NPTEL Course on

Mathematics in India: From Vedic period to Modern times

Lecture 31

Development of Calculus in India 2

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Outline

- Mādhava Series for π
- End-correction terms and Mādhava continued fraction
- Rapidly convergent transformed series for π
- History of Approximations to π
- Nīlakaṇṭha's refinement of the Āryabhaṭa relation for second-order Rsine differences
- Mādhava series for Rsine and Rcosine
- Nīlakaṇṭha and Acyuta formulae for instantaneous velocity

Mādhava Series for π

The following verses of Mādhava are cited in *Yuktibhāṣā* and *Kriyākramarī*:

व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
त्रिशरादिविषमसङ्ख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात्॥ १॥
यत्सङ्ख्यायाऽत्र हरणे कृते निवृत्ता हृतिस्तु जामितया।
तस्या ऊर्ध्वगता या समसङ्ख्या तद्दलं गुणोऽन्ते स्यात्॥ २॥
तद्वर्गो रूपयुतो हारो व्यासाब्धिघाततः प्राग्वत्।
ताभ्यामाप्तं स्वमृणे कृते धने क्षेप एव करणीयः॥ ३॥
लब्धः परिधिः सूक्ष्मो बहुकृत्वो हरणतोऽतिसूक्ष्मः स्यात्॥ ४॥

The first verse gives the Mādhava series (rediscovered by Leibniz in 1674)

$$Paridhi = 4 \times Vy\bar{a}sa \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right)$$

Mādhava Series for π

Mādhava also gave the *cāpīkaraṇa* series giving the arc (*cāpa*) associated with any Rsine (*jyā*)

इष्टज्यात्रिज्ययोर्घातात् कोट्याप्तं प्रथमं फलम्।
ज्यावर्गं गुणकं कृत्वा कोटिवर्गं च हारकम्॥

प्रथमादिफलेभ्योऽथ नेया फलततिर्मुहुः।
एकत्र्याद्योजसङ्ख्याभिर्भक्तेष्वेतेष्वनुक्रमात्॥

ओजानां संयुतेस्त्यक्त्वा युग्मयोगं धनुर्भवेत्।
दोःकोट्योरल्पमेवेष्टं कल्पनीयमिह स्मृतम्॥
लब्धीनामवसानं स्यात् नान्यथापि मुहुर्मुहुः।

Mādhava Series for π

$$
\begin{aligned}
s &= r\left[\frac{jy\bar{a}(s)}{ko\d{t}i(s)}\right] - \left(\frac{r}{3}\right)\left[\frac{jy\bar{a}(s)}{ko\d{t}i(s)}\right]^3 + \left(\frac{r}{5}\right)\left[\frac{jy\bar{a}(s)}{ko\d{t}i(s)}\right]^5 - \ldots. \\
s &= r\theta = r\left(\frac{r\sin\theta}{r\cos\theta}\right) - \left(\frac{r}{3}\right)\left(\frac{r\sin\theta}{r\cos\theta}\right)^3 + \left(\frac{r}{5}\right)\left(\frac{r\sin\theta}{r\cos\theta}\right)^5 - \ldots.
\end{aligned}
$$

Note: It has been clearly noted that we must ensure that numerator $<$ denominator in each term. This series for $tan^{-1}x$ was rediscovered by Gregory in 1671.

Mādhava Series for π

By using the *cāpīkaraṇa* series for an arc equal to one-twelfth of the circumference (30°), Mādhava gets a more rapidly convergent series for the ratio of the circumference to the diameter:

व्यासवर्गाद् रविहतात् पदं स्यात् प्रथमं फलम्।
तदादितस्त्रिसङ्ख्याप्तं फलं स्यादुत्तरोत्तरम्॥

रूपाद्ययुग्मसंख्याभिर्हृतेष्वेषु यथाक्रमम्।
विषमानां युतेस्त्यक्त्वा समा हि परिधिर्भवेत्॥

Mādhava Series for π

For an arc s which is one-twelfth of the diameter, corresponding to 30°, we have

$$\left(\frac{jy\bar{a}(s)}{ko\d{t}i(s)}\right)^2 = \frac{1}{3}$$

Therefore

$$\begin{aligned} C &= \left(\frac{12r}{\sqrt{3}}\right)\left[1 - \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{5}\left(\frac{1}{3}\right)^2 - \dots\right] \\ &= \sqrt{12d^2}\left[1 - \frac{1}{3.3} + \frac{1}{3^2.5} - \frac{1}{3^3.7} + \dots\right] \end{aligned}$$

This was rediscovered by Abraham Sharp in 1699.

End Correction Terms

The Mādhava series (or the so called Leibniz series) for the circumference of a circle (in terms of odd numbers $p = 1, 3, 5, \ldots$)

$$C = 4d\left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-1)}{2}}\frac{1}{p} + \ldots\right],$$

is an extremely slowly convergent series.

In fact, adding fifty terms of the series will give the value of π correct only to the first decimal place.

In order to facilitate computation, Mādhava has given a procedure of using end-correction terms (*antya-saṃskāra*), of the form

$$C = 4d\left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-1)}{2}}\frac{1}{p} + (-1)^{\frac{(p+1)}{2}}\frac{1}{a_p}\right]$$

End Correction Terms

The verses of Mādhava, which give the relation between the circumference and diameter, also include the end-correction term

$$C = 4d\left[1 - \frac{1}{3} + \ldots + (-1)^{\frac{(p-1)}{2}}\frac{1}{p} + (-1)^{\frac{(p+1)}{2}}\frac{\left\{\frac{(p+1)}{2}\right\}}{\{(p+1)^2+1\}}\right]$$

Mādhava has also given a finer end-correction term

अन्ते समसङ्ख्यादलवर्गः सैको गुणः स एव पुनः ॥
युगगुणितो रूपयुतः समसङ्ख्यादलहतो भवेद् हारः ।

$$C = 4d\left[1 - \frac{1}{3} + \ldots + \ldots + (-1)^{\frac{(p-1)}{2}}\,\frac{1}{p} \right.$$
$$\left. +(-1)^{\frac{(p+1)}{2}}\frac{\left[\left\{\frac{(p+1)}{2}\right\}^2 + 1\right]}{\left[\{(p+1)^2+5\}\left\{\frac{p+1}{2}\right\}\right]}\right]$$

End Correction Terms

To Mādhava is attributed a value of π accurate to eleven decimal places which is obtained by just computing fifty terms with the above correction.

विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः ।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः ॥

The π value given above is:

$$\pi \approx \frac{2827433388233}{9 \times 10^{11}} = 3.141592653592\ldots$$

End Correction Terms

Both *Yuktibhāṣā* and *Kriyākramakarī* give a derivation of the successive end correction terms given by Mādhava, which involve a careful estimate of the error at each stage in terms of inverse powers of the odd number p.

By carrying this process further, we find that the end-correction term $\frac{1}{a_p}$ can be expressed as a continued fraction:

$$\frac{1}{a_p} = \cfrac{1}{(2p+2) + \cfrac{2^2}{(2p+2) + \cfrac{4^2}{(2p+2) + \cfrac{6^2}{(2p+2) + \dots}}}}$$

End Correction Terms

We tabulate the accuracy achieved by end-correction terms when we sum fifty terms of the series ($p = 99$) together with successive correction terms.

Order of the correction term	None	1	2	3	4	5
Accuracy of π in number of decimal places	1	5	8	11	14	17

In fact, *Sadratnamāla* (c. 1819) of Śaṅkaravarman gives the following value of π which is accurate to 17 decimal places: $\pi \approx 3.14159265358979324$

Mādhava Continued Fraction for π

Using the above continued fraction for $\frac{1}{a_p}$ we will get a continued fraction for π minus the sum of the first p-terms in the Mādhava series for each odd number p.

In particular, for $p = 1$, we get what may be called the Mādhava continued fraction for π:

$$\frac{\mathbf{2}}{(\mathbf{4}-\pi)} = \mathbf{2} + \frac{\mathbf{1^2}}{\mathbf{2}+}\frac{\mathbf{2^2}}{\mathbf{2}+}\frac{\mathbf{3^2}}{\mathbf{2}+}\cdots$$

This may be compared with the Brouncker continued fraction (1656)

$$\frac{4}{\pi} = 1 + \frac{1^2}{2_{\ +}}\frac{3^2}{2_{\ +}}\frac{5^2}{2_{\ +}}\cdots$$

Rapidly Convergent Transformed Series for π

Adding and subtracting the end-correction terms, we can rewrite the Mādhava series for π in the form:

$$C = 4d\left[\left(1 - \frac{1}{a_1}\right) + \left(\frac{1}{a_1} + \frac{1}{a_3} - \frac{1}{3}\right) - \left(\frac{1}{a_3} + \frac{1}{a_5} - \frac{1}{5}\right) + \dots\right]$$

By choosing different correction terms, we get different transformed series many of which also converge faster than the Mādhava series.

If we choose the first order correction divisor, $a_p = 2p + 2$, we get the series involving cubes of the odd numbers:

व्यासाद् वारिधिनिहतात् पृथगाप्तं त्र्याद्ययुग्विमूलघनैः ।
त्रिघ्नव्यासे स्वमृणं क्रमशः कृत्वा परिधिरानेयः ॥

$$C = 4d\left[\frac{3}{4} + \frac{1}{(3^3 - 3)} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \dots\right]$$

Rapidly Convergent Transformed Series for π

By using the identity

$$\frac{4}{(4n-1)^3-(4n-1)}-\frac{4}{(4n+1)^3-(4n+1)}=\frac{6}{[2.(2n)^2-1]^2-(2n)^2}$$

we can transform the above series into the form mentioned in *Karaṇapaddhati*

वर्गैर्युजां वा द्विगुणैर्निरेकैर्वर्गीकृतैर्वर्जितयुग्मवर्गैः ।
व्यासं च षड्घ्नं विभजेत्फलं स्वं व्यासे त्रिनिघ्ने परिधिस्तथा स्यात् ॥

$$C=3D+6D\left\{\frac{1}{(2.2^2-1)^2-2^2}+\frac{1}{(2.4^2-1)^2-4^2}+\frac{1}{(2.6^2-1)^2-6^2}+\dots\right\}$$

We thus have a series involving fourth powers of even numbers

$$\frac{\pi-3}{6}=\frac{1}{(2.2^2-1)^2-2^2}+\frac{1}{(2.4^2-1)^2-4^2}+\frac{1}{(2.6^2-1)^2-6^2}+\dots$$

Rapidly Convergent Transformed Series for π

If we choose the second-order correction divisor, which is the first correction divisor given by Mādhava,

$$a_p = (2p+2) + \frac{4}{(2p+2)} = \frac{(2p+2)^2+4}{(2p+2)} = \frac{(p+1)^2+1}{\left\{\frac{(p+1)}{2}\right\}}$$

then we get the series involving fifth powers of the odd numbers.

समपञ्चाहतयो या रूपाद्ययुजां चतुर्घ्नमूलयुताः।
ताभिः षोडशगुणिताद् व्यासात् पृथगाहृतेषु विषमयुते।
समफलयुतिमपहाय स्यादिष्टव्याससम्भवः परिधिः॥

$$\begin{aligned} C &= 4d\left(1-\frac{1}{5}\right) - 16d\left[\frac{1}{(3^5+4.3)} - \frac{1}{(5^5+4.5)} + \frac{1}{(7^5+4.7)}\right] - \ldots \\ &= 16d\left[\frac{1}{(1^5+4.1)} - \frac{1}{(3^5+4.3)} + \frac{1}{(5^5+4.5)} - \ldots\right]. \end{aligned}$$

Rapidly Convergent Transformed Series for π

Yuktibhāṣā and *Kriyākramakarī* do not discuss the transformed series when we use the accurate correction divisor of Mādhava

$$
\begin{aligned}
a_p &= (2p+2) + \frac{4}{(2p+2)+}\ \frac{16}{(2p+2)} \\
&= \frac{\left[\left\{(p+1)^2+5\right\}\left\{\frac{(p+1)}{2}\right\}\right]}{\left[\left\{\frac{(p+1)}{2}\right\}^2+1\right]}
\end{aligned}
$$

We can easily see that it leads to the following transformed series involving terms of the order of the seventh powers of successive odd numbers.

Rapidly Convergent Transformed Series for π

$$C = \frac{28D}{9} + 144D\left[\frac{1}{\{3^3-3)(2^2+5)(4^2+5)\}} - \frac{1}{\{(5^3-5)(4^2+5)(6^2+5)\}} + \ldots\right]$$

Or, equivalently

$$\frac{\left(\frac{\pi}{4}-\frac{7}{9}\right)}{36} = \frac{1}{(3^3-3)(2^2+5)(4^2+5)} - \frac{1}{(5^3-5)(4^2+5)(6^2+5)} + \ldots$$

We can get transformed series also by considering other divisors a_p different from the optimal divisors given by Mādhava. The resultant series of course may not show as rapid a convergence as seen in the case of transformed series obtained from the optimal divisors of Mādhava.

Rapidly Convergent Transformed Series for π

If we take the correction divisor as the non-optimum divisor

$$a_p = 2p$$

then we get the transformed series which involves the squares of successive even numbers.

$$c = 4d\left[\frac{1}{2} + \frac{1}{(2^2-1)} - \frac{1}{(4^2-1)} + \frac{1}{(6^2-1)} + \dots\right].$$

This series is presented in the following verse given in *Yuktibhāṣā* and *Yuktidīpikā*.

द्व्यादियुजां वा कृतयो व्येका हाराद् द्विनिघ्नविष्कम्भे।
धनम् ऋणमन्तस्योर्ध्वगतौजकृतिर्द्विसहिता हरस्यार्धम्॥

Incidentally the verse also gives an end correction term of the form

$$(-1)^{\frac{p+2}{2}}\ \frac{1}{2\left[(p+1)^2+2\right]}$$

where, p is the last even denominator whose square appears in the series.

A History of Approximations to π

	Approximation to π	Accuracy (Decimal places)	Method Adopted
Rhind Papyrus - Egypt (Prior to 2000 BCE)	$\frac{256}{81} = 3.1604$	1	Geometrical
Babylon (2000 BCE)	$\frac{25}{8} = 3.125$	1	Geometrical
Śulvasūtras (Prior to 800 BCE)	3.0883	1	Geometrical
Jaina Texts (500 BCE)	$\sqrt{(10)} = 3.1623$	1	Geometrical
Archimedes (250 BCE)	$3\frac{10}{71} < \pi < 3\frac{1}{7}$	2	Polygon doubling ($6.2^4 = 96$ sides)
Ptolemy (150 CE)	$3\frac{17}{120} = 3.141666$	3	Polygon doubling ($6.2^6 = 384$ sides)
Lui Hui (263)	3.14159	5	Polygon doubling ($6.2^9 = 3072$ sides)
Tsu Chhung-Chih (480?)	$\frac{355}{113} = 3.1415929$ 3.1415927	6 7	Polygon doubling ($6.2^9 = 12288$ sides)
Āryabhaṭa (499)	$\frac{62832}{20000} = 3.1416$	4	Polygon doubling ($4.2^8 = 1024$ sides)

A History of Approximations to π

	Approximation to π	Accuracy (Decimal places)	Method Adopted
Mādhava (1375)	$\frac{2827433388233}{9.10^{11}} = 3.141592653592\ldots$	11	Infinite series with end corrections
Al Kasi (1430)	3.1415926535897932	16	Polygon doubling (6.2^{27} sides)
Francois Viete (1579)	3.1415926536	9	Polygon doubling (6.2^{16} sides)
Romanus (1593)	3.1415926535 . . .	15	Polygon doubling
Ludolph Van Ceulen (1615)	3.1415926535 . . .	32	Polygon doubling (2^{62} sides)
Wildebrod Snell (1621)	3.1415926535 . . .	34	Modified Polygon doubling (2^{30} sides)
Grienberger (1630)	3.1415926535 . . .	39	Modified Polygon doubling
Isaac Newton (1665)	3.1415926535 . . .	15	Infinite series

A History of Approximations to π

Abraham Sharp (1699)	3.1415926535 . . .	71	Infinite series for $tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
John Machin (1706)	3.1415926535 . . .	100	Infinite series relation $\frac{\pi}{4} = 4tan^{-1}\left(\frac{1}{5}\right) - tan^{-1}\left(\frac{1}{239}\right)$
Ramanujan (1914), Gosper (1985)		17 Million	Modular Equation
Kondo, Yee (2010)		5 Trillion	Modular Equation

A History of Exact Results for π

Mādhava (1375)	$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$ $\pi/\sqrt{12} = 1 - 1/3.3 + 1/3^2.5 - 1/3^3.7 + \ldots$ $\pi/4 = 3/4 + 1/(3^3 - 3) - 1/(5^3 - 5) + 1/(7^3 - 7) - \ldots$ $\pi/16 = 1/(1^5 + 4.1) - 1/(3^5 + 4.3) + 1/(5^5 + 4.5) - \ldots$
Francois Viete (1593)	$\frac{2}{\pi} = \sqrt{[1/2]}\sqrt{[1/2 + 1/2\sqrt{(1/2)}]}$ $\sqrt{[1/2 + 1/2\sqrt{(1/2 + 1/2\sqrt{(1/2)})}]}\ldots$ (infinite product)
John Wallis (1655)	$\frac{4}{\pi} = \left(\frac{3}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{7}{8}\right)\ldots$ (infinite product)
William Brouncker (1658)	$\frac{4}{\pi} = 1 + \frac{1^2}{2+}\ \frac{3^2}{2+}\ \frac{5^2}{2+}\ \ldots$ (continued fraction)
Isaac Newton (1665)	$\pi = \frac{3\sqrt{3}}{4} + 24\left[\frac{1}{12} - \frac{1}{5.32} - \frac{1}{28.128} - \frac{1}{72.512} - \ldots\right]$

A History of Exact Results for π

James Gregory (1671)	$tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$
Gottfried Leibniz (1674)	$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$
Abraham Sharp (1699)	$\frac{\pi}{\sqrt{12}} = 1 - \frac{1}{3.3} + \frac{1}{3^2.5} - \frac{1}{3^3.7} + \ldots$
John Machin (1706)	$\frac{\pi}{4} = 4tan^{-1}\left(\frac{1}{5}\right) - tan^{-1}\left(\frac{1}{239}\right)$

Ramanujan (1914)

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4K)!(1103 + 26390k)}{(k!)^4 396^{4k}}$$

Ramanujan's Series for π

One of Ramanujan's early papers is on the "Modular equations and approximations to π". Though published later from London in 1914 (QJM 1914, 350-372), it is said to embody "much of Ramanujan's early Indian work." Here is a sample of his results:

$$\frac{1}{3\pi\sqrt{3}} = \frac{3}{49} + \frac{43}{49^3}\frac{1}{2}\frac{1.3}{4^2} + \frac{83}{49^5}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2} + \ldots, \ldots\ldots$$

$$\frac{2}{\pi\sqrt{11}} = \frac{19}{99} + \frac{299}{99^3}\frac{1}{2}\frac{1.3}{4^2} + \frac{579}{99^5}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2} + \ldots, \ldots\ldots$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6}\frac{1}{2}\frac{1.3}{4^2} + \frac{53883}{99^{10}}\frac{1.3}{2.4}\frac{1.3.5.7}{4^2.8^2} + \ldots$$

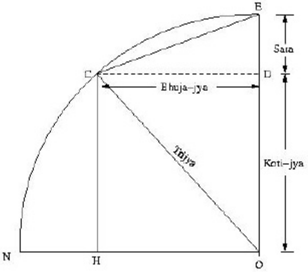
Ramanujan also notes that the last series "is extremely rapidly convergent". Indeed in late 1980s, it blazed a new trail in the saga of computation of π.

Rsine, Rcosine and Rversine

The *jyā* or *bhujā-jyā* of an arc of a circle is actually the half the chord (*ardha-jyā* or *jyārdha*) of double the arc.

In the figure below, if r is the radius of the circle, *jyā* (Rsine), *koṭi* or *koṭi-jyā* (Rcosine) and *śara* (Rversine) of the *cāpa* (arc) $EC = s = r\vartheta$, are given by:

jyā (arc EC) = $R \sin(s) = CD = r \sin \vartheta$
koṭi (arc EC) = $R \cos(s) = OD = r \cos \vartheta$
śara(arc EC) = Rvers (s)= ED = $r - r \cos \vartheta$



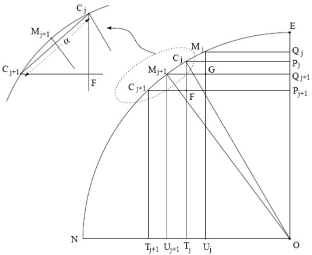
Nīlakaṇṭha's Refinement of Āryabhaṭa Relation for Second Order Sine Differences

We consider a given arc of arc-length s, which is divided into n equal arc-bits. If $s = r\vartheta$, then the j-th *piṇḍa-jyā* B_j and the corresponding *koṭi-jyā* K_j, and the *śara* S_j, are

$$B_j = R\sin\left(\frac{js}{n}\right) = r\sin\left(\frac{j\vartheta}{n}\right) = r\sin\left(\frac{js}{rn}\right) \qquad [C_jP_j \text{ in the Figure}]$$

$$K_j = R\cos\left(\frac{js}{n}\right) = r\cos\left(\frac{j\vartheta}{n}\right) = r\cos\left(\frac{js}{rn}\right) \qquad [C_jT_j \text{ in the Figure}]$$

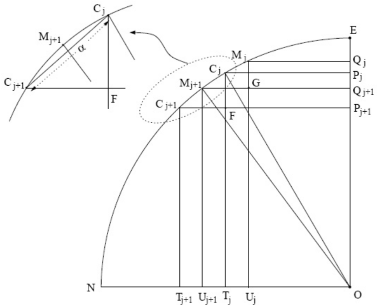
$$S_j = \mathrm{Rvers}\left(\frac{js}{n}\right) = r\left[1-\cos\left(\frac{j\vartheta}{n}\right)\right] = r\left[1-\cos\left(\frac{js}{rn}\right)\right] \quad [P_jE \text{ in the Figure}]$$



Second-Order Sine-Differences

Let M_{j+1} be the mid-point of the arc-bit C_jC_{j+1} and similarly M_j the mid-point of the previous (j-th) arc-bit.

We shall denote the *piṇḍa-jyā* of the arc EM_{j+1} as $B_{j+\frac{1}{2}}$ and clearly $B_{j+\frac{1}{2}} = M_{j+1}Q_{j+1}$. The corresponding $K_{j+\frac{1}{2}} = M_{j+1}U_{j+1}$ and $S_{j+\frac{1}{2}} = EQ_{j+1}$. Similarly, $B_{j-\frac{1}{2}} = M_jQ_j, K_{j-\frac{1}{2}} = M_jU_j$ and $S_{j-\frac{1}{2}} = EQ_j$. The full-chord of the arc-bit $\frac{s}{n}$ may be denoted α.



Second-Order Sine-Differences

Then a simple argument based on similar triangles (*trairāśika*) leads to the relations for Rsine and Rcosine differences

$$\triangle_j = B_{j+1} - B_j = \left(\frac{\alpha}{R}\right) K_{j+\frac{1}{2}}$$

$$K_{j-\frac{1}{2}} - K_{j+\frac{1}{2}} = \left(S_{j+\frac{1}{2}} - S_{j-\frac{1}{2}}\right) = \left(\frac{\alpha}{R}\right) B_j$$

Thus, we obtain the relation for second-order sine differences

$$\triangle_{j+1} - \triangle_j = -\left(\frac{\alpha}{R}\right)^2 B_j = -\frac{(\triangle_1 - \triangle_2)}{B_1} B_j$$

With $n = 24$, Āryabhaṭa used the approximation

$$(\triangle_1 - \triangle_2) \approx 1', B_1 \approx 225'$$

Second-Order Sine-Differences

Nīlakaṇṭha in *Tantrasaṅgraha* has given a better approximation

$$B_1 \approx 224'50'', \frac{(\triangle_1 - \triangle_2)}{B_1} \approx \frac{1}{233'30''}$$

Śaṅkara Vāriyar in his commentary *Laghuvivṛti* on *Tantrasaṅgraha* has given a still better approximation

$$B_1 \approx 224'50''22''', \frac{(\triangle_1 - \triangle_2)}{B_1} \approx \frac{1}{233'32''}$$

Note: We can re-express the Āryabhaṭa second-order sine difference relation in the form

$$[R\sin((j+1)h) - R\sin jh] - [R\sin jh - R\sin((j-1)h)] = -\frac{R\sin jh}{\left[\frac{1}{2}(1-\cos h)\right]}$$

Mādhava Series for Rsine

निहत्य चापवर्गेण चापं तत्तत्फलानि च।
हरेत् समूलयुग्वर्गैस्त्रिज्यावर्गहतैः क्रमात्॥
चापं फलानि चाधोऽधो न्यस्योपर्युपरि त्यजेत्।
जीवाप्त्यै संग्रहोऽस्यैव विद्वान् इत्यादिना कृतः॥

$$R\sin(s) \approx s - s\,\frac{\left(\frac{s}{r}\right)^2}{(2^2+2)} + s\,\frac{\left(\frac{s}{r}\right)^4}{(2^2+2)(4^2+4)} - \ldots$$

This can be rewritten in the form

$$R\sin(s) = s - \left(\frac{1}{R}\right)^2 \frac{s^3}{(1.2.3)} + \left(\frac{1}{R}\right)^4 \frac{s^5}{(1.2.3.4.5)} - \left(\frac{1}{R}\right)^6 \frac{s^7}{(1.2.3.4.5.6.7)} + \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{(3!)} + \frac{\theta^5}{(5!)} - \frac{\theta^7}{(7!)} + \ldots$$

Mādhavā Series for Rversine

निहत्य चापवर्गेण रूपं तत्तत्फलानि च।
हरेद् विमूलयुत्वर्गैस्त्रिज्यावर्गहतैः क्रमात्॥

किन्तु व्यासदलेनैव द्विघ्नेनाद्यं विभज्यताम्।
फलान्यधोऽधः क्रमशो न्यस्योपर्युपरि त्यजेत्॥
शरास्त्यै संग्रहोऽस्यैव स्तेनः स्त्रीत्यादिना कृतः।

$$Rver(s) = \frac{R\left(\frac{s}{R}\right)^2}{(2^2+2)} - \frac{R\left(\frac{s}{R}\right)^4}{(2^2+2)(4^2+4)} + \dots$$

Mādhavā Series for Rversine

This can be rewritten in the form

$$S = Rvers(s) = \left(\frac{1}{R}\right)\frac{s^2}{2} - \left(\frac{1}{R}\right)^3 \frac{s^4}{(1.2.3.4)} + \left(\frac{1}{R}\right)^5 \frac{s^6}{(1.2.3.4.5.6)} - \ldots$$

$$\text{vers}\theta = \frac{\theta^2}{(2!)} - \frac{\theta^4}{(4!)} + \frac{\theta^6}{(6!)} - \ldots$$

The verses giving the Rsine and Rversine series also note that the method of obtaining accurate approximations to Rsine and Rversine values, as encoded in the mnemonics (also due to Mādhava) *Vidvān* etc and *Stenaḥ* etc, indeed follow from these series.

Mādhava has also listed accurate values of the 24 tabular Rsines in a series of verses beginnig *śreṣthaṃ nāma variṣṭhānāṃ*. They coincide with the modern values up to “thirds” (corresponding to an accuracy of sines up to seventh or eighth decimal place).

Mādhava's Sine Table

θ in min.	R sin θ according to Āryabhaṭīya	Govindasvāmi	Mādhava(also Modern)
225	225	224 50 23	224 50 22
450	449	448 42 53	448 42 58
675	671	670 40 11	670 40 16
900	890	889 45 08	889 45 15
1125	1105	1105 01 30	1105 01 39
1350	1315	1315 33 56	1315 34 7
1575	1520	1520 28 22	1520 28 35
1800	1719	1718 52 10	1718 52 24
2025	1910	1909 54 19	1909 54 35
2250	2093	2092 45 46	2092 46 03
2475	2267	2266 38 44	2266 39 50
2700	2431	2430 50 54	2430 51 15
2925	2585	2584 37 43	2584 38 06
3150	2728	2727 20 29	2727 20 52
3375	2859	2858 22 31	2858 22 55
3600	2978	2977 10 09	2977 10 34
3825	3084	3083 12 51	3083 13 17
4050	3177	3175 03 23	3176 03 50
4275	3256	3255 17 54	3255 18 22
4500	3321	3320 36 02	3320 36 30
4725	3372	3371 41 01	3371 41 29
4950	3409	3408 19 42	3408 20 11
5175	3431	3430 22 42	3430 23 11
5400	3438	3437 44 19	3437 44 48

Nīlakaṇṭha's Formula for Instantaneous Velocity (c.1500)

Instead of basing the calculation of instantaneous velocity on the approximate form of *manda-phala* or equation of centre that Bhāskarācārya and others had considered, Nīlakaṇṭha Somayājī uses the exact form of the *manda-phala* :

$$\mu = M + Rsin^{-1}\left[\left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right)R\sin(M-\alpha)\right]$$

where M is the mean longitude of the planet (which varies uniformly with time) and α is the longitude of the apogee, which in the case of Moon also varies uniformly with time.

Nīlakaṇṭha's Formula for Instantaneous Velocity

Nīlakaṇṭha also gives the correct formula for the correction to the mean velocity of Moon in his treatise *Tantrasaṅgraha*.

चन्द्रबाहुफलवर्गशोधितत्रिज्यकाकृतिपदेन संहरेत्।
तत्र कोटिफललिप्तिकाहतां केन्द्रभुक्तिरिह यच्च लभ्यते॥
तद्विशोध्य मृगादिके गतेः क्षिप्यतामिह तु कर्कटादिके।
तद्भवेत्स्फुटतरा गतिर्विधोः अस्य तत्समयजा रवेरपि॥

Nīlakaṇṭha's Formula for Instantaneous Velocity

Let the product of the *koṭiphala* $[r_0 \cos(M - \alpha)]$ in minutes and the daily motion of the *manda-kendra* $\left(\frac{d(M-\alpha)}{dt}\right)$ be divided by the square root of the square of the *bāhuphala* subtracted from the square of *trijyā* $\left(\sqrt{R^2 - r_0^2 \sin^2(M - \alpha)}\right)$.

The result thus obtained has to be subtracted from the daily motion of the Moon if the *manda-kendra* lies within six signs beginning from *Mṛga* and added if it lies within six signs beginning from *Karkaṭaka*. The result gives a more accurate value of the Moon's angular velocity. In fact, the procedure for finding the instantaneous velocity of the Sun is also the same.

Nīlakaṇṭha's Formula for Instantaneous Velocity

Nīlakaṇṭha thus gives the derivative of the second term in the equation of centre noted above in the form

$$\left[\left\{\left(\frac{r_0}{R}\right) R\cos(M-\alpha)\right\}\left\{R^2-\left(\frac{r_0}{R}\right)^2 R\sin^2(M-\alpha)\right\}^{-\frac{1}{2}}\right]\left[\left(\frac{d}{dt}\right)(M-\alpha)\right]$$

This formula for the velocity, which involves the derivative of the arcsine function has been attributed by Nīlakaṇṭha to his teacher Dāmodara in *Jyotirmīmāṃsā*.

Acyuta's Formula for Instantaneous Velocity (c.1600)

Acyuta Piṣāraṭi in his *Sphuṭanirṇaya-tantra* gives the Nīlakaṇṭha formula for the instantaneous velocity. He also discusses an alternative prescription for *manda*-correction due to Muñjāla (c.932) given by

$$\mu = M + \frac{\left[\left(\frac{r}{R}\right) R \sin(M - \alpha)\right]}{\left[R - \left(\frac{r}{R}\right) R \cos(M - \alpha)\right]}$$

Acyuta notes that in this model the *manda*-correction also depends on the hypotenuse and hence the correction to the mean velocity is given by:

कृतकोटिफलं त्रिजीवया विहृतं दोःफलवर्गतस्तु यत्।
मृगकर्कटकादिकेऽमुना युतहीनं फलमत्रकोटिजम्॥
दिनकेन्द्रगतिघ्नमुद्धरेत् कृतकोटीफलया त्रिजीवया।
फलपूर्वफलैकतो दलं दिनभुक्तेरपि संस्कृतिर्भवेत्॥

Acyuta's Formula for Instantaneous Velocity

Here, Acyuta gives the derivative of the second term above (which involves the derivative of ratio of two functions) in the form

$$\left[\left\{\left(\frac{r}{R}\right) R\cos(M-\alpha)\right\} + \frac{\left\{\left(\frac{r}{R}\right) R\sin(M-\alpha)\right\}^2}{\left\{R - \left(\frac{r}{R}\right) R\cos(M-\alpha)\right\}}\right]$$

$$x\left[\frac{1}{\left\{R - \left(\frac{r}{R}\right) R\cos(M-\alpha)\right\}}\right]\left[\left(\frac{d}{dt}\right)(M-\alpha)\right]$$

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Thanks!

Thank You