NPTEL COURSE ON

MATHEMATICS IN INDIA: FROM VEDIC PERIOD TO MODERN TIMES

Lecture 5

Piṅgala's *Chandaḥśāstra*

M. D. Srinivas
Centre for Policy Studies, Chennai

Outline

- Development of Prosody or *Chandaḥśāstra*
- Long (*guru*) and short (*laghu*) syllables
- Scanning of *varṇavṛtta* (syllabic metres) and the eight *gaṇas*
- *Pratyayas* in Piṅgala's *Chandaḥśāstra*
  - *Prastāra* or enumeration in the form of an array
  - *Saṅkhyā*: The total number of metrical forms of n syllables
  - *Naṣṭa* and *Uddiṣṭa*: The association between a metrical form and the row-number in the *prastāra* through binary expansion
  - *Lagakriyā*: Number of metrical forms in the *prastāra* with a given number of *laghus*
  - *Varṇameru* and the "Pascal Triangle"

Development of *Chandaḥśāstra*

In his *Chandaḥśāstra* (c.300 BCE), Piṅgala introduces some combinatorial tools called *pratyayas* which can be employed to study the various possible metres in Sanskrit prosody. Following are some of the important texts which include a discussion of various *pratyayas*:

- Piṅgala (c.300 BCE): *Chandaḥśāstra*
- Bharata (c.100 BCE): *Nāṭyaśāstra*
- Brahmagupta (c.628 CE): *Brāhmasphuṭasiddhānta*
- Virahāṅka (c.650): *Vṛttajātisamuccaya*
- Mahāvīra (c.850): *Gaṇitasārasaṅgraha*
- Halāyudha (c.950): *Mṛtasañjīvanī* Commentary on Piṅgala's *Chandaḥśāsṭra*

Development of *Chandaḥśāstra*

- Kedārabhaṭṭa (c.1000): *Vṛttaratnākara*
- Yādavaprakāśa (c.1000): Commentary on Piṅgala's *Chandaḥśāsṭra*
- Hemacandra (c.1200): *Chandonuśāsana*
- *Prākṛta-Paiṅgala* (c.1300)
- Nārāyaṇa Paṇḍita (c.1350): *Gaṇitakaumudī*
- Dāmodara (c.1500): *Vāṇībhūṣaṇa*
- Nārāyaṇabhaṭṭa (c.1550): *Nārāyaṇī* Commentary on *Vṛttaratnākara*

Varṇa-Vṛtta

- A syllable (*akṣara*) is a vowel or a vowel with one or more consonants preceding it.
- A syllable is *laghu* (light) if it has a short vowel.
- Even a short syllable will be a *guru* if what follows is a conjunct consonant, an *anusvāra* or a *visarga*.
- Otherwise the syllable is *guru* (heavy).
- The last syllable of a foot of a metre is taken to be *guru* optionally.

The first verse of Kālidāsa's *Abhijñānaśākuntalam*:

या सृष्टिः स्रष्टुराद्या वहति विधिहुतं या हविर्या च होत्री
ये द्वे कालं विधत्तः श्रुतिविषयगुणा या स्थिता व्याप्य विश्वम्।
यामाहुः सर्वबीजप्रकृतिरिति यया प्राणिनः प्राणवन्तः
प्रत्यक्षाभिः प्रसन्नस्तनुभिरवतु वस्ताभिरष्टाभिरीशः॥

GGG GLG GLL LLL LGG LGG LGG

The Eight *Gaṇas*

आदिमध्यावसानेषु यरता यान्ति लाघवम्।
भजसा गौरवं यान्ति मनौ तु गुरुलाघवम्॥

Ya: LGG **Ra:** GLG **Ta:** GGL

Bha: GLL **Ja:** LGL **Sa:** LLG

Ma: GGG **Na:** LLL

The pattern of a metre is usually characterised in term of these *gaṇas*.

For instance the verse of Kālidāsa cited earlier is in *Sragdharā* metre:

म्रभ्नैर्यानां त्रयेण त्रिमुनियतियुता स्रग्धरा कीर्तितेयम्।

Thus *Sragdharā* is characterised by the pattern: **MaRaBhaNaYaYaYa,** with a break *(yati)* after seven syllables each.

GGGGLGG LLLLLLG GLGGLGG

A Mnemonic for the *Gaṇas*

There is the mnemonic attributed to *Pāṇini*

यमाताराजभानसलगम्
L G G G L G L L L G

If we replace G by 0 and L by 1, we obtain a binary sequence of length 10

1 0 0 0 1 0 1 1 1 0

The above linear binary sequence generates all the 8 binary sequences of length 3. We can remove the last pair 1, 0 and view the rest as a cyclic binary sequence of length eight.

In modern mathematics such sequences are referred to as De Bruijn cycles.

A Mnemonic for the *Gaṇas*

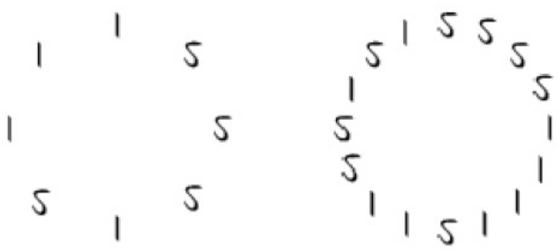


Figure: De Bruijn cycles for Patterns of three and four letters. [। and ऽ stand for L, G or 1,0]

Pratyayas in Piṅgala's *Chandaḥśāstra*

In chapter eight of *Chandaḥśāstra*, Piṅgala introduces the following six *pratyayas*:

Prastāra: A procedure by which all the possible metrical patterns with a given number of syllables are laid out sequentially as an array.

Saṅkhyā: The process of finding total number of metrical patterns (or rows) in the *prastāra*.

Naṣṭa: The process of finding for any row, with a given number, the corresponding metrical pattern in the *prastāra*.

Uddiṣṭa: The process for finding, for any given metrical pattern, the corresponding row number in the *prastāra*.

Lagakriyā: The process of finding the number of metrical forms with a given number of *laghus* (or *gurus*).

Adhvayoga: The process of finding the space occupied by the *prastāra*.

Prastāra

द्विकौ ग्लौ। मिश्रौ च। पृथग्लामिश्राः। वसवस्त्रिकाः।

(छन्दःशास्त्रम् ८.२०-२३)

- Form a G, L pair. Write them one below the other.
- Insert on the right Gs and Ls.
- [Repeating the process] we have eight (*vasavaḥ*) metric forms in the 3-syllable-*prastāra*.

Single syllable *prastāra*

1	G
2	L

Two syllable *prastāra*

1	G	G
2	L	G
3	G	L
4	L	L

Prastāra

Three syllable *prastāra*

1	G	G	G
2	L	G	G
3	G	L	G
4	L	L	G
5	G	G	L
6	L	G	L
7	G	L	L
8	L	L	L

Another Rule for *Prastāra*

पादे सर्वगुरावाद्याल्लघुं न्यस्य गुरोरधः।
यथोपरि तथा शेषं भूयः कुर्यादमुं विधिम्।
ऊने दद्याद्गुरूनेव यावत्सर्वलघुर्भवेत्। (वृत्तरत्नाकरम् ६.२-३)

Start with a row of Gs. Scan from the left to identify the first G. Place an L below that. The elements to the right are brought down as they are. All the places to the left are filled up by Gs. Go on till a row of only Ls is reached.

Example: The following are five successive rows in 4-syllable *prastāra*

G	G	G	L
L	G	G	L
G	L	G	L
L	L	G	L
G	G	L	L

Four-Syllable *Prastāra*

1	G	G	G	G
2	L	G	G	G
3	G	L	G	G
4	L	L	G	G
5	G	G	L	G
6	L	G	L	G
7	G	L	L	G
8	L	L	L	G
9	G	G	G	L
10	L	G	G	L
11	G	L	G	L
12	L	L	G	L
13	G	G	L	L
14	L	G	L	L
15	G	L	L	L
16	L	L	L	L

If we set G=0 and L=1, then we see that each metric pattern is the mirror reflection of the binary representation of the associated “row-number-1”.

Saṅkhyā

द्विरर्धे। रूपे शून्यम्। द्विःशून्ये। तावदर्धे तद्गुणितम्।

(छन्दःशास्त्रम् ८.२८-३१)

The number of metres of n-syllables is $S_n = 2^n$.

Piṅgala gives an optimal algorithm for finding 2^n by means of multiplication and squaring operations that are much less than n in number.

- Halve the number and mark “2”
- If the number cannot be halved deduct one and mark “0”
- [Proceed till you reach zero. Start with 1 and scan the seqence of marks from the end]
- If “0”, multiply by 2
- If “2”, square

Example: Six-syllable metres ($n = 6$)

- $\frac{6}{2} = 3$ and mark “2”
- 3 cannot be halved. 3-1=2 and mark “0”
- $\frac{2}{2} = 1$ and mark “2”
- $1 - 1 = 0$ and mark “0”

Sequence $2, 0, 2, 0$ yields

$$1x2, (1x2)^2, (1x2)^2x2, ((1x2)^2x2)^2 = 2^6$$

Piṅgala’s algorithm became the standard method for computing powers in Indian mathematcis.

Saṅkhyā

Next *sūtra* of Piṅgala gives the sum of all the *saṅkhyās* S_r for $r = 1, 2, \ldots n$.

द्विर्द्व्यूनं तदन्तानाम्। (छन्दःशास्त्रम् ८.३२)

$$S_1 + S_2 + S_3 + \ldots + S_n = 2S_n - 1$$

Then comes the *sūtra*:

परे पूर्णम्। (छन्दःशास्त्रम् ८.३३)

$$S_{n+1} = 2S_n$$

Together, the two *sūtras* imply

$$S_n = 2^n$$

and

$$1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$$

This clearly is the formula for the sum of a geometric series.

Saṅkhyā

- The *saṅkhyā* 2^n discussed above is for the case of syllabic metres of n-syllables which are *sama-vṛttas* – metres which have the same pattern in all the four *pādas* or quarters.
- *Ardhasama-vṛttas* are those metres which are not *sama*, but whose halves are the same.
- *Viṣama-vṛttas* are those which are neither *sama* nor *viṣama*.
- In the fifth Chapter of *Chandaḥ-śāstra*, Piṅgala has dealt with the *saṅkhyā* of *Ardhasama* and *Viṣama-vṛttas*.

समं तावत्कृत्वः कृतमर्धसमम्। विषमं च। राश्यूनम्।
(छन्दःशास्त्रम् ५.३-५)

The number of *Ardhasama-vṛttas* with *n*-syllables in each *pāda* is

$$(2^n)^2 - 2^n$$

In the same way, the number of *Viṣama-vṛttas* with *n*-syllables in each *pāda* is

$$(2^{2n})^2 - \left[\left((2^n)^2 - 2^n\right) + 2^n\right] = (2^{2n})^2 - 2^{2n}$$

लर्धे। सैके ग्। (छन्दःशास्त्रम् ८.२४-२५)

- To find the metric pattern in a row of the *prastāra*, start with the row number
- Halve it (if possible) and write an L
- If it cannot be halved, add one and halve and write a G
- Proceed till all the syllables of the metre are found

Naṣṭa

Example: Find the 7th metrical form in a 4-syllable *prastāra*

- $\frac{(7+1)}{2} = 4$ Hence G
- $\frac{4}{2} = 2$ Hence GL
- $\frac{2}{2} = 1$ Hence GLL
- $\frac{(1+1)}{2} = 1$ Hence GLLG

If we set $G = 0$ and $L = 1$, we can see that Piṅgala's *naṣṭa* process leads to the desired metric form via the binary expansion

$$7 = 0 + 1.2 + 1.2^2 + 0.2^3$$

Uddiṣṭa

प्रतिलोमगणं द्विर्लाद्यम्। ततोग्येकं जह्यात्।

(छन्दःशास्त्रम् ८.२६-२७)

To find the row number of a given metric pattern:

- Start with number 1
- Scan the pattern from the right beginning with the first L from the right
- Double it when an L is encountered
- Double and reduce by 1 when a G is encountered

Example: To find the row-number of the pattern GLLG in a 4-syllable *prastāra*:

- Start with 1.
- Skip the G and go to L. So we get $1x2 = 2$
- Then we find L. So we get $2x2 = 4$
- Finally we have G. We get $4x2 - 1 = 7$

Uddiṣṭa

Another Method

उद्दिष्टं द्विगुणानाद्यादुपर्यङ्कान् समालिखेत्।
लघुस्था ये तु तत्राङ्कास्तैः सैकैर्मिश्रितैर्भवेत्।

(वृत्तरत्नाकरम् ६.५)

- Place 1 on top of the left-most syllable of the given metrical pattern
- Double it at each step while moving right.
- Sum the numbers above L and add 1 to get the row-number

Example: To find the row-number of the pattern GLLG

1	2	2^2	2^3
G	L	L	G

Row-Number $= 0.1 + 1.2 + 1.2^2 + 0.2^3 + 1 = 7$

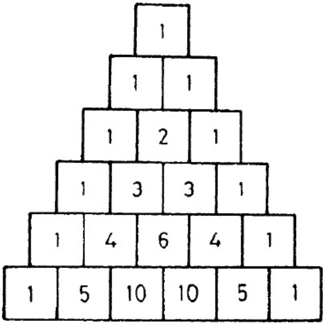
Both the *naṣṭa* and *uddiṣṭa* processes of Piṅgala are essentially based on the fact that every natural number has a unique binary representation: It can be uniquely represented as a sum of the different *saṅkhyā* S_n or the powers 2^n.

परे पूर्णमिति। (छन्दःशास्त्रम् ८.३४)

Piṅgala's *sūtra* on *lagakriyā* process is too brief. Halāyudha, the tenth century commentator explains it as giving the basic rule for the construction of a table of numbers which he refers to as the *Meru-prastāra*.

उपरिष्टादेकं चतुरस्रकोष्ठं लिखित्वा तस्याधस्तादुभयतोऽर्धनिष्क्रान्तं कोष्ठद्वयं लिखेत्। तस्याप्यधस्तात्त्रयं तस्याप्यधस्ताच्चतुष्टयं यावदभिमतं स्थानमिति मेरुप्रस्तारः। तस्य प्रथमे कोष्ठे एकसङ्ख्यां व्यवस्थाप्य लक्षणमिदं प्रवर्तयेत्। तत्र परे कोष्ठे यद्वृत्तसंख्याजातं तत् पूर्वकोष्ठयोः पूर्णं निवेशयेत्। तत्रोभयोः कोष्ठकयोरेकैकमङ्कं दद्यात् मध्ये कोष्ठे तु परकोष्ठद्वयाङ्कमेकीकृत्य पूर्णं निवेशयेदिति पूर्णशब्दार्थः। चतुर्थ्यां पङ्क्तावपि पर्यन्तकोष्ठयोरेकैकमेव स्थापयेत्। मध्यमकोष्ठयोस्तु परकोष्ठद्वयाङ्कमेकीकृत्य पूर्णं त्रिसङ्ख्यारूपं स्थापयेत्।...

Varṇa-Meru of Piṅgala



Clearly the number of metrical forms with r *gurus* (or *laghus*) in the *prastāra* of metres of n-syllables is the binomial coefficient ${}^nC_r$

The above passage of Halāyudha shows that the basic rule for the construction of the above table, is the recurrence relation

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

Pascal Triangle

The above *Varṇa-Meru* is actually a rotated version of the so called Pascal Triangle (c.1655) shown below:

1	1	1	1	1	1	1
1	2	3	4	5	6	
1	3	6	10	15		
1	4	10	20			
1	5	15				
1	6					
1						

References

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Thanks!

Thank You