NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 26

Gaṇitakaumudī of **Nārāyaṇa Paṇḍita** 2

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Outline

- Meeting of travelers
- Progressions
- $V\bar{a}rasa\dot{n}kalita$: Sum of sums. The k^{th} sum. The k^{th} sum of series in A.P.
- The Cow problem
- Diagonals of a cyclic quadrilateral - Third diagonal, area of a cyclic quadrilateral
- Construction of rational triangles with rational sides, perpendiculars, and segments whose sides differ by unity.
- Generalisation of binomial coefficients and generalized Fibonacci numbers.

Meeting of two travelers

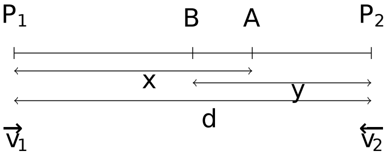
Let d be the distance between two places and suppose that two persons started from these places simultaneously in opposite direction with speeds v_1 and v_2. The following rule gives the rules for their times of meeting.

Rule 39.

> अध्वनि गतियोगहृते प्रजायते प्रथमसङ्गमे कालः।
> तस्मिन् योगे द्विगुणे योगात् तस्मात् पुनर्योगः॥ ३९ ॥

> "The distance divided by the sum of the speeds happen to be the time for the first meeting. Twice (the quotient) obtained by the division of distance by the same (is the time of) meeting again after that meeting."

Meeting of two travelers



Meeting of two travelers at $A(t_1)$ and $B(t_2)$.

Let the travelers start from P_1 and P_2 with speeds v_1 and $v_2(P_1P_2 = d)$. They meet first at A where $P_1A = x$, at time t_1. Then traveler 1 proceeds towards P_2 and 2 towards P_1 and reverse their directions and meet at again at B, where $P_2B = y$, at time t_2.
At A, '1' would have traveled a distance x at with speed v_1 and '2' would have traveled a distance $d - x$ with speed v_2.

Then

$$\frac{x}{v_1} = \frac{d-x}{v_2}$$

Meeting of two travelers

Solving for x, we find $x = \dfrac{dv_1}{v_1 + v_2}$

Timing of meeting $t_1 = \dfrac{x}{v_1} = \dfrac{d}{v_1 + v_2}$, as stated.

Let the second meeting be at B, where $P_2B = y$. Then, total distance traveled by '1' $= d + y = D$. Total distance traveled by '2' $= d + d - y = 3d - D$. As the speeds of '1' and '2' are v_1 and v_2 respectively.

$$\frac{D}{v_1} = \frac{3d - D}{v_2}$$

Solving for D, we find $D = \dfrac{3dv_1}{v_1 + v_2}$. So, time of second meeting $= t_2 = \dfrac{D}{v_1} = \dfrac{3d}{v_1 + v_2}$.

The time between the first ad second meetings, $= t_2 - t_1 = \frac{2d}{v_1+v_2}$ as stated.

Example

Example 44.

> योजनत्रिशती पन्थाः पुरयोरन्तरं तयोः।
> एकादशगतिस्त्वेको नवयोजनगः परः॥
> युगपन्निर्गतौ स्वस्वपुरतो लिपिवाहकौ।
> समागमद्वयं ब्रूहि गच्छतोश्च निवृत्तयोः॥ ४४ ॥

“The distance between two towns is 300 yojanas. Two letter carriers started from their respective towns (simultaneously), one with a speed of 11 yojanas, and the other, 9 yojanas per days. O learned, if you know, tell quickly the times of their two meetings, (the first) after their start and (the second) while returning back.”

Solution: Here $d = 300, v_1 = 11, v_2 = 9$.

$$\text{Time of first meeting}\,(t_1) = \frac{d}{v_1 + v_2} = \frac{300}{11 + 9} = 15$$

$$\text{Time of second meeting}\,(t_2) = \frac{3d}{v_1 + v_2} = 45$$

(Time between the two meetings $= 30$.)

Travellers along a circle

Next, the problem of two travelers traveling along a circle (or any closed path) with different speeds in the same direction is considered.

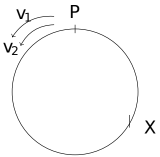
Rule 40 (a).

सङ्गमकालः परिधौ गत्यन्तरभाजिते भवति ॥ ४० *a* ॥

"Length of the circumference, divided by the difference of speeds, happens to be the time of meeting."

Let the travelers start from the same point P with speeds v_1 and v_2.

Travellers along a circle



Meeting of two travelers, traveling along a closed path.

Speeds v_1 and v_2. Let $v_1 > v_2$. Let them meet x at a distance from P. Let the circumference of the path be C. Distance traveled by '1' $= c + x$. Distance traveled by '2' $= x$.

$$\frac{C+x}{v_1} = \frac{x}{v_2}$$

Solving for x, we find $x = \dfrac{Cv_2}{v_1 - v_2}$. As x is the distance traveled by 2, Time of meeting $= \frac{x}{v_2} = \frac{C}{v_1 - v_2}$ as stated.

True and False statements as in earlier texts.

Important

Chapter 3

Earlier results on arithmetic progression stated here also, some sophisticated problems based on these discussed. Standard results on

$$\sum r, \sum r^2, \sum r^3, \sum\sum r = \sum \frac{r(r+1)}{2}$$

where the summations are from 1 to n are stated.
He consider an A.P. with terms:

$$(1+2+\cdots+a), (1+2+\cdots+a+a+1+\cdots+a+d), \cdots, [1+2+\cdots a+(n-1)d]$$

A.P. with each term a sum of A.P.

$$\text{So the r}^{\text{th}} \text{ term } = 1 + 2 + \cdots + \{a + (r-1)d\}$$

is the sum of an A.P.

$$\text{The sum of this A.P.} = \sum_{r=1}^{n} \frac{\{a+(r-1)d\}\{a+(r-1)d+1\}}{2}$$

is stated to be

$$= \frac{n(n-1)}{2}\left[\frac{d}{2}(2a+1) + \frac{d^2}{2}\right] + \frac{na(a+1)}{2} + \frac{d^2}{1 \cdot 2 \cdot 3} n(n-1)(n-2).$$

[Try this as an exercise.]

A very important advancement in *Gaṇitakaumudī*: k^{th} Sum

The following rule is an extremely important result in *Gaṇitakaumudī*. Earlier we had $\sum r = \dfrac{r(r+1)}{2}$, $\sum\sum r = \sum \dfrac{r(r+1)}{2} = \dfrac{n(n+1)(n+2)}{1\cdot 2\cdot 3}$. The last is the sum of sums or the 2^{nd} sum. Nārāyaṇa Paṇḍita generalises this to the k^{th} sum :

$$\underbrace{\sum\sum\cdots\sum r}_{k\text{ sums}} = \frac{n(n+1)(n+2)\cdots(n+k)}{1\cdot 2\cdot 3\cdots(k+1)} = {}^{n+k}C_{k+1}$$

This result is stated in *Yuktibhāṣā* also without referring to *Nārāyaṇa Paṇḍita*. It is possible that the Kerala mathematicians discovered it independently. This plays a crucial role in the infinite Taylor series for the sine and cosine functions.
This is how he states it:

k^{th} sum of n

Rule 19 (b) - 20 (a):

एकाधिकवारमिताः पदादिरूपोत्तराः पृथक् तेंऽशाः ॥ १९ *b* ॥
एकोद्येकचयहराः तद्घातो वारसङ्कलितम् ॥ २० *a* ॥

“Number of terms (say n) is the first term (of an A.P.) and 1, the common difference. Those (i.e., terms of the A.P., their numbers being) 1 more than the number of times (the sum is to be taken, i.e., $k+1$), separately, (are) the numerators. 1 (is) to first term (of another A.P.) and 1 the common difference. These are the denominators, (their number being) the same as that of the former A.P). Their product (is) the k^{th} sum of n.”

k^{th} sum of n

Proof: It is stated that the k^{th} sum of n denoted by V_n^k is ${}^{n+k}C_{k+1}$. We will now show that V_n^k satisfies:

$$V_n^{(k)} = V_1^{(k-1)} + V_2^{(k-1)} + \cdots + V_n^{(k-1)} = \sum_{r=1}^{n} V_r^{k-1}$$

$$\begin{aligned}
\text{Now, } V_n^k &= {}^{n+k}C_{k+1} \\
&= {}^{n+k-1}C_{k+1} + {}^{n+k-1}C_k, \text{ using the properties of } {}^{n}C_r. \\
\therefore\ V_n^k &= V_n^{(k-1)} + V_{n-1}^k \text{ Using this repeatedly} \\
&= V_n^{(k-1)} + V_{n-1}^{(k-1)} + V_{n-2}^k \\
&= \cdots \\
&= V_n^{(k-1)} + V_{n-1}^{(k-1)} + \cdots + V_1^{(k)}
\end{aligned}$$

k^{th} sum of n

But $V_1^{(k)} = V_1^{(k-1)} = 1$.

$$V_n^k = V_n^{(k-1)} + V_{n-1}^{(k-1)} + \cdots + V_1^{(k-1)}$$

$$\therefore\ V_n^{(k)} = \sum_{r=1}^{n} V_r^{(k-1)}$$

Proceeding in this manner,

$$V_n^{(k)} = \sum \cdots \sum V_r^{(0)}$$

Now, $V_r^{(0)} = {}^rC_1 = r$. (Zeroth of sum of r, which is r itself.) So, $V_n^{(k)}$ is indeed the k^{th} sum of first n integers.

The use of the k^{th} sum is illustrated with the “Cow problem” in *Gaṇitakaumudī*.

Cow problem

Example 16.

प्रतिवर्षं गौः सूते वर्षत्रितयेन तर्णकी तस्याः।
विद्वन् विंशतिवर्षैः गोरेकस्याश्च सन्ततिं कथय॥

“A cow gives birth to a (she) calf every year and their calves themselves begin giving birth, when 3 years old. O learned, tell me the number of progeny produced by a cow during 20 years.”

The method of solution is given in Rule 22.

Cow problem: Stated solution

Rule 22.

अब्दास्तर्ण्यब्दोनाः पृथक् पृथक् यावदल्पतां यान्ति।
तानि क्रमशश्चैकादिकवाराणां पदानि स्युः॥

“Subtract the number of years (in which a calf begins giving birth) from the number of years (successively and separately), till the remainder becomes smaller (then the subtractive). Those are the number for repeated summations. Once, (twice) etc., in order. Sum of the summations along with 1 added to the number of years is the number of progeny. (Seems to be including the original cow also).”

Cow problem

The following table would help us in computing the number of progeny in 20 years. The initial cow would give birth to calf every year per 20 years, which constitute the 'first generation' numbering 20. The calf born in the first year would produce its first offspring in the fourth year, this and the one born in the second year would together produce two offsprings in the fifth year, and so on. So, the total number of the the second generation calves would be

$$V_{17}^{(1)} = 1 + 2 + 3 + \cdots + 17$$

Similarly, the total number of third, fourth, fifth, sixth, and seventh generation calves would be $V_{14}^{(2)}, V_{11}^{(3)}$, $V_8^{(4)}, V_5^{(5)}$ and $V_2^{(6)}$. There are no more generation within 20 years, as the eighth generations would only in the 22^{nd} year.

Total progeny

So the total progeny in 20 years is:

$$
\begin{aligned}
V_{20}^{(0)} + V_{17}^{(1)} + V_{14}^{(2)} + V_{11}^{(3)} + V_{8}^{(4)} + V_{5}^{(5)} + V_{2}^{(6)} = \quad & 20 + \frac{17 \cdot 18}{1 \cdot 2} + \frac{14 \cdot 15 \cdot 16}{1 \cdot 2 \cdot 3} + \frac{11 \cdot 12 \cdot 13 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4} \\
& + \frac{8 \cdot 9 \cdot 10 \cdot 11 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \\
& + \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \\
= \quad & 2744
\end{aligned}
$$

We have to add 1 if want to include the original (initial) cow. (Note $V_n^{(0)} = n$, $V_1^{(k)} = 1$ for all k).

Year	1^{st} generation	2^{nd} generation	3^{rd} generation	4^{th} generation	5^{th} generation	6^{th} generation	7^{th} generation
1	1						
2	1						
3	1						
4	1	$V_1^{(0)} = 1$					
5	1	$V_2^{(0)} = 2$					
6	1	$V_3^{(0)} = 3$					
7	1	$V_4^{(0)} = 4$	$V_1^{(0)} = V_1^{(1)}$				
8	1	$V_5^{(0)} = 5$	$V_1^{(0)} + V_2^{(0)} = V_2^{(1)}$				
9	1	$V_6^{(0)} = 6$	$V_1^{(0)} + V_2^{(0)} + V_3^{(0)} = V_3^{(1)}$				
10	1	$V_7^{(0)} = 7$	$V_4^{(1)}$	$V_1^{(1)} = V_1^{(2)}$			
11	1	$V_8^{(0)} = 8$	$V_5^{(1)}$	$V_1^{(1)} + V_2^{(1)} = V_2^{(2)}$			

Year	1^{st} generation	2^{nd} generation	3^{rd} generation	4^{th} generation	5^{th} generation	6^{th} generation	7^{th} generation
12	1	$V_9^{(0)} = 9$	$V_6^{(1)}$	$V_1^{(1)} + V_2^{(1)} + V_3^{(1)} = V_3^{(2)}$			
13	1	$V_{10}^{(0)} = 10$	$V_7^{(1)}$	$V_4^{(2)}$	$V_1^{(2)} = V_1^{(3)}$		
14	1	$V_{11}^{(0)} = 11$	$V_8^{(1)}$	$V_5^{(2)}$	$V_1^{(2)} + V_2^{(2)} = V_2^{(3)}$		
15	1	$V_{12}^{(0)} = 12$	$V_9^{(1)}$	$V_6^{(2)}$			
16	1	$V_{13}^{(0)} = 13$	$V_{10}^{(1)}$	$V_7^{(2)}$	$V_4^{(3)}$	$V_1^{(3)} = V_1^{(4)}$	
17	1	$V_{14}^{(0)} = 14$	$V_{11}^{(1)}$	$V_8^{(2)}$	$V_5^{(3)}$	$V_1^{(3)} + V_2^{(3)} = V_2^{(4)}$	
18	1	$V_{15}^{(0)} = 15$	$V_{12}^{(1)}$	$V_9^{(2)}$	$V_6^{(3)}$	$V_3^{(4)}$	
19	1	$V_{16}^{(0)} = 16$	$V_{13}^{(1)}$	$V_{10}^{(2)}$	$V_7^{(3)}$	$V_4^{(4)}$	$V_1^{(5)}$
20	1	$V_{17}^{(0)} = 17$	$V_{14}^{(1)}$	$V_{11}^{(2)}$	$V_8^{(3)}$	$V_5^{(4)}$	$V_2^{(5)}$
Sum	$V_{20}^{(0)}$	$V_{17}^{(1)}$	$V_{14}^{(2)}$	$V_{11}^{(3)}$	$V_8^{(4)}$	$V_5^{(5)}$	$V_2^{(6)}$

Table. Seven generations of the offsprings of the cow, born in 20 years.

k^{th} sum of a series in A.P.

The next rule gives the k^{th} sum of a series in A.P.

Consider an A.P. with: $a, a+d, \cdots, a+(n-1)d$ (first term is a, common difference is d, number of terms is n).

Rule 20 b - 21.

> रूपोनितपदवारजसङ्कलितं स्याच्च ये गुणाः स पृथक्॥ २० ॥
>
> एकाधिकचारघ्नो व्येकपराप्तो मुखे गुणो भवति।
> स्वगुणघ्नाद्युत्तरयोर्योगः स्याद् वारजं गणितम्॥ २१ ॥

"The k^{th} sum of 'number of terms less 1' is the multiplier of (the common difference). That separately multiplied by '1 more than number of times' (the sum is to be taken, i.e., $k+1$ and then) divided by the 'number of terms less 1' is multiplier of the first term The first term and the common difference (are both) multiplied by their own multipliers. The sum of the products happens to be the k^{th} sum (of the series in A.P.)."

k^{th} sum of a series in A.P.

$$\text{So, } \sum \sum_{k} \cdots \sum \text{A.P.} = \left[\frac{a(k+1)}{(n-1)} + d \right] \frac{(n-1) \cdot r \cdots (n+r-1)}{1 \cdot 2 \cdots (k+1)}$$

$$\text{Rationale: First sum of A.P.} = an + \frac{n(n-1)}{2} d$$

$$k^{\text{th}} \text{ Sum of the series in A.P. } = (k-1)^{\text{th}} \text{ Sum of } \left[an + \frac{n(n-1)}{2} d \right]$$

$$= a[(k-1)^{\text{th}} \text{ Sum of } n] + d[k^{\text{th}} \text{ Sum of } (n-1)],$$

as the first sum $\sum n - 1 = \dfrac{n(n-1)}{2}$.

$$= a \frac{n(n+1) \cdots (n+k-1)}{1 \cdot 2 \cdots k} + d \frac{(n-1)n \cdots (n-1+k)}{1 \cdot 2 \cdots (k+1)}$$

$$= \left[\frac{a(k+1)}{n-1} + d \right] \frac{(n-1) \cdots (n+k-1)}{1 \cdot 2 \cdots (k+1)}$$

Example

Example 15.

आदिः समीरणमितः प्रचयस्त्रिसङ्ख्यो
गच्छेषु सप्तसु वदाशु प्रार्ड्ढबुद्धे ।
वारैः पयोनिधिमितैः परिवर्तनेन
स्यात् किं फलं गणितमत्सरताऽस्ति ते चेत् ॥

"First term of (an A.P) is 5, common difference, 3 (and) the number of terms, 7. O best among scholars, tell quickly the 4^{th} sum (of the series in A.P.). (Also,) if you have passion for mathematics, tell the sum by changing the ingredients."

Solution: $a = 5, d = 3, n = 7, k = 4$.

$$\therefore \text{ Sum } = \left[\frac{5 \times 5}{6} + 3\right] \frac{6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \left[\frac{25}{6} + 3\right] 4 \times 63 = 2 \times 25 \times 21 + 12 \times 63 = 1806.$$

One can work out changing ingredients.

The treatment of G.P is as in *Gaṇitasārasaṅgraha* and at *Līlāvatī*: nothing new.

So also, *Sama Vṛttas* as in *Līlāvatī*.

Geometry in *Gaṇitakaumudī* in chapter 4

All the results of an geometry in *Gaṇitasārasaṅgraha* and *Līlāvatī* are stated here. *Nārāyaṇa Paṇḍita* adds many results of his own especially on rational triangles and quadrilaterals, and also generalizes many earlier results. We give some interesting results in the geometry of plane figures introduced / stated in this chapter.

Rule 15 gives "gross-area of regular polygon with *n* sides.

Rule 15.

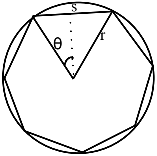
रश्म्यूनरश्मिकृतिहतभुजकृतिरिनहृत् फलं त्रिकोणादौ ॥ १५ ॥

"Subtract the number of sides from the square of the number of sides. Multiply (the difference) by the square of the side. (The product) divided by 12 is the (gross) area of a triangle."

Area of a regular polygon with n sides

The area of a regular polygon of n sides, each of length S is states to be:

$$A = (n^2 - n)\frac{S^2}{12}$$



Area of a regular polygon with side S.

Inscribe the polygon with n sides each of length S in a circle of radius r. It is clear that $\theta = \dfrac{\pi}{n}$ and $S = 2r \sin\theta$.

$$\text{The area of each triangle, as indicated } = r^2 \sin\theta \cos\theta$$

$$\therefore \text{ Area of the polygon } = nr^2 \sin\theta\cos\theta$$

$$\therefore A = n\frac{S^2\cos\theta}{4\sin\theta} = n\frac{S^2\cos\left(\frac{\pi}{n}\right)}{4\sin\left(\frac{\pi}{n}\right)}.$$

This is the exact area.

Area for large n

When n is large,

$$\sin\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n} - \frac{1}{6}\left(\frac{\pi}{n}\right)^3 \approx \frac{\pi}{n}\left[1 - \frac{1}{6}\frac{\pi^2}{n^2}\right]$$

$$\text{and } \cos\left(\frac{\pi}{n}\right) \approx 1 - \frac{\pi^2}{2n^2}$$

$$\therefore A \approx n\frac{S^2}{4\frac{\pi}{n}}\frac{(1 - \frac{\pi^2}{2n^2})}{(1 - \frac{1}{6}\frac{\pi^2}{n^2})}$$

$$\approx \frac{n^2S^2}{4\pi}\left(1 - \frac{1}{3}\frac{\pi^2}{n^2}\right)$$

Further approximation

If we put $\pi \approx 3$, (this crude approximation to π has been stated by Narayana), we obtain

$$A \approx \frac{n^2 S^2}{12}\left(1 - \frac{3}{n^2}\right) \approx \frac{(n^2 - 3)}{12} S^2$$

It is not clear what is the approximation which led the author to his result.

After many other 'gross' results Nārāyaṇa states;"The earlier gross rules have been stated for novice calculations. Due to occasional disagreement between (gross and exact) results, I have not much respect (for them)."

Diagonals of a cyclic quadrilateral

Diagonals of a cyclic quadrilateral.

Here *Nārāyaṇa* gives the standard expression for the diagonals of a cyclic quadrilateral. He also introduces the concept of third diagonal, which is very useful in deriving many results (including the expression for the area of a cyclic quadrilateral, $\sqrt{(s-a)(s-b)(s-c)(s-d)}$) which is proved in *Yuktibhāṣā*.

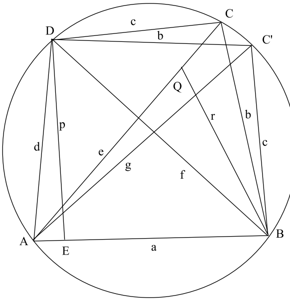
Rule 47-52 includes:

> उभयश्रवणाश्रितभुजवधयोगौ तौ परस्परं विहृतौ।
> प्रतिभुजभुजबधयोगा हतौ तु मूले चतुर्भुजे कर्णौ॥
> सर्वचतुर्बाहूनां मुखस्य परिवर्तने यदा विहिते।
> कर्णस्तदा तृतीयः पर इति कर्णत्रयं भवति॥ ४८ ॥

“Divide the sum of the products of the sides about both the diagonals by each other. Multiply the quotients by the sum of the products of opposite sides. Square roots of the, products are the diagonals in a quadrilateral.

In all (cyclic) quadrilaterals, the (new) diagonal obtained by the interchange of its face and flank side is the third diagonal.”

Cyclic quadrilateral: Third diagonal



A cyclic quadrilateral ABCD. AC, BD are the diagonals. Another cyclic quadrilateral ABC'D got by interchanging the sides BC and CD. AC' is the third 'diagonal'.

In the figure, $ABCD$ is a cyclic quadrilateral with sides, $AB = a, BC = b$, $CD = c$, and $DA = d$. $AC = e$ and $BD = f$ are its diagonals. Now on the arc BD, choose a point C' such that $arc\,DC' = arc\,BC$. Then naturally, $arc\,C'B = arc\,CD$. The corresponding chords are also. Therefore, $DC' = C'B$. So the quadrilateral $ABC'D$ is generated by interchanging the sides b and c in the original quadrilateral. Then $AC' = g$ is called the third diagonal.

Expressions for the three diagonals

We had already derived the expressions for the diagonals $AC = e$ or $BD = f$ in the material on *Brāhmasphuṭasiddhānta*. These are stated by Nārāyaṇa:

$$AC = (e) = \left[\frac{(ac+bd)(ad+bc)}{(ab+cd)}\right]^{1/2}$$

$$BD = (f) = \left[\frac{(ac+bd)(ab+cd)}{(ad+bc)}\right]^{1/2}$$

(The expressions may look slightly different, as the symbols for the sides is different here.)

The third diagonal $AC' = g$ is got by interchanging b and c in the original quadrilateral. Hence, by interchanging b and c in AC

$$AC' = (g) = \left[\frac{(ab+cd)(ad+bc)}{(ac+bd)}\right]^{1/2}$$

Circumdiameter

Now we had already seen that in a triangle, the product of sides (about a perpendicular) divided by the perpendicular is the diameter of the circumcircle. The circumdiameter D of a cyclic quadrilateral can be obtained in this way, by considering an appropriate triangular part of the cyclic quadrilateral. For instance, Let $BQ = r$ be the perpendicular to the diagonal AC. Then,

$$D = \frac{ab}{r}$$

It is perpendicular to AB

$$D = \frac{AD \cdot BD}{DE} = \frac{d \cdot f}{p} \text{ also.}$$

Area of a cyclic quadrilateral

Area of a cyclic quadrilateral is stated in the following rule;

Rule 134 a.

कर्णाश्रितभुजबधयुतिगुणिते तस्मिन् श्रवस्यऽपि विभक्तो।
चतुराहतहृदयेन द्विसमादिचतुर्भुजे गणितम्।

"Multiply the sum of the products of the sides (of a quadrilateral) lying on the same side of a diagonal by the diagonal. Divide (the product) by 4 times the circum-radius. (The result) is the area of the equilateral and other quadrilaterals (A)."

$$\text{That is, area } A = \frac{(ab+cd)e}{4R} = \frac{(ad+bc)f}{4R}$$

(Here circum-radius, $R = \dfrac{D}{2}$).

Area of a cyclic quadrilateral

Proof: Now $D = \dfrac{ab}{r}$, or $r = \dfrac{ab}{D}$.

$$\text{Area of triangle } ABC = \frac{1}{2} \cdot AC \cdot r = \frac{ab \cdot e}{2D}$$

$$\text{Similarly, Area of the triangle } ADC = \frac{cd \cdot e}{2D}$$

By adding two results, area of the quadrilateral $ABCD$

$$= \frac{(ab + cd)e}{2D} = \frac{(ab + cd)e}{4R}$$

Similarly, considering the triangles DCB and DAB flanking the diagonal, $BD = f$,

$$\text{Area } = \frac{(ad + bc)f}{4R}$$

An alternate expression for the third diagonal

An alternate expression for the third diagonal, $AC' = g$:

Rule 136:

चतुराहतहृदयहते गणिते श्रुतिभाजिते भवति।
भुजमुखपरिवर्तनजे पराभिधाना श्रुतिर्नियतम्॥ १३६ ॥

“4 times the circum-radius multiplied by the area (and then) divided by the (product of) the diagonals determines the other diagonal which is obtained by interchanging the face with a flank side.”

The diagonals, circumradius and area

$$\text{That is, } g = \frac{4RA}{ef}$$

From the expression for the diagonals $BD(f)$ and $AC'(g)$

$$gf = ab + cd$$

$$\text{But, } A = \frac{(ab+cd)e}{4R}$$

$$\therefore\ gf = \frac{4R \cdot A}{e}$$

$$\text{or } g = \frac{4RA}{ef}$$

From this it follows that

$$R = \frac{efg}{4A}$$

which is stated in the following rule:

Rule 138 a.

चतुराहतफलविहृते त्रिकर्णघातेऽथवा हृदयम् ॥ १३८ ॥

“Alternatively the product of the three diagonals divided by four times the area is the circum-radius.”

Construction of integral cyclic quadrilaterals

Construction of integral Cyclic quadrilaterals.

Remember that if we had two right-triangles with the upright, side and hypotenuse as (a_1, b_1, c_1) and (a_2, b_2, c_2), Brahmagupta had constructed a cyclic quadrilateral with sides c_2a_1, c_1b_2, c_2b_1 and c_1a_2 and diagonals: $a_1a_2 + b_1b_2$ and $b_1a_2 + a_1b_2$ which are perpendicular to each other.

Here Nārāyaṇa Paṇḍita states how one can obtain a cyclic quadrilateral using the same procedure in which not only all the sides and diagonals are integral, but also the various perpendiculars (from the vertices to the appropriate sides,or diagonals) and the various segments which the perpendiculars divide the appropriate sides and diagonal into, are integral or rational. It is stated thus:

Integral cyclic quadrilaterals

Rule 93 b - 97 a:

जात्ये चतुर्भुजे द्वे लघुकर्णघ्नावनल्पकोटिभुजौ॥ ९३ ॥

भवदनेऽनल्पश्रुतिसङ्गुणितावल्पकोटिभुजौ।

विषमचतुर्भुजजाताः सर्वभुजा अल्पकर्णसङ्गुणिताः॥ ९४ ॥

कोटिवधाबाहुवधयोः संयोगो जायते गुणश्चैकः॥ ९५ ॥

भुजकोटिवधसमासः परोऽल्पकर्णाहतौ हि तौ कर्णौ॥ ९६ ॥

व्यासः स्यात् कर्णद्वयघातो दलितः फलं सूक्ष्मम्॥ ९७ ॥

“The upright and the side of the bigger rectangle, (really triangle, among two given or assumed triangles) multiplied by the diagonal of the smaller (triangle separately) are the face and the base. The upright and side of the smaller (triangle) multiplied by the diagonal of the bigger rectangle (triangle) are the two flank sides. All sides multiplied by the diagonal of the smaller (triangle) are all the sides of a scalene quadrilateral.

Integral cyclic quadrilaterals

The sum of the products of the sides and (the product) of the uprights is the $guṇa$. The sum of the products of the side (of one triangle) and the (upright) of the other is the other $guṇa$. The $guṇa$s separately multiplied by the two diagonal of the smaller (triangle) are the diagonals of the quadrilateral. The diagonal of smaller (triangle) multiplied by the (product of) the diagonals (of the two triangles) is the (circum) diameter and half of the product of the diagonals of the quadrilaterals is the exact area of the quadrilateral."

[The smaller and the greater $guṇa$s multiplied separately by the greater and the smaller sides of the smaller rectangle (triangle), respectively are the perpendiculars and those multiplied separately, by the smaller and the greater sides are the $P\bar{ı}ṭha$s (i.e,, complements) of the segments). $P\bar{ı}ṭha$s subtracted from the base, are the $Sandhi$s (i.e., links)].

[Portion in square bracket is not in the text. It has been reconstructed by P. Singh based on the formulae used in the example solved by Narayana following the above rule.]

Integral cyclic quadrilaterals

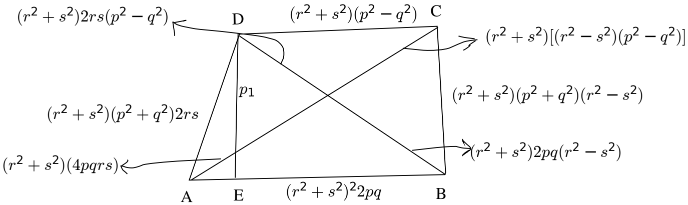
Let $(r^2 - s^2)(= b_2), 2rs(= a_2), (r^2 + s^2)(= c_2)$ be the base, upright and the diagonal of the smaller triangle and $(p^2 - q^2)(= b_1), 2pq(= a_1)$ and $(p^2 + q^2)(= c_1)$ be those of the bigger triangle in order. According to the rule, a scalene (unequal sides) quadrilateral can be formed such that its:

$$\text{face } = (r^2 + s^2)^2 (p^2 - q^2), \qquad \text{base } = (r^2 + s^2)^2\, 2pq$$

$$\text{flank sides } = (r^2 + s^2)(p^2 + q^2)(r^2 - s^2) \quad \text{and} \quad 2rs(r^2 + s^2)(p^2 + q^2)$$

$$\text{diagonals } = (r^2 + s^2)[4pqrs + (r^2 - s^2)(p^2 - q^2)] \quad \text{and} \quad (r^2 + s^2)[2pq(r^2 - s^2) + 2rs(p^2 - q^2)]$$

(So far, an extra factor $(r^2 + s^2)$ was not necessary to make the sides and diagonals integral (by choosing r, s, p, q integral). But this extra factor in the sides and diagonals is necessary to make the perpendiculars and the segments integral.)



Cyclic quadrilateral constructed according to the rule.

Integral cyclic quadrilaterals

The perpendiculars are given to be;

$$p_1 = (r^2 - s^2)[2pq(r^2 - s^2) + 2rs(p^2 - q^2)] \quad \text{and} \quad p_2 = 2rs[(p^2 - q^2)(r^2 - s^2) + 4pqrs]$$

$P\bar{\imath}\d{t}ha$s (complements of the segment) are given to be:

$$s_1 = 2rs[2pq(r^2 - s^2) + 2rs(p^2 - q^2)] \quad s_2 = (r^2 - s^2)[(r^2 - s^2)(p^2 - q^2) + 4pqrs]$$

Links are (these are appropriate base s_1 or s_2, that is the other segment)

$$l_1 = 2pq(r^2 + s^2)^2 - [2pq(r^2 - s^2) + 2rs(p^2 - q^2)] \cdot 2rs$$

$$\text{(i.e.,)}\ s_1 + l_1 = AB = (r^2 + s^2)^2 \cdot 2pq$$

$$l_2 = ?$$

$$\text{circumdiameter} = (r^2 + s^2)^2 (p^2 + q^2)^2$$

$$\text{Area} = \frac{1}{2}[2pq(r^2 - s^2) + 2rs(p^2 - q^2)][(r^2 - s^2)(p^2 - q^2) + 4pqrs](r^2 + s^2)^2$$

Exercise: Verify the expansion for p_1 and work out AE, EB using AD and BD and then find the circumdiameter. Verify the expression for the area. From the expression for the circumdiameter figure out which are the sides / diagonals involved in p_2 and indicate it. Find l_2.

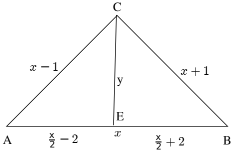
Construction of rational triangles, whose sides differ by unity

The following rule gives the procedure;

Rule 118.

द्विगुणेष्टमिष्टकृत्या त्रिहीनयाप्तं च तत्कृतिस्त्रिगुणा।
सैका मूलं द्विगुणं भूः सैकोनाऽधिका बाहुः॥ ११८ ॥

"Divide twice an optional number by the square of the optional number less 3. Add 1 to thrice the square (of the quotient). Twice the square root of the sum is the base. 1 added to and subtracted (from the base) are the flank sides."



Rational triangle whose sides differ by unity.

We can conceive of the triangle as above, with the sides $AB = x, AC = x - 1$ and $BC = x + 1$. The perpendicular $CE = y$ divides the base AB into segments $AE = \dfrac{x}{2} - 2$ and $EB = \dfrac{x}{2} + 2$. We want the perpendicular and to segments also to be the rational.

Rational triangles whose sides differ by unity

We should have,

$$(x-1)^2-\left(\frac{x}{2}-2\right)^2=(x+1)^2-\left(\frac{x}{2}+2\right)^2=y^2$$

This leads to

$$\frac{3}{4}x^2-3=y^2$$

$$\text{Choose: } y=\frac{3\cdot 2n}{(n^2-3)}, \quad n \text{ an integer.}$$

$$\begin{aligned}
\therefore x &= 2\left[3\left(\frac{2n}{n^2-3}\right)^2+1\right]^{1/2} \\
&= \left[\frac{12n^2+(n^2-3)^2}{(n^2-3)^2}\right]^{1/2} \\
&= 2\left[\frac{n^4+6n^2+9}{(n^2-3)^2}\right]^{1/2}=2\left(\frac{n^2+3}{n^2-3}\right).
\end{aligned}$$

The solution for x is as stated in the rule. Clearly, x is rational.

Rational triangles whose sides differ by unity

Rule 119-120.

प्रथमं जात्यत्र्यस्रं त्रिलम्बकं भूचतुष्कमस्माद्य।

जात्यान्युत्पद्यन्तेऽनन्तान्येकोत्तरभुजानि॥ ११९ ॥

त्रिगुणा भूमिः स्वादिमलम्बयुता लम्बकः सलम्बमही।

द्विगुणा भूमिः पुरत स्त्रिभुजं जात्यं भवेदेवम्।

सर्वेषां त्रिभुजानां एकोनयुता मही बाहुः॥ १२० ॥

“3 being the length of the perpendicular and 4, the base of the first right angled triangle, and its infinite (pairs of) right angled triangles are produced in which sides increase by unity. (In these), the perpendiculars from the vertex to the respective base is the sum of thrice the previous base added to the still previous perpendicular and the base is twice the sum of the previous perpendicular added to the previous base. Triangles in opposition (in such triangles) are right-angled and in all such triangles, 1 added to and subtracted from the base, are the flank and the sides.”

Rational triangles

We want solutions for the equation $\frac{3}{4}x^2 - 3 = y^2$, where x is the base, and y is the perpendicular. Let the solutions for the base be written as $x_1, x_2, \cdots$ and the corresponding perpendiculars $y_1, y_2, \cdots$. Suppose we have found x_j and y_j, up to $j = i - 1$. Then it is stated that new solution x_i, y_i can be found using

$$x_i = 2(x_{i-1} + y_{i-1}) \text{ and } y_i = 3x_{i-1} + y_{i-2}.$$

This is a '$bh\bar{a}van\bar{a}$' or a 'composition law'. ($Sam\bar{a}sa\ bh\bar{a}van\bar{a}$ in this particular case). From these x_{i+1}, y_{i+1} can be found, and so on. The simplest integer is '0' and this is not a solution for triangle. Next is $x_1 = 4, y_1 = 3$, which is a trivial solution. Then we can generate an infinite number of triangles.

“Integral Triangles” whose sides differ by unity

Proof: We have to solve $\frac{3}{4}x^2 - 3 = y^2$. Let $x = x_{i-1}, y = y_{i-1}$ satisfy this, that is:

$$\frac{3}{4}x_{i-1}^2 - 3 = y_{i-1}^2$$

Now take $x_i = 2(x_{i-1} + y_{i-1})$ and $y_i = 2\left(y_{i-1} + \frac{3}{4}x_{i-1}\right) = 2y_{i-1} + \frac{3}{2}x_{i-1}$

$$\frac{3}{4}x_i^2 - 3 = \frac{3}{4}(4x_{i-1}^2 + 4y_{i-1}^2 + 8x_{i-1}y_{i-1}) - 3$$

Using $\frac{3}{4}x_{i-1}^2 - 3 = y_{i-1}^2$,

We have, $\frac{3}{4}x_i^2 - 3 = 7y_{i-1}^2 + 6x_{i-1}y_{i-1} + 9$

$$y_i^2 = 4y_{i-1}^2 + \frac{9}{4}x_{i-1}^2 + 6x_{i-1}y_{i-1}$$

Again using the relation between x_{i-1} and y_{i-1}

$$y_i^2 = 7y_{i-1}^2 + 6x_{i-1}y_{i-1} + 9$$

Hence, $\frac{3}{4}x_i^2 - 3 = y_i^2$, so the equation is satisfied.

Integral traingles

Now the recurrence relation for

$$x_i = 2(x_{i-1} + y_{i-1}) \quad \text{is correct.}$$

Recurrence relation for y_i, we have used $y_i = 2\left(y_{i-1} + \frac{3}{4}x_{i-1}\right)$

$$\begin{aligned} y_i &= 2\left[2\left(y_{i-2} + \frac{3}{4}x_{i-2}\right)\right] + \frac{3}{2}x_{i-1} \\ &= y_{i-2} + 3(y_{i-2} + x_{i-2}) + \frac{3}{2}x_{i-1} \\ &= y_{i-2} + \frac{3}{2}x_{i-1} + \frac{3}{2}x_{i-1} \end{aligned}$$

$$\therefore\ y_i = 3x_{i-1} + y_{i-2}$$

This is the recurrence relation, as stated in the rule.

Generating the triangles

Generating the triangles.

Take $x_0 = 2, y_0 = 0$. These satisfy the equation.

$$\therefore\ x_1 = 2(x_0 + y_0) = 4 \quad y_1 = 2(y_0 + \frac{3}{4}x_0) = 3$$

So, $x = x_1 = 4, y = y_1 = 3$. The other sides are $x - 1 = 3$, $x + 1 = 5$ and the segments are $\frac{x}{2} - 2 = 0$, $\frac{x}{2} + 2 = 4$.

$$\text{Next}\, x_2 = 2(x_1 + y_1) = 14, y_2 = 3x_1 + y_0 = 12$$

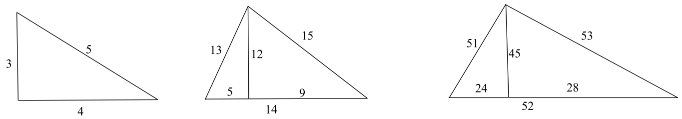
So, $x = x_2 = 14, y = y_2 = 12$ (perpendicular). The other sides are $x - 1 = 13$, $x + 1 = 15$, and the segments are $\frac{x}{2} - 2 = 5$, $\frac{x}{2} + 2 = 9$

Generating the triangles

$$\text{Next } x_3 = 2(x_2 + y_2) = 52, y_3 = 3x_2 + y_1 = 45$$

So, $x = x_3 = 52$ (base), $y = y_3 = 45$ (perpendicular). The other sides are $x - 1 = 51, x + 1 = 53$, and the segments are $\frac{x}{2} - 2 = 24$, $\frac{x}{2} + 2 = 28$.

These are depicted in the figure below:



First three triangles with sides differing by unity, generated by the algorithm stated in Rule 119-120.

These are integral solutions (for sides differing by unity, as well as the perpendicular and the segments). Clearly there are an infinite number of solutions.

Combinatorics

Chapter 13 on *aṅkapāśa* or 'combinatorics' is a very elaborate one, containing many new results on permutations and combinations. We first consider the generalised 'Fibonacci' sequence described here.

The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 12, $\cdots$. If P_n denotes the n^{th} term in the sequence, where we start with $n = 0$, it satisfies the recursion relation:

$$P_n = P_{n-1} + P_{n-2}$$

They are related to the number of ordered partitions of a number into parts containing 1 and 2 only. $P_0 = 1$, by convention. We have

$$\begin{aligned}
1 &= 1, P_1 = 1 \\
2 &= 1 + 1 = 2, P_2 = 2, \\
3 &= 1 + 1 + 1 = 1 + 2 = 2 + 1, P_3 = 3, \\
4 &= 1 + 1 + 1 + 1 = 1 + 1 + 2 = 1 + 2 + 1 = 2 + 1 + 1 = 2 + 2, P_4 = 5, \\
5 &= 1 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 2 = 1 + 1 + 2 + 1 \\
&= 1 + 2 + 1 + 1 = 2 + 1 + 1 + 1 \\
&= 1 + 2 + 1 + 1 = 1 + 2 + 2 = 2 + 1 + 2 = 2 + 2 + 1, P_5 = 8,
\end{aligned}$$

and so on.

Nārāyaṇa's *Sāmāsikī* sequence

It can be shown that

$$P_n = {}^{n}C_0 + {}^{n-1}C_1 + {}^{n-2}C_2 + \cdots + {}^{n-m}C_m,$$

where $m = \dfrac{n}{2}$ if n is even, and $m = \dfrac{n-1}{2}$ if n is odd. One can check that the numbers P_n satisfy the recursion relation mentioned earlier.

The Fibonacci numbers in fact appeared six hundred earlier in the work *Vṛttajātisamuccaya* of *Virahāṅka* (c.600), who arrived at the recurrence relation $P_n = P_{n-1} + P_{n-2}$, in the context of his discussion of *Mātrā-vṛtta*s or moric metres. Nārāyaṇa's *Sāmāsikī* sequence is essentially a generalisation of the sequence discovered by *Virahāṅka* in the context of prosody. It is essentially a generalised Fibonacci sequence, where one considers the partitions of a number n when all the digits from 1 upto q take part in the partitions. This is denoted by P_n^q.

Sāmāsikī sequence

We have the relations:

$$\begin{aligned} P_0^q &= P_1^q = 1, \\ P_n^q &= P_0^q + P_1^q + \cdots P_{n-1}^q,\ 2 \leq n \leq q, \\ P_n^q &= P_{n-q}^q + P_{n-q+1}^q + \cdots + P_{n-1}^q, n > q. \end{aligned}$$

When $q = 2$, we have the Fibonacci numbers: 1, 1, 2, 3, 5, 8, $\cdots$. When $q = 3$, the *Sāmāsikī* sequence would be 1, 1, 2, 4, 7, 13, 24, 44, $\cdots$. The members of this sequence satisfy the recurrence relation:

$$P_n^3 = P_{n-1}^3 + P_{n-2}^3 + P_{n-3}^3$$

Generalisations of binominal coefficients

Generalisations of the binomial coefficients: The binomial coefficients ${}^nC_r$ can be defined through:

$$(1+x)^n = \sum {}^nC_r\ x^r,$$

where the summation is from $r = 0$ to n.

The binomial coefficients are generalised to 'polynomial coefficients' which we write as $u(p,q,r)$, in *Gaṇitakaumudī*. They are defined through what amounts to the formula:

$$(1+x+x^2+\cdots+x^{q-1})^p = \sum u(p,q,r)\,x^r,$$

where the summation is from $r = 0$ to $r = (q-1)p$. He also gives methods to generate $u(p,q,r)$. It is obvious that when

$$q = 2, u(p,2,r) = {}^pC_r$$

Sāmāsikī sequence and polynominal coefficients

Various *meru*'s associated with these co-efficients are discussed in the text. Nārāyaṇa also gives the relations among the generalised Fibonacci numbers and the polynomial coefficients:

$$
\begin{aligned}
P_0^q &= 1 = u(0,q,0) \\
P_1^q &= 1 = u(1,q,0) \\
P_2^q &= u(2,q,0) + u(1,q,1) \\
& \quad \ldots\ldots \\
P_t^q &= u(t,q,0) + u(t-1,q,1) + \cdots + u(t-s,q,s), \\
, & \quad \textit{where } s \leq \frac{q-1}{q} t.
\end{aligned}
$$

References

1. *Gaṇitakaumudī* of Nārāyaṇa Paṇḍita, Ed. by Padmākara Dvivedi, 2 Vols, Varanasi, 1936, 1942.

2. *Gaṇitakaumudī* of Nārāyaṇa Paṇḍita, Eng. Tr. with Notes, Paramanand Singh, *Gaṇita Bhāratī* **20**, 1998, pp. 25-82; ibid. **21**, 1999, pp. 10-73; ibid. **22**, 2000, pp. 19-85; ibid. **23**, 2001, pp. 18-82; ibid. **24**, 2002, pp. 34-98.

3. Paramanand Singh, ”The so-called Fibonacci numbers in ancient and medieval India” , Historia Mathematica, **12**, 1985, pp. 229-244.

4. Raja Sridharan, R.Sridharan and M.D.Srinivas, ”Nārāyaṇa Paṇḍita’s enumeration of combinations of and associated representation of numbers as sums of Binomial coefficients, Indian Journal of History of Science, **47**, 2012, pp. 607-636.

Thanks!

Thank You