NPTEL COURSE ON

MATHEMATICS IN INDIA: FROM VEDIC PERIOD TO MODERN TIMES

Lecture 36

Proofs in Indian Mathematics 1

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Outline

- *Upapattis* or proofs in Indian mathematical tradition
- Early European scholars of Indian Mathematics were aware of *upapattis*
- Some important commentaries which present *upapattis*
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- *Upapatti* of *bhujā-koṭi-karṇa-nyāya* (Pythagoras theorem)
- *Upapatti* of *kuṭṭaka* process
- Restricted use of *tarka* (proof by contradiction) in Indian Mathematics
- The Contents of *Yuktibhāṣā*
- *Yuktibhāṣā* demonstration of *bhujā-koṭi-karṇa-nyāya*
- Estimating the circumference by successive doubling of circumscribing polygon

Upapattis in Indian Mathematics

While there have been several extensive investigations on the history and achievements of Indian mathematics, there has not been much discussion on its methodology, the Indian mathematicians' and philosophers' understanding of the nature and validation of mathematical results and procedures, their views on the nature of mathematical objects, and so on.

Traditionally, such issues have been dealt with in the detailed *bhāṣyas* or commentaries, which continued to be written till recent times, and played a vital role in the traditional scheme of learning.

It is in such commentaries that we find detailed *upapattis* or 'proofs' of the results and procedures, apart from a discussion of methodological and philosophical issues.

Early European Scholars Were Aware of *Upapattis*

In the early stages of modern scholarship on Indian mathematics, we find references to the methods of demonstration found in texts of Indian mathematics.

In 1817, H. T. Colebrooke referred to them in his pioneering and widely circulated translation of *Līlāvatī* and *Bījagaṇita* and the two mathematics chapters of *Brāhmasphuṭa-siddhānta*:

> "On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically: as is particularly noticed by Bhāskara himself, towards the close of his algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities."

Early European Scholars Were Aware of *Upapattis*

Similarly, Charles Whish, in his seminal article on Kerala School of Mathematics of 1835, referred to the demonstrations in *Yuktibhāṣā*:

> "A further account of the *Yuktibhāṣā*, the demonstrations of the rules for the quadrature of the circle by infinite series, with the series for the sines, cosines, and their demonstrations, will be given in a separate paper: I shall therefore conclude this, by submitting a simple and curious proof of the 47th proposition of Euclid [the so called Pythagoras theorem], extracted from the *Yuktibhāṣā*."

Whish does not seem to have written any further papers on the demonstrations of the infinite series as given in *Yuktibhāṣā*.

Whish's paper was widely noticed in the scholarly circles of Europe in the second quarter of nineteenth century.

But it was soon forgotten and there was no study of *Yuktibhāṣā* till 1940s, when C. T. Rajagopal and his collaborators wrote pioneering articles on the proofs outlined in that seminal text.

The Alleged Absence of Proofs in Indian Mathematics

It has been the scant attention paid, by the modern scholarship of the last two centuries, to this extensive tradition of commentaries which has led to a lack of comprehension of the methodology of Indian mathematics. This is reflected in the often repeated statements on the absence of logical rigour in Indian mathematics in works on history of mathematics such as the following:

> "As our survey indicates, the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the deductive patterns. Their name for mathematics was *gaṇita*, which means 'the science of calculation'. There is much good procedure and technical facility, but no evidence that they considered proof at all. They had rules, but apparently no logical scruples. Moreover, no general methods or new viewpoints were arrived at in any area of mathematics."[1]

[1] Morris Kline: *Mathematical Thought from Ancient to Modern Times,* Oxford 1972, p.190.

Some Important Commentaries Which Discuss *Upapattis*

1. *Bhāṣya* of Bhāskara I (c.629) on *Āryabhaṭīya* of Āryabhaṭa (c.499)
2. *Bhāṣya* of Govindasvāmin (c.800) on *Mahābhāskarīya* of Bhāskara I (c.629)
3. *Vāsanābhāṣya* of Caturveda Pṛthūdakasvāmin (c.860) on *Brāhmasphuṭasiddhānta* of Brahmagupta (c.628)
4. *Vivaraṇa* of Bhāskarācārya II (c.1150) on *Śiṣyadhīvṛddhida-tantra* of Lalla (c.748)
5. *Vāsanā* of Bhāskarācārya II (c.1150) on his own *Bījagaṇita*
6. *Mitākṣarā* or *Vāsanā* of Bhāskarācārya II (c.1150) on his own *Siddhāntaśiromaṇi*

Some Important Commentaries Which Discuss *Upapattis*

7. *Siddhāntadīpikā* of Parameśvara (c.1431) on the *Bhāṣya* of Govindasvāmin (c.800) on *Mahābhāskarīya* of Bhāskara I (c.629)
8. *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somasutvan (c.1501) on *Āryabhaṭīya* of Āryabhaṭa
9. *Yuktibhāṣā* (in Malayalam) of Jyeṣṭhadeva (c.1530)
10. *Yuktidīpikā* of Śaṅkara Vāriyar (c.1530) on *Tantrasaṅgraha* (c.1500) of Nīlakaṇṭha Somasutvan
11. *Kriyākramakarī* of Śaṅkara Vāriyar (c.1535) on *Līlāvatī* of Bhāskarācārya II (c.1150)

Some Important Commentaries Which Discuss *Upapattis*

12. *Gaṇitayuktayaḥ*, Tracts on Rationale in Mathematical Astronomy by various Kerala Astronomers (c.16th-19th century)
13. *Sūryaprakāśa* of Śūryadāsa (c.1538) on Bhāskarācārya's *Bījagaṇita* (c.1150)
14. *Buddhivilāsinī* of Gaṇeśa Daivajña (c.1545) on *Līlāvatī* of Bhāskarācārya II (c.1150)
15. *Bījanavāṅkura* or *Bījapallavam* of Kṛṣṇa Daivajña (c.1600) on *Bījagaṇita* of Bhāskarācārya II (c.1150)
16. *Vāsanāvārttika*, commentary of Nṛsiṃha Daivajña (c.1621) on *Vāsanābhāṣya* of Bhāskarācārya II on his own *Siddhāntaśiromaṇi* (c.1150).
17. *Marīci* of Munīśvara (c.1630) on *Siddhāntaśiromaṇi* of Bhāskarācārya II (c.1150).

Kṛṣṇa Daivajña on the Importance of *Upapatti*

The following passage from Kṛṣṇa Daivajña's commentary on *Bījagaṇita* brings out the general understanding of the Indian mathematicians that citing any number of favourable instances (even an infinite number of them) where a result seems to hold, does not amount to establishing it as a valid result in mathematics. Only when the result is supported by an *upapatti* or demonstration can the result be accepted as valid.

ननूपपत्त्या विना वर्गयोगो द्विघ्नघातेन युतो हीनो वा युतिवर्गोऽन्तरवर्गो वा भवतीत्येतदेव कथम्? क्वचिद्दर्शनं त्वप्रयोजकम्। अन्यथा चतुर्गुणो राशिघातो युतिवर्गो भवतीत्यपि सुवचम्। तस्यापि क्वचित्तथा दर्शनात्। तथाहि राशी २, २ अनयोर्घातः ४ चतुर्गुणः १६ अयं जातो युतिः ४ वर्गः १६ वा राशी ३,३ अनयोर्घातश्चतुर्गुणः ३६ अयमेव युति ६ वर्गश्च ३६ वा राशी ४, ४ अनयोर्घातः १६ चतुर्गुणः ६४ अयमेव युति ८ वर्गः ६४ इत्यादिषु। तस्मात् क्वचिद्दर्शनम् अप्रयोजकं क्वचिद्व्यभिचारस्यापि संभवात्। अतो वर्गयोगो द्विघ्नघातयुतोनो युतिवर्गोऽन्तरवर्गश्च भवतीत्यत्र युक्तिर्वक्तव्येति चेत् सत्यम्। इयमुपपत्तिरेकवर्णमध्यमाहरणान्ते।

Kṛṣṇa Daivajña on the Importance of *Upapatti*

"How can we state without proof (*upapatti*) that twice the product of two quantities when added or subtracted from the sum of their squares is equal to the square of the sum or difference of those quantities? That it is seen to be so in a few instances is indeed of no consequence. Otherwise, even the statement that four times the product of two quantities is equal to the square of their sum, would have to be accepted as valid. For, that is also seen to be true in some cases. For instance take the numbers 2, 2. Their product is 4, four times which will be 16, which is also the square of their sum 4. Or take the numbers 3, 3. Four times their product is 36, which is also the square of their sum 6. Or take the numbers 4, 4. Their product is 16, which when multiplied by four gives 64, which is also the square of their sum 8. Hence the fact that a result is seen to be true in some cases is of no consequence, as it is possible that one would come across contrary instances also. Hence it is necessary that one would have to provide a proof (*yukti*) for the rule that twice the product of two quantities when added or subtracted from the sum of their squares results in the square of the sum or difference of those quantities. We shall provide the proof (*upapatti*) in the end of the section on *ekavarṇa-madhyamāharaṇa*."

Bhāskara I on *Upapatti* (c.629)

In his discussion of Āryabhaṭa's approximate value of the ratio of the circumference and diameter of a circle, Bhāskara I notes that the approximate value is given, as the exact value cannot be given. He then goes on to argue that other values which have been proposed are without any justification:

एवं मन्यन्ते स उपाय एव नास्ति येन सूक्ष्मपरिधिरानीयते। ननु चायमस्ति

> विक्खंभवग्गदसगुणकरणी वट्टस्स परिरओ होवि।
> (विष्कम्भवर्गदशगुणकरणी वृत्तस्यपरिणाहो भवति)

इति। अत्रापि केवल एवागमः नैवोपपत्तिः। रूपविष्कम्भस्य दशकरण्यः परिधिरिति। अथ मन्यन्ते प्रत्यक्षेणैव प्रमीयमाणो रूपविष्कम्भक्षेत्रस्य परिधिर्दशकरण्य इति। नैतत् अपरिभाषितप्रमाणत्वात् करणीनाम्। एकत्रिविस्तारायामायतचतुरश्रक्षेत्रकर्णेन दशकरणिकेनैव तद्विष्कम्भ-परिधिर्वेष्ट्यमाणः स तत्प्रमाणो भवतीति चेत्तदपि साध्यमेव।

“It is the view of the Ācārya that there is no means by which the exact circumference can be obtained. Is it not true that there is the following?

‘The square root of the ten times the diameter is the circumference of a circle.’

Here also, there is only a tradition, and not a proof (*upapatti*), that the circumference when the diameter is one is square-root of ten. Then it is contended that ‘the circumference when diameter is unity, when measured by means of direct perception (*pratyakṣa*), is the square-root of ten (*daśa-karaṇī*)’. That is incorrect, because the magnitude of the square-roots (*karaṇī*) cannot be stated exactly. It may be further contended that ‘when the circumference associated with that diameter (one) is enclosed with the diagonal of a rectangle whose breadth and length are one and three respectively, i.e. one whose length is square root of ten only, it (the circumference) has that length.’ But that too has to be established.”

Bhāskara II on *Upapatti* (c.1150)

In *Siddhāntaśiromaṇi*, Bhāskarācārya II (1150) presents the *raison d'être* of *upapatti* in the Indian mathematical tradition:

मध्याद्यं द्युसदां यदत्र गणितं तस्योपपत्तिं विना
प्रौढिं प्रौढसभासु नैति गणको निःसंशयो न स्वयम्।
गोले सा विमला करामलकवत् प्रत्यक्षतो दृश्यते
तस्मादस्म्युपपत्तिबोधविधये गोलप्रबन्धोद्यतः॥

Without the knowledge of *upapattis*, by merely mastering the calculations (*gaṇita*) described here, from the *madhyamādhikāra* (the first chapter of *Siddhāntaśiromaṇi*) onwards, of the [motion of the] heavenly bodies, a mathematician will not be respected in the scholarly assemblies; without the *upapattis* he himself will not be free of doubt (*niḥsaṁśaya*). Since *upapatti* is clearly perceivable in the (armillary) sphere like a berry in the hand, I therefore begin the *Golādhyāya* (section on spherics) to explain the *upapattis*.'

The same has been stated by Gaṇeśa Daivajña in the introduction to his commentary *Buddhivilāsinī* (c. 1540) on *Līlāvatī* of Bhāskarācārya

व्यक्ते वाव्यक्तसंज्ञे यदुदितमखिलं नोपपत्तिं विना तत्
निर्भ्रान्तो वा ऋते तां सुगणकसदसि प्रौढतां नैति चायम्।
प्रत्यक्षं दृश्यते सा करतलकलितादर्शवत् सुप्रसन्ना
तस्मादग्र्योपपत्तिं निगदितुमखिलम् उत्सहे बुद्धिवृद्ध्यै॥

Bhāskara II on *Upapatti*

Thus, according to the Indian mathematical texts, the purpose of *upapatti* is mainly:

(i) To remove confusion and doubts regarding the validity and interpretation of mathematical results and procedures; and,

(ii) To obtain assent in the community of mathematicians.

This is very different from the ideal of "proof" in the Greco-European tradition which is to irrefutably establish the absolute truth of a mathematical proposition.

Bhāskara II on *Upapatti*

In his *Bījagaṇita-vāsanā*, Bhāskarācārya II (c.1150) refers to the long tradition of *upapattis* in Indian mathematics.

अस्योपपत्तिः। सा च द्विधा सर्वत्र स्यादेका क्षेत्रगताऽन्या राशिगतेति। तत्र क्षेत्रगतोच्यते।...अथ राशिगतोपपत्तिरुच्यते सापि क्षेत्रमूलान्तर्भूता। इयमेव क्रिया पूर्वाचार्यैः संक्षिप्तपाठेन निबद्धा। ये क्षेत्रगतां उपपत्तिं न बुद्ध्यन्ति तेषामियं राशिगता दर्शनीया।

> "The demonstration follows. It is twofold in each case: One geometrical and the other algebraic. There, the geometrical one is stated... Then the algebraic demonstration is stated, that is also geometry-based. This procedure [of *upapatti*] has been earlier presented in a concise instructional form by ancient teachers. For those who cannot comprehend the geometric demonstration, to them, this algebraic demonstration is to be presented."

Here, Bhāskara also refers to the *kṣetragata* (geometric) and *rāśigata* (algebraic) demonstrations. To understand them, we shall consider the two proofs given by Bhāskara of the *bhujā-koṭi-karṇa-nyāya*.

Upapatti of *Bhujā-Koṭi-Karṇa-Nyāya*

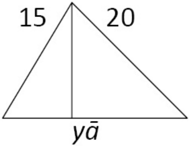
In the *madhyamāharāṇa* section Bhāskara poses the following problem

> क्षेत्रे तिथिनखैस्तुल्ये दोः कोटी तत्र का श्रुतिः।
> उपपत्तिश्च रूढस्य गणितस्यास्य कथ्यताम्।
>
> "In a right angled triangle with sides 15 and 20 what is the hypotenuse? Also give the demonstration for this traditional method of calculation."

Here Bhāskara gives two proofs. First the geometrical:

अत्र कर्णः या १। एतत् त्र्यस्रं परिवर्त्य यावत्तावत्कर्णो भूः कल्पिता। भुजकोटी तु भुजौ तत्र यो लम्बस्तदुभयतो ये त्र्यस्रे तयोरपि भुजकोटी पूर्वरूपे भवतः। अतस्त्रैराशिकं यदि यावत्तावति कर्णेऽयं १५ भुजस्तदा भुजतुल्ये कर्णे क इति लब्धो भुजः स्यात्। सा भुजाश्रिताऽऽबाधा २२५/या। पुनर्यदि यावत्तावति कर्ण इयं २० कोटिस्तदा कोटितुल्ये कर्णे केति जाता कोटयाश्रिताबाधा ४००/या। आबाधायुतिर्यावत्तावत्कर्णसमा क्रियते तावद्भुजकोटिवर्गयोगस्य पदं कर्णमानमुपपद्यते। अनेनोत्थापिते जाते आबाधे ९, १६ ततो लम्बः १२। क्षेत्रदर्शनम्।

Upapatti of *Bhujā-Koṭi-Karṇa-Nyāya*



$$y\bar{a} = \left(\frac{225}{y\bar{a}}\right) + \left(\frac{400}{y\bar{a}}\right)$$
$$y\bar{a}^2 = 625$$
$$y\bar{a} = 25$$

"Let the hypotenuse (*karṇa*) be *yā*. Take the hypotenuse as the base. Then the perpendicular to the hypotenuse from the opposite vertex divides the triangle into two triangles [which are similar to the original]. Now by the rule of proportion (*trairāśika*), if *yā* is the hypotenuse the *bhujā* is 15, then when this *bhujā* 15 is the hypotenuse, the *bhujā*, which is now the *ābādhā* (segment of the base) to the side of the original *bhujā* will be (225/*yā*). Again if *yā* is the hypotenuse, the *koṭi* is 20, then when this *koṭi* 20 is the hypotenuse, the *koṭi*, which is now the segment of base to the side of the (original) *koṭi* will be (400/*yā*). Adding the two segments (*ābādhās*) of *yā* the hypotenuse and equating the sum to (the hypotenuse) *yā*, we get *yā* = 25, the square root of the sum of the squares of *bhujā* and *koṭi*. The base segements are 9, 16 and the perpendicular is 12. See the figure".

Upapatti of *Bhujā-Koṭi-Karṇa-Nyāya*

Now the algebraic demonstration:

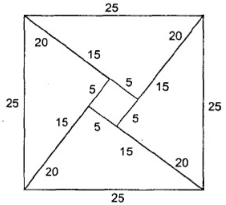
> अथाऽन्यथा वा कथ्यते कर्णः या १ दोःकोटिघातार्धं
> त्र्यस्रक्षेत्रस्य फलम् १५०। एतद्विषमत्र्यस्रचतुष्टयेन
> कर्णसमचतुर्भुजं क्षेत्रमन्यत् कर्णज्ञानार्थं कल्पितम्। एवं
> मध्ये चतुर्भुजमुत्पन्नमत्र कोटिभुजान्तरसमं भुजमानम् ५।
> अस्य फलम् २५। भुजकोटिवधो द्विगुणस्त्र्यस्राणां चतुर्णां
> फलम् ६००। एतद्योगः सर्वं बृहत्क्षेत्रफलम् ६२५
> एतद्यावत्तावद्वर्गसमं कृत्वा लब्धं कर्णमानम् २५। यत्र
> व्यक्तस्य न पदं तत्रकरणीगतः कर्णः।

As Bhāskara has noted, this algebraic demonstration is also geometrical in nature.

Upapatti of *Bhujā-Koṭi-Karṇa-Nyāya*

Let the hypotenuse be $y\bar{a}$. The area of the triangle which is half the product of the $bhuj\bar{a}$ (15) and $ko\d{t}i$ (20) is 150. Consider a square whose sides are formed out of the hypotenuse of these triangles.

In the centre is formed a square whose side is 5, the difference of $bhuj\bar{a}$ and $ko\d{t}i$, and whose area is 25. The area of the four triangles is 600. Thus, adding these, the area of the big square is 625. Taking $y\bar{a}^2 = 25$, we get the hypotenuse to be 25.



The above $upapatti$ is based on the algebraic identity

$$(a-b)^2 + 4.\frac{1}{2}ab = a^2 + b^2$$

Kṛṣṇa Daivajña's *Upapatti* of *Kuṭṭaka* Process

As an example of an *upapatti* which proceeds in a sequence of steps, we may briefly consider the detailed *upapatti* for the *kuṭṭaka* procedure given by Kṛṣṇa Daivajña (c.1600) in his commentary *Bījapallava* on *Bījagaṇita* of Bhāskara.

The *kuṭṭaka* procedure is for solving first order indeterminate equations of the form

$$\frac{(ax + c)}{b} = y$$

Here, **a, b, c** are given integers (called *bhājya, bhājaka* and *kṣepa*) and **x, y** are to be solved for in integers.

Kṛṣṇa first shows that the solutions for **x, y** do not vary if we factor all three numbers **a, b, c** by the same common factor.

He then shows that if **a** and **b** have a common factor then the above equation will not have a solution unless **c** is also divisible by the same.

He then gives the *upapatti* for the process of finding the *apavartāṅka* (greatest common divisor) of **a** and **b** by mutual division (the so-called Euclidean algorithm).

Kṛṣṇa Daivajña's *Upapatti* of *Kuṭṭaka* Process

Kṛṣṇa then provides a detailed justification for the *kuṭṭaka* method of finding the solution by making a *vallī* (table) of the quotients obtained in the above mutual division, based on a detailed analysis of the various operations in reverse (*vyasta-vidhi*).

In doing the reverse computation on the *vallī* (*vallyupasaṃhāra*) the numbers obtained, at each stage, are shown to be the solutions to the *kuṭṭaka* problem for the successive pairs of remainders (taken in reverse order from the end) which arise in the mutual division of **a** and **b**.

After analysing the reverse process of computation with the *vallī*, Kṛṣṇa shows how the solutions thus obtained are for positive and negative *kṣepa*, depending upon whether there are odd or even number of coefficients generated in the above mutual division.

And this indeed leads to the different procedures to be adopted for solving the equation depending on whether there are odd or even number of quotients in the mutual division.

Kṛṣṇa Daivajña's *Upapatti* of *Kuṭṭaka* Process

As an illustration, Kṛṣṇa considers the equation $\frac{(173x+3)}{71} = y$ with *bhājya* 173, *bhājaka* 71 and *kṣepa* 3.

In the mutual division of 173 and 71 we get the quotients 2, 2, 3 and 2 and remainders 31, 9, 4 and 1.

If we do the reverse computation on the *vallī* formed by 2, 2, 3, 2, 1, 3 and 0, we first get 6, 3 as the *labdhi* and *guṇa*, which satisfy the equation $\frac{(9.3-3)}{4} = 6$, with the remainders 9, 4 serving as *bhājya* and *bhājaka*.

In the reverse computation on the *vallī*, we then get 21, 6 as *labdhi* and *guṇa*, which satisfy the equation $\frac{(31.6+3)}{9} = 21$, with the remainders 31, 9 serving as *bhājya* and *bhājaka*.

And so on, till we get 117 and 48 as *labdhi* and *guṇa*, which satisfy the equation $\frac{(173.48+3)}{71} = 117$

Use of *Tarka* in *Upapatti*

The method of “proof by contradiction” is referred to as *tarka* in Indian logic. We see that this method is employed in order to show the non-existence of an entity.

For instance, Kṛṣṇa Daivajña essentially employs *tarka* to show the non-existence of the square-root of a negative number while commenting on the statement of Bhāskara that a negative number has no root.

वर्गस्य हि मूलं लभ्यते। ऋणाङ्कस्तु न वर्गः कथमतस्तस्य मूलं लभ्यते। ननु ऋणाङ्कः कुतो वर्गो न भवति न हि राजनिर्देशः।. . . सत्यम्। ऋणाङ्कं वर्गं वदता भवता कस्य स वर्ग इति वक्तव्यम्। न तावद्धनाङ्कस्य “समद्वि-घातो हि वर्गः” तत्र धनाङ्केन धनाङ्के गुणिते यो वर्गो भवेत् स धनमेव “स्वयोर्वधः स्वम्” इत्युक्तत्वात्। नाप्यृणाङ्कस्य। तत्रापि समद्विघातार्थ-मृणाङ्केनर्णाङ्कगुणिते धनमेव वर्गो भवेत् “अस्वयोर्वधः स्वम्” इत्युक्त-त्वात्। एवं सति कथमपि तमङ्कं न पश्यामो यस्य वर्गः क्षयो भवेत्।

Use of *Tarka* in *Upapatti*

Thus according to Kṛṣṇa

> "The square-root can be obtained only for a square. A negative number is not a square. Hence how can we consider its square-root? It might however be argued: 'Why will a negative number not be a square? Surely it is not a royal fiat'... Agreed. Let it be stated by you who claim that a negative number is a square as to whose square it is; surely not of a positive number, for the square of a positive number is always positive by the rule... not also of a negative number. Because then also the square will be positive by the rule... This being the case, we do not see any such number whose square becomes negative..."

Use of *Tarka* in *Upapatti*

While the method of "proof by contradiction" or *reductio ad absurdum* has been used to show the non-existence of entities, the Indian mathematicians do not use this method to show the existence of entities, whose existence cannot be demonstrated by other direct means. They have a "constructive approach" to the issue of mathematical existence.

It is a general principle of Indian logic that *tarka* is not accepted as an independent *pramāṇa*, but only as an aid to other *pramāṇas*.

Indian Logic Excludes *Aprasiddha* Entities from Logical Discourse

Naiyāyikas or Indian logicians do not grant any scheme of inference, where a premise which is known to be false is used to arrive at a conclusion, the status of an independent *pramāṇa* or means of gaining valid knowledge.

In fact, they go much further in exorcising the logical discourse of all *aprasiddha* terms or terms such as “rabbit’s horn” (*śaśaśṛṅga*) which are empty, non-denoting or unsubstantiated.

Indian Logic Excludes *Aprasiddha* Entities from Logical Discourse

> "*Nyāya*...(excludes) from logical discourses any sentence which will ascribe some property (positive or negative) to a fictitious entity. Vācaspati remarks that we can neither affirm nor deny anything of a fictitious entity, the rabbit's horn. Thus *nyāya* apparently agrees to settle for a superficial self-contradiction because, in formulating the principle that nothing can be affirmed or denied of a fictitious entity like rabbit's horn, *nyāya*, in fact violates the same principle. *Nyāya* feels that this superficial self-contradiction is less objectionable (than admitting fictitious entities in logical discourse)...(This can be seen from the discussion in) Udayana's *Ātmatattvaviveka*..."[2]

[2] B. K. Matilal, *Logic, Language and Reality*, Delhi 1985, p-9.

Yuktibhāṣā of Jyeṣṭhadeva

The most detailed exposition of *upapattis* in Indian mathematics is found in the Malayalam text *Yuktibhāṣā* (1530) of Jyeṣṭhadeva, a student of Dāmodara, and a junior colleague of Nīlakaṇṭha.

At the beginning of *Yuktibhāṣā*, Jyeṣṭhadeva states that his purpose is to present the rationale of the results and procedures as expounded in the *Tantrasaṅgraha*.

Many of these rationales have also been presented (mostly in the form of Sanskrit verses) by Śaṅkara Vāriyar (c.1500-1556) in his commentaries *Kriyākramakarī* (on *Līlāvatī*) and *Yuktidīpikā* (on *Tantrasaṅgraha*).

Yuktibhāṣā has 15 chapters and is naturally divided into two parts, Mathematics and Astronomy.

In the Mathematics part, the first five chapters deal with the notion of numbers, logistics, arithmetic of fractions, the rule of three and the solution of linear indeterminate equations.

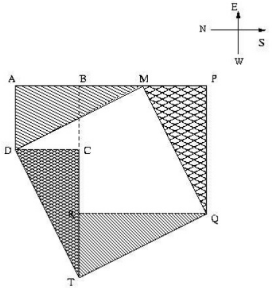
Chapter VI of *Yuktibhāṣā* deals with the *paridhi-vyāsa-sambandha* or the relation between the circumference and diameter of a circle. It presents a detailed derivation of the Mādhava series for π, including the derivation of the binomial series, and the estimate of the sums of powers of natural numbers $1^k + 2^k + \dots n^k$ for large n. This is followed by a detailed account of Mādhava's method of end correction terms and their use in obtaining rapidly convergent transformed series.

Chapter VII of *Yuktibhāṣā* is concerned with *jyānayana* or the computation of Rsines. It presents a derivation of the second order interpolation formula of Mādhava. This is followed by a detailed derivation of the Mādhava series for Rsine and Rversine.

In the end of the Mathematics section, *Yuktibhāṣā* also presents proofs of various results on cyclic quadrilaterals, as also the formulae for the surface area and volume of a sphere.

The Astronomy part of *Yuktibhāṣā* has seven chapters which give detailed demonstrations of all the results of spherical astronomy.

Yuktibhāṣā Proof of *Bhujā-Koṭi-Karṇa-Nyāya*



ABCD, a square with its side equal to the *bhujā*, is placed on the north. The square BPQR, with its side equal to the *koṭi*, is placed on the South. It is assumed that the *bhujā* is smaller than the *koṭi*. Mark M on AP such that AM = BP = *koṭi*. Hence MP = AB = *bhujā* and MD = MQ = *karṇa*. Cut along MD and MQ, such that the triangles AMD and PMQ just cling at D, Q respectively. Turn them around to coincide with DCT and QRT. Thus is formed the square DTQM, with its side equal to the *karṇa*. It is thus seen that

karṇa-square MDTQ = *bhujā*-square ABCD + *koṭi*-square BPQR

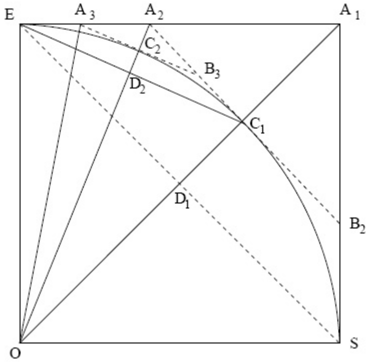
Successive Doubling of Circumscribing Polygon

It appears that the Indian mathematicians (at least in the Āryabhaṭan tradition) employed the method of successive doubling of the circumscribing square (leading to an octagon, etc.) to find successive approximations to the circumference of a circle. This method has been described in the *Yuktibhāṣā* and also in the *Kriyākramakarī*.

The latter cites the verses of Mādhava in this connection

चतुर्भुजे दोःकृतिनागभागमूलं हरो हारभुजाङ्घ्रिभेदात्।
भुजाहताद्धारहृतं तु कोणान्नीत्वा विलिख्याष्टभुजाः प्रसाध्याः॥ १॥
अष्टाश्रदोरर्धकृतिर्निधेया व्यासार्धवर्गे पदमत्र कर्णः।
तेनाऽऽहरेद्दोर्दलवर्गहीनं व्यासार्धवर्गं यदतः फलं स्यात्॥ २॥
तदूनकर्णौ दलितो हराख्यो गुणस्तु विष्कम्भदलोनकर्णः।
भुजार्धमेतेन हतं गुणेन हरेण भुक्त्वा यदिहापि लब्धम्॥ ३॥
तत्कोणतः पार्श्वयुगेषु नीत्वा छिन्नेऽन्तरे स्यादिह षोडशाश्रम्।
अनेन मार्गेण भवेदतश्च रदाश्रकं वृत्तमतश्च साध्यम्॥ ४॥

Successive Doubling of Circumscribing Polygon



In the figure, $EOSA_1$ is the first quadrant of the square circumscribing the given circle. EA_1 is half the side of the circumscribing square. Let OA_1 meet the circle at C_1. Draw $A_2C_1B_2$ parallel to ES. EA_2 is half the side of the circumscribing octagon.

Similarly, let OA_2 meet the circle at C_2. Draw $A_3C_2B_3$ parallel to EC_1. EA_3 is now half the side of a circumscribing regular polygon of 16 sides. And so on.

Successive Doubling of Circumscribing Polygon

Let half the sides of the circumscribing square, octagon etc., be denoted

$$l_1 = EA_1,\ l_2 = EA_2,\ l_3 = EA_3, \ldots$$

The corresponding *karṇas* (diagonals) are

$$k_1 = OA_1,\ k_2 = OA_2,\ k_3 = OA_3, \ldots$$

and the *ābādhās* (intercepts) are

$$a_1 = D_1A_1,\ a_2 = D_2A_2,\ a_3 = D_3A_3, \ldots$$

Now

$$l_1 = r,\ k_1 = \sqrt{2}r,\ \text{and } a_1 = \frac{r}{\sqrt{2}}.$$

Using the *bhujā-koṭi-karṇa-nyāya* (Pythagoras theorem) and *trai-rāśika-nyāya* (rule of three for similar triangles), it can be shown that

$$\begin{aligned} l_2 &= l_1 - (k_1 - r)\left(\frac{l_1}{a_1}\right) \\ k_2^2 &= r^2 + l_2^2 \\ a_2 &= \frac{\left[k_2^2 - \left(r^2 - l_2^2\right)\right]}{2k_2} \end{aligned}$$

Successive Doubling of Circumscribing Polygon

In the same way $l_{n+1},\ k_{n+1}$ and a_{n+1} can be obtained from $l_n,\ k_n$ and a_n.

These can be shown equivalent to the recursion relation for l_{n+1} which is the half side of the circumscribing polygon of 2^{n+1} sides:

$$l_{n+1} = \left(\frac{r}{l_n}\right)\left[\left(r^2 + l_n^2\right)^{\frac{1}{2}} - r\right]$$

We of course have the initial value $l_1 = r$.

This leads to $l_2 = (\sqrt{2} - 1)\ r$ and so on.

Kriyākramakarī notes that

> एवं यावदभीष्टं सूक्ष्मतामापादयितुं शक्यम्।
>
> Thus, one can obtain an approximation (to the circumference of the circle) to any desired level of accuracy.

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Thanks!

$\boxed{\text{Thank You}}$