NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 25

***Gaṇitakaumudī* of Nārāyaṇa Paṇḍita 1**

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Outline

- Importance of *Gaṇitakaumudī*
- Solutions of quadratic equations
- Double equations of second and higher degree - rational solutions
- Determinations pertaining to the mixture of things
- Interest calculations - payment in installments

Nārāyaṇa Paṇdita's *Gaṇitakaumudī*

Gaṇitakaumudī was composed in 1356 CE by Nārāyaṇa Paṇḍita as indicated in the final verses of the work. It is not clear where he was born, or where he flourished. It was published by Padmakar Dvivedi in two volumes, based on a single manuscript, which belonged to his late father, the legendary Sudhakar Dvivedi, in 1930's. There is another work of Nārāyaṇa Paṇḍita entitled '*Bījagaṇitāvataṃśa*'. Only the first portion of this has been published, based on a single and incomplete manuscript at Benares.

Gaṇitakaumudī has been translated with explanatory notes by the late Paramanand Singh of Bihar. The translation and notes are published in Volumes 20-24 of *Gaṇita Bhāratī*, an Indian journal devoted to history of mathematics, during 1998-2001.

Post Bhāskara Indian Mathematics

After Bhāskara-II, there were two major developments in Indian mathematics: (i)Nārāyana Paṇḍita's *Gaṇitakaumudī* and (ii) Kerala school of mathematics and astronomy, mainly during $14^{\text{th}} - 17^{\text{th}}$ centuries, wherein calculus concepts were developed.

Nārāyaṇa Paṇḍita carries forward the tradition substantially. There are more formulae, generalisations of earlier results, and systematisation. There is a big leap in the treatment of combinatorics and magic squares in chapters 13 and 14, which will be dealt with separately. It was meant to a comprehensive text, covering most of the prevalent mathematical knowledge in India at the time of its composition, and making substantial additions to it.

Contents of Gaṇitakaumudī

I will give a brief summary of the 14 chapters below.

- Chapter 1 is on measures of weight, length, area, volume, capacity etc., 8 operations namely, addition, subtraction, multiplication, division, square, square root, cube and cube root described. Solutions of more complex (compared to earlier works) linear and quadratic equations are discussed.
- Chapter 2 is on '*Vyavahāra gaṇita*' or 'mathematics pertaining to daily life'. Calculation pertaining to mixture of materials, interest on a principal, payment in instalments, mixing gold objects with different purities and other problems pertaining to linear indeterminate equations for many unknowns are considered. There is substantial progress compared to earlier treatments. Example: Interest calculations where payment by instalments are considered.

Contents of Gaṇitakaumudī contd.

- Chapter 3 on arithmetic and geometric progressions, where the concept of 'sum of sums' is generalised to the 'k^{th} sum' of the first n integers for arbitrary k. This is a very important generalisation, which is needed for finding the infinite series for sine and cosine functions, among other things.
- Chapter 4 is on plane geometry where properties of triangles, quadrangles, circles, cyclic quadrilaterals etc., are considered. The 'third diagonal' associated with a cyclic quadrilateral and the area and circumference of the same are discussed.
- Chapter 5 to 8 on three dimensional geometry, where rules for solid figures and capacities of excavated volumes are presented.
- Chapter 9 is on the *Kuṭṭaka* procedure for solving linear indeterminate equations of the first degree.

Contents of Gaṇitakaumudī contd.

- Chapter 10 is on quadratic indeterminate equations or '*Vargaprakṛti*'. Here a variant of the '*Cakravāla*' procedure of Jayadeva and Bhāskara is also considered.
- Chapter 11 and 12 is on the divisors of a number and fractions.
- Chapter 13 is on '*aṅkapāśa* or combinatorics, partitions of numbers, sequences of binomial and polynomial coefficients, and various '*Meru*'s.
- Chapter 14 is on Magic squares.
  I take up some important mathematical results of *Gaṇitakaumudī*, pertaining to chapters 1-12 in what follows.

Arithmetical operations and *saṅkramaṇa*

Chapter 1

After discussing the eight operations, various simplifying manipulations involving fractions are discussed. In the earlier texts, '*saṅkramaṇa*' (or concurrence) problems involved solving for x and y; given $x+y=a$ and $x-y=b$, $\implies x=\dfrac{a+b}{2}$, $y=\dfrac{a-b}{2}$. In *Gaṇitakaumudī*, problems which can be reduced to *saṅkramaṇa* are discussed.

For example. Rule 31: Given $x^2-y^2=a, x-y=b$, we find $x+y=\dfrac{x^2-y^2}{x-y}=\dfrac{a}{b}$ and x,y can now be solved as we know $x-y$ and $x+y$. Similarly, given $x^2-y^2=a$, $x+y=b$, $x-y=\dfrac{a}{b}$ and solution by *saṅkramaṇa*.

Rule 33.

> वर्गसमासाद् द्विगुणात् अन्तरवर्गोनितात् पदं योगः॥ ३३ ॥
>
> "The square-root of the difference between 'twice the sum of squares (of two numbers' and) the 'square of (their) difference' is their sum."

If $x^2 + y^2 = a, x - y = b, x + y = \sqrt{2a - b^2}$ and solution by *saṅkramaṇa*.

Rule 34. Given $x^2 - y^2 = a, x^2y^2 = b, x^2 + y^2 = [(x^2 - y^2)^2 + 4x^2y^2]^{1/2} = c$, say. From a and c, x^2 and y^2 are obtained, and then x and y.

Rule 35. Given $x - y = b, xy = c$, then $x + y = (b^2 + 4c)^{1/2}$ and solution for x, y by *saṅkramaṇa*. Similarly, given $x + y, xy = c, x - y = (a^2 - 4c)^{1/2}$ and solution by *saṅkramaṇa*.

Rule 37(a). If $x + y$ and $x^2 + y^2$ are known, $x - y = [2(x^2 + y^2) - (x + y)^2]^{1/2}$, and solution by *saṅkramaṇa*.

Solution of the quadratic equations

Rule 39-40 (a).

रूपोत्थहृतपदाग्रे स्यातामन्तरवधौ ततस्ताभ्यम्।

प्राग्वद् योगः साध्यः स्यातां सङ्क्रामतो राशी॥ ३९ ॥

क्षयगे मूलेऽनल्पं तत्कृती राशिः। ४० ।

" '*Pada*' and '*agra*' (*dṛśya*) divided by '*rūpottha*' happens to be the (so called) difference and the (so called) 'product' respectively. Calculate the (so called) 'sum' from them by the method stated earlier. The numbers are (obtained from them) by the method of concurrence. (In case) *Pada* is negative, the greater number is to be taken. The square of that is the number."

The rule gives the solution of a quadratic equation of the type $ax + b\sqrt{x} = c$. a is the '*rūpottha*', b is the '*Pada*' and c is the '*dṛśya*', also '*agra*'. Find $\frac{b}{a}$ and $\frac{c}{a}$. The root of the equation is $\sqrt{x}$. Consider an auxiliary quantity $\sqrt{y}$, when b is negative. $b = -|b|$. In this case,

$$\sqrt{x} - \sqrt{y} = \frac{|b|}{a}, \ \sqrt{x}\sqrt{y} = \frac{c}{a}$$

Quadratic equation

Then, from rule 35,

$$\sqrt{x} + \sqrt{y} = \frac{(b^2 + 4ac)^{1/2}}{a}$$

$$\sqrt{x} = \frac{|b| + \sqrt{b^2 + 4ac}}{2} = \frac{-b + \sqrt{b^2 + 4ac}}{2}$$

In this case, $\sqrt{x}$ is the greater quantity out of $\sqrt{x}$ and $\sqrt{y}$. When b is positive, it is implied that

$$\sqrt{y} - \sqrt{x} = \frac{b}{a}, \;\; \sqrt{x}\sqrt{y} = \frac{c}{a}$$

$$\text{and } \sqrt{x} = \frac{-b + \sqrt{b^2 + 4ac}}{2}$$

(In this case $\sqrt{x}$ is the smaller quantity out of $\sqrt{x}$ and $\sqrt{y}$.)

[In either case, $\sqrt{x}$ is the correct solution using 'modern' standard way. Here c comes in the RHS, so we get, $b^2 + 4ac$ inside the square root, instead of $b^2 - 4ac$. It appears that Nārāyaṇa is giving only the positive root, so only one root is presented.]

Example

Example 27.

आकर्ण्य ध्वनिमद्रिमूर्ध्नि शिखिनोऽब्दानां स्फुरद्विद्युतां
वृन्दार्धात्रिलवौ तदन्तरचतुर्भागैस्त्रिभिः संयुतौ।
अध्यर्धैकपदाधिकौ ननृततुः प्रीत्याऽऽप्तवृन्दौ सखे
जातं तत्र शतत्रयं प्रवद मे तत्पूर्ववृन्ते कति॥ २७ ॥

“A group of peacocks along with its half, its one-third and $\frac{3}{4}$ of their difference (i.e, difference between its half and its one-third) together with $1\frac{1}{2}$ times the square root (of the group) standing on top of a mountain, hearing the thunder of clouds surcharged with electricity, are dancing with joy. The peacocks are 300 in all. O friend, tell me (the number of peacocks) in the earlier group.”

Solution

Let x be the number of Peacocks in the 'earlier' group. Then

$$x + \frac{1}{2}x + \frac{1}{3}x + \frac{3}{4}\left(\frac{1}{2} - \frac{1}{3}\right)x + \frac{3}{2}\sqrt{x} = 300$$

$$\text{or} \quad \frac{47}{24}x + \frac{3}{2}\sqrt{x} = 300$$

$$\text{or} \quad \frac{47}{12}x + 3\sqrt{x} = 600$$

Hence

$$\sqrt{x} = \frac{-3 + \sqrt{9 + 4 \cdot \frac{47}{12} \cdot 600}}{2 \cdot \frac{47}{12}} = \frac{-3 + 97}{47} \cdot 6 = 12$$

$$\therefore\ x = 144$$

This is the number of peacocks in the earlier group.

Quadratic equations involving successive remainders

Rule 40 b.

उक्तनिजविधिवदन्त्याच्छेषविधौ जायते राशिः। ४० ब्।

"Starting from the end, by the method stated earlier, the process (on being performed on) the remainders, gives the number."

This involves solving equations of the type:

$$(x - a\sqrt{x}) - b\left(\sqrt{x - a\sqrt{x}}\right) = c$$

and more complicated equations. This is solved by putting $x - a\sqrt{x} = y$. Then $y - b\sqrt{y} = c$. Then the problem is solved for y first and substituting this value for y in the equation for x, x is solved.

This is best illustrated by an example.

Example

Example 40.

कान्तायाः सुरतप्रसङ्गसमये भिन्ना च मुक्तावली
मुक्तानां च पदद्वयं विचरणं शय्यापटस्योपरि।
तच्छेषस्य पदं त्रिभागयुगलेनाऽऽढ्यं प्रियेणाऽऽहृतं
तच्छेषस्य पदं क्षितौ निपतितं सूत्रे द्वयं किं वद॥

"During the amorous tussle, (the beloved's) garland of pearls was broken twice the square root less $\frac{1}{4}$ (of the root) of the pearls were on the cover of the bed. The square root of the rest along with $\frac{2}{3}$ (of this root) were seized by the lover. The square root of the rest fell down on the earth and 2 (pearls) were in the string (of the garland). Tell, how many pearls were in the garland."

Solution

Solution: Let x be the number of pearls in the garland.

$$\text{Cover of the bed: } = \left(2 - \frac{1}{4}\right)\sqrt{x} = \frac{7}{4}\sqrt{x} = y$$

So remaining $= x - y$

$$\text{Seized by the lover } : \sqrt{x-y} + \frac{2}{3}\sqrt{x-y} = \frac{5}{3}\sqrt{x-y} = z$$

So remaining $= x - y - z = z'$

$$\text{On the earth } = \sqrt{z'} = \sqrt{x-y-z}$$

Remaining (on the string) $= 2$.

$$\therefore \sqrt{z'} + y + z + 2 = x$$

$$\sqrt{z'} + 2 = x - y - z = z'$$

Solution contd.

This is solved first: $\sqrt{z'} = \dfrac{1+\sqrt{1+8}}{2} = 2$. $\therefore\ z' = 4$. Now, $x - y - z = z'$, and $z = \dfrac{5}{3}\sqrt{x-y}$.

$$\therefore\ x - y - \frac{5}{3}\sqrt{x-y} = 4$$

Solving for this $\sqrt{x-y}$,

$$\sqrt{x-y} = \frac{\frac{5}{3} + \sqrt{\frac{25}{9} + 16}}{2} = 3$$

$$\therefore\ x - y = 9$$

Now $y = \dfrac{7}{4}\sqrt{x}$

$$\therefore\ x - y = x - \frac{7}{4}\sqrt{x} = 9.$$

Solving this,

$$\sqrt{x} = \frac{\frac{7}{4} + \sqrt{\frac{49}{16} + 36}}{2} = \frac{\frac{7}{4} + \frac{\sqrt{625}}{4}}{2} = \frac{7+25}{8} = 4.$$

$$\therefore\ x = 16.$$

This is the number of pearls originally in the garland.

Remainder type equation again

Rule 41 a.

> रूपोत्थघ्नाद्याग्रं योज्यान्त्याग्रे विधिः प्राग्वत्॥ ४१अ ॥
>
> "Multiply the "*rūpottha*" by the first remainder and add the product to the last remainder. (After that solve the problem) by the method stated earlier."

Here author is referring to an equation of the form:

$$x - a - \frac{b}{f}(x-a) - \frac{c}{g}\left\{x - a - \frac{b}{f}(x-a)\right\} - \frac{e}{h}\left[x - a - \frac{b}{f}(x-a) - \frac{c}{g}\{\qquad\}\right] - j\sqrt{x} = d$$

We find

$$x\left(1-\frac{b}{f}\right)\left(1-\frac{c}{g}\right)\cdots - j\sqrt{x} = a\underbrace{\left(1-\frac{b}{f}\right)\left(1-\frac{c}{g}\right)\cdots\left(1-\frac{e}{h}\right)}_{R\bar{u}pottha} + d$$

$a \rightarrow$ first remainder, $d \rightarrow$ last remainder.

Example

Example 41.

गणेशं पद्मेन त्रिनयनहरिब्रह्मादिनपान्
विलोमैः शेषांशैः विषयलवपूर्वैश्च कमलाम्।
पदेनाऽऽपूज्यैकेन च गुरुपदाम्भोजयुगलं
सरोजेनाऽऽचक्ष्व द्रुतमखिलमम्भोजनिचयम्॥ ४१अ ॥

"Gaṇeśa (was worshipped) with 1 lotus flower. Śiva, Hari, Brahma (and) the sun (were worshipped) with $\frac{1}{5}, \frac{1}{4}, \frac{1}{3}$ and $\frac{1}{2}$ of what remains successively. *Kamalā* was worshipped with the square root of the lotus flowers (in the beginning) and the two feet of the teacher resembling lotus flowers (were worshipped) with 1 lotus flower. Quickly say the number of lotus flowers in the collection."

Solution

Here let x be the number of lotus flowers.

$$\text{Gaṇeśa} \rightarrow 1$$

After worshipping Gaṇeśa $\rightarrow R_1 : x - 1$.

$$\text{Śiva} \rightarrow \frac{1}{5}(x-1)$$

After worshipping Gaṇeśa, Śiva $\rightarrow R_2 : (x-1) - \dfrac{1}{5}(x-1)$

$$Hari \rightarrow \frac{1}{4}\left\{(x-1) - \frac{1}{5}(x-1)\right\}$$

Solution contd.

After worshipping Ganesa, Siva, Hari, $\rightarrow R_3 : (x-1)-\frac{1}{5}(x-1)-\frac{1}{4}\left\{x-1-\frac{1}{5}(x-1)\right\}$

$$\text{Brahma} \rightarrow \frac{1}{3}\left[x-1-\frac{1}{5}(x-1)-\frac{1}{4}\left\{x-1-\frac{1}{5}(x-1)\right\}\right]$$

After worshipping Gaṇeśa, Śiva, Hari, and Brahma R_4 :

$$(x-1)-\frac{1}{5}(x-1)-\frac{1}{4}\left\{(x-1)-\frac{1}{5}(x-1)\right\}-\frac{1}{3}\left[x-1-\cdots\right]$$

$$\text{Sun} \rightarrow \frac{1}{2}R_4$$

After worshipping Gaṇeśa, Śiva, Hari, Brahma and sun $\rightarrow R_4 - \frac{1}{2}R_4$.

$$\textit{Kamalā} \text{ was worshipped with } \rightarrow \sqrt{x}$$

$$\text{Remaining Last remainder } = 1$$

Finding the Solution

$$\therefore (x-1) - \frac{1}{5}(x-1) - \frac{1}{4}\left\{(x-1) - \frac{1}{5}(x-1)\right\} - \frac{1}{3}[\qquad] - \frac{1}{2}\cdots - \sqrt{x} = 1$$

or $$(x-1)\left(1-\tfrac{1}{5}\right)\left(1-\tfrac{1}{4}\right)\left(1-\tfrac{1}{3}\right)\left(1-\tfrac{1}{2}\right) - \sqrt{x} = 1$$

or $$\underbrace{x\left(1-\frac{1}{5}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right)}_{R\bar{u}pottha=\frac{1}{5}} - \sqrt{x} = 1\cdot\underbrace{\left(1-\frac{1}{5}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right)}_{R\bar{u}pottha=\frac{1}{5}} + 1$$

$$\therefore \frac{1}{5}x - \sqrt{x} = \frac{1}{5} + 1 = \frac{6}{5}$$

$$\therefore \sqrt{x} = \frac{1+\sqrt{1+4\cdot\frac{6}{5}\cdot\frac{1}{5}}}{2\cdot\frac{1}{5}} = \frac{1+\frac{\sqrt{25+24}}{5}}{2\cdot\frac{1}{5}} = \frac{5+7}{2} = 6$$

$$\therefore x = 36$$

So, there were 36 lotus flowers in the original collection.

Another type of quadratic equation

Consider a quadratic equation of the form

$$x - \left[\frac{x}{(m/n)} - h\right]^2 = d.$$

One can easily check that this reduces to $x^2 - bx + c = 0$,

where $b = \left(\frac{m}{n}\right)^2 + 2\left(\frac{m}{n}\right)h$ and $c = \left(\frac{m}{n}\right)^2 (h^2 + d)$

$$\text{Then, Sum of the roots } x + y = b,$$

$$\text{Product of the roots } xy = c,$$

Then we obtain the solutions x, y by *saṅkramaṇa* with

$$x - y = \sqrt{(x+y)^2 - 4xy} = \sqrt{b^2 - 4ac},$$

so that

$$x = \frac{b + \sqrt{b^2 - 4c}}{2}, \quad y = \frac{b - \sqrt{b^2 - 4c}}{2}$$

Nārāyaṇa calls $\dfrac{m}{n}$ as the *rūpahara*, h the subtractive, and d, the *dṛśya* or the observed quantity and states the solution thus:

Rule

Rule 42 (b) - 43:

स्वांशकभक्तश्छेदो रूपहराख्यो द्विनिघ्नहीनहतः॥ ४२ ब् ॥

रूपहरवर्गयुक्तौ योगः स्याद् हीनवर्गयुतदृश्यः।
रूपहरवर्गगुणितो घातो राशिर्द्विधा प्राग्वत्॥ ४३ ॥

“The divider in one’s own part is called *Rūpahara*. This multiplied by twice the subtractive is added to the square of *rūpahara*. (The result) happens to be the sum. The square of the subtractive is added to the observed quantity (i.e., added to *dṛśya*). The (sum) multiplied by the square of *rūpahara* is the product of two numbers. (The numbers should be obtained) by the method stated earlier.”

So, he is stating that,

$$\text{Sum } = x + y = b = \left(\frac{m}{n}\right)^2 + 2\left(\frac{m}{n}\right) h \text{ and Product } = xy = \left(\frac{m}{n}\right)^2 (h^2 + d).$$

[It appears that Nārāyaṇa considers both the roots when both are positive.]

Example

यूथात् विंशांशकस्यैकवर्जितस्य कृतिः सखे।
प्रयाता मानसं हंसाः खं पञ्चोनशतं कति॥ ४५ ॥

"The square of $\dfrac{1}{20}$ of a swarm of swans less 1 flew away in sky towards $M\bar{a}nasarovara$ and 95 (remained there). O friends, (tell) how many (of them) were there?"

If the number of swans is x,

$$x - \left(\frac{x}{20} - 1\right)^2 = 95$$

So here $\dfrac{m}{n} = 20, h = 1, d = 95.$

$$b = \left(\frac{m}{n}\right)^2 + 2\left(\frac{m}{n}\right)h = 20^2 + 2 \cdot 20 \cdot 1 = 440$$

$$c = \left(\frac{m}{n}\right)^2 [h^2 + d] = 20^2(1 + 95) = 20^2 \cdot 96$$

Example

Now $x + y = b$, $xy = c$,

$$\therefore\ x = \frac{b + \sqrt{b^2 - 4c}}{2}, \quad y = \frac{b - \sqrt{b^2 - 4c}}{2}$$

We find $b^2 - 4c = 4 \times 20^2 \times 25$ $\qquad \therefore\ \sqrt{b^2 - 4c} = 200.$

$$\therefore\ x = \frac{440 + 200}{2} = 320, \quad y = \frac{440 - 200}{2} = 120.$$

Both are solutions.

“Double equations of second and higher order degree equations: Rational solutions”

Rules 45-57 give rational solutions of some second and higher degree equations involving two unknown quantities, x, y. In the following, m, n, p, u, v, a, b, etc., are integers.

Rule 45. A solution of $x^2 + y^2 + 1 = u^2, x^2 - y^2 + 1 = v^2$ is $x = \dfrac{m^2}{2}, y = m$.

Rule 46. A solution of $x^2 \pm y^2 - 1 = u^2$ or v^2 is $x = \dfrac{m^4}{2} + 1$, $y = m^3$.

Rule 47 A solution of $x + y = u^2, x - y = v^2, xy + 1 = w^2$ is $x = 2(m^4 + m^2)$, $y = 2(m^4 - m^2)$.

Rule 48. A solution of $x + y = u^2, x - y = v^2$ is $x = m^2 + n^2, y = 2mn$.

Rule 49. A solution of $x + y = u^2, x - y = v^2, xy = w^3$ is

$$x = \frac{(m^2 + n^2)pb}{[2mn(m^2 + n^2)]^2}, y = \frac{2mnpb}{[2mn(m^2 + n^2)]^2}$$

Rule 50. A solution of $x^2 + y^2 = u^3, x^3 + y^3 = v^2$ is $x = \dfrac{mb}{25}$, $y = \dfrac{2mb}{25}$.

$\left(\& \, u = \dfrac{m^4}{5} \text{ and } v = \dfrac{3m^3}{25}\right)$

Double equations, contd.

Rule 51. A solution of $ax+1=u^2, bx+1=v^2$ is $x=\dfrac{8(a+b)}{(a-b)^2}$

Rule 52. $x=\left[\dfrac{1}{2}\left(\dfrac{a+b}{m}-m\right)\right]^2+b$ is a solution of $x+a=u^2, x-b=v^2$

$\left[\text{with } u=\frac{1}{2}\left(\frac{a+b}{m}\right)+\frac{1}{2}m, \quad v=\frac{1}{2}\left(\frac{a+b}{m}-\frac{1}{2}m\right)\right]$

Rule 53. A solution of $x+a=u^2, x+b=v^2$ is $x=\left(\dfrac{a+b+1}{2}\right)^2-a$.

Rule 54. A solution of $x-a=u^2, x-b=v^2$ is $x=\left(\dfrac{a-b-1}{2}\right)^2-a$.

Rule 55. A solution of $x^2+xy+y^2=u^2$ is $x=\dfrac{1}{2}\left(\dfrac{3m^2}{4n}-n-m\right), y=m$

[with $u=\dfrac{3m^2}{8n}+\dfrac{1}{2}n$].

Rule 56. A solution of $x+y=xy$ is $x=\dfrac{m+n}{m}, y=\dfrac{m+n}{n}$.

Rule 57. A solution of $x+y=x^2+y^2$ is $x=\dfrac{m(m+n)}{(m^2+n^2)}, \; y=\dfrac{n(m+n)}{(m^2+n^2)}$.

Rule

Rule 58.

घनयुतिभक्ते कृतियुति युतिकृतिघाताहते त्विष्टे।
घनयुतियुतिघनतुल्या कृतियुतियुतिकृतिवधां राश्योः॥ ५८ ॥

“Divide the sum of squares, square of the sum and the product of any two assumed numbers by the sum of (their) cubes and the cube of (their) sum and (then) multiply by the two numbers. The (results will) be the two numbers, the sum of (whole) cubes and the cube (of whose) sum will be equal to the sum of their squares, the square of (their) sum (and their) product.”

Rational solution for Rule: 58

Let m and n be any two assumed numbers. Then according to the rule, solution of

1) $x^3 + y^3 = x^2 + y^2$ is $x = \frac{(m^2+n^2)m}{m^3+n^3}$, $y = \frac{(m^2+n^2)n}{m^3+n^3}$
2) $x^3 + y^3 = (x+y)^2$ is $x = \frac{(m+n)^2 m}{m^3+n^3}$, $y = \frac{(m+n)^2 n}{m^3+n^3}$
3) $x^3 + y^3 = xy$ is $x = \frac{m^2+n}{m^3+n^3}$, $y = \frac{m+n^2}{m^3+n^3}$
4) $x^3 + y^3 = x^2 + y^2$ is $x = \frac{(m^2+n^2)m}{(m+n)^3}$, $y = \frac{(m^2+n^2)n}{(m+n)^3}$
5) $(x+y)^3 = (x+y)^2$ is $x = \frac{(m+n)^2 m}{(m+n)^3}$, $y = \frac{(m+n)^2 n}{(m+n)^3}$
6) $(x+y)^3 = xy$ is $x = \frac{m^2 n}{(m+n)^3}$, $y = \frac{mn^2}{(m+n)^3}$

Nārāyaṇa considers the 'Rule of inversions' (Remember *Līlāvatī*) in Rule 59, and Rules of 3, 5, 7, 9, 11 and inverse rules in Rule 60 onwards. Barter of commodities and the sale of living beings etc., are also considered.

Chapter 2 on determination pertaining to mixture of things

Rule 1 (a).

प्रक्षेपास्तद्युतिहृतमिश्रेण हताः पृथक् फलानि स्युः ॥ १*a* ॥

“The contributions severally divided by their sum (and) multiplied by the mixed amount happen to be (the respective) fruits.”

This is understood with the solution for the following problem:

Example

Example 3.

सखे चतुर्णां वणिजां क्रमेण पञ्चादिकाश्वैर्दिवसैः षदश्वैः।
संचारितैः क्षेत्रधनं पणानां सहस्रमेकं वद किं पृथक् स्यात्॥

"Four traders obtain 1000 $paṇa$s by tilling a land for 6 etc., days with 5 etc., horses in order. O friend, tell what are (their shares) respectively."

Solution

Solution:

					Contribution
Trader	1:	6 days	5 horses	$\rightarrow$	$6 \times 5 = 30$
	2:	7 days	6 horses	$\rightarrow$	$7 \times 6 = 42$
	3:	8 days	7 horses	$\rightarrow$	$8 \times 7 = 56$
	4:	9 days	8 horses	$\rightarrow$	$9 \times 8 = 72$
					Sum $= 200$

So shares:

$$\text{Trader 1: } \frac{30}{200} \times 1000 = 150; \text{ Trader 2: } \frac{42}{200} \times 1000 = 210.$$

$$\text{Trader 3: } \frac{56}{200} \times 1000 = 280; \text{ Trader 4: } \frac{72}{200} \times 1000 = 360.$$

Use of rule of proportion

In rules 1(b)-3(a), rule of proportion is used cleverly to solve problems. Suppose there is one item and x quantity of something is purchased at r quantity / coin and sold at $r^{'}$ quantity / coin.

$$\therefore \ \text{Principal } = \frac{x}{r} = \frac{x}{rr'} \cdot r'$$

. Sold: Principal $+$ Profit $=$ Amount $A = \dfrac{x}{r'} = \dfrac{x}{rr'} \cdot r$.

$$\text{Profit } I = A - P = \frac{x}{r'} - \frac{x}{r} = \frac{x(r - r')}{rr'}$$

$$\therefore \ \text{Principal, P : Amount, A: Profit, I} :: \ r' : r \, : (r - r')$$

That is why, the rule which is applicable when several items are there, also states:

Rule

Rule 1(b)-2:

मिश्रं भवेत् क्रयार्घः क्रयश्च मिश्रं तु विक्रयो मूलम्॥ १ ॥
मूलं च विक्रयार्घो विक्रयहीनः क्रयो लाभः।
ज्ञेयमनुपातविधिना यद्यदविदितं फलं तत् तत्॥ २ ॥

"The rate of purchase is the mixed (amount) and the purchase-price is also the mixed amount. The sale-price is the principal and the rate of sale is also the principal. The sale-price less the purchase-price is the profit. The (unknown) fruit should be obtained from the knowns by the process of proportion (and from the mixed amount) and the method of partnership stated earlier."

Mixed quantities and rule of proportion

It is written somewhat confusingly, but refers to the proportionality among P, A, and I which has been stated by us. Now, if different types of objects are involved. Let us say:

x_1 amount of 1 is purchased at the rate of r_1 and sold at rate r_1'.

x_2 amount of 2 is purchased at the rate of r_2 and sold at rate r_2', etc.,

Then Principals of 1, 2, 3, $\cdots$ are: P_i are: $\frac{x_1}{r_1 r_1'} \cdot r_1', \frac{x_2}{r_2 r_2'} \cdot r_2', \cdots$

Amounts (Principal + Profit) odf 1, 2, 3, $\cdots A_i$ are: $\frac{x_1}{r_1 r_1'} \cdot r_1, \frac{x}{r_2 r_2'} \cdot r_2, \cdots$

Profits $= A_i - P_i$: $\frac{x_1}{r_1 r_1'}(r_1 - r_1'), \frac{x_2}{r_2 r_2'}(r_2 - r_2'), \cdots$

For different items,

$$\frac{\text{Principal}}{\text{Amount}} = \frac{\text{Principal}}{\text{Principal + Profit}} \text{ are in ratio } : \frac{r_1'}{r_1} : \frac{r_2'}{r_2}, \cdots$$

$$\frac{\text{Principal}}{\text{Profit}} \text{ are in ratio } : \frac{r_1'}{r_1 - r_1'} : \frac{r_2'}{r_2 - r_2'} : \frac{r_3'}{r_3 - r_3'} \cdots$$

$$\frac{\text{Principal}}{\text{Principal -Profit}} \text{ are in ratio } : \frac{r_1'}{2r_1' - r_1} : \frac{r_2'}{2r_2' - r_2} : \cdots$$

Example

Example 5.

शालिगोधूमकुल्माषखार्यः सखे रामबाणाऽद्रिसंख्याः क्रयाः सप्ततिः।
रूपहीना धनं विक्रया भूकराग्न्युन्मितास्तुल्यलाभं धनं किं पृथक्॥
लाभयुक्तानि तुल्यानि वित्तानि वा लाभहीनानि वा स्युः कथं ब्रूहि मे।

"*Śāli* rice, wheat and a half-ripe pulse were purchased in proportion of 3, 5, and 7 (per coin) for an amount of 69 (and then) sold in the proportion of 1, 2, and 3 (per coin). If the profits be equal or the profit added. or subtracted from the principals be equal, tell the principal separately."

Solution

Now, $\frac{\text{Principal}}{\text{Profit}}$ are in the ratio : $\frac{r_1'}{r_1 - r_1'} : \frac{r_2'}{r_2 - r_2'} : \frac{r_3'}{r_3 - r_3'}$ (a) If the profits are equal, Principals are themselves are in this ratio. Here $r_1 = 3, r_2 = 5, r_3 = 7$, $r_1' = 1, r_2' = 2, r_3' = 3$.

So, here: Principal of 1 : 2 : 3 :: $\frac{r_1'}{r_1 - r_1'} : \frac{r_2'}{r_2 - r_2'} : \frac{r_3'}{r_3 - r_3'}$

$$= \frac{1}{3-1} : \frac{2}{5-2} : \frac{3}{7-3} = \frac{1}{2} : \frac{2}{3} : \frac{3}{4}$$

$$\text{Sum} = \frac{6+8+9}{12} = \frac{23}{12}$$

$$\text{Total Principal} = 69.$$

$$\therefore \text{ Principal 1} = \frac{69}{\frac{23}{12}} \times \frac{1}{2} = 18$$

$$\text{Principal 2} = \frac{69}{\frac{23}{12}} \times \frac{2}{3} = 24$$

$$\text{Principal 3} = \frac{69}{\frac{23}{12}} \times \frac{3}{4} = 27.$$

Solution contd.

(b) Now if Principal $+$ Profit $=$ Amount are equal,

$$\text{Principal}\ 1 : 2 : 3 :: \frac{r_1'}{r_1} : \frac{r_2'}{r_2} : \frac{r_3'}{r_3} = \frac{1}{3} : \frac{2}{5} : \frac{3}{7}$$

$$\begin{aligned}
\text{Sum} &= \frac{35+42+45}{105} = \frac{122}{105} \\
\text{Total Principal} &= 69 \\
\therefore \frac{\text{Total Principal}}{\text{Sum}} &= \frac{69}{122} \times 105 \\
\text{Principal 1} &= \frac{69}{122} \times 105 \times \frac{1}{3} = \frac{69}{122} \times 35 \\
\text{Principal 2} &= \frac{69}{122} \times 105 \times \frac{2}{5} = \frac{69}{122} \times 42 \\
\text{Principal 3} &= \frac{69}{122} \times 105 \times \frac{3}{7} = \frac{69}{122} \times 45
\end{aligned}$$

Solution contd.

(c) If Principal $-$ Profit are equal. Then

$$\text{Principal}\, 1:2:3 :: \frac{r_1'}{r_1 - 2r_1'} : \frac{r_2'}{r_2 - 2r_2'} : \frac{r_3'}{r_3 - 2r_3'} = \frac{1}{1} : \frac{2}{1} : \frac{3}{1}$$

$$\text{Total} = 6.$$

Principal 1 $= \frac{69}{6} \times 1 = \frac{23}{2}$, Principal 2 $= \frac{69}{6} \times 2 = 23$, Principal 3 $= \frac{69}{6} \times 3 = \frac{69}{2}$

Several interesting problems involving interest for a principal are discussed.

Mixed problem involving interest

Let the principals (capitals) P_1 and P_2 be lent out for t_1 and t_2 months at the same rate of interest (r percent per month) and let the corresponding interests be I_1 and I_2. Then out of the seven quantities $(P_1, P_2, r, t_1, t_2, I_1, I_2)$, two can be determined from the sum of them and other four (r consider fixed).
In particular, Rule 5(b)-7(a) says:

विहिते विपरीतफले पक्षद्वितये मिथस्तु फलयोर्वा॥ ५ ॥
धनयोश्च कालयोर्वा धनफलयोः कालफलयोर्वा।
मूलधनकालयोर्वा मिश्रं यदि दृश्यते तदा तत्र॥ ६ ॥
तत्पक्षयोश्च घातौ ताभ्यां प्रक्षेपतो जाती॥ ७*a* ॥

Mixed problem involving interest

> "In two prescribed different sides, if the sum of either the interest or the capitals or the times or the capital and the time (of different sides) or the capital and the interest or time and the interest (of the same side) is observed as mixed, their shares in the sum can be determined, separately by the method of partnership considering the products of two (known ingredients) in the two sides as contribution."

So this rule says: when any one sum out of $I_1+I_2, t_1+t_2, P_1+P_2, P_1+t_2, P_2+t_1, I_1+t_1+I_2+t_2, I_1+P_1$ or I_2+P_2 is known and the rest 4 ingredients are known, then two (comprising the sum) can be found separately.

Example

Given $I_1 + I_2, P_1, t_1, P_2, t_2$ to find I_1 and I_2 separately. Now,

$$I_1 = \frac{P_1 t_1 r}{100}, \;\; I_2 = \frac{P_2 t_2 r}{100}$$

$$\therefore \; \frac{I_1}{I_2} = \frac{P_1 t_1}{P_2 t_2}$$

$$\therefore \; \frac{I_1 + I_2}{I_2} = \frac{P_1 t_1 + P_2 t_2}{P_2 t_2}$$

$$\therefore \; I_2 = \frac{(I_1 + I_2) P_2 t_2}{(P_1 t_1 + P_2 t_2)}.$$

$$\text{Similarly } I_1 = \frac{(I_1 + I_2) p_1 t_1}{P_1 t_1 + P_2 t_2}$$

Example.: Suppose P_2, t_1, I_1, I_2 and $P_1 + t_2$ are given:

$$\text{Now, } \frac{I_1}{I_2} = \frac{P_1 t_1}{P_2 t_2} \qquad \frac{P_1}{t_2} = \frac{I_1 P_2}{I_2 t_1} \qquad \frac{P_1 + t_2}{t_2} = \frac{I_1 P_2 + I_2 t_1}{I_2 t_1}$$

$$\therefore \; t_2 = \frac{(P_1 + t_2) I_2 t_1}{I_1 P_2 + I_2 t_1}, \qquad P_1 = (P_1 + t_2) - t_2$$

Example

Example 17.

मासेन शतस्य क्रियत् षष्टेर्वर्षस्य यत् फलं फलयोः
योगे चत्वारिंशद्रूपयुतं मे फलं कथय॥
ताभ्यां पक्षद्वितये मिथो विमिश्रे पृथक् कृते ब्रूहि॥ १७ ॥

"The sum of interest of 100 in a month, added to that of 60 in a year (at the same rate) is 41. Tell me the interest separately."

Here $P_1 = 100, P_2 = 60, t_1 = 1, t_2 = 12$. Given $I_1 + I_2 = 41$.

$$\text{We have } I_2 = \frac{(I_1 + I_2)P_2 t_2}{P_1 t_1 + P_2 t_2} = \frac{41 \times 60 \times 12}{100 + 60 \times 12} = \frac{41 \times 60 \times 12}{820} = 36$$

.

$$I_1 = (I_1 + I_2) - I_2 = 41 - 36 = 5.$$

Now let any one out of $P_1 + I_2, P_2 + I_1, t_1 + I_2, t_2 + I_1, P_1 + t_1 P_2 + t_2$ be known and the other four quantities are known separately. Then the product of the two quantities can be obtained and the quantities themselves can be determined according to Rule 7(b)-8.

Example

Example. a) P_2, t_1, t_2, I_1 and $P_1 + I_2$. To find P_1, I_2.

Now as the rates of interest are equal,

$$\frac{I_1}{I_2} = \frac{P_1 t_1}{P_2 t_2}$$

$$\therefore\ P_1 I_2 = \frac{I_1 P_2 t_2}{t_1}$$

$$P_1 - I_2 = \sqrt{(P_1 + I_2)^2 - 4P_1 I_2}$$

From, $P_1 + I_2$ and $P_1 - I_2$, P_1 and I_2 are found from '$sa\dot{n}kramaṇa$'.

(b) Similarly, Let P_1, P_2, t_2, I_1 and $I_2 + t_1$ are given. Then

$$I_2 t_1 = \frac{I_1 P_2 t_2}{P_1} \qquad \therefore\ I_2 - t_1 = \sqrt{(I_2 + t_1)^2 - 4I_2 t_1}$$

From, $I_2 + t_1$ and $I_2 - t_1$, I_2 and t_1 can be determined.

Example

Example 18.

मासि शतस्य फलं यद् वर्षेण च षड्कृतिः फलं यस्य।
तद्योगे पञ्चगुणाः त्रयोदश सखे पृथक् कथय॥
काले निजधनयुक्ते सहफलयुक्तेऽथवा गणितम्॥ १८ ॥

"The sum of the interest of 100 for a month and the capital (principal) whose interest in a year is 36, is 65. O friend, tell (them) separately (knowing) the sum of the capital and its time or that of the interest of 100 and the time (tell them separately)."

Solution: Here $P_1 = 100, I_2 = 36, t_1 = 1, t_2 = 12$, and $P_2 + I_1 = 65$.

$$\text{Now}, P_2 I_1 = \frac{I_2 P_1 t_1}{t_2} = \frac{36 \times 100 \times 1}{12} = 300.$$

Solution

$$\therefore P_2 - I_1 = \sqrt{(P_2 + I_1)^2 - 4P_2I_1} = \sqrt{65^2 - 1200} = 5\sqrt{13^2 - 48} = 55$$

$$\therefore P_2 + I_1 = 65, P_2 - I_1 = 55 \qquad \therefore P_2 = 60, I_1 = 5.$$

Now P_2, t_2, I_1, I_2 and $P_1 + t_1$ be given. To find P_1, t_1.

$$\text{Now, } P_1t_1 = \frac{I_1P_2t_2}{I_2} \qquad \therefore P_1 - t_1 = \sqrt{(P_1 + t_1)^2 - 4P_1t_1}$$

In this example, $P_1t_1 = 101, P_1t_1 = \dfrac{5 \times 60 \times 12}{36} = 100.$

$$\therefore P_1 - t_1 = \sqrt{101^2 - 400} = \sqrt{100^2 - 200 + 1} = \sqrt{(100 - 1)^2} = 99$$

$$\therefore P_1 + t_1 = 101, P_1 - t_1 = 99.$$

$$P_1 = 100, t_1 = 1.$$

Payment in Instalments

The following rule pertains to the payment in instalments of a debt, based on simple interest.

Rule 10.

> स्कन्धककालकलान्तर हीनस्कन्धेन भाजिते वित्तम्।
> स्कन्धककालविगुणिते नियतं निर्मुक्तकालः स्यात्॥ १० ॥
>
> "The capital (principal) is multiplied by the fixed period of instalment (the product is) divided by the amount of instalment less the interest (on the total amount) for the period of instalment. (The quotient) determines the time of being free from debt."

Amount lent $= P$, Amount of instalment $= a$. Period of instalment (No. of months / instalments) $= t$ Total time for being free from debt $= T = nt$ let us say.

Payment in Instalments

Interest / period of Instalment (for time t) $= I'$

$\therefore$ Interest for total duration of debt (time $T = nt$) $= nI'$.

Now, the amount due after time $T = nt$ is $P + nI'$ and amount paid is na. Equating them,

$$P + nI' = na$$

$$\therefore\ n = \frac{P}{a - I'}$$

$$\therefore\ T = nt = \frac{Pt}{a - I'}$$

This is what is stated in the rule.

Example

Example 20.

मासेन पञ्चकशतेन शतं दशोनं
दत्तं कुसंकट इहाप्यधमर्णकाय।
मासद्वयं प्रति सखे दशपञ्चयुक्तात्
स्कन्धं द्रुतं कथय मे परिमुक्तकालम्॥ २० ॥

"In his bad days a debtor was given an amount (principal P) of 90 at the rate of 5 percent per month to be paid back in bimonthly ($t = 2$) instalments of 15 (each). O friend, tell me quickly, the time of being free from the debt."

Here $P = 90, t = 2$ months (period of instalments)

I' (payment interest per period of instalment) $= \frac{5}{100} \times 90 \times 2 = 9$, Instalment amount, $a = 15$.

$$\therefore\ T = \frac{Pt}{a - I'} = \frac{90 \times 2}{15 - 9} = \frac{90 \times 2}{6} = 30\,\text{months}$$

This is the time for being free from debt.

Payment in instalments: Time for being free from debt

The next rule gives the expression for the Principal (capital) P, in terms of the instalment amount, a, time being free from debt, T, the instalment period, t and the rate of interest (in fact interest on unity) for the time of being free from debt.

Rule 11.

निर्मुक्तकालवृद्ध्या रूपस्य हि सैकया हतेन भजेत्।
स्कन्धककालेन च गतकालस्कन्धाहतिर्मूलम्॥ ११ ॥

"The time of being free from debt is multiplied by the amount of instalment. (The product) is divided by (the product of the sum of) the interest on unity for the time of being free from debt added to 1, multiplied by the period of instalment. The (quotient) is the capital."

We had $P + nI' = na$

$$\therefore P\left(\frac{nI'}{P} + 1\right) = na = \frac{aT}{t}$$

$$\therefore P = \frac{aT}{\left(\frac{n}{p}I' + 1\right)t}$$

This is what being stated, when it is realised that $\dfrac{nI'}{P}$ is the interest for the period $T = nt$.

Example

Example 21.

पञ्चकशतेन वित्तं मासद्वितयेन सदलेन।
स्कन्धः पञ्चदशाऽथ त्रिंशान्मासा विनिर्मुक्तः॥
कालस्त्विह वद मूलं किं वृद्धिः का च यदि वेत्सि॥ २१ ॥

"An amount is lent at the the rate of 5 percent per month. The period of instalment is $2\frac{1}{2}$ months, the amount of interest is 15, and the time of being free from debt is 30 months. If you know, tell the capital and the interest."

Solution: Here $t = 2.5, a = 15, T = 30$. Rate of interest $=$ 5 percent per month.

$$\text{Interest on unity for } T = \frac{5}{100} \times 30 = 1.5 \left(= \frac{nI'}{P} \right)$$

$$\therefore P = \frac{15 \times 30}{(1.5 + 1) \times 2.5} = 72.$$

$$\text{Total interest paid } = 1.5 \times 72 = 108$$

$$\text{Alternatively, total interest } = na - P = \frac{30}{2.5} \times 15 - 72 = 108.$$

Payment in instalments: More realistic computation

The above is based on simple interest. The debtor owes the amount $P + nI'$ at the end of the time $nt = T$ based on simple interest, but he is actually paying an instalment amount a after each instalment period t. He is not getting any benefit for paying back the debt in instalments at earlier times. The next few rules take this into account.

The rule in 12-14(a) pertains to the instalment amount a being adjusted only towards payment of capital. The rule gives the expression for the amount of interest to be paid in addition to the 'monthly instalments' being paid to clear the capital.

Let amount lent (Principal or capital) $= P$, Instalment amount $= a$, Instalment time period $= t$ months. Rate of interest $= r$ percent per month. If the amount is cleared in n instalments $P = n'a$ or $n' = \dfrac{P}{a}$.

$$\therefore \text{ Total time for being free from debt } = n't = \frac{Pt}{a}$$

Payment in instalments contd.

Here n' indeed not be an integer. Let

$$n' = n + \frac{f}{a}$$

$$\therefore \frac{P}{a} = n + \frac{f}{a}$$

$$\text{or } P = na + f, \text{ where } f < a.$$

Consider the amount, P as made up of na and f. Consider the integral part na, first. At time t instalment amount a is being paid. The interest should be the entire amount, na.

$$\therefore \text{Interest at time } t = na \cdot \frac{rt}{100}$$

Here an amount a is paid towards clearing the principal. So the amount due at the beginning of the second instalment is $na - a = (n-1)a$. So, at time $2t$, we have to calculate interest for the amount $(n-1)a$.

$$\therefore \text{Interest at time } 2t = (n-1)a \cdot \frac{rt}{100}$$

Payment in instalments: Interest to be paid

Proceeding in this manner,

$$\text{Interest at time } rt = [n-(r-1)]\cdot a\cdot \frac{rt}{100}$$

$$\text{Interest at time } \; nt = a\cdot \frac{rt}{100}$$

Hence, total interest for the integral part

$$= [n+(n-1)+\cdots+2+1]a\cdot \frac{rt}{100} = \frac{n(n+1)}{2}a\cdot \frac{rt}{100}$$

At the end of time duration nt, Principal amount due is f. (Integral part na having been cleared). This will be cleared in time $\frac{f}{a}t$. But the debtor has to pay the interest for f for a period $nt + \frac{f}{a}t = \frac{P}{a}t$, which is $f\cdot \frac{Pt}{a}\cdot \frac{r}{100}$. Hence,

$$\text{Total interest to be paid } = \left[\frac{n(n+1)}{2}a + \frac{Pf}{a}\right]\frac{rt}{100}$$

n is called the '*pada*'. f/a is the '*agrā*. $\left[\frac{n(n+1)}{2}a + \frac{Pf}{a}\right]$ is called '*mūla piṇḍa*'.

The total interest to be paid and the time for being free from debt are stated in the following rule:

Rule

Rule 12-14 a:

स्कन्धकभक्तं वित्तं लब्धं पदसंज्ञकं च शेषांशः।

अग्राख्यः पदवर्गः पदयुक् स्कन्धार्धसंगुणो युक्तः॥ १२ ॥

अग्रांशघ्नघनेन प्रजायते मूलपिण्डाख्यः।

तस्य स्कन्धककालात् समानयेद् वृद्धिमानमथ॥ १३ ॥

स्कन्धककालघ्नधने स्कन्धहृतं मुख्यकालः स्यात्॥ १४ अ ॥

Total interest and time for being free from debt

> "The amount (capital or principal lent) is divided by the amount of instalment. (The integral part of) the quotient is called '$Pada$' and the remaining part, '$Agra$'. '$Pada$' is added to its square. (The sum) is multiplied by half the amount of instalment. (The product) is added to (the product) of 'agra' (i.e. the remaining part) multiplied by amount (lent. The sum) is called '$M\bar{u}la\ Pi\d{n}\d{d}a$'. The interest should be obtained from that (i.e. from the $m\bar{u}la\ pi\d{n}\d{d}a$) by taking the time equal to the period of instalment. After that, the period of instalment multiplied by the amount (lent, the product) divided by the amount of instalment happens to be the time of being free from the debt."

[Note: Let $\frac{rt}{100} = r'$. Probably, the debtor is paying $a + nar'$ at time t, $a + (n-1)ar'$ at time $2t, \cdots, a + ar'$ at time nt, and at time $nt + \frac{f}{a}t$, he is paying $f + \frac{Af}{a}r'$. In all these, the first part is payment towards the principal and the second part is the interest.]

Example

Example 22.

पञ्चकशतेन दत्त्वा पञ्चयुताः सप्ततिः केन।
स्कन्धन च प्रयच्छति मासाभ्यां ग्राहकवृद्धिभयात्॥
वृद्धिं विमुक्तिकालं कथय सखे त्वं पुरा वेत्सि॥ २२ ॥

"Somebody lent 75 at the rate of 5 percent per month. The debtor, for fear of (high) interest, returns (the amount) in instalments of 9 (along with the interest due), bimonthly. O friend, if you know, tell the interest and the time of being free from debt."

Solution: Here $P = 75, r = 5$, instalment $a = 9, t = \dfrac{1}{2}$.

$$\frac{A}{a} = \frac{75}{9} = 8 + \frac{3}{9} \qquad (f = 3, a = 9)$$

$$\text{So}, n = 8.$$

$$\therefore \text{ Interest paid } = \left[\frac{8 \times 9}{2} \times 9 + 75 \times \frac{1}{3}\right] \times \frac{1}{2} \times \frac{5}{100}$$

$$= (36 \times 9 + 25) \times \frac{2 \cdot 5}{100} = 8\frac{29}{40}$$

$$\text{Time } = \frac{75}{9} \times \frac{1}{2} = 4\frac{1}{6} \text{ months}$$

Equated monthly instalments

Now we consider the case when the instalment *a* is the interest on the outstanding amount plus the payment of a part of the principal. Then the following rule gives the time of being free from the debt.

Rule 16 b - 17.

प्रतिमासिकफलशुद्धौ मूलं मूलात् पृथक् पृथग् जह्यात्॥ १६ ब् ॥
शेषस्य मासिकफलं विशोधयेद् मासिकोपनयात्।
शेषेणाऽनेन मूलविशेषमाप्तं तु मासयुक् कालः॥ १७ ॥

"Subtract the principal part of each monthly instalment free from the capital, successively. (This gives the number of complete months and a residue of the capital, if any). Subtract the interest of residue for a month from the amount of monthly instalment. Divide the residue of the capital by the remainder (The quotient) added to the number of (complete) months is the time (of being free from the debt)."

Equated monthly instalments

Explanation: Lent amount,principal $= P$. Rate of interest $= r$ percent per month. Let $r' = \dfrac{rt}{100}$, where t is the period of instalment. At the time of payment of first instalment, interest due is $r'P$. Monthly instalment $= a$. $\therefore$ Principal part of monthly instalment $= a - r'P$. Hence, Amount due after first instalment paid $P_1 = P - (a - r'P)$ or $P_1 = P(1 + r') - a$.

The interest for this at the time of second instalment $= r'P_1$. Amount paid $= a$. $\therefore$ Principal part of monthly instalment $= a - r'P_1$ Amount due after payment of second instalment
$= P_1 - (a - r'P_1) = (1 + r')P_1 - a = (1 + r')^2P - (1 + r')a - a$.

Amount due after payment of n instalments
$P_n = (1 + r')^nP - [(1 + r')^{n-1} + \cdots + 1]a$. According to the rule, we go on till $R = P_n < a$. R is the remainder or the 'Residue' he talks about. It takes less than an instalment period $= t$ to clear R with its interest. Let this be $\mathcal{R}t$. (or $\mathcal{R}$ is the fraction of the instalment period).

Equated monthly instalments

The interest on R for the fraction, f of the instalment period $= R \cdot \mathcal{R} \cdot r$. The amount the debtor pays for the fraction of the instalment period $\mathcal{R}a$. Equating these,

$$R + R\mathcal{R}r' = \mathcal{R}a.$$

$$\therefore \mathcal{R} = \frac{R}{a - Rr'} = \frac{R}{a - I'}$$

where $I' = Rr'$ is the interest on R for one instalment. This is what has been stated.

[Though the rule appears to state the result for the case when the instalment period is a month, it is clearly applicable for any value of the period, t].

Nārāyaṇa's concept of instalments is very similar to the modern concept of equated monthly instalments. The only difference is that in modern times the time for clearing the debt is fixed, and the equated instalment worked out, whereas here, the monthly instalment is fixed and the time for clearing the debt is worked out.

Example

Example 24.

दत्तं दशकशतेन च शतं च कस्यापि केनचिद्धनिना।
प्रतिमासिकफलसहिता पञ्चाशत् स्कन्धकं प्रयच्छति च।
अनृणी कालेन सखे केन भवेद् ग्राहकस्य वद॥ २४ ॥

"A rich man lent somebody 100 at the rate of 10 percent per month. (The debtor) gives a monthly instalment of 50 including the interest. O friend, tell me the debtor's time of being free from debt."

Solution: $P = 100, r' = \frac{10}{100} = \frac{1}{10}, a = 50.$

At the time of first monthly instalment: Interest $= 100 \times \frac{1}{10} = 10$. Payment $a = 50$.

$$\begin{aligned}
\text{Payment towards principal part } &= 50 - 10 = 40 \\
\text{Amount due after first instalment } &= 100 - 40 = 60 \\
\text{Interest on this for month } 60\frac{10}{100} &= 6 \\
\text{Payment } &= 50 \\
\text{Payment towards principal part } 50 - 6 &= 44
\end{aligned}$$

Solution

Amount due after payment of second instalment $= 60 - 44 = 16 = R$

as it is less than $a = 50$.

$$\text{Interest on } R = 16, \text{ for a month } 16 \times \frac{10}{100} = 1.6$$

Fraction of a month required to clear this debt with interest, $\mathcal{R}$.

$$\mathcal{R} = \frac{16}{50 - 1.6} = \frac{16}{48.4} = \frac{4}{12.1}$$

This is the fraction of a month required to clear the debt. At this time the debtor pays $\mathcal{R} \times a = \frac{4}{12.1} \times 50$ to clear the debt completed.

References

1. *Gaṇitakaumudī* of Nārāyaṇa Paṇḍita, Ed. by Padmākara Dvivedi, 2 Vols, Varanasi, 1936, 1942.

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Thanks!

Thank You