NPTEL COURSE ON

MATHEMATICS IN INDIA:
FROM VEDIC PERIOD TO MODERN TIMES

Lecture 33

Trigonometry and Spherical Trigonometry 1

M. S. Sriram
University of Madras, Chennai.

Outline

- Crucial role of trigonometry in astronomy problems
- Indian sines, cosines, *bhujajyā*, *koṭijyā*, sine tables
- Interpolation formulae
- Determination of the exact value of 24 sines
- Bhāskara's *jyotpatti* - $sin(18^\circ)$, $sin(36^\circ)$

Non-uniform motion of planets

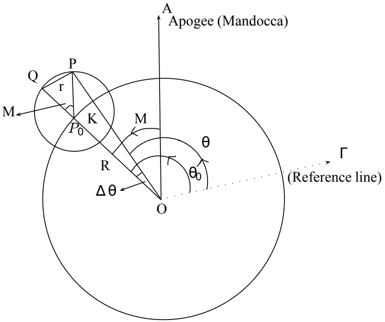
Ancients had observed regularity in the motion of celestial bodies (Stars, Sun, Moon and Planets) in the sky. Stars : Extremely regular. Others : Not Completely. Departures from complete regularity observed over millenia. Ancients : Sun , Moon also considered as planets. So : Non-uniform Motion of Planets.

Trigonometry is needed to explain the non-uniform motion of the planets. This was the historical context for developing trigonometry both in Indian and Greek astronomy.

Now, we know that the planets move in elliptical orbits around the Sun. Moon moves in an elliptical orbit around the Earth. In a geocentric framework, One can say that the Sun moves in an elliptical orbit around the Earth. So, the orbits have an eccentricity. How was this taken into account in ancient astronomy?

Epicycle model

One had an ‘epicycle’ model for the motion of a planet both in Indian and Greek astronomy. The details are different, but the basic idea is as follows:



Epicycle model for the eccentricity correction.

P_0 : Mean planet moving aound O at a uniform rate in a circle called the ‘*Kakśyāvṛtta*’ or ‘ Deferent’. Γ is a reference line (like the direction of the first point of ‘*Meṣa*’ *rāśi*.)

$\Gamma\hat{O}P = \theta_0$ is called the ‘mean planet’.

Finding the true planet

To find the true position of the planet, draw a circle of radius r around P_0 (the radius of the Deferent is R.) This is the 'epicycle, or '$Mandav\d{r}tta$'. Now there is what is known as the direction of the 'apogee' shown as OA in the figure. A is called the '$mandocca$' in Indian texts: Draw a line P_0P parallel to OA, intersecting the epicycle ($Mandav\d{r}tta$) at P. Then O is the true position of the planet. $\Gamma\hat{O}A$ is the longitude of the 'apogee' and $M = A\hat{O}P_0 = \Gamma\hat{O}P_0 - \Gamma\hat{O}A = \theta_0 - \Gamma\hat{O}A$, is called the '$Mandakendra$'.

$$\text{True Longitude}\,\theta = \Gamma\hat{O}P = \Gamma\hat{O}P_0 - P_0\hat{O}P = \theta_0 - \Delta\theta$$

where $\Delta\theta$ is the correction to be applied to θ_0, the mean longitude to obtain the true longitude. It is called the "Equation of Centre."

Appearence of Sine function, Enter Trigonometry

Let $K = OP$. This is called the *manda-karṇa*. Extend P_0 to Q such that PQ is perpendicular to P_0Q. As P_0P is parallel to OA (by construction), $P\hat{P}_0Q = M$ and $PM = r\sin M$.

$$\text{In triangle } POQ, P\hat{O}Q = P\hat{O}P_0 = \Delta\theta, \text{ and so,}$$

$$OP\sin\Delta\theta = PQ = r\sin M$$

$$\therefore\ K\sin\Delta\theta = r\sin M$$

$$\therefore\ \sin\Delta\theta = \frac{r}{K}\sin M = \frac{r}{K}\sin(\theta_0 - A)$$

$$\therefore\ \Delta\theta = \sin^{-1}\left(\frac{r}{K}\sin M\right) = \sin^{-1}\left(\frac{r}{K}\sin(\theta_0 - A)\right)$$

where $K = OP = [(R + r\cos M)^2 + r^2\sin^2 M]^{1/2}$.

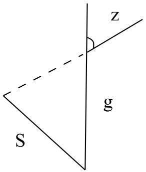
To know the correction $\Delta\theta$, one needs the sine function. One should also know how to find the inverse sine function, that is to find the arc from the sine.

This is how the trigonometric functions enter astronomy.

To find $\Delta\theta$ for any θ_0 and A, we should know $\sin(\theta_0 - A) = \sin M$, either by explicit construction or tabulated values.

Shadows and Trigonometry

Again, to find the time from the shadow of a gnomon.



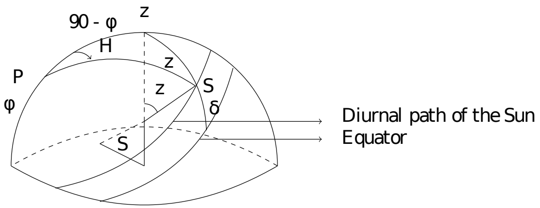
Shadow of a gnomon.

The light rays are slanted at an angle z to the vertical. z is the 'Zenith distance' of the sun. g is the "gnomon" height and S is the shadow.

$$S = g \tan z = g \frac{\sin z}{\cos z}$$

z depends upon how much time has elapsed since the Sun has crossed the meridian, through the 'hour angle' H.

Time from shadows: Spherical Trigonometry



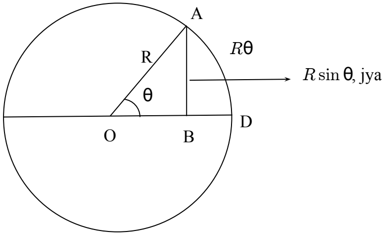
Sun at zenith distance z on the celestial sphere.

In the figure, the position of the Sun in the sky is shown. z is the Zenith distance of the Sun, H is its hour angle, which indicates how much time has elapsed from the 'noon' when the Sun crosses the meridian. ϕ is the latitude of the place and δ is Sun's declination (how much it is above or below the equator). One can show, using spherical trigonometry that

$$\cos z = \sin\phi \sin\delta + \cos\phi \cos\delta \cos H.$$

So, one determines z from the shadow, and H from the above relation. So lot of trigonometry (plane and spherical) are involved! So determination of sine and cosine functions very critical to calculation in astronomy.

Indian $jy\bar{a}$

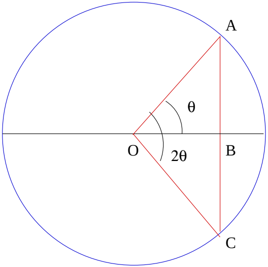


The Indian $Jy\bar{a}$.

In Indian astronomical and mathematical works, the circumference of a circle is taken to be $360^\circ = 21600'$. The radius $R = (21600'/2\pi) \approx 3438'$. This is the '$Trijy\bar{a}$'. Then for an angle θ, or an arc $R\theta$, the $jy\bar{a}$ or $j\bar{\imath}v\bar{a}$ is $AB = R\sin\theta$ as shown in the figure. $OB = R\cos\theta$ is the $ko\d{t}ijy\bar{a}$ or $kojy\bar{a}$ and $BD = R(1-\cos\theta)$ is called $Utkramajy\bar{a}$ or Versed R Sine, or '$\acute{S}ara$'.

The Greek Chord and the Indian Sine

Greeks worked with the chords. Indians, with the Rsines, as defined just now.



Chord and Sine

$AC = \text{Chord}\ (2\theta) = 2AB = 2R\sin(\theta)$

In all calculations, it is the sine the appears. The Indian sine is perfectly suited for writing formulae and performing calculations. The chord is far less so.

The terms ‘Sine’, ‘Cosine’, can be traced to India

$jy\bar{a}$: Also, $j\bar{\imath}v\bar{a}$. Adopted by the Arabs.

$J\bar{\imath}v\bar{a} \rightarrow j\bar{\imath}b\bar{a} \rightarrow$ In Arabic, read as ‘jayb’.

‘Jayb’ (‘pocket’ or ‘fold’): Translated into Latin as ‘Sinus’$\rightarrow$ Sine.

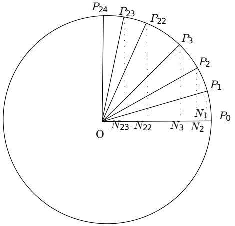
So the term ‘Sine’ is derived from Indian ‘$j\bar{\imath}v\bar{a}$’.

Now in India, the complement of the $jy\bar{a}$ is $ko\d{t}ijy\bar{a}$.

So complement of Sine $\rightarrow$ Cosine.

24-fold division of the quadrant

For a Sine table, the quadrant is divided into n equal divisions. Typically, in most texts $n = 24$, that is, the quadrant is divided into 24 parts. Each segment corresponds to $\dfrac{90^\circ}{24} = 3^\circ 45'$ or 225'. In the following figure, the points $P_i (i = 1, 2, 3, \cdots, 24)$ represent the end points of the 24 segments. The set of *jyās*, $J_i = P_i N_i (i = 1, 2, \cdots, 24)$ corresponding to the *Cāpa*s $P_0 P_i$ are explicitly stated in many texts, such as *Āryabhaṭīya*, *Sūryasiddānta*, *Tantrasaṅgraha* etc. Later values of n other than 24 are also discussed in some works. for instance, we will consider $n = 30$ or 90, as discussed by Bhāskara-II in his '*Jyotpatti*' section of '*Siddhāntaśiromani*'.



jyā's corresponding to arc lengths wjich are multiples of 225'.

In the 24-fold division, we have to find $R \sin i\alpha$, where $x = 225' = 3^\circ 45'$ and $i = 1, 2, \cdots, 24$.

Āryabhaṭīya: Finding Rsine

In his *Āryabhaṭīya*, Āryabhaṭa gives the following second-order difference equation for finding $R \sin i\alpha$:

$$R \sin\{(i+1)\alpha\} - R \sin(i\alpha) \approx R \sin(i\alpha) - R \sin\{(i-1)\alpha\} - \frac{R \sin i\alpha}{R \sin \alpha}$$

The whole table of sines can be generated from this, with $R \sin \alpha = R\alpha = 225$ (as α is small), as the only input. For instance, $R \sin 2\alpha = 449, R \sin 3\alpha = 671$ from this (We have to divide by $R = \dfrac{21600}{2\pi} \approx 3438$ to get the modern sine.)

It is amazing that Āryabhaṭa realised that the second-order difference is proportional to R sine itself, as far back 499 CE itself. The second order relation is essentially the equivalent of

$$\frac{d^2 \sin x}{dx^2} = -\sin x$$

Correct difference equation, Nīlakaṇṭha

The correct finite difference equation of the second order is

$$R\sin\{(i+1)\alpha\} - R\sin(i\alpha) = R\sin(i\alpha) - R\sin\{(i-1)\alpha\} - 2(1-\cos\alpha)R\sin i\alpha$$

$$\text{while } 2(1-\cos\alpha) = 0.0042822, \quad \frac{1}{R\sin\alpha} = \frac{1}{225} = 0.0044444$$

The exact recursion relation is stated in Nīlakaṇṭha's *Tantrasaṅgraha* (1500 CE.) He also uses a better value for $2(1-\cos\alpha)$. Also the first sine, $R\sin\alpha$ is taken to be $224'50''$ or $(224 + \frac{50}{60})'$. This is based on the better approximation $\sin\alpha \approx \alpha - \frac{\alpha^3}{3!}$.

(For $\alpha = 225'$, we have $2(1-\cos\alpha) \approx 0.004282153$). This is approximated in the text by $\frac{1}{233\frac{1}{2}} \approx 0.004282655$).

Obviously, Nīlakaṇṭha gets a much better sine table. The topic of sine tables generated in this manner will be taken up separately.

Sine of an intermediate angle, Interpolation

What about sines of angles which are not multiples of α, that is, intermediate angles? This is done by interpolation, as stated:

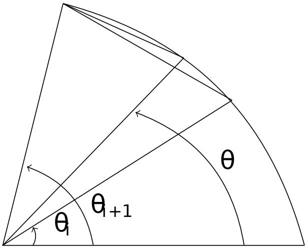


Figure: Rsine of an intermediate angle.

$$R\sin\theta = R\sin\theta_i + R(\theta-\theta_i)\left[\frac{R\sin(\theta_{i+1}) - R\sin(\theta_i)}{R(\theta_{i+1}-\theta_i)}\right] \qquad (\theta_i = i\alpha)$$

In his *Khaṇḍakhādyaka*, Brahmagupta gives a second order interpolation formula in the context of sine and cosine functions, but which is valid for an arbitrary function too.

Second order Interpolation due to Brahmagupta

गतभोग्यखण्डकान्तरदलविकलघातशतैर्नवभिराप्तया।
तद्युतिदलं युतोनं भोग्यादूनाधिकं भोग्यम्॥

"Multiply the residual arc left after division by 900' (α) by half the difference of the tabular difference passed over and that to be passed over and divide by 900' (α); by the result increase or decrease, as the case may be, half the sum of the same two tabular differences; the result which, less or greater than the tabular difference to be passed, is the true tabular difference to be passed over."

Suppose one is given $f[(i-1)\alpha]$, $f(i\alpha)$, $f[(i+1)\alpha]$ etc. (Brahmagupta : $\alpha = 900'$. Residual arc left after division by $900' = \beta\alpha$).

Second order Interpolation

Then, according to the interpolation formula,

$$f(i\alpha+\beta\alpha)=f(i\alpha)+\frac{\beta\alpha}{\alpha}\left[\frac{\Delta_{i+1}+\Delta_i}{2}+\frac{\beta(\Delta_{i+1}-\Delta_i)}{2}\right]$$

where

$$\Delta_{i+1}=f[(i+1)\alpha]-f(i\alpha)$$

$$\Delta_i=f(i\alpha)-f[(i-1)\alpha].$$

Compare with Taylor series:

$$f(i\alpha+\beta\alpha)=f(i\alpha)+\left.\frac{df}{dx}\right|_{x=i\alpha}\beta\alpha+\frac{1}{2}\left.\frac{d^2f}{dx^2}\right|_{x=i\alpha}\beta^2\alpha^2$$

Second order Interpolation

So Brahmagupta is taking

$$\begin{aligned}
\frac{df}{dx} &= \frac{1}{2}\left(\frac{\Delta_{i+1}}{\alpha} + \frac{\Delta_{i-1}}{\alpha}\right) \\
&= \frac{1}{2}\left[\frac{f[(i+1)\alpha] - f(i\alpha)}{\alpha} + \frac{f(i\alpha) - f[(i-1)\alpha]}{\alpha}\right]
\end{aligned}$$

(Average of the rate of change at $(i+1)\alpha$ and $i\alpha$) and

$$\begin{aligned}
\frac{d^2f}{dx^2} &= \frac{\Delta_{i+1} - \Delta_i}{\alpha^2} \\
&= \frac{\left[\frac{f[(i+1)\alpha] - f(i\alpha)}{\alpha} - \frac{f(i\alpha) - f[(i-1)\alpha]}{\alpha}\right]}{\alpha}
\end{aligned}$$

("Derivative" of rate of change.) as should it be.

Exact values of Sines

Exact values of Sines.
Apart from finding Sines from the second order difference equation, there is a method of finding the 24 Rsines exactly.

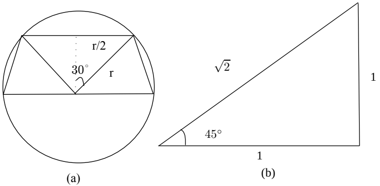


Fig.19. Finding $\sin 30^\circ$ and $\sin 45^\circ$.

One knows that a regular hexagon inscribed in a circle has a side which is equal to the radius of the circle, and that the angle subtended by a side at the centre is 60°, half of it which is 30°. Then in Fig. 19 a,

$$r \sin 30^\circ = \frac{r}{2}$$

$$\therefore \sin 30^\circ = \frac{1}{2}$$

Exact values of Sines

Similarly, if we take a right triangle whose sides are 1 and 1 and the hypotenuse $\sqrt{2}$, then from Fig. 19 b,

$$\sin 45^\circ = \frac{1}{\sqrt{2}}.$$

Also it was known that $\sin^2\theta + \cos^2\theta = 1$. So, if one knows $\sin\theta$

$$\cos\theta = \sqrt{1 - \sin^2\theta}$$

In particular, in the 24-fold division, if we know the i^{th} Rsine, that is $R\sin\alpha$, We also know $(24-i)^{\mathrm{th}}$ Rsine, that is $R\sin[(24-i)\alpha]$, as $24\alpha = 90^\circ$, and

$$R\sin[(24-i)\alpha] = R\sin[90^\circ - i\alpha] = R\cos i\alpha = \sqrt{R^2 - R^2\sin^2 i\alpha}$$

Finding $Rsin(\theta/2)$ from $Rsin(\theta)$

It was realised that we can find $R\sin(\theta/2)$ from $R\cos\theta$ which can be found from $R\sin\theta$. In his *Brāhmasphuṭasiddhānta*, Brahmagupta says:

> उत्क्रमसमखण्डगुणव्यासात् अथवा चतुर्थभागाद्धम्।
> कृत्वा उत्तखण्डकानि ज्यार्द्धानयनं नलघ्वस्मात्॥
>
> "The square root of the fourth part of the Versed Rsine of an arc multiplied by the diameter is the Rsine of half that arc."

That is,

$$R\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{D}{4}R(1-\cos\theta)} = \sqrt{\frac{R}{2}R(1-\cos\theta)} \qquad (D = 2R)$$

$$\text{or } \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1-\cos\theta)$$

$Rsin(\theta/2)$ from $Rsin(\theta)$: Varāhamihira

In fact, this had been stated by Varāhamihira earlier in his *Pañcasiddhāntikā* in Verse 5, Chapter 4, thus :

इष्टांशद्विगुणोनत्रिभज्ययोना त्रयस्य चापज्या।
षष्टिगुणा सा करणी तया ध्रुवोनाऽवशेषस्य॥

"Twice any desired arc is subracted from three signs (i.e. 90°),the Rsine of the remainder is subtracted from the Rsine of three signs. The result multiplied by sixty is the square of the Rsine of that arc."

Here, he is again essentially saying :

$$(R\sin\theta)^2 = \frac{R}{2}R(1-\cos\theta),$$

with $R = 120$.

Finding the 24 Rsines

With the knowledge of the 8^{th} sine which is $\sin 30^\circ = 1/2$, the 12^{th} sine which is $\sin 45^\circ = \frac{1}{\sqrt{2}}$, $(i/2)^{th}$ sine from the i^{th} sine, $(24 - i)^{th}$ sine from the i^{th} sine, the whole table of Rsines can be generated. This is indicated thus, from the 8^{th} sine:

$$8 \to 16,$$

$$8 \to 4, 20;\; 4 \to 2, 22;\; 2 \to 1, 23;\; 22 \to 11, 13;$$

$$20 \to 10, 14;\; 10 \to 5, 19,\; 14 \to 7, 17$$

From the 12^{th} sine

$$12 \to 6, 18;\; 6 \to 3, 21;\; 18 \to 9, 15$$

Of course $R\sin(24\alpha) = R$. So, 24 Rsines are found.

There would be lots of square roots on the way. So the method is exact, but cumbersome.

Bhāskara's *jyotpatti*: Finding $sin(18°)$

Bhāskara's '*Jyotpatti*' (Generation of Rsines) is a part of '*Golādhyāya*' which ia a part of '*Siddhāntaśiromaṇi*'. It gives the value of $\sin 18°$ and $\sin 36°$.

Verse 9.

त्रिज्याकृतीषुघातात् मूलं त्रिज्योनितं चतुर्थभक्तम्।
अष्टादशभागानां जीवा स्पष्टा भवत्येवम्॥

"Deduct the radius from the square root of the product of the square of radius and 5 and divide the remainder by 4; the quotient thus found will give the exact Rsine of 18°."

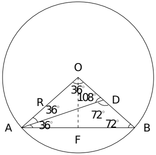
So, it states:

$$R \sin 18° = R \frac{[\sqrt{5} - 1]}{4}$$

Proof of expression for $sin(18^\circ)$

Proof: Refer to the following figure, (with circle of radius R), where $A\hat{O}B = 36^\circ$, and $O\hat{A}B = O\hat{B}A = 72^\circ$. Let AD (D on OB) bisect the angle $O\hat{A}B$. So, $O\hat{A}D = 36^\circ$. Both the triangles AOD and DAB are isosceles triangles, so

$$OD = AD = AB$$



Finding $\sin 18^\circ$

OF bisects the angle $A\hat{O}B = 36^\circ$. OF is perpendicular to AB, $A\hat{O}F = 18^\circ$. Let $x = R \sin 18^\circ$.

$$AB = 2AF = 2R \sin 18^\circ = 2x$$

Now triangle, ABD is similar to the triangle OAB.

$$\therefore \frac{AB}{BD} = \frac{OA}{AB}$$

$$\therefore AB^2 = OA.BD.$$

Proof of expression for $sin(18^\circ)$

Now, $BD = OB - OD = OB - AB = R - 2x$.

$$OA = R$$

$$\therefore \ (2x)^2 = R(R - 2x)$$

$$4x^2 + 2Rx - R^2 = O$$

$$\therefore x = \frac{-2R + \sqrt{4R^2 + 16R^2}}{2 \cdot 4} = R\frac{[\sqrt{5} - 1]}{4}$$

Hence,

$$R \sin 18^\circ = R\frac{[\sqrt{5} - 1]}{4}$$

$sin(36^\circ)$ in *jyotpatti*

In Verse 7 of *Jyotpatti*, Bhāskara says:

त्रिज्याकृतीषुघातात् त्रिज्याकृतिवर्गपञ्चघातस्य।
मूलोनात् अष्टहृतात् मूलं षड्त्रिंशदंशज्या॥

"Deduct the square root of five times the fourth power of the radius, from 5 times the square of radius, and divide the remainder by 8; the square root of the quotient will be the Rsine of 36°."

$sin(36^\circ)$ in *jyotpatti*

$$\text{So, he says: } R\sin 36^\circ = \sqrt{\frac{5R^2 - \sqrt{5R^4}}{8}}$$

$$\text{or } \sin 36^\circ = \sqrt{\frac{5-\sqrt{5}}{8}}$$

This can be easily understood as follows:

$$\begin{aligned}
\sin 36^\circ &= \sqrt{\frac{1}{2}(1-\cos 72^\circ)} = \sqrt{\frac{1}{2}(1-\sin 18^\circ)} \\
&= \sqrt{\frac{1}{2}\left\{1-\frac{\sqrt{5}-1}{4}\right\}} = \sqrt{\frac{4-(\sqrt{5}-1)}{8}} \\
&= \sqrt{\frac{5-\sqrt{5}}{8}}
\end{aligned}$$

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Thanks!

Thank You