NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 6

Decimal place value system

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Outline

Decimal Place Value System & Representation of numbers

- Introduction
- Numerals found in the inscriptions (Brāhmī & Kharoṣṭhi)
- Use of Zero as a symbol by Piṅgala (2nd cent BCE)
- Earliest evidences for the use of Place value system
  - From Indian Philosophical, Mathematical literature
  - From various Inscriptions
- Different systems of numeration employing place value system:
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  - *Kaṭapayādi* system
  - *Bhūtasaṅkhyā* system
- A few useful charts (Vowels & Consonants)
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Introduction

Origin of numbers and Place value system

- Generally most of us do not get to know or have opportunities to get to know answers to questions like
  - When did we start counting?
  - Were there other systems of counting?,
  - What are the different ways of representing numbers? etc.,
- As we keep using decimal system of numeration right from our childhood, we are so familiar with that, that we tend to think that it has been there for ever.
- It is indeed pretty old. But how old?
- One of the most ancient literature $\dot{R}g\text{-}veda$ presents the number 3339 using word numeration:

  त्रीणि शता त्रीसहस्राण्यग्निं त्रिंशश्च देवा नव चासपर्यन्।
  औक्षन् घृतैरसृणन् बर्हिरस्मा ... *[$\dot{R}g$-veda 3.9.9.]*

- From such quotes it is evident that decimal place value system has been in vogue amongst the Vedic seers.

Introduction

Ingenuity of the advent of Place value system & Zero

- Laplace[1] while describing the contribution of Indians to mathematics observes:

> The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions. The importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of Antiquity, Archimedes and Appolonius.

[1] A renowned French Scientist of the 18th-19th century who made phenomenal contributions to the fields of mathematics and astronomy

Origin of numerals: Which hypothesis to choose?

- Several hypotheses have been put forth on the orgin of the so-called modern/(Arabic?) numerals.
- Commenting on it the French G. Beaujouan observes:[2]

  > the origin of the so-called 'Arabic' numerals have been written about so often that every view on the question seems plausible, and the only way of choosing between them is by personal conviction

- Generally origins are traced through citations. Though citations are testimonies, it is quite possible that it could be marred due to gaps in perceptions and unavoidable distortions of human memory.
- Hence, collating several evidences becomes essential.

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[2] cited by George Ifrah in *The Universal History of Numbers II, p. 1., Penguin Books India, 2005.*

Origin of the present day numeration?

- In tracing the origin of the present day numeration, George Ifrah has done a splendid job in his book *The Universal History of Numbers*.[3]
- While introducing the topic of representation of numbers by Indian astronomers, Georges Ifrah observes (pp. 107-108):

  > We are now going to look at truly remarkable method of expressing numbers which is frequently found on mathematical and astronomical texts written in Sanskrit . . . added to all other evidence, it allows us not only to prove beyond doubt that our present day numeration is of Indian origin and Indian alone . . .

- Besides the evidences provided by Ifrah, we will be providing quotes from original Sanskrit texts to prove the point.

[3] This book is an outcome of his attempt to answer his pupil's question.

Various hypotheses on the origin of ‘Arabic’ Numerals

- The various hypotheses that have been put forward regarding the origin of the so-called Arabic numerals include:
  - the number of angles each numeral possesses
  - the number of lines the numeral possesses
  - the number of points, and so on.

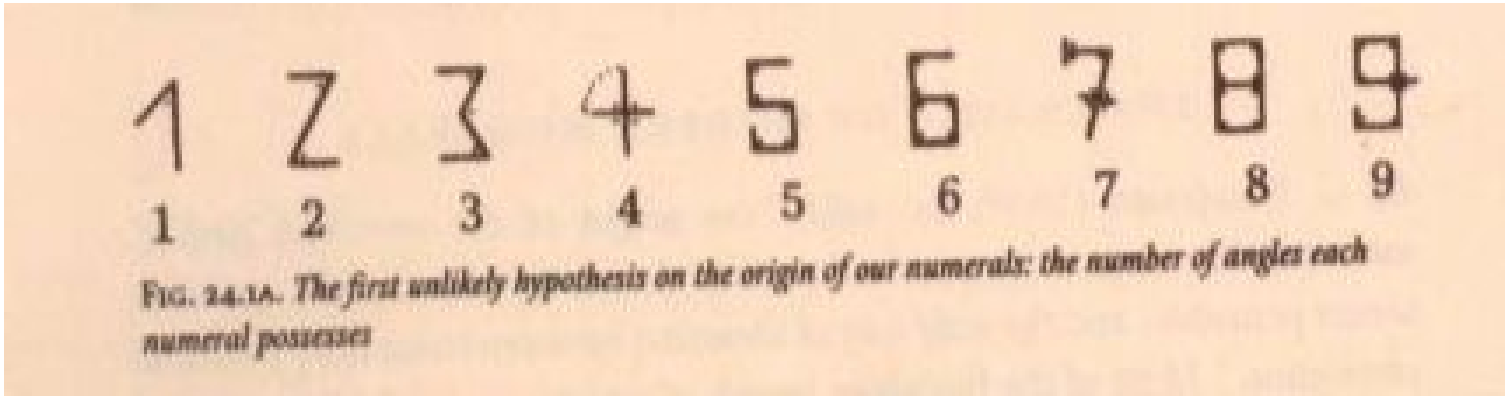


FIG. 24.1A. *The first unlikely hypothesis on the origin of our numerals: the number of angles each numeral possesses*

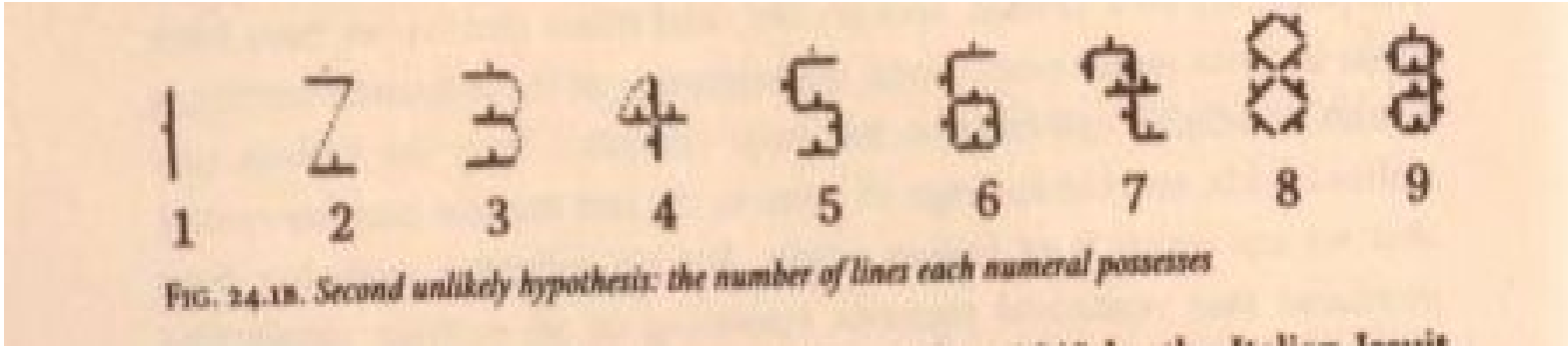


FIG. 24.1B. *Second unlikely hypothesis: the number of lines each numeral possesses*

‘Arabic’ Numerals & its spread in Western Asia

- Georges Ifrah, presenting several such hypotheses, finally dismisses all of them as merely fanciful explanations.
- He authentically quotes from dozens of works authored by Arabo-Muslim scholars as well as European scholars – spanning over almost a millenia (9–19th cent.) – to prove the point that discovery of decimal place value system was made by Indians.
- The Syrian Monophysite Bishop Serus Sebokht in 662 CE[4] notes:

> “the science of the Indians,” including “their subtle discoveries in astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians, and of their valuable methods of calculation which surpass description,” which he described as “done by means of nine signs” presumably ignoring the zero.

[4] Datta and Singh pp. 95-6, Kim Plofker p. 255.

Quotation from al-Khwarizmi's work

- One of the earliest historical records (c. 810 CE) available to us attributing the discovery is the work of al-Khwarizmi.
- In his work *Kitab al jam' wa'l tafriq bi hisab al hind* (Indian tradition of addition and subtraction) Abu Ja 'far Muhammad ibn Musa al Khwarizmi right at the beginning of the work observes:[5]

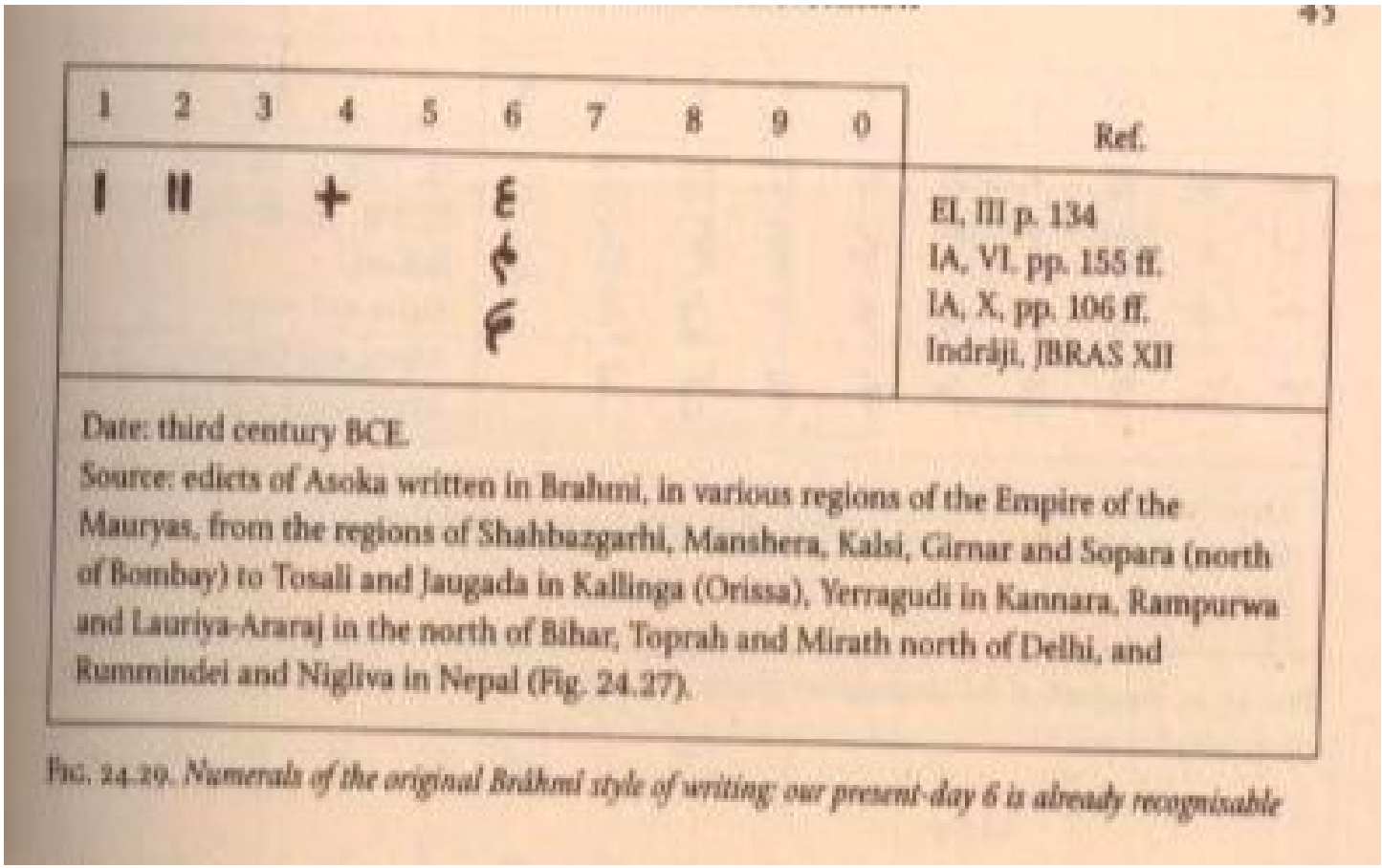
  > *. . . we have decided to explain Indian calculating techniques using the nine characters and to show how, because of their simplicity and conciseness, these characters are capable of expressing any number . . . the tenth figure in the shape of the circle (zero), . . .*

- al Khwarizmi in his work further advises that this be used 'so as not to confuse the positions'.

[5] Georges Ifrah, p. 18–19.

Numerals found in the Inscriptions (Brāhmī)

- Some of the rock edicts of Asoka (Mauryan dynasty) dating back to 3rd cent. BCE contain numerals in Brāhmī script.



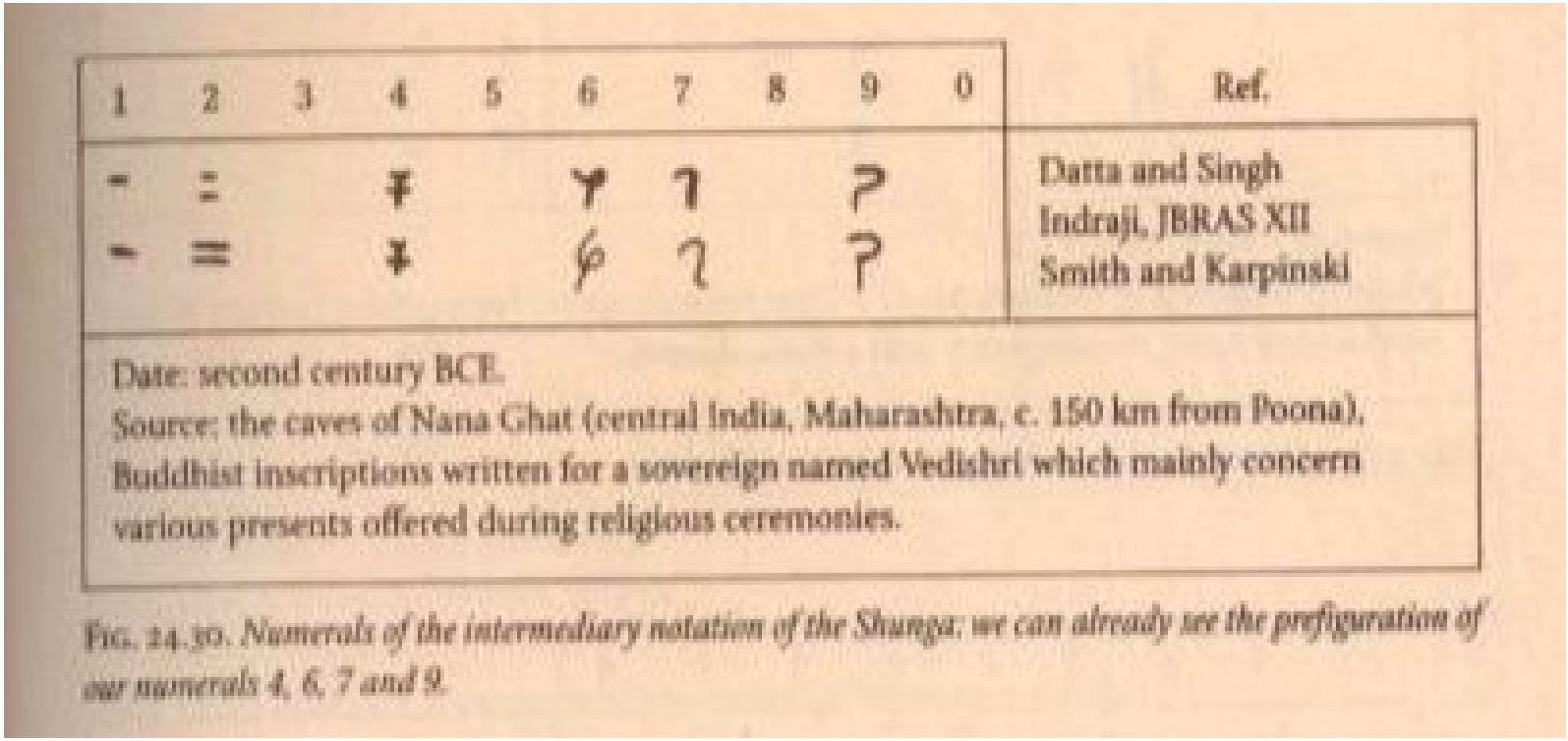
1	2	3	4	5	6	7	8	9	0	Ref.
𑁒	𑁓		+		𑁕					EI, III p. 134 IA, VI, pp. 155 ff. IA, X, pp. 106 ff. Indrāji, JBRAS XII

Date: third century BCE.
Source: edicts of Asoka written in Brahmi, in various regions of the Empire of the Mauryas, from the regions of Shahbazgarhi, Mansherа, Kalsi, Girnar and Sopara (north of Bombay) to Tosali and Jaugada in Kallinga (Orissa), Yerragudi in Kannara, Rampurwa and Lauriya-Araraj in the north of Bihar, Toprah and Mirath north of Delhi, and Rummindei and Nigliva in Nepal (Fig. 24.27).

FIG. 24.29. *Numerals of the original Brāhmī style of writing: our present-day 6 is already recognizable*

Numerals found in Buddhist Inscriptions (2nd cent. BCE)

- Buddhist inscriptions on the walls of the grottoes of Nana Ghat contain numerical symbols in Brāhmī script.

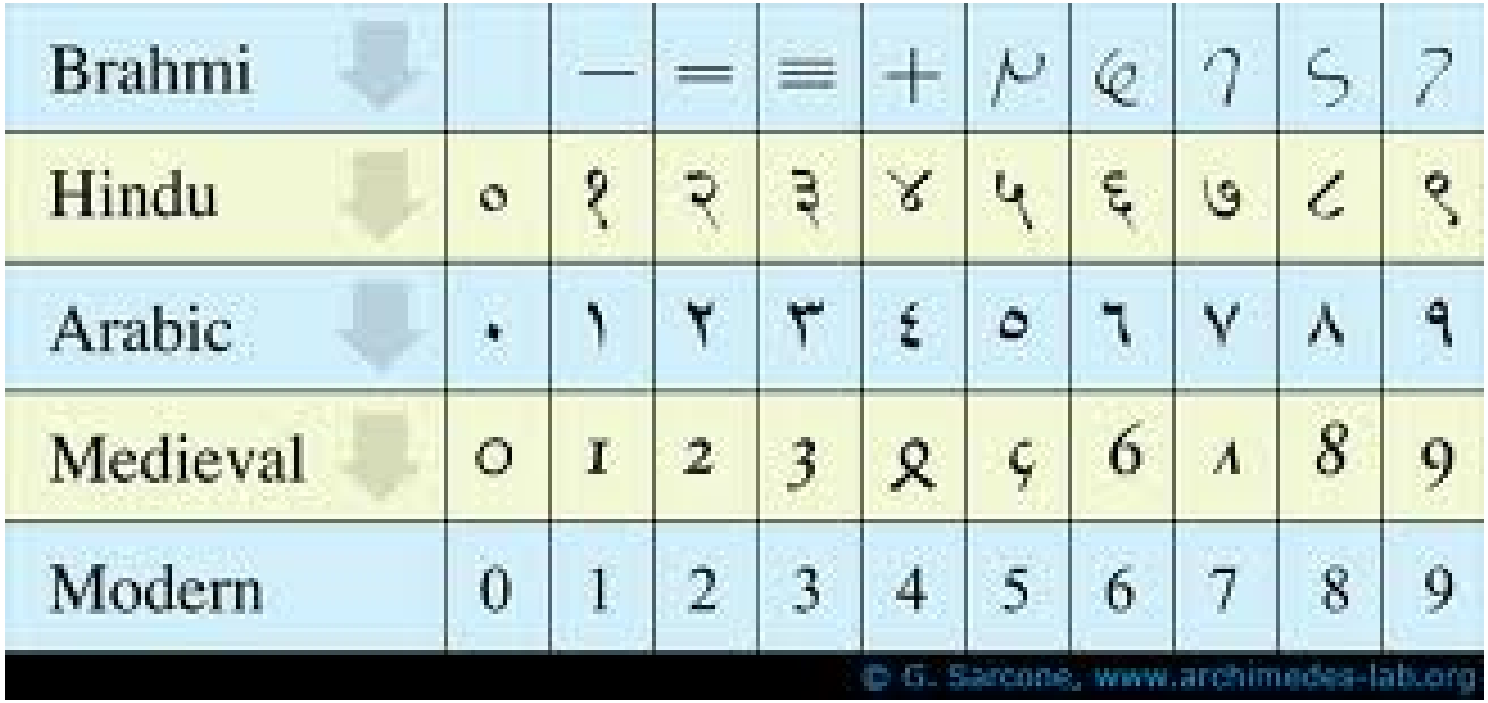


1	2	3	4	5	6	7	8	9	0	Ref.
										Datta and Singh Indraji, JBRAS XII Smith and Karpinski

Date: second century BCE.
Source: the caves of Nana Ghat (central India, Maharashtra, c. 150 km from Poona). Buddhist inscriptions written for a sovereign named Vedishri which mainly concern various presents offered during religious ceremonies.

Fig. 14.30. Numerals of the intermediary notation of the Shunga: we can already see the prefiguration of our numerals 4, 6, 7 and 9.

Evolution of Numerals: Brāhmī → Modern



- It has taken more than 18 centuries (3rd BCE – 15th CE) for the numerical notation to acquire the present form.
- The present form seems to have got adopted ‘permanently’ with the advent of printing press in Europe. However, there are as many as 15 different scripts used in India even today (Nāgāri, Bengali, Tamil (Grantha), Punjabi, Malayalam, etc.).

Use of zero as a symbol by Piṅgala (2nd cent CE)

- Pāṇini's *Aṣṭādhyāyī* (c. 500 BCE) has the notion of *lopa* which functions as a null-morpheme.

  अदर्शनं लोपः। *(1.1.60).*

- *Śūnya* appears as a symbol in Piṅgala's *Chandaḥ-sūtra* (c. 300 BCE). In Chapter VIII, while enunciating an algorithm for evaluating any positive integral power of 2 in terms of an optional number of squaring and multiplication (duplication) operations, *śūnya* is used as a marker.

  रूपे शून्यम्। द्विः शून्ये। *(8.29-30).*

- Different schools of Indian philosophy have related notions such as the notion of ***abhāva*** in Nyāya School, and the ***śūnyavāda*** of the Bauddhas.

Description of decimal place value system

Indian philosophical literature

- In *Vyāsa-bhāṣya* on the *Yogasūtra* of Patañjali, we find an interesting description of the place value system:

  यथैका रेखा शतस्थाने शतं दशस्थाने दश एका च एकस्थाने;
  Just as the same line in the hundreds place [means] a hundred, in the tens place ten, and one in the ones place;

- In the same vein, Śaṅkara in his BSSB (2.2.17) observes:

  यथा एकोऽपि सन् देवदत्तः लोके स्वरूपं सम्बन्धिरूपं च अपेक्ष्य अनेकशब्दप्रत्ययभाग्भवति – मनुष्यः, ब्राह्मणः, श्रोत्रियः, वदान्यः, बालः, युवा, स्थविरः, पिता, पुत्रः, पौत्रः, भ्राता, जामाता इति। यथा च एकापि सती रेखा (अङ्कः) स्थानान्यत्वेन निविशमाना एक-दश-शत-सहस्रादि शब्दप्रत्ययभेदम् अनुभवति, तथा सम्बन्धिनोरेव ...

Earliest explicit use of decimal place value system

Indian mathematical and astronomical texts

- The earliest comprehensive astronomical/mathematical work that is available to us today is $\bar{A}ryabhaṭīya$ (499 CE).
- The degree of sophistication with which $\bar{\text{A}}\text{ryabhaṭa}$ has presented the number of revolutions made by the planets etc., clearly points to the fact that they had perfect knowledge of zero and the place value system.
- Moreover, his algorithms for finding square-root, cube-root etc. are also based on this.
- The system developed by $\bar{\text{A}}\text{ryabhaṭa}$ is indeed unique in the whole history of written numeration.
- Not only unique but also quite ingenious and sophisticated. Numbers of the order of 10^{16} can be represented by a single character.
- However, it was not made use of by anybody other than $\bar{\text{A}}\text{ryabhaṭa}$ — as it is too complicated to read!

Vowels employed in *Devanāgarī* script

- The chart below presents a summary of the vowels and used in the *Devanāgarī* script:[6]

Primary vowels

	Short				Long				Diphthongs			
	Initial		Diacritic		Initial		Diacritic		Initial		Diacritic	
Unrounded low central	अ	a	प	pa	आ	ā	पा	pā				
Unrounded high front	इ	i	पि	pi	ई	ī	पी	pī				
Rounded high back	उ	u	पु	pu	ऊ	ū	पू	pū				
Syllabic variants	ऋ	ṛ	पृ	pṛ	ॠ	ṝ	पॄ	pṝ				
	ऌ	ḷ	पॢ	pḷ	ॡ	ḹ	पॣ	pḹ				

Secondary vowels

Unrounded front					ए	e	पे	pe	ऐ	ai	पै	pai
Rounded back					ओ	o	पो	po	औ	au	पौ	pau

[6] http://www.omniglot.com/writing/devanagari.htm

Consonants employed in *Devanāgarī* script

- The chart below presents a summary of the vowels and used in the *Devanāgarī* script:

Occlusives

	Voiceless plosives				Voiced plosives				Nasals	
	unaspirated		aspirated		unaspirated		aspirated			
Velar	क	ka	ख	kha	ग	ga	घ	gha	ङ	ṅa
Palatal	च	ca	छ	cha	ज	ja	झ	jha	ञ	ña
Retroflex	ट	ṭa	ठ	ṭha	ड	ḍa	ढ	ḍha	ण	ṇa
Dental	त	ta	थ	tha	द	da	ध	dha	न	na
Labial	प	pa	फ	pha	ब	ba	भ	bha	म	ma

Sonorants and fricatives

	Palatal		Retroflex		Dental		Labial	
Sonorants	य	ya	र	ra	ल	la	व	va
Sibilants	श	śa	ष	ṣa	स	sa		

Other letters

ह	ha	ळ	ḷa

Assigning numerical values to consonants

- Āryabhaṭa's scheme of assigning numerical values to the 33 consonants can be represented as:

gutturals	क *ka* = 1	ख *kha* = 2	ग *ga* = 3	घ *gha* = 4	ङ *ṅa* = 5
palatals	च *cha* = 6	छ *chha* = 7	ज *ja* = 8	झ *jha* = 9	ञ *ña* = 10
cerebrals	ट *ṭa* = 11	ठ *ṭha* = 12	ड *ḍa* = 13	ढ *ḍha* = 14	ण *ṇa* = 15
dentals	त *ta* = 16	थ *tha* = 17	द *da* = 18	ध *dha* = 19	न *na* = 20
labials	प *pa* = 21	फ *pha* = 22	ब *ba* = 23	भ *bha* = 24	म *ma* = 25
semivowels	य *ya* = 30	र *ra* = 40	ल *la* = 50	व *va* = 60	
sibilants	श *śha* = 70	ष *sha* = 80	स *sa* = 90		
aspirates	ह *ha* = 100				

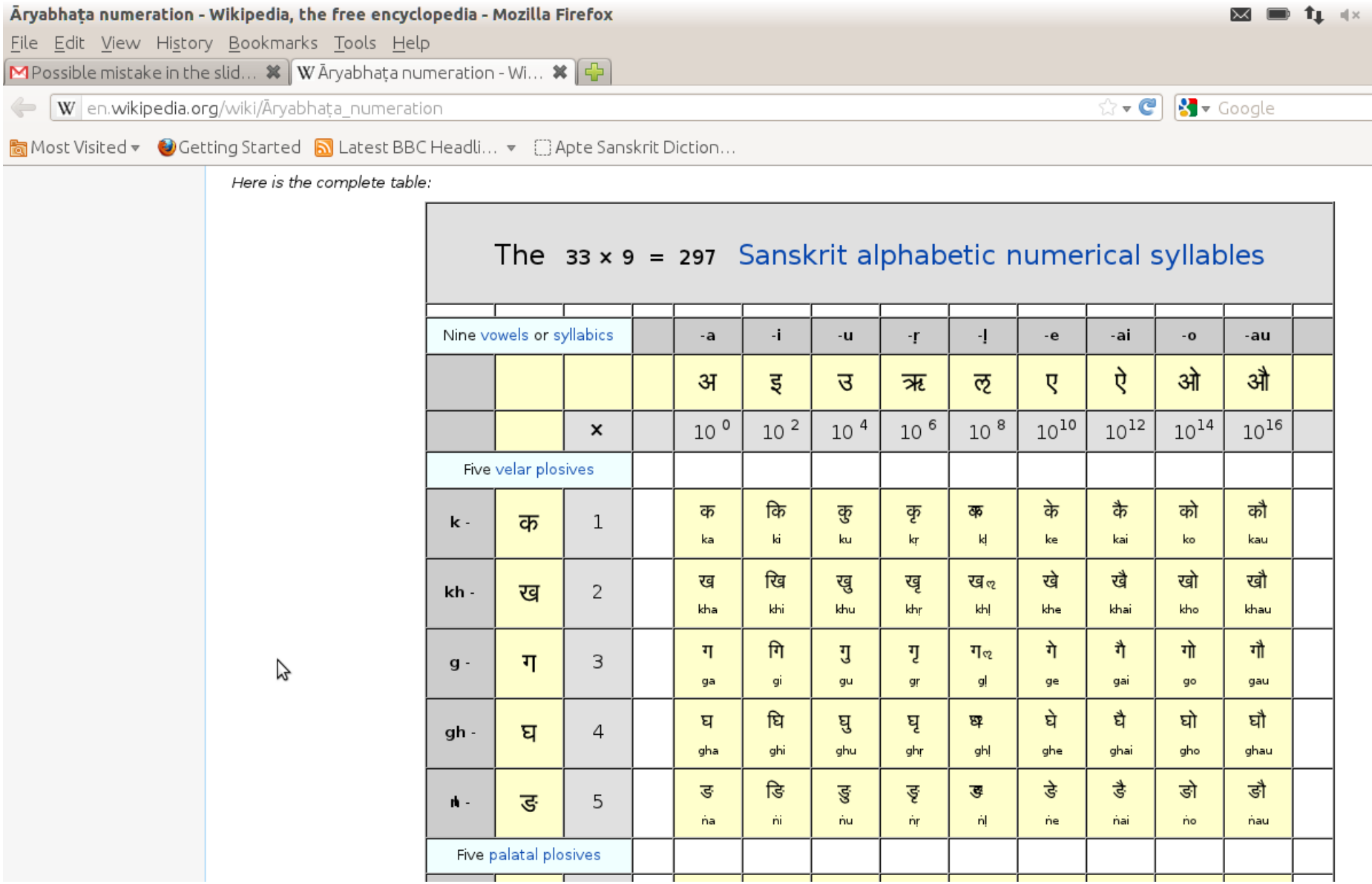
Vocalizing (making vowels to decide) the place value

- Āryabhaṭa's idea of making vowels decide the place value of the 33 consonants is quite novel and ingenious.
- The scheme proposed by him is as follows:
  - First he classifies the consonants into two groups (*varga* and *avarga*);
  - Assigns values to them, and then states that their place value is decided by the vowel that is tagged to them.
  - Here the value of an isolated consonant (generally pronounced/vocalized with *a*), is taken as such.
  - However if they are tagged with other vowels (i, u, ṛ, etc.) we need to multiply them by successive powers 100.

Vowels	au	ai	o	e	ḷ	ṛ	u	i	a
Multipled by	10^{16}	10^{14}	10^{12}	10^{10}	10^{8}	10^{6}	10^{4}	10^{2}	10^{0}

- In a way, because of *varga*, *avarga* classification Āryabhaṭa's system can be viewed as centesimal system.

Flaw in representing Āryabhaṭa 's system



Āryabhaṭa numeration - Wikipedia, the free encyclopedia - Mozilla Firefox

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Here is the complete table:

The 33 × 9 = 297 Sanskrit alphabetic numerical syllables

Nine vowels or syllabics				-a	-i	-u	-ṛ	-ḷ	-e	-ai	-o	-au	
				अ	इ	उ	ऋ	ऌ	ए	ऐ	ओ	औ	
		×		10^0	10^2	10^4	10^6	10^8	10^{10}	10^{12}	10^{14}	10^{16}	
Five velar plosives													
k -	क	1		क ka	कि ki	कु ku	कृ kṛ	क्ऌ kḷ	के ke	कै kai	को ko	कौ kau	
kh -	ख	2		ख kha	खि khi	खु khu	खृ khṛ	ख्ऌ khḷ	खे khe	खै khai	खो kho	खौ khau	
g -	ग	3		ग ga	गि gi	गु gu	गृ gṛ	ग्ऌ gḷ	गे ge	गै gai	गो go	गौ gau	
gh -	घ	4		घ gha	घि ghi	घु ghu	घृ ghṛ	घ्ऌ ghḷ	घे ghe	घै ghai	घो gho	घौ ghau	
ṅ -	ङ	5		ङ ṅa	ङि ṅi	ङु ṅu	ङृ ṅṛ	ङ्ऌ ṅḷ	ङे ṅe	ङै ṅai	ङो ṅo	ङौ ṅau	
Five palatal plosives													

Āryabhaṭa's representation: Illustrative examples

- Example 1: Decode the alpha-numeral ख्युघृ (*khyughṛ*).

Category	V	A	V	A	V	A	V	A	V
Vowel	ḷ	ṛ	ṛ	u	u	i	i	a	a
Given syllable	-	-	- gh	y	kh	-	-	-	-
Its value			4	3	2	0	0	0	0

The given number is: 43,20,000.

- *Note:* In this scheme NULL SYLLABLE ≡ ZERO VALUE.
- The verse in *Āryabhaṭīya* that lays down the procedure is:

  वर्गाक्षराणि वर्गे अवर्गे अवर्गाक्षराणि कात् ङ्यौ यः।
  खद्विनवके स्वराः नव वर्गे अवर्गे नवान्त्यवर्गे वा॥

- In the above verse, we have to make आवृत्ति of the word वर्गाक्षराणि.

  1. वर्गाक्षराणि वर्गे *[स्थाने स्थाप्यानि]*
  2. कात् वर्गाक्षराणि *[सङ्ख्यां भजन्ते]* – *the varga letters take values 1,2,3 . . . 25 starting from 'k'.*

- Also ङ्यौ यः defines the value of *ya*. That is, $ya = ṅ + ma$.

Āryabhaṭa's representation: Examples & Exercises

- *Ex. 2*: Find the no. given by चयगियिङ्शुछ्लृ (*cayagiyiṅuśuchṛlṛ*)

Category	V	A	V	A	V	A	V	A	V
Vowel	ऌ	ऋ	ॠ	उ	ऊ	इ	ई	अ	आ
Given syllable		l	ch	ś	ṅ	y	g	y	c
Its value		5	7	7	5	3	3	3	6

The given number is: 5,77,53,336.

- *Ex. 3*: Find the no. given by ढुङ्घ्व (*ḍhuṅvighva*).

Category	V	A	V	A	V	A	V
Vowel	ॠ	उ	ऊ	इ	ई	अ	आ
Given syllable			ḍh	v	ṅ	v	gh
Its value			14	6	5	6	4

The given number is: 1,46,564.

- *Exercises*: Find the numbers represented by

ङिंशिबुण्लृष्खृ (*ṅiśibuṇlṛṣkhṛ*) & र्जुष्खिध (*rjuṣkhidha*)

The sine-table of Āryabhaṭa

(Algorithm for constructing this table will be discussed later)

- Verse 12 of *Gītikā-pāda* of *Āryabhaṭīya*[7] gives the value of Rsine-differences (expressed in arc-minutes):

  मखि भखि फखि धखि णखि ञखि

  ङखि हस्झ स्ककि किष्ग स्यकि किघ्व।

  घ्लकि किग्र हक्य धकि किच

  स्ग श्झ ङ्व क्ल त फ छ कलार्धज्याः॥

  225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, and 7—these are the Rsine-differences [at intervals of 225′ of arc] in terms of the minutes of arc.

- Recalling that in Āryabhaṭa's notation: म → 25 & खि → 200, the word *makhi* represents jyā 225 = *Rsine* 225 = 225.

[7] This verse is perhaps one of the most terse verses in the entire Sanskrit literature. Only after several trials would it be ever possible to read the verse properly, let alone deciphering its content.

Kaṭapayādi system

- The name '*Kaṭapayādi*' stems from the fact that here the Sanskrit alphabets $ka,\ ṭa,\ pa,\ ya$ etc. are used to denote the numbers.
- According to this system,
  - the vowels standing alone, represent the number zero.
  - the same vowels in conjunction with the consonants have no numerical significance.
  - the 33 consonants $k,\ kh,\ g,\ gh,\ \ldots,\ ś,\ ṣ,\ s,\ h$ are associated with numbers.
- The mapping of the consonants with numbers is as follows:

Number	1	2	3	4	5	6	7	8	9	0
Consonants used to represent numbers	k	kh	g	gh	$ṅ$	c	ch	j	jh	$ñ$
	$ṭ$	$ṭh$	$ḍ$	$ḍh$	$ṇ$	t	th	d	dh	n
	p	ph	b	bh	m	–	–	–	–	–
	y	r	l	v	$ś$	$ṣ$	s	h	$ḷ$[8]	–

[8] This is a special character—denoted in *Devanāgarī* script as ळ—rarely employed by the Kerala astronomers to represent the number nine.

Kaṭapayādi system

- The following verse in *Sadratnamālā* of Śaṅkaravarman (c. 1830 CE) succinctly summarizes the system:

  नञावचश्च शून्यानि सङ्ख्याः कटपयादयः।
  मिश्रे तूपान्तहल्संख्या न च चिन्त्यो हलः स्वरः॥
  [The letters] *n*, *ñ* and the vowels [when standing alone] denote zeros. [The consonants] commencing from *ka*, *ṭa*, *pa* and *ya* denote the numbers [1, 2, 3, . . .] in order. In the case of conjunct consonants (*miśre tu*) only the last consonant represents the number. The vowel suffixed to a consonant should not be counted.

- In this system, it is the least significant decimal place that is given first, and the highest the last.
- For example, the word आयुरारोग्यसौख्यं, represents 1712210.

ā	*yu*	*rā*	*ro*	*gya*	*sau*	*khyam*
0	1	2	2	1	7	1

- Vararuci's *cāndravākyas* of unknown antiquity and Haridatta's ***Mahāmārganibandha*** (c. 600 CE) form the earliest examples.

Kaṭapayādi system

The table below presents a few illustrative examples chosen from the texts on Indian mathematics and astronomy.

Word/Words	Number represented
विद्वान्	44
तुन्नबलः	3306
कवीशनिचयः	160541
सर्वार्थशीलस्थिरः	2735747
निर्विद्धाङ्गनरेन्द्ररुक्	22203940
भद्राङ्गभव्यासनः	714324
ऊनधनकृद्भूरेव	42410900
धीगापाङ्गजळाङ्गस्त्री	23983139
नानाज्ञानतपोधरः	29160000
हे विष्णो निहितं कृत्स्नम्	1680548
लक्ष्मीशनिहितध्यानैः	1680553

Bhūtasaṅkhyā system

- The word *Bhūtasaṅkhyā* is a compound word which has two constituents, namely *bhūta* and *saṅkhyā*—referring to a 'being' and a 'number' respectively.
- Here words commonly employed in Sanskrit such as:
  1. The physical entities such as Earth, Moon (1), planets, stars, ocean, mountain, fire, sky, direction etc.
  2. The parts of a human body such as eyes, ears(2), jaws, knees, hands, fingers, teeth, nails etc.
  3. The animals, such as serpent, horse, elephant etc.
  4. The names of the gods, such as Śiva, Indra etc. and sometimes historical figures such as Manu, Rāma (3) etc.
  5. The season, fortnight, month (12), week, etc.

  whose meaning has the potential to evoke a certain number in the reader's mind, were used to denote specific numbers.

Bhūtasaṅkhyā system

Illustrative example: Approximation to π given by Mādhava

- The commentary *Kriyākramakarī* while presenting several values of π given by different *Ācāryas*, also lists the one due to Māadhava. We give this as an illustrative example of the use of *Bhūtasaṅkhyā* system:

  माधवाचार्यः पुनः अतोप्यासन्नतमां परिधिसङ्ख्यामुक्तवान् –

  विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः ।
  नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः ॥[9]

- The values of π given by the above verse amounts to:

$$\pi = \frac{2827433388233}{9 \times 10^{11}} = 3.141592653592$$ (correct to 11 places)

- How did Mādhava arrive at this value? This will be covered in the later part of the Workshop!

[9] *Vibudha=33, Netra=2, Gaja=8, Ahi=8, Hutāśana=3, Triguṇa=3, Veda=4, Bha=27, Vāraṇa=8, Bāhu=2, Nava-nikharva*=9×10^{11}. (The word *nikharva* represents 10^{11}).

Bhūtasaṅkhyā system

- The table below presents a few examples.

Word	Number represented
खाद्रिरामाग्नयः	3370
वेदवेदाङ्कचन्द्राः	1944
वेदचन्द्रद्विवेदाब्धिनागाः	844214
भुजङ्गनन्दद्विनगाङ्गबाणषट्कृतेन्दवः	146567298

- Advantages
  - As the language is extremely rich in synonyms, an author could choose any synonym that would suit the metre.
  - From the reader's view point, since the words are familiar, it enormously enhances the readability.
- However, lack of familiarity with the connotation of a specific $bh\bar{u}ta$ and/or the lack of knowledge of synonyms could pose a problem, not to mention improper splitting of the words.
- This of course has to do with the ignorance on the part of a reader, and is no reason to blame the system!

Thanks!

THANK YOU