NPTEL Course on

Mathematics in India:
From Vedic period to Modern times

Lecture 15

Mahāvīra's *Gaṇitasārasaṅgraha*

M. S. Sriram
University of Madras, Chennai.

Outline

Gaṇitasārasaṅgraha of Mahāvīra I

- Introduction
- Arithmetical operations, operations with zero
- Squares, cubes, square roots, cube roots
- Arithmetical and Geometric progressions, *Citi* (summation)
- Manipulations with fractions and solutions of equations
- Mixed problems including interest calculations

Mahāvīra's *Gaṇitasārasaṅgraha*

Mahāvīra was a Digambara Jain who lived in the later part of the rule of *Amoghavarṣa Nṛpatuṅga* (815-877 CE), a great king of the *Raṣṭrakūṭa* dynasty, ruling over north Karnataka, parts of Andhra, Maharastra and other parts of India also. Mahāvīra wrote an extensive Sanskrit treatise called *'Gaṇitasārasaṅgraha'* (Compendium of the essence of Mathematics) about 850 CE. *Pātīgaṇita* of Śridhara (750 CE) is another important work of the same era. I will not be covering that separately, as Mahāvīra's work includes most of the material of *Pāṭīgaṇita*, as also older works like *Brāhmasphuṭasiddhānta* of Brahmagupta. It also introduces several new topics. It is written in the style of a textbook and provides a rich source of information on ancient Indian mathematics.

Mahāvīra's *Gaṇitasārasaṅgraha*

A commentary called *Bālabodha* in Kannada was written by Daivajña Vallabha. Also there is a Sanskrit commentary by Varadarāja. Dates of these are not known. Also these are not available in print.

Based on the manuscripts in Sanskrit, as also in Kannada script and explanation in Kannada in some manuscripts, M. Rangacharya published the edited version of *Gaṇitasārasaṅgraha* (GSS), with English translation and explanatory notes in 1912. My account of GSS is based on this.

In Chapter 1 as terminology, Mahāvīra waxes eloquent on the use of mathematics in verses 9-16.

Importance of Mathematics

Verses 9-16:

लौकिके वैदिके वापि तथा सामायिकेऽपि यः।
व्यापारस्तत्र सर्वत्र सङ्ख्यानमुपयुज्यते॥ ९ ॥
कामतन्त्रेऽर्थशास्त्रे च गान्धर्वे नाटकेऽपि वा।
सूपशास्त्रे तथा वैद्ये वास्तुविद्यादिवस्तुषु॥ १० ॥
छन्दोऽलङ्कारकाव्येषु तर्कव्याकरणादिषु।
कलागुणेषु सर्वेषु प्रस्तुतं गणितं परम्॥ ११ ॥
सूर्यादिग्रहचारेषु ग्रहणे ग्रहसंयुतौ।
त्रिप्रश्ने चन्द्रवृत्तौ च सर्वत्राङ्गीकृतं हि तत्॥ १२ ॥

Importance of Mathematics

Verses 9-16:

द्वीपसागरशैलानां सङ्ख्याव्यासपरिक्षिपः।
भवनव्यन्तरज्योतिर्लोककल्पाधिवासिनाम्॥ १३ ॥
नारकाणां च सर्वेषां श्रेणीबन्धेन्द्रकोत्कराः।
प्रकीर्णकप्रमाणाद्या बुध्यन्ते गणितेन ते॥ १४ ॥
प्राणिनां तत्र संस्थानमायुरष्टगुणादयः।
यात्राद्यास्संहिताद्याश्च सर्वे ते गणिताश्रयाः॥ १५ ॥
बहुभिर्विप्रलापैः किं त्रैलोक्ये सचराचरे।
यत्किञ्चिद्वस्तु तत्सर्वं गणितेन विना न हि॥ १६ ॥

Importance of Mathematics

"In all those transactions which relate to worldly Vedic or (other) similarly religious affairs, calculation is of use. In the science of love, in the science of wealth, in music and in the drama, in the art of cooking, and similarity in medicine and in things like the knowledge of architecture. In prosody, in poetics and poetry, in logic and grammar and such other things and in relation to all that constitutes the peculiar value of (all) the various arts, the science of computation is held in high esteem. In relation to the movements of the Sun and other heavenly bodies, in connection with eclipses and conjuction of planets and in connection with *tripraśna* (diurnal problems) and the course of the Moon indeed in all these (connections) it is utilised. The number, the diameter and the perimeter of islands, oceans and mountains; the extensive dimensions of the rows of habitation and halls belonging to the inhabitants of the (earthly) world.

Importance of Mathematics

Of the interspace (between the worlds), of the world of light and of the world of Gods; (as also the dimensions of those belonging) to the dwellers in the hell: and (other) miscellaneous measurements of all sorts - all these are made out by means of computation. The configuration of living beings therein, the length of their lives, their eight attributes and other similar things, their progress and other such things - all these are dependent upon computation (for their due measurement and comprehension). What is the good of saying much in vain? Whatever there is in all the three worlds, which are possessed by moving and non-moving beings- all that indeed cannot exist as apart from measurement."

Arithmatical Operations

He talks of 8 arithmetical operations.

(i) *Guṇakāra* or *Pratyutpanna*, their multiplication

(ii) *Bhāgahāra*; Division

(iii) *Kṛti*: Squaring

(iv) *Vargamūla*: Square root

(v) *Ghana*: Cubing

(vi) *Ghanamūla*: Cube root

(vii) *Citi* (or *Saṅkalita*): summation and

(viii) *Vyutkalita* (or *Śeṣa*): subtraction of a part of a series, from the whole series.

Operations with Zero

In particular, consider his understanding of the Operations with zero:
Verse 49:

> ताडितः खेन राशिः खं सोऽविकारी हृतो युतः।
> हीनोऽपि खवधादिः खं योगे कं योज्यरूपकम्॥ ४९ ॥

> “A number multiplied by zero is zero, and that (number) remains unchanged when it is divided by, combined with (or) diminished by zero. Multiplication and other operations in relation to zero (give rise to) zero; and in the operation of additions the zero becomes the same as what is added to it.”

So, for any number

$$a \times 0 = 0 \times a = 0,\ a \div 0 = 0,\ 0 \div a = 0,$$

$$a + 0 = 0 + a = a,\ a - 0 = a.$$

Obviously, $a \div 0 = 0$ is wrong. Bhāskara call this *khahara* and assigns the value, infinity to it.

Operations with Positive and Negative Quantities

Verse 50-52:

ऋणयोर्धनयोर्घाते भजने च फलं धनम्।
ऋणं धनर्णयोस्तु स्यात् स्वर्णयोर्विवरं युतौ॥ ५० ॥
ऋणयोर्धनयोर्योगो यथासङ्ख्यमृणं धनम्।
शोध्यं धनमृणं राशेः ऋणं शोध्यं धनं भवेत्॥ ५१ ॥
धनं धनर्णयोर्वर्गौ मूले स्वर्णे तयोः क्रमात्।
ऋणं स्वरूपतोऽवर्गो यतस्तस्मान्न तत्पदम्॥ ५२॥

“In multiplying as well as dividing two negative (or) two positive (quantities, one by the other), the result is a positive (quantity). But it is a negative quantity in relation to two (quantities), one (of which) is positive and the other negative. In adding a positive and negative quantity, the result is their difference.

Positive and Negative Quantities

The addition of two negative (quantities) or of two positive (quantities give rise to) a negative or positive (quantity) in order. A positive (quantity) which has to be subtracted from a (given) number becomes negative, and a negative (quantity) which has to be (so) subtracted becomes positive.

The square of a positive as well as of a negative (quantity) is positive; and the square root of those (square quantities) are positive and negative in order. As in the nature of things a negative (quantity) is not a square (quantity), it has therefore no square root."

Positive and Negative Quantities and Notational Places

$$(+a)\times(+b) = (-a)\times(-b) = (+ab); (+a)\times(-b) = (-a)\times(+b) = -(ab).$$

$$a + (-b) = (a \sim b) \text{ [means } a - b \text{ if } a > b \text{ and } -(b - a) \text{ if } b > a].$$

$$(+a) + (+b) = +(a + b), \qquad (-a) + (-b) = -(a + b).$$

$$b - a = b + (-a); a - (-b) = a + b$$

$$(+a)^2 = (-a)^2 = +a^2; \quad \sqrt{+a^2} = (+a) \text{ or } (-a).$$

A negative number $(-b)$ has no square root, because a square is always positive.

In Verses 63-68, names of rotational places:

1-*Eka*, ... 10 - *Daśa* ,... 10^{23}: *Mahākṣobha*.

Chapter 2: elaborates on the arithmetical operations (*parikarma*) and simplification of various kinds.

Multiplication

1. Multiplication: "The multiplicand and the multiplier are placed one below the other in the manner of the hinges of a door" - *Kapāṭasandhi*.

$$\begin{array}{ll} (a_n\, a_{n-r} & a_1) \\ & (b_m \ldots b_1) \leftarrow \text{ moved} \end{array}$$

$$\text{We can use:} (a \times b) \times (c \times d) = \left(\frac{a \times b}{a}\right) \times (a \times cd)$$

$$\text{or } = ((a \times b) \times c) \times \left(\frac{c \times d}{c}\right) \text{ etc.,}$$

Necklace of *Narapāla*: $12345679 \times 9 = 111111111$

Royal necklace: $142857143 \times 7 = 1000000001$

Division, Squaring, Square root

2. Division $= \dfrac{\text{Dividend}}{\text{Divisor}}$ from left to right. Remove common factors.
3. Squaring:

$$a^2 = (a-b)(a+b) + b^2$$
$$n^2 = 1 + 3 + 5 + \cdots + (2n-1)$$

(Arithmetic Progression with 1 as the first term, 2 as the common difference and n terms.)

$$(a+b+c+\cdots)^2 = a^2 + b^2 + c^2 + \cdots + 2ab + 2ac + \cdots$$

$$\begin{aligned}
(a_n a_{n-1} \ldots a_2 a_1)^2 &= (a_n \times 10^{n-1} + a_{n-1} \times 10^{n-2} + \cdots + a_2 \times 10 + a_1)^2 \\
&= a_n^2 10^{2(n-1)} + a_{n-1}^2 10^{2(n-2)} + \cdots + 2a_n a_{n-1} 10^{n-1+n-2} + \cdots \\
&= a_1^2 + a_1 \times 10(a_2 + a_3 \times 10 + \cdots + a_n \times 10^{n-2}) + a_2^2 \times 10^2 \\
&\quad +2a_2 \times 10^3 (a_3 + a_4 \times 10 + \cdots + a_n \times 10^{n-3}) + \cdots
\end{aligned}$$

4. Square root: Earlier Indian procedure.

Cube and Cube root

5. Cube:

   (i) $a^3 = a(a-b)(a+b) + b^2(a-b) + b^3$ or

   (ii) $a^3 = a + 3a + 5a + \cdots + (2a-1)a = a[1+3+\cdots+(2a-1)] = a.a^2$
   That is, an A.P with a as the first term, $2a$ as the common difference and a as the number of terms. or,

   (iii) $a^3 = a^2 + (a-1)[1+3+\cdots(2a-1)]$. or,

   (iv)

$$\begin{aligned}
a^3 &= 3[1\cdot 2 + 2\cdot 3 + \cdots + (a-1)a] \times a. \\
\text{Check. RHS} &= 3[2^2 - 2 + 3^2 - 3 + \cdots + a^2 - a] \times a \\
&= 3[(1^2 + \cdots + a^2) - (1 + \cdots + a)] + a \\
&= 3\left[\frac{a(a+1)(2a+1)}{6} - \frac{a(a+1)}{2}\right] + a = a^3
\end{aligned}$$

Cube and Cube root

$$(a_1 + a_2 + \cdots + a_n)^3 = 3 \sum_{i \neq j} a_i^2 \times a_j + \sum a_i^3$$

or

$$(a_1 + a_2 + \cdots + a_n)^3 = a_n^3 + 3a_n^2(a_1 + \cdots + a_{n-1}) + 3a_n(a_1^2 + \cdots + a_{n-1}^2)$$
$$+ a_{n-1}^3 + 3a_{n-1}^2(a_1 + \cdots + a_{n-2}) + 3a_{n-1}(a_1^2 + \cdots + a_{n-2}^2) + \cdots$$

6. Cube root: Earlier Indian procedure (Āryabhaṭa, Brahmagupta)

Verse 60: "O mathematician, who are clever in calculation, give out after examination, the root of 8590113699459488864, which is a cubic quantity." [Try this as an exercise].

Citi Summation

एकविहीनो गच्छः प्रचयगुणो द्विगुणितादिसंयुक्तः।
गच्छाभ्यस्तो द्विहृतः प्रभवेत् सर्वत्र सङ्कलितम्॥ ६२ ॥

"The number of terms (in the series) as diminished by one and (then) multiplied by the common difference is combined with twice the first term in the series; and when this (combined sum) is multiplied by the number of terms (in the series) and is (then) divided by two, it becomes the sum of the series in all cases."

If a : First term, d : Common difference , n: No. of terms, S : Sum, of an A.P.,

$$S = a + (a + d) + \cdots + [a + (n-1)d] = \left[\frac{2a + (n-1)d}{2}\right] n,$$

Example

Verse 67 (Example):

> आदिस्त्रयश्चयोऽष्टौ द्वादश गच्छस्त्रयोऽपि रूपेण।
> आसप्तकान्प्रवृद्धास्सर्वेषां गणक भण गणितम्॥ ६७ ॥
>
> "The first term is 3; the common difference is 8; and the number of terms is 12. All the three (quantities) are (gradually) increased by 1, until (there) are 7 (series). O mathematician, give out the sums of all (those series)."

Expression for number of terms, n from a, d and S. Same as in *Brāhmasphuṭasiddhānta* (BSS).

Example

Verse 71 (Example):

आदिर्द्वौ प्रचयोऽष्टौ द्वौ रूपेणात्रयात् क्रमात् वृद्धौ।
खाङ्कौ रसाद्रिनेत्रं खेन्दुहरा वित्तमत्र को गच्छः॥ ७१ ॥

“The first term is 2, the common difference is 8; these two are increased successively by 1 till three (series are so made up). The sums of the three series are 90, 276 and 1110, in order. What is the number of terms in each series.” [Try this as an exercise].

Also gives a, d in terms of d, n, S and a, n, S respectively.

$$a = \frac{S}{n} - \frac{(n-1)d}{2}; \quad d = \frac{\frac{S}{n} - a}{\frac{n-1}{2}}.$$

Mixed Problems

He discusses some “mixed problems”. An example:

First series: I term = a, common difference = d, No. of terms = n, Sum = S

Second series: I term = $a_1 = d$; common difference $= d_1 = a$, No. of terms, n_1, Sum $= S_1$.

The first term and common difference of the two series are interchanged. The ratio of the sums is given (S/S_1) and the no. of terms n and n_1 are given. To find a, d (S, a_1, d_1, S_1 are automatically determined.)

Verse 86 gives the solution:

$$a = n(n-1)p - 2n_1, d = n_1^2 - n_1 - 2pn, \text{ where } p = \frac{S_1}{S}$$

.

Mixed Problems

or,

$$d = n_1(n_1 - 1) - 2pn_1.$$

$$\text{Now}\, S = n\left[a + \frac{(n-1)}{2}d\right], S_1 = n_1\left[d + \frac{(n_1-1)}{2}a\right]$$

There are two equations for a and d, in terms of n, n_1, S and S_1. Actually only $\frac{S}{S_1}$ is given. So it is an indeterminate equations for a and d. One can show that for given solutions for a and d, $\frac{S_1}{S} = p$. So, it is correct. It is an 'ansatz'.

Exercise: Find the value of S and S_1 for this ansatz.

Mixed Problems

Example given in GSS.

Verse 88:

द्वादशषोडशपदयोर्व्यस्तप्रभवोत्तरे समानधनम्।
द्व्यादिगुणभागघनमपि कथय त्वं गणितशास्त्रज्ञ॥ ८८ ॥

"In relation to two series (in A.P) having 12 and 16 for their number of terms, the first term and the common difference are interchangeable. The sums (of the series) are equal, or the sum (of one of them) is twice or any such multiple, or half or such fraction (of that of the other). You who are versed in the science of calculation, give out (the value of these sums and the interchangeable first terms and common difference)."

[Try this as an exercise. Here $n = 12$, $n_1 = 16$. Take $p = 1, 2, 1/2, 3/2,$ and $2/3$ and find a, d, S, S_1. Check that $\dfrac{S_1}{S} = p$ in each case.]

Geometric Progressions

Next Geometric progressions are considered: $a, ar, ar^2, \ldots, ar^{n-1}$.

$$S = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{(r - 1)}.$$

[The procedure for finding r^n, as in BSS is given.]

An important method to find r, the common ratio, given the first term, a, number of terms, n and the sum, S.

Verse 101 a:

असकृद्व्येकं मुखहृतवित्तं येनोद्धृतं भवेत्स चयः।

“That (quantity) by which the sum of the series divided by the first term and (then) lessened by one is divisible throughout (when this process of division after the subtraction of one is carried on in relation to all the successive quotients) time after time (that quantity) is the common ratio.”

Explanation

$$S' = \frac{S}{a} = \frac{r^n - 1}{r - 1}, \quad S' - 1 = \frac{r^n - 1}{r - 1} - 1 = \frac{r^n - r}{r - 1} = \frac{r(r^{n-1} - 1)}{(r - 1)}$$

$$\therefore\ S' - 1 \text{ is divisible by } r : \frac{1}{r}(S' - 1) = \frac{r^{n-1} - 1}{r - 1} = S''$$

Subtract 1 from this: $S'' - 1 = \dfrac{r^{n-1} - 1}{r - 1} - 1 = \dfrac{r(r^{n-2} - 1)}{(r - 1)}$ is divisible by r

.

$$\text{Again } \frac{1}{r}(S'' - 1) = \frac{r^{n-3} - 1}{r - 1} = S'''.\ S''' - 1 \text{ is divisible by } r$$

One should check r such that at each stage of the process, $S' - 1$, $S'' - 1$, $S''' - 1, \ldots,$ etc., is divisible by r, till one gets 1.

Note: $r^n - 1 = (r - 1)(r^{n-1} + r^{n-2} + \cdots + 1)$; $r^n - 1$ is divisible by $r - 1$.

Example

Example. Let $a = 3, n = 6, S = 4095$.

$$\frac{4095}{3} = 1365. \quad 1365 - 1 = 1364.$$

Choose by trial $r = 4, \frac{1364}{4} = 341$.

$$\begin{aligned} 341 - 1 &= 340, \frac{340}{4} = 85; 85 - 1 = 84, \frac{84}{4} = 21; 21 - 1 \\ &= 20, \frac{20}{4} = 5; 5 - 1 = 4, \frac{4}{4} = 1. \end{aligned}$$

Hence 4 is the common ratio.

Finding *n*, given *r* and *S*

Verse 103

एकोनगुणाभ्यस्तं प्रभवहृतं रूपसंयुतं वित्तम्।
यावत्कृत्वो भक्तं गुणेन तद्वारसम्मितिर्गच्छः॥ १०३ ॥

"Multiply the sum (of the given series in geometrical progression) by the common ratio lessened by one; (then) divided this (product) by the first term and (then) add one to this (quotient). The number of times that this (resulting quantity) is (successively) divisible by the common ratio - that gives the measure of the number of terms (in the series)."

Finding n, given r and S

$$\frac{S(r-1)}{a} + 1 = r^n.$$

Keep dividing $\frac{S(r-1)}{a} + 1$ by r. The number of times that this (resulting quantity) is (successively) divisible by r (till one gets 1), that is 'n'.

Example. In Verse 104: $a = 3, r = 6, S = 777$. What is n?

$$\frac{S(r-1)}{a} + 1 = \frac{777 \times 5}{3} + 1 = 259 \times 5 + 1 = 1296.$$

$$\frac{1296}{6} = 216, \frac{216}{6} = 36, \frac{36}{6} = 6, \frac{6}{6} = 1.$$

$$\therefore n = 4.$$

Example

Exercise from Verse 105.

Verse 105

त्र्यास्ये पञ्चगुणाधिके हुतवहोपेन्द्राक्षवह्निद्विप-
श्वेतांशुद्विरदेभकर्मकरदृङ्मानेऽपि गच्छः कियान्॥ १०५ ॥

What is n, when the first term is 3, the common ratio is 5 and the sum is $S = 22888183593$? [Try this as an exercise.]

Vyutkalita

$$\underbrace{a, a+b, \cdots, a+(d-1)b}_{\textit{iṣṭa}\text{: d terms}} + \underbrace{a+db+\cdots a+(n-1)b}_{\textit{vyutkalita}\text{: n−d terms}}$$

$$S_i = \left[a + \frac{(d-1)b}{2}\right] d$$

$$S_v = \left[a + db + \frac{(n-d-1)b}{2}\right](n-d) = \left[a + \frac{(n+d-1)b}{2}\right](n-d)$$

Multiplication, Division, root, etc.

Number of terms in an A.P can be fractional!

$$S = a, a + b, \cdots, a + (n' - 1)d = n'\left[a + \frac{(n' - 1)}{2}d\right]$$

where n' can be a fraction.

Very detailed exposition of manipulations with fractions.

Arithmatic Progressions

Verse 25.

$$a + (a + d) + \cdots + [a + (n-1)d] = n^2,$$

for arbitrary a, and $d = \dfrac{(n-a)}{(\frac{n-1}{2})}$.

Similarly $a + (a + d) + \cdots + [a + (n-1)d] = n^3$, for $a = 2n$ and $b = \dfrac{(n-2)n}{(n-1)/2}$.

In general if $a = \frac{x}{4}, d = 2a$ and $n = 2x$,

$$S = \left[\frac{x}{4} + \frac{(2x-1)}{2} \cdot \frac{x}{2}\right] 2x = x^3$$

Verse 29. If a, d, n yield S, $a_1 = \dfrac{S_1}{S} a, d_1 = \dfrac{S_1}{S} d$ yield S_1.

Relations between A.P and G.P

An interesting rule. To find the (common) first term of two series of having the same sum, one of them being in arithmetical progression and the other in geometrical progression, their optionally chosen number of terms being equal and similarly, the chosen common difference and the common ratio also being 'equal' in value.

The essential content of Verse 43:

Consider a G.P with a as the first term, r as the common ratio, n terms and sum $\dfrac{a(r^n-1)}{(r-1)}$.

Consider an A.P with the same number of terms n, a as the first term, common difference $d = r$, and sum

$$S_A = \left[a + \frac{(n-1)d}{2}\right] n.$$

Then $S_G = S_A$ if $a = \dfrac{\dfrac{n(n-1)r}{2}}{\left[\dfrac{r^n-1}{r-1} - n\right]}$.

Ingenious Manipulations with Fractions to Solve Problems

A problem of '*śeṣa*' variety: Consider a number x and fractions $b_1, b_2, \cdots, b_n$. A fraction $b_1 x$ is subtracted from x. Remainder: $R_1 = x - b_1 x = x(1 - b_1)$. A fraction b_2 of the remainder is subtracted. Remainder $R_2 = x - b_1 x - b_2(x - b_1 x) = (1 - b_1)(1 - b_2)x$. A fraction b_3 of the remainder is subtracted. Remainder R_3 is given by

$$
\begin{aligned}
R_3 &= x - b_1 x - b_2(x - b_1 x) - b_3[x - b_1 x - b_2(x - b_1 x)] \\
&= (1 - b_1)(1 - b_2)(1 - b_3)x
\end{aligned}
$$

Manipulations with Fractions

This goes on till $R_n = y$ a known quantity.

To solve for x in terms of y.

We have $R_n = (1 - b_1)(1 - b_2)(1 - b_3) \cdots (1 - b_n)x = y$,

$$\therefore x = \frac{y}{(1 - b_1)(1 - b_2)(1 - b_3) \cdots (1 - b_n)}.$$

Example

Example. Verse 32 of Chapter 4.

कोष्ठस्य लेभे नवमांशमेकः परेऽष्टभागादिदलान्तिमांशान्।
शेषस्य शेषस्य पुनः पुराणा दृष्टा मया द्वादश तत्प्रमा का॥ ३२ ॥

"Of the contents of a treasury, one man obtained $\frac{1}{9}$ part; others obtained from $\frac{1}{9}$ in order to $\frac{1}{2}$ in the end, by the successive remainders; and (at last) 12 $purāṇas$ were seen by me (to remain). What is the (numerical) measure (of the $purāṇa$s) contained n the treasury?"

Solution: Here $b_1 = \frac{1}{9}$, $b_2 = \frac{1}{9}$, $b_3 = \frac{1}{8}$, $b_4 = \frac{1}{7}$, $b_5 = \frac{1}{6}$, $b_6 = \frac{1}{5}$, $b_7 = \frac{1}{4}$, $b_8 = \frac{1}{3}$, $b_9 = \frac{1}{2}$, $y = 12$.

$$\begin{aligned}
\therefore x &= \frac{12}{\left(1-\frac{1}{9}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{6}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right)} \\
&= \frac{12}{\frac{8}{9}\cdot\frac{8}{9}\cdot\frac{7}{8}\cdot\frac{6}{7}\cdot\frac{5}{6}\cdot\frac{4}{5}\cdot\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}} \\
&= \frac{12\cdot 81}{8} = \frac{243}{2} = 121\frac{1}{2}\ purāṇa\text{s}.
\end{aligned}$$

Problems of the *Śeṣamūla* variety

Suppose we have problems of the form: $x - bx - c\sqrt{x} - a = 0$. Then

$$\sqrt{x} = \frac{c/2}{1-b} + \sqrt{\left(\frac{c/2}{1-b}\right)^2 + \frac{a}{1-b}}; \qquad x = (\sqrt{x})^2$$

Example

Verse 49 (Example):

सिंहाश्चत्वारोऽद्रौ प्रतिशेषषडंशकादिमार्धान्ताः।
मूले चत्वारोऽपि च विपिने दृष्टा कियन्तस्ते॥ ४९ ॥

“Four (out of a collection of) lions were seen on a mountain; and fractional parts commencing with $\frac{1}{6}$ and ending with $\frac{1}{2}$ of the successive remainders (of the collection), and (lions equivalent in number to) twice the square root (of the numerical value of the collection) as also (the finally remaining) four (lions) were seen in a forest. How many are those (lions in the collections) ?”

Solution

Solution:

$$\left(1-\frac{1}{6}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{2}\right)(x-4)-2\sqrt{x}-4=0.$$

$$\text{or,} \qquad \frac{1}{6}(x-4)-2\sqrt{x}-4=0$$

$$\frac{1}{6}x-2\sqrt{x}-4\cdot\frac{7}{6}=0$$

Here $1-b=\frac{1}{6}, c=2, a=4\cdot\frac{7}{6}, \quad \frac{c}{2}=1.$

$$\therefore, \sqrt{x}=\frac{1}{\frac{1}{6}}+\sqrt{\left(\frac{1}{\frac{1}{6}}\right)^2+\frac{4\cdot\frac{7}{6}}{6}}=6+\sqrt{36+28}=6+8=14.$$

$$\therefore x=196$$

.

Rule of Proportions

Chapter 5 is on the Rule of three, five, $\cdots$.

Example. Verse 31:

द्वात्रिंशद्धस्तदीर्घः प्रविशति विवरे पञ्चभिस्सप्तमार्धैः
कृष्णाहीन्द्रो दिनस्यासुखपुरजितः सार्धसप्ताङ्गुलानि।
पादेनाहोऽङ्गुले द्वे त्रिचरणसहिते वर्धते तस्य पुच्छं
रन्ध्रं कालेन केन प्रविशति गणकोत्तंस मे ब्रूहि सोऽयम्॥३१ ॥

"A powerful unvanquished excellent black snake, which is 32 *hastas* in length, enters into a hole (at the rate of) $7\frac{1}{2}$ $a\dot{n}gulas$ in $\frac{5}{14}$ of a day; and in the course of $\frac{1}{4}$ a day its tail grows by $2\frac{3}{4}$ of an $a\dot{n}gula$. O ornament of mathematicians, tell me by what time this same (serpent) enters into the hole" [1 $hasta$ = 24 $a\dot{n}gulas$. Try this as exercise].

Mixed Problems

Chapter 6 is on “mixed problems”. Mahāvīra shows ingenuity in solving determinate and indeterminate equations.

Principal (capital), Interest, Mixed amount (A), etc.

Now, if I is the interest on the principal, P (rate principal) in time T months and i is the interest on a capital p in t months,

$$i = \frac{t \times p \times I}{T \times P}.$$

Suppose we are given $m = p + t$ along with P, T, I and i. How to find p and t? The solution is given in Verse 29 :

Mixed Problems

Verse 29:

स्वफलोद्धृतप्रमाणं कालचतुर्वृद्धिताडितं शोध्यम्।
मिश्रकृतेस्तन्मूलं मिश्रे क्रियते तु सङ्क्रमणम्॥ २९ ॥

“From the square of the given mixed sum (of the capital and the time), the rate-capital divided by its rate-interest and multiplied by the rate-time and by four times the given interest is to be subtracted. The square root of this (resulting remainder) is then used in relation to the given mixed sum so as to carry out the process of *saṅkramaṇa*.”

Solutions

Now

$$t = \frac{i \times T \times P}{p \times I}$$

Now

$$m = p + t = p + \frac{i \times T \times P}{p \times I}$$

$$\therefore\ p^2 - mp + \frac{i \times T \times P}{p \times I} = 0$$

$$p = \frac{m \pm \sqrt{m^2 - \frac{P \times T}{I} \times 4i}}{2}, \quad t = m - p.$$

Example

Example. Verse 31:

त्रिकषष्ट्या दत्त्वैकः किं मूलं केन कालेन।
प्राप्तोऽष्टादशवृद्धिं षड्वृष्टिः कालमूलमिश्रं हि॥ ३१ ॥

“By lending out what capital for what time at the rate of 3 per 60 (per month) would a man obtain 18 as interest, 66 being the mixed sum of that time and that capital?”

Solutions

Solution: Here $m = 66$, $P = 60$, $I = 3$, $T = 1$, $i = 18$.

$$\begin{aligned} p &= \frac{66 \pm \sqrt{66^2 - \frac{60\times 1}{3} \times 4 \times 18}}{2} \\ &= \frac{66 \pm 6\sqrt{11^2 - 40}}{2} = \frac{66 \pm 6 \times 9}{2} = 60 \text{ or } 6. \end{aligned}$$

Correspondingly $t = 6$ or 60.

Mixed Problems

There are many more mixed problems involving money lent out at various rates involving the contributions of several persons to the principal and so on, as also proportionate division in other situations.

Example: Suppose there are various quantities purchased at specified different rates and the total amount of the purchase is specified. To find the amounts of quantities purchased.

The Rule for Mixed Quantities

The rule:

Verse 87 a, Chapter 6:

भक्तं शेषैर्मूलं गुणगुणितं तेन युजितं प्रक्षेपम्।
तद्द्रव्यं मूल्यघ्नं क्षेपविभक्तं हि मूल्यं स्यात्॥ ८७ १/२ ॥

“The (number representing the) rate - price is divided by (the number representing) the thing purchasable herewith; (it) is (then) multiplied by the (given) proportional number; by means of this, (we get at) the sum of the proportionate parts (through) the process of addition. Then the given amount multiplied by the (respective) proportionate parts and then divided by (this sum of) the proportionate parts gives rise to the value (of the various things in the required proportion).”

Mixed Quantities

Rate price, R'_i (per unit quantity of i) $= \frac{R_i}{n_i}$(where R_i is the price for quantity, n_i).

Let the quantities of $1, 2, 3, \cdots, n$ be $x_1, x_2, \cdots, x_n$ (x_i for i).

$\therefore$ Total amount of money for purchase,

$$A = \sum x_i R'_i = x_1 R'_1 + x_2 R'_2 + \cdots + x_n R'_n.$$

If $x_1 : x_2 : x_3 \cdots$ is given as $\alpha_1 : \alpha_2 : \alpha_3 \cdots$, that is, $x_i = \alpha_i X$,

$$\left(\sum \alpha_i R'_i \right) X = A, \text{given total amount.}$$

$$\therefore \left(\sum \alpha_i \frac{R_i}{n_i} \right) X = A$$

$$\therefore x_i = \alpha_i X = \frac{\alpha_i A}{\sum \alpha_i \frac{R_i}{n_i}}, \text{ which is the rule.}$$

Example

Example.Verse 90 b and 91:

द्वाभ्यां त्रीणि त्रिभिः पञ्चभिस्तप्त मानकैः।
दाडिमाम्रकपित्थानां फलानि गणितार्धवित्॥ ९० १/२ ॥
कपित्थात् त्रिगुणं ह्याम्रं दाडिमं षड्गुणं भवेत्।
क्रीत्वानय सखे शीघ्रं त्वं षड्सप्ततिभिः पणैः॥ ९१ १/२ ॥

"Pomegranates, mangoes and wood apples are obtainable at the (respective) rates of 3 for 2, 5 for 3 and 7 for 5 respectively. O you friend, who know the principles of computation, come quickly having purchased fruits for 76 $paṇas$, so that mangoes may be three times as the wood-apples, and pomegranates six times as much."

Solutions

1 (Pomegranates), $n_1 = 3, R_1 = 2$; 2 (Mangoes), $n_2 = 5, R_2 = 3$; 3 (Wood apples) $n_3 = 7, R_3 = 5$

$$\alpha_1 = 6, \alpha_2 = 3, \alpha_3 = 1$$

Amount $A = 76$.

$$\begin{aligned}\sum \alpha_i \frac{R_i}{n_i} &= 6 \times \frac{2}{3} + 3 \times \frac{3}{5} + 1 \times \frac{5}{7} = 4 + \frac{9}{5} + \frac{5}{7} \\ &= \frac{140 + 63 + 25}{35} = \frac{228}{35}.\end{aligned}$$

$$\frac{A}{\sum \alpha_i \frac{R_i}{n_i}} = \frac{76}{\frac{228}{35}} = \frac{35}{3}.$$

$$\therefore\ x_1(\text{No.of pomegranates}) = 6 \times \frac{35}{3} = 70,$$

$$x_2(\text{No.of mangoes}) = 3 \times \frac{35}{3} = 35,$$

$$x_3(\text{No.of wood-apples}) = 1 \times \frac{35}{3} = \frac{35}{3}.$$

.

References

1. *Gaṇitasārasaṅgraha* of Mahāvīra : Ed. and Tr. by M.Rangacharya, Govt. Press, Madras, 1912.

2. *Pāṭīgaṇita* of *Śrīdhara* : Ed. with an ancient comm. and Tr. by K.S.Shukla, Lucknow, 1959.

3. *Gaṇitasārasaṅgraha* of Mahāvīra : Ed. with Hindi Tr. by L.C.Jain, Jain Samskrit Samrakshaka Sangha, Sholapur, 1963.

4. *Gaṇitasārasaṅgraha* of Mahāvīra : with Eng. Tr. of Rangacharya and Kannada Tr. by Padmavatamma, Sri Hombuja Jain Math, Shimoga Dt., 2000.

Thanks!

Thank You