NPTEL COURSE ON

MATHEMATICS IN INDIA:
FROM VEDIC PERIOD TO MODERN TIMES

Lecture 28

Magic Squares - Part 1

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Outline

- Introduction
- Classification of Magic squares
- Purpose of studying (as stated by Nārāyaṇa)
- ***Kacchapuṭa*** of **Nāgārjuna** (c.100 BCE)
- ***Sarvatobhadra*** of **Varāhamihira** (550 CE)
- The ***Turagagati*** method of obtaining magic squares
- Possible no. of 4×4 PD squares (with elements 1,2...16)?
- Ancient Indian method for odd squares
- ***Kuṭṭaka*** and magic squares
- Properties of 4×4 magic squares
- Construction of magic squares using these properties

Introduction

Background and Relevance

- Today there is a lot of discussion going on all around the world to see how to make mathematics learning more interesting.
- As far as arithmetic is concerned, certainly one way to make it interesting is to introduce the topic of Magic Squares—called *Bhadra-gaṇita* in Indian Mathematics.
- The nomenclature stems from the fact it was considered to fetch ***bhadra***—all round prosperity/well-being—just like *yantras*, wherein we have various letters inscribed.
- The earliest extant mathematical text in India that presents some detailed treatment on the topic of magic squares is *Gaṇitasārakaumudī* of Thakkura Pherū (c. 1300 CE).
- A more detailed mathematical treatment, by way of exclusively devoting a chapter (chap. 14, consisting of 75+ verses), is provided by Nārāyaṇa in his *Gaṇitakaumudī* (c. 1356).

Normal and Pan-diagonal Magic squares

- Depending on the number of variant ways in which one can get the desired sum, magic squares have been classified into:
  - semi-magic (only rows and columns sum up to the no.)
  - magic, (rows, columns & principal diagonals)
  - pan-diagonal magic (the above, plus the broken diagonals)
- Example of a normal and a pan-diagonal (PD) magic squares:

A normal Magic Square

(Sum = 34)

12	3	6	13
14	5	4	11
7	16	9	2
1	10	15	8

PD Sum: $6 + 5 + 7 + 8 \neq 34$

A Pan-diagonal Magic Square

(Sum = 34)

10	3	13	8
5	16	2	11
4	9	7	14
15	6	12	1

PD Sum: $13 + 16 + 4 + 1 = 34$

Classification of Magic squares

- Thakkura Pherū in his *Gaṇitasārakaumudī* classifies $n \times n$ magic squares into the following types:
  - *Samagarbha* (n doubly-even or of the form $4m$)
  - *Viṣamagarbha* (n singly-even or of the form $4m+2$)
  - *Viṣama* (n is odd)
- Having made this classification, Pherū presents a few examples of magic squares—that are non pan-diagonal.
- Moreover, they are "*normal*" magic squares of order $n = 3, 4, 5, 6, \ldots,$ whose magic sum are $S = 15, 34, 65, 111, \ldots.$
- In these squares, the entries in the n^2 cells will be sequence of natural numbers $1, 2, \ldots n^2$ and the magic sum will be $\boxed{S = \frac{n(n^2+1)}{2}.}$
- However, in the pan-diagonal magic square described by Nārāyaṇa the sum S need not be magic sum given above.

Purpose as laid down by Nārāyaṇa

- The purpose of magic squares has been delineated thus:

  सद्गणितचमत्कृतये यन्त्रविदां प्रीतये कुगणकानाम्।
  गर्वक्षित्यै वक्ष्ये तत्सारं भद्रगणिताख्यम्॥[1]

- Classifying the magic squares Nārāyaṇa observes:

  समगर्भविषमगर्भे विषमञ्चेति त्रिधा भवेद् भद्रम्।

- Defines them as follows:

  भद्राङ्के चतुराप्ते निरग्रके तद्भवेच्च समगर्भम्।
  द्व्यग्रे तु विषमगर्भं त्र्येकाग्रे केवलं विषमम्॥

  When the order of the magic square is divided by 4, if the remainder $r = 0$, then it is *samagarbha*; if $r = 2$, then it is *viṣamagarbha*; and if $r = 3$ or 1, then it is *viṣama*.

[1] *Gaṇitakaumudī* 14.2.

Avatārikā to *Bhadragaṇita* by Nārāyaṇa

- One of the notable features of Nārāyaṇa is that he methodically introduces all topics that he discusses.
- For instance, in the chapter on Magic squares he sets apart 5 verses right at the beginning to introduce the topic.

सर्वेषां भद्राणां श्रेढीरीत्या भवेद् गणितम्।
येषां गणितमभीष्टं साध्यौ तेषां मुखप्रचयौ। . . .

यद्यावन्ति गृहाणि श्रेढीविषये भवेद् गच्छः।

भद्रे कृतिगतकोष्ठे तन्मूलं जायते चरणः।
इह नारायणविहिता परिभाषा भद्रगणिते च॥

- In all magic squares, it is through arithmetic progression . . .
- By those desirous . . . the first term and the common difference have to be determined.
- As many as the number of boxes in the square will be equal to the number of terms (n^2).

Popularity of Magic squares in India

- The first chapter of Srinivasa Ramanujan's Notebooks is on Magic Squares. It is said to be "much earlier than the remainder of the notebooks".
- T. Vijayaraghavan, in his article on Jaina Magic Squares (1941) notes: "The author of this note learnt by heart at the age of nine the following pan-diagonal square which was taught to him by an elderly person who had not been to school at all."

8	11	2	13
1	14	7	12
15	4	9	6
10	5	16	3

- This clearly indicates the popularity of Magic Squares in India.
- Indian mathematicians specialized in the construction of a special class of magic squares called *sarvatobhadra*.

Kacchapuṭa of Nāgārjuna (c.100 BCE)

- The elements in the magic square are given using the ***Kaṭapayādi*** system of specifying numbers by the string *arka indunidhānārī* . . . .
- Half the blocks are filled with zeros.
- These blocks can be simply filled with $n - x$, where x is alternate element across the diagonal.

अर्क इन्दुनिधानारी
तेन लब्धा विनासनम्।

0	1	0	8
0	9	0	2
6	0	3	0
4	0	7	0

$n-3$	1	$n-6$	8
$n-7$	9	$n-4$	2
6	$n-8$	3	$n-1$
4	$n-2$	7	$n-9$

Pan-diagonal with total $2n$

$n-3$	1	$n-5$	8
$n-6$	9	$n-4$	2
6	$n-7$	3	$n-1$
4	$n-2$	7	$n-8$

Total $2n + 1$

Kacchapuṭa of Nāgārjuna (c.100 BCE)

The following pan-diagonal magic square totaling to 100 has also been called *Nāgārjunīya*

30	16	18	36
10	44	22	24
32	14	20	34
28	26	40	6

Sarvatobhadra of Varāhamihira (550 CE)

In the Chapter on *Gandhayukti* of *Bṛhatsaṃhitā*, Varāhamihira describes the *Sarvatobhadra* perfumes

द्वित्रिइन्द्रियअष्टभागैः अगुरुः पत्रं तुरुष्कशैलेयौ।
विषयअष्टपक्षदहनाः प्रियङ्गुमुस्तारसाः केशः॥
स्पृक्कात्वक्तगराणां मांस्याश्च कृतएकसप्तषड्भागाः।
सप्तऋतुवेदचन्द्रैः मलयनखश्रीककुन्दुरुकाः॥

2	3	5	8
5	8	2	3
4	1	7	6
7	6	4	1

षोडशके कच्छपुटे यथा तथा मिश्रिते चतुर्द्रव्ये।
येऽष्टादश भागास्तेऽस्मिन् गन्धादयो योगाः॥
नखतगरतुरुष्कयुता जातीकर्पूरमृगकृतोब्दोधाः।
गुडनखधूप्या गन्धाः कर्तव्याः सर्वतोभद्राः॥

In the *Kacchapuṭa* with sixteen cells, [are placed] two parts of *agaru*, three parts of *patra*, five parts of *turuṣka* and eight parts of *śaileya* in the cells of the first row,....
When these are mixed in whatever way, there will be 18 parts. To such a mixture are added *nakha* ..., in equal measures, and in this way the *sarvatobhadra* are produced.

Sarvatobhadra of Varāhamihira (550 CE)

As the commentator *Bhaṭṭotpala* (c.950) explains:

अस्मिन् षोडशके षोडशककोष्ठके
कच्छपुटे यथा तथा येन केन
प्रकारेण चतुर्द्रव्ये मिश्रिते
एकीकृते।

2	3	5	8
5	8	2	3
4	1	7	6
7	6	4	1

चतुर्भिर्द्रव्यैः यथाभागपरिकल्पितैः मिश्रीकृतैरत्र येऽष्टादश भागा भवन्ति तेऽस्मिन् कच्छपुटे गन्धादय ऊर्ध्वाधःक्रमेण तिर्यग्वा चतुर्षु कोणेषु वा मध्यमचतुष्कोणे वा कोणकोष्ठचतुष्टये वा प्राक्पङ्क्तौ वा मध्यमकोष्ठद्वये वा अन्त्यपङ्क्तौ। मध्यमकोष्ठद्वये या द्वितीयतृतीयपङ्क्तौ वाद्यन्तकोष्ठके वा येन केन प्रकारेण, चतुर्षु मिश्रितेषु अष्टादशभागा भवन्ति।...तस्माद्यतस्ततो गृह्यमाणा अष्टादशभागा भवन्ति अतः सर्वतोभद्रसंज्ञाः।

In this *kacchapuṭa* with 16 cells, when four substances are mixed in whatever way: When the four substances with their mentioned number of parts are mixed, then the total will be 18 parts; this happens in the above *kacchapuṭa* when the perfumes are mixed from top to bottom (along the columns) or horizontally (along the rows), along the four directions, or the central quandrangle, or the four corner cells, or the middle two cells of the first row together with those of the last row; the middle two cells of the second and third row or the first and last cells of the same, or in any other manner. If the substances in such four cells are added there will be 18 parts in all. ... Since, in whatever way they are mixed, they lead to 18 parts, they are called *Sarvatobhadra*.

Sarvatobhadra of Varāhamihira $\rightarrow$ Nārāyaṇa's

This square

8	0	8	0
0	8	0	8
0	8	0	8
8	0	8	0

$+$

Varāhamihira's

2	3	5	8
5	8	2	3
4	1	7	6
7	6	4	1

$\rightsquigarrow$

Nārāyaṇa's

10	3	13	8
5	16	2	11
4	9	7	14
15	6	12	1

- It is to be noted that all the three squares are pan-diagonal.
- This belongs to a class of 4×4 pan-diagonal magic squares studied by Nārāyaṇa Paṇḍita in *Gaṇitakaumudī* (c.1356).
- We'll see later that there are 384 possible ways of constructing such (4×4) magic squares.

Jaina magic square (inscriptional reference)

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Pan-diagonal magic square found in the inscriptions at Dudhai in Jhansi District (c.11^{th} century) and at the Jaina temple in Khajuraho (c.12^{th} century).

Obtaining 4 $\times$ 4 PD squares: Horse-move method

चतुरङ्गतुरगगत्या द्वौ द्वौ श्रेढीसमुद्भवावङ्कौ।
न्यस्य क्रमोत्क्रमेण च कोष्ठैक्य-एकान्तरेण च॥ १० ॥
सव्यासव्यतुरङ्गमरीत्या कोष्ठान् प्रपूरयेदङ्कैः।
समगर्भे षोडशगृहभद्रे प्रोक्तो विधिश्चायम्॥ ११ ॥
तिर्यक्कोष्ठगतानां ऊर्ध्वस्थानां च कर्णगानां च।
अङ्कानां संयोगः पृथङ्मितो जायते तुल्यः॥ १२ ॥
इह समगर्भानामप्यन्येषां उद्भवश्चतुर्भद्रात्।

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

चतुरङ्गतुरगगत्या $\rightarrow$ like the movement of horse in chess
द्वौ द्वौ $\rightarrow$ choose pairs of numbers [from the sequence]
कोष्ठैक्य $\rightarrow$ in two adjacent cells
एकान्तरेण च $\rightarrow$ and at an interval of one cell
सव्यासव्यतुरङ्गमरीत्या $\rightarrow$ by the method of the horse moving to the left and right
षोडशगृहभद्रे $\rightarrow$ in a magic square with 16 cells
समगर्भानामप्यन्येषां $\rightarrow$ other magic squares of order $4m$

Possible no. of 4 $\times$ 4 PD squares (with elements 1,2...16)?

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

- Nārāyaṇa now poses the question:

  एकाद्येकोत्तरके षोडशगृहकेऽपि कति चतुर्भद्रे।
  भेदा वद यदि गणिते गणकवर, अस्त्यत्र गर्वस्ते॥

- Having displayed 24 pan-diagonal 4 $\times$ 4 magic squares, with the top left entry being 1, Nārāyaṇa states:

  एवं चतुर्भद्रस्य चतुर्भिः यमलैः चतुरशीत्यधिक-शतत्रयभेदा भवन्ति।
  Thus there are 384 possibilities in a magic square . . .

- This has been proved by B. Rosser and R. J. Walker (1938); Much simpler proof was provided by T. Vijayaraghavan (1941).

Ancient Indian method for odd squares

8	1	6
3	5	7
4	9	2

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

- This method of proceeding along small diagonals (***alpaśruti***) is described as an ancient method by Nārāyaṇa Paṇḍita in *Gaṇitakaumudī*.
- Nārāyaṇa actually also displays the eight — and only eight — 3×3 magic squares that can be constructed this way.

De La Laubere, French Ambassador in Siam, wrote in 1693 that he learnt this Indian method from a French doctor M. Vincent who had lived in Surat.

Ancient Indian method for odd squares

Verses presented by Nārāyaṇa

इष्टाशाप्रथमे कोष्ठे श्रेढ्यङ्कं प्रथमं न्यसेत्।
तत्प्रत्याशाप्रान्त्यकोष्ठसमीपभवने ततः ॥४३॥
अस्माद् अल्पश्रुतिगृहेषु अङ्कानेकादिकान् लिखेत्।
कर्णकोष्ठे पुरः साङ्के तत्स्यात् पादप्रपूरणम्॥४४॥
तत्पृष्ठगान् पुनश्चैवं पादानां पूरणं क्रमात्।
अथवा एवं भवन्त्यस्मिन् भेदा भद्रे च वैषमे॥४५॥

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

इष्टाशा $\rightarrow$ desired direction
प्रथमे कोष्ठे $\rightarrow$ in the top cell
श्रेढ्यङ्कं प्रथमं $\rightarrow$ the first no. of the sequence
न्यसेत् $\rightarrow$ may be placed
तत्प्रत्याशा $\rightarrow$ the opp. direction
अल्पश्रुतिगृहेषु $\rightarrow$ in the cells along the small diagonals
कर्णकोष्ठे पुरः साङ्के $\rightarrow$ if the next cell is already filled with number

Obtaining the magic sum

- Right at the beginning of the chapter, Nārāyaṇa presents the formula for finding the magic sum (S).

  सपदः पदवर्गोऽर्धं रूपादिचयेन भवति सङ्कलितम्।
  तत् पदमूलेन हृतं फलं भवेदिष्टभद्रे वै॥

- The term *padaṃ* is used to refer to the number of terms. Denoting it by N, the formula given may be written as:

$$\textit{saṅkalita} = \frac{1}{2}(N^2 + N)$$

- Now the magic sum is given by

$$\text{magic sum} \quad S = \frac{\textit{saṅkalita}}{\sqrt{N}}.$$

- Taking $N = 16$, we will get $S = 34$.

Kuṭṭaka and magic squares

- Given the magic sum S, and the order of the magic square n, the first thing to be done to construct the magic square is to obtain the *śreḍhi*—defined by $(a,\ d)$.
- Having obtained $(a,\ d)$, the *śreḍhī* having n^2 elements is constructed and this will be used to fill in $n \times n$ square. Nārāyaṇa makes use of the following *kuṭṭākāra* to obtain $(a,\ d)$.

$$nS = n^2\left[\frac{1}{2}\{a + (a + (n^2 - 1)d)\}\right] \tag{1}$$

$$\text{or} \quad S = na + \left(\frac{n}{2}\right)(n^2 - 1)d \tag{2}$$

- It is well known that in a *kuṭṭākāra* problem there exists an infinite number of integral solutions for $(a,\ d)$ if S is divisible by the GCD of $\left(n, \left(\frac{n}{2}\right)(n^2 - 1)\right)$. In other words, S should be divisible by n when n is odd, and by $\frac{n}{2}$ for n even.
- Nārāyaṇa's example: Construct a 4×4 magic square with $S = 40$. Now we have the equation $40 = 4a + 30d$ which is satisfied by the pairs $(a, d) = (-5, 2)\ (10, 0)\ (25. - 2)$, and so on.

Properties of 4×4 pan-diagonal magic squares

Property 1: Let M be a pan-diagonal 4×4 magic square with entries $1, 2, \ldots, 16$, which is mapped on to the torus by identifying opposite edges of the square. Then the entries of any 2*x*2 sub-square formed by consecutive rows and columns on the torus add up to 34.

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

$$1 + 12 + 15 + 6 = 1 + 12 + 14 + 7 = 34$$

Property 2: Let M be a 4×4 pan-diagonal magic square with entries $1, 2, \ldots, 16$, which is mapped on to the torus. Then, the sum of an entry on M with another which is two squares away from it along a diagonal (in the torus) is always 17.

$$1 + 16 = 6 + 11 = 15 + 2 = 4 + 13 = 14 + 3 = 9 + 8 = 17$$

Properties of 4×4 pan-diagonal magic squares

1	12	13	8
15	6	3	10
4	9	16	5
14	7	2	11

The "neighbours" of an element of a 4×4 pan-diagonal magic square (which is mapped on to the torus as before) are the elements which are next to it along any row or column. For example, $3, 5, 2$ and 9 are the "neighbours" of 16 in the magic square given in the LHS..

Property 3: Let M be a 4×4 pan-diagonal magic square with entries $1, 2, \ldots, 16$, which is mapped on to the torus. Then the neighbours of the entry 16 have to be the entries $2, 3, 5$ and 9 in some order.

We can use the above properties, and very easily construct 4×4 pan-diagonal magic squares starting with 1, placed in any desired cell.

Thanks!

THANK YOU

More of Magic squares in the next lecture!