NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 23

Bījagaṇita of **Bhāskarācārya** 1

M. D. Srinivas
Centre for Policy Studies, Chennai

Outline

- Development of *Bījagaṇita* or *Avyaktagaṇita* (Algebra) and Bhāskara's treatise on it.
- Understanding of negative quantities.
- Development of algebraic notation.
- The *Vargaprakṛti* equation $X^2 - D\,Y^2 = K$, and Brahmagupta's *bhāvanā* process
- The *Cakravāla* method of solution of Jayadeva and Bhāskara.
- Bhāskara's examples $X^2 - 61\,Y^2 = 1,\ X^2 - 67\,Y^2 = 1$
- The equation $X^2 - DY^2 = -1$.
- Solution of general quadratic indeterminate equations
- Bhāskara's solution of a biquadratic equation.

Development of *Bījagaṇita*

- The notion of a variable quantity, *yāvat-tāvat* (as many as), goes back to *Śulvasūtras*. The *Kātyāyana-Śulvasūtra* deals with the problem of constructing a square whose area is *n*-times that of a given square
- *Āryabhaṭīya* (c.499) uses the term *gulikā* for the unknown. There, we also find the solution of linear and quadratic equations as well as the *kuṭṭaka* process for the solution of linear indeterminate equations.
- Bhāskara I (c.629) uses the notion of *yāvat-tāvat* in his commentary of *Āryabhaṭīya*.
- Brahmagupta has given a detailed exposition of *Bījagaṇita* in the Chapter XVIII, *Kuṭṭakādhyāya*, of his *Brāhmasphuṭasiddhānta* (c.628). This work has been commented upon by Pṛthūdakasvāmi (c.860).
- Śrīpati deals with *avyakta-gaṇita* in Chapter XIV of his *Siddhāntaśekhara* (c.1050)

Development of *Bījagaṇita*

- Bhāskarācārya II has written the most detailed available treatise on *Bījagaṇita* (1150), where he states that he has only compiled and abridged from the treatises of Śrīdhara (c.750) and Padmanābha, which are not available to us.
- *Bījagaṇita* has about 100 verses giving various rules and about 110 verses givng various examples. Bhāskara has written his own commentary *Vāsanā*, which gives details of solutions of the various examples.
- *Bījagaṇita* has also been commented upon by Sūryadāsa (c.1540) and Kṛṣṇa Daivajña (c.1600). The commentary *Bījapallava* or *Bījanavāṅkurā* of Kṛṣṇa provides detailed proofs (*upapattis*) of various results presented in *Bījagaṇita*.
- Nārāyaṇa Paṇḍita (c.1350) has also composed a treatise, *Bījagaṇitāvataṃsa*, of which only the first few chapters are available.

Bījagaṇita of Bhāskarācārya II (c.1150)

The following are the topics dealt with in the *Bījagaṇita* of Bhāskarācārya:

- Six operations with positive and negative numbers (*dhanarṇa-ṣaḍvidha*)
- Arithmetic of zero (*kha-ṣaḍvidha*)
- Six operations with unknowns (*avyakta-ṣaḍvidha*)
- Arithmetic of surds (*karaṇī-ṣaḍvidha*)
- Linear indeterminate equations (*kuṭṭaka*)
- Second order indeterminate equation $x^2 - D\,y^2 = 1$ (*vargaprakṛti*)
- The cyclic method (*cakravāla*)

Bījagaṇita of Bhāskarācārya II

- Equations with single unknown (*ekavarṇa-samīkaraṇa*)
- Elimination of middle term in quadratic equations (*madhyamāharaṇa*)
- Equations with several unknowns (*anekavarṇa-samīkaraṇa*)
- Elimination of middle term in equations with several unknowns (*anekavarṇa-madhyamāharaṇa*)
- Equations with products of unknowns (*bhāvita*)

According to Bhāskarācārya, the topics dealt with in the first few chapters, upto *cakravāla*, are said to be *"bījopayogi"* —useful for the consideration of equations that are to follow.

Invocation to *Avyakta-Gaṇita*

Bījagaṇita or *avyakta-gaṇita*, is computation with seeds, or computation with unmanifest or unknown quantities, which are usually denoted by *varṇas*, colours or symbols.

The following invocatory verse of *Bījagaṇita* of Bhāskarācārya has been interpreted in three different ways by Kṛṣṇa Daivajña:

उत्पादकं यत्प्रवदन्ति बुद्धेरधिष्ठितं सत्पुरुषेण साङ्ख्याः।
व्यक्तस्य कृत्स्नस्य तदेकबीजमव्यक्तमीशं गणितं च वन्दे॥

तदव्यक्तम् ईशं गणितं च वन्दे।...अव्यक्तं प्रधानम्। साङ्ख्यशास्त्रे जगत्कारणतया प्रसिद्धम् ।ईशं सच्चिदानान्दरूपं वेदान्तवेद्यम्। गणितमव्यक्तमेव।

Meaning I: I salute that *avyakta* (*prakṛti* or primordial nature), which the philosophers of the *Sāṅkhya* School declare to be the producer of *buddhi* (the intellectual principle *mahat*), while it is being directed by the immanent *Puruṣa* (the Being). It is the sole *bīja* (seed or the cause) of all that is manifest.

Invocation to *Avyakta-Gaṇita*

Meaning II: I salute that *Īśa* (the ruling power, *Brahman*), which the *Sāṅkhyas* (those who have realised the Self) declare to be the producer of *buddhi* (*tattvajñāna* or true knowledge of reality), which arises in a distinguished person (who has accomplished the four-fold *sādhanas* of *viveka*, etc.). It is the sole *bīja* (seed or the cause) of all that is manifest.

Meaning III: I salute that *avyakta-gaṇita* (computations with unmanifest or indeterminate quantities), which the *Sāṅkhyas* (who are proficient in numbers) declare to be the producer of *buddhi* (mathematical knowledge), which arises in a distinguished person (proficient in mathematics). It is the sole *bīja* (seed or the cause) of all *vyakta-gaṇita* (computations with manifest quantities, such as arithmetic, geometry etc.)

Kṛṣṇa on the Notion of Negative Quantities

The verse of *Bījagaṇita*

संशोध्यमानं स्वमृणत्वमेति स्वत्वं क्षयः तद्युतिरुक्तवच्च॥

gives the rule of signs that, in subtraction, a negative quantity becomes positive and vice versa.

Kṛṣṇa Daivajña, in his commentary explains how negativity is to be understood in different contexts. He then goes on to show that this physical interpretation of negativity can be used to demonstrate the rule of signs in algebra in different situations.

Kṛṣṇa on the Notion of Negative Quantities

ऋणत्वमिह त्रिधा तावदस्ति देशतः कालतः वस्तुतश्चेति ।तच्च वैपरीत्यमेव। यत उक्तमाचार्यैर्लीलावत्यां क्षेत्रव्यवहारे “दशसप्तदशप्रमौ भुजौ …” इत्यस्मिन्नुदाहरणे। ऋणगता आबाधा दिग्वैपरीत्येनेत्यर्थ इति। तत्रैकरेखास्थिता द्वितीया दिक् विपरीता दिगित्युच्यते। यथा पूर्वविपरीता पश्चिमा दिक्। यथा वोत्तरदिग्विपरीता दक्षिणा दिगित्यादि। तथा च पूर्वापरदेशयोर्मध्ये एकतरस्य धनत्वे कल्पिते तं प्रति तदितरस्य ऋणत्वम्।…

एवं पूर्वोत्तरकालयोरन्योन्यम् ऋणत्वं वारप्रवृत्यादिषु प्रसिद्धम्। एवं यस्मिन् वस्तुनि यस्य स्वस्वामिभावः सम्बन्धः तस्य तद्धनमिति व्यवह्रियते। तस्मिन् वैपरीत्यं तु परस्य स्वस्वामिभावः सम्बन्धः।

Kṛṣṇa on the Notion of Negative Quantities

"Negativity (*ṛṇatva*) here is of three types: Spatial, temporal and that pertaining to objects. In each case, [negativity] is indeed the *vaiparītya* or the oppositeness. As has been clearly stated by the Ācārya in *Līlāvatī* in the example 'The *bhujās* are ten and seventeen etc.' The negative base intercept (*ābādhā*) is to be understood to be in the opposite direction. There, the other direction in the same line is called the opposite direction (*viparītā dik*); just as west is the opposite of east... Further, between two stations, if one way of traversing is considered positive, then the other is negative...

In the same way past and future time intervals will be mutually negative of each other... Similarly, when one possesses the said objects they would be called his *dhana* (wealth). The opposite would be the case when another owns the same objects..."

Bhāskara's Example of Negative Intercept

The example discussed by Bhāskara in *Līlāvatī* has to do with the calculation of the base-intercepts (*ābādhās*) in a triangle. There, Bhāskara explains that if the calculation leads to a negative intercept, it should be interpreted as going in the opposite direction. This happens when the foot of the altitude falls outside the base.

दशसप्तदशप्रमौ भुजौ त्रिभुजे यत्र नवप्रमा मही।
अबधे वद लम्बकं तथा गणितं गाणितिकाऽऽशु तत्र मे॥

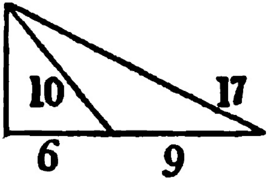
"In a triangle, which has sides 10, 17 and 9, tell me quickly, Oh mathematician, the base intercepts and the area."

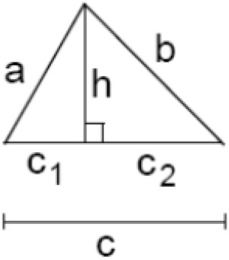
Bhāskara’s Example of Negative Intercept

In his *Vāsanā*, Bhāskara gives the solution of this problem as follows

> अत्र त्रिभुजे भुजयोर्योग इत्यादिना लब्धम् २१।
> अनेन भूरुना न स्यात्। तस्मादेव भूरपनीता शेषार्धं
> ऋणगताऽऽबाधा दिग्वैपरीत्येनेत्यर्थः। तथा जाते आबाधे ६•,
> १५।
>
> Here, using the rule, “The sum of the sides...”, we get (the difference of the intercepts to be) 21. We cannot subtract this from the base (9). Hence, the base has to be subtracted from this (difference) only and the half of the result is the intercept which is negative, and it is to be understood to be in the opposite direction. Thus the intercepts are (-6, 15).

Bhāskara's Example of Negative Intercept





In a triangle with sides a, b, and base c, let the base intercepts be $c_1,\ c_2$. Now $(c_1 + c_2) = c$ and $(b^2 - a^2) = (c_2^2 - c_1^2)$. Therefore,

$$c_2 - c_1 = \frac{(b^2 - a^2)}{(c_1 + c_2)} = \frac{(b^2 - a^2)}{c}$$

Thus, in our problem,

$$c_1 + c_2 = 9 \ \text{ and } \ c_2 - c_1 = \frac{(17^2 - 10^2)}{9} = 21$$

Hence $c_2 = 15$ and $c_1 = -\frac{(21-9)}{2} = -6$

Algebraic Notation

In the *Āryabhaṭīyabhāṣya* of Bhāskara I (c.629) we find references to the algebraic notation used in Indian mathematics. Various features of the notation are more clearly known from the *Bakhshālī* manuscript (c. 700).

The system of algebraic notation is explained and fully exemplified in the *Bījagaṇita* of Bhāskara II (c.1150), together with his own auto-commentary *Vāsanā*.

In *Vāsanā* on verse 3 of *Bījagaṇita*, Bhāskara says:

> अत्र रूपाणामव्यक्तानां चाद्याक्षराण्युपलक्षणार्थं लेख्यानि।
> तथा यान्यूनगतानि तान्यूर्ध्वबिन्दूनि च।
>
> Here (in algebra), the initial letters of both the known and unknown quantities should be written as their signs. Similarly those (quantities) which are negative, they have (to be shown with) a dot over them.

Algebraic Notation

- The symbols used for unknown numbers are the initial syllables $y\bar{a}$ of $y\bar{a}vat\text{-}t\bar{a}vat$ (as much as), $k\bar{a}$ of $k\bar{a}laka$ (black), $n\bar{\imath}$ of $n\bar{\imath}laka$ (blue), $p\bar{\imath}$ of $p\bar{\imath}ta$ (yellow) etc.
- The product of two unknowns is denoted by the initial syllable $bh\bar{a}$ of $bh\bar{a}vita$ (product) placed after them. The powers are denoted by the initial letters va of $varga$ (square), gha of $ghana$ (cube); $vava$ stands for $vargavarga$, the fourth power. Sometimes the initial syllable $gh\bar{a}$ of $gh\bar{a}ta$ (product) stands for the sum of powers.
- A coefficient is placed next to the symbol. The constant term is denoted by the initial symbol $r\bar{u}$ of $r\bar{u}pa$ (form).
- A dot is placed above the negative integers.
- The two sides of an equation are placed one below the other. Thus the equation $x^4 - 2x^2 - 400x = 9999$, is written as:

यावव	१	याव	२•	या	४००•	रू	०
यावव	०	याव	०	या	०	रू	९९९९

Vargaprakṛti

In the *Kuṭṭakādhyāya* of his *Brāhmasphuṭasiddhānta* (c.628), Brahmagupta considered the problem of solving for integral values of $X, \; Y,$ the equation

$$\mathbf{X^2 - D\, Y^2 = K}$$

given a non-square integer $D > 0$, and an integer K.

X is called the *jyeṣṭha-mūla*, Y is called the *kaniṣtha-mūla*

D is the *prakṛti*, K is the *kṣepa*

One motivation for this problem is that of finding rational approximations to square-root of D. If X, Y are integers such that $X^2 - D\, Y^2 = 1$, then,

$$\left| \sqrt{D} - \left(\frac{X}{Y}\right) \right| \leq \frac{1}{2XY} < \frac{1}{2Y^2}$$

The *Śulva-sūtra* approximation $\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} = \frac{577}{408}$ is an example, as $(577)^2 - 2(408)^2 = 1$.

Brahmagupta's *Bhāvanā*

मूलं द्विधेष्टवर्गाद् गुणकगुणादिष्टयुतविहीनाच्च।
आद्यवधो गुणकगुणः सहान्त्यघातेन कृतमन्त्यम्॥
वज्रवधैक्यं प्रथमं प्रक्षेपः क्षेपवधतुल्यः।
प्रक्षेपशोधकहृते मूले प्रक्षेपके रूपे॥

If $\mathbf{X_1^2 - D\,Y_1^2 = K_1}$ and $\mathbf{X_2^2 - D\,Y_2^2 = K_2}$ then
$\mathbf{(X_1X_2 \pm DY_1Y_2)^2 - D(X_1Y_2 \pm X_2Y_1)^2 = K_1K_2}$

In particular, given $X^2 - D\,Y^2 = K$, we get the rational solution

$$\left[\frac{(X^2 + D\,Y^2)}{K}\right]^2 - D\left[\frac{(2XY)}{K}\right]^2 = 1$$

Also, if one solution of the equation $X^2 - D\,Y^2 = 1$ is found, an infinite number of solutions can be found, via

$$(X, Y) \to (X^2 + D\,Y^2, 2XY)$$

Use of $Bh\bar{a}van\bar{a}$ when $K = -1, \pm 2, \pm 4$

The $bh\bar{a}van\bar{a}$ principle can be used to obtain a solution of equation

$$x^2 - D\,y^2 = 1,$$

if we have a solution of the equation

$$x_1^2 - D\,y_1^2 = K, \text{ for } K = -1, \pm 2, \pm 4.$$

$$\begin{aligned}
K = -1\ : x &= x_1^2 + Dy_1^2,\ y = 2x_1y_1.\\
K = \pm 2 : x &= \frac{(x_1^2 + Dy_1^2)}{2},\ y = x_1y_1.\\
K = -4 : x &= (x_1^2 + 2)\left[\frac{1}{2}(x_1^2 + 1)(x_1^2 + 3) - 1\right],\\
y &= \frac{x_1y_1(x_1^2 + 1)(x_1^2 + 3)}{2}.\\
K = 4 : x &= \frac{(x_1^2 - 2)}{2},\ y = \frac{x_1y_1}{2},\ \text{if } x_1 \text{ is even},\\
x &= \frac{x_1(x_1^2 - 3)}{2},\ y = \frac{y_1(x_1^2 - 1)}{2},\ \text{if } x_1 \text{ is odd}.
\end{aligned}$$

Brahmagupta's Examples

राशिकलाशेषकृतिं द्विनवतिगुणितां त्र्यशीतिगुणितां वा।
सैकां ज्ञदिने वर्गं कुर्वन्नावत्सराद्गणकः॥

To solve, $\mathbf{x^2 - D\,y^2 = 1}$, **for** $\mathbf{D = 92, 83}$

$10^2 - 92.1^2 = 8$

Doing the *bhāvanā* of the above with itself,

$192^2 - 92.20^2 = 64 \quad [10^2 + 92.1^2 = 192 \text{ and } 2.10.1 = 20]$

Dividing both sides by 64, we get $24^2 - 92.\left(\frac{5}{2}\right)^2 = 1$

Doing the *bhavanā* of the above with itself,

$\mathbf{1151^2 - 92.120^2 = 1} \quad \left[24^2 + 92.\left(\frac{5}{2}\right)^2 = 1151 \text{ and } 2.24.\left(\frac{5}{2}\right) = 120\right]$

Similarly, $\quad 9^2 - 83.1^2 = -2$

Doing the *bhāvanā* of the above with itself,

$164^2 - 83.18^2 = 4$ and hence, $\mathbf{82^2 - 83.9^2 = 1}$

Cakravāla : The Cycle Method

The first known reference to *Cakravāla* or the Cyclic Method occurs in a work of Udayadivākara (c.1073), who cites the following verses of Ācārya Jayadeva:

ह्रस्वज्येष्ठक्षेपान् प्रतिराश्या क्षेपभक्तयोः क्षेपात्।
कुट्टाकारे च कृते कियद्गुणं क्षेपकं क्षिप्त्वा॥
तावत्कृतेः प्रकृत्या हीने प्रक्षेपकेण संभक्ते।
स्वल्पतरावाप्तिः स्यादित्याकलितोऽपरः क्षेपः॥
प्रक्षिप्तप्रक्षेपककुट्टाकारे कनिष्ठमूलहते।
सज्येष्ठपदे प्रक्षेपकेण लब्धं कनिष्ठपदम्॥
क्षिप्तक्षेपककुट्टागुणितात् तस्मात्कनिष्ठमूलहतम्।
पाश्चात्यं प्रक्षेपं विशोध्य शेषं महन्मूलम्॥

कुर्यात् कुट्टाकारं पुनरनयोः क्षेपभक्तयोः पदयोः।
तत्सेष्टहतक्षेपे सदृशगुणेऽस्मिन् प्रकृतिहीने॥
प्रक्षेपः क्षेपाप्ते प्रक्षिप्तक्षेपकाद्य गुणकारात्।
अल्पघ्नात् सज्येष्ठात् क्षेपावाप्तं कनिष्ठपदम्॥
एतत्क्षिप्तक्षेपककुट्टकघातादनन्तरक्षेपम्।
हित्वाऽल्पहतं शेषं ज्येष्ठं तेभ्यश्च गुणकादि॥
कुर्यात्तावद्यावत् षण्णामेकद्विचतुर्णां पततिं।
इति चक्रवालकरणेऽवसरप्राप्तानि योज्यानि॥

Cakravāla according to Jayadeva

Given $X_i,\ Y_i,\ K_i$ such that $X_i^2 - D\ Y_i^2 = K_i$

First find P_{i+1} as follows:

(I) Use *kuṭṭaka* **process to solve**

$$\frac{(Y_i P_{i+1} + X_i)}{K_i} = Y_{i+1}$$

for integral $P_{i+1},\ Y_{i+1}$

(II) Of the solutions of the above, choose P_{i+1} **such that**

$$\frac{(P_{i+1}^2 - D)}{K_i} \text{ has the least value}$$

Cakravāla according to Jayadeva

Then set

$$K_{i+1} = \frac{(P_{i+1}^2 - D)}{K_i} \qquad Y_{i+1} = \frac{(Y_i\ P_{i+1} + X_i)}{K_i}$$

$$X_{i+1} = P_{i+1} Y_{i+1} - K_{i+1} Y_i$$

These satisfy $X_{i+1}^2 - D\ Y_{i+1}^2 = K_{i+1}$

Iterate the process till $K_{i+1} = \pm 1, \pm 2$ or ± 4, and then solve the equation using *bhāvanā* is necessary.

Jayadeva's verses do not reveal how condition II is to be interpreted.

Cakravalā according to Bhāskara (c.1150)

In his *Bījagaṇita*, Bhāskarācārya gives the following description of *cakravāla*:

ह्रस्वज्येष्ठपदक्षेपान् भाज्यप्रक्षेपभाजकान्।
कृत्वा कल्प्यो गुणस्तत्र तथा प्रकृतितश्च्युते॥
गुणवर्गे प्रकृत्योनेऽथवाल्पं शेषकं यथा।
तत्तु क्षेपहृतं क्षेपो व्यस्तः प्रकृतितश्च्युते॥
गुणलब्धिः पदं ह्रस्वं ततो ज्येष्ठमतोऽसकृत्।
त्यक्त्वा पूर्वपदक्षेपांश्चक्रवालमिदं जगुः॥
चतुर्द्व्येकयुतावेवमभिन्ने भवतः पदे।
चतुर्द्विक्षेपमूलाभ्यां रूपक्षेपार्थभावना॥

Bhāskara has given the Condition II in the precise form:

(II) Choose P_{i+1} such that $|(P_{i+1}^2 - D)|$ has the least value

Cakravāla according to Bhāskara

In 1930, Krishnaswami Ayyangar showed that the *cakravāla* procedure always leads to a solution of the *vargaprakṛti* equation with $K = 1$. He also showed that condition (**I**) is equivalent to the simpler condition

$(\mathbf{I}')$ $\mathbf{P_i + P_{i+1}}$ **is divisible by** $\mathbf{K_i}$

Thus, we use the *cakravāla* algorithm in the following form:

To solve $\mathbf{X^2 - D\ Y^2 = 1}$: Set $X_0 = 1, Y_0 = 0, K_0 = 1,\ P_0 = 0.$

Given X_i, Y_i, K_i such that $X_i^2 - D\ Y_i^2 = K_i$

First find $P_{i+1} > 0$ so as to satisfy:

$(\mathbf{I}')\mathbf{P_i + P_{i+1}}$ **is divisible by** $\mathbf{K_i}$

$(\mathbf{II})|\mathbf{P_{i+1}^2 - D}|$ **is minimum.**

Cakravāla according to Bhāskara

Then set

$$K_{i+1} = \frac{(P_{i+1}^2 - D)}{K_i} \qquad Y_{i+1} = \frac{(Y_i P_{i+1} + X_i)}{|K_i|} = a_i Y_i + \varepsilon_i Y_{i-1}$$

$$X_{i+1} = \frac{(X_i P_{i+1} + D\ Y_i)}{|K_i|} = P_{i+1} Y_{i+1} - sign\,(K_i) K_{i+1} Y_i = a_i X_i + \varepsilon_i X_{i-1}$$

These satisfy $X_{i+1}^2 - D\ Y_{i+1}^2 = K_{i+1}$

Iterate till $K_{i+1} = \pm 1, \pm 2$ or ± 4, and then use *bhāvanā* if necessary.

Note: We also need $\mathbf{a_i} = \frac{(\mathbf{P_i} + \mathbf{P_{i+1}})}{|\mathbf{K_i}|}$ **and** $\varepsilon_{\mathbf{i}} = \frac{(\mathbf{D} - \mathbf{P_i^2})}{|\mathbf{D} - \mathbf{P_i^2}|}$, with $\varepsilon_0 = 1$

Bhāskara's Examples

का सप्तषष्टिगुणिता कृतिरेकयुक्ता का चैकषष्टिनिहता च सखे सरूपा।
स्यान्मूलदा यदि कृतिप्रकृतिर्नितान्तं त्वच्चेतसि प्रवद तात तता लतावत्॥

द्वितीयोदाहरणे न्यासः

प्र ६१ क १ ज्ये ८ क्षे ३ ।

कुट्टार्थं न्यासः भा १ हा ३ क्षे ८।

हरतष्टे घनक्षेपे इति लब्धिगुणौ ३,१। इष्टाहतेति द्वाभ्यामुत्थाप्य जातौ लब्धिगुणौ ५, ७। गुणवर्गे ४९। प्रकृतेः शोधिते १२ व्यस्त इति ऋणम् १२• इदं क्षेपहृतं जातः क्षेपः ४•। अतः प्राग्वज्ज्ञाते चतुःक्षेपमूले क ५ ज्ये ३९।

इष्टवर्गहृतः क्षेप स्यादित्युपपन्नरूपशुद्धिमूलयोर्भावनार्थं न्यासः

क $\frac{5}{2}$ ज्ये $\frac{39}{2}$ क्षे १

क $\frac{5}{2}$ ज्ये $\frac{39}{2}$ क्षे १•

Bhāskara's Example: $X^2 - 61\ Y^2 = 1$

$\mathbf{i}$	$\mathbf{P_i}$	$\mathbf{K_i}$	$\mathbf{a_i}$	$\varepsilon_\mathbf{i}$	$\mathbf{X_i}$	$\mathbf{Y_i}$
0	0	1	8	1	1	0
1	8	3	5	-1	8	1
2	7	-4	4	1	39	5
3	9	-5	-3	-1	164	21

To find $P_1 :\ 0+7,\ 0+8,\ 0+9\ \ldots$ divisible by 1. Of them 8^2 closest to 61. Hence $P_1 = 8,\ K_1 = 3$

To find $P_2 :\ 8+4,\ 8+7,\ 8+10\ \ldots$ divisible by 3. Of them 7^2 closest to 61. Hence $P_2 = 7,\ K_2 = -4$

After the second step, we have: $39^2 - 61.5^2 = -4$

Now, since have reached $K = -4$, we use $bh\bar{a}van\bar{a}$ principle to get

$X = (39^2+2)\left[\left(\frac{1}{2}\right)(39^2+1)(39^2+3) - 1\right] = \mathbf{1,766,319,049}$

$Y = \left(\frac{1}{2}\right)(39.5)(39^2+1)(39^2+3) = \mathbf{226,153,980}$

$1766319049^2 - 61.226153980^2 = 1$

Bhāskara's Example: $X^2 - 61\ Y^2 = 1$

i	P_i	K_i	a_i	ε_i	X_i	Y_i
0	0	1	8	1	1	0
1	8	3	5	-1	8	1
2	7	-4	4	1	39	5
3	9	-5	3	-1	164	21
4	6	5	3	1	453	58
5	9	4	4	-1	1,523	195
6	7	-3	5	1	5,639	722
7	8	-1	16	-1	29,718	3,805
8	8	-3	5	-1	469,849	60,158
9	7	4	4	1	2,319,527	296,985
10	9	5	3	-1	9,747,957	1,248,098
11	6	-5	3	1	26,924,344	3,447,309
12	9	-4	4	-1	90,520,989	11,590,025
13	7	3	5	1	335,159,612	42,912,791
14	8	1	16	-1	**1,766,319,049**	**226,153,980**

Bhāskara's Example: $X^2 - 67\ Y^2 = 1$

i	P_i	K_i	a_i	ε_i	X_i	Y_i
0	0	1	8	1	1	0
1	8	-3	5	1	8	1
2	7	6	2	1	41	5
3	5	-7	2	1	90	11
4	9	-2	9	-1	221	27

To find P_1 : $0+7,\ 0+8,\ 0+9\ldots$ divisible by 1. Of them 8^2 closest to 67. Hence $P_1 = 8,\ K_1 = -3$

To find P_2 : $8+4,\ 8+7,\ 8+10\ldots$ divisible by 3. Of them 7^2 closest to 67. Hence $P_2 = 7,\ K_2 = 6$

To find P_3 : $7+5,\ 7+11,\ 7+17\ldots$ divisible by 6. Of them 5^2 closest to 67. Hence $P_3 = 5,\ K_3 = -7$

To find P_4 : $5+2,\ 5+9,\ 5+16\ldots$ divisible by 7. Of them 9^2 closest to 67. Hence $P_4 = 9,\ K_4 = -2$

Now, since have reached $221^2 - 67.27^2 = -2$, we do *bhāvanā* to get

$$\mathbf{48842^2 - 67.\ 5967^2 = 1}$$

Bhāskara's Example: $X^2 - 67\ Y^2 = 1$

i	P_i	K_i	a_i	ε_i	X_i	Y_i
0	0	1	8	1	1	0
1	8	-3	5	1	8	1
2	7	6	2	1	41	5
3	5	-7	2	1	90	11
4	9	-2	9	-1	221	27
5	9	-7	2	-1	1,899	232
6	5	6	2	1	3,577	437
7	7	-3	5	1	9,053	1,106
8	8	1	16	1	**48,842**	**5,967**

The Equation $X^2 - D\ Y^2 = -1$

Bhāskara states that the equation $X^2 - D\ Y^2 = -1$ cannot be solved unless D is a sum of two squares.

रूपशुद्धौ खिलोद्दिष्टं वर्गयोगो गुणो न चेत्।

Taking $D = m^2 + n^2$ Bhāskara gives two rational solutions

$$(X,\ Y) = \left(\frac{n}{m},\ \frac{1}{m}\right) \text{ and } (X,\ Y) = \left(\frac{m}{n},\ \frac{1}{n}\right)$$

From these, it is sometimes possible to get integral solutions by ingenious use of *bhāvanā* and *cakravāla*, as Bhāskara shows in the case of $X^2 - 13Y^2 = -1$. He obtains $X = 18,\ Y = 5$.

The Equation $X^2 - D\ Y^2 = -1$

While considering the equation $X^2 - 8\ Y^2 = -1$, Bhāskara merely gives the rational solutions $X = 1$, $Y = \frac{1}{2}$.

Here, the commentator Kṛṣṇa Daivajña seems to imply that in this case also we can obtain integral solutions by *bhāvanā* and *cakravāla*.

> प्राग्वच्चक्रवालेनाभिन्ने कार्ये।

This is incorrect as it can be shown that $X^2 - 8Y^2 = -1$ has no integral solutions.

General Quadratic Indeterminate Equations

Bhāskara has shown how the solution of *vargaprakṛti* equation with $K = 1$ can be used in determining solutions of general quadratic indeterminate equations.

We illustrate the method with a couple of examples.

को राशिर्द्विगुणो राशिवर्गैः षड्भिः समन्वितः।
मूलदो जायते बीजगणितज्ञ वदाशु तम्॥

अत्र यावत्तावद्राशिर्द्विगुणो वर्गैः षड्भिः समन्वितः याव ६ या २।
एष वर्ग इति कालकवर्गेण समीकरणार्थं न्यासः

याव	६	या	२	काव	०।
याव	०	या	०	काव	२।

General Quadratic Indeterminate Equations

अत्र समशोधने जातौ पक्षौ

याव ६ या २, काव १।

अथैतौ षड्भिः संगुण्य रूपं प्रक्षिप्य प्राग्वत् प्रथमपक्षमूलम्

या ६ रू १।

अथ द्वितीयपक्षस्यास्य काव ६ रू १। वर्गप्रकृत्या मूले क २ ज्ये ५, वा क २० ज्ये ४९। ज्येष्ठं प्रथमपक्षपदेनानेन, या ६ रू १ समं कृत्वा लब्धं यावत्तावन्मानम् २/३ वा ८। ह्रस्वं प्रकृतिवर्णस्य कालकस्य मानम् २ वा २०।

एवं कनिष्ठज्येष्ठवशाद्बहुधा।

General Quadratic Indeterminate Equations

Here the two sides are $6.x^2 + 2.x + 0.y^2$ and $0.x^2 + 0.x + 1.y^2$

Equating and clearing the sides, the equation is

$$6x^2 + 2x = y^2$$

Multiplying both sides by 6 and adding 1, we get

$$(6x + 1)^2 - 6\ y^2 = 1$$

The solutions of the above equation are, for instance,

$y = 2, 20$ and correspondingly $6x + 1 = 5, 49$ and so on.

In this way we get the integral solutions $x = 8,\ y = 20$.

General Quadratic Indeterminate Equations

त्रिकादिद्व्युत्तरश्रेढ्यां गच्छे क्वापि च यत्फलम्।
तदेव त्रिगुणं कस्मिन्नन्यगच्छे भवेद्वद॥

Say what is the number of terms of a sequence (in AP) whose first term is 3 the common difference is 2; but whose sum multiplied by three is equal to the sum of a different number of terms (of the same AP).

Let x and y be the number of terms. Then

$$3(x^2 + 2x) = y^2 + 2y$$

Multiplying both sides by 3 and adding 9,

$$(3x + 3)^2 = z^2 = 3y^2 + 6y + 9$$

Again multiplying by 3 and subtracting 18

$$3z^2 - 18 = t^2 = (3y + 3)^2$$

We find that a solution is $z = 9$ and $t = 15$.

Other solutions can be found by doing $bh\bar{a}van\bar{a}$ with solutions $(2,\ 1)$ of $t^2 - 3\ z^2 = 1$.

Thus we get $z = 33$ and $t = 57$.

Hence we get $x = 2, y = 4$ and $x = 10, y = 18$, and so on.

Bhāskara's Solution of a Biquadratic

Bhāskara II has given an example of the method of solution of a biquadratic equation of the special form

$$x^4 + px^2 + qx + r = 0$$

which involves adding $ax^2 - qx + b$ to both sides, choosing a and b such that both sides are perfect squares.

This can be done in general, but it could involve solving a cubic equation for a. In his example, Bhāskara seems to have guessed the values of $a, \; b$.

Bhāskara's example is in the *madhyamāharaṇa* section of *Bījagaṇita*:

को राशिर्द्विशतीक्षुण्णो राशिवर्गयुतो हतः।
द्वाभ्यां तेनोनितो राशिवर्गवर्गोऽयुतं भवेत्।
रूपोनं वद तं राशिं वेत्सि बीजक्रियां यदि॥

$$x^4 - 2(x^2 - 200x) = 10000 - 1$$

The way Bhāskara solves this equation is given in his *Vāsanā* commentary:

अत्र राशिः या १। द्विशतीक्षुण्णः २००। राशिवर्गयुतो जातः यावव १ या २०० अयं द्वाभ्यां गुणितः या २ या ४००। अनेनायं यावत् १ राशिवर्गवर्ग ऊनितो जातः यावव १ याव २• या ४००•। अयं रूपोनायुत सम इति समशोधने कृते जातौ पक्षौ

यावव	१	याव	२•	या	४००•	रू	०
यावव	०	याव	०	या	०	रू	९९९९

Bhāskara's Solution of a Biquadratic

अत्राद्यपक्षे किल यावत्तावच्चतुःशतीं रूपाधिकां प्रक्षिप्य मूलं लभ्यते परं तावति क्षिप्ते नान्यपक्षस्य मूलमस्ति एवं क्रिया न निर्वहति। अतोऽत्र स्वबुद्धिः।

इह पक्षयोर्यावत्तावद्वर्गचतुष्टयं यावच्चतुःशतीं रूपं च प्रक्षिप्य मूले

याव	१	या	०	रू	१	।
याव	०	या	२	रू	१००	।

पुनरनयोः समीकरणेन प्राग्वल्लब्धं यावत्तावन्मानम् ११।

इत्यादि बुद्धिमता ज्ञेयम्॥

Bhāskara's Solution of a Biquadratic

Bhāskara first obtains the two sides of the equation in the form

$$x^4 - 2x^2 - 400x = 9999$$

He then remarks that if we add $400x + 1$ to the left side we get a complete square, but the same thing added to the right hand side will not produce one, and hence proceeding in this way we cannot accomplish anything.

Hence, says Bhāskara, here one has to apply one's intellect.

If we add $4x^2 + 400x + 1$ to both sides, we get the roots

$$x^2 + 1 = 2x + 100$$

This can be solved in the usual way to obtain

$$x = 11$$

Bhāskara remarks that this is how the intelligent should attempt such problems.

Krishnaswami Ayyangar has noted that Bhāskara's is indeed the earliest attempt at the solution of a non-trivial biquadratic equation.

References

1. *Bījagaṇita* of Bhāskara with Comm. of Kṛṣṇa Daivajña, Ed. Radhakrishna Sastri, Saraswati Mahal, Thanjavur 1958.
2. *Bījagaṇita* of Bhāskara with *Vāsanā* and Hindi Tr. by Devachandra Jha, Chaukhambha, Varanasi 1983.
3. H. T. Colebrooke, *Algebra with Arthmetic and Mensuration from the Sanskrit of Brahmagupta and Bhāskara*, London 1817.
4. *Bījagaṇita* of Bhāskara with *Vāsanā*, with notes by T. Hayashi, SCIAMVS, 10, December 2009, pp. 3-303.
5. A. A. Krishnaswami Ayyangar, New Light on Bhāskara's *Cakravāla* or Cyclic method, J.Indian Math. Soc. 18, 1929-30, pp.225-248.

References

6. A. A. Krishnaswami Ayyangar, The Earliest Solution of the Biquadratic, Current Sci. Oct 1938, p.179.
7. B. Datta and A. N. Singh, *History of Hindu Mathematics, Vol II, Algebra,* Lahore 1938; Reprint, Asia Publishing House, Bombay 1962.
8. K. S. Shukla, Acārya Jayadeva the Mathematician, *Gaṇita*, 5, 1954, pp.1-20.
9. Sita Sundar Ram, *Bījapallava* of *Kṛṣṇa Daivajña*: *A Critical Study,* KSR Institute, Chennai 2012.

Thanks!

Thank You