NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 27

Gaṇitakaumudī of **Nārāyaṇa Paṇḍita** 3

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Outline

- Outline of *Gaṇitakaumudī*
- *Vargaprakṛti*
  - Nārāyaṇa's variant of *cakravāla* algorithm.
  - Solutions of *Vargaprakṛti* and approximation of square roots
- *Bhāgādāna*: Nārāyaṇa's method of factorisation of numbers.
- *Aṅkapāśa* (Combinatorics)
  - Enumeration (*prastāra*) of generalised *mātrā-vṛttas* (moric metres with syllabic units such as *pluta* etc., in addition to *laghu* and *guru*).
  - Some sequences (*paṅkti*) and tabular figures (*meru*) used in combinatorics.
  - Enumeration (*prastāra*) of permutations with repetitions.
  - Enumeration (*prastāra*) of combinations.

Gaṇitakaumudī of Nārāyaṇa Paṇḍita (c.1356)

Gaṇitakaumudī, with about 475 *sūtra* verses (rules) and 395 *udāharaṇa* verses (examples), is a much bigger work than Bhāskarācārya's *Līlāvatī*. It is divided into the following 14 *vyavahāras*:

1. *Prakīrṇaka-vyavahāra* (Weights and measures, logistics) – 63 rules and 82 examples
2. *Miśraka-vyavahāra* (Partnership, sales, interest etc) – 42 rules and 49 examples
3. *Śreḍhī-vyavahāra* (Sequences and series) – 28 rules and 19 examples
4. *Kṣetra-vyavahāra* (Geometry of planar figures) – 149 rules and 94 Examples
5. *Khāta-vyavahāra* (Excavations) – 7 rules and 9 examples
6. *Citi-vyavahāra* (Stacks) – 2 rules and 2 examples
7. *Rāśi-vyavahāra* (Mounds of grain) – 2 rules and 3 examples

8. *Chāyā-vyavahāra* (Shadow problems) – 7 rules and 6 examples
9. *Kuṭṭaka* (Linear indeterminate equations) – 69 rules and 36 examples
10. *Vargaprakṛti* (Quadratic indeterminate equations) – 17 rules and 10 examples
11. *Bhāgādāna* (Factorisation) – 11 rules and 7 examples
12. *Rūpādyaṃśāvatāra* (Partitioning unity into unit-fractions) – 22 rules and 14 examples
13. *Aṅkapāśa* (Combinatorics) – 97 rules and 45 examples
14. *Bhadragaṇita* (Magic squares) – 60 rules and 17 examples

Nārāyaṇa has written his own commentary, *Vāsanā*, which presents the working and solution of all the examples.

Cakravāla According to Nārāyaṇa

Nārāyaṇa Paṇḍita has described the *cakravāla* process in both of his works *Gaṇitakaumudī* and *Bījagaṇitāvataṃsa* as follows:

ह्रस्वबृहत्प्रक्षेपान् भाज्यप्रक्षेपभाजकान् कृत्वा।
कल्प्यो गुणो यथा तद्वर्गात् संशोधयेत् प्रकृतिम्॥
प्रकृतेर्गुणवर्गे वा विशोधिते जायते तु यच्छेषम्।
तत् क्षेपहृतम् क्षेपो गुणवर्गविशोधिते व्यस्तम्।
लब्धिः कनिष्ठमूलं तन्निजगुणकाहतं वियुक्तं च।
पूर्वाल्पपदपरप्रक्षिप्त्योर्घातेन जायते ज्येष्ठम्॥
प्रक्षेपशोधनेष्वप्येकद्विचतुर्ष्वभिन्नमूले स्तः।
द्विचतुःक्षेपपदाभ्यां रूपक्षेपाय भावना कार्या॥

Cakravāla According to Nārāyaṇa

Nārāyaṇa's version of the *Cakravāla* algorithm to solve the equation

$$X^2 - D\,Y^2 = 1$$

is essentially the same as that given by Bhāskara in *Bījagaṇita*. Nārāyaṇa prescribes that given

$$X_i^2 - D\,Y_i^2 = K_i$$

we should obtain P_{i+1}, by solving the *kuṭṭaka* problem

$$\mathbf{Y_{i+1} = \frac{(Y_i P_{i+1} + X_i)}{K_i}}$$

Cakravāla According to Nārāyaṇa

Then Nārāyaṇa states that P_{i+1} is to be so chosen that

$$K_{i+1} = \frac{(P_{i+1}^2 - D)}{K_i} \quad \textbf{or} \quad K_{i+1} = \frac{(D - P_{i+1}^2)}{K_i}$$

adding that, in the later case, the *kṣepa* (K_{i+1}) should be taken with the opposite sign (*vyasta*). Thus, according to Nārāyaṇa,

$\mathbf{P_{i+1}^2}$ may be chosen to be greater than or lesser than D.

Nārāyaṇa also gives the relation

$$X_{i+1} = P_{i+1} Y_{i+1} - K_{i+1} Y_i$$

Nārāyaṇa's Example: $X^2 - 103\ Y^2 = 1$

i	P_i	K_i	a_i	ε_i	X_i	Y_1
0	0	1	10	1	1	0
1	10	-3	7	1	10	1
2	11	-6	3	-1	71	7
3	7	9	2	1	203	20
4	11	2	11	-1	477	47
5	11	9	2	-1	5044	497
6	7	-6	3	1	9611	947
7	11	-3	7	-1	33877	3338
8	10	1	20	1	227528	22419

At step 4, we can use *bhāvanā* to obtain the result directly

$$X = \frac{(477^2 + 103.47^2)}{2} = \frac{455056}{2} = 227528$$

$$Y = 2.477.\frac{47}{2} = \frac{44838}{2} = 22419$$

The sequence of steps in this example is the same as would follow from Bhāskara's prescription that P_{i+1} is so chosen that $\mid P_{i+1}^2 - D \mid$ is minimum.

Nārāyaṇa's Example: $X^2 - 97\ Y^2 = 1$

i	P_i	K_i	a_i	ε_i	X_i	Y_1
0	0	1	10	1	1	0
1	10	3	7	-1	10	1
2	11	8	3	-1	69	7
3	13	9	2	-1	197	20
4	14	11	3	-1	522	53
5	8	-3	2	1	847	86
6	10	-1	6	-1	5604	569

Note that, in step 3, following Bhāskara's prescription, P_3 can be either 5 or 13.

Nārāyaṇa's Example: $X^2 - 97\ Y^2 = 1$

In step 4, if we had used Bhāskara's prescription that $\left| P_4^2 - 97 \right|$ is minimum, along with the condition that $13 + P_4$ be divisible by 9, we would obtain $P_4 = 5$ and not 14 as found above.

While

$$|5^2 - 97| < |14^2 - 97|,$$

we also have

$$|14 - \sqrt{97}| < |5 - \sqrt{97}|.$$

In the above example, Nārāyaṇa seems to be using the prescription

(II'') $\left| \mathbf{P_{i+1}} - \sqrt{\mathbf{D}} \right|$ **is minimum.**

This leads to the so called Nearest Integer Continued Fraction expansion for $\sqrt{D}$.

Thus, by giving these two examples, Nārāyaṇa seems to be indicating the possibility of there being different variants of the *cakravāla* process.

Rational Approximation of Square-Roots

We had noted that if X, Y are integers such that $X^2 - D\,Y^2 = 1$, then,

$$\left|\sqrt{D} - \frac{x}{y}\right| \leq \frac{\mathbf{1}}{2xy},$$

Nārāyaṇa Paṇḍita gives an instance of how better and better approximations are obtained by successive applications of *bhāvanā*.

Now, $x^2 - 10y^2 = 1$ has solutions $x = 19$ $y = 6$.

By doing *bhāvanā* of this solution with itself

$x_1 = 19^2 + 10.6^2 = 721$ and $y_1 = 2.19.6 = 228$

By doing *bhāvanā* of these two sets of solutions, we get

$x_2 = 19.721 + 10.6.228 = 27379$ and $y_2 = 19.228 + 6.721 = 8658$

Thus, we have successive approximations

$\sqrt{10} \approx \frac{19}{6},\ \frac{721}{228},\ \frac{27379}{8658}$

Nārāyaṇa's Method of Factorisation

Chapter XI of *Gaṇitakaumudī* deals with *bhāgādāna* or methods of factoring a number.

Here, Nārāyaṇa begins by stating the usual method of checking whether the successive primes (*acchedyas*) 2, 3, 5, 7, ... divide the given number.

Then he comes up with the following interesting methods:

अपदप्रदस्तु भाज्यः कयेष्टकृत्या युतात् पदं भाज्यात्।
पदयोः संयुतवियुती हारौ परिकल्पितौ भाज्यौ॥...

अपदप्रदस्य राशेः पदमासन्नं द्विसङ्गुणं सैकम्।

मूलावशेषहीनं वर्गश्चेत् क्षेपकश्च कृतिसिद्धौ॥

वर्गो न भवेत् पूर्वासन्नपदं द्विगुणितं त्रिसंयुक्तम्।
आद्याद्युत्तरवृद्ध्या तावद्यावद्भवेद्वर्गः॥

Nārāyaṇa's Method of Factorisation

The first method is the following: If N is a non-square positive integer, find a square integer k^2 such that

$$N + k^2 = b^2$$

Then

$$N = (b + k)(b - k)$$

In the second method, find the square number a^2 which is close to N, so that

$$N = a^2 + r.$$

If

$$2a + 1 - r = b^2,$$

then

$$N = (a + 1 + b)(a + 1 - b)$$

The above method is then generalised as follows:

Keep adding the arithmetic sequence of numbers $2a+1, 2a+3, \ldots$, with common difference 2, such that their sum minus r becomes a perfect square, b^2. That is

$$(2a+1)+(2a+3)+\ldots+(2a+2m+1)-r=b^2$$

Then we can easily see that

$$N=(a+m+1+b)(a+m+1-b)$$

The above method, which is useful when the factors of N are large, was rediscovered by Fermat in 1640.

Nārāyaṇa's Method of Factorisation

Nārāyaṇa's Example 1: To Factorise N = 1161

Here $1161 = 34^2 + 5$

Further $2.34 + 1 - 5 = 64 = 8^2$

Hence, 1161 = (34+1+8) (34+1-8) = 43.27 = 3.3.3.43

Nārāyaṇa's Example 2: To Factorise N = 1001

Here $1001 = 31^2 + 40$

We first note, $2.31 + 1 - 40 = 22$ is not a square

We hence calculate

$$63 + 65 + 67 + ... + 89 - 40 = 32^2$$

Therefore

$1001 = (31 + 13 + 1 + 32)(31 + 13 + 1 - 32) = 77.13 = 7.11.13$

Aṅkapāśa: Combinatorics

In Chapter XIII of *Gaṇitakaumudī* on *Aṅkapāśa*, Nārāyaṇa Paṇḍita gives a general mathematical formulation of most of the combinatorial problems considered in earlier literature.

After listing the various *pratyayas*, Nārāyaṇa defines various *paṅktis* (sequences) and *merus* (tabular figures) that are going to be useful in different combinatorial problems.

Nārāyaṇa then considers different kinds of *prastāras* which generalise those considered in prosody and music.

He formulates the problem as one of enumerating the various possibilities which arise when there are p slots or places (*sthānas*) in which the q digits $1,\ 2,\ \ldots q$, are being placed, subject to various conditions.

The first *prastāra* considered is that of permutations without repetitions.

Nārāyaṇa then considers the *prastāra* of permutations where some of the digits are repeated.

Next Nārāyaṇa considers the *prastāra* of a general class of *varṇa-vṛttas* where, apart from *laghu* and *guru*, there could be other types of syllables (such as *pluta*, etc)–say q-types of syllables in all, which may be denoted by the digits $1, 2, \ldots, q$.

Here, Nārāyaṇa is led to the theory of representation of each natural number as a polynomial in the radix (or base) q, which is a generalisation of the binary representation discussed by Piṅgala.

Aṅkapāśa: Combinatorics

Similarly, Nārāyaṇa considers the *prastāra* of a general class of *mātrā-vṛttas* (moric metres) where, apart from the syllabic units *laghu* and *guru* (of values 1, 2), there could be other syllabic units (such as *pluta*, etc) of values $3,\ 4,\ \ldots,$ *q*.

This is also a general form of *tāla-prastāra*, but it does not subsume the specific *tāla-prastāra* considered by Śārṅgadeva in *Saṅgītaratnākara*, where the *tāla*-units have values 1, 2, 4 & 6.

Finally, Nārāyaṇa discusses the *prastāra* of combinations.

Prastāra of Generalised *Mātrā-Vṛttas*

Nārāyaṇa refers to this case as

नियतयोग-अनियतस्थान-नियतान्तिमाङ्क-भेदानयनम्

Nārāyaṇa states the rule for *prastāra* as follows:

अन्तिमाङ्कं लिखेदादौ वामे चाङ्कैक्यपूरणम्।
न्यस्याल्पमाद्यान्महतोऽधस्ताच्छेषं यथोपरि॥
अङ्कैक्यपूरणं वामे यावत्सर्वैकको भवेत्।
प्रस्तारोऽयं समाख्यातो भरतज्ञैः पुरातनैः॥

Write the final number at the beginning and fill out the sum of the numbers on the left. After one has put the smaller number below the first larger number, the rest is [brought down] as above. The filling out of number on the left and this continues till a row with all 1s is obtained. This is the enumeration as declared by the ancients well-versed in Bharata.

Prastāra of Generalised *Mātrā-Vṛttas*

Nārāyaṇa has given the following example of a *prastāra* where the total value $n = 7$ and the highest digit is 3

1	133	12	2122	23	2131	34	3211
2	223	13	11122	24	11131	35	12211
3	1123	14	1312	25	1321	36	21211
4	313	15	2212	26	2221	37	111211
5	1213	16	11212	27	11221	38	13111
6	2113	17	3112	28	3121	39	22111
7	11113	18	12112	29	12121	40	112111
8	232	19	21112	30	21121	41	31111
9	1132	20	111112	31	111131	42	121111
10	322	21	331	32	2311	43	211111
11	1222	22	1231	33	11311	44	1111111

This is nothing but 7-*mātrā-prastāra* with *laghu*, *guru* and *pluta* being the syllabic elements.

Sāmāsikī-Paṅkti: Generalised Virahāṇka Sequence

Nārāyaṇa defines the *sāmāsikī-paṅkti* (additive sequence), which is a generalised version of the Virahāṅka (or the so called Fibonacci) sequence, as follows:

एकाङ्कौ विन्यस्य प्रथमं तत्संयुतिं पुरो विलिखेत्।
उत्क्रमतोऽन्तिमतुल्यस्थानयुतिं तत्पुरस्ताच्च॥
अन्तिमतुल्यस्थानाभावे तत्संयुतिं पुरस्ताच्च।
एवं सैकसमासस्थाना सामासिकीयं स्यात्॥

The *sāmāsikī-paṅkti* of order q is thus defined by the relations

$$
\begin{aligned}
s_1^q &= s_2^q = 1 \\
s_r^q &= s_{r-1}^q + s_{r-2}^q + \ldots s_1^q \text{ when } 2 < r < q \\
s_n^q &= s_{n-1}^q + s_{n-2}^q + \ldots s_{n-q}^q \text{ when } n > q.
\end{aligned}
$$

Sāmāsikī-Paṅkti: Generalised Virahāṇka Sequence

s_n^q is the *saṅkhyāṅka* or the total number of rows in the *prastāra* of total value n with the highest digit being q

Nārāyaṇa gives the example of the generalised Virahāṅka sequence of order three:

1, 1, 2, 4, 7, 13, 24, 44,...

This is the sequence of *saṅkhyāṅkas*, or the sequence of the total number of rows in the *prastāra* of generalised *mātrā-vṛttas* (moric metres), where we include apart from the syllabic units *laghu* and *guru* (of values 1, 2), a third syllabic unit, *pluta*, which has value 3.

Sūcī-Paṅkti: Needle Sequence

Nārāyaṇa defines the *sūcī-paṅkti* (needle sequence) or the *nārācikā-paṅkti* (arrow-head sequence) as follows:

> अन्तिममितवैश्लेषस्थानाङ्कमिताश्च ताः पृथक् स्थाप्याः।
> तासां घातः सूचीपङ्क्तिर्नाराचिका वा स्यात्॥

If p is the number of places and q is the final digit, then the sequence is defined by the *vaiśleṣikī-paṅkti*, 1, 1, 1, ...1 (repeated q times), multiplied by itself p times, by the *kapāṭa-sandhi* (door-junction) method, or the algebraic method of multiplying keeping in mind the different place-values.

The $(r+1)$-th element of the sequence is a sum of multinomial co-efficients:

$U_{p,q}(r)$ = Coefficient of x^r in $(1 + x + x^2 + \ldots + x^{q-1})^p$

Sūcī-Paṅkti

Nārāyaṇa gives the example of the needle-sequence when $p = q = 3$, which is worked out as follows:

$$
\begin{array}{ccccccc}
 & & & & 1 & 1 & 1 \\
 & & & & 1 & 1 & 1 \\
\cline{5-7}
 & & & & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & \\
 & & 1 & 1 & 1 & & \\
\cline{3-7}
 & & 1 & 2 & 3 & 2 & 1 \\
 & & & & 1 & 1 & 1 \\
\cline{3-7}
 & & 1 & 2 & 3 & 2 & 1 \\
 & 1 & 2 & 3 & 2 & 1 & \\
1 & 2 & 3 & 2 & 1 & & \\
\hline
1 & 3 & 6 & 7 & 6 & 3 & 1
\end{array}
$$

These are nothing but the coefficients of different powers of x in the expansion of $(1 + x + x^2)^3$.

Sūcī-Paṅkti

The tabular figure called the *Matsyameru* is constructed where each row is filled by the elements of the needle sequence $U_{p,q}(r)$ with suitable values of $p,\ q,\ r$. While mentioning the sums of the rows and columns of *Matsyameru*, Nārāyaṇa gives some important properties of the $U_{p,q}(r)$:

> नाराच्यस्तिर्यगाः स्थानसम्मितास्तद्युतिः पृथक्।
> गुणोत्तरा भवेत्पङ्क्तिरूर्ध्वा अङ्कैक्यसम्मितः॥
> पृथक्तदूर्ध्वकोष्टाङ्कयोगात् सामासिका भवेत्।

Here, Nārāyaṇa first mentions that the sum of the needle-sequence (for a given value of q) form a geometrical sequence. This is essentially the relation

$$\sum_{r=0}^{p(q-1)} U_{p,q}(r) = (1 + 1 + ... + 1)^p = q^p$$

Sūcī-Paṅkti and *Sāmāsikī-Paṅkti*

Nārāyaṇa then gives the following important relation between the coefficients $U_{p,q}(r)$ and the *sāmāsikī-paṅkti* or the generalised Virahāṅka sequence:

$$\sum_{r=0}^{n-t} U_{n-r,q}(r) = S_n^q,$$

where t is such that $(t-1)q < n \leq tq$.

This relation has the following interpretation in the *prastāra* of the generalised *mātrā-vṛttas*:

We can show that $U_{n-r,q}(r)$ is the total number of generalised *mātrā-vṛttas* of value n, and of length $(n-r)$, i.e., have $(n-r)$ syllables in all.

Therefore the sum of $U_{n-r,q}(r)$ for various possible lengths $(n-r)$, should be equal to s_n^q, which is the *saṅkhyāṅka* or the total number of the generalised *mātrā-vṛttas* of value n.

The *Unmeru*

Nārāyaṇa introduces the following *unmeru*, which helps in carrying out the *naṣṭa* and *uddiṣṭa* processes in the case of the generalised *mātrā-vṛtta-prastāra*.

The bottom row of *unmeru* has a number of entries which is one more than total value, and subsequent rows have one entry less at each step. The bottom row is filled with the *saṅkhyāṅkas*. In the subsequent rows the numbers $1, 2, \ldots q$ are written from the right end.

The following is the *unmeru* when $n = 7$ and $q = 3$

1							
2	1						
3	2	1					
	3	2	1				
		3	2	1			
			3	2	1		
				3	2	1	
1	1	2	4	7	13	24	44

Naṣṭa Process with *Unmeru*

Nārāyaṇa's Example: To Find the 36^{th} row of the 7-*mātrā-prastāra* with highest digit 3

$44 - 36 = 8, 8 - 7 = 1$, and $1 - 1 = 0$

Thus the second 1 and 7 are *patita* and the rest are *apatita-saṅkhyāṅkas*.

1							
2	1						
3	2	**1**					
	3	2	1				
		3	**2**	1			
			3	2	**1**		
				3	2	**1**	
1	**1**	2	4	**7**	13	24	44

Note the entry (1) in the first row above the last *apatita* 24. In the row above the topmost entry of the column of that *apatita*, move left till you reach the column of the next *apatita* 13. Note the corresponding entry (1). And so on. Thus the 36^{th} row is 21211.

Uddiṣṭa Process with *Unmeru*

Example: Find the row where 2212 appears in the 7-*mātrā-prastāra* with highest digit 3

1							
2	1						
3	2	1					
	3	**2**	1				
		3	2	**1**			
			3	2	1		
				3	**2**	1	
1	**1**	2	**4**	7	13	**24**	44

From the right, we first identify the *apatita* 13 above which 2 appears. Then we move left in the row above the topmost entry of the column of that *apatita* till we get 1 and note the *apatita* number 7. And so on.

Row number $=$ 44 - sum of the *patitas* $= 44 - (24 + 4 + 1) = 15$

Uddiṣṭa Process: Alternative Method

We can devise an alternative method for *uddiṣṭa*, which is similar to the one we discussed for the usual *mātrā-vṛttas* (with only *laghus* and *gurus*):

- Write the sequence of *saṅkhyāṅkas* such that three are written above 3, two above 2 and one above 1.
- Sum the second and third entries above each 3, the second entry above each 2, and subtract the sum from the *saṅkhyāṅka* of the *prastāra*, to obtain the row number.

Example 1: Find the row where 2212 appears in the 7-*mātrā-prastāra* with highest digit 3

<table>
<tr><td>1</td><td>1</td><td>2</td><td>4</td><td>7</td><td>13</td><td>24</td><td>44</td></tr>
<tr><td colspan="2">2</td><td colspan="2">2</td><td>1</td><td colspan="2">2</td><td></td></tr>
</table>

Row-number $= 44 - (1 + 4 + 24) = 15$

Uddiṣṭa Process: Alternative Method

Example 2: Find the row where 322 appears in the 7-*mātrā-prastāra* with highest digit 3

1	**1**	**2**	4	**7**	13	**24**	44
	3		2		2		

Row-number $= 44 - (1 + 2 + 7 + 24) = 10$

The above *naṣṭa* and *uddiṣṭa* processes are based on the following representations of numbers as sums of the generalised Virahāṅka numbers.

$$8 = 1 + 7,\ 29 = 1 + 4 + 24,\ 34 = 1 + 2 + 7 + 24.$$

These are particular cases of the interesting property that every number is either a generalised Virahāṅka number (s_n^q) or can be uniquely expressed as a sum of generalised Virahāṅka numbers with the condition that q consecutive Virahāṅka numbers do not appear in the sum.

Prastāra of Permutations with Repetitions

Nārāyaṇa Paṇḍita explains that the same rule which is used to generate the enumeration (*prastāra*) of all the permutations of q different digits can be used to generate the enumeration (*prastāra*) of permutations where all the digits are not different. This is the same rule which we found in *Saṅgītaratnākara* for *tāna-prastāra*.

Nārāyaṇa gives the following example of the *prastāra* of permutations of 1, 1, 2, 4:

1	1124	4	1142	7	1241	10	4121
2	1214	5	1412	8	2141	11	2411
3	2114	6	4112	9	1421	12	4211

Of course, the total number of permutations is given by $\frac{4!}{2!} = 12$.

Prastāra of Combinations

Towards the end of the chapter on *Aṅkapāśa*, Nārāyaṇa discusses the *prastāra* of combinations (*mūlakramabheda-prastāra*)

> न्यस्याल्पमाद्यान् महतोऽधस्ताच्छेषं यथोपरि।
> ऊने तदुत्क्रमादङ्कानेकैकोनान्समालिखेत्॥
> चयपङ्क्तिर्भवेद्यावत्तावत् प्रस्तारजो विधिः।
>
> [Starting from] the beginning (i.e, from the left), place the [next] lesser digit below the greater one; the remaining digits [to the right] are to be placed as in the row above. If there are gaps [to the left], then place in reverse order, digits which are successively one less than the previous. This method of generating *prastāra* is to be followed till the [smallest] arithmetic sequence is obtained.

Prastāra of Combinations

Nārāyaṇa's rule for generating the *prastāra* of the $C(n, r)$ combinations of r symbols selected from among the (ordered) set of symbols $1, 2, \ldots, n$ may be stated as follows:

- The first row of the *prastāra* is given by the sequence of symbols $n - r + 1, n - r + 2, \ldots, n$.
- To go from any row to the next row, scan the row from the left and below the first entry $i > 2$, such that $i - 1$ does not appear earlier in the row, place the symbol $i - 1$.
- The symbols to the right of i are brought down to the next row and placed in the same order to the right of $i - 1$.
- To the left of $i - 1$, place the symbols $i - 2, i - 3$, and so on in order, till the next row also has r symbols.
- The process is repeated till we reach the last row of the *prastāra*, given by the sequence $1, 2, \ldots, r$.

Prastāra of Combinations

Nārāyaṇa gives the following example of the *prastāra* of the C(8,3) combinations obtained when we select three digits out of the eight digits 1, 2, . . . , 8.

1	**678**	15	**158**	29	**257**	43	**146**
2	**578**	16	**348**	30	**157**	44	**236**
3	**478**	17	**248**	31	**347**	45	**136**
4	**378**	18	**148**	32	**247**	46	**126**
5	**278**	19	**238**	33	**147**	47	**345**
6	**178**	20	**138**	34	**237**	48	**245**
7	**568**	21	**128**	35	**137**	49	**145**
8	**468**	22	**567**	36	**127**	50	**235**
9	**368**	20	**467**	37	**456**	51	**135**
10	**268**	24	**367**	38	**356**	52	**125**
11	**168**	25	**267**	39	**256**	53	**234**
12	**458**	26	**167**	40	**156**	54	**134**
13	**358**	27	**457**	41	**346**	55	**124**
14	**258**	28	**357**	42	**246**	56	**123**

Laḍḍūka-Cālana

Nārāyaṇa also discusses another way of generating the *prastāra*, but now in the inverse order, by moving *laḍḍūkas* (sweetmeats). He indicates that this method could be used for *naṣṭa* and *uddiṣṭa*.

The process of *laḍḍūka-cālana* is as follows:

We have n-slots in a row in which r-*laḍḍūkas* have to be placed.

- Start with *laḍḍūkas* placed sequentially in the extreme left.
- At each stage, starting from the left, move the first *laḍḍūka* which can be moved to the right by one step. Leave the *laḍḍūkas* to the right as they are.
- If there are *laḍḍūkas* to the left, move them to the extreme left.
- Go on till all *laḍḍūkas* are in the extreme right.

As an illustration of the process, *Nārāyaṇa* again considers the *prastāra* of the C(8,3) combinations obtained by selecting three digits from among the eight digits1, 2,...,8. The process of *laḍḍūka-cālana* generates the *prastāra*, as shown below, which is nothing but the *prastāra* displayed earlier, but enumerated in the reverse order or from the bottom.

Laḍḍūka-Cālana

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	○	○	○					
2	○	○		○				
3	○		○	○				
4		○	○	○				
5	○	○			○			
6	○		○		○			
7		○	○		○			
8	○			○	○			
9		○		○	○			
10			○	○	○			
11	○	○				○		
12	○		○			○		
13		○	○			○		
14	○			○		○		
15		○		○		○		
16			○	○		○		
56						○	○	○

A Binomial Representation

We can show that, both the *naṣṭa* and *uddiṣṭa* processes, of associating the k-th row of the *prastāra* with a certain combination, are related to a certain decomposition of the number k as a sum of binomial co-efficients.

In fact, depending on the way we number the rows of the *prastāra*, from the top or from the bottom, there arise two different combinatorial decompositions of each number $k < C(n, r)$.

If we number the rows of the *prastāra* from the bottom as 0, 1,... , then it can be shown that the number of the row in which the combination $p_1 < p_2 < \ldots < p_r$ occurs in the *prastāra* of the $C(n, r)$ combinations of r symbols selected from $1, 2, \ldots, n$, is given by

$$C(p_r - 1, r) + C(p_{r-1} - 1, r - 1) + \ldots + C(p_1 - 1, 1),$$

where it is understood that $C(p, q) = 0$ if $p < q$.

A Binomal Representation

Example : Binomial Representation for numbers $k < C(8,3)$

No	Combination	Binomial Representation
0	123	0
1	124	C(3,3)
2	134	C(3,3)+C(2,2)
3	234	C(3,3)+C(2,2)+C(1,1)
4	125	C(4,3)
5	135	C(4,3)+C(2,2)
6	235	C(4,3)+C(2,2)+C(1,1)
7	145	C(4,3)+C(3,2)
8	245	C(4,3)+C(3,2)+C(1,1)
9	345	C(5,3)+C(3,2)+C(2,1)
10	126	C(5,3)
11	136	C(5,3)+C(2,2)
12	236	C(3,3)+C(2,2)+C(1,1)

A Binomal Representation

No	Combination	Binomial Representation
13	146	C(5,3)+C(3,2)
14	246	C(5,3)+C(3,2)+C(1,1)
15	346	C(5,3)+C(3,2)+C(2,1)
16	156	C(5,3)+C(4,2)
17	256	C(5,3)+C(4,2)+C(1,1)
18	356	C(5,3)+C(4,2)+C(2,1)
19	456	C(5,3)+C(4,2)+C(3,1)
20	127	C(6,3)
21	137	C(6,3)+C(2,2)
22	237	C(6,3)+C(2,2)+C(1,1)
23	147	C(6,3)+C(3,2)
24	247	C(6,3)+C(3,2)+C(1,1)
25	347	C(6,3)+C(3,2)+C(2,1)
26	157	C(6,3)+C(4,2)
27	257	C(6,3)+C(4,2)+C(1,1)

A Binomal Representation

No	Combination	Binomial Representation
28	357	C(6,3)+C(4,2)+C(2,1)
29	457	C(6,3)+C(4,2)+C(3,1)
30	167	C(6,3)+C(5,2)
31	267	C(6,3)+C(5,2)+C(1,1)
32	367	C(6,3)+C(5,2)+C(2,1)
33	467	C(6,3)+C(5,2)+C(3,1)
34	567	C(6,3)+C(52)+C(4,1)
35	128	C(7,3)
36	138	C(5,3)+C(4,2)+C(1,1)
37	238	C(7,3)+C(2,2)+C(1,1)
38	148	C(7,3)+C(3,2)
39	248	C(7,3)+C(3,2)+C(1,1)
40	348	C(7,3)+C(3,2)+C(2,1)

A Binomal Representation

No	Combination	Binomial Representation
41	158	C(7,3)+C(4,2)
42	258	C(7,3)+C(4,2)+C(1,1)
43	358	C(7,3)+C(4,2)+C(2,1)
44	458	C(7,3)+C(4,2)+C(3,1)
45	168	C(7,3)+C(5,2)
46	268	C(7,3)+C(5,2)+C(1,1)
47	368	C(7,3)+C(5,2)+C(2,1)
48	468	C(7,3)+C(5,2)+C(3,1)
49	568	C(7,3)+C(5,2)+C(4,1)
50	178	C(7,3)+C(6,2)
51	278	C(7,3)+C(6,2)+C(1,1)
52	378	C(7,3)+C(6,2)+C(2,1)
53	478	C(7,3)+C(6,2)+C(3,1)
54	578	C(7,3)+C(6,2)+C(4,1)
55	678	C(7,3)+C(3,2)+C(5,1)

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Thanks!

Thank You