NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 13

Brāhmasphuṭasiddhānta of Brahmagupta - Part 3

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Outline

- The $Kuṭṭākāra$ method
  - Importance of $Kuṭṭākāra$
  - Verses presenting the algorithm
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  - Some obervations
- Operations with positive, negative and zero (*dharṇaśūnya*)
- Dealing with surds (sq. root of a non-square number)
- Equations with one unknown (*ekavarṇasamīkaraṇam*)
- Equations with many unknowns (*anekavarṇasamīkaraṇam*)
- *Bhāvita*: Equations with products of unknowns
- *Varga-prakṛti* and *Bhāvanā* principle
- Examples

Importance of *Kuṭṭākāra*

- The *Kuṭṭakādhyāya* of *Brāhmasphuṭasiddhānta* consists of about 100 verses.
- Through the very first verse Brahmagupta coveys the importance of the technique of *kuṭṭaka* that is employed in solving a large class of problems in astronomy.

प्रायेण यतः प्रश्नाः कुट्टाकारादृते न शक्यन्ते।
ज्ञातुं वक्ष्यामि ततः कुट्टाकारं सह प्रश्नैः॥ १ ॥

Mostly it may not be possible [for mathematicians] to solve problems without knowing the technique of *kuṭṭakāra*. Therefore, I am going to explain the *kuṭṭākāra* method, along with [a few] illustrative problems.

Gaining recognition amongst mathematicians

Presenting a list of the topics to be covered in the chapter, Brahmagupta mentions that one would gain recognition as ‘*ācārya*’ amongst mathematicians only by gaining mastery over them.

कुट्टकखर्णधन-अव्यक्तमध्यहरणैकवर्णभावितकैः।
आचार्यस्तन्त्रविदां ज्ञातैः वर्गप्रकृत्या च॥ २ ॥

कुट्टकेन	–	by knowing to solve *kuṭṭaka* problem
खर्णधनेन	–	operations with zero, -ve and +ve quatities
अव्यक्त-सङ्कलनेन	–	doing mathematical operation with unknowns
मध्यमाहरणेन	–	eliminationn of middle term in a quadratic
एकवर्णसमीकरणेन	–	dealing with equation with single unknown
भावितेन	–	solving equations with products of unknowns
वर्गप्रकृत्या च	–	solving second order indeterminate equation of the form $x^2 - Dy^2 = 1$

Statement of the *kuṭṭākāra* problem (+ terminologies)

- Suppose we have a number N, that satisfies the following equations

$$\begin{aligned} N &= ax + r_a \\ &= by + r_b. \end{aligned}$$

- Here (a, b) are known as *chedas*[1] and (r_a, r_b) the *agrās*, or *śeṣas*.
- If the remanider $r_a > r_b$, then,

$a \rightarrow$ अधिकाग्रहार (or) अधिकाग्रच्छेद

$b \rightarrow$ ऊनाग्रहार (or) ऊनाग्रच्छेद

otherwise, vice versa.

- The problem is to find N, x and y (all integers) given a, b, r_a and r_b (also integers)
- This forms an example of *first* order indeterminate equation, since we have 3 unknowns and only two equations.

[1] They (a, b) are called *chedas* (divisors) in the sense that, when they divide N, they leave remainders r_1 and r_2.

The *kuṭṭākāra* algorithm

Suppose $N = ax + r_a = by + r_b$, with $r_a > r_b$. (a is *adhikāgrabhāgahāra*)

अधिकाग्रभागहारात् ऊनाग्रभागहारितात् शेषम्।
यत् तत् परस्परहृतं लब्धं अधोऽधः पृथक् स्थाप्यम्॥ ३ ॥
शेषं तथेष्टगुणितं यथा अग्रयोरन्तरेण संयुक्तम्।
शुद्ध्याति गुणकः स्थाप्यः लब्धं च अन्त्यादुपान्त्यगुणः॥ ४ ॥
स्वोर्ध्वेऽन्त्ययुतो अग्रान्तो हीनाग्रच्छेदभाजितः शेषम्।
अधिकाग्रच्छेदहतं अधिकाग्रयुतं भवत्यग्रम्॥ ५ ॥

- First $a \div b$. With the remainders keep doing mutual division.
- Arrange the quotients one below the other.
- Multiply the last remainder (r_{2n}) by the desired number (*iṣṭaguṇitaṃ*) 't' such that this plus $r_a \sim r_b$ in divisible by r_{2n-1}.
- The multiplier t has to be placed [below the quotients].
- And so too the *labdhi* 'l'.[2]

[2] Thus we have to form the *vallī* with the quotients $(r_1, r_2, \ldots) + t + l$.

The *kuṭṭākāra* algorithm

Suppose $N = ax + r_a = by + r_b$, with $r_a > r_b$. (a is *adhikāgrabhāgahāra*)

शुद्ध्याति, गुणकः स्थाप्यः लब्धं च अन्त्याद् उपान्त्यगुणः।
स्वोर्ध्वेऽन्त्ययुतो अग्रान्तो हीनाग्रच्छेदभाजितः शेषम्।
अधिकाग्रच्छेदहतं अधिकाग्रयुतं भवत्यग्रम्॥ ५ ॥

- Starting from *antya*—which is *labdhi*—we have to carry out operations to complete the *vallī* (अन्त्यादारभ्य कर्म कर्तव्यम्).
- The penultimate has to be multiplied with the one above and added to the ultimate (स्वोर्ध्वे उपान्त्यगुणः अन्त्ययुतः).
- This is to be carried out till we exhaust all the quotients, and we are left with only two numbers (*rāśis*).
- At this stage, [the number at the top *ūrdhvarāśi*] has to be divided by *hīnāgraccheda*.
- This remainder (s) obtained is to be multiplied by *adhikāgraccheda*, and the result is to be added to *adhikāgra* to get N.

The *kuṭṭakāra* algorithm: Example

To solve $N = 34x + 2 = 13y + 10$; Here $a = 34$, $b = 13$, $r_a = 2$, $r_b = 10$

$$
\begin{array}{llllllllll}
34) & 13 & (0 & & & & & & & \\
 & 0 & & & & & & & & \\
\hline
 & 13) & 34 & (2 & & & & & & \\
 & & 26 & & & & & & & \\
\cline{3-4}
 & & 8) & 13 & (1 & & & & & \\
 & & & 8 & & & & & & \\
\cline{4-5}
 & & & 5) & 8 & (1 & & & & \\
 & & & & 5 & & & & & \\
\cline{5-6}
 & & & & 3) & 5 & (1 & & & \\
 & & & & & 3 & & & & \\
\cline{6-6}
 & & & & & 2 & & & &
\end{array}
$$

Since the no. of quotients is even, t is to be chosen such that $(2 \times t + 8)$ should be divisibe by 3. This $\Rightarrow t = 2$.
$2 \times 2 + 8 = 12$. This divided by 3 gives the quotient *labdhi* 4.

Vallī

2	2	2	2	36	(r_u - *ūrdhvarāśi*)
1	1	1	14	14	(r_a - *adhorāśi*)
1	1	8	8		
1	6	6			
2	2				
4					

Having obtained the *vallī*, the operations that remain are:

- $36 \div 34$, gives the remainder 2.
- $N = 2 \times 13 + 10 = 36$ is the desired number.

The *kuṭṭakāra* algorithm: Example

To solve $N = 34x + 2 = 13y + 10$; Here $a = 34$, $b = 13$, $r_a = 2$, $r_b = 10$

34)	13	(0					
	0						
	13)	34	(2				
		26					
		8)	13	(1			
			8				
			5)	8	(1		
				5			
				3)	5	(1	
					3		
					2)	3	(1
						2	
						1	

Since the no. of quotient is odd, t is to be chosen such that $(1 \times t - 8)$ should be divisibe by 2. This $\Rightarrow t = 10$. $1 \times 10 - 8 = 2$. This divided by 2 gives the quotient *labdhi* 1.

2	2	2	2	2	138
1	1	1	1	53	53
1	1	1	32	32	
1	1	21	21		
1	11	11			
10	10				
1					

- $138 \div 34$, gives the remainder 2.
- $2 \times 13 + 10 = 36$ is the desired number.
- It is noted that whether we carry on the division till we get the remainder as 1 or not, the process works.

The *kuṭṭakāra* algorithm: Example

To solve $N = 34x + 2 = 13y + 10$; Here $a = 34, b = 13, r_a = 2, r_b = 10$

Suppose we terminate the division two steps ahead of reaching the remainder 1 in the abnove example.

34)	13	(0			
	0				
	13)	34	(2		
		26			
		8)	13	(1	
			8		
			5)	8	(1
				5	
				3	

Here again, since the no. of quotient is odd, $(3 \times t - 8)$ should be divisible by 5. This $\Rightarrow$ $t = 6$. $3 \times 6 - 8 = 10$. This divided by 5 gives the quotient *labdhi* 2.

2	2	2	36
1	1	14	14
1	8	8	
6	6		
2			

What is the lesson?

- The mutual division process can be terminated at any stage of the division as we like. By choosing an appropriate '*mati*', the *vallī* can be constructed.
- Though the *vallī* may be different, it leads to the same solution.
- In fact, various artifices have been invented by later mathematicians like Mahāvīra, Bhāskaracārya to achieve more simplification.

Some general observations

- Āryabhaṭa seems to be the first mathematician to have provided systematic procedure for solving indeterminate equation of the first degree of the form

$$ax \pm c = by \qquad (a, b, c \text{ are +ve integers})$$

- Considering his period, (5th cent CE), this indeed is a landmark achievement in the field of pure mathematics.
- Āryabhaṭa's verses (2) are terse, and hence require to be supplimented by commentary, whereas Brahmagupta's verses (3) are comparitively easier, and are complete by themselves.
- If (a, b) are not coprime, their common factor should be a factor of c too. Otherwise, the equation has no solution.
- On the other hand, if the equation has one solution $(x, y) = (\alpha, \beta)$, then $(x, y) = (\alpha + bm,\ \beta + am)$, is also a solution for any integer 'm': Thus the existence of one solution $\Rightarrow$ the existence of infinite solutions.

Mathematics of positive, negative and zero

- *Brāhmasphuṭa-siddhānta* (c. 628 CE) is the first available text that discusses the mathematics of zero (*śūnya-parikarma*) along with operations with positives and negatives (*dhanarṇa*).[3]
- The first of the six verses presenting rules for *saṅkalana* goes as:

धनयोर्धनम्ऋणमृणयोः धनर्णयोरन्तरं समैक्यं खम्।
ऋणमैक्यं च धनमृणधनशून्ययोः[4] शून्ययोः शून्यम्॥ ३० ॥

positive + positive → positive
negative + negative → negative
positive + negative → positive/negative
positive + negative → zero (when of same magnitude (सम))

positive + zero → positive
negative + zero → negative
zero + zero → zero

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[3] *Brāhmasphuṭasiddhānta* of Brahmagupta, Ed. with his own commentary by Sudhakara Dvivedi, Benaras 1902, verses 18.30–35, pp. 309–310.

[4] ऋणशून्ययोः ऐक्यम् ऋणम्, धनशून्ययोः ऐक्यम् धनम्।

Mathematics of positive, negative and zero

- The next couple of verses present rules for *vyavakalana*:

  ऊनमधिकाद्विशोध्यं धनं धनादृणमृणादधिकमूनात्।
  व्यस्तं तदन्तरं स्यादृणं धनं धनमृणं भवति॥ ३१ ॥
  शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम्।
  शोध्यं यदा धनमृणादृणं धनाद्वा तदा क्षेप्यम्॥ ३२ ॥

positive − positive → positive
negative − negative → negative

positive − zero → positive
negative − zero → negative
zero − zero → zero

negative − positive → negative (simply to be added (क्षेप्यम्))
positive − negative → positive (simply to be added (क्षेप्यम्))

Mathematics of positive, negative and zero

- The next verse presents rules for *guṇana* (multiplication):

  ऋणमृणधनयोर्घातो धनमृणयोः धनवधो धनं भवति।
  शून्यर्णयोः खधनयोः खशून्ययोर्वा वधः शून्यम्॥ ३३ ॥

negative × positive → negative
negative × negative → positive
positive × positive → positive

negative × zero → zero
positive × zero → zero

zero × zero → zero

Mathematics of positive, negative and zero

- The next couple of verses present rules for *haraṇa*:

धनभक्तं धनम् ऋणहृतमृणं धनं भवति खं खभक्तं खम्।
भक्तमृणेन धनमृणं धनेन हृतम् ऋणमृणं भवति॥
खोद्धृतमृणं धनं वा तच्छेदं खमृणधनविभक्तं वा।
ऋणधनयोर्वर्गः स्वं खं खस्य पदं कृतिर्यत् तत्॥

positive ÷ positive → positive
negative ÷ negative → positive
zero ÷ zero → zero (defined the undefined ?)

positive ÷ negative → negative
negative ÷ positive → negative

positive/negative ÷ zero → taccheda
zero ÷ positive/negative → taccheda/zero

(positive/negative)^2 → positive

Operations with *karaṇī* or surds

- Brahmagupta devotes a few verses in the same section to discuss various operations with surds.
- In what follows we present one example.

  इष्टोद्धृतकरणी पदयुतिकृतिः इष्टगुणिता अन्तरकृतिर्वा।
  गुण्यस्तिर्यगधोऽधो गुणकसमस्तद्गुणः सहितः॥

  The surds being divided by a desired (i.e., suitable optional) number, the square of the sum of the square-roots of the quotients is multiplied by the desired number . . .

$$\sqrt{a} \pm \sqrt{b} = \sqrt{c\left\{\sqrt{\left(\frac{a}{c}\right)} \pm \sqrt{\left(\frac{b}{c}\right)}\right\}^2}$$

- Examples:

$$\begin{array}{rrcl} & \sqrt{8}+\sqrt{2} & = & \sqrt{18} \\ \text{and} & \sqrt{8}-\sqrt{2} & = & \sqrt{2}. \end{array}$$

Solution of linear and quadratic equations

- The section on arithmetical operations is followed by the section dealing with solutions of linear and quadratiic equations (*ekavarṇasamīkaraṇam*).
- Brahmagupta first gives the general rule for obtaining solutions:

  अव्यक्तान्तरभक्तं व्यस्तं रूपान्तरं समे, अव्यक्तः ।

  While dealing with linear equation in one unknown, the difference of the known terms taken in reverse order, divided by the difference [of the coefficients] of the unknown is the value of the unknown.

- In the verse above, the word *rūpa* (one with form) is used to refer to numbers of known magnitude (*vyaktāṅka*), as against *avyaktāṅka* ('formless' number, or number with 'unmanifest form').
- Suppose we have an equation,

$$\begin{aligned} Ax + C &= Bx + D \\ \text{then,} \qquad x &= \frac{(D - C)}{(A - B)} \end{aligned}$$

Solution of linear and quadratic equations

- Brahmagupta presents the solution to a quadratic as follows:

  वर्गचतुर्गुणितानां रूपाणां मध्यवर्गसहितानाम्।
  मूलं मध्येनोनं वर्गाद्विगुणोद्धृतं मध्यः॥ ४४ ॥

  The absolute quantities multiplied by four times the coefficient of the square of the unknown are increased by the square of the co-efficient of the middle (i.e., unknown); the square root of the result being diminished by the coefficient of the middle term and divided by twice the coefficient of the square of the unknown (is the value of the) unknown.

- Suppose we conceive of quadratic written in the form (with constant on one side and the unknown on the other),

$$
\begin{aligned}
Ax^2 + Bx &= C \\
\text{then,} \qquad x &= \frac{\left[\sqrt{(4AC + B^2)} - B\right]}{2A}
\end{aligned}
$$

Example of *ekavarṇasamīkaraṇam* (I order)

द्व्यूनमधिमासशेषं त्रिहृतंसप्ताधिकं द्विसङ्गुणितम्।
अधिमासशेषतुल्यं यदा, तदा युगगतं कथय॥ ४७ ॥

- The concept of *adhimāsa* is extremely important in Indian astronomy, as its computation plays a crucial role in our calendrical system.
- It x refers too *adhimāsaśeṣa*, then the content of the above verse, when expressed in modern notation amounts to solving the equation

$$2\left(\frac{x-2}{3}+7\right) = x$$

- This gives the solution $x = 38$

Example of *ekavaṇasamīkaraṇam* (II order)

अधिमासशेषपादात् त्र्यूनात् वर्गोऽधिमासशेषसम्।
अवमावशेषतो वा अवमशेषसमः कदा भवति॥ ५० ॥

- Let x refer to *adhimāsaśeṣa*. The problem posed in the verse when expressed in modern notation translates to

$$\left(\frac{x}{4} - 3\right)^2 = x$$

- Thus we have the following quadratic to be soved

$$x^2 - 40x + 144 = 0$$

- This gives $x = 36$ or 4.

Example of *anekavarṇasamīkaraṇam*

गतभगणयुतात् द्युगणात् तच्छेषयुतात् तदैक्यसंयुक्तात्।
तद्योगात् द्युगणं वा यः कथयति कुट्टकज्ञः सः॥

Let $(N, A) \rightarrow$ (revolution, civil days) in a *kalpa*
$(n', a) \rightarrow$ (revolution, civil days) in a given period.

- Now the no.of revolution made by the planet in *ahargana* a is given by

$$n' = \frac{a \times N}{A} \qquad \text{(integer + fraction)}$$

- This may be rewritten as

$$\begin{aligned} n &= \frac{(a \times N)}{A} + \frac{x}{A} \\ \text{or} \quad n + a &= \frac{a(N + A) + x}{A} = y \\ \text{or} \quad x &= Ay - C, \end{aligned}$$

which is a *kuṭṭākāra* problem.

Bhāvita or equations with products of unknowns

भावितकरूपगुणना सा अव्यक्तवधा इष्टभाजिता इष्टाप्त्योः।
अल्पेऽधिकोऽधिकेऽल्पः क्षेप्यः भावितहृतौ व्यस्तम्॥ ६० ॥

- In the verse above, Brahmagupta presents a rule for finding rational solutions to equations of the form

$$Axy = Bx + Cy + D$$

- The constants in the above equation are referred to as,

$$A \rightarrow \mathit{bh\bar{a}vita} \text{ or } \mathit{bh\bar{a}vitaka}$$
$$D \rightarrow \mathit{r\bar{u}pa}$$
$$B, C \rightarrow \mathit{avyakta}$$

- The first step in finding the solution to the above equation is to choose any number m (*iṣṭa*), and obtain the quotient q (*āpti*).

$$\frac{(BC + AD)}{m} = q \quad (\bar{a}pti)$$

- With m, q, B and C, we obtain the values of x and y. How?

Bhāvita or equations with products of unknowns

- The equation $Axy = Bx + Cy + D$ to be solved may be written as

$$\begin{array}{rrcl} & (Ax - C)y & = & Bx + D \\ \text{or} & (Ay - B)x & = & Cy + D \end{array}$$

- Multiplying the above and simplifying we get,

$$(Ax - C)(Ay - B) = AD + BC$$

- Now, by setting $(Ay - B) = m$, we have

$$(Ax - C) = \frac{(BC + AD)}{m} = q \ \ (\bar{a}pti)$$

- This straightaway gives the solution:

$$x = \frac{(C + q)}{A} \quad \text{and} \quad y = \frac{(B + m)}{A}$$

- Now, by setting $(Ax - C) = m$, we get the other solution

$$x = \frac{(C + m)}{A} \quad \text{and} \quad y = \frac{(B + q)}{A}$$

Bhāvita problem: Illustrative example

भानो राश्यंशवधात् त्रिचतुर्गुणितान् विशोध्य राश्यंशान्।
नवतिं दृष्ट्वा सूर्यं कुर्वन्नावत्सराद्गणकः॥ ६१ ॥

- The two variables appearing in the problem are *rāśi* and *aṃśa* pertaining to the sun.
- The problem posed here may be represented as follows:

$$xy - 3x - 4y = 90 \qquad (A = 1, B = 3, C = 4, D = 90)$$

- In finding solution, the first step is to rewrite the equation in the form

$$(Ax - C)(Ay - B) = AD + BC$$

- Substituting the values we get

$$(x - 4)(y - 3) = 90 + 12 = 6 \times 17$$

- This gives the solution: $x = 10$ and $y = 20$.
- Obviously the solution is not unique. When we factor the RHS as 2×51, we get $x = 6$ and $y = 54$.

Introduction to *vargaprakṛti*

- Brahmagupta discusses the problem of solving for integral values of x, y for equation of the form

$$x^2 - D\,y^2 = K \qquad (D > 0, \text{a non-square integer})$$

- In equation of the above form,

$$x \rightarrow \textit{jyeṣṭha-mūla}, \qquad y \rightarrow \textit{kaniṣṭha-mūla}$$
$$D \rightarrow \textit{prakṛti} \quad \text{and} \quad K \rightarrow \textit{kṣepa}$$

- One motivation for solving problem of this type is to find rational approximation to $\sqrt{D}$.
- If x, y are integers such that $x^2 - D\,y^2 = 1$, then we have

$$\left|\sqrt{D} - \left(\frac{x}{y}\right)\right| \leq \frac{1}{2xy} < \frac{1}{2y^2}$$

- The *Śulva-sūtra* approximation $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} = \frac{577}{408}$ in fact, forms an example of $(577)^2 - 2(408)^2 = 1$.

The *Bhāvanā* principle

मूलं द्विधेष्टवर्गाद् गुणकगुणादिष्टयुतविहीनाच्च।
आद्यवधो गुणकगुणः सहान्त्यघातेन कृतमन्त्यम्॥
वज्रवधैक्यं प्रथमं प्रक्षेपः क्षेपवधतुल्यः।
प्रक्षेपशोधकहृते मूले प्रक्षेपके रूपे॥

- If $(\mathbf{x_1^2} - \mathbf{D\ y_1^2}) = \mathbf{K_1}$ and $(\mathbf{x_2^2} - \mathbf{D\ y_2^2}) = \mathbf{K_2}$ then evidently

$$(\mathbf{x_1 x_2} \pm \mathbf{D\ y_1 y_2})^2 - \mathbf{D}(\mathbf{x_1 y_2} \pm \mathbf{x_2 y_1})^2 = \mathbf{K_1 K_2}$$

- In particular, given $x^2 - D\,y^2 = K$, we get the rational solution

$$\left[\frac{(x^2 + D\,y^2)}{K}\right]^2 - D\left[\frac{(2xy)}{K}\right]^2 = 1$$

- Also, if one solution of the equation $x^2 - D\,y^2 = 1$ is found, an infinite number of solutions can be found, via

$$(x, y) \to (x^2 + D\,y^2, 2xy)$$

Use of *Bhāvanā* when $K = -1, \pm 2, \pm 4$

The *bhāvanā* principle can be used to obtain a solution of equation

$$x^2 - D\,y^2 = 1,$$

if we have a solution of the equation

$$x_1^2 - D\,y_1^2 = K, \text{ for } K = -1, \pm 2, \pm 4.$$

$$\begin{aligned}
K = -1 : x &= x_1^2 + Dy_1^2,\ y = 2x_1y_1. \\
K = \pm 2 : x &= \frac{(x_1^2 + Dy_1^2)}{2},\ y = x_1y_1. \\
K = -4 : x &= (x_1^2 + 2)\left[\frac{1}{2}(x_1^2 + 1)(x_1^2 + 3) - 1\right], \\
y &= \frac{x_1y_1(x_1^2 + 1)(x_1^2 + 3)}{2}. \\
K = 4 : x &= \frac{(x_1^2 - 2)}{2},\ y = \frac{x_1y_1}{2}, \text{ if } x_1 \text{ is even}, \\
x &= \frac{x_1(x_1^2 - 3)}{2},\ y = \frac{y_1(x_1^2 - 1)}{2}, \text{ if } x_1 \text{ is odd}.
\end{aligned}$$

Illustrative examples (given by Brahmagupta)

राशिकलाशेषकृतिं द्विनवतिगुणितां त्र्यशीतिगुणितां वा।
सैकां ज्ञदिने वर्गं कुर्वन्नावत्सराद्गणकः॥ ७५ ॥

- The content of the verse may be translated into the following equations:

$$x^2 - 92\,y^2 = 1 \tag{1}$$
$$x^2 - 83\,y^2 = 1 \tag{2}$$

- Considering equation (1), we start with $(x, y) = (10, 1)$. This gives,

$$10^2 - 92.1^2 = 8 \tag{3}$$

- Doing *bhāvanā*[5] of (3) with itself amounts to

$$(10, 1) \;\rightarrow\; (10^2 + 92.1^2,\; 2.10.1)$$

- Thus we have

$$192^2 - 92.20^2 = 64 \tag{4}$$

- Dividing both sides by 64, we get

$$24^2 - 92.\left(\frac{5}{2}\right)^2 = 1 \tag{5}$$

[5] Recall: $(x, y) \rightarrow (x^2 + D\,y^2, 2xy)$

Illustrative examples (given by Brahmagupta)

- Considering equation (5), and applying *bhāvanā*[6] amounts to

$$\left(24, \left(\frac{5}{2}\right)\right) \rightarrow \left(24^2 + 92.\left(\frac{5}{2}\right)^2, \; 2.24.\left(\frac{5}{2}\right)\right)$$

- Thus we have

$$1151^2 - 92.120^2 = 1. \tag{6}$$

- Similarly, for the other problem $x^2 - 83\,y^2 = 1$, we start with $(x, y) = (9, 1)$. This gives

$$9^2 - 83.1^2 = -2 \tag{7}$$

- Doing the *bhāvanā* of the above with itself amounts to

$$(9, 1) \;\rightarrow\; (9^2 + 83.1^2, \; 2.9.1) \tag{8}$$

- And hence we get

$$164^2 - 83.18^2 = 4 \;\rightsquigarrow\; \mathbf{82^2 - 83.9^2 = 1}$$

[6] Recall: $(x, y) \rightarrow (x^2 + D\,y^2, 2xy)$

Thanks!

THANK YOU