NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 34

Trigonometry and Spherical Trigonometry 2

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Outline

- Sine of difference of two angles
- Sines at the intervals of 3°, 1.5° and 1°
- *Jīve paraspara nyāya*
- Bhāskara I formula for the sine function
- Sines and Cosines of multiples and submultiples of angles
- Plane trigonometry formulae
- A bit of spherical trigonometry

$\sin(\theta_1 - \theta_2)$ from $\sin\theta_1,\ \sin\theta_2$

In Verse 13, *Bhāskara* gives the method to find the sine of the difference of two angles given the individual sines.

Verse 13.

यद्दोर्ज्ययोरन्तरमिष्टयोर्यत् कोटिज्ययोस्तत्कृतियोगमूलम् ।
दलीकृतं स्याद् भुजयोर्वियोगखण्डस्य जीवैवमनेकधा वा ॥

“Take the sines of two arcs and find their difference, then find also the difference of their cosines, square these differences, add these squares, extract their square root and halve it. This half will be sine of half the difference of the arcs. Thus sines can be determined by several ways.”

So, $R\sin\left(\frac{\theta_1-\theta_2}{2}\right) = \frac{1}{2}\sqrt{(R\sin\theta_1 - R\sin\theta_2)^2 + (R\cos\theta_1 - R\cos\theta_2)^2}$

Proof

Proof:

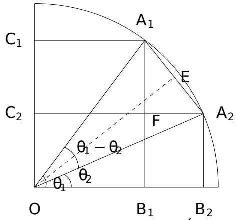


Diagram to find $\sin\left(\frac{\theta_1-\theta_2}{2}\right)$

We drop R.

In the figure,

$$\begin{aligned} A_1\hat{O}B_1 &= \theta_1, A_1B_1 = \sin\theta_1, \quad A_1C_1 = OB_1 = \cos\theta_1, \\ A_2\hat{O}B_2 &= \theta_2, A_2B_2 = \sin\theta_2, \quad A_2C_2 = OB_2 = \cos\theta_2. \end{aligned}$$

Let OE bisect the angle $A_1\hat{O}A_2 = \theta_1 - \theta_2$.

Hence $E\hat{O}A_2 = \frac{\theta_1-\theta_2}{2}$. $A_1A_2 = 2EA_2 = 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)$.

Getting $\sin(\frac{\theta_1-\theta_2}{2})$ from $\sin\theta_1,\ \sin\theta_2$

Now, $A_1F = A_1B_1 - FB_1 = A_1B_1 - A_2B_2 = \sin\theta_1 - \sin\theta_2$

$FA_2 = C_2A_2 - FC_2 = OB_2 - OB_1 = \cos\theta_2 - \cos\theta_1$.

In the right triangle A_1FA_2,

$$A_1A_2^2 = A_1F^2 + FA_2^2$$

$$\therefore \left[2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right]^2 = [\sin\theta_1 - \sin\theta_2]^2 + [\cos\theta_1 - \cos\theta_2]^2$$

$$\text{or } \sin\left(\frac{\theta_1-\theta_2}{2}\right) = \frac{1}{2}\sqrt{(\sin\theta_1 - \sin\theta_2)^2 + (\cos\theta_1 - \cos\theta_2)^2},$$

which is the desired result.

$$\text{From } \sin\left(\frac{\theta_1-\theta_2}{2}\right) \to \cos(\theta_1-\theta_2) \to \sin(\theta_1-\theta_2).$$

Getting $\sin(\frac{\theta_1-\theta_2}{2})$ from $\sin\theta_1, \ \sin\theta_2$

We had discussed how to find 24 Rsines at the interval of $225' = 3^\circ 45'$. Similarly we can find sines at the interval of 3°, that is 30-fold division of the quadrant. From this sines at the interval of 1.5°, that is 60-fold division of the quadrant can be determined.

We know $\sin 30^\circ, \sin 15^\circ$ from this, $\sin 45^\circ, \sin 18^\circ, \sin 36^\circ$. We write the angles in degrees whose sines can be found. If we know, $\sin\theta_1, \sin\theta_2$, we can find $\sin(\theta_1 - \theta_2)$. From $\sin\theta$, $\sin(90-\theta)$, $\sin(\frac{\theta}{2})$ can be found. The scheme is outlined here.

30-fold and 60-fold divisions of the quadrant

$$18 \to 9, 81;\ \ 72 \to 36,\ \ 54 \to 27$$

$$\text{From}\, 15, 9 \to 6 \to 3,\ \ 15, 3 \to 12$$

$$27, 6, \to 21, 27, 3 \to 24, 36, 3 \to 33$$

$$45, 3 \to 42, 42, 3 \to 39.$$

So, we have sines of 3, 6, 9, 12, 15, 18, 21, 24,127, 30, 33, 36, 39, 42, 45 degrees. Sines of 48, 51, 54 degrees etc., found from $\sin(48) = \cos(42) = \sqrt{1 - \sin^2 42}$, etc.

Sines of angles at the interval of 1.5° (60-fold division of the quadrant) can be found from the above values and the formula for finding $\sin\left(\frac{\theta}{2}\right)$ from $\sin\theta$.

“*Jīve paraspara nyāya*”

Now, Bhāskara goes on to find the sines at the interval of 1°, using a different method. For that he needs to use

$$R\sin(A \pm B) = \frac{R\sin A\cos B}{R} \pm \frac{R\cos R\sin B}{R}$$

This is the famous “***Jīve paraspara nyāya***” in Indian trigonometry. This is the composition law. With the positive sign, it is the *Samāsa-bhāvanā* and with the negative sign, it is *Antara-bhavanā*. This is how, Bhāskara states it in Verses 21 and 22 of (*Jyotpatti*):

चापयोरिष्टयोर्दोर्ज्ये मिथःकोटिज्यकाहते।
त्रिज्याभक्ते तयोरैक्यं स्याच्चापैक्यस्य दोर्ज्यका॥ २१ ॥
चापान्तरस्य जीवा स्यात् तयोरन्तरसम्मिता।
अन्यज्यासाधने सम्यगियं ज्याभावनोदिता॥ २२॥

“If the sines of any two arcs of a quadrant be multiplied by their cosines reciprocally and the products divided by the radius then the quotients, will when added together, be the sine of the sum of the two arcs, and the difference of these quotients will be the sine of their difference. This excellent rule called *Jyā-Bhāvanā* has been prescribed for ascertaining the other sines.”

“*Jīve paraspara nyāya*”

Kamalākara’s सिधान्ततत्त्वविवेक (1658 CE) also gives the *Jīve paraspara nyāya* and its proof in his own commentary.

मिथः कोटिज्यक्रान्तिघ्नयौ त्रिज्याप्ते चापयोर्ज्यके।
तयोर्योगान्तरे स्यातां चापयोगान्तरज्यके॥ ६८ ॥

“The quotients of the Rcosines of any two arcs of a circle divided by its radius are reciprocally multiplied by their Rsines; the sum and difference of them (the products) are equal to the Rsine of the sum and difference respectively of the two arcs.”

$$jy\bar{a}(A \pm B) = \frac{jy\bar{a}(A)}{Trijy\bar{a}} Kojy\bar{a}(B) \pm \frac{Kojy\bar{a}(A)}{Trijy\bar{a}} jy\bar{a}(B)$$

or $$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

Cosine of sum and difference of angles

दोर्ज्योः कोटिमौर्व्योश्च घातौ त्रिज्योद्धृतौ तयोः।
वियोगयोगौ जीवे स्तः चापैक्यान्तरकोटिजे॥ ६९ ॥

"The product of the Rcosines of and of the Rsines of the two arcs of a circle are divided by its radius; the difference and sum of them (the quotients) are equal to the Rcosine of the sum and difference (respectively) of the two arcs."

$$Kojy\bar{a}(A \pm B) = \frac{Kojy\bar{a}(A)}{Trijy\bar{a}} Kojy\bar{a}(B) \pm \frac{jy\bar{a}(A)}{Trijy\bar{a}} jy\bar{a}(B)$$

or $$\cos(A \pm B) = \cos A \cos B - \sin A \sin B.$$

Finding the Sine at the Interval of 1°

Bhāskara-II then gives the method for generating the sine table for a 90-fold division of the quadrant. That is sine of multiples of 1° $(1^\circ, 2^\circ, \cdots, 90^\circ)$. He tells us how to find $\sin(\theta + 1^\circ)$ gives $\sin\theta$.

Verse 16,17:

स्वगोऽङ्गेषुषडंशेन वर्जिता भुजशिञ्जिनी।
कोटिज्या दशभिः क्षुण्णा त्रिसप्तेषु विभाजिता॥ १६ ॥
तदैक्यमग्रजीवा स्यादन्तरं पूर्वशिञ्जिनी।
प्रथमज्या भवेदेवं षष्टिरन्यास्ततस्ततः॥ १७ ॥
व्यासार्धेऽष्टगुणाब्ध्यग्नितुल्ये स्युर्नवतिर्ज्यकाः। १८*a* ।

Finding the Sine at the Interval of 1°

"Deduct from the sine of $bhuj\bar{a}$ its $\dfrac{1}{6569}$ part and divide the tenfold of koti by 573. The sum of two results will give the following sine (i.e, the sine of $bhuj\bar{a}$ one degree more than the original $bhuj\bar{a}$ and the difference between the same results will give the preceding sine i.e.,the sine of bhuja one degree less than original $bhuj\bar{a}$). Here the first-sine, ie., the sine of 1° will be 60 and the sines of the remaining arcs may be successively found."

Verse 18 a: "The rule, however supposes that radius is $3438'$. Thus the 90 sines (multiples of 1°) may be found."

Sines at the Interval 1°

Bhāskara takes the Rsine of $1^\circ = 60'$ to be the arc itself, that is, 60.

$$\text{So, } R\sin 1^\circ = \text{ Jya } = \text{ Arc } = 60$$

$$\therefore \ \sin 1^\circ = \frac{60}{R} = \frac{60}{3438} = \frac{10}{573}$$

To find $\sin(\theta + 1^\circ)$ from $\sin\theta$, he gives the formula:

$$\sin(\theta + 1^\circ) = \sin\theta\left(1 - \frac{1}{6569}\right) + \cos\theta\frac{10}{573}$$

$$= \sin\theta\cos 1^\circ + \cos\theta\sin 1^\circ$$

We have seen that $\sin 1^\circ = \dfrac{10}{573}$ according to him.

[Bhāskara's $\sin(1^\circ) = \dfrac{10}{573} = 0.017452007$. Modern $\sin(1^\circ) = 0.01745245$]

$$\text{According to Bhāskara, } \cos 1^\circ = 1 - \frac{1}{6559} = 0.99984777$$

$$\text{Modern } \cos 1^\circ = 0.99984769$$

Sines at the Interval 1°

How did Bhāskara state the value of $\cos 1^\circ$ as $1 - \dfrac{1}{6569}$?

Now,

$$\cos 1^\circ = \sqrt{1 - \sin^2(1^\circ)} = \sqrt{1 - \left(\frac{10}{573}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{10}{573}\right)^2 = 1 - \frac{1}{6566.58},$$

if we stop at the first term in the binomial expression. If we consider the next term also,

$$\cos 1^\circ = \sqrt{1 - \left(\frac{10}{573}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{10}{573}\right)^2 + \frac{3}{8}\left(\frac{10}{573}\right)^4 \approx \frac{1}{6568.08}.$$

Bhāskara has taken: $\cos 1^\circ = 1 - \dfrac{1}{6569}$.

Better Value of Rsines

Bhāskara takes

$$R\sin(225') = 3438 \times \frac{100}{1529} = 224\frac{6}{7},$$

This is an improvement over the Āryabhaṭan value of $R\sin 225' = 225'$. There Āryabhaṭa takes the Rsine of the arc 225' to be arc itself. Here Bhāskara takes the Rsine of a smaller arc 60' to be arc itself. So, naturally, Rsines of the larger arcs will be less than arcs themselves.

Bhāskara I formula for Sine

Actually, in his *Āryabhaṭīyabhāṣyā*, Bhāskara-I ascribes the formula to Āryabhaṭa himself :

$R\sin(\theta) = \frac{R(180-\theta)\theta}{(40500-(180-\theta)\theta)/4}$, where θ is in degrees.

$40500 = 5/4 \times 180 \times 180$.

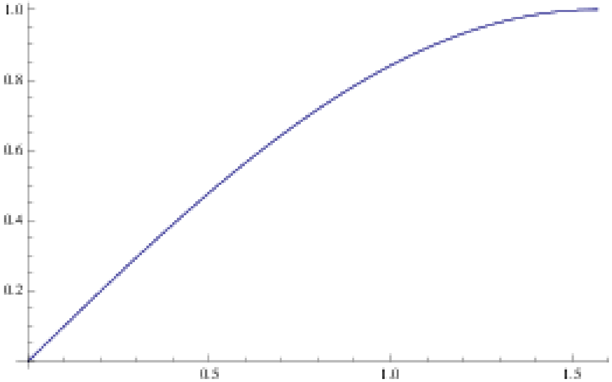
If θ is in radians,

$\sin(\theta) = \frac{4(\pi-\theta)\theta}{5/4\pi^2-(\pi-\theta)\theta}$

Comparison of actual Sine and Bhāskara formula

$\frac{\theta}{\pi/2}$	$Sin(\theta)$	Bhāskara formula	Percent Error
0.1	.156434	.158004	+1
0.2	.309017	.310345	+.43
0.3	.453990	.454343	+.04
0.4	.587785	.587156	-.11
0.5	.707107	.705882	-.17
0.6	.809017	.807692	-.16
0.7	.891007	.889976	-.12
0.8	.951057	.950495	-.06
0.9	.987688	.987531	-.01
1.0	1.0	1.0	0

Bhāskara I formula and actual Sine $\theta = (0, \frac{\pi}{2})$



Sines of multiples of angles

In his '*Jyotpatti*' Bhāskara-II explains how sines of multiples of angles can be found using the above *bhāvanā* principles:

इयं सिद्धज्यातोऽन्यज्यासाधने भावना।
तद्यथा – तुल्यभावनया प्रथमज्यार्धस्य प्रथमज्यार्धेन सह
समासभावनया द्वितीयं द्वितीयस्य द्वितीयेनैवं चतुर्थं इत्यादि।
अथातुल्यभावनया। द्वितीयतृतीययोः समासभावनया पञ्चमम्।
अन्तरभावनया प्रथमं स्यादित्यादि।

"This being proved, it becomes an argument for determining the values of other functions. For example, take the case of the combination of functions of equal arcs: by combining the functions of any arc with those of itself, we get the functions of twice that arc; by combining the functions of twice the arc with those of twice the arc, we get functions of four times that arc; and so on. Next take the case of combination of function of unequal arcs: on combining the functions of twice an arc with those of thrice that arc, by the addition theorem, we set the functions of five times that arc; but by the subtraction theorem, we set the functions of arc time that arc; and so on."

Sines of multiples of angles

Sines and cosines of multiples and submultiples of angles are discussed in Kamalākara's "*Siddhāntatattvaviveka*" (1658 CE).

अथात्र दोर्ज्यावगमाद्वदामि
द्वित्र्यब्धिपञ्चघ्नभुजांशजीवाम्।
दोःकोटिजीवाभिहतिर्द्विनिघ्नो
त्रिज्योद्धृता सा द्विगुणांशजीवा॥ ७३ ॥

"Hereafter I shall describe how to find the Rsine of twice, thrice, four times or five times as arc, having known the Rsine of the sum of two arcs. The product of the Rsine and Rcosine of an arc is multiplied by 2 and divided by the radius; the result is the Rsine of twice that arc."

Sines of multiples of angles

यद्बाहुकोटिज्यकयोश्च वर्ग-
वियोगमानं त्रिभजीवयाऽऽप्तम्।
नूनं च तत्कोटिगुणस्य मानं
द्विसंगुणानां च तदंशकानाम्॥ १०॥

“The difference of the squares of the Rsine and Rcosine of an arc is divided by the radius; the quotient is certainly the Rcosine of twice that arc.”

Sines of multiples of angles

The relations explicitly stated by him are:

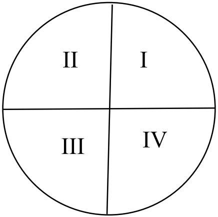
$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta; \quad \cos 2\theta = \cos^2\theta - \sin^2\theta \\
\sin 3\theta &= 3\sin\theta - 4\sin^3\theta; \quad \cos 3\theta = 4\cos^2\theta - 3\cos\theta \\
\sin 4\theta &= 4(\cos^3\theta\sin\theta - \sin^3\theta\cos\theta) \\
\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\
\sin 5\theta &= \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^2\theta \\
\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.
\end{aligned}$$

Sines of submultiples of angles

The following Sines of submultiples of angles are also stated by him:

$$
\begin{aligned}
\sin\left(\frac{\theta}{2}\right) &= \sqrt{\frac{1}{2}(1-\cos^2\theta)} \\
\sin\left(\frac{\theta}{3}\right) &= \frac{1}{3}\sin\theta + \frac{4}{81}\sin^3\theta \\
\sin\left(\frac{\theta}{4}\right) &= \frac{1}{2}\sqrt{2-\frac{\sin\theta}{\sin(\theta/2)}} \\
\sin\left(\frac{\theta}{5}\right) &= \frac{1}{5}\sin\theta + \frac{4}{5}\left(\frac{\sin\theta}{5}\right)^3 - \frac{16}{5}\left(\frac{\sin\theta}{5}\right)^5
\end{aligned}
$$

Signs of Sines and Cosines in the four quadrants



The four quadrants.

Signs of sines and cosines in the four quadrants are correctly understood in all the texts. It is explicitly stated in Mañjula's *Laghumānasa* in the context of *Manda*-correction (Equation of Centre) "The (mean) planet when diminished by its apogee or aphelion is the kendra (mean anomaly). Its Rsine is positive or negative in the upper or lower halves (of the quadrants) and its Rcosine is positive, negative, negative and positive (respectively) according to the quadrants. So, sin θ is positive $0 \leq \theta \leq 180^\circ$; negative, $180^\circ \leq \theta \leq 360^\circ$, cos θ is positive, $0^\circ < \theta < 90^\circ$, $270^\circ < \theta < 360^\circ$ and negative $90^\circ \leq \theta \leq 270^\circ$.

Brahmagupta again: Plane Trigonometry Formula

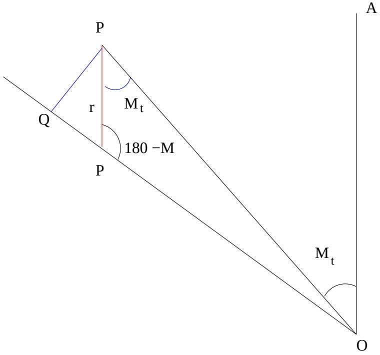
In Chapter 14 of his *Brāhmasphuṭasiddhānta*, Brahmagupta gives the relations among the sines and cosines of a plane triangle essentially:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

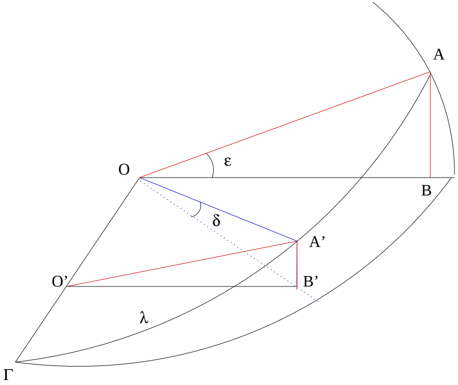
and

$$a^2 = b^2 + c^2 - 2bc \cos A$$

in the context of the triangle associated with the *manda* correction.



Spherical Triangle and Declination formula



$$R \sin \delta = R \sin \epsilon \sin \lambda$$

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Thanks!

Thank You