NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 18

Development of Combinatorics 1

M. D. Srinivas

Centre for Policy Studies, Chennai

Outline

- Combinatorics in *Āyurveda*
- *Gandhayukti* of Varāhamihira
- *Mātrā-vṛttas* or moric metres
  - *Prastāra*: Enumeration of metres of n-*mātrās* in the form of an array
  - *Saṅkhyā*: The total number of metrical forms of given number of *mātrās* – the Virahāṅka sequence (so called Fibonacci sequence)
  - *Naṣṭa* and *Uddiṣṭa* processes: Finding the metrical form given the row-number and vice versa in a *prastāra*.
  - *Mātrā-meru*: To determine the number of metrical forms with a given number of *gurus*.
  - Representation of any number as a sum of Virahāṅka numbers.

Combinatorics in *Āyurveda*

The ancient Indian medical treatises of Caraka and Suśruta (prior to 500 BCE) deal with certain combinatoric questions in relation to the six *rasas* and the three *doṣas*. For instance, Caraka (*Sūtrasthāna* Ch. 26) discusses the 63 combinations that are possible from the 6 *rasas*:

भेदश्चैषां त्रिषष्टिविधविकल्पो द्रव्यदेशकालप्रभवाद्भवति तमुपदेक्ष्यामः।

स्वादुरम्लादिभिर्योगं शेषैरम्लादयः पृथक्।
यान्ति पञ्चदशैतानि द्रव्याणि द्विरसानि हि॥
पृथगम्लादियुक्तस्य योगः शेषैः पृथग्भवेत्।
मधुरस्य तथाम्लस्य लवणस्य कटोस्तथा॥
त्रिरसानि यथा सङ्ख्यं द्रव्याण्युक्तानि विंशतिः।
वक्ष्यन्ते च चतुष्केण द्रव्याणि दशपञ्च च॥

Combinatorics in *Āyurveda*

स्वाद्वम्लौ सहितौ योगं लवणाद्यैः पृथग्गतम्।
योगं शेषैः पृथग्यातश्चतुष्करससंख्यया॥
सहितौ स्वादुलवणौ तद्वत्कट्वादिभिः पृथक्।
युक्तौ शेषैः पृथग्योगं यातः स्वादूषणौ तथा॥
कट्वाद्यैरम्ललवणौ संयुक्तौ सहितौ पृथक्।
यातः शेषैः पृथग्योगं शेषैरम्लकटू तथा॥
युज्यते तु कषायेण सतिक्तौ लवणोषणौ।
षट् तु पञ्चरसान्याहुरेकैकस्यापवर्जनात्॥
षट् चैवेकरसानि स्युरेकं षड्रसमेव च।
इति त्रिषष्टिर्द्रव्याणां निर्दिष्टा रससंख्यया॥

Combinatorics in *Āyurveda*

The number of *bheda* or combinations that can be obtained by combining different number of *rasas* is 63 as seen below:

The number of combinations of 2 *rasas* selected from the 6 is 15

The number of combinations of 3 *rasas* selected from the 6 is 20

The number of combinations of 4 *rasas* selected from the 6 is 15

The number of combinations of 5 *rasas* selected from the 6 is 6

The number of combinations of 1 *rasa* selected from the 6 is 6

The number of combinations of 6 *rasas* selected from the 6 is 1

Hence the total number of *bhedas* is 63

This is a particular case of the relation

$$^nC_1 +^n C_2 + \ldots +^n C_n = 2^n - 1$$

The *Suśruta-saṃhitā* (*Uttarasthāna* Ch. 63) actually lists each of these possibilities in a sequential enumeration akin to a *prastāra*.

Gandhayukti of Varāhamihira

Chapter 76 of the great compilation *Bṛhatsaṃhitā* of Varāhamihira (c.550) is devoted to a discussion of perfumery. In verse 20, Varāha mentions that there are 1,820 combinations which can be formed by choosing 4 perfumes from a set of 16 basic perfumes (${}^{16}C_4 = 1820$).

षोडशके द्रव्यगणे चतुर्विकल्पेन भिद्यमानानाम्।
अष्टादश जायन्ते शतानि सहितानि विंशत्या॥

In verse 22, Varāha gives the method of construction of a *meru* (or a tabular figure) which may be used to calculate the number of combinations. This verse also very briefly indicates a way of arranging these combinations in an array or a *prastāra*.

पूर्वेण पूर्वेण गतेन युक्तं स्थानं विनान्त्यं प्रवदन्ति सङ्ख्याम्।
इच्छाविकल्पैः क्रमशोऽभिनीय नीते निवृत्तिः पुनरन्यनीतिः॥

Bhaṭṭotpala (c.950) in his commentary has explained both the construction of the *meru* and the method of *loṣṭa-prastāra* of the combinations.

16			
15	120		
14	105	560	
13	91	455	1820
12	78	364	1365
11	66	286	1001
10	55	220	715
9	45	165	495
8	36	120	330
7	28	84	210
6	21	56	126
5	15	35	70
4	10	20	35
3	6	10	15
2	3	4	5
1	1	1	1

Gandhayukti of Varāhamihira

- In the first column the natural numbers are written.
- In the second column, their sums, in the third the sums of sums, and so on. One row is reduced at each step.
- The top entry in each column gives the number of combinations.

The above *meru* is based on the relation

$$^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + \ldots + {}^{r-1}C_{r-1}$$

which is equivalent to Piṅgala's relation

$$^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

Mātrā-vṛtta

- By assigning the values 1 for *laghu* and 2 for *guru*, we can obtain the total value or *mātrās*, associated with each metrical form.
- This leads to the notion of *mātrā-vṛtttas* or moric metres, where the metrical patterns are classified by their total value or *mātrā*.
- Piṅgala has only briefly touched upon *mātrā-vṛttas* in Chapter IV of *Chandaḥ-śāstra* while discussing the various forms of *Āryā* and *Vaitālīya vṛttas*.

Mātrā-vṛtta

- *Mātrā-vṛttas* are more commonly met with in *Prākṛta* and regional languages.
- The *Prākṛta* work *Vṛttajāti-samuccaya* of Virahāṅka (c.600) discusses the *pratyayas* of *prastāra* and *saṅkhyā* for *mātrā-vṛttas*.
- A more detailed discussion of *mātrā-vṛttas* is available in *Chandonuśāsana* of Hemacandra (c.1200), *Prākṛta-Paiṅgala*, *Vāṇībhūṣaṇa* of Dāmodara (c.1500) and the commentary of Nārāyaṇabhaṭṭa (c.1550) on *Vṛttaratnākara* of Kedāra (c.1000).

Mātrā-Prastāra

एष एव प्रस्तारो मात्रावृत्तानां साधितः किन्तु।
मात्रा यत्र न पूर्यते प्रथमं स्पर्शं तत्र देहि॥

(वृत्तजातिसमुच्चयः ६.२०)

The *prastāra* in the case of a *mātrā-vṛtta* of *n*-*mātrās* is to be generated following the same procedure as in the case of a *varṇa-vṛtta* except for the following:

- The first row consists of all Gs if *n* is even and an L followed by all Gs if *n* is odd.
- [Given any row in the *prastāra*, to generate the next, scan from the left to identify the first G. Place an L below that. The elements to the right are brought down as they are.]
- The remaining *mātrās* to the left are filled in by all Gs, and by placing an L at the beginning, if need be, to keep the total number of *mātrās* the same.

Mātrā-Prastāra

1-*Mātrā Prastāra*

1	L

2-*Mātrā Prastāra*

1		G
2	L	L

3-*Mātrā Prastāra*

1		L	G
2		G	L
3	L	L	L

Mātrā-Prastāra

4-*Mātrā Prastāra*

1			G	G
2		L	L	G
3		L	G	L
4		G	L	L
5	L	L	L	L

5-*Mātrā Prastāra*

1			L	G	G
2			G	L	G
3		L	L	L	G
4			G	G	L
5		L	L	G	L
6		L	G	L	L
7		G	L	L	L
8	L	L	L	L	L

Mātrā-Prastāra

6-*Mātrā Prastāra*

1				G	G	G
2			L	L	G	G
3			L	G	L	G
4			G	L	L	G
5		L	L	L	L	G
6			L	G	G	L
7			G	L	G	L
8		L	L	L	G	L
9			G	G	L	L
10		L	L	G	L	L
11		L	G	L	L	L
12		G	L	L	L	L
13	L	L	L	L	L	L

द्वौ द्वौ पूर्वविकल्पौ या मेलयित्वा जायते सङ्ख्या।
सा उत्तरमात्राणां सङ्ख्याया एष निर्देशः॥

(वृत्तजातिसमुच्चयः ६.४१)

The number of metrical forms S_n in the n-*mātrā prastāra* is the sum of the number of metrical forms S_{n-1}, S_{n-2}, in the *prastāras* of $(n-1)$ and $(n-2)$ *mātrās* respectively:

$$S_n = S_{n-1} + S_{n-2}$$

The above rule follows from the fact that the n-*mātrā prastāra* is generated as follows:

The first S_{n-2} rows are obtained by adding a G to the right of each row of the *prastāra* of $(n-2)$-*mātrās*.

The next S_{n-1} rows are obtained by adding an L to the right of each row of the *prastāra* of $(n-1)$-*mātrās*.

The *prastāra* of one *mātrā* has only one metrical form and that of 2-*mātrās* has just two metrical forms (G and LL).

$S_1 = 1$ and $S_2 = 2$

Therefore we get the following Virahāṅka sequence of *saṅkhyāṅkas* —the so called Fibonacci (c.1200) sequence.

n	1	2	3	4	5	6	7	8	9	10
S_n	1	2	3	5	8	13	21	34	55	89

Uddiṣṭa

दत्वा पूर्वयुगाङ्कं गुरुशीर्षाङ्कं विलुप्य शेषाङ्के।
अङ्कैरितोऽवशिष्टैः शिष्टैरुद्दिष्टमुद्दिष्टम्॥

(वाणीभूषणम् १.३१)

Given a moric metric form, write down from the left the *saṅkhyāṅkas* sequentially in the following way:

- For each G write two *saṅkhyāṅkas*, the first above and the next below.
- For each L, write one *saṅkhyāṅka* above.
- Add all the *saṅkhyāṅkas* above the Gs. This sum subtracted from the total *saṅkhyā* associated with the *mātrā-vṛtta* gives the row number.

Example: To find the row-number of the metrical form GLLG in the 6-*Mātrā-prastāra*:

1	3	5	**8**
G	L	L	G
2			13

The row number is $13 - (1 + 8) = 4$

Naṣṭa

नष्टे कृत्वा कलाः सर्वाः पूर्वयुग्माङ्कयोजिताः।
पृष्टाङ्कहीनशेषाङ्कं येन येनैव पूर्यते॥
परां कलामुपादाय तत्र तत्र गुरुर्भवेत्।
मात्राया नष्टमेतत्तु फणिराजेन भाषितम्॥ (वाणीभूषणम् १.३२-३३)

To find the moric metric form associated with a given row-number in the n-*mātrā-prastāra*

- Write down n Ls with the sequence of *saṅkhyāṅkas* above them.
- Subtract the given row number from the *saṅkhyāṅka* S_n.
- From the result, subtract S_{n-1} if possible. Otherwise, subtract S_{n-2} and so on till the end.
- The moric metric form is obtained by converting each L below a *saṅkhyāṅka* which has been subtracted, together with the L to the right of it, into a G.

Naṣṭa

Example: To find the seventh metrical form in the 6-*mātrā-prastāra*

1	2	3	**5**	8	13
L	L	L	**L**	L	L

- 13-7 = 6.
- 8 cannot be subtracted form 6
- 6-5 = 1
- 3, 2 cannot be subtracted from 1
- 1-1=0

Thus 5 and 1 are the *saṅkhyāṅkas* which have been subtracted.

Hence the metric form is GLGL

Virahāṅka Representation of Numbers

The above examples of *naṣṭa* and *uddiṣṭa* are based upon the following representations of the numbers 9,6 as sums of Virahāṅka numbers.

$$9 = 1 + 8$$

$$6 = 1 + 5$$

In fact it can be shown the *naṣṭa* and *uddiṣṭa* processes are based on the following very interesting property:

Every integer is either a Virahāṅka number or can be expressed uniquely as a sum of non-consecutive Virahāṅka (or the so called Fibonacci) numbers.

Mātrā-Meru

The number of metric forms with different number of *laghus* (*gurus*) can be found from the following table known as the *Mātrā-meru*:

द्वयं द्वयं समं कोष्ठं कृत्वान्त्येष्वेकमर्पयेत्।
एकद्विकत्रिकचतुःक्रमेण प्रथमेष्वपि॥
शीर्षाङ्कात्तपराङ्काभ्यां शेषकोष्ठान् प्रपूरयेत्।
मात्रामेरुरयं दुर्गः सर्वेषामतिदुर्गमः॥ (वाणीभूषणम् १.३७-३८)

The successive rows of the *mātrā-meru* have 1, 2, 2, 3, 3, etc cells.

Place 1 in the top row and in the end of each row.

Place 1 in the beginning of all the even rows and 2, 3, 4 etc in the beginning of odd rows.

The other cells are filled by the sum of the number in the row above which is above and the number in the row further above which is to the right.

Mātrā-Meru

						1	$S_1 = 1$
					1	1	$S_2 = 2$
					2	1	$S_3 = 3$
				1	3	1	$S_4 = 5$
				3	4	1	$S_5 = 8$
			1	6	5	1	$S_6 = 13$
			4	10	6	1	$S_7 = 21$
		1	10	15	7	1	$S_8 = 34$
		5	20	21	8	1	$S_9 = 55$
	1	15	35	28	9	1	$S_{10} = 89$
	6	35	56	36	10	1	$S_{11} = 144$
1	21	70	84	45	11	1	$S_{12} = 233$
7	56	126	120	55	12	1	$S_{13} = 377$

From the above *meru* we can see that, in the 6-*mātrā-prastāra*, there is 1 metrical form with 0L, 6 with 2L, 5 with 4L and 1 with 6L. We can also infer that in the same *prastāra* there is one form with 3Gs, 6 with 2Gs, 5 with 1G and 1 with 0G.

Mātrā-Meru

In the n-*mātrā-vṛtta-prastāra*, if a metrical form has i *gurus*, then it will have $(n-2i)$ *laghus*.

Hence the metrical forms which have i *gurus* will all have a total of $(n-i)$ syllables, of which i will be *gurus*.

Therefore, it follows that G_n^i the number of metrical forms with i Gs, in the n-*mātrā-vṛtta prastāra*, is given by:

$$G_n^i = {}^{(n-i)}C_i = \frac{(n-i)!}{(i)!(n-2i)!} \text{ for } 0 \leq i \leq \frac{n}{2}$$

and

$$G_n^i = 0 \text{ otherwise.}$$

Note that G_n^i satisty the recurrence relation:

$$G_n^i = G_{n-1}^i + G_{n-2}^{i-1}$$

References

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Thanks!

Thank You