NPTEL COURSE ON

MATHEMATICS IN INDIA: FROM VEDIC PERIOD TO MODERN TIMES

Lecture 30

Development of Calculus in India 1

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Outline

Background to the Development of Calculus (c.500-1350)

- The notions of zero and infinity
- Irrationals and iterative approximations
- Second order differences and interpolation in computation of Rsines
- Summation of infinite geometric series
- Instantaneous velocity (*tātkālika-gati*)
- Surface area and volume of a sphere
- Summations and repeated summations (*saṅkalita* and *vārasaṅkalita*)

Outline

The Kerala School of Astronomy and the Development of Calculus

- Kerala School: Mādhava (c. 1340-1420) and his successors to Acyuta Piṣāraṭi (c. 1550-1621)
- Nīlakaṇṭha (c.1450-1550) on the irrationality of π
- Nīlakaṇṭha and the notion of the sum of infinite geometric series
- Binomial series expansion
- Estimating the sum $1^k + 2^k + ...n^k$ for large n

Notions of Zero and Infinity

Background

- The concept of *pūrṇa* in the invocatory verse of *Īśopaniṣad* is closely related to the notion of infinite.

  पूर्णमदः पूर्णमिदं पूर्णात्पूर्णमुदच्यते।
  पूर्णस्य पूर्णमादाय पूर्णमेवावशिष्यते॥

  That (*Brahman*) is *pūrṇa*; this (universe) is *pūrṇa*; this *pūrṇa* emanates from that *pūrṇa*. Even when *pūrṇa* is drawn out of *pūrṇa*, what remains is *pūrṇa*.
- The concepts of *lopa* in Pāṇini, *abhāva* in *Nyāya* and *śūnya* in *Bauddha* philosophy are closely related to the idea of zero.
- Zero (*śūnya*) is introduced as a symbol in *Chandaḥsūtra* (VIII.29) of Piṅgala (c.300BC)

Notions of Zero and Infinity

The *Brāhmasphuṭasiddhānta* (c.628) of Brahmagupta is the first available text which discusses the mathematics of zero. The six operations with zero (*śūnya-parikarma*) are discussed in six verses of the Chapter XVIII (*Kuṭṭakādhyāya*), which also discuss the six operations with positive and negative numbers (*dhanarṇa-ṣaḍvidha*).

Bhāskarācārya II while discussing the mathematics of zero in his *Bījagaṇita*, explains that the infinite magnitude, which results when some number is divided by zero, is called *khahara*. He also mentions the characteristic property of infinity that it remains unaltered even if ”many” are added to or taken away from it, in terms similar to what we saw in the invocatory verse of *Īśopaniṣad*.

Notions of Zero and Infinity

खहरो भवेत् खेन भक्तश्च राशिः॥

द्विघ्नं त्रिहृत् खं खहृतं त्रयं च शून्यस्य वर्गं वद मे पदं च॥

```अयमनन्तो ३/० राशिः खहरः इत्युच्यते।

अस्मिन्विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु।
बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत्॥
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Notions of Zero and Infinity

Bhāskarācārya, while discussing the mathematics of zero in *Līlāvatī*, notes that when further operations are contemplated, the quantity being multiplied by zero should not be changed to zero, but kept as is; and that, when the quantity which is multiplied by zero is also divided by zero, then it remains unchanged.

He follows this up with an example and declares that this kind of calculation has great relevance in astronomy.

शून्ये गुणके जाते खं हारश्चेत्पुनस्तदा राशिः।
अविकृत एव ज्ञेयस्तथैव खेनोनितश्च युतः॥

खं पञ्चयुग्भवति किं वद खस्य वर्गं मूलं घनं घनपदं खगुणाश्च पञ्च।

खेनोद्धृता दश च कः खगुणो निजार्धयुक्तस्त्रिभिश्च गुणितः खहृतस्त्रिषष्टिः॥

```अज्ञातो राशिः तस्य गुणः ०। सार्धं क्षेपः १/२ ।गुणः ३।हरः ०।दृश्यं ६३।

ततो वक्ष्यमाणेन विलोमविधिना इष्टकर्मणा वा लब्धो राशिः १४।
अस्य गणितस्य ग्रहगणिते महानुपयोगः।
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Notions of Zero and Infinity

> What is the number which when multiplied by zero, being added to half of itself multiplied by three and divided by zero, amounts to sixty-three?

Bhāskara works out his example as follows:

$$\begin{aligned} 0\left[\left(x+\frac{x}{2}\right)\ \frac{3}{0}\right] &= 63 \\ \left(\frac{3x}{2}\right)3 &= 63 \\ x &= 14 \end{aligned}$$

Notions of Zero and Infinity

Bhāskara, it seems, had not fully mastered this kind of "calculation with infinitesimals" as is clear from some of the examples he considers in *Bījagaṇita*, while solving quadratic equations by eliminating the middle term (*ekavarṇa-madhyamā-haraṇa*).

कः स्वार्धसहितो राशिः खगुणो वर्गितो युतः।
स्वपदाभ्यां खभक्तश्च जाताः पञ्चदशोच्यताम्॥

$$\frac{\left[\left\{0(x+\left(\frac{x}{2}\right)\right\}^2+2\left\{0(x+\left(\frac{x}{2}\right))\right\}\right]}{0}=15.$$

Bhāskara in his *Vāsanā* just cancels out the zeroes and obtains $x=2$.

Irrationals and Iterative Approximations

- Background
  1. *Śulva-sūtra* approximation for square-root of 2.
  2. *Śulva-sūtra* approximation for π.
- Systematic algorithms for finding the square-root and cube-root of any number, based on the decimal place value system, have been known at least from the time of *Āryabhaṭīya* of Āryabhaṭa (c.499).
- *Āryabhaṭīya* also gives the value

$$\pi \approx 62832/20000 = 3.1416$$

  and mentions that it is approximate (*āsanna*). This value seems to have been obtained by the method of circumscribing the circle successively by a square, octagon etc., by a process of doubling, and cutting of corners which is explained in *Yuktibhāṣā* and *Kriyākramakarī*.

Irrationals and Iterative Approximations

Śrīdhara (c.850) in his *Triśatikā* has explained how the Āryabhaṭa method can be used to get better and better approximations to the square-root of a non-square number.

> राशेरमूलदस्याहतस्य वर्गेण केनचिन्महता।
> मूलं शेषेण विना विभजेद्गुणवर्गमूलेन॥
>
> Multiply the non-square number by some large square number, take the square-root [of the product] neglecting the remainder, and divide by the square-root of the multiplier.

For instance if D is a non-square number, we can use

$$\sqrt{D} = \frac{[\sqrt{(D.10^{2n})}]}{10^n}$$

to calculate $\sqrt{D}$ to any desired accuracy.

Irrationals and Iterative Approximations

Nārāyaṇa Paṇḍita in his *Gaṇitakaumudī* has noted that the solutions of the *varga-prakṛti* equation $X^2 - D\,Y^2 = 1$ can be used to obtain successive approximations to $\sqrt{D}$

> मूलं ग्राह्यं यस्य च तद्रूपक्षेपजे पदे तत्र।
> ज्येष्ठं ह्रस्वपदेन च समुद्धरेत् मूलमासन्नम्॥
>
> [With the number] whose square-root is to be found as the *prakṛti* and unity as the *kṣepa* [obtain the greater and smaller] roots. The greater root divided by the lesser root is an approximate value of the squre-root.

Nārāyaṇa considers the example $X^2 - 10\,Y^2 = 1$ and gives the successive approximate values

$$\sqrt{10} \approx \frac{19}{6}, \frac{721}{228}, \frac{27379}{8658}.$$

which are obtained from the solution $x = 19,\ y = 6$, by *bhāvanā*.

Second-Order Differences and Interpolation in Computation of Rsines

Computation of Rsine-table (accurate to minutes in a circle of circumference 21,600 minutes), by the method of second-order Rsine-differences, is outlined in the *Āryabhaṭīya* (c.499).

The tabular Rsines are given by $B_j = Rsin(jh)$, where h is usually taken as an arc of 225'. Then the Rsine-differences are given by

$$\triangle_j = B_{j+1} - B_j$$

The second-order differences satisfy the relation

$$\triangle_{j+1} - \triangle_j = -B_j \left[\frac{(\triangle_1 - \triangle_2)}{B_1} \right]$$

Āryabhaṭa makes use of the approximation $\triangle_1 - \triangle_2 \approx 1'$ to obtain

$$\triangle_{j+1} - \triangle_j \approx \frac{-B_j}{B_1}$$

The Rsine table is then computed by taking the first tabular sine $B_1 \approx 225'$

Second-Order Differences and Interpolation

In his *Khaṇḍakhādyaka* (c. 665), Brahmagupta has given the second-order interpolation formula for finding arbitrary Rsine values from the tabular Rsines.

> गतभोग्यखण्डकान्तरदलविकलवधात् शतैर्नवभिराप्त्या।
> तद्युतिदलं युतोनं भोग्यादूनाधिकं भोग्यम्॥
>
> Multiply the residual arc after division by 900’ by the half the difference of the tabular Rsine difference passed over (*gata-khaṇḍa*) and to be passed over (*bhogyakhaṇḍa*) and divide by 900’. The result is to be added to or subtracted from half the sum of the same tabular sine differences according as this [half-sum] is less than or equal to the Rsine tabular difference to be passed. What results is the true Rsine-difference to be passed over.

Second-Order Differences and Interpolation

If the arc for which the Rsine is to be obtained is $jh+\varepsilon$, then $\mathrm{Brahmagupta}$ interpolation formula is

$$\begin{aligned} Rsin(jh+\varepsilon) &= B_j + \left(\frac{\varepsilon}{h}\right)\left[\left(\frac{1}{2}\right)(\triangle_j + \triangle_{j+1}) - \left(\frac{\varepsilon}{h}\right)\frac{(\triangle_j - \triangle_{j+1})}{2}\right] \\ &= B_j + \left(\frac{\varepsilon}{h}\right)\frac{(\triangle_{j+1} + \triangle_j)}{2} + \left(\frac{\varepsilon}{h}\right)^2\frac{(\triangle_{j+1} - \triangle_j)}{2} \\ &= B_j + \left(\frac{\varepsilon}{h}\right)\triangle_{j+1} + \left(\frac{\varepsilon}{h}\right)\left(\left(\frac{\varepsilon}{h}\right) - 1\right)\frac{(\triangle_{j+1} - \triangle_j)}{2} \end{aligned}$$

Summation of Infinite Geometric Series

- The geometric series $1 + 2 + \dots 2^n$ is summed in Piṅgala's *Chandaḥ-sūtra* (c.300 BCE). Piṅgala also gave an algorithm for evaluating a positive integral power of a number in terms of an optimal number of squaring and multiplication operations.
- Mahāvīrācārya (c.850), in his *Gaṇita-sāra-saṅgraha*, gives the sum of a geometric series.
- Vīrasena (c. 816), in his Commentary *Dhavalā* on the *Ṣaṭkhaṇḍāgama*, has made use of the sum of the following infinite geometric series in his evaluation of the volume of the frustrum of a right circular cone:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \dots \qquad \left(\frac{1}{4}\right)^n + \dots = \frac{1}{3}$$

Tātkālika-Gati: Instantaneous Velocity

In astronomy, in order to determine the true longitude of a planet, a *manda-phala* which corresponds to the so called "equation of centre" is added to the mean longitude. While the mean longitude itself varies uniformly with time, the *manda-phala*, in the first approximation, is proportional to the Rsine of the mean longitude. The velocity of the planet therefore varies continuosly with time.

- An approximate formula for velocity (*manda-gati*) of a planet in terms of Rsine-differences was given by Bhāskara I (c.630) and he also commented on its limitation (*Laghu-bhāskarīya* 2.14-15).
- The expression for the true velocity (*sphuṭa-manda-gati*) in terms of Rcosine (the derivative of Rsine) appears for the first time in the *Laghu-mānasa* of *Muñjāla* (c. 932) and *Mahā-siddhānta* of *Āryabhaṭa* II (c. 950).

कोटिफलघ्नी भुक्तिर्गज्याभक्ता कलादिफलम्॥

The *koṭiphala* multiplied by the [mean] daily motion and divided by the radius gives the minutes of the correction [to the rate of the motion].

Tātkālika-Gati: Instantaneous Velocity

In his *Siddhānta-śiromaṇi*, Bhāskara II (c.1150) discusses the notion of instantaneous velocity (*tātkālika-gati*) and contrasts it with the so-called true daily rate of motion which is the difference of the true longitudes on successive days. He emphasises that the instantaneous velocity is especially relevant in the case of Moon.

> समीपतिथ्यन्तसमीपचालनं विधोस्तु तत्कालजयैव युज्यते।
> सुदूरसञ्चालनमाद्यया यतः प्रतिक्षणं सा न समा महत्यतः॥
>
> In the case of the Moon, the ending moment of a *tithi* which is about to end or the beginning time of a *tithi* which is about to begin, are to be computed with the instantaneous rate of motion at the given instant of time. The beginning moment of a *tithi* which is far away can be calculated with the earlier [daily] rate of motion. All this is because the Moon's rate of motion is large and varies from moment to moment.

Tātkālika-Gati: Instantaneous Velocity

In his commentary, *Vāsanā*, Bhāskara emphasises the above point still further.

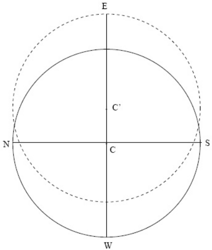
तात्कालिक्या भुक्त्या चन्द्रस्य विशिष्टं प्रयोजनम्। तदाह ‘समीपतिथ्यन्तसमीपचालनम्’ इति। यत्कालिकश्चन्द्रस्तस्मात् कालाद्गतो वा गम्यो वा यदासन्नस्तिथ्यन्तस्तदा तात्कालिक्या गत्या तिथिसाधनं कर्तुं युज्यते। तथा समिपचालनं च। यदा तु दूरतरस्तिथ्यन्तो दूरचालनं वा चन्द्रस्य तदाद्यया स्थूलया कर्तुं युज्यते। स्थूलकालत्वात्। यतश्चन्द्रगतिर्महत्वात् प्रतिक्षणम् समा न भवति। अतस्तदर्थमयं विशेषोऽभिहितः।

In the case of the Moon, this instantaneous rate of motion is especially useful. ...Because of its largeness, the rate of motion of Moon is not the same every instant. Hence, in the case [of Moon] the special [instantaneous] rate of motion is instructed.

Velocity Correction Vanishes at the Maxima

Bhāskara II also notes the relation between maximum equation of centre (correction to displacement) and the vanishing of velocity correction, both of which happen when the mean planet is on the line perpendicular to the line of apsides.

> कक्ष्यामध्यगतिर्यग्रेखाप्रतिवृत्तसंपाते।
>
> मध्यैव गतिः स्पष्टा परं फलं तत्र खेटस्य॥
>
> Where the [North-South] line perpendicular to the [East-West] line of apsides through the centre of the concentric meets the eccentric, there the mean velocity itself is true and the equation of centre is extremum.



Velocity Correction Vanishes at the Maxima

In his *Vāsanā*, Bhāskara explains why the correction to the velocity vanishes when the equation of centre is maximum.

कक्ष्यावृत्तमध्ये या तिर्यग्रेखा तस्याः प्रतिवृत्तस्य च यः संपातस्तत्र मध्यैव गतिः स्पष्टा। गतिफलाभावात्। किंच तत्र ग्रहस्य परमं फलं स्यात्। यत्र ग्रहस्य परमं फलं तत्रैव गतिफलाभावेन भवितव्यम्। यतोऽद्यतनश्वस्तन-ग्रहयोरन्तरं गतिः। फलयोरन्तरं गतिफलम्। ग्रहस्य गतेर्वा फलाभाव-स्थानमेव धनर्णसन्धिः।

The mean rate of motion itself is exact at the points where the line perpendicular [to the line of apsides], at the middle of the concentric circle, meets the eccentric circle, because there is no correction to the rate of motion [at those points]. Also because, there the equation of centre is extreme. Wherever the equation of centre is maximum, there the correction to the velocity should be absent. Because, the rate is the difference between the longitudes today and tomorrow. The correction to the velocity is the difference between the equations of centre. The place where the correction to the velocity vanishes, there is a change over from positive to the negative.

Surface Area and Volume of a Sphere

- In *Āryabhaṭīya*, the volume of a sphere is incorrectly estimated as the product of the area of a great circle by its square-root.
- Bhāskarācārya II (c.1150) has given the correct relation between the diameter, the surface area and the volume of a sphere in his *Līlāvatī*.

वृत्तक्षेत्रे परिधिगुणितव्यासपादः फलं यत्
क्षुण्णं वेदैरुपरि परितः कन्दुकस्येव जालम्।
गोलस्यैवं तदपि च फलं पृष्ठजं व्यासनिघ्नं
षड्भिर्भक्तं भवति नियतं गोलगर्भे घनाख्यम्॥

Surface Area and Volume of a Sphere

> In a circle, the circumference multiplied by one-fourth the diameter is the area, which, multiplied by four, is its surface area going around it like a net around a ball. This [surface area] multiplied by the diameter and divided by six is the volume of the sphere.

In his *Vāsanā* commentary on *Siddhānta-śiromaṇi*, Bhāskara has also presented justifications for these results.

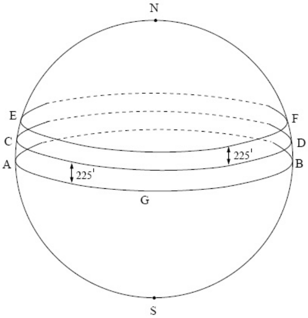
The volume of the sphere is estimated by summing the volumes of pyramids with apex at the centre.

Surface Area of a Sphere

As regards the surface area of a sphere, Bhāskara's justification is the following:

अथ बालावबोधार्थं गोलस्योपरि दर्शयेत्। भूगोलं मृण्मयं दारुमयं वा कृत्वा तं चक्रकलापरिधिं (२१६००) प्रकल्प्य तस्य मस्तके बिन्दुं कृत्वा तस्माद्बिन्दोर्गोलषण्णवतिभागेन शरद्विदस्रसङ्ख्येन (२२५) धनूरूपेणैव वृत्तरेखामुत्पादयेत्। पुनस्तस्मादेव बिन्दोः तेनैव द्विगुणसूत्रेणान्यां त्रिगुणेनान्यामेवं चतुर्विंशतिगुणं यावच्चतुर्विंशतिर्वृत्तानि भवन्ति। एषां वृत्तानां शरनेत्रबाहवः (२२५) इत्यादीनि ज्यार्धानि व्यासार्धानि स्युः। तेभ्योऽनुपाताद्वृत्तप्रमाणानि। तत्र तावदन्त्यवृत्तस्य मानं चक्रकलाः (२९६००)। तस्य व्यासार्धं त्रिज्या ३४३८। ज्यार्धानि चक्रकलागुणानि त्रिज्याभक्तानि वृत्तमानानि जायन्ते। द्वयोर्द्वयोर्वृत्तयोर्मध्य एकैकं वलयाकारं क्षेत्रम्। तानि चतुर्विंशतिः। बहुज्यापक्षे बहूनि स्युः। तत्र महदधोवृत्तं भूमिमुपरितनं लघुमुखं शरद्विदस्रमितं लम्बं प्रकल्प्य लम्बगुणं कुमुखयोगार्धमित्येवं पृथक् पृथक् फलानि। तेषां फलानां योगो गोलार्धपृष्ठफलम्। तद्द्विगुणं सकलगोलपृष्ठफलम्। तद्व्यासपरिधिघाततुल्यमेव स्यात्।

Surface Area of a Sphere



Here, Bhāskara is taking the circumference to be $C = 21,600'$ and the corresponding radius to be $R = 3,438'$. As shown in the figure, circles are drawn parallel to the equator, each separated in latitude by 225'. This divides the northern hemisphere into 24 strips, each of which can be cut and spread across as a trapezium. If $B_1, B_2, \ldots, B_{24}$ are the tabulated Rsines, then the area of the j^{th} trapezium will be

$$A_j = \left(\frac{C}{R}\right) \frac{(B_j + B_{j+1})}{2} 225.$$

Surface Area of a Sphere

Therefore, the surface area S of the sphere is estimated to be

$$S = 2\left(\frac{C}{R}\right)\left[B_1 + B_2 + \ldots B_{23} + \left(\frac{B_{24}}{2}\right)\right](225).$$

Now, Bhāskara states that the by substituting the values of the tabulated Rsines the right hand side can be found to be 2CR. In fact, according to Bhāskara's Rsine table

$$\begin{aligned}\left[B_1 + B_2 + \ldots B_{23} + \left(\frac{B_{24}}{2}\right)\right](225) &= 52514 \times (225)\\ &= 11815650\\ &\approx (3437.39)^2.\end{aligned}$$

Surface Area of a Sphere

Taking this as $R^2 = (3438)^2$, Bhāskara obtains the surface area of the sphere to be

$$S = 2\left(\frac{C}{R}\right) R^2 = 2CR.$$

The grossness of the approximation used in deriving this result is due to the fact that the quadrant of the circumference was divided into 24 bits.

Bhāskara himself notes that we can consider dividing the quadrant to many more (*bahūni*) arc-bits. This is indeed the approach taken by *Yuktibhāṣā*, where the circumference of the circle is divided in to a large number (n) of arc-bits. *Yuktibhāṣā* also uses the relation between the Rsines and second-order Rsine-differences to evaluate the sum of the areas of the trapezia for large n.

Saṅkalita and *Vārasaṅkalita*

Āryabhaṭa gives the sum of the sequence of natural numbers

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

He further gives the sums of squares and cubes of natural numbers

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \ldots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1^3 + 2^3 + 3^3 + \ldots + n^3 &= [1 + 2 + 3 + \ldots + n]^2 \\ &= \left[\frac{n(n+1)}{2}\right]^2. \end{aligned}$$

Āryabhaṭa also gives the repeated sum (*vārasaṅkalita*) of the sum of the sequence of natural numbers

$$\frac{1.2}{2} + \frac{2.3}{2} + \ldots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

Nārāyaṇa Paṇḍita on *Vārasaṅkalita* (c.1350)

Āryabhaṭa's result for repeated summation was generalised to arbitrary order by Nārāyaṇa Paṇḍita (c.1350):

Let

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = V_n^{(1)}.$$

Then, Nārāyaṇa's result is

$$\begin{aligned} V_n^{(r)} &= V_1^{(r-1)} + V_2^{(r-1)} + \ldots + V_n^{(r-1)} \\ &= \frac{[n(n+1)\ldots(n+r)]}{[1.2.\ldots(r+1)]} \end{aligned}$$

$$\sum_{m=1}^{n} \frac{[m(m+1)\ldots(m+r-1)]}{[1.2\ldots r]} = \frac{[n(n+1)\ldots(n+r)]}{[1.2\ldots(r+1)]}$$

Nārāyaṇa's above result can be used to estimate the behaviour of $V_n^{(r)}$, as also the sums of powers of natural numbers $1^r + 2^r + \ldots n^r$, for large n.

The Kerala School of Astronomy (c.1350-1825)

Kerala traces its ancient mathematical traditions to Vararuci. There are speculations that Āryabhaṭa hailed from Kerala. In the classical period, there were many great Astronomer-Mathematicians in Kerala such as Haridatta (c.650-700), Devācārya (c.700), Govindasvāmin (c.800), Śaṅkaranārāyaṇa (c.850) and Udayadivākara (c.1100).

However it was Mādhava of Saṅgamagrāma (near Ernakulam) who pioneered a new School of Astronomy and Mathematics

Mādhava (c.1340-1425): Of his works, only *Veṇvāroha, Sphuṭa-candrāpti*, and a few tracts are available. Most of his celebrated results, such as the infinite series and fast convergent approximations for π, Rsine and Rcosine functions, are available only through citations in later works.

Parameśvara of Vaṭasseri (c.1360-1455), a disciple of Mādhava: His major works are *Dṛggaṇita, Goladīpikā*, and commentaries on *Sūryasiddhānta, Āryabhaṭīya, Mahābhāskarīya, Laghubhāskarīya, Laghumānasa* and *Līlāvatī* and *Siddhāntadīpikā* on Govindasvāmin's commentary on *Mahābhāskarīya*.

Parameśvara is reputed to have carried out detailed observations for over 50 years and come up with his *Dṛggaṇita* system.

Nīlakaṇṭha Somayājī of Tṛkkaṇṭiyūr (c.1444-1555), student of Dāmodara (son of Parameśvara): He is the most celebrated member of the Kerala School after Mādhava. His major works are *Tantrasaṅgraha* (c.1500), *Āryabhaṭīyabhāṣya, Golasāra, Candracchāyāgaṇita, Siddhāntadarpaṇa, Jyotirmīmāṃsā* and *Grahasphuṭānayane Vikṣepavāsanā*.

In *Tantrasaṅgraha*, Nīlakaṇṭha presents a major revision of the traditional planetary theory, which, for the first time in the history of astronomy, gives a correct formulation of the equation of centre and the motion in latitude of the interior planets. In his later works he discusses the geometrical picture corresponding to his modified planetary theory, according to which the five planets, Mercury, Venus, Mars, Jupiter and Saturn go around the mean Sun, which itself goes around the earth.

Jyeṣṭhadeva of Parakroḍa (c.1500-1610), student of Dāmodara: His works are *Yuktibhāṣā* (c.1530) and *Dṛkkaraṇa*.

Yuktibhāṣā, written in Malayalam prose, gives detailed proofs (*yukti*) for all the results on infinite series and their transformations discovered by Mādhava and also the astronomical results and procedures outlined in *Tantrasaṅgraha*. It has been hailed as the "First Textbook of Calculus".

The Kerala School of Astronomy (c.1350-1825)

Citrabhānu (c.1475-1550), student of Nīlakaṇṭha: His works are *Karaṇāmṛta, Ekaviṃśatipraśnottara*.

Śaṅkara Vāriyar of Tṛkkuṭaveli (c.1500-1560), student of Citrabhānu: His works are *Karaṇasāra*, commentaries *Kriyākramakarī* (c.1535) on *Līlāvatī, Yuktidīpikā, Kriyākalāpa* (in Malayalam) and *Laghuvivṛti* on *Tantrasaṅgraha*. The commentaries *Kriyākramakarī* and *Yuktidīpikā* present most of the proofs contained in *Yuktibhāṣā* in Sanskrit verses.

Acyuta Piṣāraṭi (c.1550-1621), student of Jyeṣṭhadeva and teacher of Nārāyaṇa Bhaṭṭātiri: His works are *Sphuṭanirṇaya-tantra, Karaṇottama, Rāśigola-sphuṭanīti*, and a Malayalam commentary on *Veṇvāroha*.

The Kerala School of Astronomy (c.1350-1825)

The Kerala School continued to flourish till early nineteenth century. Some of the later works are *Karaṇapaddhati* (c.1700?) of **Putumana Somayājī** and *Sadratnamālā* of **Śaṅkaravarman** (c.1774-1839).

Modern scholarship came to know of the work of the Kerala School and the demonstrations contained in *Yuktibhāṣā* through an article of Charles Whish in the Transactions of the Royal Asiatic Society in 1835.

However, most of these works got published only in the later part of 20th century.

Nīlakaṇṭha on the Irrationality of π

One of the main motivations of the mathematical work of the Kerala school is *paridhi-vyāsa-sambandha*, obtaining accurately the relation between the circumference of a circle and its diameter.

Āryabhaṭa (c.499) had given the following approximate value for π:

> चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम्।
> अयुतद्वयविष्कम्भस्यासन्नो वृत्तपरिणाहः॥
>
> One hundred plus four multiplied by eight and added to sixty-two thousand: This is the approximate measure of the circumference of a circle whose diameter is twenty thousand.

Thus, according to Āryabhaṭa, $\pi \approx \frac{62832}{20000} = 3.1416$

Nīlakaṇṭha on the Irrationality of π

Nīlakaṇṭha Somayaji in his *Āryabhaṭīya-bhāṣya*, while discussing square-roots, remarks that only an approximate value of a *karaṇī* can be known and, consequently, only an approximate value is given for π, as the traditional methods for its evaluation involve computation of square-roots:

एवं कृतोऽप्यासन्नमेव मूलं स्यात्। न पुनः करणीमूलस्य तत्त्वतः परिच्छेदः कर्तुं शक्य इत्यभिप्रायः।...

वक्ष्यति च "अयुतद्वयविष्कम्भस्य आसन्नो वृत्तपरिणाहः" इति। तत्र व्यासेन परिधिज्ञाने अनुमानपरम्परा स्यात्। तत्कर्मण्यपि मूलीकरणस्य अन्तर्भावादेव तस्य आसन्नत्वम्। तत्सर्वं तदवसरे एव प्रतिपादयिष्यामः।

Nīlakaṇṭha on the Irrationality of π

Later, Nīlakaṇṭha states that the ratio of the circumference to the diameter of a circle cannot be expressed as the ratio of two integers exactly.

कुतः पुनः वास्तवीं संख्याम् उत्सृज्य आसन्नैव इहोक्ता? उच्यते। तस्याः वक्तुमशक्यत्वात्। येन मानेन मीयमानो व्यासः निरवयवः स्यात् तेनैव मीयमानः परिधिः पुनः सावयव एव स्यात्।

…इति एकेनैव मीयमानयोः उभयोः क्वापि न निरवयवत्वं स्यात्। महान्तम् अध्वानं गत्वापि अल्पावयवत्वम् एव लभ्यम्। निरवयत्वं तु क्वापि न लभ्यम् इति भावः।

Nīlakaṇṭha on the Irrationality of π

"Why then has an approximate value been mentioned here instead of the actual value? This is the explanation. Because the actual value cannot be expressed. Why? Given a certain unit of measurement in which the diameter has no fractional part, the same measure when applied to measure the circumference will certainly have a fractional part.

...Thus when both are measured by the same unit they cannot both be without fractional parts. Even if you go a long way (by choosing smaller and smaller units of measure) a small fractional part will remain. The import [of $\bar{a}sanna$] is that there will never be a situation where both are integral."

Nīlakaṇṭha on the Sum of Infinite Geometric Series

Vīrasena (c. 816), had made use of the sum of the following infinite geometric series

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots \qquad \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}$$

This is proved in the *Āryabhaṭīya-bhāṣya* by Nīlakaṇṭha Somayājī, who makes use of this series for deriving an approximate expression for a small arc in terms of the corresponding chord in a circle. Nīlakaṇṭha begins his discussion of the sum of the infinite geometric series by posing the issue as follows:

> चतुरंशपरम्परासमुदायः कृत्स्नोऽपि त्र्यंशत्वमेवापाद्यते।
> कथं पुनः तावदेव वर्धते तावद्वर्धते च?
>
> "The entire series of powers of $\frac{1}{4}$ adds up to just $\frac{1}{3}$. How is it known that [the sum of the series] increases only up to that [limiting value] and that it actually does increase up to that [limiting value]?"

Nīlakaṇṭha on the Sum of Infinite Geometric Series

Nīlakaṇṭha obtains the sequence of results

$$\begin{aligned} \frac{1}{3} &= \frac{1}{4} + \frac{1}{(4.3)} \\ \frac{1}{(4.3)} &= \frac{1}{(4.4)} + \frac{1}{(4.4.3)} \\ \frac{1}{(4.4.3)} &= \frac{1}{(4.4.4)} + \frac{1}{(4.4.4.3)} \end{aligned}$$

and so on, from which he derives the general result

$$\frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right)$$

Nīlakaṇṭha on the Sum of Infinite Geometric Series

Nīlakaṇṭha then goes on to present the following crucial argument to derive the sum of the infinite geometric series: As we sum more terms, the difference between $\frac{1}{3}$ and sum of powers of $\frac{1}{4}$ (as given by the right hand side of the above equation), becomes extremely small, but never zero. Only when we take all the terms of the infinite series together do we obtain the equality

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}$$

Incidentally, Nīlakaṇṭha uses the above series to prove the following relation between the *cāpa* (arc), *jyā* (Rsine) and *śara* (Rversine) for small arc:

$$C\bar{a}pa \approx \left[\left(\frac{4}{3}\right) \acute{S}ara^2 + Jy\bar{a}^2\right]^{\frac{1}{2}}$$

Binomial Series Expansion

In obtaining the accurate relation between circumference and diameter, the binomial series expansion plays a crucial role. The following derivation of the series is found in $Yuktibh\bar{a}\d{s}\bar{a}$ and $Kriy\bar{a}kramakar\bar{\imath}$.

Given three positive numbers a, b, c, with $b > c$. we have the identity

$$a\left(\frac{c}{b}\right) = a - a\frac{(b-c)}{b}$$

In the right hand side, we may replace b in the denominator by c by making use of the identity

$$a\frac{(b-c)}{b} = a\frac{(b-c)}{c} - \left(a\frac{(b-c)}{c} \times \frac{(b-c)}{b}\right)$$

Binomial Series Expansion

Substituting for $\frac{(b-c)}{b}$ on the right and iterating we get

$$a\frac{c}{b} = a - a\frac{(b-c)}{c} + a\left[\frac{(b-c)}{c}\right]^2 - \ldots + (-1)^m a\left[\frac{(b-c)}{c}\right]^m + (-1)^{m+1} a\left[\frac{(b-c)}{c}\right]^m \frac{(b-c)}{b}$$

Both *Yuktibhāṣā* and *Kriyākramakarī* mention that logically there is no termination of the iteration process, so that

$$a\frac{c}{b} = a - a\frac{(b-c)}{c} + a\left[\frac{(b-c)}{c}\right]^2 - \ldots + (-1)^m a\left[\frac{(b-c)}{c}\right]^m + (-1)^{m+1} a\left[\frac{(b-c)}{c}\right]^{m+1} + \ldots$$

Binomal Series Expansion

It is also noted that one may stop after having obtained results to the desired accuracy if the later terms can be shown get smaller and smaller, and that this will happen only when $(b - c) < c$ (which is the condition for the convergence of the binomial expansion).

> एवं मुहुः फलानयने कृतेऽपि युक्तितः क्वापि न समाप्तिः।
> तथापि यावदपेक्षं सूक्ष्मतामापाद्य पाश्चात्यान्युपेक्ष्य
> फलानयनं समापनीयम्। इहोत्तरफलानां न्यूनत्वं तु
> गुणहारान्तरे गुणकारान्न्यून एव स्यात्।

If we set $\left[\frac{(b-c)}{c}\right] = x$, then the above series takes the form

$$\frac{a}{1+x} = a - ax + ax^2 - \ldots + (-1)^m ax^m + \ldots$$

Sum of Integral Powers of Natural Numbers

The derivation of the Mādhava series for π also involves estimating, for large n, the value of the *sama-ghāta-saṅkalita*

$$S_n^{(k)} = 1^k + 2^k + \ldots n^k$$

We had noted earlier the formulae given by Āryabhaṭa for $k = 1, 2$ and 3.

$$\begin{aligned}
S_n^{(1)} &= 1 + 2 + \ldots n = \frac{n(n+1)}{2} \\
S_n^{(2)} &= 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \\
S_n^{(3)} &= 1^3 + 2^3 + \ldots + n^3 = [1 + 2 + \ldots + n]^2 = \left[\frac{n(n+1)}{2}\right]^2
\end{aligned}$$

Sum of Integral Powers of Natural Numbers

Yuktibhāṣā and *Kriyākramakarī* derive the following estimate for the general *sama-ghāta-saṅkalita*:

$$S_n^{(k)} = 1^k + 2^k + \ldots + n^k \approx \frac{n^{k+1}}{k+1} \text{ for large } n$$

They also give an estimate for the repeated summation (*vāra-saṅkalita*)

$$\begin{aligned} V_n^{(1)} &= 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \\ V_n^{(k)} &= V_1^{(k-1)} + V_2^{(k-1)} + \ldots + V_n^{(k-1)} \approx \frac{n^{k+1}}{(k+1)!} \text{ for large } n \end{aligned}$$

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Thanks!

Thank You