NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 1

Indian Mathematics: An Overview

M. D. Srinivas
Centre for Policy Studies, Chennai

Outline

- Mahāvīrācārya on the all-pervasiveness of *Gaṇita*
- The algorithmic approach of Indian Mathematics
- Development of Indian Mathematics I: Ancient and Early Classical Period (till 500 CE)
  - *Śulva-sūtra* methods of construction
  - Piṅgala and combinatorial methods
  - Development of decimal place-value system
  - Indian place value system acclaimed universally
  - *Gaṇitapāda* of *Āryabhaṭīya* (c 499 CE)
- Development of Indian Mathematics II: Later Classical Period (500 -1250)
  - *Gandhayukti* of Varāhamihira
  - Mathematics of Brahmagupta (c 628)
  - *Vargaprakṛti* and *Cakravāla*: From Brahmagupta to Bhāskara (c 1150)
  - Instantaneous velocity (*tātkālika-gati*) in Astronomy

Outline

- Development of Indian Mathematics III: Medieval Period (1250 - 1850)
  - *Vārasaṅkalita* of Nārāyaṇa Paṇḍita (c.1356)
  - Folding method of Nārāyaṇa for the construction of magic squares
  - Kerala School and development of calculus (1350-1825)
  - Mādhava Series for π and end-correction terms
  - A history of approximations and exact expressions for π
  - Nīlakaṇṭha's formula for instantaneous velocity (c.1500)
- Proofs in Indian Mathematics
  - Bhāskara on *Upapatti*
  - *Upapatti* and "Proof"
- The genius of Srinivasa Ramanujan (1887-1920)
- Lessons from History
- Summary

Mahāvīrācārya on the all-pervasiveness of *Gaṇita*

लौकिके वैदिके वापि तथा सामायिकेऽपि यः।
व्यापारस्तत्र सर्वत्र संख्यानमुपयुज्यते॥
कामतन्त्रेऽर्थशास्त्रे च गान्धर्वे नाटकेऽपि वा।
सूपशास्त्रे तथा वैद्ये वास्तुविद्यादिवस्तुषु॥
छन्दोऽलङ्कारकाव्येषु तर्कव्याकरणादिषु।
कलागुणेषु सर्वेषु प्रस्तुतं गणितं परम्॥
सूर्यादिग्रहचारेषु ग्रहणे ग्रहसंयुतौ।
त्रिप्रश्ने चन्द्रवृत्तौ च सर्वत्राङ्गीकृतं हि तत्॥

Mahāvīrācārya on the all-pervasiveness of *Gaṇita*

द्वीपसागरशैलानां संख्याव्यासपरिक्षिपः।
भवनव्यन्तरज्योतिर्लोककल्पाधिवासिनाम्॥
नारकाणां च सर्वेषां श्रेणीबन्धेन्द्रकोत्कराः।
प्रकीर्णकप्रमाणाद्या बुध्यन्ते गणितेन ते॥
प्राणिनां तत्र संस्थानमायुरष्टगुणादयः।
यात्राद्याः संहिताद्याश्च सर्वे ते गणिताश्रयाः॥
बहुभिर्विप्रलापैः किं त्रैलोक्ये सचराचरे।
यत्किंचिद्वस्तु तत्सर्वं गणितेन विना न हि॥[1]

[1]महावीराचार्यविरचित-गणितसारसङ्ग्रहः, १.९-१६

Mahāvīrācārya on the all-pervasiveness of *Gaṇita*

"All activities which relate to worldly, vedic or religious affairs make use of enumeration (*saṅkhyāna*). In the art of love, economics, music, dramatics, in the art of cooking, in medicine, in architecture and such other things, in prosody, in poetics and poetry, in logic, grammar and such other things, and in relation to all that constitute the peculiar value of the arts, the science of calculation (*gaṇita*) is held in high esteem. In relation to the movement of the sun and other heavenly bodies, in connection with eclipses and conjunction of planets, and in the determination of direction, position and time (*tripraśna*) and in (knowing) the course of the moon – indeed in all these it (*gaṇita*) is accepted (as the sole means)."

Mahāvīrācārya on the all-pervasiveness of *Gaṇita*

"The number, the diameter and perimeter of the islands, oceans and mountains; the extensive dimensions of the rows of habitations and halls belonging to the inhabitants of the world, of the interspaces between the worlds, of the world of light, of the world of the Gods and of the dwellers in hell, and other miscellaneous measurements of all sorts all these are understood by the help of *gaṇita*. The configuration of living beings therein the length of their lives, their eight attributes and other similar things, their staying together, etc. – all these are dependent on *gaṇita*.

Why keep talking at length? In all the three worlds involving moving and non-moving entities, there is nothing that can be without the science of calculation (*gaṇita*)."[2]

[2] *Gaṇitasārasaṅgraha* of Mahāvīrācārya (c.850), 1.9-16.

Gaṇita: Indian Mathematics of Computation

गण्यते संख्यायते तद् गणितम्। तत्प्रतिपादकत्वेन तत्संज्ञं शास्त्रमुच्यते।

As noted by Gaṇeśa Daivajña, in his commentary *Buddhivilāsinī* (c.1540) on *Līlāvatī* (c.1150), *Gaṇita* (Indian Mathematics) is the science (art) of computation. Indian Mathematical Texts give rules to describe systematic and efficient procedures of calculation.

Here is an ancient rule for squaring as cited by Bhāskara I (c.629 AD)

अन्त्यपदस्य वर्गं कृत्वा द्विगुणं तदेव चान्त्यपदम्।
शेषपदैराहन्यात् उत्सार्योत्सार्य वर्गविधौ॥

In the process for calculating the square, the square of the last digit is found (and placed over it). The rest of the digits are multiplied by twice the last digit (and the results placed over them). Then (omitting the last digit), moving the rest by one place each, the process is repeated again and again.

An Example: To calculate 125^2

$$
\begin{array}{ccccc|l}
1 & 5 & 6 & 2 & 5 & \\
\hline
 & & & & 25 & 5^2 = 25 \\
 & & 4 & 20 & & 2^2 = 4,\ 5.2.2 = 20 \\
1 & 4 & 10 & & & 1^2 = 1,\ 2.2.1 = 4,\ 5.2.1 = 10 \\
\hline
1 & 2 & 5 & & &
\end{array}
$$

Note: This ancient rule for squaring, uses $\frac{n(n-1)}{2}$ multiplications for squaring an n-digit number.

The modern word **algorithm** derives from the medieval word algorism, which referred to the Indian methods of calculation based on the place value system. The word algorism itself is a corruption of the name of the Central Asian mathematician al Khwarizmi (c.825) whose *Hisab al Hindi* was the source from which the Indian methods of calculation reached the Western world.

Śāstras: Present Systematic Procedures

Most of the canonical texts on different disciplines (*śāstras*) in Indian tradition do not present a series of propositions; instead they present a series of rules, which serve to characterize and carry out systematic procedures to accomplish various ends.

These systematic procedures are variously referred to as *vidhi*, *kriyā* or *prakriyā*, *sādhana*, *karma* or *parikarma*, *karaṇa*, etc., in different disciplines.

These rules are often formulated in the form of *sūtras* or *kārikās*.

Śāstras: Present Systematic Procedures

Pāṇini's *Aṣṭādhyāyī* is acknowledged to be the paradigmatic example of a canonical text in Indian tradition. All other disciplines, especially mathematics, have been deeply influenced by its ingenious symbolic and technical devices, recursive and generative formalism and the system of conventions governing rule application and rule interaction. In recent times it has had a deep influence on modern linguistics too.

> "Modern linguistics acknowledges it as the most complete generative grammar of any language yet written and continues to adopt technical ideas from it".[3]

[3] P. Kiparsky, Pāṇinian Linguistics, in *Encyclopaedia of Language and Linguistics, VI,* 1994

Pāṇini and Euclid

"In Euclid's geometry, propositions are derived from axioms with the help of logical rules which are accepted as true. In Pāṇini's grammar, linguistic forms are derived from grammatical elements with the help of rules which were framed ad hoc (i.e. *sūtras*)....

Historically speaking, Pāṇini's method has occupied a place comparable to that held by Euclid's method in Western thought. Scientific developments have therefore taken different directions in India and in the West....

In India, Pāṇini's perfection and ingenuity have rarely been matched outside the realm of linguistics. Just as Plato reserved admission to his Academy for geometricians, Indian scholars and philosophers are expected to have first undergone a training in scientific linguistics...."[4]

Note: The word "derived" means "demonstrated" in the case of Euclidean Geometry; it means "generated" in the case of Pāṇini's Grammar (*upapatti* and *niṣpatti*)

---

[4] J. F. Staal, Euclid and Pāṇini, Philosophy East and West, 15, 1965, 99-116

Development of Indian Mathematics I

Ancient Period (Prior to 500 BCE)

- *Śulvasūtras* (prior to 800 BCE): The oldest texts of geometry. They give procedures for construction and transformation of geometrical figures and alters (*vedi*) using rope (*rajju*) and gnomon (*śaṅku*).
- The ancient astronomical *siddhāntas* are from this period.

Early Classical Period (500 BCE - 500 CE)

- Pervasive influence of the methodology of Pāṇini's *Aṣṭādhyāyī*
- Piṅgala's *Chandaḥsūtra* (c.300 BCE) and the development of binary representation and combinatorics
- Mathematical ideas in Bauddha and Jaina Texts
- The notion of zero and the decimal place value system
- Mathematics and Astronomy in *Āryabhaṭīya* (c.499 CE): Most of the standard procedures in arithmetic, algebra, geometry and trigonometry are perfected by this time.

Baudhāyana-Śulvasūtra (Prior to 800 BCE)

- Units of measurement (*Bhūmiparimāṇa*)
- Marking directions and construction of a square of a given side (*Samacaturaśra-karaṇa*)
- Construction of a rectangle and isosceles trapezium of given sides
- Construction of $\sqrt{2}$ (*Dvikaraṇī*), $\sqrt{3}$ and $\left(\frac{1}{\sqrt{3}}\right)$ times a given length
- The square of the diagonal of a rectangle is the sum of the squares of its sides (*Bhujā-Koṭi-Karṇa-Nyāya* – Oldest Theorem in Geometry)

  दीर्घचतुरश्रस्याक्ष्णयारज्जुः पार्श्वमानी तिर्यङ्मानी च यत् पृथग्भूते कुरुतस्तदुभयं करोति।

Baudhāyana-Śulvasūtra

- Construction of squares which are the sum and difference of two squares
- Transforming a square into a rectangle, isosceles trapezium, isosceles triangle and a rhombus of equal area and vice versa
- Approximate conversion of a square of side **a** into a circle of radius

$$r \approx \left(\frac{a}{3}\right)(2+\sqrt{2}). \quad [\pi \approx \mathbf{3.0883}]$$

- An approximation for $(2)^{\frac{1}{2}}$ (*dvikaraṇī*):

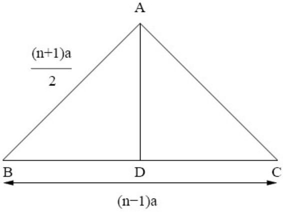
$$\sqrt{\mathbf{2}} \approx \mathbf{1} + \frac{\mathbf{1}}{\mathbf{3}} + \frac{\mathbf{1}}{\mathbf{3.4}} - \frac{\mathbf{1}}{\mathbf{3.4.34}} = \mathbf{1.4142156}$$

- Positions, relative distances and areas of altars. Shapes of different altars and their construction.

To construct a square which is n-times a given square

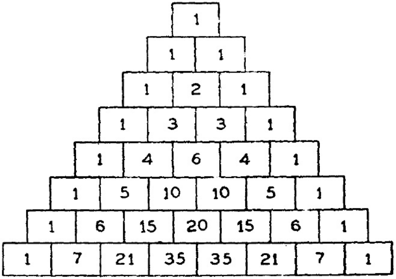
यावत्प्रमाणानि समचतुरश्राण्येकीकर्तुं चिकिर्षेत् एकोनानि तानि भवन्ति तिर्यक् द्विगुणान्येकत एकाधिकानि। त्र्यस्रिर्भवति तस्येषुस्तत्करोति। (कात्यायनशुल्बसूत्रम् ६.७)

As many squares as you wish to combine into one, the transverse line will be one less than that. Twice the side will be one more than that. That will be the triangle. Its arrow (altitude) will produce that.



$$\begin{aligned} AD^2 &= AB^2 - BD^2 \\ &= \left[\frac{(n+1)a}{2}\right]^2 - \left[\frac{(n-1)a}{2}\right]^2 \\ &= na^2. \end{aligned}$$

Varṇa-Meru of Piṅgala



The number of metrical forms with r *gurus* (or *laghus*) in the *prastāra* of metres of n-syllables is the binomial coefficient ${}^nC_r$.

Halāyudha's commentary (c.950) on *Piṅgala-sūtras* (c.300 BCE) explains the basic rule for the construction of the above table, which is the recurrence relation

$$ {}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r $$

Decimal Place Value System

The Indian Mathematicians developed the decimal place value system along with the notion of the zero-number.

The place value system is essentially an algebraic concept:

$5203 = 5.10^3 + 2.10^2 + 0.10 + 3$ is analogous to $5x^3 + 2x^2 + 0x + 3$

It is this algebraic technique of representing all numbers as polynomials of a base number, which makes all the calculations systematic and simple.

The algorithms developed in India for multiplication, division and evaluation of square, square-root, cube and cube-root, etc., have become the standard procedures. They have contributed immensely to the simplification and popularisation of mathematics the world over.

Sometimes, the Indian texts also discuss special techniques of calculation which are based on the algebraic formalism underlying the place value system. For instance, the *Buddhivilāsinī* (c.1540) commentary of Gaṇeśa Daivajña discusses the "vertical and cross-wise" (*vajrābhyāsa*) technique of multiplication.

Development of Decimal Place Value System

- The *Yajurveda-Saṃhitā* talks of powers of 10 up to 10^{12} (*parārdha*).
- The *Upaniṣads* talk of zero (*śūnya, kha*) and infinity (*Pūrṇa*).
- Pāṇini's *Aṣṭādhyāyī* uses the idea of zero-morpheme (*lopa*).
- The Bauddha and Naiyāyika philosophers discuss the notions of *śūnya* and *abhāva*.
- Piṅgala's *Chandaḥśāstra* uses zero as a marker (*Rupe śūnyam*).
- Philosophical works such as the works of Vasumitra (c.50 CE) and *Vyāsabhāṣya* on *Yogasūtra* refer to the way the same symbol acquires different meanings in the place value system.

  यथैका रेखा शतस्थाने शतं दशस्थाने दश एका च एकस्थाने यथा चैकत्वेपि स्त्री माता चोच्यते दुहिता च स्वसा चेति।
- Amongst the works whose dates are well established, decimal place value system occurs for the first time in the *Vṛddhayavanajātaka* (c.270 CE) of Sphūjīdhvaja.
- *Āryabhaṭīya* (499 CE) of Āryabhaṭa presents all the standard methods of calculation based on the place value system.

Development of Decimal Place Value System

An eighth century inscription in a Viṣṇu Temple in Gwalior, depicting the number 270 in decimal place value format. There are inscriptions of early 7th century in Southeast Asia which depict numbers in place value format.

Indian Place Value System Acclaimed Universally

"I will omit all discussion of the science of the Hindus, a people not the same as Syrians, their subtle discoveries in the science of astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians; their computing that surpasses description. I wish only to say that this computation is done by means of nine signs. If those who believe because they speak Greek, that they have reached the limits of science should know these things, they should be convinced that there are also others who know something."[5]

[5] Syrian Monophysite Bishop Severus Sebokht (c.662)

Indian Place Value System Acclaimed Universally

"By the time I was ten I had mastered the Koran and a great deal of literature, so that I was marveled at for my aptitude. . . Now my father was one of those who has responded to the Egyptian propagandist (who was an Ismaili); he, and my brother too, had listened to what they had to say about the Spirit and the Intellect, after the fashion in which they preach and understand the matter. . . Presently they began to invite me to join the movement, rolling on their tongues talk about philosophy, geometry, Indian arithmetic: and my father sent me to a certain vegetable-seller who used the Indian arithmetic, so that I might learn it from him."[6]

[6] From *The Autobiography* of the Islamic Philosopher Scientist Ibn Sina (980-1037)

Indian Place Value System Acclaimed Universally

"It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to all computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity."[7]

"To what height would science now have been if Archimedes made that discovery [place value system]!"[8]

[7] Pierre-Simon Laplace

[8] Carl Friedrich Gauss

Gaṇitapāda of *Āryabhaṭīya* (499 CE)

The following topics are dealt with in 33 verses of *Gaṇitapāda* of *Āryabhaṭīya*:

- *Saṃkhyāsthāna*: Place values.
- *Vargaparikarma, ghanaparikarma*: Squaring and cubing.
- *Vargamūlānayana*: Obtaining the square-root.
- *Ghanamūlānayana*: Obtaining the cube-root.
- Area of a triangle and volume of an equilateral tetrahedron.
- Obtaining the area of a circle, volume of a sphere.
- Obtaining the area of a trapezium.
- Chord of a sixth of the circumference.
- Approximate value of the circumference ($\pi \approx 3.1416$)

Gaṇitapāda of *Āryabhaṭīya*

- *Jyānayana*: Computing table of Rsines
- *Chāyā-karma*: Obtaining shadows of gnomons.
- *Karṇānayana*: Square of the hypotenuse is the sum of the squares of the sides.
- *Śarānayana*: Arrows of intercepted arcs
- *Śreḍhī-gaṇita*: Summing an AP, finding the number of terms, repeated summations
- *Varga-ghana-saṅkalanānayana*: Obtaining the sum of squares and cubes of natural numbers.
- *Mūlaphalānayana*: Interest and principal
- *Trairāśika*: Rule of three

Gaṇitapāda of *Āryabhaṭīya*

- *Bhinna-parikarma*: Arithmetic of fractions.
- *Pratiloma-karaṇa*: Inverse processes
- *Samakaraṇa-uddeśaka-pradarśana*: Linear equation with one unknown
- *Yogakālānayana*: Meeting time of two bodies
- *Kuṭṭākāra-gaṇita*: Solution of linear indeterminate equation

Thus, by the time of *Āryabhaṭīya*, Indian mathematicians had systematised most of the basic procedures of arithmetic, algebra, geometry and trigonometry that are generally taught in schools to-day, and many more that are more advanced (such as *kuṭṭaka* and sine-tables) and are of importance in astronomy.

Computation of Sines From Second Order Sine-Differences

Computation of Rsine-table (accurate to minutes in a circle of circumference 21,600 minutes), by the method of second-order Rsine-differences, in *Āryabhaṭīya* of Āryabhaṭa (c.499)

प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम्।
तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि॥

$$\begin{aligned} B_j &= Rsin(jh), j = 1, 2, \ldots, 24, h = 225' \\ \triangle_j &= B_{j+1} - B_j \end{aligned}$$

Rsines are to be computed from the relations:

$$\begin{aligned} \triangle_{j+1} - \triangle_j &= -B_j \left[\frac{(\triangle_1 - \triangle_2)}{B_1} \right] \\ &\approx \frac{-B_j}{B_1} \\ B_1 &\approx 225' \end{aligned}$$

27

Āryabhaṭa's Sine Table

θ in min.	R $\sin\theta$ according to		
	Āryabhaṭīya	Govindasvāmi	Mādhava(also Modern)
225	225	224 50 23	224 50 22
450	449	448 42 53	448 42 58
675	671	670 40 11	670 40 16
900	890	889 45 08	889 45 15
1125	1105	1105 01 30	1105 01 39
1350	1315	1315 33 56	1315 34 7
1575	1520	1520 28 22	1520 28 35
1800	1719	1718 52 10	1718 52 24
2025	1910	1909 54 19	1909 54 35
2250	2093	2092 45 46	2092 46 03
2475	2267	2266 38 44	2266 39 50
2700	2431	2430 50 54	2430 51 15
2925	2585	2584 37 43	2584 38 06
3150	2728	2727 20 29	2727 20 52
3375	2859	2858 22 31	2858 22 55
3600	2978	2977 10 09	2977 10 34
3825	3084	3083 12 51	3083 13 17
4050	3177	3175 03 23	3176 03 50
4275	3256	3255 17 54	3255 18 22
4500	3321	3320 36 02	3320 36 30
4725	3372	3371 41 01	3371 41 29
4950	3409	3408 19 42	3408 20 11
5175	3431	3430 22 42	3430 23 11
5400	3438	3437 44 19	3437 44 48

Development of Indian Mathematics II

Later Classical Period (500 CE - 1250 CE)

- Works of Varāhamihira: *Pañcasiddhāntikā* (c.505). *Bṛhatsaṃhitā*
- Works of Bhāskara I (c.629): *Āryabhaṭīyabhāṣya*, *Mahābhāskarīya* and *Laghubhāskarīya*
- Works of Brahmagupta: *Brāhmasphuṭasiddhānta* (c.628 CE) *Khaṇḍa-khādyaka* (c.665): Mathematics of zero and negative numbers. Development of algebra.
- Bakhshālī Manuscript (c. 7-8th century?)
- Works of Śrīdhara, Lalla (c.750), Govindasvāmin (c.800)
- *Gaṇitasārasaṅgraha* of Mahāvīrācārya (c.850)
- Works of Pṛthūdakasvāmin (c.860), Muñjāla (c.932), Āryabhaṭa II (c.950), Śrīpati (c.1039) and Jayadeva (c.1050)
- Works of Bhāskarācārya II (c.1150): *Līlāvatī*, *Bījagaṇita*, and *Siddhānta-Śiromaṇi*: They became the canonical texts of Indian mathematics and astronomy. *Upapattis* (proofs) in Bhāskara's *Vāsanābhāṣyas*

Gandhayukti of Varāhamihira

Chapter 76 of the great compilation *Bṛhatsaṃhitā* of Varāhamihira (c.550) is devoted to a discussion of perfumery. In verse 20, Varāha mentions that there are 1,820 combinations which can be formed by choosing 4 perfumes from a set of 16 basic perfumes $\left({}^{16}C_4 = 1820\right)$.

षोडशके द्रव्यगणे चतुर्विकल्पेन भिद्यमानानाम्।
अष्टादश जायन्ते शतानि सहितानि विंशत्या॥

In verse 22, Varāha gives a method of construction of a *meru* (or a tabular figure) which may be used to calculate the number of combinations. This verse also very briefly indicates a way of arranging these combinations in an array or a *prastāra*.

पूर्वेण पूर्वेण गतेन युक्तं स्थानं विनान्त्यं प्रवदन्ति सङ्ख्याम्।
इच्छाविकल्पैः क्रमशोऽभिनीय नीते निवृत्तिः पुनरन्यनीतिः॥

Bhaṭṭotpala (c.950) in his commentary has explained both the construction of the *meru* and the method of *loṣṭaprastāra* of the combinations.

Gandhayukti of Varāhamihira

16			
15	120		
14	105	560	
13	91	455	1820
12	78	364	1365
11	66	286	1001
10	55	220	715
9	45	165	495
8	36	120	330
7	28	84	210
6	21	56	126
5	15	35	70
4	10	20	35
3	6	10	15
2	3	4	5
1	1	1	1

In the first column the natural numbers are written. In the second column, their sums, in the third the sums of sums, and so on. One row is reduced at each step. The above *meru* is based on the relation.

$$^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + \ldots\ldots + {}^{r-1}C_{r-1}$$

Brāhmasphuṭasiddhānta of Brahmagupta (c.628)

Topics dealt with in Chapter XII, *Gaṇitādhyāya* (Arithmetic and Geometry)

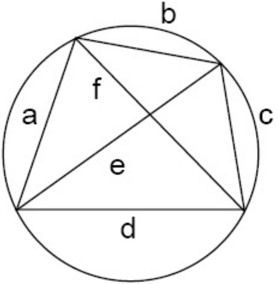
- Arithmetic of fractions
- Cube root
- Reduction of fractions
- Rule of three
- Interest problems
- Area of a triangle, diagonals and area of cyclic quadrilateral
- Rational triangles and cyclic quadrilaterals
- Circumference, area and chords of a circle
- Excavations, piles etc
- Shadow problems

Brāhmasphuṭasiddhānta of Brahmagupta

Topics dealt with in Chapter XVIII, *Kuṭṭakādhyāya* (Algebra)

- Solutions of linear indeterminate equations by *kuṭṭaka* process and its applications in astronomical problems
- Rule of signs and arithmetic of zero
- Surds (*karaṇī*)
- Operations with unknowns (*varṇa-ṣaḍvidha* or *avyakta-ṣaḍvidha*)
- Equations with single unknown (*ekavarṇa-ṣamīkaraṇa*)
- Elimination of middle term in quadratic equations (*madhyamāharaṇa*)
- Equations with several unknowns (*anekavarṇa-samīkaraṇa*)
- Equations with products of unknowns (*bhāvita*)
- *Vargaprakṛti*: Second order indeterminate equation $x^2 - Dy^2 = 1$. *Bhāvanā* and applications to finding rational and integral solutions.
- Various problems

Brahmagupta's Formulae for Cyclic Quadrilaterals



The diagonals e, f are given in terms of the sides a,b,c,d, by the formulae

$$e = \sqrt{\frac{(ab+bc)(ac+bd)}{ab+cd}}, \quad f = \sqrt{\frac{(ab+cd)(ac+bd)}{ad+bc}}$$

The area is given by

$$A = [(s-a)(s-b)(s-c)(s-d)]^{\frac{1}{2}} \ with \ \ s = \frac{(a+b+c+d)}{2}$$

Brahmagupta's *Bhāvanā*

मूलं द्विधेष्टवर्गाद् गुणकगुणादिष्टयुतविहीनाच्च।
आद्यवधो गुणकगुणः सहान्त्यघातेन कृतमन्त्यम्॥
वज्रवधैक्यं प्रथमं प्रक्षेपः क्षेपवधतुल्यः।
प्रक्षेपशोधकहृते मूले प्रक्षेपके रूपे॥

If $X_1^2 - D\ Y_1^2 = K_1$ and $X_2^2 - D\ Y_2^2 = K_2$ then

$$(X_1 X_2 \pm D\ Y_1\ \ Y_2)^2 - D(X_1 Y_2 \pm X_2 Y_1)^2 = K_1 K_2$$

In Particular given $X^2 - DY^2 = K$, we get the rational solution

$$[(X^2 + DY^2)/K]^2 - D[(2XY)/K]^2 = 1$$

Also, if one solution of the equation $X^2 - DY^2 = 1$ is found, an infinite number of solutions can be found, via

$$(X, Y) \to (X^2 + DY^2, 2XY)$$

$Cakrav\bar{a}la$ Algorithm of $\mathrm{Bh\bar{a}skar\bar{a}c\bar{a}rya}$ II (c.1150)

To solve $\mathbf{X^2 - DY^2 = 1}$

Set $X_o = 1$, $Y_0 = 0, K_0 = 1$ and $P_0 = 0$.

Given X_i, Y_i, K_i such that $X_i^2 - DY_i^2 = K_i$

First find P_{i+1} so as to satisfy:

(I) $\mathbf{P_i + P_{i+1}}$ is divisible by $\mathbf{K_i}$

(II) $|\mathbf{P_{i+1}}^2 - \mathbf{D}|$ is minimum.

Then set

$$\begin{aligned}
K_{i+1} &= \frac{(P_{i+1}{}^2 - D)}{K_i} \\
Y_{i+1} &= \frac{(Y_i P_{i+1} + X_i)}{|K_i|}, \; X_{i+1} = \frac{(X_i P_{i+1} + DY_i)}{|K_i|}
\end{aligned}$$

These satisfy $X_{i+1}{}^2 - D \;\; Y_{i+1}{}^2 = K_{i+1}$

Iterate till $K_{i+1} = \pm 1, \pm 2$ or ± 4, and then use $bh\bar{a}van\bar{a}$ if necessary.

Bhāskara's Example: $X^2 - 61Y^2 = 1$

i	P_i	K_i	a_i	ε_i	X_i	Y_i
0	0	1	8	1	1	0
1	8	3	5	-1	8	1
2	7	-4	4	1	39	5
3	9	-5	3	-1	164	21

To find $P_1 : 0+7, 0+8, 0+9 \ldots$ divisible by 1. Of them 8^2 closest to 61. Hence, $P_1 = 8, K_1 = 3$

To find $P_2 : 8+4, 8+7, 8+10 \ldots$ divisible by 3. Of them 7^2 closest to 61. Hence, $P_2 = 7, K_2 = -4$

After the second step, we have: $39^2 - 61 \times 5^2 = -4$

Since $K = -4$, we can use *bhāvanā* principle to obtain

$$X = (39^2+2)\left[\left(\tfrac{1}{2}\right)(39^2+1)(39^2+3) - 1\right] = \mathbf{1,766,319,049}$$

$$Y = \left(\tfrac{1}{2}\right)(39 \times 5)(39^2+1)(39^2+3) = \mathbf{226,153,980}$$

$$1766319049^2 - 61 \times 226153980^2 = 1$$

Tātkālika-gati: Instantaneous Velocity of a Planet

- Approximate formula for velocity (*manda-gati*) in terms of Rsine-differences was given by Bhāskara I (c.630) and he also comments on its limitation (*Laghu-bhāskarīya* 2.14-15).
- True velocity (*sphuṭa-manda-gati*) in terms of Rcosine (as the derivative of Rsine) is given in *Laghu-mānasa* of Muñjāla (c. 932) and *Mahā-siddhānta* of Āryabhaṭa II (c. 950).
- Bhāskara II (c.1150) discusses the notion of instantaneous velocity (*tātkālika-gati*) and contrasts it with the so-called true daily motion. He also evaluates the *manda-gati* and *śīghra-gati* (*Vāsanā* on *Siddhānta-Śiromaṇi* 2.37-39).
- Bhāskara II notes the relation between maximum equation of centre (correction to displacement) and the vanishing of velocity correction (*Vāsanā* on *Siddhānta-śiromaṇi, Gola* 4.3).

Development of Indian Mathematics III

Medieval Period (1250 -1850)

- *Gaṇitasārakaumudī* (in Prakrita) of Ṭhakkura Pherū (c.1300) and other works in regional languages such as *Vyavahāragaṇita* (in Kannada) of Rājāditya and *Pāvulūrigaṇitamu* (in Telugu) of Pāvulūri Mallana.
- *Gaṇitakaumudī* and *Bījagaṇitāvataṃsa* of Nārāyaṇa Paṇḍita (c. 1350)
- Mādhava (c.1350): Founder of the Kerala School. Infinite series for π, sine and cosine functions and fast convergent approximations to them.
- Works of Parameśvara (c.1380-1460)
- Works of Nīlakaṇṭha Somayājī (c.1444-1540): Revised planetary model
- Systematic exposition of Mathematics and Astronomy with proofs in *Yuktibhāṣā* (in Malayalam) of Jyeṣṭhadeva (c.1530) and commentaries *Kriyākramakarī* and *Yuktidīpikā* of Śaṅkara Vāriyar (c.1540).

Development of Indian Mathematics III (contd.)

Medieval Period (1250 - 1850)

- Works of Jñānarāja (c.1500), Gaṇeśa Daivajña (b.1507), Sūryadāsa (c.1541) and Kṛṣṇa Daivajña (c.1600): Commentaries with *upapattis*
- Works of Munīśvara (b.1603) and Kamalākara (b.1616)
- Mathematics and Astronomy in the Court of Savai Jayasiṃha (1700-1743). Translation from Persian of Euclid and Ptolemy.
- Works of later Kerala astronomers Acyuta Piṣāraṭi (c.1550-1621), Putumana Somayājī (c.1700) and Śaṅkaravarman (c. 1830)
- Candraśekhara Sāmanta of Orissa: All the major lunar inequalities (1869)

Nārāyaṇa Paṇḍita on *Vārasaṅkalita* (c.1350)

Āryabhaṭīya, gives the sum of the sequence of natural numbers

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

as also the result of first order repeated summation:

$$\frac{1.2}{2} + \frac{2.3}{2} + \ldots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

Āryabhaṭa's result for repeated summation was generalised to arbitrary order by Nārāyaṇa Paṇḍita (c.1350). Let

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = V_n^{(1)}$$

Then, **Nārāyaṇa**'s result is

$$\begin{aligned} V_n^{(r)} &= V_1^{(r-1)} + V_2^{(r-1)} + \ldots + V_n^{(r-1)} \\ &= \frac{[n(n+1)\ldots(n+r)]}{[1.2\ldots(r+1)]} \end{aligned}$$

Nārāyaṇā's Folding Method for *Samagarbha* (4nx4n) Magic Squares

समगर्भे द्वे कार्ये छादकसंज्ञं तयोर्भवेदेकम्।
छाद्याभिधानमन्यत्करसंपुटवच्च संपुटो ज्ञेयः॥
इष्टादीष्टचयाङ्का भद्रमिता मूलपङ्क्तिसंज्ञाद्या।
तद्वदभीप्सितमुखचयपङ्क्तिश्चान्या पराख्या स्यात्॥
मूलाख्यपङ्क्तियोगोनितं फलं परसमाससंभक्तम्।
लब्धहता परपङ्क्तिर्गुणजाख्या सा भवेत् पङ्क्तिः॥
मूलगुणाख्ये पङ्क्ती ये ते भद्रार्धतस्तु परिवृत्ते।
ऊर्ध्वस्थितैस्तदङ्कैश्छादकसंछाद्ययोः पृथग्यानि॥
तिर्यक्कोष्ठान्याद्येऽन्यतरस्मिन्नूर्ध्वगानि कोष्ठानि।
भद्रस्यार्धं क्रमगैरुत्क्रमगैः पूरयेदर्धम्॥
भद्राणामिहसम्पुटविधिरुक्तो नृहरितनयेन।

Nārāyaṇa's Folding Method

Nārāyaṇa's Example: To construct 4x4 square adding to 40

Choose a *mūlapaṅkti*: 1, 2, 3, 4

Choose a *parapaṅkti*: 0, 1, 2, 3

The find the *guṇa* = $\frac{[40-(1+2+3+4)]}{[0+1+2+3]} = 5$

Form the *guṇapaṅkti* by multiplying the *parapaṅkti* by *guṇa*: 0,5,10,15

Then form the *chādya* (covered) and *chādaka* (coverer) squares:

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

5	0	10	15
10	15	5	0
5	0	10	15
10	15	5	0

Nārāyaṇa's Folding Method

To construct 4x4 square adding to 40

Sampuṭīkaraṇa (folding) gives

2+15	3+10	2+0	3+5
1+0	4+5	1+15	4+10
3+15	2+10	3+0	2+5
4+0	1+5	4+15	1+10

=

17	13	2	8
1	9	16	14
18	12	3	7
4	6	19	11

Nārāyaṇa also displays the other square which is obtained by interchanging the coverer and the covered.

Note: This method leads to a pan-diagonal magic square. That is, the broken diagonals also add up to the same magic sum.

Mādhava Series for π and End-correction Terms

The following verses of Mādhava are cited in *Yuktibhāṣā* and *Kriyākramakarī*, which also present a detailed derivation of the relation between diameter and the circumference:

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
त्रिशरादिविषमसङ्ख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात्॥ १॥
यत्सङ्ख्ययाऽत्र हरणे कृते निवृत्ता हृतिस्तु जामितया।
तस्या ऊर्ध्वगता या समसङ्ख्या तद्दलं गुणोऽन्ते स्यात्॥ २॥
तद्वर्गो रूपयुतो हारो व्यासाब्धिघाततः प्राग्वत्।
ताभ्यामाप्तं स्वमृणे कृते धने क्षेप एव करणीयः॥ ३॥
लब्धः परिधिः सूक्ष्मो बहुकृत्वो हरणतोऽतिसूक्ष्मः स्यात्॥ ४॥

The first verse gives the Mādhava series

$$Paridhi = 4 \times Vy\bar{a}sa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right)$$

Mādhava Series for π and End-correction Terms

The Mādhava series for the circumference of a circle (in terms of odd numbers $p = 1, 3, 5, \ldots$) can be written in the form

$$C = 4d\left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-1)}{2}}\frac{1}{p} + \ldots\right]$$

This is an extremely slowly convergent series. In order to facilitate computation, Mādhava has given a procedure of using end-correction terms (*antya-saṃskāra*), of the form

$$C = 4d\left[1 - \frac{1}{3} + \cdots + (-1)^{\frac{(p-1)}{2}}\frac{1}{p} + (-1)^{\frac{(p+1)}{2}}\frac{1}{a_p}\right]$$

In fact, the famous verses of Mādhava, which give the relation between the circumference and diameter, also include an end-correction term

$$C = 4d\left[1 - \frac{1}{3} + \ldots + \ldots (-1)^{\frac{(p-1)}{2}}\frac{1}{p}\right] + (-1)^{\frac{(p+1)}{2}}\frac{\left\{\frac{p+1}{2}\right\}}{\{(p+1)^2 + 1\}}$$

Mādhava Series for π and End-correction Terms

Mādhava has also given a finer end-correction term

अन्ते समसङ्ख्यादलवर्गः सैको गुणः स एव पुनः ॥
युगगुणितो रूपयुतः समसङ्ख्यादलहतो भवेद् हारः ।

$$C = 4d\left[1 - \frac{1}{3} + \ldots + \ldots (-1)^{\frac{(p-1)}{2}}\frac{1}{p}\right]$$

$$+(-1)^{\frac{(p+1)}{2}} \frac{\left[\left(\frac{p+1}{2}\right)^2 + 1\right]}{\left[((p+1)^2+5)\left(\frac{(p+1)}{2}\right)\right]}$$

Mādhava Series for π and End-correction Terms

To Mādhava is attributed a value of π accurate to eleven decimal places which is obtained by just computing fifty terms with the above correction.

विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः॥

The π value given above is:

$$\pi \approx \frac{2827433388233}{9 \times 10^{11}} = 3.141592653592\ldots$$

A History of Approximations to π

	Approximation to π	Accuracy (Decimal places)	Method Adopted
Rhind Papyrus - Egypt (Prior to 2000 BCE)	$\frac{256}{81} = 3.1604$	1	Geometrical
Babylon (2000 BCE)	$\frac{25}{8} = 3.125$	1	Geometrical
Śulvasūtras (Prior to 800 BCE)	3.0883	1	Geometrical
Jaina Texts (500 BCE)	$\sqrt{(10)} = 3.1623$	1	Geometrical
Archimedes (250 BCE)	$3\frac{10}{71} < \pi < 3\frac{1}{7}$	2	Polygon doubling ($6.2^4 = 96$ sides)
Ptolemy (150 CE)	$3\frac{17}{120} = 3.141666$	3	Polygon doubling ($6.2^6 = 384$ sides)
Lui Hui (263)	3.14159	5	Polygon doubling ($6.2^9 = 3072$ sides)
Tsu Chhung-Chih (480?)	$\frac{355}{113} = 3.1415929$ 3.1415927	6 7	Polygon doubling ($6.2^9 = 12288$ sides)
Āryabhaṭa (499)	$\frac{62832}{20000} = 3.1416$	4	Polygon doubling ($4.2^8 = 1024$ sides)

A History of Approximations to π

	Approximation to π	Accuracy (Decimal places)	Method Adopted
Mādhava (1375)	$\frac{2827433388233}{9.10^{11}}$ $= 3.141592653592\ldots$	11	Infinite series with end corrections
Al Kasi (1430)	3.1415926535897932	16	Polygon doubling (6.2^{27} sides)
Francois Viete (1579)	3.1415926536	9	Polygon doubling (6.2^{16} sides)
Romanus (1593)	3.1415926535 . . .	15	Polygon doubling
Ludolph Van Ceulen (1615)	3.1415926535 . . .	32	Polygon doubling (2^{62} sides)
Wildebrod Snell (1621)	3.1415926535 . . .	34	Modified Polygon doubling (2^{30} sides)
Grienberger (1630)	3.1415926535 . . .	39	Modified Polygon doubling
Isaac Newton (1665)	3.1415926535 . . .	15	Infinite series

A History of Approximations to π

Abraham Sharp (1699)	$3.1415926535\ldots$	71	Infinite series for $tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
John Machin (1706)	$3.1415926535\ldots$	100	Infinite series relation $\frac{\pi}{4} = 4tan^{-1}\left(\frac{1}{5}\right) - tan^{-1}\left(\frac{1}{239}\right)$
Ramanujan (1914), Gosper (1985)		17 Million	Modular Equation
Kondo, Yee (2010)		5 Trillion	Modular Equation

A History of Exact Results for π

Mādhava (1375)	$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$ $\pi/\sqrt{12} = 1 - 1/3.3 + 1/3^2.5 - 1/3^3.7 + \ldots$ $\pi/4 = 3/4 + 1/(3^3-3) - 1/(5^3-5) + 1/(7^3-7) - \ldots$ $\pi/16 = 1/(1^5+4.1) - 1/(3^5+4.3) + 1/(5^5+4.5) - \ldots$
Francois Viete (1593)	$\frac{2}{\pi} = \sqrt{[1/2]}\sqrt{[1/2+1/2\sqrt{(1/2)}]}$ $\sqrt{[1/2+1/2\sqrt{(1/2+1/2\sqrt{(1/2)})}]} \ldots$ (infinite product)
John Wallis (1655)	$\frac{4}{\pi} = \left(\frac{3}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{7}{8}\right) \ldots$ (infinite product)
William Brouncker (1658)	$\frac{4}{\pi} = 1 + \frac{1^2}{2+}\ \frac{3^2}{2+}\ \frac{5^2}{2+}\ \ldots$ (continued fraction)
Isaac Newton (1665)	$\pi = \frac{3\sqrt{3}}{4} + 24\left[\frac{1}{12} - \frac{1}{5.32} - \frac{1}{28.128} - \frac{1}{72.512} - \ldots\right]$

A History of Exact Results for π

James Gregory (1671)	$tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$
Gottfried Leibniz (1674)	$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$
Abraham Sharp (1699)	$\frac{\pi}{\sqrt{12}} = 1 - \frac{1}{3.3} + \frac{1}{3^2.5} - \frac{1}{3^3.7} + \ldots$
John Machin (1706)	$\frac{\pi}{4} = 4tan^{-1}\left(\frac{1}{5}\right) - tan^{-1}\left(\frac{1}{239}\right)$

Ramanujan (1914)

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4K)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

Nīlakaṇṭha's Formula for Instantaneous Velocity

Instead of basing the calculation of instantaneous velocity on the approximate form of *manda*-correction, Nīlakaṇṭha Somayājī uses the exact form of the *manda* correction

$$\mu = m + Rsin^{-1}\left[\left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right) Rsin(m-\alpha)\right]$$

In his treatise *Tantrasaṅgraha*, Nīlakaṇṭha gives the correct formula for the correction to the mean velocity which involves the derivative of the arc-sine function.

Nīlakaṇṭha's Formula for Instantaneous Velocity

चन्द्रबाहुफलवर्गशोधितत्रिज्यकाकृतिपदेन संहरेत्।
तत्र कोटिफललिप्तिकाहतां केन्द्रभुक्तिरिह यच्च लभ्यते॥
तद्विशोध्य मृगादिके गतेः क्षिप्यतामिह तु कर्कटादिके।
तद्भवेत्स्फुटतरा गतिर्विधोः अस्य तत्समयजा रवेरपि॥

Nīlakaṇṭha gives the derivative of the second term above in the form

$$\left[\frac{\left(\left(\frac{r_0}{R}\right) R\cos(m-\alpha)\right)}{\left(R^2-\left(\frac{r_0}{R}\right)^2 R\sin^2(m-\alpha)\right)^{\frac{1}{2}}}\right]\left[\left(\frac{d}{dt}\right)(m-\alpha)\right]$$

Upapattis in Indian Mathematics

While there have been several extensive investigations on the history and achievements of the Indian mathematics, there has not been much discussion on the Indian mathematicians' and philosophers' understanding of the nature and validation of mathematical results and procedures, their views on the nature of mathematical objects, and so on.

Traditionally, such issues have been dealt with in the detailed *bhāṣyas* or commentaries, which continued to be written till recent times and played a vital role in the traditional scheme of learning. It is in such commentaries that we find detailed *upapattis* or "proofs" of the results and procedures, apart from a discussion of methodological and philosophical issues.

Amongst the available texts of Indian mathematics, a discussion of the way of validating the results (*pratyayakaraṇa*), or of demonstrating them (*upapatti*) is found first in the *Āryabhaṭīyabhāṣya* of Bhāskara I (c.629)

Yuktibhāṣā of Jyeṣṭhadeva (c.1530)

The most detailed exposition of *upapattis* in Indian mathematics is found in the Malayalam text *Yuktibhāṣā* of Jyeṣṭhadeva, a student of Dāmodara.

At the beginning of *Yuktibhāṣā*, Jyeṣṭhadeva states that his purpose is to present the rationale of the procedures given in the *Tantrasaṅgraha*. Many of these rationales have also been presented (mostly in the form of Sanskrit verses) by Śaṅkara Vāriyar (c.1500-1556) in his commentaries *Kriyākramakarī* (on *Līlāvatī*) and *Yuktidīpikā* (on *Tantrasaṅgraha*)

Yuktibhāṣā comprising 15 chapters is naturally divided into two parts, Mathematics and Astronomy.

In the Mathematics part, the first five chapters deal with logistics, arithmetic of fractions, the rule of three and the solution of linear indeterminate equations.

Chapter VI presents a derivation of the Mādhava series for π, his estimate of the end-correction terms and their use in transforming the series to ensure faster convergence.

Chapter VII discusses the derivation of the Mādhava series for Rsine and Rversine, followed by a derivation of various results on cyclic quadrilaterals and the surface area and volume of a sphere.

The Astronomy part of *Yuktibhāṣā* gives a detailed derivation of all the spherical trigonometrical results used in spherical astronomy.

Bhāskara on *Upapatti* (c.1150)

In *Siddhāntaśiromaṇi*, Bhāskarācārya II (1150) presents the *raison d'être* of *upapatti* in the Indian mathematical tradition:

मध्याद्यं द्युसदां यदत्र गणितं तस्योपपत्तिं विना
प्रौढिं प्रौढसभासु नैति गणको निःसंशयो न स्वयम्।
गोले सा विमला करामलकवत् प्रत्यक्षतो दृश्यते
तस्मादस्म्युपपत्तिबोधविधये गोलप्रबन्धोद्यतः॥

Without the knowledge of *upapattis*, by merely mastering the calculations (*gaṇita*) described here, from the *madhyamādhikāra* (the first chapter of *Siddhāntaśiromaṇi*) onwards, of the [motion of the] heavenly bodies, a mathematician will not be respected in the scholarly assemblies; without the *upapattis* he himself will not be free of doubt (*niḥsaṃśaya*). Since *upapatti* is clearly perceivable in the (armillary) sphere like a berry in the hand, I therefore begin the *Golādhyāya* (section on spherics) to explain the *upapattis*.

Gaṇeśa on *Upapatti* (c.1540)

The same has been stated by Gaṇeśa Daivajña in the introduction to his commentary *Buddhivilāsinī* (c.1540) on *Līlāvatī* of Bhāskarācārya

> व्यक्ते वाव्यक्तसंज्ञे यदुदितमखिलं नोपपत्तिं विना तत्
> निर्भ्रान्तो वा ऋते तां सुगणकसदसि प्रौढतां नैति चायम्।
> प्रत्यक्षं दृश्यते सा करतलकलितादर्शवत् सुप्रसन्ना
> तस्मादग्र्योपपत्तिं निगदितुमखिलम् उत्सहे बुद्धिवृद्ध्यै॥

Thus, according to the Indian mathematical texts, the purpose of *upapatti* is mainly:

i To remove confusion and doubts regarding the validity and interpretation of mathematical results and procedures; and,

ii To obtain assent in the community of mathematicians.

This is very different from the ideal of “proof” in the Greco-European tradition which is to irrefutably establish the absolute truth of a mathematical proposition.

Upapatti and "Proof"

The following are some of the important features of *upapattis* in Indian mathematics:

1. The Indian mathematicians are clear that results in mathematics, even those enunciated in authoritative texts, cannot be accepted as valid unless they are supported by *yukti* or *upapatti*. It is not enough that one has merely observed the validity of a result in a large number of instances.

2. Several commentaries written on major texts of Indian mathematics and astronomy present *upapattis* for the results and procedures enunciated in the text.

3. The *upapattis* are presented in a sequence proceeding systematically from known or established results to finally arrive at the result to be established.

Upapatti and “Proof”

4. In the Indian mathematical tradition the *upapattis* mainly serve to remove doubts and obtain consent for the result among the community of mathematicians.

5. The *upapattis* may involve observation or experimentation. They also depend on the prevailing understanding of the nature of the mathematical objects involved.

6. The method of *tarka* or “proof by contradiction” is used occasionally. But there are no *upapattis* which purport to establish existence of any mathematical object merely on the basis of *tarka* alone.

Upapatti and "Proof"

7. The Indian mathematical tradition did not subscribe to the ideal that *upapattis* should seek to provide irrefutable demonstrations establishing the absolute truth of mathematical results.

8. There was no attempt made in Indian mathematical tradition to present the *upapattis* in an axiomatic framework based on a set of self-evident (or arbitrarily postulated) axioms which are fixed at the outset.

9. While Indian mathematicians made great strides in the invention and manipulation of symbols in representing mathematical results and in facilitating mathematical processes, there was no attempt at formalisation of mathematics.

The Genius of Srinivasa Ramanujan (1887-1920)

In a recent article commemorating the 125^{th} birth-day of Ramanujan, Bruce Berndt has presented the following overall assessment of the results contained in his notebooks (which record his work prior to leaving for England in 1914):

"Altogether, the notebooks contain over three thousand claims, almost all without proof. Hardy surmised that over two-thirds of these results were rediscoveries. This estimate is much too high; on the contrary, at least two-thirds of Ramanujan's claims were new at the time that he wrote them, and two-thirds more likely should be replaced by a larger fraction. Almost all the results are correct; perhaps no more than five to ten are incorrect."

“The topics examined by Ramanujan in his notebooks fall primarily under the purview of analysis, number theory and elliptic functions, with much of his work in analysis being associated with number theory and with some of his discoveries also having connections with enumerative combinatorics and modular forms. Chapter 16 in the second notebook represents a turning point, since in this chapter he begins to examine the q-series for the first time in these notebooks and also to begin an enormous devotion to theta functions.”[9]

[9] B. Berndt, Notices of AMS 59, December 2012, p.1533.

Ongoing Work on Ramanujan's "Lost Notebook"

The manuscript of Ramanujan discovered in the Trinity College Library (amongst Watson papers) by G. E. Andrews in 1976, is generally referred as Ramanujan's "Lost Notebook". This seems to pertain to work done by Ramanujan during 1919-20 in India. This manuscript of about 100 pages with 138 sides of writing has around 600 results. G. E. Andrews and B. Berndt have embarked on a five volume edition of all this material. They note in the preface of the first volume that:

Ongoing Work on Ramanujan's "Lost Notebook"

"...only a fraction (perhaps 5%) of the notebook is devoted to the mock theta functions themselves. A majority of the results fall under the purview of q-series. These include mock theta functions, theta functions, partial theta function expansions, false theta functions, identities connected with the Rogers-Fine identity, several results in the theory of partitions, Eisenstein series, modular equations, the Rogers-Ramanujan continued fraction, other q-continued fractions, asymptotic expansions of q-series and q-continued fractions, integrals of theta functions, integrals of q-products, and incomplete elliptic integrals. Other continued fractions, other integrals, infinite series identities, Dirichlet series, approximations, arithmetic functions, numerical calculations, Diophantine equations, and elementary mathematics are some of the further topics examined by Ramanujan in his lost notebook."

The Enigma of Ramanujan's Mathematics

For the past hundred years, the problem in comprehending and assessing Ramanujan's mathematics and his genius has centred around the issue of "proof". In 1913, Hardy wrote to Ramanujan asking for proofs of his results. Ramanujan responded by asserting that he had a systematic method for deriving all his results, but that could not be communicated in letters.

Ramanujan's published work in India, and a few of the results contained in the note books have proofs, but they have often been said to be sketchy, not rigorous or incomplete. Ramanujan had never any doubts about the validity of his results, but still he was often willing to wait and supply proofs in the necessary format so that his results could be published. But, all the time, he was furiously discovering more and more interesting results.

The Enigma of Ramanujan's Mathematics

The Greco-European tradition of mathematics does almost equate mathematics with proof, so that the process of discovery of mathematical results can only be characterised vaguely as "intuition", "natural genius" etc. Since mathematical truths are believed to be non-empirical, there are no systematic ways of arriving at them except by pure logical reason. There are some philosophers who have argued that this philosophy of mathematics is indeed barren: it seems to have little validity when viewed in terms of mathematical practice–either in history or in our times.

In the Indian mathematical tradition, as is known from the texts of the last two to three millennia, mathematics was not equated with proof. Mathematical results were not perceived as being non-empirical and they could be validated in diverse ways. Proof or logical argumentation to demonstrate the results was important. But proofs were mainly for the purpose of obtaining assent for one's results in the community of mathematicians.

Ramanujan: Not a Newton but a Mādhava

In 1913, Bertrand Russell had jocularly remarked about Hardy and Littlewood having discovered a "second Newton in a Hindu clerk". If parallels are to be drawn, Ramanujan may indeed be compared to the legendary Mādhava.

It is not merely in terms of his methodology and philosophy that Ramanujan is clearly in continuity with the earlier Indian tradition of mathematics. Even in his extraordinary felicity in handling iterations, infinites series, continued fractions and transformations of them, Ramanujan is indeed a successor, a very worthy one at that, of Mādhava, the founder of the Kerala School and a pioneer in the development of calculus.

Lessons from History

“It is high time that the full story of Indian mathematics from vedic times through 1600 became generally known. I am not minimizing the genius of the Greeks and their wonderful invention of pure mathematics, but other peoples have been doing math in different ways, and they have often attained the same goals independently. Rigorous mathematics in the Greek style should not be seen as the only way to gain mathematical knowledge. In India where concrete applications were never far from theory, justifications were more informal and mostly verbal rather than written. One should also recall that the European enlightenment was an orgy of correct and important but semi-rigorous math in which Greek ideals were forgotten. The recent episodes with deep mathematics flowing from quantum field theory and string theory teach us the same lesson: that the muse of mathematics can be wooed in many different ways and her secrets teased out of her. And so they were in India...”[10]

[10] David Mumford, Review of Kim Plofker, *Mathematics in India*, Notices of AMS 2010, p.390.

Lessons from History

Ever since the seminal work of Needham, who showed that till around the sixteenth century Chinese science and technology seem to have been more advanced than their counterparts in Europe, it has become fashionable for historians of science to wonder “Why modern science did not emerge in non-western societies?”

In the work of the Kerala School, we notice clear anticipations of some of the fundamental discoveries which are associated with the emergence of modern science, such as the mathematics of infinite series and the development of new geometrical models of planetary motion.

It seems therefore more appropriate to investigate “Why science did not flourish in non-western societies after the 16th Century?”

It would be worthwhile to speculate “What would have been the nature of modern science (and the modern world) had sciences continued to flourish in non-western societies?” In this way we could gain some valuable insights regarding the sources and the nature of creativity of geniuses such as Srinivasa Ramanujan, Jagadish Chandra Bose, Prafulla Chandra Roy, Chandrasekhara Venkata Raman, and others, in modern India.

Summary

The most striking feature of the long tradition of Indian mathematics is the efficacy with which complex mathematical problems were handled and solved.

The basic theorems of plane geometry had already been discovered in *Śulvasūtras*.

By the time of *Āryabhaṭīya* (c.499), a sophisticated theory of numbers including the concepts of zero, and negative numbers had also been established and simple algorithms for arithmetical operations had been formulated using the place-value notation. By then, the Indian tradition of mathematics was aware of all the basic mathematical concepts and procedures that are today taught at the high school level and much more.

By the 11th century sophisticated problems in algebra, such as quadratic indeterminate equations, were solved.

By the 14th century, infinite series for trigonometric functions like sine and cosine were written down. By the same time, irrational character of π was recognised, and its value was determined to very high levels of approximation.

Summary

The reason for this spectacular success of the Indian mathematicians probably lies in the explicitly algorithmic and computational nature of Indian mathematics.

Indian mathematicians were not trying to discover the ultimate axiomatic truths in mathematics; they were interested in finding methods of solving specific problems that arose in the astronomical and other contexts.

Therefore, Indian mathematicians were prepared to work with simple algorithms that may give only approximate solutions to the problem at hand, and to evolve theories of error and recursive procedures so that the approximations may be kept in check.

This algorithmic methodology persisted in the Indian mathematical consciousness till recently. Srinivasa Ramanujan in the twentieth century seems to have made his impressive mathematical discoveries through the use of this traditional Indian methodology.

It is important that we teach at least the highlights of this great tradition of mathematics to all our students in Schools and Colleges.

References

1. B. Datta and A. N. Singh, *History of Hindu Mathematics*, 2 Parts, Lahore 1935, 1938; Reprint, Asia Publishing House, Bombay 1962; Reprint, Bharatiya Kala Prakashan, Delhi 2004.

   [Supplementary material (revised by K. S. Shukla): Hindu Geometry, IJHS, 15, 121-188, 1980; Hindu Trigonometry, IJHS, 18, 39-108, 1983; Use of Calculus in Hindu Mathematics, IJHS, 19,95-104, 1984; Magic Squares in India, IJHS, 27, 51-120, 1992; Use of Permutations and Combinations in India, IJHS, 27, 231-249, 1992; Use of Series in India, IJHS, 28, 103-129, 1993; Surds in Hindu Mathematics, IJHS, 28, 253-275, 1994]
2. C. N. Srinivasa Iyengar, *History of Indian Mathematics*, The World Press, Calcutta 1967.
3. A. K. Bag, *Mathematics in Ancient and Medieval India*, Choukhambha, Varanasi 1979.

References

4. T. A. Saraswati Amma, *Geometry in Ancient and Medieval India*, Motilal Banarsidass, Varanasi 1979; 2^{nd} Ed., 1999.
5. G. G. Joseph, *The Crest of the Peacock: The Non-European Roots of Mathematics*, Penguin 1990; 3^{rd} Ed., Princeton 2010.
6. R. C. Gupta, *Pracina Kalin Bharatiya Ganit ki Aitihasik va Sanskritik Jhalakiyan* and *Madhya Kalin Bharatiya Ganit ki Aitihasik va Sanskritik Jhalakiyan* (in Hindi), NCERT, New Delhi 1997.
7. S. Parameswaran, *The Golden Age of Indian Mathematics*, Swadeshi Science Movement, Kochi 1998.
8. S. Balachandra Rao, *Indian Mathematics and Astronomy: Some Landmarks*, 3^{rd} Ed. Bhavan's Gandhi Centre, Bangalore 2004.

References

9. G. G. Emch, M. D. Srinivas and R. Sridharan Eds., *Contributions to the History of Mathematics in India*, Hindustan Publishing Agency, Delhi, 2005.
10. Venugopal D. Heroor, *The History of Mathematics and Mathematicians of India*, Vidya Bharati, Bangalore 2006.
11. H. Selin Ed., *Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures*, Springer, New York 2008.
12. Kim Plofker, *Mathematics in India*, Princeton University Press, Princeton, 2010.
13. C. S. Seshadri Ed., *Studies in History of Indian Mathematics*, Hindustan Book Agency, Delhi 2010.

Thanks!

Thank You