NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 32

Jyānayanam: Computation of Rsines

K. Ramasubramanian
IIT Bombay

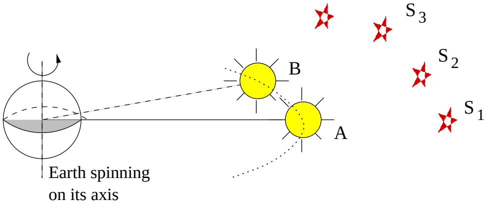
Outline

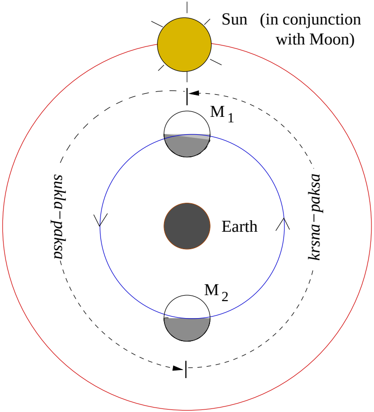
- Introduction: Motivation for finding sine values
- Approach in earlier *Siddhāntas*
- Construction of sine table: Āryabhaṭa's approach
  - Geometric & Analytic
  - The terse verse in *Āryabhaṭīya*
- Bhāskara's formula (his invention?)
  - A brilliant rational approximation
  - Plausible derivation of the result
- The sine table in *Sūryasiddhānta*
- Govindasvamin's improvised tabular Rsines
- Rule for obtaining the sine values in *Tantrasaṅgraha* (+ comm.)
- Mādhava's series for sine
- His simplified formula (*vidvānityādi*)

Introduction

Motivation for finding accurate sine values

- The positions of the planets in the background of stars forms the basis for reckoning time.
- Determination of their positions crucially depends upon accurate knowledge of sine function.





Introduction

Determination of time from shadow measurement

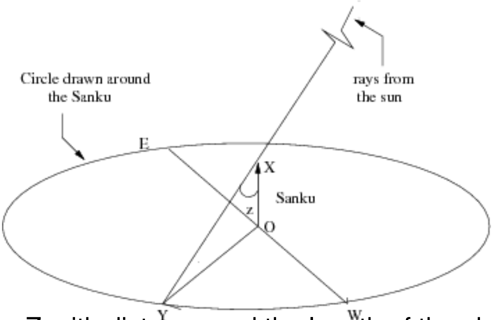


Figure: Zenith distance and the length of the shadow.

$$t = (R\sin)^{-1}\left[\frac{R\cos z}{\cos\phi\cos\delta} \pm R\sin\Delta\alpha\right] \mp \Delta\alpha.$$

If ϕ and δ are known ($\Delta\alpha = f(\phi,\delta)$), then t is known.

Computing Rsines: Approach in earlier *siddhāntas*

- Varāhamihira has given the following Rsine values and relations in his *Pañcasiddhāntikā* (c. 505 CE):[1]

$$
\begin{aligned}
R\sin(30^\circ) &= \frac{R}{2} && (1a)\\
R\sin(45^\circ) &= \frac{R}{\sqrt{2}} && (1b)\\
R\sin(60^\circ) &= \frac{\sqrt{3}}{2}R && (1c)\\
R\sin(90^\circ) &= R && (1d)
\end{aligned}
$$

$$
\begin{aligned}
R\sin(\theta) &= R\cos(90-\theta) && (2)\\
R\sin^2(\theta) + R\cos^2(\theta) &= R^2 && (3)\\
R\sin\left(\frac{\theta}{2}\right) &= \left(\frac{1}{2}\right)\left[R\sin^2(\theta) + R\,\mathrm{vers}^2(\theta)\right]^{\frac{1}{2}} && (4)
\end{aligned}
$$

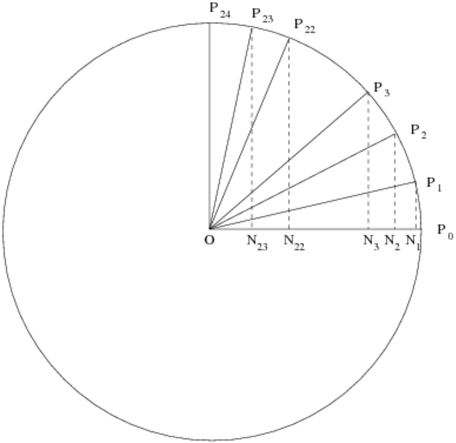
The above Rsine values (1) and relations (2)–(4) can be derived using the *bhujā-koṭi-karṇa-nyāya* and *trairāśika* (rule of three for similar triangles). Equations (2)–(4) can be used to compute all the 24 tabular Rsine values.

---

[1] *Pañcasiddhāntikā* of Varāhamihira, Ed. by T. S. Kuppanna Sastry and K. V. Sarma, Madras 1993, verses 4.1–5, pp. 76–80.

Construction of the sine-table

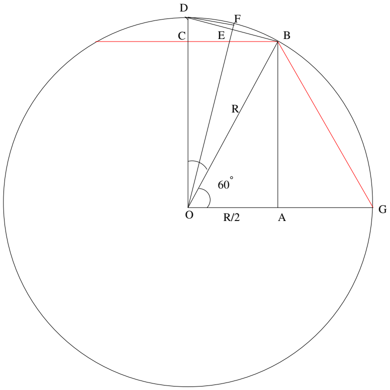
A quadrant is divided into 24 equal parts, so that each arc bit $\alpha = \frac{90}{24} = 3^{\circ}45' = 225'$.



- Āryabhaṭa presents two different methods for finding $R \sin i\theta, (P_i N_i)$ $i = 1, 2, \ldots 24$.
- The Rsines of the intermediate angles are determined by interpolation (I order or II order).

Finding tabular sines: Geometrical approach

समवृत्तपरिधिपादं छिन्द्यात् त्रिभुजाच्चतुर्भुजाच्चैव।
समचापज्यार्धानि तु विष्कम्भार्धे यथेष्टानि॥



- We know, chord $60^\circ = R = 3438$
- From the triangle OBC,

$$R \sin 30 = BC = R/2 = 1719$$

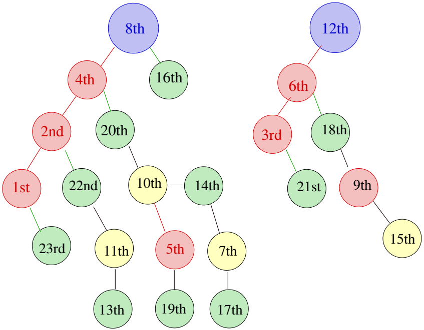
- Using ***bhujā-koṭi-karṇa-nyāya*** from 8th Rsine (30°) we can get 16th Rsine (60°).
- In other words, $R \sin \theta \rightsquigarrow R \sin(90 - \theta)$.
- Further, noting that

$$R \sin \theta \;\rightsquigarrow\; R \cos \theta \;\rightsquigarrow\; R \text{ vers } \theta$$
$$R \sin \theta \;\&\; R \text{ vers } \theta \rightsquigarrow R \sin \frac{\theta}{2},$$

it can be seen that all the 24 Rsines can be calculated.

Finding tabular sines: Geometrical approach (contd.)

- Most of the Indian astronomers have presented their sine tables by dividing the quadrant (90°) into 24 parts.
- By the principle outlined above, it can be easily shown that all the 24 Rsines can be obtained provided the 24th, 12th and 8th Rsines are known.



- The circumference of the circle was taken by Āryabhaṭa to be 21600 units.
- From that using the approximation for π given by him, we get $R = 24th$ Rsine ≈ 3438.
- Once this it known, it is noteworthy that in the proposed scheme of constructing the table, all that is required is extraction of square root, for which Āryabhaṭa had clearly evolved an efficient algorithm.

Finding tabular sines: Analytic approach

प्रथमाच्चापज्यार्धात् यैरूनं खण्डितं द्वितीयार्धम्।
तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि॥

Let $B_k = R\sin(k \times 225'),\ (k = i, 2, \ldots, 24)$, be the twenty-four Rsines, and let $\Delta_k = B_k - B_{k-1},\ (k = i, 2, \ldots, 24)$... be the Rsine-differences. Then, the above rule may be expressed as[2]

$$\Delta_2 = B_1 - \frac{B_1}{B_1} \tag{5}$$

$$\Delta_{k+1} = B_1 - \frac{(B_1 + B_2 + \ldots + B_k)}{B_1} \qquad (k = 1, 2, \ldots, 23). \tag{6}$$

This second relation is also sometimes expressed in the equivalent form

$$\Delta_{k+1} = \Delta_k - \frac{(\Delta_1 + \Delta_2 + \ldots + \Delta_k)}{B_1} \qquad (k = 1, 2, \ldots, 23). \tag{7}$$

From the above the discrete version of the harmonic equation follows

$$\Delta_{k+1} - \Delta_k = \frac{-B_k}{B_1} \qquad (k = 1, 2, \ldots, 23). \tag{8}$$

[2] Āryabhaṭa is using the approximation $\Delta_2 - \Delta_1 \approx 1'$.

Āryabhaṭa's table for computing Rsines

- Using either/both the approaches, Āryabhaṭa having obtained Rsine values has presented a table in *Gītikā-pāda* of *Āryabhaṭīya* (verse 12).
- This verse[3] lists the 24 first order Rsine-differences (in arc-minutes):

  मखि भखि फखि धखि णखि ञखि

  ङखि हस्झ स्ककि किष्ग स्घकि किघ्व।

  घ्लकि किग्र हक्य धकि किच
  स्ग श्झ ङ्व क्ल प्त फ छ कलार्धज्याः ॥

  225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, and 7—these are the Rsine-differences [at intervals of 225′ of arc] in terms of the minutes of arc.

- In Āryabhaṭa's notation: म $\rightarrow$ 25; & खि $\rightarrow$ 200;

[3] This verse is one of the most terse verse in the entire Sanskrit literature that I have ever come across. Only after several trials would it be ever possible to read the verse properly, let also deciphering its content.

Bhāskara's approximation to sine function

In his *Mahābhāskarīya*, [4] while presenting planetary parameters—the dimension of epicycles, their application, etc. —Bhāskara (c. 628) presents a brilliant rational approximation to sine function.

मख्यादिरहितं कर्म वक्ष्यते तत् समासतः ।
चक्रोर्धांशकसमूहात् विशोध्या ये भुजांशकाः ॥
तच्छेषगुणिता द्विष्ठाः शोध्याः खाभ्रेषुखाब्धितः ।
चतुर्थांशेन शेषस्य द्विष्ठं अन्त्यफलं हतम् ॥
बाहुकोटयोः फलं कृत्स्नं क्रमोत्क्रमगुणस्य वा ।
लभ्यते चन्द्रतीक्ष्णांश्वोः ताराणां वापि तत्त्वतः ॥

This in modern notation translates to

$$\sin x = \frac{x(180 - x)}{\frac{1}{4}[40500 - x(180 - x)]} \qquad (0 \leq x \leq 180)$$

[4] *Mahābhāskarīya*, VII.17-19.

Bhāskara's approximation to sine function

मख्यादिरहितं कर्म वक्ष्यते तत् समासतः ।
चक्रोर्धांशकसमूहात् विशोध्या ये भुजांशकाः ।
तच्छेषगुणिता द्विष्ठाः शोध्याः खाभ्रेषुखाब्धितः ।
चतुर्थांशेन शेषस्य द्विष्ठं . . .

मख्यादिरहितं कर्म	–	operations without मख्यादि
वक्ष्यते	–	is being stated
समासतः	–	completely
चक्रोर्धांशकसमूहात्	–	from 180°
भुजांशकाः	–	arg. of sine in degrees (x)
ये विशिध्याः	–	that which is to be subtracted
तच्छेषगुणिता (भुजांशकाः)	–	$x(180 - x)$
द्विष्ठाः	–	kept in two places
शोध्याः (एकत्र)	–	have to be subracted (in one place)
खाभ्रेषुखाब्धितः	–	from 40500
चतुर्थांशेन (विभजेत्)	–	with $\frac{1}{4}^{th}$ of the result (divide)
द्विष्ठं	–	the value placed in the other place

How good is Bhāskara's approximation?

x	Bhāskara's value	Modern value
0	0.000	0.000
15	0.260	0.259
30	0.500	0.500
45	0.706	707
60	0.865	0.866
75	0.966	0.966
90	1.000	1.000

- The formula gives exact value for certain arguments.
- However, it is noted that for the entire range $[0 \rightarrow 90^\circ]$ it is almost 99% accurate
- This clearly speaks of:
  - The brilliance of the discoverer in arriving at an excellent approximation.
  - a novel attempt to obtain a rational approximation to sine function.
  - the beauty and sophistication of ancient Indian mathematics

Properties mirrored by Bhāskara's approximation

- We know that sine function
  1. is symmetric about 90° point
  2. is concave over the range $0^\circ \to 180^\circ$
- The formula given by Bhāskara clearly satisfies these properties

$\boxed{\text{Bhāskara's approximation}}$

$$\sin \mathbf{x} = \frac{\mathbf{4x(180 - x)}}{\mathbf{40500 - x(180 - x)}}$$

- Isn't a mathematician's delight to arrive at an expression for sine function that at once captures the properties as well as serves as a very good approximation ($\approx$ 99%) for the entire range ($0 - 180^\circ$)?
- Here I may quote the statement made by Hardy[5]

> The Greeks were the first mathematicians who are still 'real' to us today. Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. . . .

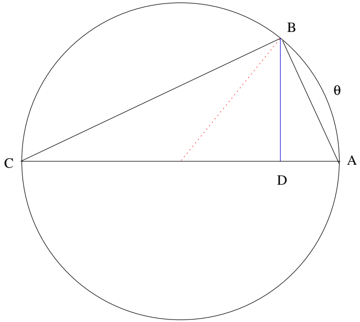
[5] G. H. Hardy, *A Mathematician's Apology* Cambridge, 2nd ed. (1967) p. 80.

Proof of Bhāskara's formula

In the figure, the area of the triangle ABC can be expressed in two ways:

$$A = \frac{1}{2} AB.BC = \frac{1}{2} AC.BD$$

$$\text{or} \qquad \frac{1}{BD} = \frac{AC}{AB.BC} \qquad (9)$$



Since the length of the chord $<$ that of the arc, (9) may be expressed as an inequality

$$\frac{1}{BD} > \frac{AC}{\overset{\frown}{AB} \,.\, \overset{\frown}{BC}}$$

$$\text{or} \qquad \frac{1}{BD} = \frac{x.AC}{\overset{\frown}{AB} \,.\, \overset{\frown}{BC}} + y$$

$$= \frac{2xR}{\theta(180 - \theta)} + y$$

$$\text{or} \qquad R \sin\theta = \frac{\theta(180 - \theta)}{2xR + \theta(180 - \theta)y} \qquad (10)$$

Proof of Bhāskara's formula

Substituting $\theta = 30$ and $\theta = 90^\circ$ in (10) we have,

$$2xR + 4500y = \frac{9000}{R} \tag{11}$$

$$2xR + 8100y = \frac{8100}{R} \tag{12}$$

Soving the above equations for x and y we have,

$$y = -\frac{1}{4R} \qquad \text{and} \qquad 2xR = \frac{40500}{4R} \tag{13}$$

Using the above values in (10), we have

$$R\sin\theta = \frac{4\theta(180-\theta)R}{40500-\theta(180-\theta)} \tag{14}$$

which is the same expression given by Bhāskara.[6]

[6] The above proof has been given by K. S. Shukla in his edition of the text *Mahābhāskarīya* with translation and annotation.

The sine table given in *Sūryasiddhānta* (II. 17–22)

The values presented are in *Bhūtasaṅyā* system and the same as that of *Āryabhaṭīya*

तत्त्वाश्विनः अङ्काब्धिकृता रूपभूमिधरर्त्तवः।
खाङ्काष्टौ पञ्चशून्येशा बाणरूपगुणेन्दवः॥
शून्यलोचनपञ्चैकाः छिद्ररूपमुनीन्दवः।
वियच्चन्द्रातिधृतयो गुणरन्ध्राम्बराश्विनः॥
मुनिषड्यमनेत्राणि चन्द्राग्निकृतदस्रकाः।
पञ्चाष्टविषयाक्षीणि कुञ्जराश्विनगाश्विनः॥
रन्ध्रपञ्चाष्टकयमाः वस्वद्र्यङ्कयमास्तथा।
कृताष्टशून्यज्वलनाः नागाद्रिशशिवह्नयः॥
षट्पञ्चलोचनगुणाः चन्द्रनेत्राग्निवह्नयः।
यमाद्रिवह्निज्वलनाः रन्ध्रशून्यार्णवाग्नयः॥
रूपाग्निसागरगुणाः वस्वग्निकृतवह्नयः।

225, 449, 671, 890, 1105, 1315, 1520, 1710 . . . 3178, 3256, 3321, 3372, 3409, 3431, 3438.

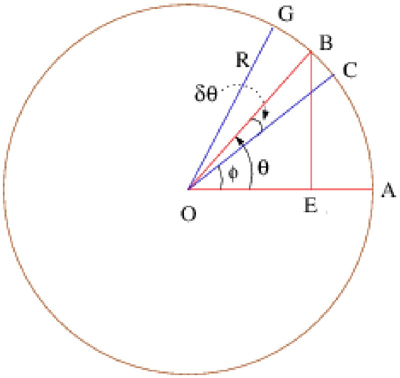
How to find intermediate sine values?

Prescription of first order interpolation (based on *trairāśika-nyāya*)

लिप्ताभ्यस्तत्त्वनेत्राप्ताः गता ज्याः शेषतः पुनः ।
गतगम्यान्तरघ्नाद्य हृतास्तत्त्वयमैः क्षिपेत् ॥ २ ॥
दोःकोटिज्ये नयेदेवं ज्याभ्यश्चापं विपर्ययात् ।

- Suppose we want to find the $jy\bar{a}$ of θ where $i \times 225 < \theta < (i+1) \times 225$.
- Let $\phi = i \times 225$, whose $jy\bar{a}$ is known (J_i).
- Now, $jy\bar{a}\,\theta$ is given by

$$jy\bar{a}\,\theta = J_i + \frac{\delta\theta \times (J_{i+1} - J_i)}{225}.$$



Improvised values of tabular Rsines

- Govindasvamin (9th cent.) in his commentary on *Mahābhāskarīya* presents a set of corrections to the values of the tabular Rsine presented by Āryabhaṭa.
- He observes:

  मख्यादयो हि न्यूनाधिकावयवाः। तेन ज्याच्छेदविधानात्
  अवयवावगतिः। तथा च अवगता एते तत्पराद्याः –

  सप्ताग्निरन्ध्राणि वियद्गुणागं नेत्राब्धिनेत्रं मुनिपञ्चवेदाः।
  द्वाक्ष्यष्टयः षण्णयनद्विरामा वेदाग्निभूतं रविष्टकृशानुः॥
  रन्ध्राभ्रपक्षं गुणापावकाष्टौ चक्षुर्वियत्सप्तखचन्द्रसूर्याः।
  रुद्राग्निचन्द्रा मनुसप्तसोमा दस्राभ्रनेत्रं नयनं द्विसूर्यम्॥
  अक्ष्ब्धिपक्षं वसुनेत्ररन्ध्रं चन्द्राग्निविद्या वसुखाष्टचन्द्रम्।
  रन्ध्रेषु वेदं नवरूपमिध्मं खाभ्राग्नयस्सप्तगुणेध्मसंख्यम्॥

- The values listed in the verses are: $9'' \, 37'''$, $7'' \, 30'''$, $2'' \, 42'''$, $4'' \, 57'''$...

Improvised values of tabular Rsines

θ in min.	R sin θ according to *Āryabhaṭīya*	Govindasvāmi	Mādhava(also Modern)
225	225	224 50 23	224 50 22
450	449	448 42 53	448 42 58
675	671	670 40 11	670 40 16
900	890	889 45 08	889 45 15
1125	1105	1105 01 30	1105 01 39
1350	1315	1315 33 56	1315 34 07
1575	1520	1520 28 22	1520 28 35
1800	1719	1718 52 10	1718 52 24
2025	1910	1909 54 19	1909 54 35
2250	2093	2092 45 46	2092 46 03
2475	2267	2266 38 44	2266 39 50
2700	2431	2430 50 54	2430 51 15
2925	2585	2584 37 43	2584 38 06
3150	2728	2727 20 29	2727 20 52
3375	2859	2858 22 31	2858 22 55
3600	2978	2977 10 09	2977 10 34
3825	3084	3083 12 51	3083 13 17
4050	3177	3175 03 23	3176 03 50
4275	3256	3255 17 54	3255 18 22
4500	3321	3320 36 02	3320 36 30
4725	3372	3371 41 01	3371 41 29
4950	3409	3408 19 42	3408 20 11
5175	3431	3430 22 42	3430 23 11
5400	3438	3437 44 19	3437 44 48

Computation of tablular Rsines (*Tantrasaṅgraha*)

विलिप्तादशकोना ज्या राश्यष्टांशधनुःकलाः ॥
आद्यज्यार्धात् ततो भक्ते सार्धदेवाश्विभिस्ततः ।
त्यक्ते द्वितीयखण्डज्या द्वितीया ज्या च तद्युतिः ॥
ततस्तेनैव हारेण लब्धं शोध्यं द्वितीयतः ।
खण्डात् तृतीयखण्डज्या द्वितीयस्तद्युतो गुणः ॥
तृतीयः स्यात् ततश्चैवं चतुर्थाद्याः क्रमाद् गुणाः ।

- The first $pi\d{n}\d{d}ajy\bar{a}$ is often taken to be $225'$ based on the approximation,

$$R\sin\alpha \approx R\alpha = 225'. \tag{15}$$

- In contrast to the above, it is stated to be $225' - 10'' = 224'\,50''$.
- This is based on the approximation $\sin\alpha \approx \alpha - \frac{\alpha^3}{3!}$. Thus we have,

$$R\sin\alpha \approx \frac{21600}{2\pi}\left(\alpha - \frac{\alpha^3}{6}\right) = 224.8389' \approx 224'50''. \tag{16}$$

Computation of tablular Rsines (*Tantrasaṅgraha*)

विलिप्तादशकोना ज्या राश्यष्टांशधनुःकलाः ॥
आद्यज्यार्धात् ततो भक्ते सार्धदेवाश्विभिस्ततः ।
त्यक्ते द्वितीयखण्डज्या द्वितीया ज्या च तद्युतिः ॥
ततस्तेनैव हारेण लब्धं शोध्यं द्वितीयतः ।
खण्डात् तृतीयखण्डज्या द्वितीयस्तद्युतो गुणः ॥
तृतीयः स्यात् ततश्चैवं चतुर्थाद्याः क्रमाद् गुणाः ।

- The above verses essentially present the following equations for generating the successive *jyā* values.

$$\mathbf{J_{i+1}} = \mathbf{J_i} + \Delta_{i+1} \qquad (0 \leq i \leq 23) \tag{17}$$

$$\Delta_{i+1} = \Delta_i - \frac{\mathbf{J_i}}{\mathbf{233\frac{1}{2}}} \qquad (1 \leq i \leq 23), \tag{18}$$

- Since $\Delta_1 = J_1$, is known, all the *jyās* can be generated using the above equations recursively.

Evolution in the recursion relation

- From the time of Āryabhaṭa (499 CE), the astronomers have been using the recursion relation, to obtain the values of the 24 *Rsine – differences*.
- Either these values themselves, or the *jyā* values have been listed explicitly in the form verses.
- The exact recursion relation for the Rsine differences is:

$$\Delta_{i+1} = \Delta_i - J_i \; 2(1 - \cos\alpha) \qquad (\alpha = 225) \tag{19}$$

- While, $2(1 - \cos\alpha) = 0.0042822$, the different. approximations that have been employed are:

$$2(1-\cos\alpha) \approx \frac{1}{225} \quad = \quad 0.0044444 \qquad (\text{Āryabhaṭa})$$

$$2(1-\cos\alpha) \approx \frac{1}{233.5} \quad = \quad 0.0042827 \qquad (\text{Nīlakaṇṭha})$$

$$2(1-\cos\alpha) \approx \frac{1}{233\frac{32}{60}} \quad = \quad 0.00428204 \qquad (\text{Śaṅkara Variyar}).$$

Rsine table due to Mādhava

The values presented are in *Kaṭapayādi* system and are correct to thirds

श्रेष्ठं नाम वरिष्ठानां हिमाद्रिर्वेदभावनः ।
तपनो भानुसूक्तज्ञो मध्यमं विद्धि दोहनम् ॥ १ ॥
धिगाज्यो नाशनं कष्टं छन्नभोगाशयाम्बिका ।
मृगाहारो नरेशोऽयं वीरो रणजयोत्सुकः ॥ २ ॥
मूलं विशुद्धं नालस्य गानेषु विरला नराः ।
अशुद्धिगुप्ता चोरश्रीः शङ्कुकर्णो नगेश्वरः ॥ ३ ॥······
धीरो युवा कथालोलः पूज्यो नारीजनैर्भगः ।
कन्यागारे नागवल्ली देवो विश्वस्थली भृगुः ॥ ६ ॥
तत्परादिकलान्तास्तु महाज्या माधवोदिताः ।
स्वस्वपूर्वविशुद्धे तु शिष्टास्तत्खण्डमौर्विकाः ॥ ७ ॥ इति ॥

- The first and the last values are: $224' \; 50'' \; 22'''$ and $3437' \; 44 \; 48'''$.
- These values have been arrived at by considering terms up to θ^{11} in the series expansion of $\sin\theta$ – ascribed to Mādhava.

Comparing the *jyā* values of different texts

Āryabhaṭīya, *Sūryasiddhānta*, *Tantrasaṅgraha*, *Laghuvivṛtti*, Mādhava & modern

Dhanu or *Cāpa*		Notation	Value of the *jyā* (in minutes, seconds and thirds)				
Symbol used	Length (min)	Notation used	As in AR/SS	From TS	From LV	Given by Mādhava	Modern
S_1	225	J_1	225	224 50	224 50 22	224 50 22	224 50 21
S_2	450	J_2	449	448 42	448 42 58	448 42 58	448 42 57
S_3	675	J_3	671	670 39	670 40 16	670 40 16	670 40 16
S_4	900	J_4	890	889 44	889 45 17	889 45 15	889 45 15
S_5	1125	J_5	1105	1105 00	1105 01 41	1105 01 39	1105 01 38
S_6	1350	J_6	1315	1315 32	1315 34 11	1315 34 07	1315 34 07
S_7	1575	J_7	1520	1520 26	1520 28 41	1520 28 35	1520 28 35
S_8	1800	J_8	1719	1718 49	1718 52 32	1718 52 24	1718 52 24
S_9	2025	J_9	1910	1909 51	1909 54 46	1909 54 35	1909 54 35
S_{10}	2250	J_{10}	2093	2092 42	2092 46 19	2092 46 03	2092 46 03
S_{11}	2475	J_{11}	2267	2266 35	2266 40 10	2266 39 50	2266 39 50
S_{12}	2700	J_{12}	2431	2430 45	2430 51 40	2430 51 15	2430 51 14
S_{13}	2925	J_{13}	2585	2584 32	2585 38 37	2584 38 06	2584 38 05
S_{14}	3150	J_{14}	2728	2727 14	2727 21 31	2727 20 52	2727 20 52
S_{15}	3375	J_{15}	2859	2858 15	2858 23 42	2858 22 55	2858 22 55
S_{16}	3600	J_{16}	2978	2977 02	2977 11 30	2977 10 34	2977 10 33
S_{17}	3825	J_{17}	3084	3083 03	3083 14 23	3083 13 17	3083 13 16
S_{18}	4050	J_{18}	3177	3175 53	3176 05 07	3176 03 50	3176 03 49
S_{19}	4275	J_{19}	3256	3255 06	3255 19 50	3255 18 22	3255 18 21
S_{20}	4500	J_{20}	3321	3320 24	3320 38 11	3320 36 30	3320 36 30
S_{21}	4725	J_{21}	3372	3371 27	3371 43 24	3371 41 29	3371 41 29
S_{22}	4950	J_{22}	3409	3408 05	3408 22 20	3408 20 11	3408 20 10
S_{23}	5175	J_{23}	3431	3430 07	3430 25 35	3430 23 11	3430 23 10
S_{24}	5400	J_{24}	3438	3437 27	3437 47 29	3437 44 48	3437 44 48

Infinite series for the sine function

- The verses giving the ∞ series for the sine function is[7] –

निहत्य चापवर्गेण चापं तत्तत्फलानि च ।
हरेत् समूलयुग्वर्गैः त्रिज्यावर्गहतैः क्रमात् ॥
चापं फलानि चाधोऽधो न्यस्योपर्युपरि त्यजेत् ।
जीवाप्त्यै, सङ्ग्रहोऽस्यैव विद्वानित्यादिना कृतः ॥

- $N_0 = R\theta \qquad D_0 = 1$
- $N_1 = R\theta \times (R\theta)^2 \qquad N_{i+1} = N_i \times (R\theta)^2$
- $D_1 = R^2(2 + 2^2) \qquad D_i = D_{i-1} \times R^2(2i + (2i)^2)$
- जीवा $= \frac{N_0}{D_0} - [\frac{N_1}{D_1} - (\frac{N_2}{D_2} - \{\frac{N_3}{D_3} - \ldots\})]$
- जीवाप्त्यै = For obtaining the *jīva* (Rsine)

[7] *Yuktidīpikā* (16th cent) and attributed to Mādhava (14th cent. AD).

Infinite series for the sine function

- Expressing the series using modern notation as described as described in the above verse –

$$J\bar{\imath}v\bar{a} = R\theta - \frac{R\theta \times (R\theta)^2}{R^2(2+2^2)} + \frac{R\theta \times (R\theta)^2 \times (R\theta)^2}{R^2(2+2^2)\; R^2(4+4^2)} - \ldots$$

- Simplifying the above we have –

$$J\bar{\imath}v\bar{a} = R\theta - \frac{(R\theta)^3}{R^2 \times 6} + \frac{(R\theta)^5}{R^4 \times 6 \times 20} - \frac{(R\theta)^7}{R^6 \times 6 \times 20 \times 42} + \ldots$$

- Further simplifying –

$$J\bar{\imath}v\bar{a} = R\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right) = R\sin\theta$$

- Thus the given expression $\equiv$ well known sine series.

How to use the series to evaluate Rsines?

- Though Mādhava came up with the infinite series, it would certainly be impossible to use the series expansion

$$R \sin\theta = R\theta - \frac{(R\theta)^3}{3!R^2} + \frac{(R\theta)^5}{5!R^4} - \frac{(R\theta)^7}{7!R^6} + \frac{(R\theta)^9}{9!R^8} - \frac{(R\theta)^{11}}{11!R^{10}} + \ldots,$$

  for obtaining the values of Rsines.
- However, since the series converges pretty fast—because of the factorial in the denominator—it would suffice to use a few terms in it.
- Noting this, Mādhava gave explicit values of the magnitudes of the terms starting in the reverse from sixth and up to the second in (the RHS of) the above equation when the arc $R\theta = 5400' = 90^\circ$.
- These values are mentioned in *Yuktidīpikā* using the *kaṭapayādi* notation in the following verse.

  विद्वांस्तुन्नबलः कवीशनिचयः सर्वार्थशीलस्थिरः
  निर्विद्धाङ्गनरेन्द्ररुङ्निगदितेष्वेषु क्रमात् पञ्चसु ।

How to use the series to evaluate Rsines?

Term no. in RHS	Sanskrit equivalent in *kaṭapayādi*	Mādhava's value according to *Yuktidīpikā*	Modern value
VI	विद्वान्	$0'0''44'''$	$0'0''44.54'''$
V	तुन्नबलः	$0'33''6'''$	$0'33''5.6'''$
IV	कवीशनिचयः	$16'05''41'''$	$16'05''40.87'''$
III	सर्वार्थशीलस्थिरः	$273'57''47'''$	$273'57''47.11'''$
II	निर्विद्धाङ्गनरेन्द्ररुक्	$2220'39''40'''$	$2220'39''40.10'''$

We find that the values given by Mādhava are indeed very accurate. Thus for finding arbitrary value of sine we use the equation

$$\begin{aligned} R\sin\theta &= R\theta - \beta^3(2220'39''40''') + \beta^5(273'57''47''') - \beta^7(16'5''41''') \\ &\quad + \beta^9(0'33''6''') - \beta^{11}(0'0''44'''), \end{aligned}$$

where $\beta = \frac{R\theta}{5400}$.

Concluding Remarks

- It was quite interesting to know the evolution of different techniques in India to evaluate the sine function – which is ubiquitous – as accurately as possible.
- Broadly speaking, the approach taken by Indian astronomers and mathematicians can be classfied as:
  - to improve the accuracy of the sine table (which forms basis for interpolation as well as $\sin(A+B)$)
  - to obtain a good rational approximation
  - arrive at infinite series.
- The absolute logical rigor with which the results have been arrived at is indeed remarkable.
- Why were they worried about very accurate values of sines ?
- Accuracy of *Trijyā* R $\rightarrow$ Accuracy in the computation of sines $\rightarrow$ Accuracy in planetary positions $\rightarrow$ Accuracy in the determination of tithis, and so on, $\rightarrow$ Avoid incompleteness.[8]

[8] नास्ते कालावयवकलना ...श्रौतस्मार्तव्यवहृतिरपि छिद्यते तत्र धर्माः ।

Thanks!

THANK YOU