NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 14

Brāhmasphuṭasiddhānta of Brahmagupta - Part 4

and

The Bakhshālī Manuscript

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Outline

- Rational approximation to surds
  - The value of $\sqrt{2}$ in *Śulbasūtra*
  - The *Bhāvanā* principle
  - Its application to find the value of surds
- The Bakhshali Manuscript
  - Its discovery and period
  - Content of the Manuscript (that is extant)
  - Problems in deciphering & making overall assessment
  - Mathematical notation employed
- The square root formula
  - Kaye's misinterpretation
  - Channabasappa's interpretation
  - His plausible derivation (consistent with BM's period)
- Other interesting problems in BM

How did *Sulvakāras* specify the value of $\sqrt{2}$?

- The following *sūtra* gives an approximation to $\sqrt{2}$:

  प्रमाणं तृतीयेन वर्धयेत्, तच्चतुर्थेन, आत्मचतुस्त्रिंशेनोनेन, सविशेषः। *[BSS 2.12]*

$$\sqrt{2} \approx 1+\frac{1}{3}+\frac{1}{3\times 4}\left(1-\frac{1}{34}\right)$$
$$= \frac{577}{408} = 1.414215686$$

- What is noteworthy here is the use of the word सविशेषः in the *sūtra*, which literally means 'that which has some speciality' (speciality $\equiv$ being approximate!)
- How did the *Śulbakāras* arrive at the above expression?
- As there are no evidences, it is anybody's guess! Presently our aim is to show how this can be arrived using the *bhāvanā* principle — enunciated by Brahmagupta much later!.

Vargaprakṛti and *Bhāvanā* principle

- It was mentioned that one of the motivations for solving equation of the form

$$x^2 - D\,y^2 = K \qquad (D > 0, \text{a non-square integer})$$

  for integral values of x, y is to find rational approximation to $\sqrt{D}$.

- If x, y are integers such that $x^2 - D\,y^2 = 1$, then we have

$$\left|\sqrt{D} - \left(\frac{x}{y}\right)\right| \leq \frac{1}{2xy} < \frac{1}{2y^2} \tag{1}$$

- Also, using the *Bhāvanā* principle, from one solution of the equation $x^2 - D\,y^2 = 1$, an infinite number of solutions can be generated, via

$$(x, y) \longrightarrow (x^2 + D\,y^2, 2xy) \tag{2}$$

- From the inequality given by (1), it can be seen that the successive solutions generated by ***bhāvanā*** given by (2), will yield better and better approximations for $\sqrt{D}$.

Bhāvanā and rational approximation for surds

- When we do *bhāvanā* of

$$x^2 - D\,y^2 = 1 \tag{3}$$

with itself, then as first approximation to $\sqrt{D}$ we get,

$$\frac{x_1}{y_1} = \frac{(x^2 + D\,y^2)}{(2xy)} \tag{4}$$

- Now using (3) in (4), we have

$$\frac{x_1}{y_1} = \frac{(2x^2 - 1)}{(2xy)} = \left(\frac{x}{y}\right) - \frac{1}{y.2x}$$

- The second approximation to $\sqrt{D}$ is obtained by doing *bhāvanā* of (x_1, y_1) with itself. That is,

$$\begin{aligned}\frac{x_2}{y_2} &= \left(\frac{x_1}{y_1}\right) - \frac{1}{y_1.2x_1}\\ &= \left(\frac{x}{y}\right) - \left(\frac{1}{y.2x}\right) - \left[\frac{1}{y.2x.(4x^2 - 2)}\right]\end{aligned}$$

Bhāvāna and rational approximation for surds

Thus, we have a series of better approximations that may be written as

$$\left(\frac{x_r}{y_r}\right) = \left(\frac{x}{y}\right) - \left(\frac{1}{y.n_1}\right) - \left(\frac{1}{y.n_1.n_2}\right) - \dots - \left(\frac{1}{y.n_1.n_2\dots n_r}\right), \qquad (5)$$

where $n_1 = 2x$ and $n_i = n_{i-1}^2 - 2$, for $i = 2, 3, \dots, r$.

Example: For $D = 2$, we start with $x = 3$ and $y = 2$. We have

$$\begin{aligned}\frac{x_2}{y_2} &= \left(\frac{3}{2}\right) - \frac{1}{2.6} - \frac{1}{2.6.(6^2-2)} \\ &= \left(\frac{3}{2}\right) - \frac{1}{2.6} - \frac{1}{2.6.34}\end{aligned}$$

By re-writing the first two terms, the above approximation can be seen to be the same as in *Śulva-sūtras*. Generating further terms in the series, we've

$$\sqrt{\mathbf{2}} = \mathbf{1} + \frac{\mathbf{1}}{\mathbf{3}} + \frac{\mathbf{1}}{\mathbf{3.4}} - \frac{\mathbf{1}}{\mathbf{3.4.34}} - \frac{\mathbf{1}}{\mathbf{3.4.34.1154}} - \frac{\mathbf{1}}{\mathbf{3.4.34.1154.1331714}} - \dots$$

where $1154 = 34^2 - 2$, $1331714 = 1154^2 - 2$, and so on.

Bhāvāna and rational approximation for surds

Yet another example (non-textual)

- In an identical manner (to that of $\sqrt{2}$), the series for $\sqrt{3}$ can also be constructed.
- Substituting $x = 7$ and $y = 4$ in the वर्गप्रकृति equation, we have

$$7^2 - 3.\ 4^2 = 1$$

- Using *bhāvanā* principle as earlier we get

$$\sqrt{3} = \frac{7}{4} - \frac{1}{2.8} - \frac{1}{2.8.62} - \frac{1}{2.8.62.3842}$$

- This can be regrouped and written in the form

$$\sqrt{3} = 1 + \frac{3}{4} - \frac{1}{4.4} - \frac{1}{4.4.62} - \frac{1}{4.4.62.3842}$$

Discovery of the Bakhshali Manuscript (BM)

- BM was discovered—purely by a stroke of luck—by a farmer in the year 1881 CE as he was excavating the soil, in a place called Bakhshali.[1]
- It is in the form of birch bark, and only 70 folios are available. It is hard to estimate as what fraction would have got lost and what is available(?)
- Providentially, the discovered manuscript reached the right hands, and after passing through several hands, finally reached F R Hoernle, an indologist who had interest in unearthing its contents—for whatever purposes!
- It was first edited and published in 1922 by G. R. Kaye.[2]
- Another edition has been brought out more recently by Takao Hayashi in 1995.

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[1] This place is identified as a village $\approx$ 80 km from Peshawar (currently in Pakistan).

[2] It has been unambigously shown by scholars (Datta and others) that the views expressed by Kaye were highly biased.

More detailed account of the discovery (Gupta)

An Inspector of Police named Mian An-Wan-Udin (whose tenant actually discovered the manuscript while digging a stone enclosure in a ruined place) took the work to the Assistant Commissioner at Mardan who intended to forward the manuscript to Lahore Museum. However, it was subsequently sent to the Lieutenant Governor of Punjab who, on the advice of General A Cunningham, directed it to be passed on to Dr Rudolf Hoernle of the Calcutta Madrasa for study and publication. Dr Hoernle presented a description of the BM before the Asiatic Society of Bengal in 1882, and this was published in the Indian Antiquary in 1883. He gave a fuller account at the Seventh Oriental Conference held at Vienna in 1886 and this was published in its Proceedings. A revised version of this paper appeared in the Indian Antiquary of 1888. In 1902, he presented the Bakhshali Manuscript to the Bodleian Library, Oxford, where it is still (Shelf mark: MS. Sansk. d. 14).

Date of the Manuscript

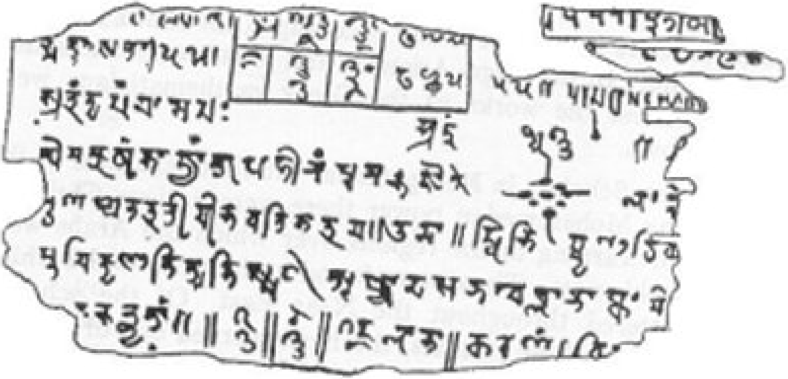
- There has been considerable debate regarding the date of BM. The variation is *almost a millenium*!
  - Kaye[3] unscrupulously attempted to place the BM around 12 century CE.
  - Datta and others[4]—based on a more careful analysis—place it anywhere between 200–400 CE.
  - Hayashi places somewhere in 7–8th century.
- It is interesting to note that all of them try to arrive at the dates mentioned above based on the analysis of language, script and content of the manuscript.
  - Language – $G\bar{a}th\bar{a}$ (a variation of Sanskrit & Prākṛt)
  - Script – $\acute{S}\bar{a}rad\bar{a}$ (used in the Gupta period $\sim$ 350 CE).
  - Content – the nature of problems discussed in BM.

[3]Unlike Colebrooke, the interpretations of Kaye are generally prejudiced and distorted—as we shall see soon.

[4]Martin Levey and Marvin Petruck, *Translation of Kushyar Ibn Labban's Principles of Hindu Reckoning*, The University of Wisconsin Press, pp. 6-7, 1965.

Folio of the manuscript (from web)

Refined - Folio of the manuscript (from web)



- The vertical and horizontal lines are used to segregate numerals and symbols from the main text.
- They at times represent fractions, but without a horizontal line as we keep using nowadays.

Problems in deciphering the content of the Ms.

- The style of writing in palm-leaf or birch bark is completely different from the contemporary style for writing.
- One can hardly find any clear marker (sentence, paragraph, chapter, etc.). There will be about 8-10 lines per folio (in just 2 inhces), very tightly packed..
- This itself makes deciphering the content difficult.
- In the case of BM, the problem is all the more acute:
  - due to the lonely copy of the Manuscript available today
  - it is in a deteriorated condition and
  - it is completely disordered.
- As regards the authorship of BM, we hardly have any idea or clue whatsoever.
- The only information that is available in the form of colophon mentions that it was written by a Brāhmaṇa, identified as *Chājaka* (king of calculator).

The square root formula

- An interesting piece of mathematics found in BM concerns with the formula for finding square root of a non-square number.
- Any non-square number N may be expressed as $\sqrt{A^2+b}$. The following formula is given in the manuscript

$$\sqrt{N} = \sqrt{A^2+b} = A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A+\frac{b}{2A}\right)} \tag{6}$$

- This is the famous Bakhshali formula about which we will discuss for a short while.
- The formula due to Heron[5] is:

$$\sqrt{N} = \sqrt{A^2+b} = A + \frac{b}{2A} \tag{7}$$

- Evidently Bakhshali formula is an improvised version of the Heron formula. What would have been the route taken by the author of BM to arrive at the formula?

[5] A Greek mathematician who lived in the later half of 1st century AD.

Kaye's version of the square root formula

- Kaye has reproduced the *sūtra* as follows:

  akṛte śliṣṭa kṛtyūnā śeṣa cchedo dvisaṃguṇaḥ |
  tadvarga dala saṃśliṣṭa hṛti śuddhi kṛti kṣayaḥ ||

- We write the above formula with a minor emendment in Devanāgarī:

  अकृते श्लिष्टकृत्यूनात् शेषच्छेदो द्विसंगुणः ।
  तद्वर्गदल संश्लिष्टहृति शुद्धिकृति क्षयः ॥

- We accept the emendment for the following reasons:
  - in its feminine form it cannot go as an adjective ...
  - in yet another place in the BM it is found 'correctly'
  - by all probability the scribe might not have 'heard' properly and hence dropped 'त्'.

Kaye's (incorrect) version of translation

- The correct rendering of the *sūtra* seems to be:

  अकृते श्लिष्टकृत्यूनात् शेषच्छेदो द्विसंगुणः।
  तद्वर्गदलसंश्लिष्टहृतिः शुद्धिकृतिः क्षयः।

- The above *sūtra* has been translated by Kaye as

  > The mixed surd is lessened by the square portion and the difference divided by twice that. The difference is divided by the quantity and half that squared is the loss.

- Datta describes the above translation as 'wrong and meaningless' as this NO way leads to $A + \frac{b}{2A}$.
- However, Kaye somehow tries to map the description—rather unscrupulously–to the Heron formula.
- Why do we call it unscrupulous?

Channabasappa's observations (on Kaye's interpretation)

- The reasons have been neatly brought out by M N Channabasappa (MNC) in his scholarly article published in *Gaṇita Bhāratī* in 1975.
- According to MNC, following the *Bakhshāli-sūtra* (BS) there is a numerical example provided as illustration.

$$\sqrt{41} = 6 + \frac{5}{12} - \frac{\frac{25}{144}}{2\left(6 + \frac{5}{12}\right)} \tag{8}$$

- Here MNC argues: If the author of the *Bakhshāli-sūtra* (BS) had Heron's formula in the mind, then why would he present a numerical example that is disconnected with the BS?
- MNC also quoting Kaye's general observations on the nature of the text,

> No general rule is preserved, but the solution itself indicates the rule

observes: "Kaye fails to apply the above logic to BS for square roots. He thus violates his own norms in compromising with his wrong translation of the *sūtra*".

MNC's unconventional interpretation (yet convincing!)

Regarding the use of the word कृति

- MNC first points out that his discussion entirely rests upon his unconventional interpretation of the words कृति and ऊन.
- The line of argument goes as follows:
  1. The word *kṛti* generally refers to 'square'.[6]
  2. But the Bakhshali Author (BA) being prior to Āryabhaṭa, Brahmagupta, Bhāskara is not compelled to use it in the same sense defined by the later authors.
  3. BA uses the word only in BS and nowhere else in the text.
  4. For referring to 'squarring' – in half a dozen places – the BA consistently uses only the word ***varga***.
  5. Moreover, ***kṛti*** literally means 'a deed or process', and hence the meaning 'square' is purely assigned one; there is nothing compelling to take it that way.
  6. Hence, based on the context he says, the word *kṛti* in BS should be taken to refer to square root or ***mūla***.
  7. He also corroborates his thesis by citing *Śulbasūtra* texts wherein we find the usage of the word *dvikaraṇi*, etc.

[6] Bhāskara defines: समद्विघातः कृतिरुच्यते।

MNC's unconventional interpretation (yet convincing!)

Regarding the use of the word ऊन

- MNC thesis is: the word *ūna* in BS should be taken to refer to the operation of 'division' though it is unconventional.
- The arguments presented in support of this are as follows:
  1. The word हरण and परिहा are derived from the same root 'ह'.
  2. One of the *sūtras* of Pāṇini (3.3.29) explicitly states ऊन as synonym of परिहा.
  3. Hence, ऊन can be taken to refer to division.
- So much so, now कृत्यूनात् means by dividing by approx. square root. In the earlier notation employed this translates to

$$\frac{A^2+b}{A} = A + \frac{b}{A}.$$

- Now since शेष is $\frac{b}{A}$, the phrase शेषच्छेदो द्विसङ्गुणः means divisor of $\frac{b}{A}$ multiplied by 2. This translates to the expression

$$A + \frac{b}{2A}$$

MNC's unconventional interpretation (yet convincing!)

Corroborating evidences

- He corroborates this interpretation further by taking the notation employed in BM. For instance, in giving the expression for $\sqrt{41}$, the preliminary steps involved are represented as:

$$\begin{array}{|c|}6\\5\\6\\\hline\end{array} \quad \text{and} \quad \begin{array}{|r|}6\\5\\12\\\hline\end{array} \qquad \text{means} \qquad 6+\frac{5}{6} \quad \text{and} \quad 6+\frac{5}{12}$$

- Now, we present the meanings of few other words appearing in the verse

श्लिष्टकृतिः $\xrightarrow{\text{means}}$ approximate square root A

तद्वर्गदलः $\xrightarrow{\text{means}}$ half of the square of that $\frac{1}{2}\left(\frac{b}{2A}\right)^2$

संश्लिष्टहृतिः $\xrightarrow{\text{means}}$ division by the composite $\div\left(A+\frac{b}{2A}\right)$

(तस्य) क्षयः $\xrightarrow{\text{means}}$ subtraction of that

शुद्धिकृतिः $\xrightarrow{\text{means}}$ (is) the refined square root

- Thus, all the words in the verse fit so well to convey the intended meaning of the verse.

Derivation of the formula

- Let N be the surd, whose approximate value is desired to be found. To begin with, we choose a number A such that $A^2 < N$, and as close as possible to N.[7]
- This number A is taken as the zeroth order approximation to $\sqrt{N}$. Now, the error 'b' is given by

$$b = N - A^2 \tag{9}$$

- The first order approximation is given by

$$A_1 = A + \frac{b}{2A} \tag{10}$$

- At this stage, the error is given by

$$\begin{aligned} b_1 &= N - A_1^2 \\ &= A^2 + b - \left(A + \frac{b}{2A}\right)^2 = \frac{-b^2}{4A^2} \end{aligned} \tag{11}$$

- Thus, we have moved from $(A,\ b) \to (A_1,\ b_1)$.

[7] In principle, it is not necessary that A should be an integer.

Derivation of the formula (contd.)

- In other words,

$$(A_1,\ b_1) \longrightarrow \left(A + \frac{b}{2A},\ -\left(\frac{b}{2A}\right)^2 \right) \tag{12}$$

- The second order approximation is obtained by

$$A_2 = A_1 + \frac{b_1}{2A_1} \tag{13}$$

- Using (12) in (13) we have

$$(A_2,\ b_2) \longrightarrow \left(A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A + \frac{b}{2A}\right)},\ \ldots \right) \tag{14}$$

- It is to be noted that the first term in the parenthesis of RHS of (14) is same as the Bakhshali formula for finding surds.
- How accurate is this approximation?

Numerical example

- Let the surd $N = 83$. The integer whose square is closest to N is 9. Hence we choose $A = 9$. This $\Rightarrow b = 2$.
- The Bakhshali formula is

$$\sqrt{N} = \sqrt{A^2 + b} = A + \frac{b}{2A} - \frac{\left(\frac{b}{2A}\right)^2}{2\left(A + \frac{b}{2A}\right)} \tag{15}$$

- Substituting the values in the above formula we have

$$\begin{aligned} \sqrt{83} &= \sqrt{9^2 + 2} = 9 + \frac{2}{2.9} - \frac{\left(\frac{2}{2.9}\right)^2}{2\left(9 + \frac{2}{2.9}\right)} && (16) \\ &= 9 + \frac{1}{9} - \frac{1}{18.82} = 9.110433604 && (17) \end{aligned}$$

- The actual value is: 9.110433579, which is correct to 7 decimal places.
- The accuracy depends upon how close we choose the initial value A to be, or how small b is. However, even if b is large (within 'permissible limits'), successive iterates would lead to the exact value.

Interesting problems

पञ्चानां वणिजां मध्ये मणिर्विक्रीयते किल। तत्रोक्ता मणिविक्रीत्रा मणिमूल्यं कियद्भवेत्॥...अर्धं त्रिभाग पादांश पञ्चभाग षडंश च।

A jewel is sold among five merchants together. The price of the jewel is equal to half the money possessed by the first together with the moneys possessed by the others, or $\frac{1}{3}$rd the money possessed by the second together with the moneys possessed by the others, or $\frac{1}{4}$th the money possessed by the third together with the moneys possessed by the others, or $\frac{1}{5}$th the money possessed by the fourth together with the moneys possessed by the others, or $\frac{1}{6}$th the money possessed by the fifth together with the moneys possessed by the others. Find the cost of the jewel, and the money possessed by each merchant.[8]

[8] Here it may be mentioned that though the solution to the problem is available in greater detail, the statement as such is not fully decipherable from the manuscript (see for instance, Hayashi, 174-75.), and hence what has been presented above is a partially—yet faithfully—reconstructed version of it given by CNS (38-39).

Interesting problems

Solution: If m_1, m_2, m_3, m_4, m_5 be the money possessed by the five merchants, and p be the price of the jewel, then the given problem may be represented as

$$\begin{aligned}
\frac{1}{2}m_1 + m_2 + m_3 + m_4 + m_5 &= m_1 + \frac{1}{3}m_2 + m_3 + m_4 + m_5 \\
&= m_1 + m_2 + \frac{1}{4}m_3 + m_4 + m_5 \\
&= m_1 + m_2 + m_3 + \frac{1}{5}m_4 + m_5 \\
&= m_1 + m_2 + m_3 + m_4 + \frac{1}{6}m_5 \\
&= p.
\end{aligned}$$

Hence we have

$$\frac{1}{2}m_1 = \frac{2}{3}m_2 = \frac{3}{4}m_3 = \frac{4}{5}m_4 = \frac{5}{6}m_5 = q \text{ (say)}.$$

Substituting this in any of the previous equations we get $\frac{377}{60}q = p$. For integral solutions we have to take $p = 377\,r$ and $q = 60\,r$, where r is any integer. In fact, the answer provided in Bakhshālī manuscript is $p = 377$ and $m_1, m_2, m_3, m_4, m_5 = 120, 90, 80, 75, 72$ respectively.

Use of mathematical notations

- BM is one the most important sources to know the kind of notations employed in those times.
- There are at least three different kinds of notations:
  - Notation to represent fractions – This is done by placing one number below the other without a horizontal bar.
  - Notation to represent negative quantities – A -ve quantity is denoted by a small 'cross' resembling the '+' sign to the right of it. This probably could be the deformed version of the character ऋ, used in Devanagari.
  - Notation/Abbreviation for representing operations – The operations line $+ \sqrt{}$ are denoted by the characters such as यु, मू, which are abbreviations of the words denoting those operations such as युति, मूल।
- In *Amarakośa* (c. 400 CE), we have the statement –

  यदृच्छा विन्यसेत् शून्ये *(place zero . . .)*

- Taking यदृच्छा = यावत्तावत्, we find '0' for unknowns (x).

Thanks!

THANK YOU