NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 24

Bījagaṇita of **Bhāskarācārya** 2

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Outline

- Review of the *Cakravāla* method
- Analysis of the *Cakravāla* method by Krishnaswami Ayyangar
- History of the solution of the so called “Pell’s Equation” $X^2 - D\,Y^2 = 1$
- Solution of “Pell’s equation” by expansion of $\sqrt{D}$ into a simple continued fraction.
- Bhāskara semi-regular continued fraction expansion of $\sqrt{D}$
- Optimality of the *Cakravāla* method.

Cakravāla according to Bhāskara

In 1930, Krishnaswami Ayyangar showed that the *cakravāla* procedure always leads to a solution of the *vargaprakṛti* equation with $K = 1$. He also showed that the *kuṭṭaka* condition (I) is equivalent to the simpler condition

(I′) $\mathbf{P_i + P_{i+1}}$ is divisible by $\mathbf{K_i}$

Thus, we shall use the *cakravāla* algorithm in the following form:

To solve $\mathbf{X^2 - D\ Y^2 = 1}$: Set $X_0 = 1, Y_0 = 0, K_0 = 1,\ P_0 = 0$.

Given $X_i,\ Y_i,\ K_i$ such that $X_i^2 - D\ Y_i^2 = K_i$

First find $P_{i+1} > 0$ so as to satisfy:

(I′) $\mathbf{P_i + P_{i+1}}$ is divisible by $\mathbf{K_i}$

(II) $\mathbf{|P_{i+1}^2 - D|}$ is minimum.

Cakravāla according to Bhāskara

Then set

$$K_{i+1} = \frac{(P_{i+1}^2 - D)}{K_i} \quad Y_{i+1} = \frac{(Y_i P_{i+1} + X_i)}{|K_i|} = a_i Y_i + \varepsilon_i \; Y_{i-1}$$

$$X_{i+1} = \frac{(X_i P_{i+1} + D \; Y_i)}{|K_i|} = P_{i+1} Y_{i+1} - sign(K_i) K_{i+1} Y_i = a_i X_i + \varepsilon_i X_{i-1}$$

These satisfy $X_{i+1}^2 - D \; Y_{i+1}^2 = K_{i+1}$

Iterate till $K_{i+1} = \pm 1, \; \pm 2$ or ± 4, and then use *bhāvanā* if necessary.

Note: We also need $\mathbf{a_i} = \frac{(\mathbf{P_i} + \mathbf{P_{i+1}})}{|\mathbf{K_i}|}$ **and** $\varepsilon_\mathbf{i} = \frac{(\mathbf{D} - \mathbf{P_i^2})}{|\mathbf{D} - \mathbf{P_i^2}|}$ with $\varepsilon_0 = 1$

Bhāskara's Example: $X^2 - 67\ Y^2 = 1$

i	$\mathbf{P_i}$	$\mathbf{K_i}$	$\mathbf{a_i}$	$\varepsilon_\mathbf{i}$	$\mathbf{X_i}$	$\mathbf{Y_i}$
0	0	1	8	1	1	0
1	8	-3	5	1	8	1
2	7	6	2	1	41	5
3	5	-7	2	1	90	11
4	9	-2	9	-1	221	27
5	9	-7	2	-1	1,899	232
6	5	6	2	1	3,577	437
7	7	-3	5	1	9,053	1,106
8	8	1	16	1	**48,842**	**5,967**

$$48842^2 - 67.\ 5967^2 = 1$$

Analysis of the *Cakravāla* Process

In 1930, Krishnaswami Ayyangar presented a detailed analysis of the *cakravāla* process. He explained how it is different from the Euler-Lagrange process based on the simple continued fraction expansion of $\sqrt{D}$. He also showed, for the first time, that the *cakravāla* process always leads to a solution of the *vargaprakṛti* equation with $K = 1$.

Let us consider the equations

$X_i^2 - D\, Y_i^2 = K_i$

$P_{i+1}^2 - D.1^2 = P_{i+1}^2 - D$

By doing *bhāvanā* of these, we get

$$\left[\frac{(X_i P_{i+1} + D\, Y_i)}{|K_i|}\right]^2 - D\left[\frac{(Y_i P_{i+1} + X_i)}{|K_i|}\right]^2 = \frac{(P_{i+1}^2 - D)}{K_i}$$

If we assume that X_i, Y_i and K_i are mutually prime, and if we choose P_{i+1} such that $Y_{i+1} = \left[\frac{(Y_i P_{i+1} + X_i)}{|K_i|}\right]$ is an integer, then it can be shown that $X_{i+1} = \left[\frac{(X_i P_{i+1} + D\, Y_i)}{|K_i|}\right]$ and $K_{i+1} = \frac{(P_{i+1}^2 - D)}{K_i}$ are both integers.

Analysis of the *Cakravāla* Process

Further, we have

$$\begin{aligned} X_{i+2} &= \left[\frac{(X_{i+1} P_{i+2} + D\, Y_{i+1})}{|K_{i+1}|}\right] \\ &= X_{i+1}\left[\frac{(P_{i+2} + P_{i+1})}{|K_{i+1}|}\right] + \frac{X_i\,(D - P_{i+1}^2)}{|K_i|\,|K_{i+1}|} \\ &= a_{i+1}\, X_{i+1} + \varepsilon_{i+1}\, X_i \end{aligned}$$

and similarly for Y_{i+2}.

Therefore, instead of using the *kuṭṭaka* process for finding P_{i+2}, we can use the condition that

(I′) $\mathbf{P_{i+1} + P_{i+2}}$ is divisible by $\mathbf{K_{i+1}}$.

Analysis of the *Cakravāla* Process

Krishnaswami Ayyangar, then proceeds to a study of the quadratic forms $(K_i,\ P_{i+1},\ K_{i+1})$ which satisfy

$$P_{i+1}^2 - K_i\ K_{i+1} = D.$$

The form $(K_{i+1},\ P_{i+2},\ K_{i+2})$, which is obtained from $(K_i,\ P_{i+1},\ K_{i+1})$ by the *cakravāla* process, is called the successor of the latter.

Ayyangar defines a quadratic form

$$(A,\ B,\ C)\ \equiv\ Ax^2 + 2Bxy + Cy^2$$

to be a **Bhāskara form** if

$$\mathbf{A^2} + \left(\frac{\mathbf{C^2}}{\mathbf{4}}\right) < \mathbf{D}\ \textbf{and}\ \mathbf{C^2} + \left(\frac{\mathbf{A^2}}{\mathbf{4}}\right) < \mathbf{D}$$

He shows that the successor of a Bhāskara form is also a Bhāskara form and that two different Bhāskara forms cannot have the same successor.

Analysis of the *Cakravāla* Process

Krishnaswami Ayyangar considers the general case when we start the *cakravāla* process with an arbitrary initial solution

$X_0^2 - D\,Y_0^2 = K_0$

He shows that if $|K_0| > \sqrt{D}$, then the absolute values of the successive K_i decrease monotonically, till say K_m, after which we have $|\mathbf{K_i}| < \sqrt{\mathbf{D}}$ for $i > m$. He also shows that $|\mathbf{P_i}| < 2\sqrt{\mathbf{D}}$ for $i > m$.

Since $|K_i|$ cannot go on decreasing, for some $r > m$ we have $|K_{r+1}| > |K_r|$. It can then be shown that (K_r, P_{r+1}, K_{r+1}) and all the succeeding forms will be Bhāskara forms.

It can also be shown that the P_i's do not change sign and they can all be taken to be positive.

Analysis of the *Cakravāla* Process

If we start with $X_0 = 1,\ Y_0 = 0$ and $K_0 = 1$, then we see that *cakravāla* process leads to $P_1 = X_1 = d$, where $d > 0$ is the integer such that d^2 is the square nearest to D. Also $Y_0 = 1$ and $K_1 = d^2 - D$.

Ayyangar shows that $(K_0,\ P_1,\ K_1) \equiv (1,\ d,\ d^2 - D)$ is a Bhāskara form. So is the form $(d^2 - D, d, 1)$ equivalent to it.

Since the values of $K_i,\ P_i$ are bounded, the Bhāskara forms will have to repeat in a cycle and the first member of the cycle is the same as the first Bhāskara form which is obtained in the course of *cakravāla*.

Finally, Ayyangar shows that two different cycles of Bhāskara forms are non-equivalent, and that all equivalent Bhāskara forms belong to the same cycle. To show this, he sets up an association between a Bhāskara form $(K_i,\ P_{i+1},\ K_{i+1})$ and an equivalent **Gauss form**

$$(K_i^{'},\ P_{i+1}^{'},\ K_{i+1}^{'}),\ \textbf{which satisfies}\ \sqrt{D} - P_{i+1}^{'} < |K_i^{'}| < \sqrt{D} + P_{i+1}^{'}.$$

Analysis of the *Cakravāla* Process

If $P_{i+1} < \sqrt{D}$, then

$$(K_i^{'},\ P_{i+1}^{'},\ K_{i+1}^{'}) \equiv (K_i, P_{i+1},\ K_{i+1})$$

If $P_{i+1} > \sqrt{D}$, then

$$K_i^{'} = K_i,\ P_{i+1}^{'} = P_{i+1} - |K_i| \text{ and } K_{i+1}^{'} = 2P_{i+1} - |K_i| - |K_{i+1}|$$

In this way a Bhāskara cycle can be converted to a unique Gauss cycle and vice versa, from which the above results follow.

Thus, whatever initial solution we may start with, the *cakravāla* process takes us to a cycle of equivalent Bhāskara forms and since the Bhāskara form $(d^2 - D,\ d, 1)$ is in this equivalence class, the *cakravāla* process invariably leads to a solution corresponding to $K = 1$.

Fermat's Challenge to British Mathematicians (1657)

In February 1657, Pierre de Fermat (1601-1665) wrote to Bernard Frenicle de Bessy asking him for a general rule "for finding, when any number not a square is given, squares which, when they are respectively multiplied by the given number and unity added to the product, give squares." If Frenicle is unable to give a general solution, Fermat said, can he at least give the smallest values of x and y which will satisfy the equations $\mathbf{61x^2 + 1 = y^2}$ and $109x^2 + 1 = y^2$.

At the same time Fermat issued a general challenge, addressed to the mathematicians in northern France, Belgium and England:

> "...I propose the following theorem to be proved or problem to be solved... Given any number whatever which is not a square, there are also given infinite number of squares such that, if the square is multiplied into the given number and unity is added to the product, the result is a square.
>
> Eg. Let it be required to find a square such that, if the product of the square and the number 149, or 109, or 433 etc. be increased by 1, the result is a square."

Brounker-Wallis Solution

Fermat's Challenge was addressed to William Brouncker (1620-1684) and John Wallis (1616-1703). Brouncker's first response merely contained rational solutions and this led to Fermat complaining (in a letter to the interlocutor Kenelm Digby in August 1657) that they were no solutions at all to the problem that he had posed.

Brouncker then worked out his method of integral solutions which he sent to Wallis to be communicated to Fermat. Wallis describes the method of solution in two letters dated December 17, 1657 and January 30, 1658. Later in 1658, Wallis published the entire correspondence as *Commercium Epistolicum.* He also outlined the method in his *Algebra* published in English in 1685 and in Latin in 1693.

We do not know what method Fermat had for the solution of the problem he posed. Of course, he communicated to the English mathematicians that he "willingly and joyfully acknowledges" the validity of their solutions. He however wrote to Huygens in 1659 that the English had failed to give "a general proof", which according to him could only be obtained by the "method of descent".

Euler-Lagrange Method of Solution

In a paper *“De solution problematum Diophanterum per numerous integros”* written in 1730, Euler describes Wallis method, but ascribes it to John Pell. He also shows that from one solution of “Pell’s equation” an infinite number of solutions can be found and also remarks that they give good approximations to square-roots.

In a paper, read in 1759 but published in 1767, entitled *“De Usu novi algoritmi in problemate Pelliano solvendo”*, Euler describes the method of solving $X^2 - DY^2 = 1$ by the simple continued fraction expansion of $\sqrt{D}$. He gives a table of partial quotients for all non-square integers from 2 to 120 and also notes their various properties.

In a paper which was published earlier in 1764 Euler proved the $bh\bar{a}van\bar{a}$ principle and called it **“Theorema Elegantissimum”.**

In a set of three papers presented to the Berlin Academy in 1768, 1769 and 1770, Joseph Louis Lagrange (1736-1813) worked out the complete theory of simple continued fractions and their applications to “Pell’s equation” along with all the necessary proofs.

Relation with Continued Fraction Expansion

A simple continued fraction (a_i are positive integers for $i > 0$)

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

is also denoted by $[a_0,\ a_1,\ a_2,\ \ldots]$ or by $a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots$

Given any real number α, to get the continued fraction expansion, take $a_0 = [\alpha]$, the integral part of α.

Let $\alpha_1 = \frac{1}{(\alpha-[\alpha])}$. Then we take $a_1 = [\alpha_1]$

Let $\alpha_2 = \frac{1}{(\alpha_1-[\alpha_1])}$. Then we take $a_2 = [\alpha_2]$, and so on.

$a_0,\ a_1,\ a_2, \ldots$ are called partial quotients; $\alpha_1,\ \alpha_2, \ldots$ are the complete quotients.

Relation with Continued Fraction Expansion

The j-th convergent of the continued fraction

$$[a_0,\ a_1,\ a_2,\ a_3, \ldots]$$

is given by

$$\frac{A_j}{B_j} = [a_0,\ a_1,\ a_2,\ a_3, \ldots, a_j]$$

$A_j,\ B_j$ satisfy the recurrence relations:

$$\begin{aligned} A_0 &= a_0, A_1 = a_1 a_0 + 1, \\ A_j &= a_j A_{j-1} + A_{j-2} \text{ for } j \geq 2 \\ B_0 &= 1,\ B_1 = a_1, \\ B_j &= a_j B_{j-1} + B_{j-2} \text{ for } j \geq 2 \end{aligned}$$

The convergents also satisfy

$$A_j\, B_{j-1} - A_{j-1} B_j = (-1)^{j-1}$$

Relation with Continued Fraction Expansion

Example:

$$\frac{149}{17} = [8, 1, 3, 4]$$

The convergents are $\frac{A_0}{B_0} = \frac{8}{1}$, $\frac{A_1}{B_1} = \frac{9}{1}$, $\frac{A_2}{B_2} = \frac{35}{4}$, $\frac{A_3}{B_3} = \frac{149}{17}$

We have $A_3\ B_2 - A_2\ B_3 = 149.4 - 35.17 = 1$

This is very similar to the $kuṭṭaka$ method for solving $149x - 17y = 1$.

Note: The simple continued fraction expansion of a real number does not terminate if the number is irrational. For instance

$$\frac{(1+\sqrt{5})}{2} = [1, 1, 1, 1, ...]$$

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots]$$

Relation with Continued Fraction Expansion

It was noted by Euler that the simple continued fraction of $\sqrt{D}$ is always periodic and is of the form

$\sqrt{D} = [a_0, \overline{a_1, \ldots, a_{h-1}, a_{h-1}, \ldots, a_1, 2a_0}]$ if $k = 2h - 1$,

$\sqrt{D} = [a_0, \overline{a_1, \ldots, a_{h-1}, a_h, a_{h-1}, \ldots, a_1, 2a_0}]$ if $k = 2h$,

where k is the length of the period, and that the associated convergents $A_{k-1},\ B_{k-1}$ satisfy

$A_{k-1}^2 - DB_{k-1}^2 = (-1)^k$

Further, all the solutions of, $X^2 - D\ Y^2 = 1$ can be obtained by composing ($bh\bar{a}van\bar{a}$) of the above solution with itself.

These results were later proved by Lagrange.

Example: To solve $X^2 - 13\ Y^2 = 1$

$$\sqrt{13} = 3 + \overline{\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{6+}}$$

$\frac{A_4}{B_4} = \frac{18}{5}$ and we have $18^2 - 13.5^2 = -1$

Doing $bh\bar{a}van\bar{a}$ of this solution with itself, we get $649^2 - 13.180^2 = 1$

Semi-Regular Continued Fractions

A semi-regular continued fraction is of the form

$$a_0 + \frac{\varepsilon_1}{a_1+}\ \frac{\varepsilon_2}{a_2+}\ \frac{\varepsilon_3}{a_3+}\ \ldots$$

where $\varepsilon_i = \pm 1,\ a_i \geq 1$ for $i \geq 1$, and $a_i + \varepsilon_{i+1} \geq 1$ for $i \geq 1$.

Then the convergents satisfy the relations

$$\begin{aligned}
A_0 &= a_0,\ A_1 = a_1 a_0 + \varepsilon_1, \\
A_j &= a_j\, A_{j-1} + \varepsilon_j A_{j-2} \quad \text{for } \ j \geq 2 \\
B_0 &= 1,\ B_1 = a_1, \\
B_j &= a_j B_{j-1} + \varepsilon_j B_{j-2} \quad \text{for } \ j \geq 2
\end{aligned}$$

Bhāskara Semi-Regular Continued Fractions

Krishnaswami Ayyangar showed that the *cakravāla* method of Bhāskara corresponds to a periodic semi-regular continued function expansion

$$\sqrt{\mathbf{D}} = \mathbf{a_0} + \frac{\varepsilon_{\mathbf{1}}}{\mathbf{a_1}+}\ \frac{\varepsilon_{\mathbf{2}}}{\mathbf{a_2}+}\ \frac{\varepsilon_{\mathbf{3}}}{\mathbf{a_3}+}\ \ldots$$

where

$$a_i = (P_i + P_{i+1})/\mid K_i \mid,\ \varepsilon_i = (D - P_i^2)/\mid D - P_i^2 \mid$$

and the convergents are related to the solutions $A_j = X_{j+1}$ and $B_j = Y_{j+1}$.

Note: The Simple Continued Fraction and the Nearest Integer Continued Fraction can also be generated by a *cakravāla* type of algorithm if we replace the condition II respectively by

(II') $\mathbf{D} - \mathbf{P}^{\mathbf{2}}_{\mathbf{i+1}} > \mathbf{0}$ and is minimum

(II") $\mid P_{i+1} - \sqrt{D} \mid$ **is minimum**

Euler-Lagrange Method for $X^2 - 67Y^2 = 1$

i	P_i	K_i	a_i	ε_i	X_i	Y_i
0	0	1	8	1	1	0
1	8	-3	5	1	8	1
2	7	6	2	1	41	5
3	5	-7	1	1	90	11
4	**2**	**9**	**1**	**1**	**131**	**16**
5	7	-2	7	1	221	27
6	**7**	**9**	**1**	**1**	**1678**	**205**
7	2	-7	1	1	1899	232
8	5	6	2	1	3577	437
9	7	-3	5	1	9053	1106
10	8	1	16	1	48842	5967

The *cakravāla* algorithm is significantly more optimal than the Euler-Lagrange algorithm as it skips several steps of the latter.

In the above table the steps which are skipped in *cakravāla* are highlighted.

Euler-Lagrange Method for $X^2 - 67Y^2 = 1$

The corresponding simple continued fraction expansion is

$$\sqrt67 = 8 + \overline{\tfrac{1}{5+}\ \tfrac{1}{2+}\ \tfrac{1}{1+}\ \tfrac{1}{1+}\ \tfrac{1}{7+}\ \tfrac{1}{1+}\ \tfrac{1}{1+}\ \tfrac{1}{2+}\ \tfrac{1}{5+}\ \tfrac{1}{16+}}$$

The $\mathrm{Bh\bar{a}skara}$ nearest square continued fraction is given by

$$\sqrt67 = 8 + \overline{\tfrac{1}{5+}\ \tfrac{1}{2+}\ \tfrac{1}{2+}\ \tfrac{-1}{9+}\ \tfrac{-1}{2+}\ \tfrac{1}{2+}\ \tfrac{1}{5+}\ \tfrac{1}{16+}}$$

Euler-Lagrange Method for $X^2 - 61Y^2 = 1$

i	$\mathbf{P_i}$	$\mathbf{K_i}$	$\mathbf{a_i}$	ε_i	$\mathbf{X_i}$	$\mathbf{Y_i}$
0	0	1	7	1	1	0
1	7	-12	1	1	**7**	**1**
2	5	3	4	1	8	1
3	7	-4	3	1	39	5
4	5	9	1	1	**125**	**16**
5	4	-5	2	1	164	21
6	6	5	2	1	453	58
7	4	-9	1	1	**1070**	**137**
8	5	4	3	1	1523	195
9	7	-3	4	1	5639	722
10	5	12	1	1	**24079**	**3083**
11	7	-1	14	1	29718	3805
12	7	12	1	1	**440131**	**56353**

The steps which are skipped in *cakravāla* are highlighted.

Euler-Lagrange Method for $X^2 - 61Y^2 = 1$

i	**P_i**	**K_i**	**a_i**	**ε_i**	**X_i**	**Y_i**
13	5	-3	4	1	469849	60158
14	7	4	3	1	2319527	296985
15	5	-9	1	1	**7428430**	**951113**
16	4	5	2	1	9747967	1248098
17	6	-5	2	1	26924344	3447309
18	4	9	1	1	**63596645**	**8142716**
19	5	-4	3	1	90520989	11590025
20	7	3	4	1	335159612	42912791
21	5	-12	1	1	**1431159437**	**183241189**
22	7	1	14	1	1766319049	226153980

The Corresponding simple continued fraction expansion is

$$\sqrt{61} = 7 + \overline{\frac{1}{1+}\ \frac{1}{4+}\ \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{4+}\ \frac{1}{1+}\ \frac{1}{14+}}$$

The $\mathrm{Bh\bar{a}skara}$ nearest square continued fraction is given by

$$\sqrt{61} = 8 + \overline{\frac{-1}{5+}\ \frac{1}{4+}\ \frac{-1}{3+}\ \frac{1}{3+}\ \frac{-1}{4+}\ \frac{1}{5+}\ \frac{-1}{16+}}$$

Bhāskara or Nearest Square Continued Fraction

In the continued fraction development of $\sqrt{D}$, the complete quotients are quadratic surds which may be expressed in the standard form $\frac{(P+\sqrt{D})}{Q}$, where $P,\ Q$ and $\frac{(D-P^2)}{Q}$ are integers prime to each other.

If $a = \left[\frac{(p+\sqrt{D})}{Q}\right]$ is the integral part of $\frac{(P+\sqrt{D})}{Q}$, then we can have

$$\frac{(P+\sqrt{D})}{Q} = a + \frac{Q^{'}}{(P' + \sqrt{D})} \tag{1}$$

$$\frac{(P+\sqrt{D})}{Q} = (a+1) - \frac{Q^{''}}{(P'' + \sqrt{D})} \tag{2}$$

where the surds in the rhs are also in the standard form.

Bhāskara or Nearest Square Continued Fraction

In the Bhāskara or nearest square continued fraction development we choose **a** as the partial quotient if

$$\text{(i) } \mid P'^2 - D \mid < \mid P''^2 - D \mid, \quad \text{or}$$
$$\text{(ii) } \mid P'^2 - D \mid \ = \ \mid P''^2 - D \mid \text{ and } Q < 0.$$

Then we set $\varepsilon = \mathbf{1}$.

Otherwise, we choose $\mathbf{a} + \mathbf{1}$ and set $\varepsilon = -\mathbf{1}$.

Note: If we start with $\sqrt{D}$, we always have $P_i \geq 0$ and $Q_i > 0$ and $K_i = (-1)^i \varepsilon_1 \varepsilon_2 \ldots \varepsilon_i \, Q_i$

Bhāskara or Nearest Square Continued Fraction

Krishnaswami Ayyangar showed that the Bhāskara or nearest square continued fraction of $\sqrt{D}$ is of the form

$$\sqrt{D} = a_0 + \overline{\frac{\varepsilon_1}{a_1+}\ \frac{\varepsilon_2}{a_2+}\ \frac{\varepsilon_3}{a_3+}\ \cdots\ \frac{\varepsilon_{k-1}}{a_{k-1}+}\ \frac{\varepsilon_k}{2a_0+}}$$

where k is the period. It has the following symmetry properties:

Type I: There is no complete quotient of the form

$$\frac{[p+q+\sqrt(p^2+q^2)]}{p},$$

where $p > 2q > 0$ are mutually prime inters. Then, the Bhāskara continued fraction for $\sqrt{D}$ has same symmetry properties as in the case of simple continued fraction expansion.

$$\begin{aligned}
a_\nu &= a_{k-\nu}, \quad 1 \le \nu \le k-1, \\
Q_\nu &= Q_{k-\nu}, \quad 1 \le \nu \le k-1, \\
\epsilon_\nu &= \epsilon_{k+1-\nu}, \quad 1 \le \nu \le k, \\
P_\nu &= P_{k+1-\nu}, \quad 1 \le \nu \le k.
\end{aligned}$$

Bhāskara or Nearest Square Continued Fraction

Examples of Type I

$$\begin{aligned}
\sqrt{61} &= 8 + \overline{\frac{-1}{5+}\ \frac{1}{4+}\ \frac{-1}{3+}\ \frac{1}{3+}\ \frac{-1}{4+}\ \frac{1}{5+}\ \frac{-1}{16+}} \\
\sqrt{67} &= 8 + \overline{\frac{1}{5+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{-1}{9+}\ \frac{-1}{2+}\ \frac{1}{2+}\ \frac{1}{5+}\ \frac{1}{16+}}
\end{aligned}$$

Type II: There is a complete quotient of the form

$$\frac{[p + q + \sqrt{(p^2 + q^2)}]}{p}$$

where $p > 2q > 0$ are mutually prime integers. In such a case, the period $k \geq 4$ and is even, and there is only one such complete quotient which occurs at $\frac{k}{2}$.

Bhāskara or Nearest Square Continued Fraction

The symmetry properties are same as for Type I, except that

$$a_{\frac{k}{2}} = 2,\ \varepsilon_{\frac{k}{2}} = -1,\ \varepsilon_{\frac{k}{2}+1} = 1,\ a_{\frac{k}{2}-1} = a_{\frac{k}{2}+1} + 1,\ P_{\frac{k}{2}} \neq P_{\frac{k}{2}+1}$$

Examples of Type II

$$\begin{aligned}
\sqrt{29} &= 5 + \overline{\frac{1}{3+}\ \frac{-1}{2+}\ \frac{1}{2+}\ \frac{1}{10+}} \\
\sqrt{53} &= 7 + \overline{\frac{1}{4+}\ \frac{-1}{2+}\ \frac{1}{3+}\ \frac{1}{14+}} \\
\sqrt{58} &= 8 + \overline{\frac{-1}{3+}\ \frac{-1}{2+}\ \frac{1}{2+}\ \frac{-1}{16+}} \\
\sqrt{97} &= 10 + \overline{\frac{-1}{7+}\ \frac{-1}{3+}\ \frac{-1}{2+}\ \frac{1}{2+}\ \frac{-1}{7+}\ \frac{-1}{20+}}
\end{aligned}$$

Clearly Type II situation is possible only when D is of the form $(p^2 + q^2)$ with $p > 2q$.

Mid-point Criteria

In the case of simple continued fraction expansion of $\sqrt{D}$, the mid-point criteria were given by Euler:

If $Q_{h-1} = Q_h$ (or $\mid K_{h-1} \mid \; = \; \mid K_h \mid$), then the period $k = 2h - 1$ and

$$\begin{aligned} A_{k-1} &= A_{h-1}\, B_{h-1} + A_{h-2}\, B_{h-2} \\ B_{k-1} &= B_{h-1}^2 + B_{h-2}^2 \end{aligned}$$

which satisfy $A_{k-1}^2 - DB_{k-1}^2 = -1$

If $P_h = P_{h+1}$, then the period $k = 2h$ and

$$\begin{aligned} A_{k-1} &= A_{h-1} B_h + A_{h-2} B_{h-1} \\ B_{k-1} &= B_{h-1}(B_h + B_{h-2}) \end{aligned}$$

which satisfy $A_{k-1}^2 - DB_{k-1}^2 = 1$

Mid-point Criteria

Recently, Mathews, Robertson and White (2010) have worked out the mid-point criteria for the Bhāskara continued fraction expansion of $\sqrt{D}$:

If $Q_{h-1} = Q_h$ (or $\mid K_{h-1} \mid \; = \; \mid K_h \mid$) then the period $k = 2h - 1$ and

$$\begin{aligned} A_{k-1} &= A_{h-1}\, B_{h-1} + \epsilon_h\, A_{h-2}\, B_{h-2} \\ B_{k-1} &= B_{h-1}^2 + \epsilon_h\, B_{h-2}^2 \end{aligned}$$

which satisfy $A_{k-1}^2 - DB_{k-1}^2 = -\epsilon_h$

If $P_h = P_{h+1}$, then the period $k = 2h$ and

$$\begin{aligned} A_{k-1} &= A_{h-1}\, B_h + \epsilon_h\, A_{h-2}\, B_{h-1} \\ B_{k-1} &= B_{h-1}\,(B_h + \epsilon_h\, B_{h-2}) \end{aligned}$$

which satisfy $A_{k-1}^2 - DB_{k-1}^2 = 1$

In the Type I case, the mid-point will invariably satisfy one of the above two criteria.

Mid-point Criteria

In the Type II case, the following is the mid-point criterion:

When $Q_h = \ |K_h\,|$ is even and

$P_h = Q_h + \left(\frac{1}{2}\right)\ Q_{h-1}\ =\ |\ K_h\,| + \left(\frac{1}{2}\right)\ |\ K_{h-1}\,|$ and $\epsilon_h = 1$,

then $k = 2h$ and

$$\begin{aligned} A_{k-1} &= A_h\, B_{h-1} - B_{h-2}(A_{h-1} - A_{h-2}) \\ B_{k-1} &= 2B_{h-1}^2 - B_h\, B_{h-2} \end{aligned}$$

which satisfy $A_{k-1}^2 - DB_{k-1}^2 = 1$

These mid-point criteria serve to further simplify the computation of the solution.

Optimality of the *Cakravāla* Method

We have already remarked that the *cakravāla* process skips certain steps in the Euler-Lagrange process. Sometimes the period of the Euler-Lagrange continued fraction expansion could be double (or almost double) the period of Bhāskara expansion. This is seen for instance, for D=13, 44, 58:

BCF: $\sqrt{13} = 4 + \overline{\frac{-1}{2+}\ \frac{1}{2+}\ \frac{-1}{8+}}$

SCF: $\sqrt{13} = 3 + \overline{\frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{6+}}$

BCF: $\sqrt{44} = 7 + \overline{\frac{-1}{3+}\ \frac{-1}{4+}\ \frac{-1}{3+}\ \frac{-1}{14+}}$

SCF: $\sqrt{44} = 6 + \overline{\frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{12+}}$

BCF: $\sqrt{58} = 8 + \overline{\frac{-1}{3+}\ \frac{-1}{2+}\ \frac{1}{2+}\ \frac{-1}{16+}}$

SCF: $\sqrt{58} = 7 + \overline{\frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{14+}}$

Optimality of the *Cakravāla* Method

- We may note that whenever there is a 'unisequence' (1,1,...,1) of partial quotients of length n, the Bhāskara process skips exactly $\frac{n}{2}$ steps if n is even, and $\frac{(n+1)}{2}$ steps if n is odd.
- Selenius has shown that the *cakravāla* process is 'ideal' in the sense that, whenever there is such a 'unisequence', only those convergents $\frac{A_i}{B_i}$ are retained for which $B_i \mid A_i - B_i\sqrt{D} \mid$ are minimal.

Optimality of the $\mathit{Cakrav\bar{a}la}$ Method

Mathews *et al* have shown that the period of Bhāskara or nearest square continued fraction is the same as that of the nearest integer continued fraction. They estimate that the ratio of this period to that of simple continued fraction is $\mathbf{log}\left[\frac{(1+\sqrt{5})}{2}\right] \approx \mathbf{0.6942419136}\ldots$

n	$\Pi\ (n)$	$P\ (n)$	$\Pi\ (n)/P\ (n)$
1,000,000	152,198,657	219,245,100	0.6941941
2,000,000	417,839,927	601,858,071	06942499
3,000,000	755,029,499	1,087,529,823	0.6942609
4,000,000	1,149,044,240	1,655,081,352	0.6942524
5,000,000	1,592,110,649	2,293,328,944	0.6942356
6,000,000	2,078,609,220	2,994,112,273	0.6942322
7,000,000	2,604,125,007	3,751,067,951	0.6942356
8,000,000	3,165,696,279	4,559,939,520	0.6944208
9,000,000	3,760,639,205	5,416,886,128	0.6942437
10,000,000	4,387,213,325	6,319,390,242	0.6942463

$\Pi(n)$ is the sum of the NSCF period lengths of $\sqrt{D}$ up to n, D not a square, and $P(n)$ is the same for RCF.

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Thanks!

Thank You