NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 37

Proofs in Indian Mathematics - Part 2

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Outline

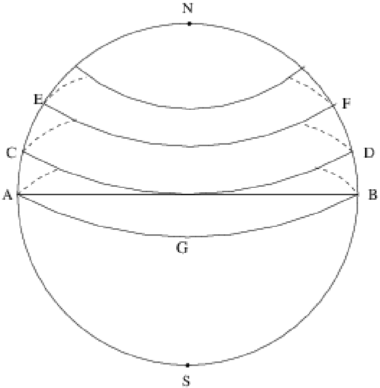
Proofs in Indian Mathematics - Part 2

- Volume of a sphere
  - The principle involved
  - Area of a circle
  - Volume of a slice of a given thickness
  - Total volume
- A couple of theorems
  - *Jyā-saṃvarga-nyāya* (Theorem on product of chords)
  - *Jyā-vargāntara-nyāya* (diff. of the squares of chords)
  - *Jyā-saṃvarga-nyāya* → *Jyotpatti*
- The cyclic quadrilateral
  - Expressing diagonals in terms of sides
  - Area in terms of sides
  - Circumradius in terms of sides

Volume of a sphere

The principle involved

- To find the volume of a sphere, it is divided into large number (say n) of slices of equal thickness.
- In the figure the sphere *NASBN* is divided into slices by planes parallel to the equatorial plane *AGB*.



- Then the volume of each slice is obtained.
  - Volume $=$ Area $\times$ thickess.
  - Area is obtained by finding the average radius of circles at the top and bottom of the slice.
  - If d is diameter, then the thickess of the slice $= \frac{d}{n}$.
- Thus, first we need to obtain an expression for finding the the area of a circle.
- Sum up the elementary volumes of the slices, that will add up to the sphere.

Volume of a sphere

Obtaining the area of a circular slice

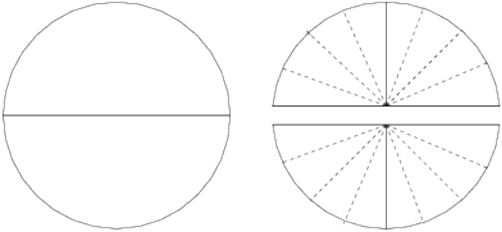
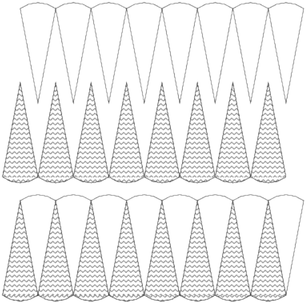


Figure: Circular slice cut into two pieces.



- In the figure above we have indicated a circular slice being turned into a rectangle by appropriately sectioning it, and inserting one half of the circular slice into the other.
- The length of this rectangular strip corresponds to half the circumference C. If the radius is r', then the area of this slice is

$$Area = \frac{1}{2}\,C \times r'. \tag{1}$$

Volume of a sphere

Obtaining the elementary volume of the sphere (= volume of the circular slice

- Let 'r' be the radius of the sphere and C the circumference of a great circle on this sphere..
- The radius of the j-th slice—into which the sphere has been divided into—can be conceived of as the half-chord B_j (*bhujā*).
- Now, the circumference of this slice is given by

$$C_{jth\ slice} = \left(\frac{C}{r}\right) B_j$$

- Hence the area of this circular slice is

$$A_{jth\ slice} = \frac{1}{2} \times \left(\frac{C}{r}\right) B_j \times B_j$$

.

- Therefore, the elementary volume is given by

$$\Delta V = \frac{1}{2}\left(\frac{C}{r}\right) B_j^2 \times \Delta, \tag{2}$$

where Δ is the thickness of the slices.

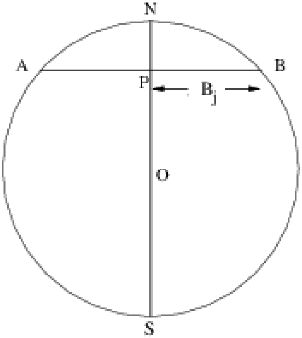
Volume of a sphere

Summing up the elementary volumes of the circular slices)

- The volume of the sphere is obtained by finding the sum of the elementary volumes constituted by the slices:

$$V = \sum \Delta V = \sum_{j=1}^{n} \frac{1}{2} \left(\frac{C}{r} \right) B_j^2 \times \Delta \qquad (3)$$

- In the above expression, the thickness can be expressed in terms of the radius as $\Delta = \frac{2r}{n}$.
- If B_j can also be expressed in terms of r, then we can get $V = V(r)$.
- For this we invoke the *Jyā-śara-saṃvarga-nyāya* given by Āryabhaṭa.



Jyā-śara-saṃvarga-nyāya of Āryabhaṭa

Theorem on the square of chords

In his *Āryabhaṭīya*, Āryabhaṭa has presents the theorem on the product of chords as follows (in half *āryā*):

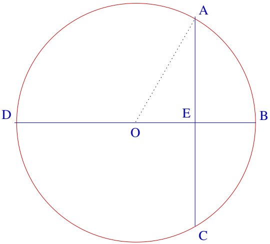
वृत्ते शरसंवर्गः अर्धज्यावर्गः स खलु धनुषोः ॥

(Āryabhaṭīya, Gaṇita 17)

- The words *varga* and q*samvarga* refer to square and product respectively.
- Similarly, *dhanus* and *śara* refer to arc and arrow respectively.

Using modern notations the above *nyāya* may be expressed as:

$$\text{product of } \acute{s}aras = \text{Rsine}^2$$
$$DE \times EB = AE^2$$



Volume of a sphere

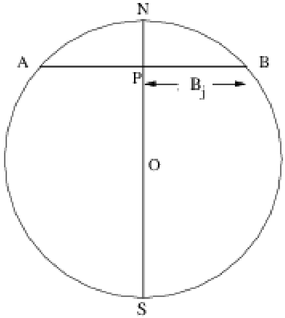
Summing up the elementary volumes of the circular slices)

- It was shown that the volume of the sphere may be expressed as:

$$V = \sum_{j=1}^{n} \frac{1}{2}\left(\frac{C}{r}\right) B_j^2 \times \Delta$$

- In the figure $AP = PB = B_j$ is the jth half-chord, starting from N.
- Applying *jyā-śara-saṃvarga-nyāya*, we have

$$\begin{aligned} B_j^2 &= AP \times PB = NP \times SP \\ &= \frac{1}{2}\left[(NP+SP)^2 - (NP^2+SP^2)\right] \\ &= \frac{1}{2}\left[(2r)^2 - (NP^2+SP^2)\right]. \qquad (4) \end{aligned}$$



- It may be noted in the figure that the j-th Rversine $NP = j\Delta$ and its complement $SP = (n-j)\Delta$. Hence, while summing the squares of the Rsines B_j^2, both NP^2 and SP^2 add to the same result.

Volume of a sphere

Summing up the elementary volumes of the circular slices)

- Thus the expression for the volume of the sphere given by

$$V \approx \sum_{j=1}^{n} \frac{1}{2}\left(\frac{C}{r}\right) B_j^2 \times \Delta,$$

reduces to

$$V \quad \approx \quad \left(\frac{C}{2r}\right)\left[\frac{1}{2}[n.\,(2r)^2 - \left(\frac{2r}{n}\right)^2 2.\,[1^2 + 2^2 + \ldots n^2]\right] \times \left(\frac{2r}{n}\right).$$

- It was known to Kerala mathematicians, that for large n

$$1^2 + 2^2 + \ldots n^2 = \frac{n^3}{3}.$$

- Using this, the expression for the volume of the sphere becomes

$$\begin{aligned} V &= \left(\frac{C}{2r}\right)\left(4r^3 - \frac{8}{3}r^3\right) \\ &= \left(\frac{C}{6}\right) d^2. \qquad (5) \end{aligned}$$

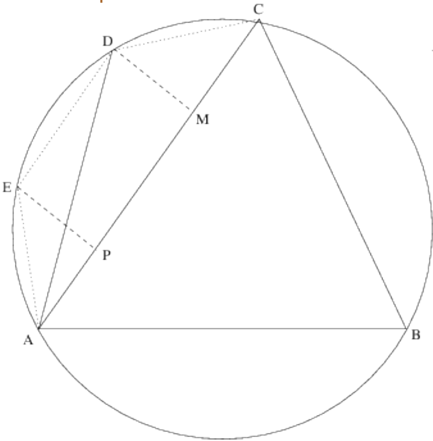
Jyāsaṃvarga-nyāya and *Jyā-vargāntara-nyāya*

Theorem on product of chords and the difference of the square of chords

- In the figure DM is the perpendicular from the vertex D onto the diagonal AC of the cyclic quadrilateral.
- Let us consider the two triangles AMD and CMD, that is formed by DM in the triangle ADC. It can be easily seen that

$$AD^2 - DC^2 = AM^2 - MC^2. \qquad (6)$$

- In other words, the difference in the squares of the *jyās* $AD,\ DC$ is equal to the difference in the square of the base segments (*ābādhās*) $AM,\ MC$.



This result may be noted down for later use as

$$\boxed{\textit{jyāvargāntara} = \textit{ābādhāvargāntara}}$$

Jyāsaṃvarga-nyāya and *Jyā-vargāntara-nyāya*

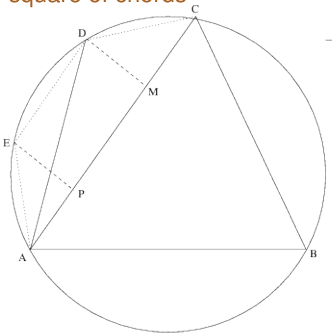
Theorem on product of chords and the difference of the square of chords

- In order to derive a certain property, we rewrite (6) as

$$AD^2 - DC^2 = (AM + MC)(AM - MC) \qquad (7)$$

- It may be noted that

$AM + MC \rightarrow jy\bar{a}$ of sum of the arcs
$AM - MC \rightarrow jy\bar{a}$ of diff. of the arcs



- Denoting the chords AD and DC as J_1 and J_2, and the arcs associated with them as c_1 and c_2, we may rewrite (7) as

$$J_1^2 - J_2^2 = Jy\bar{a}(c_1 + c_2)\ Jy\bar{a}(c_1 - c_2)$$

- In otherwords,

ज्यावर्गान्तरम् = चापद्वययोगवियोगज्यासंवर्गः

Jyāsaṃvarga-nyāya and *Jyā-vargāntara-nyāya*

Theorem on product of chords and the difference of the square of chords

- Recalling the equation,

$$J_1^2 - J_2^2 = Jy\bar{a}(c_1 + c_2)\ Jy\bar{a}(c_1 - c_2) \tag{8}$$

- It can also be shown that

$$J_1\ J_2 = \left[Jy\bar{a}\left(\frac{c_1 + c_2}{2}\right)\right]^2 - \left[Jy\bar{a}\left(\frac{c_1 - c_2}{2}\right)\right]^2 \tag{9}$$

- In otherwords,

ज्यासंवर्ग = चापद्वययोगवियोगार्धज्या-वर्गान्तरम्

- The two equations (8) and (9) are equivalent to the trigonometric relations:

$$\begin{aligned}
\sin^2(\theta_1) - \sin^2(\theta_2) &= \sin(\theta_1 + \theta_2)\sin(\theta_1 - \theta_2), \\
\sin(\theta_1) sin(\theta_2) &= \sin^2\left[\frac{(\theta_1 + \theta_2)}{2}\right] - \sin^2\left[\frac{(\theta_1 - \theta_2)}{2}\right].
\end{aligned}$$

with our convention that $c_1 > c_2$

The cyclic quadrilateral & the third diagonal

- Consider the equation

$$\sin\theta_1 \; sin\theta_2 = \sin^2\left[\frac{(\theta_1+\theta_2)}{2}\right] - \sin^2\left[\frac{(\theta_1-\theta_2)}{2}\right].$$

- If we put $\theta_1 = (n+1)\theta$, and $\theta_2 = (n-1)\theta$ in the above equation, then we immediately obtain the following equation

$$\sin(n+1)\theta = \frac{\sin^2 n\,\theta - \sin^2\theta}{\sin(n-1)\theta}$$

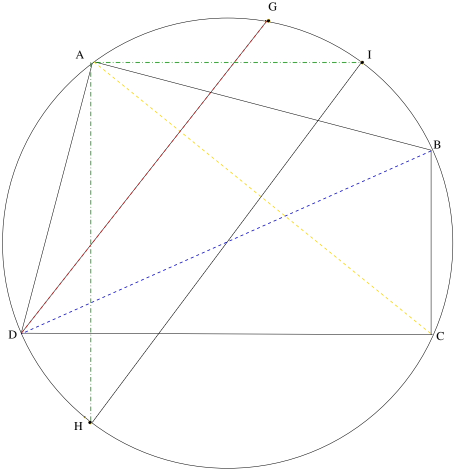
- This is precisely the equation that is presented in the following verse given by Śaṅkara in his *Kriyākramakarī*:

तत्तज्यावर्गम् आद्यज्यावर्गहीनं हरेत् पुनः।
आसन्नाधस्थशिञ्जिन्या लब्धा स्यादुत्तरोत्तरा॥

The cyclic quadrilateral & the third diagonal

- **Notation:**
- the arcs (AB, BC, CD, DA) $\rightarrow c_1, c_2, c_3, c_4$
- the chords (AB, BC, CD, DA) $\rightarrow J_1, J_2, J_3, J_4$
- the diagonals (BD, AC, DG) $\rightarrow K_1, K_2, K_3$

- Having drawn a quadrilateral, obviously we can have only two diagonals.
- G is a point chosen such that $BC = AG$.
- By swapping any two sides – could be adjacent, or opposite – we would be affecting only one of the two diagonal of the quadrilateral, while the other diagonal remains fixed.
- This 'new' diagonal that gets generated due to this swapping of sides is referred to as the third diagonal or *bhāvikarṇa*



The elegant results we would like to prove

- The three diagonals of the cyclic quadrilateral would be referred to as
  1. इष्टकर्ण (*iṣṭakarṇa*) chosen/first diagonal
  2. इतरकर्ण (*itarakarṇa*) other/second diagonal
  3. भाविकर्ण (*bhāvikarṇa*) future/third diagonal
- The results that we would prove:
  1. इष्टकर्णाश्रितभुजघातैक्यम् = इष्टकर्ण × भाविकर्ण
  2. इतरकर्णाश्रितभुजघातैक्यम् = इतरकर्ण × भाविकर्ण
  3. भुजप्रतिभुजघातैक्यम् = इष्टकर्ण × इतरकर्ण
- The above results essentially express the product of the diagonals in terms of the sum of the product of the sides *jyās*.
- Making use of them we express the diagonals in terms of sides.
- Then by making use of yet another result

$$\frac{\text{product of two sides of a triangle}}{\text{circum-diameter}} = \text{the altitude,} \tag{10}$$

we show that the area can be expressed in terms of the diagonals and in turn, in terms of the sides.

Thanks!

THANK YOU