NPTEL Course on

Mathematics in India:
From Vedic period to Modern times

Lecture 7

Āryabhaṭīya of Āryabhaṭa - Part 1

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Āryabhaṭīya of Āryabhaṭa – Part 1

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Introduction

Āryabhaṭa and his period

- The term Āryabhaṭa is a compound word which literally means *the warrior of the noble* (आर्यस्य भटः आर्यभटः).[1]
- The name Āryabhaṭa appears twice in *Āryabhaṭīya* (in the opening verses of *Gītikā* and *Gaṇita-pāda*).
- The time period of Āryabhaṭa gets unambiguously fixed based on the following verse (*Kālakriyā*, 10), wherein he himself has stated that he was 23 years at 3600 Kali.

  षष्ट्यब्दानां षष्टिर्यदा व्यतीताः त्रयश्च युगपादाः ।
  त्र्यधिका विंशतिरब्दाः तदेह मम जन्मनोऽतीताः ॥

  When sixty times sixty years had elapsed, . . . then, twenty plus three years had passed since my birth.

- Since 3600 Kali-years corresponds to the 499 CE, it follows that Āryabhaṭa was born in 476 CE.

[1] '*Ārya*' meaning 'respectable' has been defined as (*Śabdakalpadruma*):

कर्तव्यमाचरन् कामं अकर्तव्यमनाचरन् ।
तिष्ठति प्राकृताचारे स एवार्य इति स्मृतः ॥

Introduction

The work *Āryabhaṭīya*

- *Āryabhaṭīya*, the *magnum opus* of Āryabhaṭa, is the earliest clearly datable Indian astronomical treatise extant today.[2]
- In this treatise, Āryabhaṭa systematically discusses the procedure for planetary computations.
- The style of composition of this text is akin to the earlier classic works on other *śāstras*. The verses are so cryptic at places, that commentators refer to them as *Āryabhaṭa-sūtra*.
- *Āryabhaṭīya* is made up of four parts namely:
  1. *Gītikāpāda* (in 13 verses)
  2. *Gaṇitapāda* (in 33 verses)
  3. *Kālakriyāpāda* (in 25 verses)
  4. *Golapāda* (in 50 verses)
- Thus, *Āryabhaṭīya* just consists 121 verses, all of them composed in *āryā* metre. (origin of the term *āryāṣṭaśatī?*)

[2]The other treatises that would have prevalent during his period, are available only in the form of an executive summary in *Pañcasiddhāntikā*.

Introduction

The four parts of the text *Āryabhaṭīya*

Gītikāpāda Having laid down his own scheme of representing numbers, here Āryabhaṭa essentially presents the values of all parameters that would be necessary for astronomical and planetary computations.

Gaṇitapāda This succincntly presents all the mathematical formulae and techniques that would be employed in computing planetary positions, which includes procedure for extracting square roots, solving first order indeterminate equation *kuṭṭaka*, etc.

Kālakriyāpāda This deals with procedures for computing planetary positions as well as presents a geometrical picture implied by the computational procedure.

Golapāda The fourth and the final part discusses several things that includes shape of the earth, source of light on planets, procedure for the calculation of eclipses, the visibility of planets, and so on..

Brevity, clarity and 'novelty' are hallmarks of the text *Āryabhaṭīya*. Āryabhaṭa himself towards the end of work mentions – सदसज्ज्ञानसमुद्रात् . . .

Introduction

A note on Bhāskara — the renowned commentator of *Āryabhaṭīya*

- Bhāskara's commentary throws a flood of light on the work *Āryabhaṭīya*, not only from the view point of
  - astronomical theories put forth by *Āryabhaṭīya*, but also
  - in terms of the mathematical knowledge prevalent around that period (c. 629 CE).
- It is a remarkable work of great scholarship, which made later astronomers describe him as 'all-knowing' commentator.
- In his commentary on *Gītikāpāda*, mentioning about the time elapsed since the beginning of *Kalpa*, he observes:

  कल्पादेरब्दनिरोधादयं अब्दराशिरितीरितः
  खाग्न्यद्रिरामार्करसवसुरन्ध्रेन्दवः। *1986123730*

- Based on this, Prof. K. S. Shukla proposes that Bhāskara wrote his commentary on *Āryabhaṭīya* when 3730 years had elapsed since the beginning of the current *Kaliyuga*, which corresponds to 629 CE.

A note on *Gītikāpāda* of *Āryabhaṭīya*

- *Gītikāpāda* consists of 13 verses:

Verse 1	Invocation and introduction
Verse 2	Scheme of representing numbers
Verse 3-12	List of parameters
Verse 13	Concluding remarks

- The first and the last verse go as:

प्रणिपत्यैकमनेकं कं सत्यां देवतां परं ब्रह्म।
आर्यभटस्त्रीणि गदति गणितं कालक्रियां गोलम्॥
Having paid obeissance to . . . Āryabhaṭa *sets forth the three, viz., mathematics, reckoning of time and celestial sphere.*

दशगीतिकसूत्रमिदं भूग्रहचरितं भपञ्जरे ज्ञात्वा।
ग्रहभगणपरिभ्रमणं स याति भित्वा परं ब्रह्म॥
Knowing this daśagītikasūtra [giving] the motion of . . .

- The phrases *trīṇi gadati*, and independent *phalaśruti* starting with *daśagītikasūtramidaṃ*, plus the presence of invocation once again at the beginning of *Gaṇitapāda*, $\Rightarrow$ *Āryabhaṭīya* is made up of only three parts.

Invocatory verse of *Gaṇitapāda*

- Āryabhaṭa commences his work with the following verse wherein the authorship as well as the place of learning is mentioned.

  ब्रह्म-कु-शशि-बुध-भृगु-रवि-कुज-गुरु-कोण-भगणान् नमस्कृत्य।
  आर्यभटस्त्विह निगदति कुसुमपुरेऽभ्यर्चितं ज्ञानम्॥

- The first word ब्रह्म does not refer to the creator *Brahmā*, but the primordial entity.
- Similarly, the word भगण used here is not रूढि but यौगिक। i.e., it does not refer to the number of revolutions but the group of stars.
- Commenting on the word कुसुमपुर Bhāskara observes:

  कुसुमपुरं पाटलिपुत्रं,[3] तत्र अभ्यर्चितं ज्ञानं निगदति। एवमनुश्रूयते –
  अयं हि स्वायम्भुवसिद्धान्तः कुसुमपुरनिवासिभिः कृतिभिः पूजितः
  सत्स्वापि पौलिश-रोमक-वसिष्ठ-सौरेषु। तेनाह –
  कुसुमपुरेऽभ्यर्चितमिति।

- The phrases ***abhyarcitaṃ*** and ***satsvapi*** are indeed noteworthy.

[3] This is modern Patna in Magadha (modern Bihar) — a 'great' center of learning in those days — where Nalanda University was situated.

Names of the notational places and their significance

- The following verse, introduces the notational places:

  एकं दश शतं च सहस्रं त्वयुतनियुते तथा प्रयुतम्।
  कोटयर्बुदं च बृन्दं स्थानात् स्थानं दशगुणं स्यात्॥

- Bhāskara after listing of names of the notational places – one (10^0) to billion (10^9) – poses an interesting question and replies:

  अत्रेदं प्रष्टव्यम् – कैषां स्थानानां शक्तिः?

  Here one must ask – what potential do these notational places have?

  यत् एकं, रूपं दश शतं सहस्रं च भवति। सत्यां चैतस्यां स्थानशक्तौ क्रायकाः विशेषेष्टक्रय्यभाजनाः स्युः।

  The potential is that, one and the same entity (symbol for one) can connote one, ten, hundred or thousand. Once this potential gets established, it is easy for the traders to [conveniently] tag prices to their commodities.

Square and Squaring

- The following verse defines the term *varga* to be referring to both geometrical object as well as the operation

  वर्गः समचतुरश्रः फलं च सदृशद्वयस्य संवर्गः।

- समचतुरश्रम् $\equiv$ object whose four sides are equal.
- If we take this literal sense, the word *varga* can also mean rhombus. Posing himself this question Bhāskara observes:

  नैव लोके एवमाकारविशिष्टस्य समचतुरश्रक्षेत्रस्य समचतुरश्रसंज्ञा सुसिद्धा।

  The four-sided figure having this shape has certainly not gained curreny to be described by the term *samacutraśra* in the world.

  This is a very important statement made by Bhāskara, that presents a deeper understanding about *śabdārtha-sambandha*.

- Bhāskara also lists synonyms of *varga* as: करणी, कृतिः, वर्गणा, यावकरणम् (=यावतः वर्गीकरणम्) इति पर्यायाः।

Algorithm for finding the square root

- The algorithm for obtaining the square root is succinctly presented by Āryabhaṭa in the following verse:

भागं हरेत् अवर्गात् नित्यं द्विगुणेन वर्गमूलेन।
वर्गाद्वर्गे शुद्धे लब्धं स्थानान्तरे मूलम्॥

[Having subtracted the greatest possible square from the last odd place and then having written down the square root of the number subtracted in the line of the square root] always divide[4] *the even place [standing on the right] by twice the square root. Then, having subtracted the square [of the quotient] from the odd place [standing on the right], set down the quotient at the next place (i.e., on the right of the number already written in the line of the square root). This is the square root. [Repeat the process if there are still digits on the right].*[5] *[tr. K. S. Shukla]*

[4] While dividing, the quotient should be taken as great, as will allow the subtraction of its square from the next odd place.

[5] *Cf. GSS,* ii.36; *PG,* rule 25-26; *GT,* p.9, vs.23; *MSi,* xv.6(c-d)-7; *SiŚe*, *xii* 5; *L* (ASS). p.21, rule 22; *GK*, I, p.7, lines 2-9.

Algorithm for finding the square root

भागं हरेत् अवर्गात् नित्यं द्विगुणेन वर्गमूलेन।
वर्गाद्वर्गे शुद्धे लब्धं स्थानान्तरे मूलम्॥

While commenting on the above verse, Bhāskara makes certain clarifying notes that are not only found to be quite useful in understanding the content of this verse, but are quite edifying.

1. Considering the first word $bh\bar{a}ga$ which literally means 'part', it is said – भागः, हृतिः, भजनं, अपवर्तनं इति पर्यायाः।
2. Thus भागं हरेत् = भजेत् means *may you divide*.
3. He then clarifies that in the word वर्गाद्वर्गे the first $varga$ refers to an odd place. अत्र गणिते विषमं स्थानं वर्गः।
4. Then obviously $avarga$ is even place. तस्यैव नञा विषमत्वे प्रतिषिद्धे अवर्ग इति समं स्थानम्।

Algorithm for finding the square root

भागं हरेत् अवर्गात् नित्यं द्विगुणेन वर्गमूलेन।
वर्गाद्वर्गे शुद्धे लब्धं स्थानान्तरे मूलम्॥

The algorithm presented here essentially consists of three steps:

1. Starting from the least significant digit, group the digits of the given number into two. Then from the remaining (1 or 2) most significant digit(s), which constitutes *varga-sthāna*, subtract the square of the max. no. that is possible.
2. Having done that (वर्गाद्वर्गे शुद्धे), along with the remainder bring down the next digit from the *avarga* place. This has to be divided by twice the *varga-mūla*—that is currently in the sq. root line—and the quotient has to be taken to that line.
3. Along with the remainder bring down the next digit in the *varga* place and subtract from it the square of the previous quotient (वर्गात् वर्गशोधनम्).

Illustrative Example

Āryabhaṭa's algorithm for finding the square root

Example 1: Find the square root of 55225.

Starting from the unit's place, pairs of notational places called *varga* (V), and *avarga* (A), are to be marked. The digits of the given number are arranged below them appropriately.

		V	A	V	A	V	235
		5	5	2	2	5	(line of square root)
Subtract 2^2		4					
Divide by 2.2	4)	1	5	(3			
		1	2				
			3	2			
Subtract 3^2				9			
Divide 2.23	46)		2	3	2	(5	
			2	3	0		
					2	5	
Subtract 5^2					2	5	
						0	

Illustrative Example

Āryabhaṭa's algorithm for finding the square root

Example 2: Find the square root of 2989441.

		V	*A*	*V*	*A*	*V*	*A*	*V*	1729
		2	9	8	9	4	4	1	(line of square root)
Subtract 1^2		1							
Divide by 2.1	2)	1	9	(7					
		1	4						
			5	8					
Subtract 7^2			4	9					
Divide 2.17	34)			9	9	(2			
				6	8				
				3	1	4			
Subtract 2^2						4			
Divide 2.172	344)			3	1	0	4	(9	
				3	0	9	6		
							8	1	
Subtract 9^2							8	1	
								0	

Cube and Cubing

- The following verse defines the term *ghana* to be referring to both geometrical object as well as the operation

  सदृशत्रयसंवर्गः घनः तथा द्वादशाश्रिः स्यात्।

- First Āryabhaṭa says: सदृशत्रयसंवर्गः घनः $\equiv$ product of three equals. This definition has to do with the cubing process (purely as an arithmetical operation), which is stripped off from the geometry that can be associated with it.
- He then quickly highlights the other aspect too: द्वादशाश्रिः (घनः) – The term *ghana* refers to an object having 12 corners or edges. That is, a cube.
- Bhāskara in his commentary lists synonyms of *ghana* as: घनो वृन्दम् सदृशत्रयाभ्यास इति पर्यायाः।

Algorithm for finding the cube root

- The algorithm for obtaining the cube root is presented by Āryabhaṭa in the following *āryā*:

  अघनात् भजेत् द्वितीयात् त्रिगुणेन घनस्य मूलवर्गेण।
  वर्गास्त्रिपूर्वगुणितः शोध्यः[6] प्रथमात् घनश्च घनात्॥

- A few observations before we explain the verse above:
  1. If a number has n digits, the number of digits in the cube of that number will be $\geq 3n - 2$ and $\leq 3n$.
  2. With this in mind Āryabhaṭa prescribes to group the number of digits—starting from the unit's place of the given number whose cube root is to be found—into three.
  3. The groups of the three notational places are called
     - *Ghana* (G)
     - *Prathama-Aghana* (A_1)
     - *Dvitīya-Aghana* (A_2)

[6] The prose order is: द्वितीयात् (अघनात् भजनेन लब्धस्य) वर्गः त्रिपूर्वगुणितः (प्रथमात् अघनात्) शोध्यः। घनश्च घनात् शोध्यः।

Algorithm for finding the cube root

The verse and its translation

अघनात् भजेत् द्वितीयात् त्रिगुणेन घनस्य मूलवर्गेण।
वर्गस्त्रिपूर्वगुणितः शोध्यः प्रथमात् घनश्च घनात्॥

[Having subtracted the greatest possible cube from the last cube place and then having written down the cube root of the number subtracted in the line of the cube root], divide the second non-cube place (standing on the right of the last cube place) by thrice the square of the cube root [already obtained]; (then) subtract from the first non-cube place (standing on the right of the second non-cube place) the square of the quotient multiplied by thrice the previous (cube-root); and (then subtract) the cube (of the quotient) from the cube place (standing on the right of the first non-cube place) [and write down the quotient on the right of the previous cube root in the line of the cube root, and treat this as the new cube root. Repeat the process if there are still digits on the right].[7] *[tr. K. S. Shukla]*

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[7] *Cf.BrSpSi,* xii.7; *GSS,* ii.53-54; *PG,* rule 29-31; *MSi,* xv.9-10 (a-b); *GT,* p.13 lines 18-25; *SiŚe*, xiii.6-7; *L* (ASS), rule 28-29, pp. 27-28; *GK.* I, pp.8-9, vv. 24-25.

Algorithm for finding the cube root

The algorithm presented by Āryabhaṭa essentially consists of four steps:

1. Starting from the unit's place, having grouped the digits of the given number into three, from the remaining (1, 2 or 3) most significant digit(s), which constitutes *ghana-sthāna*, subtract the cube of the max. digit that is possible. This digit forms the first (most significant) digit of the cube root to be determined.
2. Then, along with the remainder bring down the next digit from the *dvitīya-agana* place. This has to be divided by thrice the square of the *ghana-mūla* obtained so far. The quotient forms the next digit of the cube root.
3. Along with the remainder bring down the next digit in the *prathama-aghana* place, and subtract from it the square of the previous quotient multiplied by 3 and the *pūrva*, the cube root determined previously, that is till now (वर्गस्त्रिपूर्वगुणितः).
4. Then from the successive *ghana* place we have to subtract the cube of the quotient that was determined previously (second step), and the whole process has to be repeated.

Illustrative Example

Āryabhaṭa's algorithm for finding the cube root

Example 1: Find the cube root of 1,771,561.

		G	A_2	A_1	G	A_2	A_1	G	121
		1	7	7	1	5	6	1	(line of cube root)
Subtract 1^3		1							
Divide by 3.1^2	3)	0	7	(2					
		0	6						
			1	7					
Subtract $3.1.2^2$			1	2					
				5	1				
Subtract 2^3				0	8				
Divide by 3.12^2	432)			4	3	5	(1		
				4	3	2			
						3	6		
Subtract $3.12.1^2$						3	6		
							0	1	
Subtract 1^3								1	
								0	

Rationale behind Āryabhaṭa's cube root algorithm

- The rationale can be readily seen by grouping the terms together.
- Any three digit number may be represented as,

$$ax^2 + bx + c, \qquad \text{where } a, b, c \text{ are integers \& } x = 10$$

- The cube of this number may be expressed as:

terms	operation	significance of it
$x^6(a^3)$	$(-)\ a^3$	cube of max. digit
$+x^5(3a^2b)$	$(\div)\ 3a^2$	to get the value of b
$+x^4(3a^2c + 3ab^2)$	$(-)\ 3ab^2$	
$+x^3(6abc + b^3)$	$(-)\ b^3$	we are left with $3c(a+b)^2$
$+x^2(3b^2c + 3ac^2)$	$(\div)\ 3(a+b)^2$	to get the value of c
$+x^1(3bc^2)$	$(-)\ 3(a+b)c^2$	
$+x^0(c^3)$	$(-)\ c^3$	remainder zero $\Rightarrow$ perfect cube.

- The algorithm presumes (i) a thorough understanding of decimal place value system, and (ii) skill in algebraic manipulation.

Thanks!

$\boxed{\text{THANK YOU}}$

More of *Āryabhaṭīya* in the next lecture!