z[ab + cd + (a+b)(c+d)]$$

$$\text{Volume of a pyramid / cone } = \frac{1}{3} \times \text{(Area at base) } \times \text{Height}$$

Chapter 8. Stacks Straightforward.

$$\text{Area} \times \text{Height } = \text{Volume; Number of bricks } = \frac{\text{Volume}}{\text{Volume of one brick}}$$

$$\text{Number of layers } = \frac{\text{Height of stalk}}{\text{Height of one brick}}$$

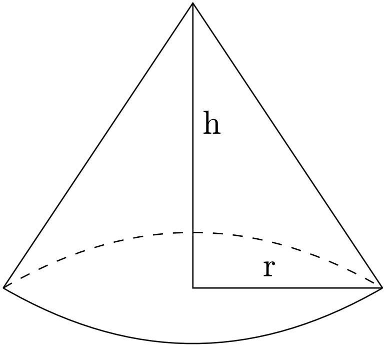
Sawing

Chapter 9 on sawing. Volume with a trapezoidal cross-section. Suppose the cross-section is a trapezium with base, b summit, S and height (perpendicular), p.

$$\therefore \text{ Area } = \frac{1}{2}(b+s) \times p$$

Total sawing in area $=$ (Area of the section) $\times$ Number of sections. Carpenter's wages are calculated based on the above.

Calculations pertaining to Mounds of grain



A mound of grain

$$\text{The Volume } = \frac{1}{3}\pi r^2 h = \frac{(2\pi r)^2}{3 \times 4\pi} \times h = (\text{Circumference})^2 \frac{h}{12\pi}$$

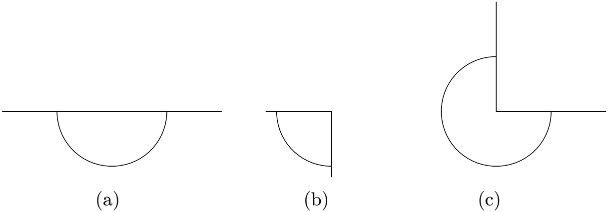
Mounds of grain

It appears that Bhāskara gives an approximate expression for the volume by taking $\pi \approx 3$, in which case:

$$\text{Volume} = \left(\frac{\text{Circumference}}{6}\right)^2 h$$

which is what is stated.
In his commentary, Gaṇeśa Daivajña states that this is a rough calculation, in which the diameter is taken at one-third of the circumference. Greater precision by taking a more nearly correct proportion between the circumference and diameter.

Different configurations



Mounds of grain piled against (a) the side of a wall (b) the inside corner (c) or outside corner.

In the above cases, given circumference in cases (a), (b) and (c) mean the relevant portion of the circumference in each case. Then the volumes of the mounds of grain piled against (a) the side of a wall (b) the circumference and (c) outside corner are stated to be:

Volume for different configurations

(a)

$$\text{Volume} = \left(\frac{\text{Circumference of circle}}{6}\right)^2 \times \frac{h}{2} = \left(\frac{\text{Given Circumference} \times 2}{h}\right)^2 \times \frac{h}{2}$$

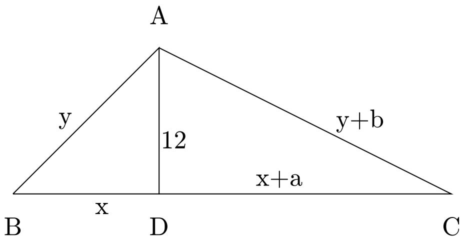
(b)

$$\text{Volume} = \left(\frac{\text{Circumference of circle}}{6}\right)^2 \times \frac{h}{4} = \left(\frac{\text{Given Circumference} \times 4}{6}\right)^2 \times \frac{h}{4}$$

(c)

$$\text{Volume} = \left(\frac{\text{Circumference of circle}}{6}\right)^2 \times \frac{h}{(4/3)} = \left(\frac{\text{Circumference} \times 4/3}{6}\right)^2 \times \frac{h}{4/3}$$

Problems related to the shadow of a gnomon, discussed by Bhāskara



Shadows and hypotenuse of a gnomon at two different times

In the figure, the shadows at two different times are x and $x+a$, and the hypotenuse at these times are y and $y+b$. The aim is to find the shadows, x and $x+a$, given the difference in shadows a and the difference in hypotenuses, b. The solution is given in Verse 238.

Problems related to shadows

छाययोः कर्णयोरन्तरे ये तयोः वर्गविश्लेषभक्ता रसाद्रीषवः।
सैकलब्धेः पदघ्नं तु कर्णान्तरं भान्तरेणोनयुक्तं दले स्तः प्रभे॥

”The number five hundred and seventy-six being divided by the difference of the squares of the differences of both shadows and of the two hypotenuses, and the quotient being added to one, the difference of the hypotenuses is mulitiplied by the square root of that sum; and the product being added to, and subtracted from, the difference of the shadows, the halves of the sum and differences are the shadows.”

Shadows

In the figure, $y^2 - x^2 = (y+b)^2 - (x+a)^2 = 144$.

$$\therefore\ by = ax + \frac{a^2 - b^2}{2}$$

$$\therefore\ b^2y^2 = a^2x^2 + \left(\frac{a^2 - b^2}{2}\right)^2 + ax(a^2 - b^2)$$

$$\therefore\ b^2(x^2 + 144) = a^2x^2 + \frac{(a^2 - b^2)^2}{4} + ax(a^2 - b^2)$$

$$\therefore\ (a^2 - b^2)x^2 + ax(a^2 - b^2) + \frac{(a^2 - b^2)^2}{4} - 144b^2 = 0$$

Shadows: Solution

Dividing by $a^2 - b^2$,

$$x^2 + ax + \left(\frac{a^2 - b^2}{4} - \frac{144b^2}{a^2 - b^2} \right) = 0$$

$$\therefore\ x = \frac{1}{2} \left\{ -a + \sqrt{a^2 - \left[a^2 - b^2 - \frac{576b^2}{a^2 - b^2} \right]} \right\}$$

$$x = \frac{1}{2} \left\{ -a + b\sqrt{1 + \frac{576}{a^2 - b^2}} \right\}$$

and

$$x + a = \frac{1}{2} \left\{ a + b\sqrt{1 + \frac{576}{a^2 - b^2}} \right\}$$

These are the expressions for the shadows in Verse 238.

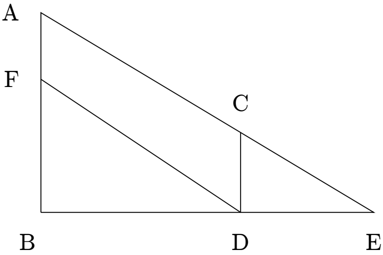
Standard results on Shadows in *Līlāvatī*

Verse 240.

शङ्कुः प्रदीपतलशङ्कुतलान्तरघ्नः।
छाया भवेद्विनरदीपशिखौच्चभक्तः॥ २४० ॥

"The gnomon multiplied by the distance of its foot from the foot of the light, and divided by the height of the torch's flame less the gnomon, will be the shadow."
In Fig. 24, AB is the height of the flame, CD is the height of the gnomon and *BD* is the distance between the foot of light and the gnomon.

Lamp, Gnomon and the shadow



Shadow (DE) of a gnomon (CD) due to light at A

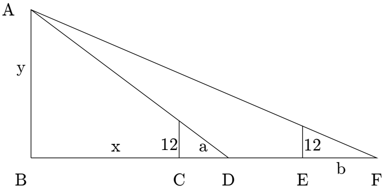
As the triangles DEC and BDF are similar

$$\frac{DE}{DC} = \frac{BD}{BF}$$

$$\text{or,} \qquad \text{Shadow}\,, DE = \frac{CD \cdot BD}{BF} = \frac{CD \cdot BD}{(AB - CD)}$$

which is the stated expression for the shadow.

Shadow at two locations



Shadows of gnomon at two locations due to a source of light

In Fig. 25, $a = CD$ and $b = EF$ are the shadows of gnomons of height 12 at the two different locations at C and E. due to the source of light at A. $y = AB =$ Elevation of the torch's flame. $c = CE =$ Distance between the gnomons. $DF =$ Distance between the shadow tips. $DF = EF + DE = EF + CE - CD = b + c - a$. Base, $BD = x + a$. From the similar triangles it is clear that

$$\frac{y}{x+a} = \frac{12}{a}, \quad \frac{y}{x+b+c} = \frac{12}{b}$$

$$\text{Hence, } \frac{x+b+c}{x+a} = \frac{b}{a}$$

Shadow at two locations

$$\therefore a(x+b+c) = b(x+a)$$

$$\text{Hence}\, x = \frac{ac}{b-a} \,\text{and}\, x+a = \frac{a(b+c-a)}{b-a})$$

$$\text{or } Base = \frac{\text{Shadow } \times \text{ (Distance between the tips of the shadows)}}{\text{Difference between the shadows}}$$

and

$$y = \frac{12(x+a)}{a}$$

$$\therefore \text{ Elevation of torch's flame} = \frac{\text{Base } \times \text{ Gnomon}}{\text{Shadow}}$$

These are stated in Verse 245.

Importance of rule of three

In arriving at all these results, similarity of triangles, which is essentially ‘the rule of three’ is used. *Bhāskara* stresses the all-pervading nature of the ‘rule of three’ at the end of the chapter:“As the Being, who relieves the minds of his worshippers from suffering and who is the sole cause of the production of this universe, pervades the whole, and does so with his various manifestations as worlds, paradises, mountains, rivers, gods, demons, men, trees and cities; so is all this collection of instructions for computations pervaded by the rule of three terms” Further, in Verse 247 :

यत्किञ्चिद् गुणभागहारविधिना बीजेऽत्र वा गण्यते
तत् त्रैराशिकमेव निर्मलधियामेवाऽवगम्या भिदा।
एतद्यद् बहुधाऽस्मदादिजड्धीबुद्धिप्रवृद्ध्यै बुधैः
तद्भेदानुगमान् विधाय रचितं प्राज्ञैः प्रकीर्णादिकम्॥

Rule of three, *kuṭṭaka* etc.

"Whatever is computed either in algebra or this (arithmetic) by means of a multiplier and a divisor, may be comprehended by the sagacious learned as the rule of three terms. Yet has it been composed by wise instructors in miscellaneous and other manifold rules, teaching its easy variations, thinking thereby to increase the intelligence of such full comprehensions as ours."

Chapter 12 on *Kuṭṭaka* (Pulverizer) is not discussed here, as it has already been discussed in the context of Āryabhaṭa's *Āryabhaṭīya*, Brahmagupta's *Brāhmasphuṭasiddhānta* and Mahāvīra's *Gaṇitasārasaṅgraha*.

Results in chapter 13 on combination of digits

Let a number be composed of n digits: $d_1, \cdots d_n$. Verse 267 tells us the sum of all the numbers which have these digits including all the possible permutations.

> स्थानाङ्क एकादिचयाङ्कघातः सङ्ख्याविभेदा हि स एव घातः।
> स्थनाङ्कभक्तोऽङ्कसमासनिघ्नस्थानेषु युक्तो मितिसंयुतिः स्यात्॥
>
> “The product of multiplication of the arithmetical series beginning and increasing by unity and continued to the number of places will be the variations of the number with specific figures; that divided by the number of digits and multiplied by the sum of the digits; being repeated in the places of figures and added together, will be the sum of the permutations.”

Combinations of digits

Let the digits be $d_1, \cdots d_n$.

No. of ways of arranging $= 1 \cdot 2 \cdots n = n!$.

Sum of the numbers $d_1 \cdots d_n$ permuted in all ways
$= \dfrac{n!}{n}(d_1 + \cdots + d_n)(1 + \cdots + 10^{n-1})$.

Proof:

In $(n-1)!$ cases d_1 is in units palce, in $(n-1)!$ cases it is in 10's place, $\cdots$, in $(n-1)!$ cases d_1 is in 10^{n-1} place.

Thus the sum arising out of d_1 alone is
$(n-1)!d_1(1 + \cdots + 10^{n-1}) = \dfrac{n!}{n}d_1(1 + \cdots + 10^{n-1})$.

Similarly for other digits. Hence,

$$\text{Sum } = \frac{n!}{n}(d_1 + \cdots + d_n)\underbrace{(1 + \cdots + 10^{n-1})}_{\underbrace{111\cdots 1}_{\text{n digits}}}$$

Example

Example in Verse 278. How many variations of number can be there with the eight digits, $2, \cdots, 9$. Tell promptly the sum of these numbers.

Here $n = 8$. $d_1 + \cdots + d_n = 2 + \cdots + 9 = 44$.

No. of ways permutations $n! = 8! = 40320$.

Hence

$$\text{Sum} = \frac{n!}{n}(d_1 + \cdots + d_n)\underbrace{(11\cdots 1)}_{\text{n digits}} = \frac{40320}{8} \times 44 \times (111\cdots 1) = 2463999975360.$$

Śambhu, Hari and combinatorics

The following problem in Verse 269 involves only the permutations and not the sum of the numbers arising out of the permutations.

पाशाङ्कुशाहिडमरूककपालशूलैः
खट्वाङ्गशक्तिशरचापयुतैर्भवन्ति ।

अन्योन्यहस्तकलितैः कति मूर्तिभेदाः
शम्भोर्हरेरिव गदारिसरोजशङ्खैः ॥ २६९ ॥

"How many of the variations of the form of the god Śambhu by the exchange of his ten attributes held reciprocally in his several hands, namely, the rope, the elephant's hook, the serpent, the tabor, the skull, the trident, the bedstead, the dagger, the arrow, and the bow, as those of Hari by the exchange of the mace, the discuss, the lotus and the conch?"

Śambhu: 10! Hari: 4! =24.

When the members of subsets are alike

The next verse (270) gives the number of permutations when the various subsets of the given set are alike. Let there be n digits. p of them are d_1, q of them are d_2, r of them are $d_3 \cdots$ etc:

यावत्स्थानेषु तुल्याङ्काः तद्भवैस्तु पृथक्कृतैः।
प्राग्भेदा विहृतावेताः तत्सङ्ख्यैक्यं च पूर्ववत्॥ २७० ॥

"The permutations found as before, being divided by the permutations separately computed for as many places as are filled by like digits, will be the variations of the number, from which the sum of the numbers will be found as before."

Permutations of sum of numbers

$$\text{Variations (Permutations)} = \frac{n!}{p!q!r!\cdots}.$$

[When p digits which are all d_1 are permutated among themselves, it does not give rise to any new variation, but the total number of permutations($n!$) takes these into account, hence $n!$ has to be divided by $p!$. Similarly, $q!, r!, \cdots$ etc., should also come as divisors.]

Sum of the numbers is found as before.

$$\text{Sum} = \frac{n!}{np!q!\cdots} \times (\text{Sum of digits}) \underbrace{(1 + \cdots + 10^{n-1})}_{\underbrace{11\cdots1}_{\text{n digits}}}$$

Sum of all permuted numbers

Check: Number of cases in which d_1 is in units or hundreds etc., place is

$$\frac{(n-1)!}{(p-1)!q!\cdots}$$

($\because$ now, after fixing one d_1 in a place, the permutations of the rest of the $p-1, d_1$'s will not give rise to a new variation. Hence, the factor $(p-1)!$ in the denominator).

$$\therefore \text{ Sum arising from } d_1 \text{ alone is } \frac{(n-1)!d_1}{(p-1)!q!\cdots}(1+\cdots+10^{n-1}).$$

$$\text{Similarly, Sum arising from } d_2 \text{ alone is } \frac{(n-1)!d_2}{p!(q-1)!\cdots}(1+\cdots+10^{n-1})$$

$$\text{Hence, Sum of all numbers } = (1+\cdots+10^{n-1})\left[\frac{d_1}{(p-1)!q!\cdots}+\frac{d_2}{p!(q-1)!\cdots}+\cdots\right]$$

$$= \frac{n!}{np!q!\cdots}(pd_1+qd_2+\cdots)(1+10^1+\cdots+10^{n-1}).$$

Example

Example in Verse 271:

द्विद्व्येकभूपरिमितैः कति सङ्ख्यकाः स्युः
तेषां युतिं च गणकाऽऽशु मम प्रचक्ष्व।
अम्भोधिकुम्भिशरभूतशरैस्तथाङ्कैः
छेदाङ्कपाशयुतिजातमथ प्रचक्ष्व॥ २७१॥

”How many are the numbers with 2, 2, 1 and 1? and tell me quickly, mathematician their sum; also with 4, 8, 5, 5, and 5 if thou be conversant with the rule of permutation of numbers.”

Solution

Solution:

1)

$$2,2,1,1. \quad \text{No. of ways } = \frac{4!}{2!2!} = 6.$$

$$\text{Sum of all numbers } = \frac{4!}{4 \cdot 2!2!}(2\times 2+2\times 1) = \frac{6}{4}(6)\times 1111 = 9999.$$

2) 4, 8, 5, 5, 5. No. of ways $= \dfrac{5!}{3!} = 4 \times 5 = 20.$.

$$\text{Sum of all numbers } = \frac{20}{5}(5 \times 3 + 4 + 8)(11111) = 1199988.$$

Choice of r out of n

Verse 272. Permutation of n things taken r at a time

$$= n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

Example in Verse 273 :

स्थानषट्कस्थितैरङ्कैः अन्योन्यं खेनवर्जितैः।
कति सङ्ख्याविभेदाः स्युः यदि वेत्सि निगद्यताम्॥ २७३ ॥

“How many are the variations of a number with any digits, except zero, exchanged in 6 places of figures? If thou know, declare them” .

Here, $n = 9, r = 6$. $\quad \therefore$ No. of ways $= \dfrac{9!}{3!} = 60480.$

Number of permutation when sum of digits is fixed

Verse 274 discusses the following problem:

Let there be n digits, the sum of which is $S = n + m$ with $m < 9$.(This restriction is put so that even if all the $(n - 1)$ of the digits are 1, remainder of the sum, $m + 1$, being not greater than 9, can form the remaining digit). The rule to discuss the number of possible permutations is stated in the verse 274, 275.

निरेकमङ्कैक्यमितं निरेकस्थानान्तमेकापचितं विभक्तम्।
रूपादिभिः सन्निहितैः समाः स्युः सङ्ख्याविभेदा नियतेऽङ्कयोगे॥
नवान्वितस्थानकसङ्ख्यकाया ऊनेऽङ्कयोगे कथितं तु वेद्यम्।
संक्षिप्तमुक्तं पृथुताभयेन नान्तोऽस्ति यस्माद् गणितार्णवस्य॥

Number of permutations

> "If the sum of the digits be determinate, the arithmetical series of numbers from one less than the sum of the digits, decreasing by unity and continued to one less than the places, being divided by one and so forth, and the quotient being multiplied together, the product will be equal to the variations of the number. This rule must be understood to hold good, provided the sum of the digits be less than the number of places added to nine. This has been stated briefly, for fear of prolixity, since the ocean of calculation has no bounds. "

No. of ways is stated as:

$$\frac{(S-1)(S-2)\cdots(S-\overline{n-1})}{1\cdot 2\cdots(n-1)}$$

$$=\frac{(n+m-1)(n+m-2)\cdots(n+m-1-\overline{n-2})}{(n-1)!}=\frac{(n+m-1)!}{(n-1)!\,m!}$$

Explanation

Explanation: The number, the sum of whose digits is $S = n + m$ can be considered as composed of n 1's are m 1's. $(n+m)$ 1's must be accommodated. Then n 1's are written as $1^n, 1^{n-1}, \cdots, 1^1$, the m 1's are written as $1_1, 1_2, \cdots, 1_m$.
So the number $x_n, \cdots x_1$ can be visualized as

$$\underbrace{(1^n 1_1 1_2 \cdots)}_{x_n} \; \underbrace{(1^{n-1} 1_3 \cdots)}_{x_{n-1}} \; \underbrace{1^1, \cdots}_{x_1}$$

Each of the digits has one 1^r and some 1_s coming from m 1's. While finding the number of ways, x_n has 1^n. This can be fixed. Then the number of ways is the one in which $(n+m-1)$ objects are permuted, in which $(n-1)$ objects are alike and m are alike separately. Therefore,

$$\text{No. of permutation } = \frac{(n+m-1)!}{(n-1)!m!} = \frac{(n+m-1)(n+m-2)\cdots(n+m-1-\overline{n-2})}{(n-1)!}$$

$$= \frac{(S-1)(S-2)\cdots(S-\overline{n-1})}{(n-1)!}$$

Another Explanation

Another way of looking at it. The 1^r (making up each digit x_i) can be represented by a partition] and m 1's can be represented by X.

X	XX	$\cdots \quad \cdots$	

(n-1) partitions and m objects:X

The i$^{\text{th}}$ box represents the digits x_i which has one 1^r and $(x_i - 1) : 1_t$, that is $x_i - 1$ crosses: X.

Hence, number of ways is the total number of ways in which $(n-1)$ parti- tions] and m objects X can be permuted together. In these, $n-1$ partitions are alike and m crosses are alike. Hence,

$$\text{Number of permutations } = \frac{(n-1+m)!}{(n-1)!m!}$$

which is the desired result.

This is essentially the way, counting is done in Bose-statistics where m identical objects are distributed among n energy levels. (Here of course there is no restriction on m).

Yet another explanation

There is another way of arriving at the same result. The desired number is actually the coefficient of x^S in the product

$$(x+\cdots x^9+\cdots)(x+\cdots)(x+\cdots)$$

Here we have n factors with all the powers of x figuring in each factor. The coefficient of x^S in the product

$$(x+x^2+\cdots x^9+\cdots)^n = x^n(1+x+x^2+\cdots)^n$$

$$\begin{aligned}
= \text{Co-efficient of } x^{S-n} \text{ in } (1-x)^{-n} \quad &= \quad \frac{n(n+1)\cdots(n+S-n-1)}{(S-n)!} \\
&= \quad \frac{(S-1)!}{(n-1)!(S-n)!} \\
&= \quad \frac{(S-1)(S-2)\cdots(S-\overline{n-1})}{1\cdot 2\cdots(n-1)}.
\end{aligned}$$

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Thanks!

Thank You