NPTEL COURSE ON

MATHEMATICS IN INDIA: FROM VEDIC PERIOD TO MODERN TIMES

Lecture 19

Development of Combinatorics 2

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Outline

- *Saṅgīta-ratnākara* of Śārṅgadeva (c.1225)
- *Tāna-Prastāra*: Enumeration of permutations or *tānas* of *svaras*
  - *Prastāra*: Rule of enumeration of permutations in the form of an array
  - *Khaṇḍameru* and the processes of *naṣṭa* and *uddiṣṭa*
  - Factorial representation of Śārṅgadeva
- *Tāla-Prastāra*: Enumeration of *tāla* forms
  - The *tālāṅgas*: *Druta, Laghu, Guru* and *Pluta* and their values
  - *Prastāra*: Rule of enumeration of all *tāla*-forms of a given value
  - *Saṅkhyā*: The Śārṅgadeva-sequence of numbers
  - The processes of *naṣṭa* and *uddiṣṭa*
  - Representation of natural numbers as sums of Śārṅgadeva-numbers
  - *Laghu-Meru*
- The general relation between *prastāra* and representation of numbers

Pratyayas in *Saṅgītaratnākara*

The study of combinatorial questions in music was undertaken by Śārṅgadeva (c.1225) in his celebrated treatise on music, *Saṅgītaratnākara*.

In the first chapter of *Saṅgītaratnākara*, there is a discussion of *Tāna-Prastāra* which generates all the possible *tānas* that can be formed from the seven *svaras*.

Later, in Chapter V, there is a very elaborate discussion of the more complicated *Tāla-Prastāra*.

Tāna-Prastāra

In *tāna-prastāra*, Śārṅgadeva considers permutations or *tānas* of subsets of the seven basic musical notes which we denote as S, R, G, M, P, D, N. The *saṅkhyā* or the total number of rows in the *prastāra* is the factorial of the number of elements in *tāna*.

Śārṅgadeva gives the following rule for *tāna-prastāra*:

> क्रमं न्यस्य स्वरः स्थाप्यः पूर्वः पूर्वः परादधः।
> स चेदुपरि तत्पूर्वः पुरस्तूपरिवर्तिनः॥
> मूलक्रमक्रमात् पृष्ठे शेषाः प्रस्तार ईदृशः।

(सङ्गीतरत्नाकरः १.४.६२-६३)

Tāna-Prastāra

- The first row has all the *svaras* in the original order. Successive lines in the *prastāra* are generated as follows.
- Starting from the left, identify the first *svara* which has at least one lower *svara* to the left. Below that is placed the highest of these (lower) *svaras* to the left.
- Then the *svaras* to the right are brought down as they are. The *svaras* left out are placed in the original order to the left, thus completing the next line of the *prastāra*.

Śārṅgadeva's rule for the construction of the *prastāra* is applicable for the enumeration of the permutations of n elements with a natural order.

It generates all the permutations in the so-called colex order (mirror image of lexicographic order in reverse).

Example: *Tāna-Prastāra* of SRG

1	S	R	G
2	R	S	G
3	S	G	R
4	G	S	R
5	R	G	S
6	G	R	S

Example: *Tāna-Prastāra* of SRGM

1	S	R	G	M
2	R	S	G	M
3	S	G	R	M
4	G	S	R	M
5	R	G	S	M
6	G	R	S	M
7	S	R	M	G
8	R	S	M	G
9	S	M	R	G
10	M	S	R	G
11	R	M	S	G
12	M	R	S	G

Example: *Tāna-Prastāra* of SRGM (contd.)

13	S	G	M	R
14	G	S	M	R
15	S	M	G	R
16	M	S	G	R
17	G	M	S	R
18	M	G	S	R
19	R	G	M	S
20	G	R	M	S
21	R	M	G	S
22	M	R	G	S
23	G	M	R	S
24	M	G	R	S

Khaṇḍa-Meru

In order to discuss the *naṣṭa* and *uddiṣṭa* processes, Śārṅgadeva introduces the so called *Khaṇḍa-Meru*:

सप्ताद्येकान्तकोष्ठानामधोऽधः सप्त पङ्क्तयः॥
तास्वाद्यायामाद्यकोष्ठे लिखेदेकं परेषु खम्।
वेद्यतानस्वरमितान् न्यसेत् तेष्वेव लोष्टकान्॥
प्राक्पङ्क्त्यन्त्याङ्कसंयोगमूर्ध्वाधः स्थितपङ्क्तिषु।
शून्यादधो लिखेदेकं तं चाधोऽधः स्वकोष्ठकान्॥
कोष्ठसङ्ख्यागुणं न्यस्येत् खण्डमेरुरयं मतः।(सङ्गीतरत्नाकरः १.४.६३-६६)

- Place 1 followed by 0s in the first row.
- Place the factorials of 1, 2, 3 etc., in the next row, starting from the second column.
- Place twice, thrice etc., of the factorials in the succeeding rows, starting from a later column at each stage.

S	R	G	M	P	D	N
1	0	0	0	0	0	0
	1	2	6	24	120	720
		4	12	48	240	1440
			18	72	360	2160
				96	480	2880
					600	3600
						4320

Note that starting from the second row, each column consists of the multiples of factorials. As we shall see, the factorials play a crucial role in the *naṣṭa* and *uddiṣṭa* processes.

Uddiṣṭa

स्वरान् मूलक्रमस्यान्त्यात् पूर्वं यावतिथः स्वरः ॥
उद्दिष्टान्त्यस्तावतिथे कोष्ठेऽधौ लोष्टकं क्षिपेत्।
लोष्टचालनमन्त्यात् स्यात्त्यक्त्वा लब्धं क्रमो भवेत्॥
लोष्टाक्रान्ताङ्कसंयोगादुद्दिष्टस्य मितिर्भवेत्।

(सङ्गीतरत्नाकरः १.४.६६-६८)

- Given a *tāna* (of n *svaras*), note the rank of the last *svara* in the reverse of natural order among the given *svaras*. Mark the corresponding entry in the last or the n-th column.
- Note the rank of the next *svara* (in the reverse of natural order) among the remaining *svaras*. Mark the corresponding entry in the next or the $(n-1)$-th column. And so on.
- The *uddiṣṭa* or the rank-number of the given *tāna* will be the sum of all the marked entries.

Uddiṣṭa

Example: To find the row of the *tāna* MSRG in the *prastāra* of SRGM

- G is the second from the last among SRGM. Hence, mark 6, the entry in second row, in the last or the fourth column.
- R is the second from the last among the remaining SRM. Hence Mark 2 in the third column.
- S is the second from the last among SM. Hence, mark 1 in the second column.
- M is the only *svara* left. Mark 1 in the first column.

Row-number of MSRG in the *prastāra* = 1+ 1+ 2+ 6 = 10

S	R	G	M
1	0	0	0
	1	**2**	**6**
		4	12
			18

Naṣṭa

यैरङ्कैर्नष्टसङ्ख्या स्यान्मौलैकाङ्कसमन्वितैः॥
तेषु लोष्टं क्षिपेन्मूले लोष्टस्थानमितं भवेत्।
नष्टतानस्वरस्थानं ततो यावतिथे पदे॥
अधःक्रमादस्तिलोष्टः स्वरस्तावतिथो भवेत्।
क्रमान्तिमस्वरात्पूर्वो लब्धत्यागादि पूर्ववत्॥ (सङ्गीतरत्नाकरः १.४.६८-७०)

- To find the *tāna* (of n *svaras*) corresponding to a given row-number (*naṣṭa-saṅkhyā*), mark the entry just below the rank-number in the n-th column.
- Subtract that entry from the rank number and mark the entry, which is just below the resulting number, in the next or the $(n-1)$-th column. And so on.
- The position of the marked entry in the last column gives the rank of the last *svara* of the *tāna* in the reverse natural order.
- The position of the marked entry in the next column gives the rank of the last but one *svara*, amongst the remaining *svaras*, in the reverse natural order. And so on.

Naṣṭa

Example: To find the 18^{th} $t\bar{a}na$ in the $prast\bar{a}ra$ of SRGM

- In the fourth column, the number just below 18 is 12, which is in the third row. Hence, the fourth $svara$ is the third among SRGM in reverse order: **R**
- 18-12 = 6. In the third column, the number just below 6 is 4, which is in the third row, which is just below 6. Hence, the next $svara$ is the third among SGM in reverse order: **S**
- 6-4 = 2. In the second column, the number just below 2 is 1, which is in the second row. Hence the next $svara$ is the second among GM in reverse order: **G**
- 2-1 = 1. The other $svara$ left is **M**

The 18th $t\bar{a}na$ is **MGSR**

S	R	G	M
1	0	0	0
	1	2	6
		4	**12**
			18

Factorial Representation of Śārṅgadeva

The *naṣṭa* and *uddiṣṭa* processes are essentially based on a certain factorial representation of numbers.

In the above examples,

$$\begin{aligned} 10 = 1 + 1 + 2 + 6 &= 1.0! + 1.1! + 1.2! + 1.3! \\ 18 = 1 + 1 + 4 + 12 &= 1.0! + 1.1! + 2.2! + 2.3! \end{aligned}$$

where we have used the convention $0! = 1$.

In fact the general result may be stated as follows:

> Every integer $1 \leq m \leq n!$ can be uniquely represented in the form
>
> $$\mathbf{m} = \mathbf{d_0 0}! + \mathbf{d_1 1}! + \mathbf{d_2 2}! + \ldots + \mathbf{d_{n-1}}(\mathbf{n} - \mathbf{1})!,$$
>
> where d_i are integers such that $d_0 = 1$ and $0 \leq d_i \leq i$, for $i = 1, 2, \ldots, n-1$.

In particular,

$$n! = 1.0! + 1.1! + 2.2! + \ldots + (n-1)\,(n-1)!$$

Tāla-Prastāra

Chapter V of *Saṅgītaratnākara* is the *Tālādhyāya* with 409 verses. The first 311 verses discuss *mārga-tālas* and about 120 *deśī-tālas*. At the end of this discussion, it is noted that there are indeed very many such *tālas* and it would not be possible to display all of them. This sets the stage for the *prastāra-prakaraṇa* which takes up the remaining nearly 100 verses of the *Tālādhyāya*.

The *tālāṅgas* considered here are *Druta, Laghu, Guru* and *Pluta,* which are taken to be of duration 1, 2, 4 and 6 respectively, in *Druta* units. *Tāla-prastāra* consists in a systematic enumeration of all *tālas* with the same total duration (*kāla-pramāṇa*)

Thus the *Tāla-prastāra* is a non-trivial generalisation of *Mātrā-vṛtta-prastāra*. Nārāyaṇa Paṇḍita in his *Gaṇitakaumudī* (C.1350) has discussed the simpler generalisation of *Mātrā-vṛtta-prastāra*, which involves the elements L, G and P with relative values 1, 2 and 3 respectively.

न्यस्याल्पमाद्यान्महतोऽधस्ताच्छेषं यथोपरि।
प्रागूने वामसंस्थांस्तु संभवे महतो लिखेत्॥

अल्पानसंभवे तालपूर्त्यै भूयोऽप्ययं विधिः।
सर्वद्रुतावधिः कार्यः प्रस्तारोऽयं लघौ गुरौ॥

प्लुते व्यस्ते समस्ते च न तु व्यस्ते द्रुतेऽस्ति सः।

(सङ्गीतरत्नाकरः ५.३१६-३१८)

Tāla-Prastāra

Śārṅgadeva's procedure for the construction of *prastāra* is as follows:

- The last row of the *prastāra* has all *Drutas* only.
- In the first row, place as many Ps as possible to the right, followed, if possible (from right to left), by a G and a D or a G alone, or by an L and a D or an L alone, or by a D alone, to the left.
- To go from any row of the *prastāra* to the next, identify the first non-D element from the left. Place below that the element next to it in duration: D below an L, L below a G and G below a P.
- Bring down the elements to the right as they are.
- Make up for the deficient units (if any) by adding to the left as many Ps as possible, followed similarly by Gs, Ls and Ds in that order from right to left.

6-*Druta-Prastāra*

1						P							6
2					L	G						2	4
3				D	D	G					1	1	4
4					G	L						4	2
5				L	L	L					2	2	2
6			D	D	L	L				1	1	2	2
7			D	L	D	L				1	2	1	2
8			L	D	D	L				2	1	1	2
9		D	D	D	D	L			1	1	1	1	2
10				D	G	D					1	4	1
11			D	L	L	D				1	2	2	1
12			L	D	L	D				2	1	2	1
13		D	D	D	L	D			1	1	1	2	1
14				G	D	D					4	1	1
15			L	L	D	D				2	2	1	1
16		D	D	L	D	D			1	1	2	1	1
17		D	L	D	D	D			1	2	1	1	1
18		L	D	D	D	D			2	1	1	1	1
19	D	D	D	D	D	D		1	1	1	1	1	1

7-*Druta-Prāstra*

1						D	P
2					D	L	G
3					L	D	G
4				D	D	D	G
5					D	G	L
6				D	L	L	L
7				L	D	L	L
8			D	D	D	L	L
9					G	D	L
10				L	L	D	L
11			D	D	L	D	L
12			D	L	D	D	L
13			L	D	D	D	L
14		D	D	D	D	D	L

7-*Druta-Prāstra* (contd.)

15						P	D
16					L	G	D
17				D	D	G	D
18					G	L	D
19				L	L	L	D
20			D	D	L	L	D
21			D	L	D	L	D
22			L	D	D	L	D
23		D	D	D	D	L	D
24				D	G	D	D
25			D	L	L	D	D
26			L	D	L	D	D
27		D	D	D	L	D	D
28				G	D	D	D
29			L	L	D	D	D
30		D	D	L	D	D	D
31		D	L	D	D	D	D
32		L	D	D	D	D	D
33	D	D	D	D	D	D	D

एकद्व्यङ्कौ क्रमान्न्यस्य युञ्जीतान्त्यं पुरातनैः।
द्वितीयतुर्यषष्ठाङ्कैरभावे तुर्यषष्ठयोः॥
तृतीयपञ्चमाङ्काभ्यां क्रमात् तं योगमग्रतः।
लिखेद् दक्षिणसंस्थैवमङ्कश्रेणी विधीयते॥...
यदङ्कयोगादन्त्योऽङ्को लब्धस्तैरन्ततः क्रमात्।
भेदा द्रुतान्तलघ्वन्तगुर्वन्ताश्च प्लुतान्तकाः॥

(सङ्गीतरत्नाकरः ५.३१९, ३२०, ३२४)

Saṅkhyā

Śārṅgadeva makes the observation that among all the *tāla*-forms which appear in the n-*druta-prastāra*, S_{n-1} end in a D, S_{n-2} in a L, S_{n-4} in a G and S_{n-6} end in a P, and hence the total number of forms *tāla*-forms S_n in the n-*druta-prastāra* is just the sum of these four numbers. Thus,

$$S_n = S_{n-1} + S_{n-2} + S_{n-4} + S_{n-6}$$

Noting $S_1 = 1, S_2 = 2$, we get the Śārṅgadeva sequence of *saṅkhyāṅkas*:

n	1	2	3	4	5	6	7	8	9	10
S_n	1	2	3	6	10	19	33	60	106	191

Uddiṣṭa

To find the row-number of a given *tāla*-form in a n-*Druta-Prastāra*, write the Śārṅgadeva *saṅkhyāṅkas* $S_1, S_2, \ldots$ sequentially from the left on top of the *tāla*-form in the following way:

- Write one *saṅkhyāṅka* above a D, two above an L, four above a G and six above each P.
- Sum the following (we shall see later that these are what are called the *patita-saṅkhyāṅkas*): The first *saṅkhyāṅka* above each L, the second and third *saṅkhyāṅkas* above each G and the second, fourth and fifth *saṅkhyāṅkas* above each P.
- The row-number of the given *tāla* form is obtained by subtracting the above sum from S_n.

Uddiṣṭa

Example: To find the row-number of LDLL in 7-*druta-prastāra*

S_n	**1**	2	3	**6**	10	**19**	33
	L		D	L		L	

Total of the *patita* S_n : $19 + 6 + 1 = 26$. Row-number: $33 - 26 = 7$

Example: To find the row-number of **GDL** in 7-*druta-prastāra*

S_n	1	**2**	**3**	6	10	**19**	33
	G				D	L	

Total of the *patita* S_n : $19 + 3 + 2 + 24$. Row-number: $33 - 24 = 9$

Example: To find the row-number of **PD** in 7-*druta-prastāra*

S_n	1	**2**	3	**6**	**10**	19	33
	P						D

Total of the *patita* S_n : $10 + 6 + 2 = 18$, Row-number: $33 - 18 = 15$

Naṣṭa

If it is required to find the *tāla*-form in the r-th row of n-*druta-prastāra*, the following procedure is prescribed:

- Place sequentially the Śārṅgadeva sequence of *saṅkhyāṅkas* $S_1, S_2, \ldots, S_n$.
- Check if the *saṅkhyāṅkas* S_{n-1} can be subtracted from $(S_n - r)$. If so, mark S_{n-1} as "**p**" (*patita-saṅkhyāṅka*) and go on to check if S_{n-2} can be subtracted from $(S_n - r - S_{n-1})$ and so on.
- If S_{n-1} cannot be subtracted from $(S_n - r)$ mark it as "**a**" (*apatita-saṅkhyāṅka*) and go on to check if S_{n-2} can be subtracted from $(S_n - r)$ and so on.
- In this way mark all *saṅkhyāṅkas* as either "**p**" or "**a** ".

Use the following signatures of various *tālāṅgas* to find the *tāla*-form:

	S_{n-6}	S_{n-5}	S_{n-4}	S_{n-3}	S_{n-2}	S_{n-1}	S_n
D						(a)	a
L					(a)	p	a
G			(a)	a	p	p	a
P	(a)	a	p	a	p	p	a

Naṣṭa

Example: To find the 8th *tāla*-form in the 7-*druta-prastāra*:

$$33 - 8 = 25, 25 - 19 = 6, 6 - 6 = 0$$

The *patita* and *apatita* S_n are given below

p/a	a	a	a	p	a	p	a
S_n	1	2	3	**6**	10	**19**	33

Starting from 33, since 19 is *patita* and 10 is *apatita*, we get an **L** at the right extreme.

Starting from 10, since 6 is *patita* and 3 is *apatita*, we get another **L** to the left of the first.

Starting from 3, since 2 is *apatita*, we get a **D** to the left.

Starting from 2, since 1 is *apatita* we get one more **D** to the left.

Since 1 is *apatita*, we get one more **D**.

Thus the *tāla*-form is **DDDLL**

Naṣṭa

Example: To find the 28th *tāla* form in the 7-*druta-prastāra*:

$33 - 28 = 5, \; 5 - 3 = 2, \; 2 - 2 = 0$

p/a	a	p	p	a	a	a	a
S_n	1	**2**	**3**	6	10	19	33

33, 19, 10 and 6 are *apatita* and thus we get **DDD** from the right.

Starting from 6, 3 and 2 are *patita* and 1 is *apatita*. They give a **G**.

Thus the *tāla*-form is **GDDD**

It can be shown that both the *naṣṭa* and *uddiṣṭa* processes for the *tāla-prastāra* are based on a very interesting property that every natural number can be uniquely written as a sum of the Śārṅgadeva *saṅkhyāṅkas* $S_1, \; S_2, \ldots,$ satisfying certain conditions.

Laghu-Meru

Śārṅgadeva discusses the *lagakriyā* process for *Tāla-prastāra* in terms of various tables, *Druta-Meru, Laghu-Meru, Guru-Meru* and *Pluta-Meru*. We disply below the *Laghu-Meru*.

									1
							1	5	15
					1	4	10	20	39
			1	3	6	10	18	33	61
	1	2	3	4	7	12	21	34	54
1	1	1	2	3	5	7	10	14	21
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**

For instance, the column 7 of the above *Meru* shows that of the 33 *tāla*-forms in the 7-*Druta-prastāra* there are 7 *tāla*-forms with 0L, 12 with 1L, 10 with 2L and 4 with 3L.

If L_k^n denotes the number of *tāla*-forms with k *laghus* in the n-*druta-prastāra*, then we have the following recurrence relation:

$$L_k^n = L_k^{n-1} + L_{k-1}^{n-2} + L_k^{n-4} + L_k^{n-6} \quad (\text{for } n > 6 \text{ and } k > 0)$$

Prastāra and Representation of Numbers

- These instances of *prastāras* in prosody and music show that in each case there is associated a unique representation of the natural numbers in terms of the *saṅkhyāṅkas* associated with the *prastāra*.
- It is this representation which facilitates the *naṣṭa* and *uddiṣṭa* processes in each of these *prastāras*.
- The *varṇa-vṛtta-prastāra* has associated with it the binary representation of natural numbers.
- The *mātrā-vṛtta-prastāra* has associated with it a representation of numbers in terms of Virahāṅka (or the so called Fibonacci) numbers.

Prastāra and Representation of Numbers

- The *tāna-prastāra* of Śārṅgadeva has associated with it a factorial representation of numbers.
- The *tāla-prastāra* of Śārṅgadeva has associated with it a representation of numbers in terms of Śārṅgadeva numbers.
- The *prastāra* of combinations of r objects selected from a set of n, has been studied by Nārāyaṇa Paṇḍita in *Gaṇitakaumdī*. Here, the *saṅkhyāṅkas* are the binomial co-efficients ${}^nC_r$ and there is an associated representation of every number as a sum of such binomial co-efficients.

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Thanks!

Thank You