NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 38

Proofs in Indian Mathematics 3

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Outline

- *Yuktibhāṣā* estimate of the *samaghāta-saṅkalita* $1^k + 2^k + \ldots n^k$ for large n.
- *Yuktibhāṣā* estimate of *Vārasaṅkalita*
- *Yuktibhāṣā* derivation of Mādhava Series for π
- *Yuktibhāṣā* derivation of end-correction terms
- *Yuktibhāṣā* derivation of Mādhava Rsine and Rcosine Series
- *Upapatti* and "Proof"
- Lessons from history

Yuktibhāṣā of Jyeṣṭhadeva

The most detailed exposition of *upapattis* in Indian mathematics is found in the Malayalam text *Yuktibhāṣā* (1530) of Jyeṣṭhadeva.

At the beginning of *Yuktibhāṣā*, Jyeṣṭhadeva states that his purpose is to present the rationale of the results and procedures as expounded in the *Tantrasaṅgraha*. Many of these rationales have also been presented (mostly in the form of Sanskrit verses) by Śaṅkara Vāriyar (c.1500-1556) in his commentaries *Kriyākramakarī* (on *Līlāvatī*) and *Yuktidīpikā* (on *Tantrasaṅgraha*)

Yuktibhāṣā has 15 chapters and is naturally divided into two parts, Mathematics and Astronomy. In the Mathematics part, the first five chapters deal with logistics, arithmetic of fractions, the rule of three and the solution of linear indeterminate equations. Chapter VI presents a detailed derivation of the Mādhava series for π, his estimate of the end-correction terms and their use in transforming the series to ensure faster convergence. Chapter VII discusses the derivation of the Mādhava series for Rsine and Rversine. This is followed by derivation of various results on cyclic quadrilaterals and the surface area and volume of a sphere.

Yuktibhāṣā Estimation of *Samaghāta-Saṅkalita*

The derivation of the Mādhava series for π crucially involves the estimation, for large n, of the so called *sama-ghāta-saṅkalita*, which is the sum of powers of natural numbers

$$S_n^{(k)} = 1^k + 2^k + \ldots n^k$$

Firstly, it is noted that the *mūla-saṅkalita*

$$S_n^{(1)} = 1 + 2 + \ldots n = \frac{n(n+1)}{2} \approx \frac{n^2}{2} \text{ for large } n$$

Then, we are asked to write the *varga-saṅkalita* as

$$S_n^{(2)} = n^2 + (n-1)^2 + \ldots + 1^2$$

and subtract it from

$$n\, S_n^{(1)} = n\,[n + (n-1) + \ldots + 1]$$

and get

$$\begin{array}{lllll} n\, S_n^{(1)} - S_n^{(2)} & = 1.(n-1) & +2.(n-2) & +3.(n-3) + \ldots & +(n-1)\,.1 \\ & = (n-1) & +(n-2) & +(n-3) + \ldots & +1 \\ & & +(n-2) & +(n-3) + \ldots & +1 \\ & & & +(n-3) + \ldots & +1 + \ldots \end{array}$$

Estimation of *Samaghāta-Saṅkalita*

Thus,

$$n\,S_n^{(1)} - S_n^{(2)} = S_{n-1}^{(1)} + S_{n-2}^{(1)} + S_{n-3}^{(1)} + \ldots .$$

Since we have already estimated $S_n^{(1)} \approx \frac{n^2}{2}$, it is argued that

$$\begin{aligned} n\,S_n^{(1)} &- S_n^{(2)} \approx \frac{(n-1)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-3)^2}{2} + \ldots \\ n\,S_n^{(1)} &- S_n^{(2)} \approx \frac{S_{n-1}^{(2)}}{2} \end{aligned}$$

Therefore

$$S_n^{(2)} \approx \frac{n^3}{3} \text{ for large } n.$$

Estimation of *Samaghāta-Saṅkalita*

Similarly it is shown that

$$S_n^{(3)} \approx \frac{n^4}{4} \text{ for large } n.$$

Then follows an argument based, on mathematical induction, to demonstrate the same estimate in the case of a general *sama-ghāta-saṅkalita*.

First it is shown that the excess of $n\, S_n^{(k-1)}$ over $S_n^{(k)}$ can be expressed in the form

$$n\, S_n^{(k-1)} - S_n^{(k)} = S_{n-1}^{(k-1)} + S_{n-2}^{(k-1)} + S_{n-3}^{(k-1)} + \ldots$$

Estimation of *Samaghāta-Saṅkalita*

If the lower order *saṅkalita* $S_n^{(k-1)}$ has already been estimated to be, $S_n^{(k-1)} \approx \frac{n^k}{k}$, for large n, then the above relation leads to

$$\begin{aligned} n\, S_n^{(k-1)} - S_n^{(k)} &\approx \frac{(n-1)^k}{k} + \frac{(n-2)^k}{k} + \frac{(n-3)^k}{k} + \dots \\ &\approx \left(\frac{1}{k}\right) S_{n-1}^{(k)} \end{aligned}$$

[Note: C. T. Rajagopal and co-workers have pointed out that the above argument may be made more rigorous by using an argument analogous to the one used in the proof of the Cauchy- Stolz Theorem]

Thus we get the estimate

$$S_n^{(k)} \approx \frac{n^{k+1}}{(k+1)} \text{ for large } n.$$

Yuktibhāṣā Estimation of *Vārasaṅkalita*

The proof of the Mādhava series for Rsine and Rcosine functions, depends crucially on the estimate, for large n, of the general repeated sum $V_n^{(r)}$ (*saṅkalitaikya* or *vārasaṅkalita*) of natural numbers, given by

$$V_n^{(1)} = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

$$V_n^{(r)} = V_1^{(r-1)} + V_2^{(r-1)} + \ldots + V_n^{(r-1)}$$

In *Gaṇitakaumudī* (c.1356) of Nārāyaṇa Paṇḍita, we find the formula

$$V_n^{(r)} = \frac{n(n+1)\ldots(n+r)}{(r+1)!}$$

The above result is also known to the Kerala Astronomers, but they prefer to derive the estimate for $V_n^{(r)}$, for large n, by mathematical induction.

Estimation of *Vārasaṅkalita*

Now,

$$V_n^{(1)} = \frac{n(n+1)}{2} \approx \frac{n^2}{2} \text{ for large } n.$$

We can express $V_n^{(2)}$ in the form

$$\begin{aligned} V_n^{(2)} &= V_n^{(1)} + V_{n-1}^{(1)} + \ldots \\ &\approx \frac{n^2}{2} + \frac{(n-1)^2}{2} + \ldots = \frac{S_n^{(2)}}{2} \end{aligned}$$

Using the estimate

$$S_n^{(2)} \approx \frac{n^3}{3},$$

we get

$$V_n^{(2)} \approx \frac{n^3}{6}$$

Estimation of *Vārasaṅkalita*

Similarly, if we write the general repeated sum as

$$V_n^{(r)} = V_n^{(r-1)} + V_{n-1}^{(r-1)} + \dots$$

And, if we have already obtained

$$V_n^{(r-1)} \approx \frac{n^r}{(r)!},$$

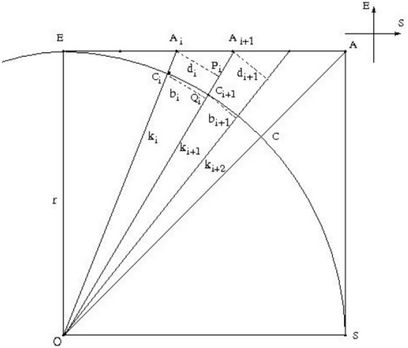
then we get,

$$\begin{aligned} V_n^{(r)} &\approx \frac{n^r}{(r)!} + \frac{(n-1)^r}{(r)!} + \dots \\ &\approx \frac{S_n^{(r)}}{(r)!} \\ &\approx \frac{n^{r+1}}{(r+1)!} \text{ for large } n. \end{aligned}$$

Yuktibhāṣā Derivation of Mādava Serires for π

Yuktibhāṣā has presented the following derivation of the Mādhava series for the ratio of the circumference of a circle to its diameter. For this purpose consider the quadrant OEAS of the square which circumscribes the circle of radius r. The eastern side of the quadrant is divided into a large number **n** of equal parts $A_i A_{i+1} = \frac{r}{n}$.

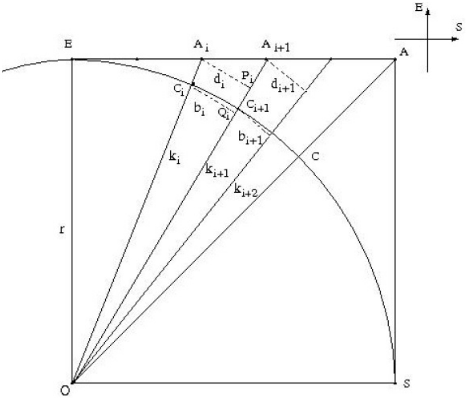
Join the hypotenuses (*karṇas*) $O\,A_1,\ O\,A_2, \ldots$ which meet the circle at $C_1,\ C_2, \ldots$ Drop the perpendiculars, $A_i P_i$ and $C_i Q_i$ onto OA_{i+1}.



Derivation of Mādhava Series for π

If we note that the triangles $A_iP_iA_{i+1}$ and OEA_{i+1} are similar and that the triangles OC_iQ_i and OA_iP_i are similar, then we get

$$C_iQ_i = A_iP_i\left(\frac{OC_i}{OA_i}\right) = A_iA_{i+1}\left(\frac{OE}{OA_{i+1}}\right)\left(\frac{OC_i}{OA_i}\right)$$

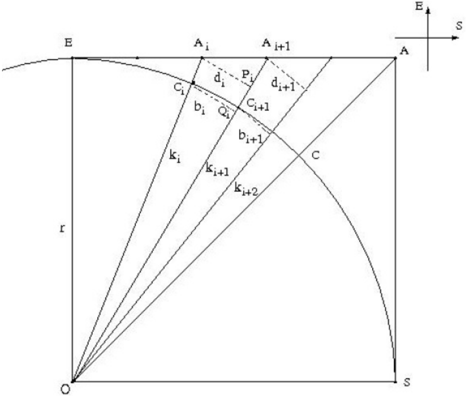


Derivation of Mādhava Series for π

We shall approximate the arc-bits C_iC_{i+1}, by the corresponding Rsines, C_iQ_i. It is noted that larger the **n** the more accurate will be the result.

If we denote the hypotenuse OA_i as k_i, then we get

$$\frac{C}{8} \approx \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0k_1}\right) + \left(\frac{r^2}{k_1k_2}\right) + \ldots + \left(\frac{r^2}{k_{n-1}k_n}\right)\right]$$



Derivation of Mādhava Series for π

It is noted that when n is large,

$$\frac{1}{k_i k_{i+1}} \approx \left(\frac{1}{2}\right)\left[\frac{1}{k_i^2} + \frac{1}{k_{i+1}^2}\right]$$

and that the earlier sum for the circumference can be replaced by

$$\frac{C}{8} \approx \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_1^2}\right) + \left(\frac{r^2}{k_2^2}\right) + \ldots + \left(\frac{r^2}{k_n^2}\right)\right]$$

If we note that

$$k_i^2 = r^2 + \left(\frac{ir}{n}\right)^2$$

then we get

$$\frac{C}{8} \approx \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{\left(r^2 + \left(\frac{r}{n}\right)^2\right)}\right) + \left(\frac{r^2}{\left(r^2 + \left(\frac{2r}{n}\right)^2\right)}\right) + \ldots + \left(\frac{r^2}{r^2 + \left(\frac{nr}{n}\right)^2}\right)\right]$$

Note: The above expression is essentially the integral of the arc-tan function from 0 to $\frac{\pi}{4}$.

Derivation of Mādhava Series for π

Each of the terms in the above sum for the circumference can be expanded as a binomial series (which has been derived earlier in *Yuktibhāṣā*) and we get, on regrouping the terms,

$$\begin{aligned}\frac{C}{8} = \; & \left(\frac{r}{n}\right)[1 + 1 + \ldots\ldots + 1] \\ & - \left(\frac{r}{n}\right)\left(\frac{1}{r^2}\right)\left[\left(\frac{r}{n}\right)^2 + \left(\frac{2r}{n}\right)^2 + \ldots + \left(\frac{nr}{n}\right)^2\right] \\ & + \left(\frac{r}{n}\right)\left(\frac{1}{r^4}\right)\left[\left(\frac{r}{n}\right)^4 + \left(\frac{2r}{n}\right)^4 + \ldots + \left(\frac{nr}{n}\right)^4\right] \\ & - \ldots\ldots\end{aligned}$$

Now, each of the *sama-ghāta-saṅkalita* or sums of powers of integers can be estimated (when n is large) in the manner explained earlier and we obtain the Mādhava series

$$\frac{C}{4d} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^n \frac{1}{(2n+1)} + \ldots$$

Yuktibhāṣā Derivation of the End-Correction Terms

The Mādhava series (or the so called Leibniz series) for the circumference of a circle (in terms of odd numbers $p = 1, 3, 5, \ldots$)

$$C = 4d\left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-1)}{2}}\frac{1}{p} + \ldots\right]$$

is an extremely slowly convergent series. Adding fifty terms of the series will give the value of π correct only to the first decimal place.

In order to facilitate computation, Mādhava has given a procedure of using end-correction terms (*antya-saṃskāra*), of the form

$$C = 4d\left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-1)}{2}}\frac{1}{p} + (-1)^{\frac{(p+1)}{2}}\frac{1}{a_p}\right]$$

Both *Yuktibhāṣā* and *Kriyākramakarī* give a derivation of the successive end correction terms given by Mādhava, which involve a careful estimate of the inaccuracy (*sthaulya*) at each stage in terms of inverse powers of the odd number p.

Derivation of the End-Correction Terms

Now, if the end-correction is made after the odd-number $p - 2$,

$$C = 4d \left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-3)}{2}} \frac{1}{(p-2)} + (-1)^{\frac{(p-1)}{2}} \frac{1}{a_{p-2}} \right]$$

If the end-correction were exact, comparing the two equations, we would have

$$\frac{1}{a_{p-2}} + \frac{1}{a_p} = \frac{1}{p}$$

It is noted that the above equation cannot be satisfied by the trivial choice

$$a_p = a_{p-2} = 2p$$

This is because if $a_p = 2p$, then a_{p-2} will have to be $2(p-2)$; or, if $a_{p-2} = 2p$, then a_p will have to be $2(p+2)$.

The method of *Yuktibhāṣā* is therefore to iteratively solve for a_p so as to minimise the inaccuracy (*sthaulya*) given by

$$E(p) = \frac{1}{a_{p-2}} + \frac{1}{a_p} - \frac{1}{p}$$

Derivation of the End-Correction Terms

The first approximation to the correction divisor is

$$a_p = 2p + 2$$

Then the inaccuracy will be

$$\begin{aligned} E(p) &= \frac{1}{(2p-2)} + \frac{1}{(2p+2)} - \frac{1}{p} \\ &= \frac{1}{(p^3 - p)} \end{aligned}$$

In fact, if we choose any other form for a_p which is linear in p, such as $a_p = 2p + 3$ or $a_p = 2p - 1$ etc., $E(p)$ will pick up a p term in the numerator also, so that for large p our inaccuracy will be much larger than in the case of $a_p = 2p + 2$.

Derivation of the End-Correction Terms

Now, the next choice for the correction divisor should be such that we add a number less than one to the earlier correction-divisor. We try

$$a_p = (2p + 2) + \frac{A}{2p + 2}$$

If we choose the correction-divisor in the form

$$a_p = (2p + 2) + \frac{\mathbf{4}}{(2p + 2)}$$

then we get the end-correction given by Mādhava

$$\frac{1}{a_p} = \frac{\left\{\frac{(p+1)}{2}\right\}}{\{(p + 1)^2 + 1\}}$$

The corresponding inaccuracy can be shown to be

$$E(p) = \frac{-4}{(p^5 + 4p)}$$

Again, if we choose $A = 3$ or 5 (or any other number), we find that the inaccuracy $E(p)$ will pick up a p term in the numerator also.

Derivation of the End-Correction Terms

The finer end-correction given by Mādhava corresponds to the correction-divisor

$$a_p = (2p+2) + \frac{4}{\left\{(2p+2) + \frac{16}{(2p+2)}\right\}}$$

The corresponding inaccuracy

$$\begin{aligned} E(p) &= \frac{2304}{(64p^7 + 448p^5 + 1792p^3 - 2304p)} \\ &= \frac{36}{[(p^3 - p)\{(p-1)^2 + 5\}\{(p+1)^2 + 5\}]} \end{aligned}$$

Again, if we choose 15, 17 (or any other number) instead of 16, we find that the inaccuracy $E(p)$ will pick up *a* p^2 term in the numerator.

Derivation of the End-Correction Terms

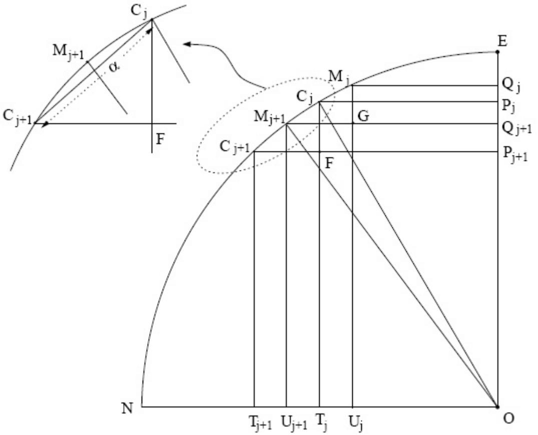
Carrying this process further, we find that the end-correction term $\frac{1}{a_p}$ can be expressed as a continued fraction:

$$\frac{1}{a_p} = \cfrac{1}{(2p+2) + \cfrac{2^2}{(2p+2) + \cfrac{4^2}{(2p+2) + \cfrac{6^2}{(2p+2) + \dots}}}}$$

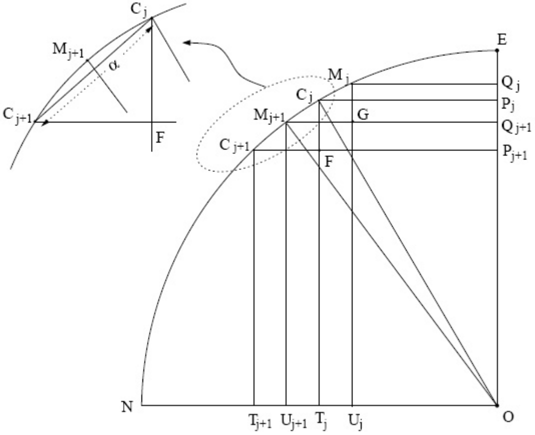
Yuktibhāṣā Derivation of the Mādhava Sine Series

Given an arc $EC = s = Rx$, divide it into n equal parts. The *piṇḍa-jyās* $B_j = C_jP_j$, *koṭi-jyās* $K_j = OP_j$ and *śaras* $S_j = P_jE$, with $j = 0,\ 1\ldots$, are given by

$$B_j = R\sin\left(\frac{jx}{n}\right), K_j = R\cos\left(\frac{jx}{n}\right), S_j = R\mathrm{vers}\left(\frac{jx}{n}\right) = R\left[1 - cos\left(\frac{jx}{n}\right)\right]$$



Let C_jC_{j+1} be the $(j+1)$-th arc-bit. Let M_{j+1} be the mid-point of the arc-bit C_jC_{j+1} and similarly M_j the mid-point of the previous (j-th) arc-bit. Let the full-chord of the equal arc-bits $\frac{s}{n}$ be denoted α. We can easily see that the triangles $C_{j+1}FC_j$ and $M_{j+1}GM_j$ are similar to $OQ_{j+1}M_{j+1}$ and OP_jC_j respectively.



Derivation of the $\mathbf{M\bar{a}dhava}$ Sine Series

We can thus show,

$$B_{j+1} - B_j = \left(\frac{\alpha}{R}\right) K_{j+\frac{1}{2}} \text{ and } K_{j-\frac{1}{2}} - K_{j+\frac{1}{2}} = S_{j+\frac{1}{2}} - S_{j-\frac{1}{2}} = \left(\frac{\alpha}{R}\right) B_j$$

Therefore, the second order Rsine differences (*jyā-khaṇḍāntaras*) are given by

$$(B_j - B_{j-1}) - (B_{j+1} - B_j) = \left(\frac{\alpha}{R}\right)\left(S_{j+\frac{1}{2}} - S_{j-\frac{1}{2}}\right) = \left(\frac{\alpha}{R}\right)^2 B_j$$

Hence

$$\begin{aligned}
S_{n-\frac{1}{2}} - S_{\frac{1}{2}} &= \left(\frac{\alpha}{R}\right)(B_1 + B_2 + \ldots + B_{n-1}) \\
B_n - n\,B_1 &= -\left(\frac{\alpha}{R}\right)^2 [B_1 + (B_1 + B_2) + \ldots + (B_1 + B_2 + \ldots + B_{n-1})] \\
&= -\left(\frac{\alpha}{R}\right)\left(S_{\frac{1}{2}} + S_{\frac{3}{2}} + \ldots + S_{n-\frac{1}{2}} - nS_{\frac{1}{2}}\right)
\end{aligned}$$

Derivation of the Mādhava Sine Series

The above relations are exact. Now, if B and S are the *jyā* and *śara* of the arc s, in the limit of very large n, we have

$$B_n \approx B,\ S_{n-\frac{1}{2}} \approx S,\ S_{\frac{1}{2}} \approx 0,\ \alpha \approx \frac{s}{n}$$

and hence

$$\begin{aligned} S &\approx \left(\frac{s}{nR}\right)(B_1 + B_2 + \ldots + B_{n-1}) \\ B - n\,B_1 &\approx -\left(\frac{s}{nR}\right)^2 [B_1 + (B_1 + B_2) + \ldots + (B_1 + B_2 + \ldots + B_{n-1})] \end{aligned}$$

In the above relations, we first approximate the Rsines (*jyā-khaṇḍas*) by the arcs (*cāpas*), $B_j \approx \frac{js}{n}$, and make use of the estimates for sums and repeated sums of natural numbers for large n, to get

$$\begin{aligned} S &\approx \left(\frac{1}{R}\right)\left(\frac{s}{n}\right)^2 (1 + 2 + \ldots + n - 1) \approx \frac{s^2}{2R} \\ B &\approx n\left(\frac{s}{n}\right) - \left(\frac{1}{R}\right)^2 \left(\frac{s}{n}\right)^3 [1 + (1 + 2) + \ldots + (1 + 2 + \ldots + n - 1)] \\ &\approx s - \frac{s^3}{6R^2} \end{aligned}$$

Derivation of the Mādhava Sine Series

We now substitute the above second approximation for *jyā-cāpāntara*

$$B_j \approx \frac{js}{n} - \frac{\left(\frac{js}{n}\right)^3}{6R^2}$$

Then we get the next approximation

$$\begin{aligned} S &\approx \frac{s^2}{2R} - \frac{s^4}{24R^2} \\ B &\approx s - \frac{s^3}{6R^2} + \frac{s^5}{120R^4} \end{aligned}$$

The above more refined approximation for *jyā-cāpāntara* is again fed back into our original equations for B and S, and so on. In this way, we are led to the series given by Mādhava for Rsine and Rversine

$$\begin{aligned} R\sin\left(\frac{s}{R}\right) &= R\left[\left(\frac{s}{R}\right) - \frac{\left(\frac{s}{R}\right)^3}{3!} + \frac{\left(\frac{s}{R}\right)^5}{5!} - \ldots\right] \\ R - R\cos\left(\frac{s}{R}\right) &= R\left[\frac{\left(\frac{s}{R}\right)^2}{2!} - \frac{\left(\frac{s}{R}\right)^4}{4!} + \frac{\left(\frac{s}{R}\right)^6}{6!} - \ldots\right] \end{aligned}$$

Upapatti and "Proof"

The following are some of the important features of *upapattis* in Indian mathematics:

1. The Indian mathematicians are clear that results in mathematics, even those enunciated in authoritative texts, cannot be accepted as valid unless they are supported by *yukti* or *upapatti*. It is not enough that one has merely observed the validity of a result in a large number of instances.
2. Several commentaries written on major texts of Indian mathematics and astronomy present *upapattis* for the results and procedures enunciated in the text.
3. The *upapattis* are presented in a sequence proceeding systematically from known or established results to finally arrive at the result to be established.

Upapatti and "Proof"

4. In the Indian mathematical tradition the *upapattis* mainly serve to remove doubts and obtain consent for the result among the community of mathematicians.
5. The *upapattis* may involve observation or experimentation. They also depend on the prevailing understanding of the nature of the mathematical objects involved.
6. The method of *tarka* or "proof by contradiction" is used occasionally. But there are no *upapattis* which purport to establish existence of any mathematical object merely on the basis of *tarka* alone. In this sense the Indian mathematical tradition takes a "constructivist" approach to the existence of mathematical objects.

Upapatti and "Proof"

7. The Indian mathematical tradition did not subscribe to the ideal that *upapattis* should seek to provide irrefutable demonstrations establishing the absolute truth of mathematical results.
8. There was no attempt made in Indian mathematical tradition to present the *upapattis* in an axiomatic framework based on a set of self-evident (or arbitrarily postulated) axioms which are fixed at the outset.
9. While Indian mathematicians made great strides in the invention and manipulation of symbols in representing mathematical results and in facilitating mathematical processes, there was no attempt at formalization of mathematics.

Lessons from History

"However vagaries of the external world were not by themselves responsible for the failure of Greek mathematics to advance materially beyond Archimedes. There were also internal factors that suffice to explain this failure. These impeding factors centred on the rigid separation in Greek mathematics between geometry and arithmetic (or algebra), and a one-sided emphasis on the former. Their analysis dealt solely with geometrical magnitudes – lengths, areas, volumes – rather than numerical ones, and their manipulation of these magnitudes was exclusively verbal or rhetorical, rather than analytic (or algebraic as we would say today)." ...

Lessons from History

"It is somewhat paradoxical that this principal shortcoming of Greek mathematics stemmed directly from its principal virtue – the insistence on absolute logical rigour. The Greeks imposed on themselves standards of exact thought that prevented them from using and working with concepts that they could not completely and precisely formulate. For this reason they rejected irrationals as numbers, and excluded all traces of the infinite, such as explicit limit concepts, from their mathematics. Although the Greek bequest of deductive rigour is the distinguishing feature of modern mathematics, it is arguable that, had all succeeding generations also refused to use real numbers and limits until they fully understood them, the calculus might never have been developed, and mathematics might now be a dead and forgotten science." [1]

[1] C. H. Edwards, *The Historical Development of the Calculus,* Springer, 1979, p.78-79.

Lessons from History

"It is high time that the full story of Indian mathematics from vedic times through 1600 became generally known. I am not minimizing the genius of the Greeks and their wonderful invention of pure mathematics, but other peoples have been doing math in different ways, and they have often attained the same goals independently. Rigorous mathematics in the Greek style should not be seen as the only way to gain mathematical knowledge. In India where concrete applications were never far from theory, justifications were more informal and mostly verbal rather than written. One should also recall that the European enlightenment was an orgy of correct and important but semi-rigorous math in which Greek ideals were forgotten. The recent episodes with deep mathematics flowing from quantum field theory and string theory teach us the same lesson: that the muse of mathematics can be wooed in many different ways and her secrets teased out of her. And so they were in India..."[2]

[2] David Mumford, Review of Kim Plofker, *Mathematics in India,* Notices of AMS 2010, p.390.

Lessons from History

Ever since the seminal work of Needham, who showed that till around the sixteenth century Chinese science and technology seem to have been more advanced than their counterparts in Europe, it has become fashionable for historians of science to wonder "Why modern science did not emerge in non-western societies?"

In the work of the Kerala School, we notice clear anticipations of some of the fundamental discoveries which are associated with the emergence of modern science, such as the mathematics of infinite series and the development of new geometrical models of planetary motion.

It seems therefore more appropriate to investigate "Why science did not flourish in non-western societies after the 16th Century?"

It would be worthwhile to speculate "What would have been the nature of modern science (and the modern world) had sciences continued to flourish in non-western societies?" In this way we could gain some valuable insights regarding the sources and the nature of creativity of geniuses such as Srinivasa Ramanujan, Jagadish Chandra Bose, Prafulla Chandra Roy, Chandrasekhara Venkata Raman, and others, in modern India.

Lessons from History

"Japanese have been looking to the West ever since the middle of the Edo period [1603-1868]. This not only holds true with the Western culture in general, but in particular in the fields of science and technology. Certainly the discipline of modern science originated in the seventeenth century in Western countries. Before that, however, perspectives of nature, as well as approaches to it, differed considerably according to place, nationality and time. This fact suggests that the modern-scientific view of, and approach to, nature is neither unique nor absolutely correct, and that there are alternatives as to the direction modern science should take.

We hope that the study of the history of sciences in India, China, and Korea, which have all had a great influence upon the Japanese culture including the indigenous science, will make us consider the past, present, and future of our own culture (and) science and enhance our understanding of neighbouring countries. It is with this view in mind that we are studying the history of exact science such as mathematics and astronomy from East-Asian and South-Asian countries."[3]

[3] Prof. Takao Hayashi, Science and Engineering Research Institute, Doshisha University http://engineering.doshisha.ac.jp/english/kenkyu/-labo/scie/sc-01/index.html.

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Thanks!

Thank You