NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 17

Mahāvīra's *Gaṇitasārasaṅgraha* 3

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Outline

- Plane figures: Circle, *Dīrghavṛtta*, Annulus
- Ratio of circumference and diameter. Segment of a circle
- *Janya* operations: rational triangles, quadrilaterals
- Excavations: Uniform and tapering cross-sections, volume of a sphere
- Time to fill a cristern
- Shadow Problems

Chapter 7. Measurement of areas

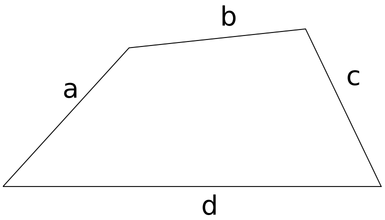


Figure: $Vi\d{s}amacatura\acute{s}ra$

$$\text{Approximate area} = \frac{(b+d)}{2}\frac{(a+c)}{2}.$$

$$\text{(Exact) Area of an annulus} = \left(\frac{c_1+c_2}{2}\right)b \quad : \text{b is the width.}$$

Circle and Annulus

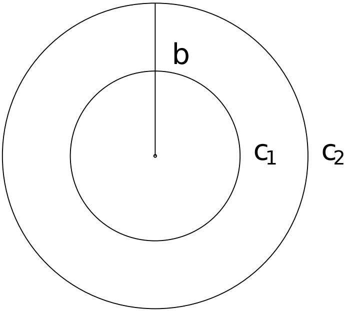


Figure: Annulus

19. Approximate circumference of a circle $= 3 \times$ diamter

Approximate area $= 3 \times \left(\frac{\text{diameter}}{2}\right)^2 \therefore \pi \approx 3$ above.

$\bar{A}yatav\d{r}tta$

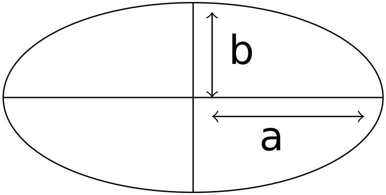


Figure: Eclipse

$$\text{Approx. Circumference} = 2(2a + b)$$

$$\text{Approx. Area} = b \cdot 2(a + \frac{1}{2}b)$$

Segments and Perpendicular

Verse 49.

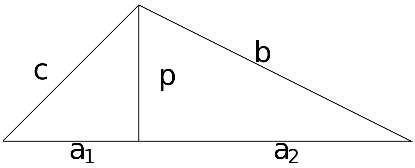


Figure:

Segments a_1, a_2 and perpendicular p in terms of a, b, c as in $Br\bar{a}hmasphu\d{t}asiddh\bar{a}nta$

(BSS). Area of a triangle, Cyclic quadrilateral as in BSS.

Diagonals of a quadrilateral as in BSS.

Circumference and Area of a Circle

Verse 60.

वृत्तक्षेत्रव्यासो दशपदगुणितो भवेत् परिक्षेपः।
व्यासचतुर्भागगुणः परिधिः फलमर्धमर्धे तत्॥ ६० ॥

"The diameter of the circular figure multiplied by the square root of 10 becomes the circumference (in measure). The circumference multiplied by one-fourth of the diameter gives the area."

Exact circumference of a circle $= \sqrt{10}$ Diameter. (so, $\pi = \sqrt{10}$: Approximate)

$$\text{Area} = \frac{1}{4}\text{Circumference} \times \text{Diameter} \quad \text{(Exact).}$$

Circumference and Area of an Ellipse

Verse 63.

व्यासकृतिः षड्गुणिता द्विसङ्गुणायामकृतियुता (पदं) परिधिः।
व्यासचतुर्भागगुणश्चायतवृत्तस्य सूक्ष्मफलम्॥ ६३ ॥

" The square of the (shorter) diameter is multiplied by 6 and square of twice the length (as measured by the longer diameter) is added to this. (This square root of the sum gives) the measure of the circumference. This measure of the circumference multiplied by one fourth of the (shorter) diameter gives the minutely accurate measure of the area of an elliptical figure."

$$\text{Circumference of an ellipse} = \sqrt{6(2b)^2 + (2 \times 2a)^2}. \quad \text{(wrong)}$$

$$[\text{Exact}: 2\pi a(1 - 1/2e^2 - (3/8)^2 e^4/3 - \cdots) \text{ with } e = \sqrt{(1 - b^2/a^2)}]$$

$$\text{Area} = b\sqrt{6b^2 + 4a^2} \quad \text{(wrong)}$$

$$[\text{Exact:} \quad = \pi ab = \sqrt{10}ab, \text{ if } \pi \text{ approximated to } \sqrt{10}]$$

Area of Segment

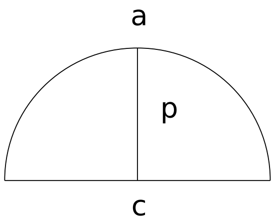
Verse $70\frac{1}{2}$: Area of segment.

इषुपादगुणश्च गुणो दशपदगुणितश्च भवति गणितफलम्।
यवसंस्थानक्षेत्रे धनुराकारे च विज्ञेयम्॥ ७० १/२ ॥

"It should be known that the measure of the string (chord) multiplied by one-fourth of the measure of the arrow, and then multiplied by the square root of 10, gives rise to the (accurate) value of the area in the case of a figure having the outline of a bow as also in the case of a figure resembling the (longitudinal) section of a $yava$ grain."

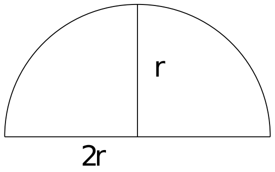
Area of Segment

$$\text{Area} = C \times \frac{p}{4} \times \underbrace{\sqrt{10}}_{\pi} \quad \text{Not correct}$$



Seems to be based on the fact that area of a semi-circle

$$= 2r \times \frac{r}{4} \times \pi = \frac{\pi r^2}{2}$$

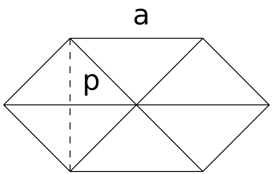


Diagonals Perpendiculars and Area of a Hexagon

Verse $86\frac{1}{2}$:

भुजभुजकृतिकृतिवर्गा द्वित्रित्रिगुणा यथाक्रमेणैव।
श्रुत्यवलम्बककृतिधनकृतयश्च षडश्रके क्षेत्रे॥ ८६ १/२ ॥

"In the case of a (regular) six-sided figure, the measure of the side, the square of the side, the square of the square of the side multiplied respectively by 2, 3 and 3 give rise, in that same order to the values of the diagonal, of the square of the perpendicular and of the square of the measure of the area."



Stated: Diagonal $= 2a$, perpendicular, $p = \sqrt{3}a$, Area $= \dfrac{3\sqrt{3}}{2}a^2$.

$Janya$ Operations

Generating figures with rational sides. $B\bar{\imath}jas$: a, b.

Verse $99\frac{1}{2}$ describes the procedure to construct an isosceles trapezium with the aid of two right triangles.

Start with two right triangles.

Then construct the isosceles trapezium thus:

Base AF = perpendicular side of first right triangle + perpendicular side of the second triangle $= (a^2 - b^2) + (c^2 - d^2)$. Topside HC = Difference of perpendicular sides $= (c^2 - d^2) - (a^2 - b^2)$.

Right Triangles

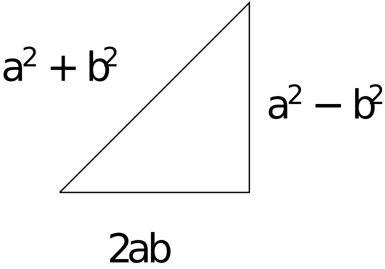


Figure: A right triangle

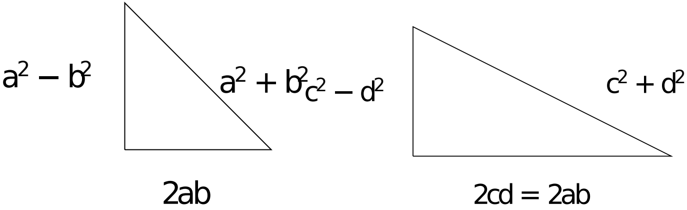


Figure: Two right triangles with the same 'side'

Isosceles Traperzium

Diagonal, HF = Diagonal of the second triangle $= c^2 + d^2$. Smaller segment of base AE = perpendicular side of first triangle $= a^2 - b^2$. Perpendicular HE = Base (side) of first or second triangle $= 2ab = 2cd$. Lateral side $AH = CF$ = Diagonal of first triangle $= a^2 + b^2$.

Then construct the isosceles trapezium thus:

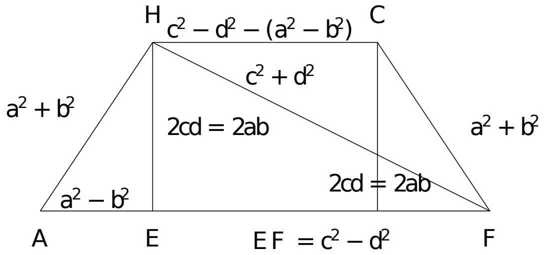


Figure: Isosceles Trapezium

The Right Triangle

Verse $101\frac{1}{2}$. Construct a right triangle with $b\bar{\imath}jas$ $\sqrt{2ab}(a+b), \sqrt{2ab}(a-b)$:

For this, upright $=$ perpendicular side

$$= 2ab(a+b)^2 - 2ab(a-b)^2 = 8a^2b^2.$$

$$\text{side } = \text{Base } = 2\sqrt{2ab}(a+b)\sqrt{2ab}(a-b) = 4ab(a^2-b^2)$$

$$\text{Diagonal } = 2ab(a+b)^2 + 2ab(a-b)^2 = 4ab(a^2+b^2)$$

An other Right Triangle

The $b\bar{\imath}jas$ for the second triangle: $a^2 - b^2$ and 2ab.

$$\text{upright} = \text{perpendicular side} = 4a^2b^2 - (a^2 - b^2)^2$$

$$\text{Base} = 4ab(a^2 - b^2)$$

$$\text{Diagonal} = 4a^2b^2 + (a^2 - b^2)^2.$$

Trapezium with Three Equal Sides

Then construct the trapezium, as earlier:

$$\text{Base} = \text{Greater perpendicular side} - \text{Smaller perpendicular sides}$$

$$= 8a^2b^2 - \{4a^2b^2 - (a^2 - b^2)^2\} = (a^2 + b^2)^2.$$

Either of the lateral sides $=$ smaller diagonal $= (a^2 + b^2)^2$.

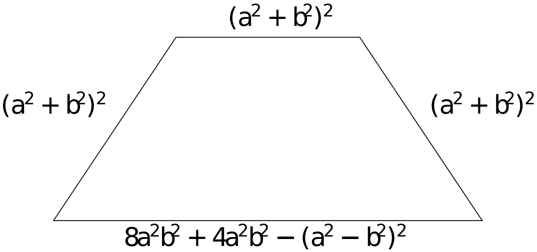


Figure: A Trapezium with 3 equal sides

Construction of a cyclic Quadrilateral

Verse 152. Given a cyclic quadrilateral with area $= A$. Let x_1, x_2, x_3, x_4 be the four chosen divisors.

$$\text{Find} \quad \frac{1}{2}\left(\frac{A^2}{x_1} + \frac{A^2}{x_2} + \frac{A^2}{x_3} + \frac{A^2}{x_4}\right) = s.$$

Then

$$a = s - \frac{A^2}{x_1},\ b = s - \frac{A^2}{x_2},\ c = s - \frac{A^2}{x_3},\ d = s - \frac{A^2}{x_4},$$

are the sides of the cyclic quadrilateral. It can be checked that

$$\frac{a+b+c+d}{2} = s, \quad \text{Now, } s - a = \frac{A^2}{x_1}, s - b = \frac{A^2}{x_2}, \text{ etc.,}$$

Construction of a cyclic Quadrilateral

$\therefore$ Area of the cyclic quadrilateral with sides a,b,c,d

$$= \sqrt{\frac{A^2}{x_1} \cdot \frac{A^2}{x_2} \cdot \frac{A^2}{x_3} \cdot \frac{A^2}{x_4}} = \frac{A^4}{\sqrt{x_1 x_2 x_3 x_4}}$$

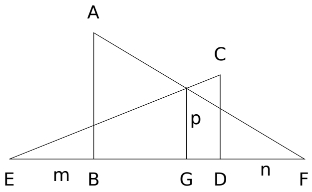
But this should be A.

$$\therefore \ \frac{A^4}{\sqrt{x_1 x_2 x_3 x_4}} = A \text{ or } x_1 x_2 x_3 x_4 = A^6.$$

So, the method works only if $x_1 x_2 x_3 x_4 = A^6$.

Pillars and Segments

Verse $180\frac{1}{2}$.



AB, CD: Pillars. With $AB = a$, $CD = b$. $BD =$ Distance between them $= c$. A and C are connected to F and E respectively. $BE = m$, $DF = n$. Then,

$$GE = C_1 = \frac{a(c+m)(c+m+n)}{a(c+m)+b(c+m)},$$

$$GF = C_2 = \frac{b(c+m)(c+m+n)}{a(c+m)+b(c+m)},$$

$$p = C_2 \times \frac{a}{c+n} = C_1 \times \frac{b}{c+m}.$$ [Exercise: Derive these]

Volumes

Circumference of a triangle/ cyclic quadrilateral, Relation between chord and s are the same as in BSS.

Chapter 8. Is on Excavations

Volume of a frustrum, same as in BSS.

Volume of a Sphere

Verse $28\frac{1}{2}$.

व्यासार्धघनार्धगुणा नव गोलव्यावहारिकं गणितम्।
तद्दशमांशं नवगुणमशेषसूक्ष्मं फलं भवति॥ २८ १/२ ॥

"The half of the cube of half the diameter, multiplied by nine, gives the approximate value of the cubical contents of a sphere. This (approximate value) multiplied by nine and divided by ten on neglecting the remainder, gives rise to the accurate value of the cubical measure."

Sphere with diameter $= d$.

Accurate Value of the Volume

More accurate volume of the sphere $= \left(\frac{d}{2}\right)^3 \frac{9}{2}\frac{9}{10}$.

$$\text{Correct value} = \frac{4\pi}{3}\left(\frac{d}{2}\right)^3$$

$$\therefore \text{ He has taken } \frac{4}{3}\pi = \frac{9}{2}\cdot\frac{9}{10}$$

$$\therefore \pi = \frac{3}{20}\times\frac{81}{4} = 3\times\frac{81}{80}.$$

Water Pipes and Cistern

Pipes leading into a Cistern / well, filling it up.

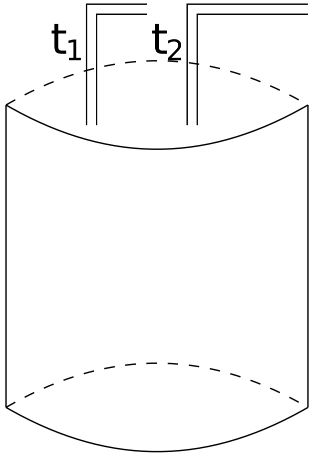


Figure: Water-pipes filling a cistern

Water Pipers and Cistern

Verse $32\frac{1}{2} - 33$.

वापीप्रणालिकाः स्वस्वकालभक्ताः सवर्णविच्छेदाः॥ ३२ १/२॥

तद्युतिभक्तं रूपं दिनांशकः स्यात् प्रणालिकायुत्या॥

तद्दिनबागहतास्ते तज्जलगतयो भवन्ति तद्वाप्याम्॥ ३३ ॥

“(The number one representing) each of the pipes is divided by the time corresponding to each of them (separately), and (the resulting quotients repesented as fractions) are reduced tso as to have a common denominator; one divided by the sum of these (fractions with the common denominator) gives the fraction of the day (within which the well would become filled) by all the pipes (pouring in their water) together. Those (fractions with the common denominator) multiplied by this resulting fraction of the day give rise to the measures of the flow of water (separately through each of the various pipes) into that well.”

Water Pipers and Cistern

Let t_i be the time (days) taken by the water flowing through the pipes to fill the volume.

$$\text{Fraction filled in one day by the} i\text{th pipe} = \frac{1}{t_i}.$$

$$\text{If all the pipes are open, amount filled in one day} = \frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}.$$

Hence the number of days or fraction of days needed to fill the cistern / well,

$$= \frac{1}{\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}} = \frac{1}{\sum \frac{1}{t_i}}$$

.

Example

Example in Verse 34.

चतस्त्रः प्रणालिकाः स्युस्तत्रैकैका प्रपूरयति वापीम्।
द्वित्रिचतुःपञ्चांशैः दिनस्य कतिभिर्दिनांशैस्ताः॥ ३४ ॥

"There are 4 pipes (leading in to a well.) Among them, each fills the well (in order) in $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ of a day. In how much of a day, will all of them (together fill the well and each of them to what extent)." [Try this as an exercise].

Volume with Uniform Cross Section

Verse $44\frac{1}{2}$: essentially states that the volume of an excavation is the product of the area of cross-section (A) multiplied by the depth (or length, l). See Fig.42.

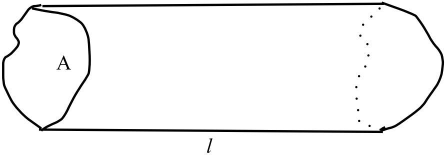


Figure: Volume generated by an uniform area of cross-section

Volume with a Trapezoidal cross-section

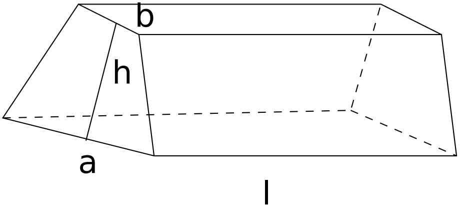


Figure: Volume with a trapezoidal cross-section

For instance, if the area of cross-section is an isosceles trapezium with base, *a*, summit, *b* and height, *h* and if the length is *l* (Fig. 43),

$$\text{Volume, V} = \left(\frac{(a+b)}{2}\right) h \times l.$$

Sloping Platform

Here, the area of cross-section is not constant, but varies uniformly. Suppose we have an isosceles trapezium, where the base is a, the summit is b, and the height is h at the end. Over a length l, it slopes uniformly to a base a, height is d, and the summit is $b + (a - b)\left(1 - \dfrac{d}{h}\right)$. What is the volume of the platform?

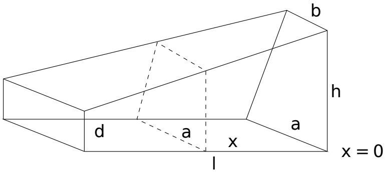


Figure: A sloping platform

Sloping Platform

At one end, it is an isosceles trapezium, with base a, summit b and height h. At the other end, base is a, summit is $b + (a - b)\left(1 - \frac{d}{h}\right)$, height is d. At the distance x from the first end, base is a, summit is $b + \dfrac{(a-h)}{l}\left(1 - \dfrac{d}{h}\right) x$ and the height is $h - \frac{x}{l}(h - d)$, by the rule of proportion.

Then, the GSS result for the volume in Verse $54\frac{1}{2}$ is

$$V = \frac{lh}{6}(2a + b + d)$$

We find the volume to be (by using integration)

$$V' = \frac{lh}{6}\left[(2a+b) + (2a+b)\frac{d}{h} + (b-a)\frac{d^2}{h^2}\right] \qquad ?\ \text{(Please Check.)}$$

Shadows

Chapter 9 is on shadows.

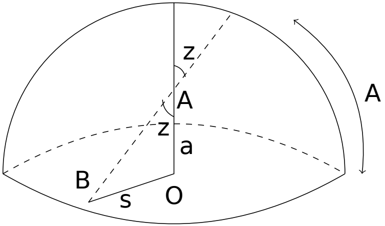


Figure: Shadow, OB =S of a gnomon, a = AO

If z is the zenith distance, altitude $A = 90^\circ - z$. Let the height of the gnomon be a. Let the shadow, OB be S.

$$\cot A = \tan z = \frac{S}{a}$$

Time from Shadow

Then it is stated that the time elapsed after the sunrise, or time, yet to elapse before sunset is

$$\text{Time, T} = \frac{1}{2\left(\frac{S}{a}+1\right)} \text{ in units of day.}$$

This is true only when $z = A = 45^\circ$, when the latitude (ϕ) and declination of the Sun (δ) are ignored. In this case, according to GSS, $t = \frac{1}{4}$ day. According to the correct formula,

$$t = \sin^{-1}(\cos z) = A = 90^\circ - z,$$

when ϕ, δ are ignored. When $z = A = 45^\circ, t = 45^\circ$ corresponds to $\frac{1}{4}$ of the day.

Shadow

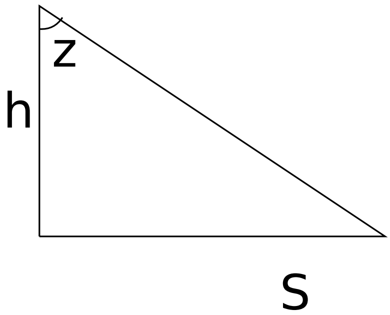


Figure: Shadow

$$\frac{S}{h} = \tan z = \text{ constant for all objects}$$

.

Shadow: Mahāvīra

Verse $12\frac{1}{2}$ says this:

कालानयनाद्दिनगतशेषसमासोनितः कालः।
स्तम्भच्छाया स्तम्भप्रमाणभक्तैव पौरुषी छाया॥ १२ १/२ ॥

“The measure of the shadow of a pillar divided by the measure of (the height of) the pillar gives rise to the measure of the shadow of a man (in terms of his own height). ”

Shadow on a Wall

Verse 21 :

नृच्छायाहतशङ्कुः भित्तिस्तम्भान्तरोनितो भक्तः।
नृच्छाययैव लब्धं शङ्कोर्भित्त्याश्रितच्छाया॥ २१ ॥

“(The height) of the pillar is multiplied by the measure of the human shadow (in terms of the man’s height). The (resulting) product is diminished by the measure of the interval between the wall and the pillar. The difference (so obtained) is divided by the very measure of the human shadow (referred to above). The quotient so obtained happens to be the measure of (that portion of) the pillar’s shadow which is on the wall.”

Shadow on a Wall

Suppose there is a pillar of height 'a'. There is a wall in front of it at a distance 'c' from it. Let the height of the shadow be h. Let b be the measure of the human shadow in terms of the man's height. Then the above verse says that

$$h = \frac{a \times b - c}{b}.$$

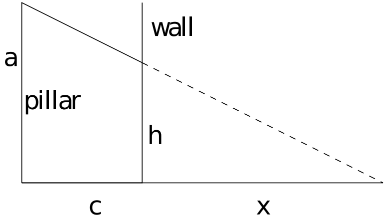


Figure: Shadow of a pillar

Shadow on a Wall

Let x be the distance between the wall and the tip of the shadow.
$x = bh$.

$$\text{Now, } \quad \frac{a}{c+x} = \frac{h}{x} = \frac{1}{b}.$$

$$\therefore\ a = \frac{c+x}{b} = \frac{c+bh}{b}.$$

$$\text{Hence,} \qquad h = \frac{ab-c}{b}.$$

Example

Example in Verse 22:

विंशतिहस्तः स्तम्भो भित्तिस्तम्भान्तरं करा अष्टौ।
पुरुषच्छाया द्विघ्ना भित्तिगता स्तम्भभा किं स्यात्॥ २२ ॥

"A pillar is 20 *hastas* (in height); the interval between (this) pillar and the wall (on which the shadow falls) in 8 *hastas*. The human shadow (at the time) is twice (the man's height). What is the measure of (that portion of) the pillar's shadow which is on the wall?". [Try this as an exercise].

The Shadow of a Slanting Pillar

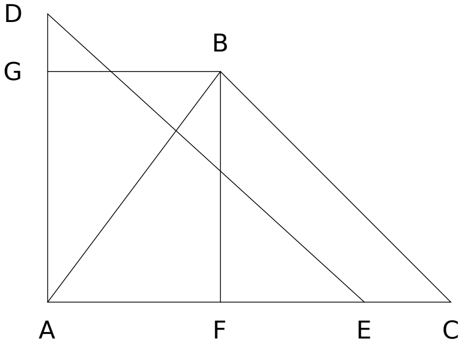


Figure: Slanting pillar

AB : slanting pillar, *AC*, its shadow. *AD*: Same pillar in vertical position. *AE* its shadow. Let r be the ratio of the shadow of a man to his height. *BG*, the perpendicular from *B* on *AD*, represents the amount of slanting of the pillar, *AB*.

Shadow of a Slanting Pillar

So,

$$r = \frac{\text{Shadow}}{\text{Height}}.$$

Now,

$$\frac{BF}{FC} = \frac{AD}{DE} = \frac{1}{r}.$$

Also,

$$BF = AG = \sqrt{AB^2 - BG^2},$$

$$FC = AC - BG.$$

$$\therefore\ BF^2 = AB^2 - BG^2 = (AC - BG)^2 \frac{1}{r^2}.$$

$$\therefore\ BG^2 \left(\frac{1}{r^2} + 1 \right) - 2\frac{AC \cdot BG}{r^2} + \left(\frac{AC^2}{r^2} - AB^2 \right) = 0$$

or

$$BG^2(1 + r^2) - 2AC \cdot BG + (AC^2 - AB^2 r^2) = 0.$$

Shadow of a Slanting Pillar

This is a Quadratic equation for BG, whose solution is given by:

$$BG = \frac{AC - \sqrt{AC^2 - (AC^2 - AB^2 r^2)(r^2 + 1)}}{r^2 + 1}.$$

The verses in 32-33 gives this formula.

Shadow of a Pillar due to Lamp

The rule for arriving at the shadow of a pillar due to (the light of) a lamp is given in

Verse 40 1/2 :

> शङ्कूनितदीपोन्नतिराप्ता शङ्कुप्रमाणेन।
> तल्लब्धहतं शङ्कोः प्रदीपशङ्क्वन्तरं छाया॥ ४० १/२ ॥
>
> “The height of the lamp is diminished by the height of the style (pillar) and is divided by the height of the style (pillar.) If, by means of the quotient so obtained, the (horizontal) distance between the lamp and the style (pillar) is divided, the measure of the shadow of the style (pillar) is arrived at”.

Lamp and Pillar

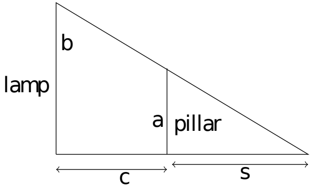


Figure: Shadow of a pillar due to a lamp at height,b

Height of the lamp is b, height of the pillar is a. Distance between them is c. s is the shadows of the pillar.
From the figure, it is clear that,

$$\frac{s}{a} = \frac{c+s}{b}.$$

$$\therefore\ bs = ac + as.$$

$$\therefore\ s = \frac{ac}{b-a}.$$

This is what is stated in verse $40\frac{1}{2}$.

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Thanks!

Thank you