NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 10

Āryabhaṭīya of Āryabhaṭa - Part 4

and

Introduction to Jaina Mathematics

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Outline

- Some algebraic identities
- The *Kuṭṭaka* problem
  - Meaning of the term + formulation of problem
  - Need for solving *kuṭṭaka* problem (Bhāskara's example)
  - Solution to the *Kuṭṭaka* problem
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The number of terms in an arithmetical series

- Consider an arithmetical series of the form –

$$a + (a + d) + (a + 2d) + (a + 3d) \ldots\ldots + (a + (n - 1)d). \qquad (1)$$

- The formula for finding the number terms n in the series, in terms of its sum S, the first term a and the common difference d is encoded in the following verse:[1]

गच्छोऽष्टोत्तरगुणिताद् द्विगुणाद्युत्तरविशेषवर्गयुतात्।
मूलं द्विगुणाद्यूनं स्वोत्तरभाजितं सरूपार्धम् ॥

- The content of the above verse can be expressed as:

$$n = \frac{1}{2}\left(\frac{\sqrt{8Sd + (2a - d)^2} - 2a}{d} + 1\right) \qquad (2)$$

[1] Āryabhaṭa, *Āryabhaṭīya*, *Gaṇitapāda*, verse 20.

Some algebraic identities

द्विकृतिगुणात् संवर्गाद् द्व्यन्तरवर्गेण संयुतान्मूलम्।
अन्तरयुक्तं हीनं तद् गुणकारद्वयं दलितम्॥[2]

Multiply the product by four, then add the square of the difference of the two (quantities), and then take the square root. [Set down this square root in two places]. (In one place) increase it by the difference (of the two quantities), and (in the other place) decrease it by the same. The results thus obtained, when divided by two, give the two factors [of the given product].

The content of the above verse may be expressed as,

$$\text{If} \qquad x - y = a; \qquad \text{and} \qquad xy = b, \qquad \text{then}$$

$$\begin{aligned} x &= \frac{\sqrt{4b + a^2} + a}{2} \\ y &= \frac{\sqrt{4b + a^2} - a}{2}. \end{aligned}$$

[2] Āryabhaṭa, *Āryabhaṭīya*, *Gaṇitapāda*, verse 24.

Kuṭṭaka algorithm

The meaning behind the nomenclature *kuṭṭaka*

- The problem of solving first order indeterminate equations was so important to Indian astronomer-mathematicians as there was a compelling need for them in choosing a given epoch, with constraints imposed by various planetary parameters.[3]
- The term ***Kuṭṭana*** in general means 'pulverising'—the act of reducing something to finer sizes by repeated operation.
- When applied in the context of mathematics, ***Kuṭṭana*** refers to repeated division by which the given numbers are made smaller and smaller by mutual division.
- This algorithm also plays a key role in finding the solution of the much more difficult problem namely, second order indeterminate equations (*varga-prakṛti*).

[3] In order to demonstrate its application in a variety of contexts, several examples have been presented by Bhāskara, ग्रहकुट्टाकार, राशिकुट्टाकार, लिप्ताकुट्टाकार, वारकुट्टाकार and so on.

Formulation of the *kuṭṭaka* problem

- Suppose there is an integer N which when divided by two integers (a, b) leaves remainders (r_1, r_2). That is,

$$\begin{aligned} N &= ax + r_1 \\ \text{also} \quad &= by + r_2. \end{aligned} \tag{3}$$

- This equation may be written as

$$by - c = ax, \qquad \text{where} \;\; c = r_1 - r_2. \tag{4}$$

- *Kuṭṭaka* problem: Given (a,b,c) that are integers, we need to find (x,y)—again integers—that will satisfy the above equation.
- Historical error: Usually the equation given above is called Diophantine equation. Diophantus who lived in 3rd cent AD was concerned with finding rational solutions—not integer solutions—of (2), which is a much simpler problem.

Kuṭṭaka problem: Bhāskara's example

- Statement of the problem

  अथेदानीं ग्रहगणिते कुट्टाकारो योज्यते। रविभगणाः केन गुणिताः मण्डलशेषं अपनीय भूदिवसानां शुद्धं भागं दद्युरिति। रविभगणाः भूदिवसाश्च न्यस्यन्ते।...

  मध्यं रवेः मृगपतौ धनुरंशकार्धे
  दृष्टं मया दिनकरोदयकालजातम्।
  आगण्यतां दिनगणो भटशास्त्रसिद्धः
  याताश्च तस्य भगणाः कलिकालसिद्धाः॥

- Consider the equation

$$by - c = ax, \qquad \text{where} \tag{5}$$

  - b – no. of *maṇḍalas* of sun in a *Mahāyuga* (4320000)
  - a – no. of *bhūdivasas* in a *Mahāyuga* (1577917500)
  - c – *maṇḍalaśeṣa*, obtained by observation (86688)

Solution to the *Kuṭṭaka* problem

- *Āryabhaṭa* presents the solution in two verses *Gaṇitapāda*, 32–33):

अधिकाग्रभागहारं छिन्द्यात् ऊनाग्रभागहारेण ।
शेषपरस्परभक्तं मतिगुणम् अग्रान्तरे क्षिप्तम् ॥
अध-उपरिगुणितमन्त्ययुगूनाग्रच्छेदभाजिते शेषम् ।
अधिकाग्रच्छेदगुणं द्विच्छेदाग्रमधिकाग्रयुतम् ॥

- अग्रं शेषः – remainder (r_1)
- अधिकाग्रं – for which the remainder is large (a)
- भागहारं – भागो ह्रियते यस्मात् the dividend (a)
- छिन्द्यात् – may you divide
- भागहारेण – भागं हरतीति the divisor (b)
- शेषपरस्परभक्तं – being mutually divided by the remainder
- मतिगुणम् – multiplied by *mati* (optional multiplier)
- अग्रान्तरे क्षिप्तम् – added to/subtracted by the diff. of the remainders
- (कथं पुनः स्वबुद्धिगुणः क्रियते? अयं राशिः केनगुणितम् इदं अग्रान्तरं प्रक्षिप्य विशोध्य वा अस्य राशेः शुद्धं भागं दास्यतीति। समेषु क्षिप्तं विषमेषु शोध्यमिति सम्प्रदायाविच्छेदात् व्याख्यायते।)

Solution to the *Kuṭṭaka* problem

- One step (left as ऊह्यं by Āryabhaṭa):

  एवं परस्परेण लब्धानि पदान्यवस्थाप्य, मतिश्च अधः, पश्चिमलब्धं च मत्या अधः

  Having placed the quotients one below the other, below them the mati, and underneath the mati the quotient just obtained

- Āryabhaṭa presents the solution in two verses *Gaṇitapāda*, 32–33):

  अधिकाग्रभागहारं . . . क्षिप्तम् ॥

  अध-उपरिगुणितम् अन्त्ययुक् उनाग्रच्छेदभाजिते शेषम् ।
  अधिकाग्रच्छेदगुणं द्विच्छेदाग्रम् अधिकाग्रयुतम् ॥

  - अध-उपरिगुणितम् – number below multiplied by the one just above it
  - (अधस्तनेन राशिना उपरिराशिः गुणितः)
  - अन्त्ययुक् – added to the last (one just below it)
  - उनाग्रच्छेदभाजिते – when divided by the divisor corresponding to the smaller remainder (÷*b*)
  - शेषम् अधिकाग्रच्छेदगुणं – the remainder multiplied by the divisor corresponding the greater remainder (×*a*)
  - अधिकाग्रयुतम् द्विच्छेदाग्रम् – when added to the greater remainder gives the number corresponding to the two divisors. ($agram = sa\dot{n}khy\bar{a}$)

Solution to the *Kuṭṭaka* problem

Translation of the verses

Below we give the translation of these verses by Datta and Singh following the interpretation of Bhāskara I:

Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder. The remainder (and the divisor corresponding to the smaller remainder) being mutually divided, the last residue should be multiplied by such an optional integer that the product being added (in case the number of quotients of the mutual division is even) or subtracted (in case the number of quotients is odd) by the difference of the remainders (will be exactly divisible by the penultimate remainder. Place the quotients of the mutual division successively one below the other in a column, below them the optional multiplier and underneath it the quotient just obtained). Any number below (i.e., the penultimate) is multiplied by the one just above it and then added to the last (one just below it). Divide the last number (obtained by doing so repeatedly) by the divisor corresponding to the smaller remainder; then multiply the remainder by the divisor corresponding the greater remainder and add the greater remainder. (The result will be) the number corresponding to the two divisors.

The *Kuṭṭaka* algorithm

- Let the two remainders be such that $r_1 > r_2$, so that (a, b) be the divisors corresponding to the greater and smaller remainders respectively. Let $c = r_1 - r_2$.
- We write down the procedure, when the number of quotients (ignoring the first one q) is even.

$$
\begin{array}{llllll}
b) & a & (q & & & \\
 & \underline{bq} & & & & \\
 & r_1) & b & (q_1 & & \\
 & & \underline{r_1 q_1} & & & \\
 & & r_2) & r_1 & (q_2 & \\
 & & & \underline{r_2 q_2} & & \\
 & & & \cdot & & \\
 & & & \cdot & & \\
 & & & \cdot & & \\
 & & & & r_{2n}) & r_{2n-1} \quad (q_{2n} \\
 & & & & & \underline{r_{2n} q_{2n}} \\
 & & & & & r_{2n+1}
\end{array}
$$

The *Kuṭṭaka* algorithm

Arranging the quotients and the choice of *mati*

- The prescription for the choice of the optimal number t (*mati*) is:
  - $r_{2n+1}t + c$ should be divisible by r_{2n}. (quotients even)
  - $r_{2n}t - c$ should be divisible by r_{2n-1} (quotients odd)

  Let s be the quotient

- Having found t and s, we have to arrange them in the form of a *vallī* (column), to generate successive columns.

$$
\begin{array}{lllll}
q_1 & q_1 & q_1 & \ldots\ldots & q_1\beta_{2n-1} + \beta_{2n-2} \\
q_2 & q_2 & q_2 & \ldots\ldots & \beta_{2n-1} \\
. & . & . & & \\
. & . & . & & \\
. & . & . & & \\
q_{2n-1} & q_{2n-1} & q_{2n-1}\ \beta_1 + t = \beta_2 & & \\
q_{2n} & q_{2n}t + s = \beta_1 & & & \\
t & t & & & \\
s & & & &
\end{array}
$$

- Divide $q_1\beta_{2n-1} + \beta_{2n-2}$ by b. The remainder is x and $N = ax + r_1$.

The *Kuṭṭaka* algorithm

Example 2: To solve $45x + 7 = 29y$. Here $a = 45,\ b = 29,\ r_1 = 7,\ r_2 = 0.$

$$
\begin{array}{llllll}
29) & 45 & (1 & & & \\
 & 29 & & & & \\
\cline{2-2}
 & 16) & 29 & (1 & & \\
 & & 16 & & & \\
\cline{3-3}
 & & 13) & 16 & (1 & \\
 & & & 13 & & \\
\cline{4-4}
 & & & 3) & 13 & (4 \\
 & & & & 12 & \\
\cline{5-5}
 & & & & 1 &
\end{array}
$$

Here the number of quotients (omitting the first) is odd. t should be chosen such that $1 \times t - 7$ is divisible by 3. Hence t is chosen to be 10. Therefore we have,

1	1	1	92
1	1	51	51
4	41	41	
10	10		
1			

Now, $92 = 29 \times 3 + 5$. Hence, $N = 49 \times 5 + 7 = 29 \times 8$. Thus $x = 5,\ y = 8$.

Introduction

- Jainas seem to have regarded mathematics as an integral part of their religion. A section of their religious literature was named *Gaṇitānuyoga* (system of calculation).
- The important Jaina mathematical works include:
  - *Jambū-dvīpa-prajñapti*
  - *Sūrya-prajñapti*
  - *Sthānaṅga-sūtra*
  - *Bhagavati-sūtra*
  - *Uttarādhyayana-sūtra*
  - *Anuyoga-dvāra-sūtra*
  - *Trilokasāra*
  - *Gaṇitasārasaṅgraha*
- Our knowledge about the earlier Jaina works is primarily based on commentaries as many of the original works are yet to come to light.

Results on mensuration

- One of Umāsvāti's (c. 219 CE) most important works, *Tattvārthādhigama-sūtra-bhāṣya*, contains mathematical formulae and results on mensuration:

  1. cirumference of a circle = $\sqrt{10} \times \textit{diameter}$
  2. area of a circle = $\frac{1}{4}\textit{circumference} \times \textit{diameter}$
  3. chord = $\sqrt{4\ \textit{sara}\ (\textit{diameter} - \textit{sara})}$
  4. sara = $\frac{1}{2}[\textit{diameter} - \sqrt{\textit{diameter}^2 - \textit{chord}^2}$
  5. arc of segment less than a semicircle = $\sqrt{6\textit{sara}^2 + \textit{chord}^2}$
  6. diameter = $\frac{\textit{sara}^2 + \frac{1}{4}\textit{chord}^2}{\textit{sara}}$

- The term *śara* employed above refers to Rversine ($R - R\cos\theta$).

- It has been observed that these results have been taken by Umāsvāti from other mathematical works extant in his time as he himself was not known to be a mathematician.

Relation between the circumfrerence, diameter and area of a circle

त्रिगुणियवासं परिही दहगुणक्त्थिरवमामूलं च।
परिहिहदवासतुस्यिं बादरसुहुमं च खेत्तफलम्॥ [4]

त्रिगुणितव्यासं परिधिः दशगुणविस्तारवर्गमूलं च।
परिधिहतव्यासतुर्यं बादरसूक्ष्मं च क्षेत्रफलम्॥

If C be the circumference of the circle, d its diameter, and A the area, then the formulae given above may be expressed as:

$$
\begin{aligned}
C &= 3d && \text{(gross)}\\
C &= \sqrt{10d^2} && \text{(subtle)}\\
A &= C \times \frac{d}{4}
\end{aligned}
$$

[4] Nemicandra's *Trilokasāra*.

Dhanur-jyā-vyāsa-bāṇayoḥ sambandhaḥ

Relation between the arc, chord, diameter and arrow (Rversine)

इसुहीणं विक्खंभं चउगुणिदिसुणा हदे दु जीवकदी।
बाणकदिं छहिगुणिदे तत्थ जुदे घणुकदी होदि॥[5]

इषुहीनं विष्कम्भं चतुर्गुणेषुणा हते तु जीवकृतिः।
बाणकृतिः षड्गुणिते तत्र युते धनुकृतिः भवति॥

If d be the diameter of the circle, a the arc length, c the corresponding chord, and h the Rversine, then the formulae given in the above verse may be expressed as:

$$
\begin{aligned}
c^2 &= 4h(d-h) \\
a^2 &= 6h^2 + c^2.
\end{aligned}
$$

[5] Nemicandra's *Trilokasāra*.

Notion of infinity (and its different types)

- Jainas had names for different positions *sthana* in the numeral system: *eka, daśa, śata, sahasra, daśa-sahasra, dasa-śata-sahasra, koṭi, daśa-koṭi, śata-koṭi*, etc.
- It has been stated that very large numbers were used for measurements of space and time.[6]
- Jainas have classified numbers as
  1. enumerables (सङ्ख्येय)
  2. unenumerable (असङ्ख्येय) and
  3. infinite (अनन्त).
- They also talk of different types of infinity:
  1. infinite in one direction (एकतोनन्तम्)
  2. infinite in two directions (द्विधानन्तम्),
  3. infinite in area (देशविस्तारानन्तम्)
  4. infinite everywhere (सर्वविस्तारानन्तम्)
  5. infinite perpetually (शाश्वतानन्तम्)

[6] No nation has used such large numbers as the Jainas and the Buddhists.

How to conceive of infinitely large numbers?

- It would be difficult—and inappropriate too—to find parallels Cantor's notions of infinity.
- However, it is evident that have conceived of infinity both spatially and temporally which by no means is crude.
- Jainas mathematicians have provided practical examples through which one can conceive of *enumerable* $\rightarrow$ *infinite*

  > *Consider a trough whose diameter is of the size of the earth (100,000* yojanas*). Fill it up with white mustard seeds counting them one after another. Similarly fill up with mustard seeds other troughs of the sizes of the various lands and seas. Still it is difficult to reach the highest enumerable number.*

- They also seem to have developed ($\sim$ first cent. BCE) a fomulation of the law of indices which is quite noteworthy considering the fact that notations had not been developed in their full blown form.

Dealing with laws of indices

Excerpts from *Uttarādhyayana-sūtra* and *Anuyoga-dvāra-sūtra*

- The text *Uttarādhyayana-sūtra* ($\sim$ third cent. BCE)[7] enumerates powers and roots of numbers:
  - *varga-varga* ($(a^2)^2 = a^4$)
  - *ghana-varga* ($(a^3)^2 = a^6$)
  - *ghana-varga-varga* ($((a^3)^2)^2 = a^{12}$)
  - *varga-mūla-ghana* ($(a^{\frac{1}{2}})^3 = a^{\frac{1}{8}}$)
- In *Anuyoga-dvāra-sūtra* we find the statement

  > *the first square root multiplied by the second square root, or the cube of the second square root; the second square root multiplied by the third square root, or the cube of the third square root,*

  which symbolically translates to

$$a^{\frac{1}{2}} \times a^{\frac{1}{4}} = (a^{\frac{1}{4}})^3; \qquad a^{\frac{1}{4}} \times a^{\frac{1}{8}} = (a^{\frac{1}{8}})^3.$$

[7] CNS, p. 25.

Typical content of Jaina mathematical text

The *Sthānaṅga-sūtra* supposed to be composed around 300 (BCE), mentions the following 10 topics:

परिकर्म : The four fundamental operations - subtraction, addition, multiplication, division

व्यवहार : Application of arithmetic to concrete problems

कलश-कर्म : Fractions

रज्जु : Geometry (called *Śulva* in the Vedic period)

राशि : This may refer to either measurement of grains or it may be referring to mensuration of plane and solid figures

यावत्-तावत् : The word of unknown quantity x, using the algebraic symbol *ya*

विकल्प : Permutations and combinations, discussed in the next section

वर्ग : Squaring

घन : Cubing

वर्ग-वर्ग : This and the previous need not necessarily mean square, cube and square-square. They may also refer to higher powers and roots.

Thanks!

THANK YOU