NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 8

Āryabhaṭīya of Āryabhaṭa - Part 2

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Outline

$\bar{A}ryabhat\d{t}\bar{\imath}ya$ of Āryabhaṭa – Part 2

- Area of a triangle
  - Bhāskara's commentary introducing $\bar{a}b\bar{a}dh\bar{a}s$
  - Numerical example
- Area of a circle
- Area of a trapezium
- Generalized approach to finding area + verification
- Chord of one-sixth of a circle
- Approximate value of π
- Bhāskara's discussion on the word $\bar{a}sanna$
- Construction of sine-table
  - General principle
  - Geometrical approach
  - Analytic approach (discrete version of harmonic equation)

Area of a triangle

- The formula for the area is presented in half *āryā*:

  त्रिभुजस्य फलशरीरं समदलकोटीभुजार्धसंवर्गः।

  The product of the perpendicular [dropped from the vertex on the base] and half the base gives the measure of the area of a triangle.

- Bhāskara in his commentary observes:
  1. the term *samadalakoṭi* means 'the perpendicular dropped from the vertex on the base of a triangle', i.e., 'the altitude of a triangle'.
  2. it should not be interpreted as 'the upright which bisects the base of the triangle into two equal parts'
  3. if we do so, the applicability of the above rule will be restricted to equilateral (*sama-tryaśra*) and isosceles triangles (*dvisama-tryaśra*).

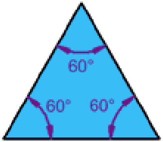
Area of a triangle

Bhāskara's commentary: Classification of triangles

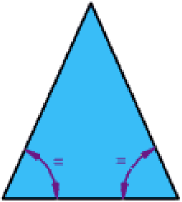
He actually commences the discussion with the classification of triangles.

भुजा बाहुः पार्श्वमिति पर्यायाः। तत्र त्रीणि क्षेत्राणि
सम-द्विसम-विषमाणि।

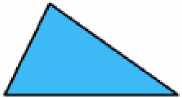
…are of three types: equilateral, isoceles and scalene.



(a) समत्र्यश्र



(b) द्विसमत्र्यश्र



(c) विषमत्र्यश्र

Area of a triangle

Bhāskara's commentary (contd.)

त्रिभुजस्य फलशरीरं समदलकोटीभुजार्धसंवर्गः।

त्रिभुजस्येति त्रिभुजक्षेत्रजातिमङ्गीकृत्य एकवचननिर्देशः। फलस्य शरीरं फलशरीरं फलप्रमाणमित्यर्थः। समे दले यस्याः सेयं समदला, समदलाचासौ कोटी च समदलकोटीति वर्णयन्ति। तेषां सम-द्विसमक्षेत्रयोरेव फलसिद्धिः ...अस्माकं पुनः समदलकोटीत्यनेन अवलम्बकव्युत्पत्त्या त्रयाणामपि फलानयनं सिद्धम्। अथवा ये व्युत्पत्तिं कुर्वन्ति तेषामपि त्रयाणां फलानयनं सिद्धमेव। कुतः? 'रूढेषु क्रिया व्युत्पत्तिकर्मार्था नार्थक्रिया'[1] इति।

Here Bhāskara explains

1. how the singular usage *tribhujasya* in the verse has to be understood.
2. how the word *śarīra* has to be understood, and
3. how the compound *samadalakoṭī* (= the perpendicular dropped from the vertex on the base) has to be interpreted in this context. In fact, he quotes a maxim to justify his interpretation.

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[1] रूढेषु व्युत्पत्तिकर्मार्था क्रिया न अवयवार्थप्रतिपादनार्था। (Ex. गौः, अश्वकर्णः)

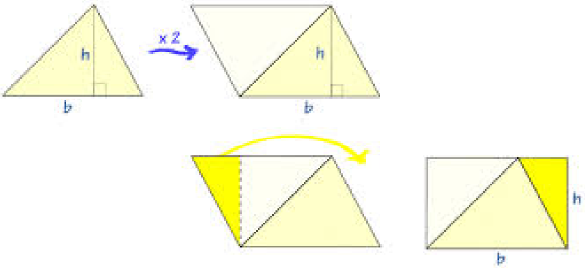
Area of a triangle

- The formula encoded in the $\bar{a}ry\bar{a}$

त्रिभुजस्य फलशरीरं समदलकोटीभुजार्धसंवर्गः । 6a ।

may be expressed as

$$\begin{aligned} \textit{tribhujaphala} &= \textit{bhujārdha} \times \textit{samadalakoṭī} \\ \textit{Areaoftriangle} &= \frac{1}{2}\textit{base} \times \textit{height}\ (\textit{altitude}) \\ &= \frac{1}{2} b \times h \end{aligned}$$



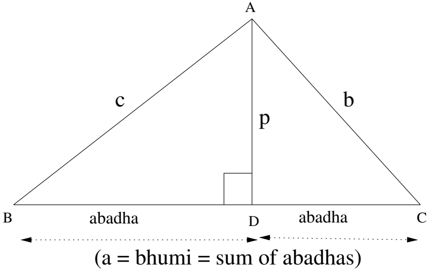
Area of a triangle

- The formula ($\frac{1}{2}$*base* $\times$ *height*) as given by Āryabhaṭa can be employed only when the altitude is known.
- If altitude is not known and only three sides are known, then how do we find the area?
- Bhāskara in his commentary presents a method by which we find segments of the base ($BD\&CD$)—called *ābādhās*—and hence the altitude (p).

It can be shown that (see next slide) the diff. in the *ābādhās* is given by

$$\begin{aligned} A_{diff} &= BD - CD \\ &= \frac{c^2 - b^2}{a} \quad (b, c : karṇas) \end{aligned}$$

$$\begin{aligned} ābādhās &= \frac{1}{2}(bhumi \pm A_{diff}) \\ altitude &= \sqrt{karṇa^2 - ābādhā^2} \end{aligned}$$



Area of a triangle [Bhāskara's commentary (contd.)]

भुजयोर्वर्गविशेषः तयोर्वा समासविशेषाभ्यासः त्रिभुजक्षेत्रे आबाधान्तरसमासविशेषाभ्यासो भवति। भूम्या आबाधान्तरसमासप्रमाणया विभज्य लब्धं भूमावेव सङ्क्रमणम् । ‘अन्तरयुक्तं हीनं दलित’मिति। अनेन क्रमेण आबाधान्तरप्रमाणे लभ्येते। ताभ्यां आबाधान्तरप्रमाणाभ्यां विषमत्रिभुजस्य समदलकोट्यानयनम्।

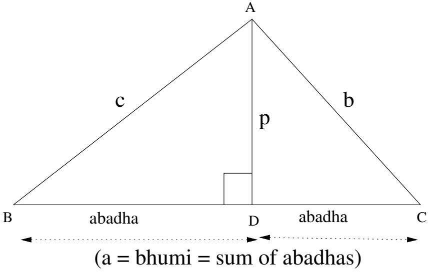
Let the sum and difference of the *ābādhās* be defined as:
$A_{sum} = BD + CD$ and $A_{diff} = BD - CD$. In the triangle ABC, $p^2 = c^2 - BD^2 = b^2 - CD^2$. Hence,

$$c^2 - b^2 = (c+b)(c-b) = BD^2 - CD^2 = A_{sum} \times A_{diff}$$

$$\text{Hence,}\quad A_{diff} = \frac{(c+b)(c-b)}{A_{sum}} = \frac{c^2 - b^2}{a}$$

$$\text{Now } \bar{a}badh\bar{a}s = \frac{1}{2}(bhumi \pm A_{diff})$$

From the *ābādhās*, the *samadalakoṭi* (altitude, p) has to be obtained.



Area of a triangle

Numerical example in Bhāskara's commentary

कर्णस्त्रयोदश स्यात् पञ्चदशान्यो मही द्विसप्तैव।
विषमत्रिभुजस्य सखे फलसङ्ख्या का भवेदस्य॥

In a scalene triangle [one of] the hypotenuse is thirteen, and the other is fifteen. The base is two times seven only. O my friend, [please tell me] what would be the measure of the area of this [triangle]?

Having posed this problem, Bhāskara presents the solution in prose as follows:

...पञ्चदशकेन कर्णेन नवप्रमाणेन च आबाधान्तरेण लब्धा समदलकोटी १२। त्रयोदशप्रमाणेन कर्णेन पञ्चप्रमाणेन च आबाधान्तरेण लब्धा सैव समदलकोटी १२। फलं 'समदलकोटीभुजार्धसंवर्गः', भुजा भूमिः, तस्य अर्धं ७, समदलकोटीभुजार्धसंवर्गः इति फलमागतम् ८४।

Now, $samadalako\d{t}\bar{\imath} = \sqrt{15^2 - 9^2} = 12$. *Also,* $\sqrt{13^2 - 5^2} = 12$; *The area is* $\frac{1}{2} \times 14 \times 12 = 84$.

Area of a circle

- The formula given in the following *āryā*

  समपरिणाहस्यार्धं विष्कम्भार्धहतमेव वृत्तफलम्। *7a* ।

  may be expressed as

$$\begin{aligned} phala &= pariṇāhārdha \times viṣkambhārdha \\ Area &= \frac{1}{2}\text{circumference} \times \text{semi-diameter} = \pi r^2. \end{aligned}$$

- Commenting on the usage of the word '*eva*' Bhāskara observes:

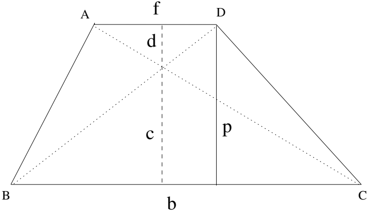
  एवकारकरणम् आर्यापूरणार्थं प्रतिपत्तव्यम्। अथवा एवकारकरणेन उपायनियमः क्रियते। समपरिणाहस्यार्धं विष्कम्भार्धहतमेव वृत्तफलम्, नान्यदुपायन्तरमिति। नैतदस्ति, उपायान्तरश्रवणात् अन्यत्र 'व्यासार्धकृतिस्त्रिसङ्गुणा गणितम्' इति। नैतदुपायान्तरं सूक्ष्मं, किन्तु व्यावहारिकमिति।

  ...Or, [to be more appropriate] by the use of the word *eva* the means is restricted. ...is the area of the circle there is no other means. This is not true, since $3 \times r^2$... This alternative method is not accurate ...

Area of a trapezium

- The latter half of the following verse in *Āryabhaṭīya* gives the formula for obtaining the area of a trapezium.

आयामगुणे पार्श्वे[2] तद्योगहृते स्वपातरेखे ते।
विस्तरयोगार्धगुणे ज्ञेयं क्षेत्रफलं आयामे॥



In this verse, as per the comm. of Bhāskara, the word *āyāma* means 'breadth'. The terms *pārśva* and *vistara* refer to the 'length' (base and face) of the trapezium.

- The formulae presented here are:

$$c = \frac{f \times p}{f+b}, \qquad d = \frac{b \times p}{f+b}, \qquad \text{Area} = \frac{1}{2}(f+b) \times p.$$

[2]Dual usage: भूः एकं पार्श्वं, मुखं इतरम्। आयामगुणे भूवदने इत्यर्थः।

Generalized approach to finding area

सर्वेषां क्षेत्राणां प्रसाध्य पार्श्वे फलं तदभ्यासः।

In the case of all plane figures, having determined the two 'sides', the area is [obtained by finding] their product.

- Here Bhāskara points out that the purpose of the verse is:

  to present a certain alternative approach by which we can verify the areas of all planar figures—trilateral, quadrilateral and circle—for which Āryabhaṭa has already given formulae.

- How is it possible to get the area of 'any' planar figure?

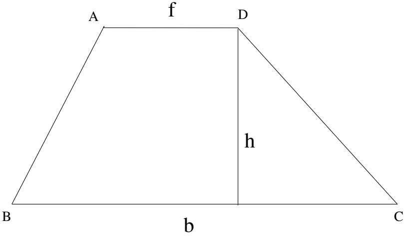
  प्रसाध्य पार्श्वे। 'प्र'-शब्दः प्रकृष्टवाची, प्रकर्षेण पार्श्वे साधयित्वा इति।
  कश्च तयोः पार्श्वयोः प्रकर्षः? उच्यते –

  Having obtained the two sides. The word 'pra' expresses some speciality; Thus it means finding the the two sides specially (prakarṣeṇa). What is the speciality of the two sides? It is said –

Generalized approach to finding area

In the case of triangle, rectangle and trapezium

- In the case of a triangle or rectangle or trapezium, its area can be found using the formula:



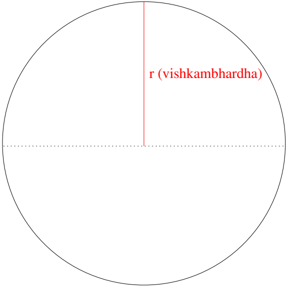
$$
\begin{aligned}
\textit{phala} &= \textit{vistarārdha} \times \textit{āyāma} \\
\textit{Area} &= \frac{1}{2}(\textit{base} + \textit{face}) \times \textit{height} \\
&= \frac{1}{2}(b + f) \times p
\end{aligned}
$$

- In the case of a triangle (equilateral, isoceles or scalene) we get the area by putting $f = 0$.
- In the case of a rectangle, we obtain the area by putting $f = b$.
- For a trapezium, the formula is to be used as such.
- How do we get the area of a circle in this generalized approach?

Generalized approach to finding area

The case of a circle

- In the case of a circle it is said:



वृत्तक्षेत्रे विष्कम्भार्धः विस्तारः परिध्यर्धं आयामः, तेदेव आयतचतुरश्रक्षेत्रम्।

In the case of a circle, the semi-diameter is the length, [and] the semi-circumference is the breadth or height, that constitutes the rectangle ($\bar{a}yatacatura\acute{s}rak\d{s}etram$)

$$\begin{aligned} \text{Area} &= vi\d{s}kambh\bar{a}rdha \times paridhyardha \\ &= \text{semi-diameter} \times \text{semi-circumference} \\ &= r \times \pi r \end{aligned}$$

- In his commentary $Bh\bar{a}skara$ also takes up more complex cases like drum, tusk of an elephant, etc. and demonstrates how to find the area of those shapes as well.

Purpose of the verse: Computation or verification?

अथ कथमेकेनैव यत्नेन फलानयनं प्रत्ययकरणं च प्रसाध्यते? अथेदं प्रत्ययकरणार्थं प्रकृतम्, स कथं फलानयनाय भवति? अथ फलानयनार्थे, कथं प्रत्ययकरणाय? नैष दोषः। अन्यार्थं प्रकृतम् अन्यार्थसाधकं दृष्टम्। तद्यथा – 'शाल्यर्थं कुल्याः प्रणीयन्ते। ताभ्यश्च पानीयं पीयते, उपस्पृश्यते च।'[3] एवमिहापि।

But how does the computation of the area and the verification happen in one stroke? If this rule is meant for verification, then how does it help in computating the area? And [conversely], if this rule is meant for computation of area, then how can it be used for verification? There is nothing wrong with that. What has been designed for one purpose, is found to be serving another purpose. That is as in the following case – Canals are constructed for the sake of [watering] paddy [field]. And from these [canals] water is drunk and also used for cleansing. It is similar in this case too.

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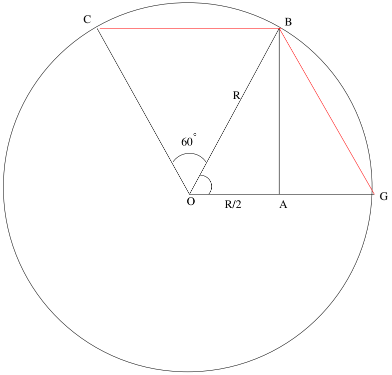
[3] अष्टाध्यायी, १-१-२२, पातञ्जलमहाभाष्यम्।

Chord of one-sixth of the circumference of a circle



परिधेः षड्भागज्या विष्कम्भार्धेन सा तुल्या।

The chord of one-sixth of the circumference [of a circle] is equal to the semi-diameter.

It is evident from the figure

$$BC = \text{chord } 60^\circ = R,$$

since OBC is an equilateral triangle. This is what is given in the verse.

- It is pointed out by Bhāskara that the purpose of the verse will become clear from the later verse '*samavṛtta-paridhipādaṃ*' :

प्रयोजनं चास्य षड्भागज्याप्रदर्शनस्य 'समवृत्त-परिधिपादं छिन्द्यात्' इत्यस्यां कारिकायां वक्ष्यति।

Approximate value of π

- The *Śulba-sūtra*-s, give the value of π close to 3.088.
- Āryabhaṭa (499 AD) gives an approximation which is correct to four decimal places.

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम्।
अयुतद्वयविष्कम्भस्य 'आसन्नो' वृत्तपरिणाहः॥

$$\pi = \frac{(100+4)\times 8 + 62000}{20000} = \frac{62832}{20000} = 3.1416$$

- The same value has been given in *Līlāvatī*[4] by removing a factor of 8 from the denominator and the numerator.

व्यासे भनन्दाग्निहते विभक्ते खबाणसूर्यैः परिधिः सुसूक्ष्मः ।
द्वाविंशतिघ्ने विहृतेऽथ शैलैः स्थूलोऽथवा स्याद् व्यवहारयोग्यः ॥

$$\pi = \frac{3927}{1250} = 3.1416$$ that's same as Āryabhaṭa's value.

[4] *Līlāvatī* of Bhāskarācārya, verse 199.

Approximate value of π

Bhāskara's edifying discussion while analysing the meaning of the term '*āsanna*'

आसन्नः निकटः। कस्यासन्नः? सूक्ष्मस्य परिणाहस्य। कथं विज्ञायते सूक्ष्मस्य आसन्न इति, न पुनः व्यावहारिकस्य आसन्नः, यावता अश्रुतपरिकल्पना[5] सूक्ष्मव्यावहारिकयोः तुल्या। नैष दोषः, सन्देहमात्रमिदम्। सर्वसन्देहेषु वा इदमवतिष्ठते – 'व्याख्यानतो विशेषप्रतिपत्तिः (न हि सन्देहादलक्षणम्)। - इति।'तस्मात् सूक्ष्मस्य आसन्नः इति व्याख्यास्यामः।

[The word] *āsanna* means close to or nearby. It is close to what? To the 'accurate' circumference. How does one understand that it is close to accurate value and not the one that is practically used (*vyāvahārikasya*), since the supposition of the unstated is equally valid in both the cases. There is nothing wrong [in the interpretation]. This is only a [valid] doubt. [But it must be remembered] that in all instances of doubts, the following principle gets invoked – Clear understanding arises out of [traditional] explanation (just because there is a doubt . . .) Therefore we would interpret as . . .

[5] In the printed edition, the text reads as श्रुतपरिकल्पना and it seems to be an editorial error while splitting सवर्णदीर्घसन्धि।

Approximate value of π

Bhāskara's edifying discussion while analysing the meaning of the term '*āsanna*'

अथवा आसन्नशब्देन तत्समीपवर्ति नाभिधीयते।[6]

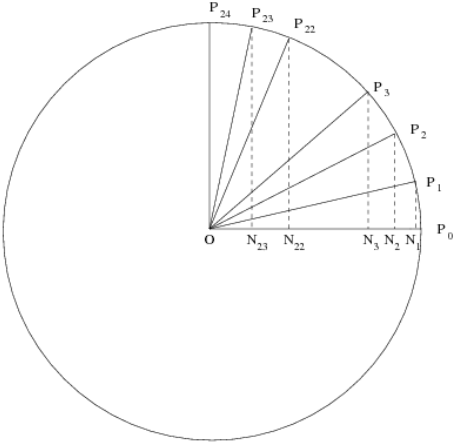
तेन च तदेव आसन्नशब्देन नोच्यते। तर्हि किञ्चित् भिन्नम्। यदि व्यावहारिकासन्नः व्यावहारिकादपि पापीयान् परिधिः, न कश्चित् पापतरं प्रयासं करोति, तेन सूक्ष्मासन्न इति न्यायसिद्धम्।
अथ आसन्नपरिधिः कस्मादुच्यते, न पुनः स्फुटपरिधिरेव उच्यते? एवं मन्यन्ते – स उपायः एव नास्ति येन सूक्ष्मपरिधिरानीयते।

Or else, by the word आसन्न, that which is close to [the accurate value] is not conveyed. . . . If an approximation to practical value [is what is conveyed], then the circumference obtained from that would be even worse (more gross). No one would make efforts to get something worse!. Therefore it is established . . . Why is . . . exact circumference is not stated? It is considered so [by scholars] — there is absolutely no means by which one can get the exact circumference.

[6] In the printed edition, the text reads as तत्समीपवर्तिनाभिधीयते। The absense of splitting—an editorial error—seems to have created a confusion, and hence in Agathe Keller's translation . . .

Construction of the sine-table

A quadrant is divided into 24 equal parts, so that each arc bit $\alpha = \frac{90}{24} = 3^{\circ}45' = 225'$.

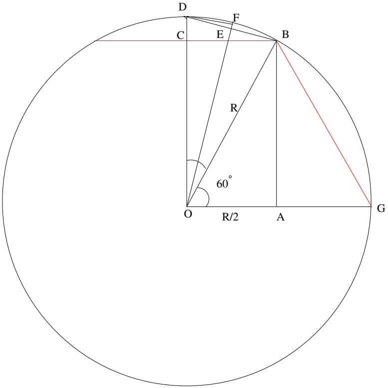


- Āryabhaṭa presents two different methods for finding $R\sin i\theta, (P_i N_i)$ $i = 1, 2, \ldots 24$.
- The Rsines of the intermediate angles are determined by interpolation (I order or II order).

Finding tabular sines: Geometrical approach

समवृत्तपरिधिपादं छिन्द्यात् त्रिभुजाच्चतुर्भुजाच्चैव।
समचापज्यार्धानि तु विष्कम्भार्धे यथेष्टानि॥

May the quadrant of the circumference of a circle be divided [into as many parts as desired]. Then, from (right) triangles and rectangles, one can find as many Rsines of equal arcs as one likes, for any given radius.



- We know, chord $60^\circ = R = 3438$
- From the triangle OBC,

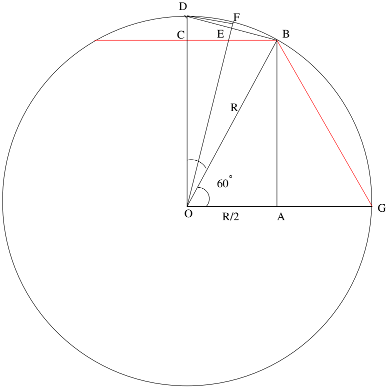
$$R \sin 30 = BC = R/2 = 1719$$

- Now consider the rectangle OABC. Here, $AB = R \sin 60^\circ = OC$.

$$OC = \sqrt{OB^2 - BC^2} = \sqrt{R^2 - \frac{R}{2}^2} = 2978$$

- Thus it may be noted that from 8th Rsine (30°) we immediately get 16th Rsine (60°)
- In other words, $R \sin \theta \rightsquigarrow R \sin(90 - \theta)$.

Finding tabular sines: Geometrical approach (contd.)



- In the triangle CBD, $BC = R\sin 30^\circ$ and $CD = OD - OC = R\,\text{vers}30^\circ$ are known. Hence, $BD = \text{chord } 30^\circ$

$$BD = \sqrt{(R\sin 30)^2 + (R\,\text{vers}30)^2},$$

is known. $R\sin 15^\circ = \frac{1}{2}BD = 890.$

- At this stage, we need to note that

$$R\sin\theta \rightsquigarrow R\cos\theta \rightsquigarrow R\,\text{vers}\,\theta$$

$$R\sin\theta \;\&\; R\,\text{vers}\,\theta \rightsquigarrow R\sin\frac{\theta}{2}$$

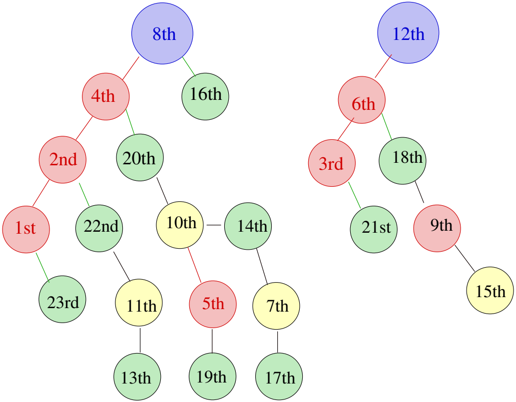
- Now considering the triangle ODE,

$$\begin{aligned} OE &= \sqrt{OD^2 - DE^2} \\ &= \sqrt{R^2 - (R\sin 15)^2} \end{aligned}$$

gives $R\sin 75^\circ$.

Finding tabular sines: Geometrical approach (contd.)

- Most of the Indian astronomers have presented their sine tables by dividing the quadrant (90°) into 24 parts.
- By the principle outlined above, it can be easily shown that all the 24 Rsines can be obtained provided the 24th, 12th and 8th Rsines are known.



- The circumference of the circle was taken by Āryabhaṭa to be 21600 units.
- From that using the approximation for π given by him, we get $R = 24th$ Rsine ≈ 3438.
- Once this it known, it is noteworthy that in the proposed scheme of constructing the table, all that is required is extraction of square root, for which Āryabhaṭa had clearly evolved an efficient algorithm.

Thanks!

THANK YOU

More of *Āryabhaṭīya* in the next lecture!