NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 20

Līlāvatī **of Bhāskarācārya I**

M. S. Sriram
University of Madras, Chennai.

Outline

- Introduction, Importance of *Līlāvatī*
- Arithmetical operations: Inversion method, rule of supposition
- Solution of quadratic equations
- Mixtures
- Combinations, progressions.
- Plane figures: Application of Right Triangles.

Introduction

Bhāsakarācārya: Perhaps the most well known name among the ancient Indian astronomer-mathematicians. Designated as Bhāskara-II to differentiate him from his earlier namesake, who lived in the seventh century CE (Bhāskara-I).

According to his own statement in his *Golādhyāya*, was born in saka 1036 or 1114 CE. He also adds that he came from *Vijjalavīda* near the *Sahyādri* mountain. $\rightarrow$ Bijapur? S.B. Dixit $\rightarrow$ Bhāskara's original home was Pāṭan in (*Khāndeś*) $\rightarrow$ Inscription lists his grandfather, father, son and grandson: Manoratha, Maheśvara, Lakṣmīdhara, Cangadeva respectively.

Bhāskara's *Līlāvatī* :A standard work of Indian Mathematics: *Patī* or *Patīgaṇita*: elementary mathematics covering arithmetic, algebra, geometry and mensuration. Still used as a textbook in Sanskrit institutions in India. Composed around 1150 CE.

Commentaries and other works

Many commentaries: Parameśvara (about 1430), Gaṇeṣa (*Buddhivilāsinī*) (1545), Sūryadāsa, Munīśvara (about 1635), and Rāmakṛṣṇa (1687). According to R. C. Gupta, a well known historian of Indian mathematics, the best traditional commentary is *Kriyākramakarī* (C. 1534) of Śaṅkara Vāriyar and Mahiṣamaṅgala Nārāyaṇa (who compiled this after the demise of *Śaṅkara*). Number of commentaries and versions in regional languages, Kannada, Telugu, etc., Three Persian translations are known: Abul-Fayd Faydi (1587 CE). My lectures based on Colebrook's translation (1817) reprinted along with the original Sanskrit text and commentaries with notes by H. C. Banerji. Others works of Bhāskara: *Bījagaṇita* (Algebra), one of the most important treatises on Indian algebra. *Siddhāntaśiromaṇi*: comprising *Grahagaṇita* and *Golādhyāya* also very popular. Other work *Karaṇakutūhala* (epoch 1183 CE).

Operations of arithmatic

$L\bar{\imath}l\bar{a}vat\bar{\imath}$: Rules and Examples in about 270 Verses.
Chapter 1 on weights and measures, money denominations.
Numeration upto 10^{14}.
Eight operations of arithmetic.

i) Addition and subtraction
ii) Multiplication
   a) $a_1 \dots a_n \times B$. Multiply a_n by A, a_{n-1} by A etc. and add (taking into account place values).
   b) $A(b+c) = Ab + Ac$.
   c) $A \times B = Ax \times \frac{B}{x}$, where x is a factor of B.
   d) $ab = a(b \pm c \mp c) = a(b \pm c) \mp ac$.
iii Division: Remove common factors.

Squaring , Square roots etc.

Verses 18-19. Squaring

i)

$$\begin{aligned}(a_1 + \cdots a_n)^2 \quad = \quad & a_n^2 + 2a_n(a_1 + \cdots + a_{n-1}) \\ & + a_{n-1}^2 + 2a_{n-1}(a_1 + \cdots + a_{n-2}) \\ & + \cdots \\ & + a_2^2 + 2a_2 a_1 \\ & + a_1^2.\end{aligned}$$

ii) $(a+b)^2 = a^2 + 2ab + b^2$.

iii) $a^2 = (a+b)(a-b) + b^2$.

Verses 21-22: Square root and example.
Verses 23-26: Cube and example.
Verse 27-28: Cube root: Standard Indian method.

Multipulation with fractions

Fractions: $\frac{a}{b}, \frac{c}{d} = \frac{ad}{bd}, \frac{cb}{db}$ for addition, subtraction etc.,

Verse 32-33:

$$\begin{aligned}
\frac{a}{b} \times \frac{c}{d} \times \cdots &= \frac{ac...}{bd....} \\
\frac{a}{b} + \frac{c}{d} \times \frac{a}{b} + \frac{e}{f}\left(\frac{a}{b} + \frac{c}{d} \times \frac{a}{b}\right) &+ \frac{g}{h}\left[\frac{a}{b} + \frac{c}{d} \times \frac{a}{b} + \frac{e}{f}(\qquad)\right]\ldots \\
&= \frac{a(d+c)(e+f)(g+h)\ldots}{bdfh\ldots}
\end{aligned}$$

Division of fractions.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

Operations with zero

Verses 44-45 :

योगे खं क्षेपसमं वर्गादौ खं खभाजितो राशिः।
खहरः स्यात् खगुणः खं खगुणश्चिन्त्यश्च शेषविधौ॥ ४४ ॥
शून्ये गुणके जाते खं हारश्चेत् पुनस्तदा राशिः।
अविकृत एव ज्ञेयस्तथैव खेनोनितश्च युतः॥ ४५ ॥

“In addition, zero makes the sum equal to the additive. In involution and (evolution) the result is zero. A definite quantity , divided by zero, is the submultiple of nought. The product of zero is nought: but it must be retained as a multiple of zero, if any operations impend. Zero having become a multiplier, should nought afterwards become a divisor, the definite quantity must be understood to be unchanged. So likewise any quantity, to which zero is added, or from which it is subtracted, (is unaltered).

Operations with zero

$a \pm 0 = a, \dfrac{a}{0} = \textit{Kha-hara}$: a fraction with zero for its denominator.

(In *Bījagaṇita*. Gaṇeśa: Infinity).

$$a \times 0 = 0$$

But it must retained as a multiple of zero, if further operations, impend. “Zero having become a multiplier, should nought afterwards become a divisor, the definite quantity must be understood to be unchanged”.

$$a \times \frac{0}{0} = a? \qquad \text{(But } \frac{0}{0} \text{ is indeterminate)}$$

.

Inversion method

Inversion method to find the solution of a problem in Verse 47-48 :

छेदं गुणं गुणं छेदं वर्गं मूलं पदं कृतिम्।
ऋणं स्वं स्वमृणं कुर्याद् दृश्ये राशिप्रसिद्धये॥ ४७ ॥
अथ स्वांशाधिकोने तु लवाढ्योने हरो हरः।
अंशस्त्वविकृतस्तत्र विलोमे शेषमुक्तवत्॥ ४८॥

"To investigate a quantity, one being given, make the divisor a multiplicator; and the multiplier, a divisor; the square, a root; and the root, a square; turn the negative into positive and the positive into negative. If a quantity is to be increased or diminished by its own proportional part, let the (lower) denominator, being the increased or diminished by its numerator, becomes the (corrected) denominator, and the numerator remain uncharged*; and then proceed with the other operations of inversion, as before directed."

(* If we have $1+\dfrac{a}{b}=\dfrac{a+b}{b}$, in the inversion, it will be $\frac{b}{a+b}$.)

Inversion method: Example

Example in Verse 49 :

यस्त्रिघ्नस्त्रिभिरन्वितः स्वचरणैर्भक्तस्ततः सप्तभिः
स्वत्र्यंशेन विवर्जितः स्वगुणितो हीनो द्विपञ्चाशता।
तन्मूलेऽष्टयुते हृते च दशभिर्जातं द्वयं ब्रूहि तं
राशिं वेत्सि हि चञ्चलाक्षि विमलां
वामे विलोमक्रियाम्॥ ४९ ॥

“Pretty girl, with tremulous eyes, if thou know the correct method of inversion, tell me the number, which multiplied by three and added to three quarters of the product, and divided by seven, and reduced by subtraction of half a third part of the quotient and then multiplied into itself, and having fiftytwo subtracted from the product and the square root of the remainder extracted, and eight added and the sum divided by ten yields two (2).”

Inversion method: Example

$$\frac{1}{10}\left[8+\sqrt{-52+\left[\underbrace{\left(1-\frac{1}{3}\right)}_{\frac{2}{3}}\frac{1}{7}\underbrace{\left(1-\frac{3}{4}\right)}_{\frac{7}{4}}\times 3\times x\right]^2}\,\right]=2$$

Answer:

$$x=\sqrt{(2\times 10-8)^2+52}\times\frac{3}{2}\times\frac{4}{7}\times\frac{1}{3}=28.$$

Rule of false position

Verse 50:

> उद्देशकालापवदिष्टराशिः क्षुण्णो हृतोंऽशै रहितो युतो वा ।
> इष्टाहतं दृष्टमनेन भक्तं राशिर्भवेत् प्रोक्तमितीष्टकर्म ॥ ५० ॥

> “Any number assumed at pleasure is treated as specified in the particular question, being multiplied and divided, raised or diminished by fractions; then the given quantity, being multiplied by the assumed and divided by that (which has been found) yields th number sought. This is called the process of supposition.”

Let some operations be done on the number x, represent it as A. Let it yield $A(x) = y$. Take any x' and do the same operations. Let it yield y', $A(x') = y'$. Then

$$x = y \times \frac{x'}{(y')}$$

Rule of false position

Example in Verse 51.

पञ्चघ्नः स्वत्रिभागोनो दशभक्तः समन्वितः।
राशित्र्यंशार्धपादैः स्यात् को राशिर्द्व्यूनसप्ततिः॥ ५१ ॥

" What is that number, which multiplied by 5 and having the third part of the product subtracted and the remainder divided by ten, and one-third, a half and a quarter of the original quantity added, gives two less than seventy?"

Solution: Multiplier 5. Subtractive $\frac{1}{3}$ of itself. Divisor 10. Additive $\frac{1}{3}, \frac{1}{2}, \frac{1}{4}$, of the quantity.

Given $68 = y$. Take $x' = 3$; $5 \times 3 = 15$, $(1 - \frac{1}{3})15 = 10$; $\frac{1}{10} \times 10 = 1$; $1 + 3(\frac{1}{3} + \frac{1}{2} + \frac{1}{4}) = \frac{17}{4}$; This is y'. So, $y' = \frac{17}{4}$.

$$\therefore, x = y \times \frac{x'}{y'} = 68 \times \frac{3}{\frac{17}{4}} = 48.$$

One gets $x = 48$, whatever value one chooses for x'. y' will be correspondingly different.

Rule of concurrence or '*Saṅkramaṇa*'.

Verse 55:

योगोऽन्तरेणोनयुतोऽर्धितस्तौ
राशी स्मृतौ सङ्क्रमणाख्यमेतत् ॥ ५५॥

"The sum with the difference added or subtracted, being halved, gives the two quantities. This is termed Concurrence.".

If $x + y = k$ and $x - y = l$ where k and l are given quantities.

$$x = \frac{1}{2}(k + l) \text{ and } y = \frac{1}{2}(k - l)$$

Verse 57. If $x - y$ and $x^2 - y^2$ are given, find $x + y = \dfrac{x^2 - y^2}{x - y}$.

Then x and y are found by *Saṅkramaṇa*.

Rational Squares

Verse 59 and 60: Examples of generation of rational squares:
Consider the pairs

a)
$$x_1 = \frac{1}{2n}(8n^2 - 1), x_2 = \frac{1}{2}\{\frac{1}{2n}(8n^2 - 1)\}^2 + 1,$$

b)
$$x_1 = \frac{1}{2n} + n, x_2 = 1,$$

c)
$$x_1 = 8n^4 + 1, x_2 = 8n^3.$$

For all these pairs, $x_1^2 + x_2^2 - 1$ are squares.

Solution of Quadratic equations

Next, Bhāskara gives the solution of a quadratic equation .
Verse 62-63: Suppose we have

$$x \pm a\sqrt{x} = b$$

Then $\sqrt{x} = \mp\frac{a}{2} + \sqrt{b + (a/2)^2}$ and $x = \left(\mp\frac{a}{2} + \sqrt{b + (a/2)^2}\right)^2$.

(At least here, only one root is being talked about)
Similarly, given: $x + \frac{c}{d}x \pm a\sqrt{x} = b$, we obtain

$$x \pm \frac{a\sqrt{x}}{1 \pm \frac{c}{d}} = \frac{b}{1 \pm \frac{c}{d}}.$$

Then proceed as before to find $\sqrt{x}$ and then x.

Quadratic equations: Example

An example in Verse 67.

पार्थः कर्णवधाय मार्गणगणं क्रुद्धो रणे सन्दधे
तस्यार्धेन निवार्य तच्छरगणं मूलैश्चतुर्भिर्हयान्॥
शल्यं षड्भिरथेषुभिस्त्रिभिरपि छत्रं ध्वजं कार्मुकं
चिच्छेदास्य शिरः शरेण कति ते यानर्जुनः सन्दधे॥ ७६ ॥

"The Son of Pṛtha, irritated in fight, shot a quiver of arrows to slay Karṇa. With half his arrows he parried those of his antagonist; with four times the square root of the quiver-full, he killed his horses; with six arrows, he slew Śalya; with three he demolished the umbrella, standard and bow; and with one, he cut off the head of the foe. How many were the arrows which Arjuna let fly?"

Solution of Quadratic equations

Let x denote the number of arrows.

$$\frac{1}{2} + 4\sqrt{x} + 6 + 3 + 1 = x$$

$$\frac{1}{2}x - 4\sqrt{x} - 10 = 0$$

$$\therefore\ x - 8\sqrt{x} - 20 = 0$$

$$\sqrt{x} = 10; x = 100.$$

(Here $\sqrt{x} = -2$ is also a solution. But in the problem $\sqrt{x}$ has to be +ve.)

Rule of 3, 5, 7, 9, ...

Investigation of mixtures

Rule in Verse 90.

अथ प्राणैर्गुणिताः स्वकालाः प्रतीतकालघ्नफलोद्धृतास्ते।
स्वयोगभक्ताश्च विमिश्रनिघ्नाः प्रयुक्तखण्डानि पृथग् भवन्ति॥

“The arguments taken into their respective times one divided by the fruit taken into the elapsed times; the several quotients, divided by their sum, and multiplied by the mixed quantity, and the parts as severally lent.”

Mixtures

Let x, y, z be the portions lent at r_1, r_2, r_3 percent per month and let I = common interest in t_1, t_2, t_3 months respectively.

Let $x + y + z = a$, a given quantity. Then,

$$\frac{x \times r_1 \times t_1}{100} = \frac{y \times r_2 \times t_2}{100} = \frac{z \times r_3 \times t_3}{100} = I.$$

$$\therefore\ x : y : z :: \frac{100 \times 1}{r_1 \times t_1} : \frac{100 \times 1}{r_2 \times t_2} : \frac{100 \times 1}{r_3 \times t_3}.$$

$$x = \frac{100 \times 1}{r_1 \times t_1} \times \frac{a}{\frac{100 \times 1}{r_1 \times t_1} + \frac{100 \times 1}{r_2 \times t_2} + \frac{100 \times 1}{r_3 \times t_3}}$$

Here: Argument 100, Time: 1, fruit: r, elapsed time : t_1.

Similar expressions for y and z.

Mixtures: Example

Example in Verse 91:

यत् पञ्चकत्रिकचतुष्कफलेन दत्तं
खण्डैस्त्रिभिर्गणक निष्कशतं षडूनम्।
मासेषु सप्तदशपञ्चसु तुल्यमाप्तं
खण्डत्रयेऽपि हि फलं वद खण्डसंख्याम्॥ ९१ ॥

"The sum of six less than a hundred *niṣkas* being lent in three proportion at interest of 5, 3, and 4 percent, an equal interest was obtained on three portions in 7, 10 and 5 months respectively. Tell mathematician, the amount of each portion."

Solutions

Here $a = 94, r_1 = 5, r_2 = 3, r_3 = 4; t_1 = 7, t_2 = 10, t_3 = 5.$.

Now $r_1t_1 = 35, r_2t_2 = 30, r_3t_3 = 20.$.

$$\therefore \frac{100}{35} + \frac{100}{30} + \frac{100}{20} = \frac{(6+7+10.5)}{210} \times 100 = \frac{235}{21}.$$

$$\therefore x(\text{portion } 1) = \frac{\frac{100}{35} \times 94}{\frac{235}{21}} = \frac{100}{35} \times \frac{21}{235} \times 94 = 24,$$

$$y(\text{portion } 2) = \frac{\frac{100}{30} \times 94}{\frac{235}{21}} = \frac{100}{30} \times \frac{21}{235} \times 94 = 28,$$

$$z(\text{portion } 3) = \frac{\frac{100}{20} \times 94}{\frac{235}{21}} = \frac{100}{20} \times \frac{21}{235} \times 94 = 42.$$

$$\text{Common interest } = I = \frac{24 \times 5 \times 7}{100} = \frac{840}{100} = 8\frac{2}{5}.$$

He considers the filling of a cistern from n fountains : Same as in $Ga\d{n}itas\bar{a}rasa\dot{n}graha$

Another kind of problem involving mixtures

Let there be a mixture of n items. Let the relative proportion of type i in the Mixture $= \beta_i$.

$\therefore$ Relative proportion $= \beta_1 : \beta_2 : \cdots \beta_n$.

(Fraction of type i in the mixture $= \beta_i' = \dfrac{\beta_i}{\sum \beta_i}; \sum \beta_i' = 1$).

Let the price of type i be x_i per measure (volume or weight).

Let the total amount of the mixture = A (volume or weight).

$$\therefore \text{ Total price } = \sum (A\beta_i') x_i = \frac{A \sum x_i \beta_i}{\sum \beta_i} = \text{ Mixed Sum } = X,$$

Formal solution

As the amount of item i in the mixture is $A\beta_i'$, and x_i is the price per measure.

$$\therefore A = \frac{X \sum \beta_i}{\sum x_i \beta_i}.$$

$$\therefore \text{ Amount (Measure) of type } i = \beta_i' A = \frac{\beta_i X}{\sum x_i \beta_i} = \frac{\text{Portion } \times \text{ Mixed sum}}{\text{Sum of quotients}}.$$

$$(\text{as } \beta_i = \beta_i' \sum \beta_i)$$

.

$$\text{Price of item of type } i = \frac{x_i \beta_i \times X}{\sum x_i \beta_i} = \frac{\text{(Price of i) } \times \text{ Portion} \times \text{Mixed Sum}}{\text{Sum of quotients}}.$$

.

Mixed quantities: Example

Example in Verse 98.

कर्पूरस्य वरस्य निष्कयुगलेनैकं पलं प्राप्यते
वैश्यानन्दनचन्दनस्य च पलं द्रम्माष्टभागेन चेत्।
अष्टांशेन तथाऽगरोः पलदलं निष्केण मे देहि तान्
भागैरेककषोडशाष्टकमितैर्धूपं चिकीर्षाम्यहम्॥ ९८ ॥

"If a pala of best camphor may be had for two $nisk'$s (=32 *drammas*) and a *pala* of sandalwood for the eighth part of a *dramma* and half a *pala* of alae wood also for eighth of a *dramma*, good merchant, give me the value for one *niṣka* (=16 *drammas*) in the proportions of 1, 16, and 8, for I wish to prepare a perfume."

Solution

$$\begin{array}{rcccl} & (1) & (2) & (3) & \\ \text{Here proportion:} & \text{Camphor:} & \text{Sandalwood:} & \text{Alaewood} & = 1:16:8 \\ \text{Prices:} & 32 & 1\frac{1}{8} & \frac{1}{4} & \\ & (x_1) & (x_2) & (x_3) & \end{array}$$

$$\sum x_i\beta_i = 32\times 1 + \frac{1}{8}\times 16 + \frac{1}{4}\times 8 = 36.$$

Total price = Mixed amount = 16 *drammas* = X =16.

$$\text{Quotient}, q = \frac{\text{Mixed sum}}{\text{Sum of quotients}} = \frac{X}{\sum x_i\beta_i} = \frac{16}{36} = \frac{4}{9}.$$

$$\text{Amounts of item } i = \frac{\beta_i X}{\sum x_i\beta_i} = \beta_i\frac{4}{9}, \text{ Price } = x_i\frac{\beta_i X}{\sum x_i\beta_i} = x_i\times\text{Amount}.$$

$\therefore$ Amount of Camphor: $1\times\frac{4}{9}$, Sandalwood $=16\times\frac{4}{9}=\frac{64}{9}$, Alaewood $=8\times\frac{4}{9}=\frac{32}{9}$

Prices : Camphor: $32\times\frac{4}{9}=\frac{128}{9}$, Sandal: $\frac{1}{8}\times\frac{64}{9}=\frac{8}{9}$, Alaewood: $=\frac{1}{4}\times\frac{32}{9}=\frac{8}{9}$.

Merchants, trading. Alligation: Mixture of gold of different quantities.

Section six on combinations

Verse 130-132.

एकाद्येकोत्तराङ्का व्यस्ता भाज्याः पृथक्स्थितैः।
परः पूर्वेण संगुण्य तत्परस्तेन तेन च॥ १३०॥
एकद्वित्र्यादिभेदाः स्युः इदं साधारणं स्मृतम्।
छन्दश्चित्त्युत्तरे छन्दस्युपयोगोऽस्य तद्विदाम्॥ १३१॥
मूषावहनभेदादौ खण्डमेरौ च शिल्पके।
वैद्यके रसभेदीये तन्नोक्तं विस्तृतेर्भयात्॥ १३२ ॥

“Let the figures of one upwards, differing by one, put in the inverse order, be divided by the same (arithmetical sequence) in direct order; and let the subsequent be multiplied by the preceding, and the next following by the foregoing (result). The several results are the changes, ones, twos, threes, etc. This is termed a general rule.”

Combinations

It serves in prosody, for those versed therein, to find the variations of metre; in the arts (as in architecture) to compute the changes upon apertures (of a building); and (in music) the scheme of musical permutations; in medicine, the combinations of different savours. For fear of prolixity, this is not (fully) set forth."

$$\frac{n}{1}, \frac{n \cdot (n-1)}{1 \cdot 2}, \ldots, \frac{n(n-1)(n-2)(n-r) \ldots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots r} = \frac{n!}{r!(n-r)!}$$

Out of n: 1 at a time: 2 at a time: 3 at a time, ... , r at a time.

Combinations: Example

Example in Verse 134.

एकद्वित्र्यादिमूषावहनमितिमहो ब्रूहि मे भूमिभर्तुः हर्म्ये
रम्येऽष्टमूषे चतुरविरचिते श्लक्ष्णशालाविशाले।
एकद्वित्र्यादियुक्त्या मधुरकटुकषायाम्लकक्षारतिक्तैः एकस्मिन्
षड्रसैः स्युर्गणक कति वद व्यञ्जने व्यक्तिभेदाः॥

“In a pleasant, spacious and elegant edifice with eight doors, constructed by a skillful architect, as a palace for the lord of the land, tell me combinations of doors taken one, two, three, etc. Say, mathematician, how many are the combinations in one composition, with ingredients of six different tastes sweet, pungent, astringent, sour, salt and bitter, taking them by ones, twos, threes etc.”

8 7 6 5 4 3 2 1

1 2 3 4 5 6 7 8

Example contd.

No. of ways: 1 at a time = $\frac{8}{1} = 8$, 2 at a time $= \frac{8\times7}{1\times2} = 28$, 3 at a time $= \frac{8\times7\times6}{1\times2\times3} = 56$, 4 at a time $= \frac{8\times7\times6\times5}{1\times2\times3\times4} = 70$. 5 at a time $= \frac{8\times7\times6\times5\times4}{1\times2\times3\times4\times5} = 56$, 6 at a time $= \frac{8\times7\times6\times5\times4\times3}{1\times2\times3\times4\times5\times\times6} = 28$, 7 at a time $= \frac{8\times7\times6\times5\times4\times3\times2}{1\times2\times3\times4\times5\times6\times7} = 8$, 8 at a time $= \frac{8\times7\times6\times5\times4\times3\times2\times1}{1\times2\times3\times4\times5\times6\times7\times8} = 8$.

Total no. of variations $= 2^8 - 1 = 255$.

Combinations of 6 different tastes:

$$\begin{array}{cccccc} 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

No. of different types of compositions: 6, 15, 20, 15, 6, 1. Total $= 2^6 - 1 = 63$.

Chapter 5. Progressions.

Standard results on $\sum r, \sum r^2, \sum \frac{r(r+1)}{2}, \sum r^3$, etc., Arithmetic and Geometric Progressions.

Number of verses with a fixed number of syllables

Verse 130-131:

पादाक्षरमितगच्छे गुणवर्गफलं चये द्विगुणे।
समवृत्तानां सङ्ख्या तद्वर्गो वर्गवर्गश्च।
स्वस्वपदोनौ स्यातां अर्धसमानां च विषमाणाम्॥ १३०, १३१
॥

“The number of syllables in a verse being taken for the period, and the increase twofold, the produce of multiplication and squaring (as above directed) will be the number of (variations) of like verses. Its square and square’s square, less their respective roots, will be (the variations of) alternately similar and of dissimilar verses.”

This rule refers specifically to the example in Verse 132.

Example

Example 132.

समानामर्धतुल्यानां विषमाणां पृथक् पृथक्।
वृत्तानां वद मे सङ्ख्यां अनुष्टुप्छन्दसि द्रुतम्॥ १४० ॥

“Tell me directly the number (of varieties) of like, alternating-like, and dissimilar Verses respectively in the metre named *anuṣṭup*.”

$$\text{4 } \mathit{caraṇas} \left[\begin{array}{cccccccc} x & x & x & x & x & x & x & x \\ x & x & x & x & x & x & x & x \\ x & x & x & x & x & x & x & x \\ x & x & x & x & x & x & x & x \end{array} \right. \begin{array}{ll} \text{8 Syllables} & \\ \quad 8 & \\ \quad 8 & \\ \quad 8 & \text{Total 32 Syllables.} \end{array}$$

- Variations of like Verses. (All the *caraṇas* are alike.) $= 2^8 = 256$.

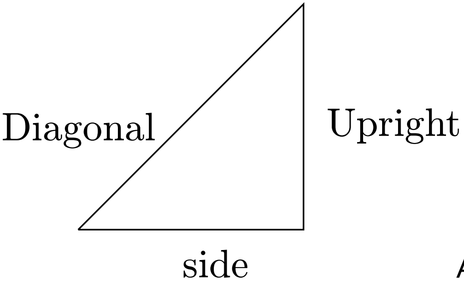
Example

- Alternatively, alike $caraṇas$. To find the number of these, we find the number of varieties of the syllables in two $caraṇas$ $= 2^{16} = 65536$. If we place each one of these under itself, we get all the cases included in "alternatively like". But of these, the number of cases of "all like" are also included. Therefore number of cases of "only alternatively like" $= 2^{16} - 2^{8} = 65536 - 256 = 65280$.
- Total no. of variations $= 2^{32}$. Subtracting the case of "all like" and "alternatively like", Number of dissimilar Verses $= 2^{32} - 2^{16}$. Note that the "dissimilar" only means it excludes "all like" and "alternatively like" and does not mean that all the $caraṇas$ are dissimilar. For instance, it includes cases in which "first two $caraṇas$ are alike, as also last two, etc."

Plane Figures

Chapter 6 is on this topic.

As expected, begins by considering Right Triangles.



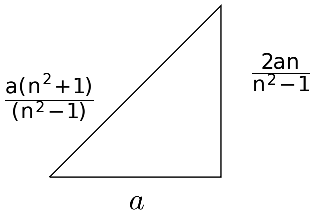
A right triangle

Concentrates on Rational triangles. One type is described in Verse 139.

इष्टेन भुजोऽस्माद् द्विगुणेननिघ्नात् इष्टस्य कृत्यैकवियुक्तयाप्तम्।
कोटिः पृथुक् सेष्टगुणाभुजोना कर्णो भवेत् त्र्यश्रमिदं तु जात्यम् ॥

A Rational right triangle

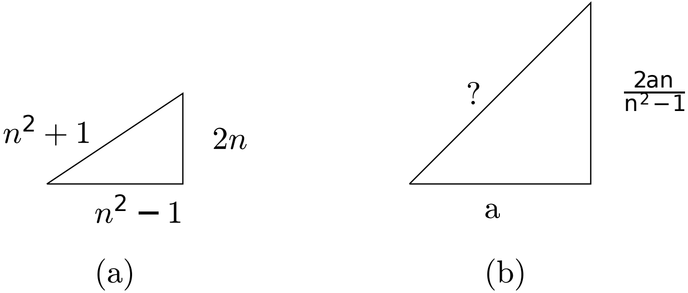
"A side is put. From this multiplied by twice some assumed number, and divided by one less than the square of the assumed number, the upright is obtained. This, being set apart, is multiplied by the arbitrary number, and the side as put is subtracted; the remainder will be the hypotenuse. Such a triangle is termed right-angled."



It is stated that side $= a$, upright $= \dfrac{2an}{n^2-1}$ yields a right triangle with hypotenuse $= \dfrac{2an}{n^2-1} \times n - a = \dfrac{a(n^2+1)}{n^2-1}$.

Explanation by Sūryadāsa in his commentary

Consider two triangles. The first triangle has side $= n^2 - 1$, upright $= 2n$ and diagonal $n^2 + 1$. Correct as $(n^2+1)^2 = (n^2-1)^2 + (2n)^2$. Let the second triangle be similar to the above, with side $= a$.



Similar right triangles

The upright of the second triangle $= \dfrac{a}{n^2-1} \times 2n$.

Now, hypotenuse of the first triangle
$= \text{upright} \times n - \text{side} = 2n \times n - (n^2 - 1) = n^2 + 1$.

Then, hypotenuse of the second triangle

$$= \text{Upright} \times n - \text{side} = \frac{2an}{n^2-1} \times n - a = \frac{a(n^2+1)}{n^2-1}.$$

References

1.H.T.Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhaskara*, London 1817; Rep. Sharada Publishing House, New Delhi, 2006.

2. Bhāskarācārya's *Līlāvatī* with Colebrooke's translation and notes by H.C.Banerji, Second Edition,The Book Company, Calcutta, 1927; Rep. Asian Educational Series, New Delhi, 1993.

3.*Līlāvatī* of Bhāsakarācārya II: Ed. with Bhāskara's *Vāsanā* and *Buddhivilāsanī* of Gaṇeśa Daivajña by V.G.Apte, 2 Vols., Pune, 1937.

4.*Līlāvatī* of Bhāsakarācārya , A Treatise of Mathematics of Vedic Tradition, Tr. by K.S.Patwardhan, S.M.Naimpally and S.L.Singh, Motilal Banarsidass, Delhi, 2001.

Thanks!

Thank You