NPTEL Course on

Mathematics in India:
From Vedic period to Modern times

Lecture 12

Brāhmasphuṭasiddhānta of Brahmagupta - Part 2

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Outline

- Diagonals of a Cyclic Quadrilateral
- Rational Triangles and Quadrilaterals
- Chords of Circle
- Volumes with Uniform and Tapering Crosssections : Pyramids and Frustrums
- Shadow Problems

Diagonals of a Cyclic Quadrilateral

Brahmagupta does not say it is a ‘Cyclic quadrilateral’.

Verse 28.

कर्णाश्रितभुजघातैक्यमुभयथान्योन्यभाजितं गुणयेत् ।
योगेन भुजप्रतिभुजवधयोः कर्णौ पदे विषमे ॥ २८ ॥

“The sums of the product of the sides about both the diagonals being divided by each other, multiply the quotients by the sum of the products of opposite sides; the square roots of the results are the diagonals is a trapezium.”

Diagonals of a Cyclic quad.

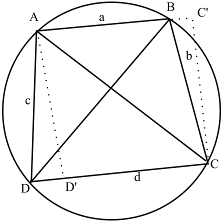
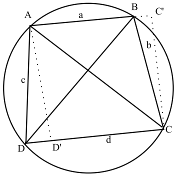


Figure: 7: Cyclic Quadrilateral

$ABCD$ is a cyclic quadrilateral with the sides a, b, c, d with the diagonals AC and BD. Sum of products of the sides about the diagonal $AC = ab + cd$.

Diagonals of a Cyclic Quadrilateral



Sum of the products of the sides about the diagonal $BD = ac + bd$. Sum of the products of opposite sides $= ad + bc$.

Then,

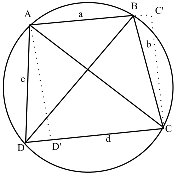
$$AC = D_1 = \sqrt{\frac{(ac+bd)}{(ab+cd)}(ad+bc)}$$

$$BD = D_2 = \sqrt{\frac{(ab+cd)}{(ac+bd)}(ad+bc)}$$

Note that $D_1 D_2 = ad + bc$. (useful later.)

Diagonals of a Cyclic quadrilateral

Proof: $Yuktibh\bar{a}\d{s}\bar{a}$ proof will be presented later in this Lecture series. For our immediate purpose :



Extend AB to C' such that AC' is perpendicular to CC'. Let AD' be perpendicular to CD. Now, $A\hat{D}D' = 180^\circ - A\hat{B}C$ (as $ABCD$ is a cyclic quadrilateral) $= C\hat{B}C'$.

So triangles ADD' and CBC' are similar. Using this, and the Theorem of right triangles , one can obtain the stated expressions for the diagonals.

Rational Triangles and Trapezia

Next, Brahmagupta gives the procedures to construct various figures mainly, triangles (isosceles, right, scalene), isosceles trapezia and trapezia with 3 equal sides, where all the sides are rational.

Verse 33. Isosceles Triangle.

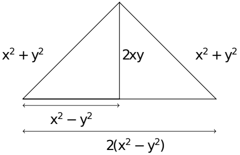


Figure: 8 : Isos. Triangle with Rational Sides

Right Triangle with Rational sides

Verse 35. Right triangle. Side a : arbitrary.

Upright $b = \frac{1}{2}\left(\dfrac{a^2}{x} - x\right)$, where x arbirtary

Diagonal $D = \frac{1}{2}\left(\frac{a^2}{x} + x\right) = \sqrt{b^2 + a^2}$.

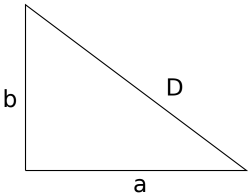


Figure: 9: Right Triangle with Rational Sides

Construction of a Rational Isosceles Trapezium

Verse 36:

Isosceles Trapezium starting from a Right Triangle with side, a, upright, b, and diagonal, c.

Take Summit $S = \frac{1}{2}\left(\frac{a^2}{x} - x\right) - b$

Base, B = $S + 2b$.

Rational Isosceles Trapezium

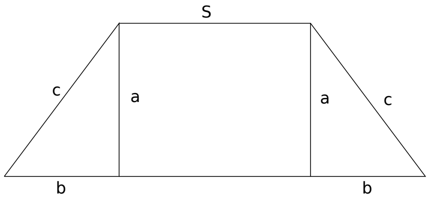


Figure: 10: Rational Isosceles Trapezium

Resulting Isosceles Trapezium shown above.
Summit, S, Base,B, Flanks,c.

Rational Trapezium with Three Equal Sides

Verse 37: Start from a right triangle : side a, upright b, diagonal, d.

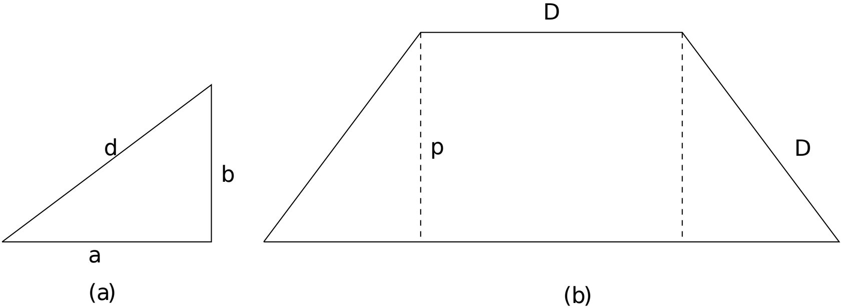


Figure: 11 : Rational Trapezium with three equal sides

3 equal sides, $= a^2 + b^2 = D$, Base $= 3a^2 - b^2$

$$\frac{\text{Base - Summit}}{2} = a^2 - b^2, \ \text{perpendicular}\, p = 2ab.$$

Ingenious Construcion of a Cyclic Quadrilateral

Verse 38.

जात्यद्वयकोटिभुजाः परकर्णगुणाः भुजाश्चतुर्विषमे ।
अधिको भूर्मुखहीनो बाहुद्वितयं भुजावन्यौ ॥ ३८ ॥

“The uprights and sides of two rectangular triangles reciprocally multiplied by the diagonals are the four dissimilar sides of a trapezium. The greatest is the base; the least is the summit; and the two others are the flanks.”

Construction of a Cyclic Quadrilateral

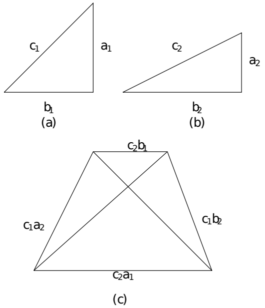


Figure: 12: Cyclic quad. from Right Triangles

Construction of a Cyclic Quadrilateral

Bhāskara-II discusses this in detail in his *Līlāvatī* and gives the diagonals.

Buddhivilāsinī of Gaṇeśa Daivajña explains the construction in more detail.

Construction of a Cyclic Quadrilateral

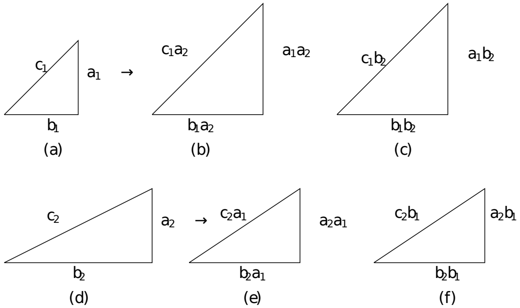


Figure: 13: Four Triangles used in the Cyclic Quad.

Ingenious Construction of a Cyclic Quadrilateral

Place the four right triangles as in figure below.

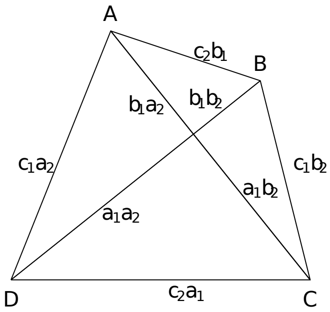


Figure: 14: The Constructed Cyclic Quad.

Ingenious Construction of a Cyclic Quadrilateral

So, the sides are $a = c_2b_1$, $b = c_1b_2$, $d = c_2a_1$, $c = c_1a_2$.

$$\text{Diagonals:}\, D_1,\, AC = a_1b_2 + a_2b_1,\;\; D_2,\, BD = a_1a_2 + b_1b_2.$$

It can be checked that circumradii of both the triangles ABC and $ACD = c_1c_2/2$. Hence the figure $ABCD$ is a cyclic quadrilateral. Expression for diagonals given earlier, coincide with the expressions above, that is,

$$\begin{aligned} D_1 &= \sqrt{(ac+bd)(ad+bc)/(ab+cd)} = a_1b_2 + a_2b_1, \\ D_2 &= \sqrt{(ab+cd)(ad+bc)/(ac+bd)} = a_1a_2 + b_1b_2. \end{aligned}$$

An Ingenious Application of the Theorem of Right Triangle

Verse 39.

इष्टगुणकारगुणितो गिर्युच्छ्रायः पुरान्तरमनष्टम् ।
द्वियुतगुणकारभाजितमुत्पातोन्यस्य समगत्योः ॥ ३९ ॥

“The height of the mountain, taken into a multiplier arbitrarily put, is the distance of the town. That result, being reserved and divided by the multiple added to two, is the height of the leap. The journey is equal.”

Application of the Theorem of Right Triangle

Pṛthūdakasvāmi explains the situation in the commentary. “On the top of a certain hill are the two ascetics. One of them being a wizard, travels through the air. Springing from the summit of the mountain, he ascends to a certain elevation, and proceeds by an oblique descent, diagonally to a neighboring town. The other walking down the hill, goes by land to the same town. Their journeys are equal. I desire to know the distance of the town from the hill and how high the wizard rose.”

Application of the Theorem of Right Triangle

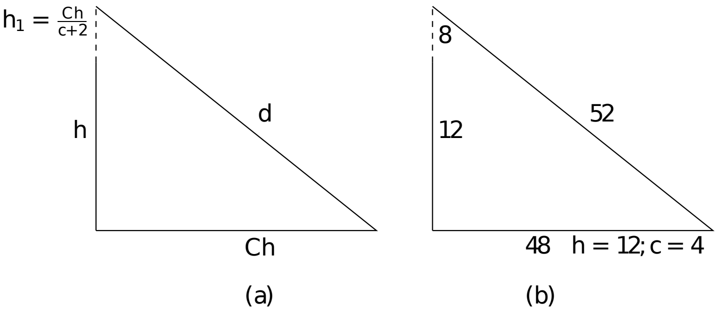
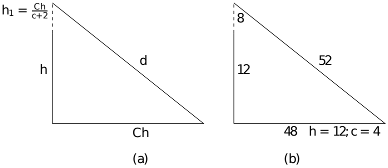


Figure: 15: Two ascetics taking different paths

In the figure, h is the height of the hill and ch is the distance of the town from the hill, where c is the multiplier. One ascetic travels a distance $h + ch$. The other ascetic jumps by an amount $\dfrac{ch}{c+2}$ and goes diagonally to the town.

The problem of two ascetics



So we have a right triangle whose upright is $h + \frac{ch}{c+2}$, side is ch and the diagonal d is $\sqrt{\left(h + \frac{ch}{c+2}\right)^2 + (ch)^2}$.

The distance traveled by the second ascetic is $\dfrac{ch}{c+2} + d$. So we should we have

$$h + ch = \frac{ch}{c+2} + \sqrt{\left(h + \frac{ch}{c+2}\right)^2 + (ch)^2}.$$

The problem of two ascetics

Does this imply a relation between c and h. No!.

It can be checked that this equation is satisfied for all c !.

Pṛthūdaka actually takes $h = 12$ and $c = 4$ to illustrate the problem. See Figure. Here one of the ascetic jumps by an amount $= \dfrac{ch}{c+2} = 8$. Clearly, the height of the right triangle is 20, side is 48 and the diagonal is 52 and we have $20^2 + 48^2 = 52^2$. The distance traveled by each of the two ascetics is 60.

Circle : Chords, Arrows

We now move to a different topic. In verse 40, the practical or the approximate value of the ratio of the circumference and the diameter (π) is stated to be 3. $\sqrt{10}$ is stated to be the correct value. The area of a circle is πr^2, where r is the radius or the semi-diameter.

Verse 41.

वृत्ते शरोनगुणितात् व्यासाच्चतुराहतात् पदं जीवा ।
ज्यावर्गश्चतुराहतशरभक्तः शरयुतो व्यासः ॥ ४१ ॥

“In a circle, the chord is the square root of the diameter less the arrow taken into the arrow and multiplied by four; The square of the chord divided by four times the arrow, and added to the arrow, is the diameter.”

Verse 42a.

ज्याव्यासकृतिविशेषात् मूलव्यासान्तरार्धमिषुरल्पः । ४२ a ।

“Half the difference of the diameter and the root extracted from the difference of the square of the diameter and the chord is the smaller arrow.”

Circle : Chords, Arrows

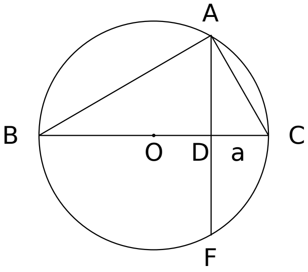


Figure: 17: The chord and the arrow

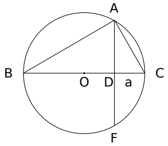
O - Centre of the circle, Diameter, $BC = d$, Arrow, $CD = a$, Chord $AF = 2AD$.

It is stated that Chord $= 2AD = \sqrt{(d - a)a \times 4}$

and Arrow, $CD = \frac{1}{2}(d - \sqrt{d^2 - \text{Chord}^2})$

Circle : Chords, Arrows

Proof:



i) Angle $B\hat{A}C = 90^\circ$.
Therefore triangles ADB and CDA are similar.
$\therefore \frac{AD}{DB} = \frac{CD}{AD}$.

$$\therefore AD^2 = DB \cdot CD = (d - a)a$$

$$\text{Hence, Chord} = AF = 2AD = \sqrt{(d-a)a \cdot 4}$$

ii) $\sqrt{d^2 - \text{Chord}^2} = \sqrt{d^2 - 4(d-a)a} = d - 2a$

$$\therefore a = \frac{1}{2}[d - (d - 2a)] = \frac{1}{2}[d - \sqrt{d^2 - \text{Chord}^2}]$$

Two Intersecting Circles: Arrows and Erosion

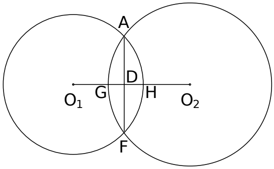


Figure: 18: Two Circles: Arrows and Erosion

Circle 1: diameter $= d_1$, Circle 2: diameter $= d_2$, Common chord $= AF$.
Arrow corresponding to circle 1, $DH = a_1$.
Arrow corresponding to circle 2, $GD = a_2$. Erosion $GH = e = a_1 + a_2$.
To find: Arrows given the diameters and the erosion.

Two Intersecting Circles: Arrows and Erosion

Verse 42b.
व्यासौ ग्रासोनगुणौ ग्रासोनैक्योद्धृतौ बाणौ ॥ ४२ ॥

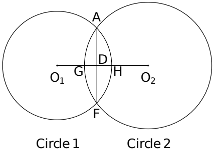
“The erosion being subtracted from the both the diameters, the remainders, multiplied by the erosion and divided by the sum of the remainders, are the arrows.”

It is stated that

$$a_1 = \frac{e(d_2 - e)}{d_1 + d_2 - 2e}, \quad a_2 = \frac{e(d_1 - e)}{d_1 + d_2 - 2e}$$

Proof :

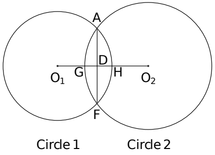
Chord $=$ *Chord*



$$
\begin{aligned}
\therefore\ (d_1 - a_1)a_1 &= (d_2 - a_2)a_2\\
\therefore\ d_1a_1 - d_2a_2 &= a_1^2 - a_2^2\\
&= (a_1 - a_2)(a_1 + a_2)\\
&= (a_1 - a_2)e.\\
\therefore\ a_1(d_1 - e) &= a_2(d_2 - e)\\
\therefore\ a_2 &= \frac{a_1(d_1 - e)}{(d_2 - e)}.
\end{aligned}
$$

Substituting this in $a_1 + a_2 = e$, we find:

Intersecting Circles: Arrows and Erosion



$$a_1 \left[1 + \frac{d_1 - e}{d_2 - e} \right] = e.$$

$$\therefore\ DH \ = a_1 = \frac{e(d_2 - e)}{d_1 + d_2 - 2e}.$$

Similarly,

$$GD \ = a_2 = \frac{e(d_1 - e)}{d_1 + d_2 - 2e}$$

.

Diameters and Erosion from Chord and Arrows

Verse 43.

इष्टशरद्वयभक्ते ज्यार्धकृती शरयुते फले व्यासौ ।
शरयोः फलयोरैक्यं ग्रासो ग्रासोनमैक्यं तत् ॥ ४३ ॥

“The square of the semichord being divided severally by the given arrows, the quotients added to the arrows respectively, are the diameters. The sum of the arrows is the erosion: and that of the quotients is the residue of subtracting the erosion.”

Essentially,

$$\text{Diameter } d_i = \frac{\text{Chord}^2}{4a_i} + a_i, \;\; i = 1, 2.$$

$$\text{Erosion}, e \;\; = \text{Sum of arrows} \; = \; a_1 + a_2.$$

Verse 44.

क्षेत्रफलं वेधगुणं समखातफलं हृतं त्रिभिः शूच्याः ।
मुखतलतुल्यभुजैक्यान्येकाग्रहृअतानि समरज्जुः ॥ ४४ ॥

“The area of the plane figure, multiplied by the depth, gives the content of the equal [or regular] excavation; and that divided by three, is the content of the needle [Pyramid or cone].”

Volumes : Regular and Pyramidal

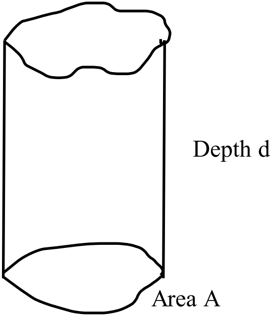


Fig 19.Regular Volume

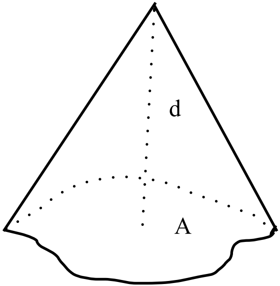


Fig.20. Pyramidal Volume

$$\text{Excavation: Volume } = Area \times Depth = A \times d.$$

$$\text{Needle (Pyramid or Cone) :Volume } = \frac{A \times d}{3}$$

Volume of a Frustrum

Verse 45-46.

मुखतलयुतिदलगुणितं वेधगुणं व्यावहारिकं गणितम् ।
मुखतलगुणितैक्यार्द्धं वेधगुणं स्याद्गणितमौत्रम् ॥ ४५ ॥
औत्रगणिताद्विशोध्य व्यवहारफलं त्रिभिर्भजेच्छेषम् ।
लब्धं व्यवहारफले प्रक्षिप्य फलं भवति सूक्ष्मम् ॥ ४६ ॥

"The area, deduced from moieties (halves) of the sums of the sides at top and at bottom, being multiplied by the depth, is the practical measure of the content. Half the sum of the areas at top and at bottom, multiplied by the depth gives the gross content. Subtracting the practical content from the other, divide the difference by three and add the quotient to the practical content. The sum is the neat content (Exact value)."

Volume of a Frustrum

Frustrum : Bottom: Square of side a ; Top: Square of side a' ; Depth : d.

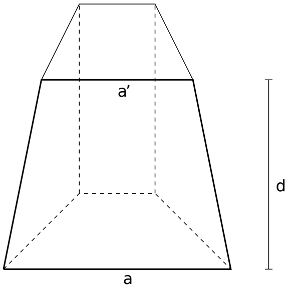


Fig 21. A Frustrum

Practical measure of Volume

$$p = \left[\frac{(a+a')}{2}\right]^2 d$$

Gross Content

$$g = \frac{(a^2 + a'^2)}{2} d$$

Exact value $= p + \dfrac{g-p}{3}$

Frustrum

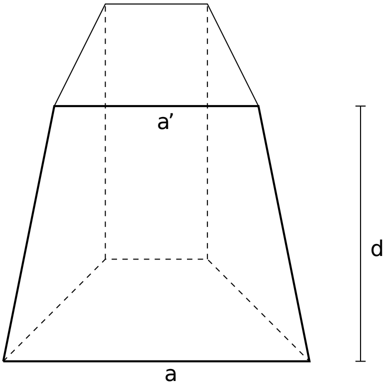


Figure: 21 : Frustrum

Volume of a Frustrum

a and $a^{'}$ are the sides of the squres at the bottom and top, respectively. Now,

$$p = \left(\frac{a^2 + a'^2 + 2aa'}{4} \right) d, \quad g = \left(\frac{(a^2 + a'^2}{2} \right) d$$

$$g - p = \left(\frac{a^2 + a'^2 - 2aa'}{4} \right) d = \frac{(a - a')^2}{4} d$$

It is easily seen that

$$\text{Exact value} = p + \frac{g - p}{3} = \left(\frac{a^2 + a'^2 + aa'}{3} \right)$$

Now we calculate the actual value using ”standard” method.

Standard calculation of Volume of Frustrum

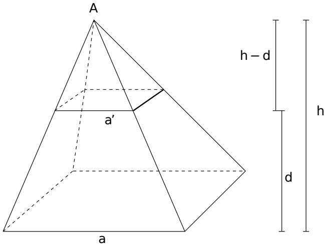


Fig 22. Pyramid -Cut off

$$
\begin{aligned}
\frac{a}{h} &= \frac{a'}{h-d}. \\
\therefore\ a(h-d) &= a'h \\
\text{or, } (a-a')h &= ad \\
\therefore\ h &= \frac{a'd}{a-a'} \\
\text{So:}\quad h-d &= \frac{a'd}{a-a'}
\end{aligned}
$$

Standard calculation of Volume of Frustrum

The actual volume is :

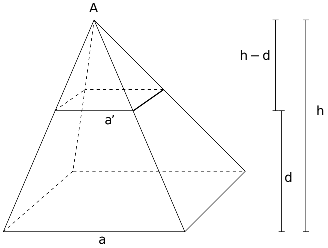


Fig 22. Pyramid -Cut off

$$
\begin{aligned}
&= \frac{1}{3}\left[a^2 h - a'^2(h-d)\right] \\
&= \frac{1}{3}\left[(a^2 - a'^2)(h-d) + a^2 d\right] \\
&= \frac{1}{3}\left[(a^2 - a'^2)\frac{a'd}{(a-a')} + a^2 d\right] \\
&= \frac{1}{3}\left[(a+a')a'd + a^2 d\right] \\
&= \frac{1}{3}\left[a^2 + a'^2 + aa'\right] d
\end{aligned}
$$

Brahmagupta’s Exact value coincides with this.

Example

A square well, measured by ten cubits at the top and by six at the bottom, is dug thirty cubits deep. Tell me the practical gross and the neat contents.

$$\text{Practical}, p : \left(\frac{10+6}{2}\right)^2 \times 30 = 1920.$$

$$\text{Gross}, g = \left(\frac{a^2 + a'^2}{2}\right) d = \frac{100+36}{2} \times 30 = 68 \times 30 = 2040.$$

$$\begin{aligned}\text{Exact volume} &= p + \frac{g-p}{3} \\ &= 1920 + \frac{120}{3} = 1960.\end{aligned}$$

Verse 47.

आकृतिफलमौच्याहतम् अग्रतलैक्यार्द्धम् औच्यदैर्घ्यगुणम् ।
घनगुणितम् इष्टकाघनफलेन हृतम् इष्टकागणितम् ॥ ४७ ॥

"The area of the form [or section] is half the sum of the breadth at bottom and at top multiplied by the height; and that multiplied by the length is the cubic content: which divided by the solid content of one brick, is the content in bricks."

Trapezoidal Stack: Volume and No. of Bricks

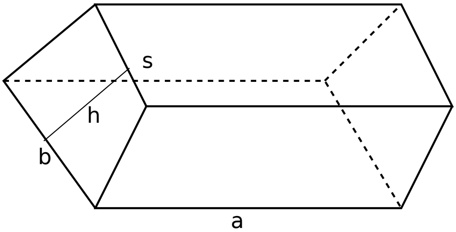


Figure: 23: Trapezoidal Stack

$$\text{Area of cross-section} = \left(\frac{b+s}{2}\right) h$$

$$\text{Stated Volume} = \left(\frac{b+s}{2}\right) h \cdot a$$

$$\text{Number of bricks} = \frac{\text{Volume}}{\text{Volume of one brick}}$$

Measures of Shadows

Verse 53.

दीपतलशङ्कुतलयोरन्तरम् इष्टप्रमाणशङ्कुगुणम् ।
दीपशिखोच्यात् शङ्कुं विशोध्य शेषोद्धृतं छाया ॥ ५३ ॥

“The distance between the foot of the light and the bottom of the gnomon multiplied by the gnomon of given length and divided by the difference between the height of the light and the gnomon, is the shadow.”

Light, Gnomon and the Shadow

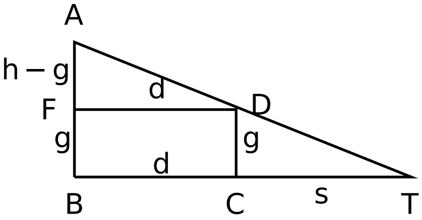


Figure: 24: Lamp and Shadow of a Gnomon

AB : Light $= h$, CD : Gnomon $= g$. Distance between the foot of the light and the bottom of the gnomon $= d$. CT Shadow $= s$.

$$\text{Clearly,} \quad \frac{CD}{CT} = \frac{AF}{FD} \quad \text{or} \quad \frac{g}{s} = \frac{h-g}{d}$$

$$\text{Hence, Shadow}, s = \frac{d \times g}{h-g}, \text{ as stated.}$$

Shadows at two different positions

Next Problem in Verse 54: To find the distance of the foot of the light and the gnomon and the height of the light, given the shadows for two different positions of the gnomon.

छायाग्रान्तरगुणिता छाया छायान्तरेण भक्ता भूः ।
भूः शङ्कुगुणा छायाविभाजिता दीपशिखौच्यम् ॥ ५४ ॥

"The Shadow multiplied by the distance between the tips of the shadows and divided by the difference of the shadows is the base. The base, multiplied by the gnomon, and divided by the shadow, is the height of the flame of the light."

Shadows at two different positions

In the following figure,
C_1, C_2 : Two positions of the gnomon and $C_1C_2 = d$.
$C_1T_1 = s_1$ and $C_2T_2 = s_2$: corresponding shadows.
Distance between tips of shadows: T_1T_2

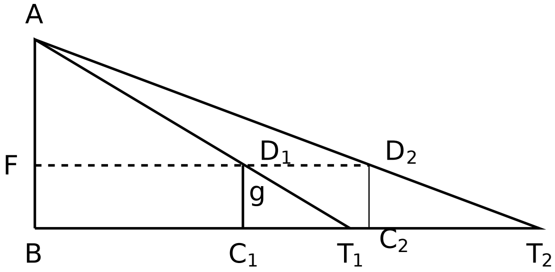
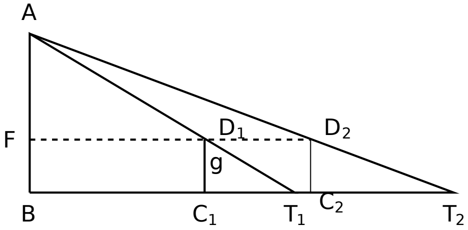


Figure: 25: Shadow of a gnomon at two different positions

Shadows at two different positions

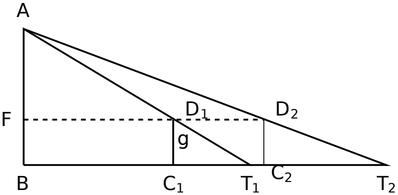


$$\begin{aligned} T_1T_2 &= C_1T_2 - CT_1 \\ &= C_1C_2 - CT_1 + C_2 \\ &= d - s_1 + s_2. \end{aligned}$$

The Base is $BT_1 = x$.
Height of the light
$AB = h$.
$C_1D_1 = C_2D_2 = g$.

Triangles ABT_1 and $D_1C_1T_1$ are similar. Similarly, ABT_2 and $D_2C_2T_2$ are similar.

$$\therefore \frac{h}{x} = \frac{g}{s_1}, \quad \frac{h}{x + T_1T_2} = \frac{g}{s_2}.$$



$$\therefore\ \frac{x + T_1T_2}{x} = \frac{s_2}{s_1}$$

$$\therefore\ \frac{T_1T_2}{x} = \frac{s_2 - s_1}{x}$$

$$\therefore\ x = \frac{s_1 \times T_1T_2}{s_2 - s_1}$$

$$\therefore\ \text{Base} = \text{Shadow} \times \frac{\text{Dist. between tips of shadows}}{\text{Difference of Shadows}},$$

as stated. We also have,

$$h = \frac{x \times g}{s_1} = \frac{\text{Base} \times \text{Gnomon}}{\text{Shadow}},$$

as stated.

References

1.H.T.Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhaskara*, London 1817; Rep. Sharada Publishing House, New Delhi, 2006.

2.*Brāhmasphuṭasiddhānta* of Brahmagupta, edited with S.Dvivedi's commentary, *Vāsanā*, and Hindi translation by R.S.Sharma in 4 Vols., Indian Institute of Astronomical and Sanskrit Research, New Delhi, 1966.