NPTEL Course on

Mathematics in India:
From Vedic period to Modern times

Lecture 16

Mahāvīra's *Gaṇitasārasaṅgraha* 2

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Outline

Gaṇitasārasaṅgraha of Mahāvīra 2

- *Vallīkāra-kuṭṭākāra* - linear indeterminate equatioons
- Two and more simultaneous indeterminate equations
- Other indeterminate equations
- *Vicitra-kuṭṭākāra* - Truthful and untruthful statements
- Sums of progressions of various types
- Variable velocity problem

Vallīkāra-kuṭṭākāra

(In the text, the term *kuṭṭīkāra* having the same meaning as *kuṭṭākāra* is used in most places).
It is the same as Āryabhaṭa's procedure. First we discuss the solutions of a single indeterminate equation

$$Bx + b = Ay,$$

where A, B and b are given integers. To find integral solutions x, y for this linear indeterminate equation (one equation for two unknowns).

Kuṭṭaka

छित्वा छेदेन राशिं प्रथमफलमपोह्याप्तमन्योन्यभक्तं
स्थाप्य ऊर्ध्वाधः यतोऽधो मतिगुणमयुजाल्पे
अवशिष्टे धनर्णम्।

छित्वाधः स्वोपरिघ्नोपरियुतहरभागाधिकाग्रस्य हारं
छित्वा छेदेन साग्रान्तरफलं अधिकाग्रान्वितं हारघातम्॥

Kuṭṭaka

"Divide the (given) group-number by the (given) divisor; discard the first quotient; then put down one below the other (various) quotients obtained by the successive division (of the various resulting divisions by the various resulting remainders; again), put down below this the optionally chosen number, with which the least remainder in the odd position of order (in the above mentioned process of successive division) is to be multiplied; and (then put down) below (this again) this product increased or decreased (as the case may be by the given known number) and then divided (by the last divisor in the above mentioned process of successive division. Thus the *Vallika* or creeper-like chain of figures is obtained. In this) the sum is obtained by adding (the lowermost number in the chain) to the product obtained by the multiplying the number above it with the number (immediately) above (this upper number; this process of addition, being in the same way continued till the whole chain is exhausted,) this sum, is to be divided by the (originally) given divisor. (The remainder in this last division becomes the multiplier(x) with which the originally given group-number is to be multiplied for the purpose of arriving at the quantity which is to be divided or distributed in the manner indicated in the problem)."

Mutual Division

To solve $Bx + b = Ay$

$$
\begin{array}{ccccccccccccc}
A &) & B & (& q_1 & & & & & & & & \\
 & & \underline{\cdots} & & & & & & & & & & \\
 & & r_1 &) & \mathrm{A} & (& q_2 & & & & & & \\
 & & & & \underline{\cdots} & & & & & & & & \\
 & & & & r_2 &) & r_1 & (& q_3 & & & & \\
 & & & & & & \underline{\cdots} & & & & & & \\
 & & & & & & r_3 &) & r_2 & (& q_4 & & \\
 & & & & & & & & \underline{\cdots} & & & & \\
 & & & & & & & & r_4 &) & r_3 & (& q_5 \\
 & & & & & & & & & & \underline{\cdots} & & \\
 & & & & & & & & & & r_5 & &
\end{array}
$$

Kuṭṭaka continued

Here it is stopped at the n^{th} remainder r_5 which is least (normally taken to be 1). (In the above $n = 5$). Discard the first quotient q_1. Now we should choose a number p, such that $pr_5 \pm b$ is divisible by r_4. Here, '$+$' is chosen , when n is odd, while '$-$' is chosen when n is even. The number p is written below the column starting with q_2, q_3 below that, ... and ending with q_n. Below this the quotient $\dfrac{pr_5 + b}{r_4} = Q$. Then the penultimate number is multiplied by the number above it to which the last number is added. Now discard the last entry Q in the column. Continue this process till we arrive at the last two top entries. The topmost T_1 is divided by A. The remainder is the least integral value of x satisfying the equation $y = \dfrac{Bx + b}{A}$.

$$
\begin{array}{llll}
q_2 & \cdot & \cdot & T_1 \\
q_3 & \cdot & \cdot & T_2 \\
q_4 & \cdot & \cdot & \\
\vdots & \vdots & \vdots & \\
q_{n-1} & \cdot & q_{n-1}(q_n p + Q) + p & \\
q_n & q_n p + Q & q_n p + Q & \\
p & p & & \\
Q & & &
\end{array}
$$

Example

Example: Solve $63x + 7 = 23y$.

$$
\begin{array}{ccccccccccccc}
23 &) & 63 & (& 2 & & & & & & & & \\
 & & \underline{46} & & & & & & & & & & \\
 & & 17 &) & 23 & (& 1 & & & & & & \\
 & & & & \underline{17} & & & & & & & & \\
 & & & & 6 &) & 17 & (& 2 & & & & \\
 & & & & & & \underline{12} & & & & & & \\
 & & & & & & 5 &) & 6 & (& 1 & & \\
 & & & & & & & & \underline{5} & & & & \\
 & & & & & & & & 1 &) & 5 & (& 4 \\
 & & & & & & & & & & \underline{4} & & \\
 & & & & & & & & & & 1 & &
\end{array}
$$

Example

We have to chose a number p such that when multiplied by the last remainder 1 and added to 7, it is divisible by the last divisor 1. We can take $p = 1$.

$Vall\bar{\imath}$ and further procedure:

1	1	1	1	51
2	2	2	38	38
1	1	13	13	
4	12	12		
1	1			
8				

Example

$51 = 23 \times 2 + 5$. So remainder is 5.

$$\therefore\ x = 5; y = \frac{63 \times 5 + 7}{23} = \frac{322}{23} = 14.$$

These are the lowest possible values for x, y. The general solutions are: $x = 5 + 23t$, $y = 14 + 63t$, where t is an arbitrary integer.

Rationale

Rationale:

$$\frac{Bx+b}{A} = y(\text{an integer}) = q_1 x + p_1,$$

where

$$p_1 = \frac{(B - Aq_1)x + b}{A}.$$

Now $B - Aq_1 = r_1$, the first remainder.

$$\therefore\ x = \frac{Ap_1 - b}{r_1}.$$

Rationale

Now

$$A = q_2 r_1 + r_2.$$

$$\therefore\ x = q_2 p_1 + p_2, \text{ where } p_2 = \frac{r_2 p_1 - b}{r_1}.$$

$$\therefore p_1 = \frac{r_1 p_2 + b}{r_2}.$$

Using

$$r_1 = r_2 q_3 + r_3,$$

$$p_1 = q_3 p_2 + p_3,$$

where

$$p_3 = \frac{r_3 p_2 + b}{r_2},$$

or

$$p_2 = \frac{r_2 p_3 - b}{r_3}.$$

Rationale

Using

$$r_2 = r_3 q_4 + r_4,$$

$$p_2 = q_4 p_3 + p_4,$$

where

$$p_4 = \frac{r_4 p_3 - b}{r_3},$$

or

$$p_3 = \frac{r_3 p_4 + b}{r_4}.$$

Using

$$r_3 = r_4 q_5 + r_5,$$

$$p_3 = q_5 p_4 + p_5 = q_5 p + p_5.$$

Rationale

If we stop at the fifth remainder r_5, we call $p_4 = p$.

$$p_5 = \frac{r_5 p_4 + b}{r_4} = \frac{r_5 p + b}{r_4}.$$

Here we are choosing $p = p_4$ such that $r_5 p + b$ is divisible by r_4.

Thus,

$$\begin{aligned}
x &= q_2 p_1 + p_2, \\
p_1 &= q_3 p_2 + p_3, \\
p_2 &= q_4 p_3 + p_4, \\
p_3 &= q_5 p_4 + p_5,
\end{aligned}$$

where we chose $p = p_4$ such that $\frac{r_5 p + b}{r_4} = p_5 = Q$ is an integer.

$\boldsymbol{Vall\bar{\imath}}$ for the Problem

We construct the $Vall\bar{\imath}$ as:

$$\begin{array}{lllll}
q_2 & q_2 & q_2 & q_2 & x = q_2 p_1 + p_2 \\
q_3 & q_3 & q_3 & p_1 = q_3 p_2 + p_3 & p_1 \\
q_4 & q_4 & p_2 = q_4 p_3 + p & p_2 & \\
q_5 & p_3 = p q_5 + p_5 & p_3 & p_3 & \\
p & p & & & \\
p_5 & & & & \\
(Q) & & & &
\end{array}$$

Solutions

So we have obtained $x = q_2 p_1 + p_2$ through this $Vall\bar{\imath}$. It has been constructed such that $\dfrac{Bx + b}{A} = y = q_1 x + p_1$ is an integer. Let the remainder of x when divided by A be x_0, that is, $x = x_0 + tA$ where t is integer. Then it is clear that $Bx_0 + b$ is also divisible by A and x_0 is the lowest solution. That is what is being said. Lowest value of y, y_0 is of course given by $\dfrac{Bx_0 + b}{A}$. Clearly if x_0, y_0 are the lowest solutions, $x_0 + tA, y_0 + tB$ are solutions for arbitrary t.

Simultaneous Indeterminate Equations

The second part of Mahāvīra's procedure for *kuṭṭaka* is for the problem: Find x such that $\frac{B_1x+b_1}{A_1}$, $\frac{B_2x+b_2}{A_2}$, $\frac{B_3x+b_3}{A_3}$ are integers. (Obviously we can go on.)

Solution:

First solve $B_1x + b_1 = A_1y_1$. Let the lowest value of x be s_1. Let the lowest value of $\frac{B_2x+b_2}{A_2}$ be the integer be s_2. When both are to be satisfied, $dA_1 + s_1 = kA_2 + s_2$ where d, k are some integers.

$$\therefore\ s_1 - s_2 = kA_2 - dA_1, \text{ that is } \frac{A_1d + (s_1 - s_2)}{A_2} = k$$

.

Simultaneous Indeterminate Equations

This is an indeterminate equation where the values of d and k are unknown. We find the lowest positive integral value of d.

Then $dA_1 + s_1$ is the lowest value of x such that $\dfrac{B_1x + b_1}{A_1} = y_1$, $\dfrac{B_2x + b_2}{A_2} = y_2$ (y_1, y_2 integers) are both satisfied.

Let the least value of d be t_1. Let the next value of x which will satisfy both the equations be t_2. Now $t_1 + nA_1 = t_2$, $t_1 + mA_2 = t_2$ where m, n are integers.

$$\therefore \frac{A_1}{A_2} = \frac{m}{n}.$$

Two Indeterminate Equations

Thus $A_1 = mp$, $A_2 = np$ where p is the highest common factor between A_1 and A_2.

$$\therefore\ m = \frac{A_1}{p}, n = \frac{A_2}{p}.$$

$$\therefore\ t_1 + \frac{A_1 A_2}{p} = t_2.$$

So the next higher value of x satisfying the two equations is obtained by adding the least common multiple of A_1 and A_2 to the lower value.

Three Indeterminate Equations

Suppose we want to find x, such that it satisfies all the three equations. Let this be v (Lowest value of x satisfying $\frac{B_3x+b_3}{A_3}$ = integer is $x = s_3$). Then

$$v = t_1 + \frac{A_1A_2}{p} \times r = t_1 + lr, \quad \left(l = \frac{A_1A_2}{p}\right)$$

$$\text{and } v = s_3 + cA_3 = t_1 + lr.$$

where r is an integer.

$\therefore\ r = \frac{cA_3+s_3-t_1}{l}$. This is solved for c. Then $v = s_3 + cA_3$ is the least value of x satisfying all the equations.

The solution of two linear indeterminate equations, $B_1x + b_1 = A_1y_1$, $B_2x + b_2 = A_2y_2$ is spelt forth in detail in GSS, that is solving the two equations, finding the lowest values of x namely s_1 and s_2, and solving $\dfrac{A_1d + (s_1 - s_2)}{A_2} = k$, an integer for d.

Then $x = A_1d + s_1 = kA_2 + s_2$ satisfies both the equations.

Example

Example involving one equation

Verse $120\frac{1}{2}$:

दृष्टास्सप्तत्रिंशत् कपित्थफलराशयो वने पथिकैः।
सप्तदशापोह्य हृते व्येकाशीत्यांशकप्रमाणं किम्॥

“In the forest 37 heaps of wood apples were seen by the travelers. After 17 fruits were removed (therefrom the remainder) was equally divided among 79 persons (so as to leave no remainders). What is the share obtained by each?” [Try this as an exercise].

Here the equation to be solved is :

$$\frac{37x - 17}{79} = y$$

Example

Example involving two equations

Verse $121\frac{1}{2}$:

दृष्ट्वाम्रराशिमपहाय च सप्त पश्चात् भक्तेऽष्टभिः पुनरपि प्रविहाय तस्मात्।
त्रीणि त्रयोदशभिरुद्दलिते विशुद्धः पान्थैर्वने गणक मे कथयैकराशिम्॥

“When, after seeing a group of mangoes in the forest and removing 7 fruits (therefrom), it was divided equally among 8 of the travelers; and when again after removing 3 (fruits) that (same) heap it was divided (equally) among 13 of them; it left no remainder in both the cases. O mathematician, tell me the numerical measure of this single group”. [Try this as an exercise].

Here the equations to be solved are :

$$\frac{x-7}{8} = y_1 \text{ and } \frac{x-3}{13} = y_2.$$

Example Involving many Equations

Example involving many equations

Verse $127\frac{1}{2}$:

दृष्टा जम्बूफलानां पथि पथिकजनै राशयस्तत्र राशी
द्वौ त्र्यग्रौ तौ नवानां त्रय इति पुनरेकादशानां विभक्ताः।
पञ्चाग्रास्ते यतीनां चतुरधिकतराः पञ्च ते सप्तकानां
कुट्टाकारार्थविन्मे कथय गणक सञ्चिन्त्य राशिप्रमाणम्॥

“The travelers saw on the way certain (equal) heaps of jambu fruits. Of them, 2 (heaps) were equally divided among 9 ascetics and left 3 (fruits) as remainder. Again 3 (heaps) were (similarly) divided among 11 persons, and the remainder was 5 fruits; then again 5 of those heaps were similarly divided among 7 and there were 4 more fruits (left out) of them.

Example Involving many Equations

> O you mathematician who know the meaning of the $kuṭṭākāra$ process of distribution, tell me after thinking out well, the numerical measure of a heap (here)." [Try this as an exercise].

Here the equations to be solved are :

$$\frac{2x+3}{9} = y_1, \quad \frac{3x+5}{11} = y_2 \text{ and } \frac{5x+4}{7} = y_3.$$

Other Indeterminate Equations: Problem of Gems

Apart from the above type of indeterminate equations, there are several other interesting ones that Mahāvīra discusses for which interesting solutions are found.

Example: Let $m_1, m_2, \ldots, m_n$ be the numbers of n kinds of gems owned by n different persons. Let $x_1, x_2, \ldots$ be the value of a single gem in each variety. Let each of them give g gems each to others. What is the value of each gem, if the wealth of all the persons become equal after the exchange.

After the exchange i^{th} person will have $m_i - (n_i - 1)g$ gems of the i^{th} kind and g gems each of other kinds. The net worth of the each person is the same.

Problem of Gems

$$\therefore x_1[m_1-(n_1-1)g]+x_2g+x_3g+\cdots+x_ng = x_1g+x_2[m_2-(n-1)g]+x_3g+\cdots = \cdots$$

$$\therefore\ x_1[m_1-ng]+[x_1+x_2+\cdots]g = x_2[m_2-ng]+[x_1+x_2+\cdots]g = \cdots$$

$$\therefore\ x_1[m_1-ng] = x_2[m_2-ng] = \cdots$$

General integer solution would be $x_1 = \dfrac{M}{m_i - ng}$ for a suitable M.

In fact Mahāvīra choses $M = (m_1 - ng)(m_2 - ng)\cdots(m_n - ng)$.
So that
$x_i = (m_1 - ng)\cdots(m_n - ng) \rightarrow$ product excluding $(m_i - ng)$.

Example

Example in Verses 165 & 166:

प्रथमस्य शक्रनीलाः षोडश दश मरकता द्वितीयस्य।
वज्रास्तृतीयपुरुषस्याष्टौ द्वौ तत्र दत्त्वैव॥ १६५ ॥
तेष्वेकैकोऽन्याभ्यां समघनतां यान्ति ते त्रयः पुरुषाः।
तच्छक्रनीलमरकतवज्राणां किंविधा अर्घाः॥ १६६ ॥

“The first man has 16 azure-blue gems, the second has 10 emeralds, and third has 8 diamonds. Each among them gives to each of the others 2 gems of the kind owned by himself; and then all three men come to be possessed of equal wealth. Of what nature are the prices of those azure-blue gems, emeralds and diamonds?”

Solutions

Here 1: Azure-blue, 2: Emerald, 3, Diamond. n=3.

$$m_1 = 16, m_2 = 10, m_3 = 8. g = 2, ng = 6.$$

$$m_1 - ng = 10, \ m_2 - ng = 4, \ m_3 - ng = 2.$$

$$\therefore x_1 = 4 \times 2 = 8, x_2 = 10 \times 2 = 20, x_3 = 10 \times 4 = 40.$$

.

Suvarṇa Kuṭṭīkāra: "Alligation"

Suvarṇa Kuṭṭīkāra: Gold of various purities. To find the purity of a mixture and so on.

If weights W_i of *varṇa* V_i are mixed and if there is no loss in weight (that is total weight of the mixture $= \sum w_i$), the *varṇa* of the mixture $= V$, where

$$(\sum w_i)V = \sum w_i V_i$$

.

$$\therefore\ V = \frac{\sum w_i V_i}{\sum w_i}.$$

Vicitra-Kuṭṭīkāra: Truthful and Untruthful Statements

There are n men. A lady likes m of them. To each of them she makes a statement: "I like you only". (These are considered as n statements to each man: one explicit and others are implicit). So, total number of statements $= n^2$. How many statements are truthful and how many are untruthful?

The answer is given in the following verse:

Truthful and Untruthful Statements

Verse 216:

पुरुषाः सैकेष्टगुणा द्विगुणेष्टोना भवन्ति असत्यानि।
पुरुषकृतिस्तैरूना सत्यानि भवन्ति वचनानि॥ २१६ ॥

"The number of men, multiplied by the number of those liked (among them) as increased by 1, and (then) diminished by twice the number of men liked, give rise to the number of untruthful statements. This, subtracted from the square of the (total) number of men, becomes the (number of) statements that are truthful."

Truthful and Untruthful Statements

Total number of statements $= n^2$. Of them m are liked. When each of the m number of persons is told, "You alone are liked", the number of untruthful statements in each case is $m - 1$. Therefore total number of untruthful statements to m persons $= m(m-1)$.

When again, the same statement is made to each of the $n - m$ persons, the untruthful statements is $m + 1$. Therefore, total number of untruthful statements to $n - m$ persons $= (n-m)(m+1)$.

$\therefore$ Total number of untruthful statements

$$= (n-m)(m+1) + m(m-1) = n(m+1) - 2m.$$

But, Total number of statements $= n^2$.

$\therefore$ Total number of truthful statements $= n^2 - [n(m+1) - 2n]$

.

Example

Verse 217:

> "There are 5 men. Among them three are in fact liked by an woman. She says (separately) to each (of them "I like you (above)." How many (of her statements, explicit as well as implicit) are true ones ? "

$$n = 5, m = 3$$

Total no. of untruthful statements $= 5 \times 4 - 2 \times 3 = 14$

Total no. of statements $= 25$

$\therefore$ Total number of truthful statements $= 11$.
Combination of r out of n objects

Combinations

Verse 218.

एकाद्येकोत्तरतः पदमूर्ध्वाधर्यतः क्रमोत्क्रमशः।
स्थाप्य प्रतिलोमघ्नं प्रतिलोमघ्नेन भाजितं सारम्॥ २१८ ॥

"Beginning with one and increasing by one, let the numbers going upto the given number of things be written down in regular order and in the inverse order (respectively) in an upper and lower (horizontal) row. (If) the product (of one, two, three, or more of the numbers in the upper row) taken from right to left (be) divided by the (corresponding) product (of one, two, three, or more of the numbers in the lower row) also taken from right to left, (the quantity required in each such case of combination) is (obtained as) the result."

It says that combination of r out of n objects

$$= \frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdots r} = \frac{n!}{(n-r)!r!}$$

An Interesting Solution of a set of Indeterminate Equations

An interesting solution of a set of indeterminate equations.

Problem: “Suppose there are n merchants with each having some money already. They find a purse. The i^{th} person says if I procure a fraction a_i of the amount in the purse, the net amount with me would be m times the sum of the amount that other merchants have. What is the amount that each of the merchant has and what is the amount in the purse ? ”

Solution of the Purse Problem

Solution: Let x_i be the amount that the merchant i has. Let the amount in the purse be P.

$$\therefore\ Pa_1 + x_1 = m(x_2 + \cdots x_n),$$

$$Pa_2 + x_2 = m(x_1 + \cdots x_n),$$

$$\cdots$$

$$Pa_n + x_n = m(x_1 + \cdots x_n).$$

These are n equations for the $(n+1)$ unknown quantities $x_1, x_2, \cdots x_n, P$. So the solution is indeterminate. Note that if $(x_1, \cdots, x_n)$ is a solution, $(\alpha x_1, \cdots, \alpha x_n, \alpha P)$ is also a solution.

Solution of the problem

So only the ratios of the quantities can be found, or in other words, we can choose an overall constant arbitrarily. Let a_i be reduced to a common denominator, that is,

$$a_i = \frac{b_i}{L}, \text{ where } L \text{ is the L.C.M of the denominator.}$$

Multiplying the equations by L,

$$Pb_1 + x_1 L = mL(x_2 + \cdots, x_n) \text{ etc.,}$$

$$\cdots$$

$$Pb_n + x_n L = mL(x_1 + \cdots, x_{n-1})$$

$$\therefore P(b_1 + \cdots + b_n) + L(x_1 + \cdots + x_n) = (n-1)mL(x_1 + \cdots + x_n)$$

$$\therefore P(b_1 + \cdots + b_n) = [(n-1)m - 1]L[x_1 + \cdots + x_n]$$

Solution of the problem

Choose $(x_1 + \cdots + x_n) = (m+1)(b_1 + \cdots + b_n)$.

$$\therefore P = L[(n-1)m-1](m+1)$$

$$\text{Now } Pb_1 + x_1 L = mL(x_2 + \cdots + x_n)$$

$$\therefore Pb_1 + (m+1)Lx_1 = mL(x_1 + \cdots x_n)$$

$$\begin{aligned}
\therefore (m+1)Lx_1 &= mL(x_1 + \cdots + x_n) - Pb_1 \\
&= mL(x_1 + \cdots + x_n) - [(n-1)m-1](m+1)b_1 L \\
&= m(m+1)L(b_1 + \cdots + b_n) - [(n-1)m-1](m+1)Lb_1
\end{aligned}$$

Solution of the Problem

$$\therefore\; x_1 = m(b_1 + \cdots + b_n) - b_1\{(n-1)m - 1\}$$

In general

$$x_i = m(b_1 + \cdots + b_n) - b_i\{(n-1)m - 1\}$$

$$\text{and} \quad P = L[(n-1)m - 1](m+1)$$

$$\text{Here} \quad L = \frac{b_i}{a_i} = \text{ L.C.M of the Denominators in } a_i.$$

This is precisely the solution that is described in Verse 238.

Mahāvīra's Statement of the Purse Problem

Verse 238:

व्येकपदघ्नव्येकगुणेष्टांशयुतिगुणघातः।
हस्तगताः स्युर्भवति हि पूर्ववदिष्टांशभाजितं पोट्टलकम्॥

”The sum of (all the specified) fractions (in the problem) - the denominator being ignored - is multiplied by the (specified common) mutiple number. From this product, the products obtained by multiplying (each of the above mentioned) fractional part (as reduced to a commmon denominator, which is the ignored), by the product of the number of cases of persons minus one and the specified multiple number, this last product being diminished bhy one, are (severally) subtracted. The resulting remainders constitute the several values of the moneys on hand. The value of the money in the purse is obtained by carrying out operations as before, and then dividing by any particular specified fractional part (mentioned in the problem).

Example

Example in Verse 239-240:

वैश्यैः पञ्चभिरेकं पोट्टलकं दृष्टमाह चैकैकः।
पोट्टलकषष्ठसप्तमनवमाष्टमदशमभागमाप्त्वैव॥ २३९ ॥
स्वस्वकरस्थेन सह त्रिगुणं त्रिगुणं च शेषाणाम्।
गणक त्वं मे शीघ्रं वद हस्तगतं च पोट्टलकम्॥ २४० ॥

"Five merchants saw a purse of money. They said one after another that by obtaining $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{8}$, and $\frac{1}{10}$ (respectively) of the contents of the purse, they would each become with what he had on hand three times as wealthy as all the remaining others with what they had on hand together. O mathematician, (you) tell me quickly what moneys they had on hand (respectively), and what the value of the money in the purse was."

Solutions

Here

$$a_1 = \frac{1}{6}, a_2 = \frac{1}{7}, a_3 = \frac{1}{9}, a_3 = \frac{1}{9}, a_4 = \frac{1}{8}, a_5 = \frac{1}{10}. \text{ L.C.M } = 2520 = L$$

$$b_1 = 420, b_2 = 360, b_3 = 280, b_4 = 315, b_5 = 252$$

$$m = 3. \qquad (n-1)m - 1 = 4 \times 3 - 1 = 11$$

$$b_1 + b_2 + b_3 + b_4 + b_5 = 1627.$$

Hence,

$$\begin{aligned}
x_1 &= 3 \times 1627 - 11b_1 = 4881 - 4620 = 261, \\
x_2 &= 3 \times 1627 - 11b_2 = 4881 - 3960 = 921, \\
x_3 &= 3 \times 1627 - 11b_3 = 4881 - 3080 = 1801, \\
x_4 &= 3 \times 1627 - 11b_4 = 4881 - 3465 = 1416, \\
x_5 &= 3 \times 1627 - 11b_5 = 4881 - 2772 = 2109. \\
\text{Purse } P &= L[(n-1)m - 1](m+1) = 2520 \times 44.
\end{aligned}$$

Arrangement of Arrows

A problem on the arrangement of arrows in quivers in the following manner:

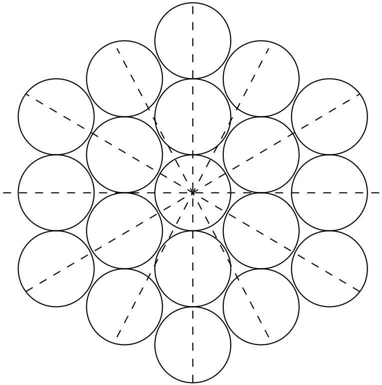

Figure: 28

Arrows

No. of arrows in the central circle $= 1$,

No. of arrows in the first neighborhood $= 6$,

No. of arrows in the second neighborhood $= 12$,

$\vdots$

No. of arrows in the p^{th} neighborhood $= 6p$.

$$\text{Total } = 1+6+2\times 6+\cdots+p\times 6 = 1+6[1+2+\cdots+p] = 1+\frac{6p(p+1)}{2}$$

Arrows Problem: Mahāvīra's Statement

If the number of arrows in the outermost periphery $= 6p = n$,

$$\text{Total number of arrows} \quad = 1 + \frac{3n}{6}\left(\frac{n}{6} + 1\right) = \frac{(n+3)^2 + 3}{12}.$$

This is what is stated in verse 288:

शरपरिधित्रिकमिलनं वर्गितमेतत्पुरस्त्रिभिस्सहितम्।
द्वादशहृतेऽपि लब्धं शरसङ्ख्या स्यात् कलापकाविष्टा॥

"Add three to the number of arrows forming the circumferential layer; then square this (resulting sum) and add again three (to this square quantity). If this be further divided by 12, the quotient becomes the number of arrows to be found in the bundle. "

Arithmetic Progression

Verse 290 gives the sum of an Arithmetic Progression, A.P

Result: Same as the one in *Āryabhaṭīya* and *Brāhmasphuṭasiddhānta*.

$$S = a + (a+b) + \cdots + [a + (n-1)b] = n\left[a + \frac{(n-1)b}{2}\right].$$

Also a in terms of S, n, b; b in terms of S, a, n.

$$\begin{aligned} a^2 + (a+b)^2 + & \cdots + [a+(n-1)b]^2 \\ & = \left[\left\{\frac{(2n-1)b^2}{6} + ab\right\}(n-1) + a^2\right] n. \end{aligned}$$

$$\left[\text{follows from } 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}; 1^2 + 2^2 + (n-1)^2 \right.$$

$$\left. = \frac{(n-1)n(2n-1)}{6}\right].$$

Sum of Cubes

Verse 301.

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Verse 303.

$$a^3 + (a+b)^3 + \cdots + [a+(n-1)b]^3 = Sa|a-b| + S^2b,$$

where

$$S = a + (a+b) + \cdots + [a+(n-1)b] = n\left[a + \frac{(n-1)b}{2}\right]$$

[Try this as an exercise.]

The first result: Same as the one in *Āryabhaṭīya* and *Brāhmasphuṭasiddhānta*.

A.P. with each Term is Sum of an A.P

Verse $305 - 305\frac{1}{2}$:

Series: $1 + \cdots + a, 1 + \cdots + a + b, \cdots, 1 + \cdots, a + (n-1)b.$

Each term is sum of series in A.P. Then Sum S is given to be

$$= \left[\left\{ \frac{(2n-1)b^2}{6} + \frac{b}{2} + ab \right\} (n-1) + a(a+1) \right] \frac{n}{2}$$

[Try this as an exercise.]

Sum of Sums

Verse $307\frac{1}{2}$:

$$\text{Let } S_1 = 1 + \cdots + n, S_2 = \sum_{r=1}^{n} \frac{r(r+1)}{2}, S_3 = n^2, S_4 = n^3.$$

$$S_1 + S_2 + S_3 + S_4 = \frac{\left[\frac{n(n+1)}{2}.7 - n\right]}{3}(n+1)$$

Geometric Progression with Additions /Subractions

Verse 314:

गुणचितिरन्यादिहृता विपदाधिकहीनसङ्गुणा भक्ता।
व्येकगुणेनान्या फलरहिता हीनेऽधिके तु फलयुक्ता

“The sum of the series in (pure) geometrical progression (with the first given term, common ratio, and the given number of terms, is writen down in two positions); one (of these sums so written down) is divided by the (given) first term. From the (resulting) quotient, the (given) number of terms is subtracted. The (resulting) remainder is (then) multiplied by the (given) quantity which is to be added to or to be subtracted (from the terms in the proposed series).

Geometric Progression with Additions /Subractions

> The quantity (so arrived at) is (then) divided by the common ratio as diminished by one .(The sum of the series in pure geometrical progression written down in) the other (position) has to be diminished by the (last) resulting quotient quantity , if the given quantity is to be subtracted (from the term in the series). If, however it is to be added , (then the sum of the series in geometrical progression written down in the other position) has to be increased by the resulting quotient (already referred to. The result in either case gives the required sum of the series)."

Consider the series $a, ar \pm m, (ar \pm m)r \pm m, [(ar \pm m)r \pm m]r \pm m$. It is stated that

$$\text{Sum } = \pm \frac{\left(\frac{S}{a} - n\right) m}{(r-1)} + S,$$

$$\text{where } S = a + ar + \cdots + ar^{n-1}.$$

Proof

Proof:

$$
\begin{aligned}
\text{Sum} &= a + ar + \cdots + ar^{n-1} \\
&\quad \pm m \pm mr \pm \cdots \pm mr^{n-2} \\
&\quad \pm m \cdots \pm mr^{n-3} \\
&\quad \cdots\cdots \\
&\qquad\qquad\quad \pm m \\
&= \frac{a(r^n - 1)}{(r-1)} \pm \frac{m(r^{n-1} - 1)}{(r-1)} \pm \frac{m(r^{n-2} - 1)}{(r-1)} \pm \cdots \pm \frac{m(r-1)}{(r-1)} \\
&= \frac{a(r^n - 1)}{(r-1)} \pm m \frac{[r + \cdots + r^{n-1} - (n-1)]}{(r-1)}
\end{aligned}
$$

Proof

$$
\begin{aligned}
&= \frac{a(r^n - 1)}{(r-1)} \pm m \frac{\left[r \left(\frac{r^{n-1}-1}{r-1} \right) - n + 1 \right]}{(r-1)} \\
&= \frac{a(r^n - 1)}{(r-1)} \pm \left[\frac{r^n - r}{(r-1)} - n + 1 \right] \\
&= S \pm m \left[\frac{r^n - 1 + 1 - r}{r-1} - n + 1 \right] \\
&= S \pm m \left[\frac{r^n - 1}{r-1} - n \right] \\
&= S \pm \frac{\left(\frac{S}{a} - n \right) m}{(r-1)}.
\end{aligned}
$$

Example

Example in Verse 315:

पञ्च गुणोत्तरमादिर्द्वौ त्रीण्यधिकं पदं हि चत्वारः।
अधिकगुणोत्तरचितिका कथय विचिन्त्याशु गणिततत्त्वज्ञ॥ ३१५ ॥

"The common ratio is 5, the first term is 2, and the quantity to be added (to the various terms) is 3, and the number of terms is 4. O you know the secret of calculation, think out and tell me quickly the sum of the series in geometrical progression, where in the terms are increased (by the specified manner)."

Here $a = 2, r = 5, n = 4, +m = 3, S = \frac{2[5^4-1]}{(5-1)} = \frac{2\cdot 624}{4} = 312$

$$\therefore\ 2 + (2\times 5+3) + (2\times 5+3)\cdot 5+3+((2\times 5+3)5+3)\cdot 5+3$$

$$\text{Sum } = 312 + \frac{\left(\frac{312}{2}-4\right)^3}{4} = 312 + \frac{152\times 3}{4} = 312 + 38\times 3 = 426$$

Example

Exercise in Verse 316:

आदिस्त्रीणि गुणोत्तरमष्टौ हीनं द्वयं च दश गच्छः।
हीनगुणोत्तरचितिका का भवति विचिन्त्य कथय गणकाशु॥

“The first term is 3, the common ratio is 8, the quantity be subtracted (from the terms) is 2, and the number of terms is 10. What is the sum of this series (where the terms are diminished by the specified quantity in the specified manner). [Try this as an exercise].”

Variable Velocity Problem

The rule for arriving at the common limits of time when one who is moving (with successive velocities representable as the terms in arithmetic progression, that is, velocity is increasing by a fixed amount after each unit of time; constant acceleration), and another moving with a constant velocity.

Verse 319 1/2:

ध्रुवगतिरादिविहीनश्चयदलभक्तस्सरूपकः कालः। ३१९ १/२ ।

“The unchanging velocity is diminished by the first term (of the velocity in series in arithmetial progression), and is (then) divided by the half of the common difference. On adding one (to the resulting quantity), the (required) time (of meeting) is arrived at.”

	*	*	*	*	*	*
I person	u	$u+a$	$u+2a$	$u+3a$	$\cdots$	$u+a(t-1)$
II person	v	v	v	v	v	v

Variable Velocity Problem

Average velocity of the accelerating person $= \frac{u + u + a(t-1)}{2} = u + \frac{a(t-1)}{2}$. If they meet after time t, this average velocity should be equal to the constant velocity of the other person, that is v.

$$\therefore\ v = u + \frac{a(t-1)}{2}$$

$$\therefore\ t = \frac{v-u}{(a/2)} + 1$$

Incidentally distance traveled by each $= vt = ut + \frac{1}{2}at(t-1)$. [Interesting, Reminds one of the distance traveled by an uniformly accelerating body $= ut + \frac{1}{2}at^2$].

Two Accelerating Travellers

One can also work out the formula for the meeting time, when both travelers are accelerating. Also works when the common difference (acceleration) of one person is negative.

Verse $325\frac{1}{2}$:

> पञ्चाद्यष्टोत्तरतः प्रथमो नाथ द्वितीयनरः।
> आदिः पञ्चघ्ननव प्रचयो हीनोऽष्ट योगकालः कः॥
>
> “The first person travels with velocities beginning with 5 and increased (successively) by 8 in the common difference. In the case of the second person; the commencing velocity is 45; and the common difference is -8. What is the time of meeting. [Try this as an exercise].

Verse $333\frac{1}{2} - 336\frac{1}{2}$: n syllables $* * * \ldots, *$. Each can be $laghu$ or $guru$. Total no. of ways $= 2^n$. The way to compute 2^n has been discussed in BSS.

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Thanks!

Thank You