Lecture II - Preliminaries from general topology:

We discuss in this lecture a few notions of general topology that are covered in earlier courses but are of frequent use in algebraic topology. We shall prove the existence of Lebesgue number for a covering, introduce the notion of proper maps and discuss in some detail the stereographic projection and Alexandroff's one point compactification. We shall also discuss an important example based on the fact that the sphere S^n is the one point compactifiaction of $\mathbb{R}^n$. Let us begin by recalling the basic definition of compactness and the statement of the Heine Borel theorem.

Definition 2.1: A space X is said to be compact if every open cover of X has a finite sub-cover. If X is a metric space, this is equivalent to the statement that every sequence has a convergent subsequence.

If X is a topological space and A is a subset of X we say that A is compact if it is so as a topological space with the subspace topology. This is the same as saying that every covering of A by open sets in X admits a finite subcovering. It is clear that a closed subset of a compact subset is necessarily compact. However a compact set need not be closed as can be seen by looking at X endowed the indiscrete topology, where every subset of X is compact. However, if X is a Hausdorff space then compact subsets are necessarily closed. We shall always work with Hausdorff spaces in this course. For subsets of $\mathbb{R}^n$ we have the following powerful result.

Theorem 2.1(Heine Borel): A subset of $\mathbb{R}^n$ is compact (with respect to the subspace topology) if and only if it is closed and bounded.

The theorem provides a profusion of examples of compact spaces.

1. The unit sphere $S^{n-1} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$ is compact.

2. The unit square $I^2 = [0,1] \times [0,1]$ is compact.

3. The set of all 3×3 matrices is clearly homeomorphic to $\mathbb{R}^9$. Then the set of all 3×3 orthogonal matrices, denoted by $O(3, \mathbb{R})$ is is compact. That is to say the orthogonal group is compact. The result readily generalizes to the group of $n \times n$ orthogonal matrices.

4. Think of the set of all $n \times n$ matrices with complex entries as $\mathbb{C}^{n^2}$ which in turn may be viewed as $\mathbb{R}^{2n^2}$. The set of all $n \times n$ unitary matrices is then easily seen to be a compact space. These matrices form a group known as the unitary group $U(n)$.

5. The set of all $n \times n$ unitary matrices with determinant one is also a closed bounded subset of $\mathbb{C}^{n^2}$ and so is compact. This is the special unitary group $SU(n)$.

Theorem 2.2: Suppose that X is a compact topological space, Y is an arbitrary Hausdorff space and $f : X \longrightarrow Y$ is a continuous surjection then

1. Y is compact.

2. If A is a closed subset of X then $f(A)$ is closed.

3. It f is bijective then f is a homeomorphism.

Proof: The first assertion is proved in courses on point set topology. We remark that the Hausdorff assumption is not necessary for (i). The second follows from the first and we shall prove the third which will be of immense use in the sequel. Let g be the inverse of f and A be closed in X then $g^{-1}(A) = f(A)$ is closed in Y from which continuity of g follows.

Definition 2.2 (The Lebsesgue number for a cover): Given an open covering $\{G_\alpha\}$ of a metric space X, a Lebesgue number for the covering is a positive number ϵ such that every ball of radius ϵ is contained in some member G_α of the cover.

Theorem 2.3: Every open covering of a compact metric space has a Lebesgue number.

Proof: The student is advised to draw relevant pictures as he reads on. Suppose that a cover $\{G_\alpha\}$ has no Lebesgue number. Then for every $n \in \mathbb{N}$, $1/n$ is not a Lebesgue number and so there is a point $x_n \in X$ such that the ball of radius $1/n$ centered at x_n is not contained in any of the open sets in the covering. By compactness the sequence $\{x_n\}$ has a convergent subsequence converging to a point $p \in X$. Choose an α such that G_α contains p and there is a $\delta > 0$ such that the ball of radius δ around p is contained in G_α. Now take n large enough that $1/n < \delta/3$ and x_n is contained in the ball of radius $\delta/3$ centered at p.

Now, since the ball of radius $1/n$ with center x_n is not contained in any of the open sets in our covering, there exists $y_n \in X$ such that $y_n \notin G_\alpha$ and $d(x_n, y_n) < 1/n$. But

$$d(p, y_n) \leq d(p, x_n) + d(x_n, y_n) < 2\delta/3 < \delta.$$

So y_n is in the ball of radius δ centered at p and so $y_n \in G_\alpha$ which is a contradiction.

Definition 2.3 (Locally compact spaces): A (Hausdorff) space X is said to be locally compact if each point of X has a neighborhood whose closure is compact.

It is an exercise for the student to check that under this hypothesis each point of X has a local base of consisting of compact neighborhoods.

Examples: The reader may easily verify the following.

1. Open subsets of $\mathbb{R}^n$ are locally compact.

2. $\mathbb{Q}$ is not locally compact.

Locally compact spaces are easily realized as dense open subsets of compact spaces. One has to merely adjoin one additional point. The idea is important in many applications and is called Alexandroff's one point compactification.

One point compactification: Let X be a locally compact, non-compact Hausdorff space and $\widehat{X} = X \cup \{\infty\}$ be the one point union of X with an additional point ∞. The topology $\mathcal{T}$ consists of all the open subsets in X as well as all the subsets of the form $\{\infty\} \cup (X - K)$, where K ranges over all the compact subsets of X. The following theorem summarizes the properties of $\widehat{X}$ and the proof is left for the reader.

Theorem 2.4: (i) The collection of sets $\mathcal{T}$ is a topology on $\widehat{X}$.

(ii) The family of sets $\widehat{X} - K$, where K ranges over all compact subsets of X, forms a neighborhood base of ∞.

(iii) X with the given topology is an open dense subset of $\widehat{X}$.

(iv) The space $\widehat{X}$ is compact.

Definition 2.4 (Proper maps): A map $f : X \longrightarrow Y$ between topological spaces is said to be proper if $f^{-1}(C)$ is a compact subset of X whenever C is a compact subset of Y.

Theorem 2.5: Suppose X and Y are locally compact spaces and $f : X \longrightarrow Y$ is a continuous proper map then it extends continuously as a map $\widehat{f} : \widehat{X} \longrightarrow \widehat{Y}$ between their one point compactifications.

Proof: Denote the adjoined points in $\widehat{X}$ and $\widehat{Y}$ as p and q respectively and extend the given map by sending p to q. We need to show that the extension is continuous at p. Let C be any compact subset of Y so that $K = f^{-1}(C)$ is compact in X. Then $N = \widehat{X} - K$ is a neighborhood of p in $\widehat{X}$ that is mapped by $\widehat{f}$ into the preassigned neighborhood $\widehat{Y} - C$ of q. This proves the continuity of the extension.

The converse is not true as the constant map shows. However the following version in the reverse direction is easy to see,

Theorem 2.6: Suppose X and Y are locally compact Hausdorff spaces and $f : \widehat{X} \longrightarrow \widehat{Y}$ is a continuous map such that $f^{-1}(q) = \{p\}$, where p and q are as in the previous theorem, then the restriction of f to X is a proper map.

Proof: If C is a compact subset of Y then $f^{-1}(C)$ being a closed subset of $\widehat{X}$ is compact. The hypothesis says that $f^{-1}(C)$ does not contain p and hence is a compact subset of X itself.

Stereographic projection: Consider the sphere

$$S^n = \Big\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\Big\}$$

and the plane $x_{n+1} = 0$ of the equator. Let $\mathbf{n} = (0, 0, \ldots, 0, 1)$ and $\mathbf{x}$ be a general point on the equatorial plane. The line through $\mathbf{n}$ and $\mathbf{x}$ is described parametrically by $(1-t)\mathbf{n} + t\mathbf{x}$ and meets the sphere at points corresponding to the roots of the quadratic equation

$$\langle (1-t)\mathbf{n} + t\mathbf{x}, (1-t)\mathbf{n} + t\mathbf{x} \rangle \;=\; 1.$$

The root $t = 0$ corresponds to the point $\mathbf{n}$ and the second root

$$t = \frac{2(1 - \mathbf{n}\cdot\mathbf{x})}{1 + \|\mathbf{x}\|^2 - 2\mathbf{n}\cdot\mathbf{x}}$$

is continuous with respect to $\mathbf{x}$ and provides a point $F(\mathbf{x}) \in S^n - \{\mathbf{n}\}$. The map F is a bijective continuous map between the plane $x_{n+1} = 0$ and $S^n - \{\mathbf{n}\}$. Note that the origin is mapped to the south pole by F. The inverse map G is called the stereographic projection. Let us now show that G is also continuous whereby it follows that F is a homeomorphism.

Well, let $\mathbf{y}$ be a point on the sphere minus the north pole $\mathbf{n}$. The ray emanating from $\mathbf{n}$ and passing through $\mathbf{y}$ meets the plane at the point

$$G(\mathbf{y}) = \Big(\frac{y_1}{1-y_{n+1}}, \frac{y_2}{1-y_{n+1}}, \ldots, \frac{y_n}{1-y_{n+1}}\Big)$$

We see that G is also continuous and so the sphere minus its north pole is homeomorphic to $\mathbb{R}^n$.

It is useful to note that the stereographic projection takes points $\mathbf{y}$ close to the north pole to points $G(\mathbf{y})$ of $\mathbb{R}^n$ such that $\|G(\mathbf{y})\| \to +\infty$. We summarize the discussion as a theorem.

Theorem 2.7: The unit sphere in S^n is homeomorphic to the one point compactification of $\mathbb{R}^n$.

Theorem 2.8: Suppose that T is a linear transformation of $\mathbb{R}^n$ into itself and not the zero map, then T extends as a continuous map of S^n to itself if and only if T is non-singular.

Proof: Note that if T is non-singular, it is a proper map and so it extends continuously as a map of S^n sending the point at infinity to itself. Conversely, if T fails to be bijective then there is a sequence of points $\mathbf{x}_n$ such that $\|\mathbf{x_n}\| \to +\infty$ but $T(\mathbf{x}_n) = 0$ for every n. Thus if T were to extend continuously as a map of S^n we would be forced to map the point at infinity namely the north pole to (the point of S^n corresponding to) the origin. On the other hand since T is not the zero map, pick a vector $\mathbf{u}$ such that $T\mathbf{u} \neq 0$ and the sequence $m\mathbf{u}$ converges (as $m \to \infty$) to the point at infinity on S^n. Thus by continuity we would have $\lim T(m\mathbf{u}) = 0$, as $m \to 0$. Hence, $m\|T\mathbf{u}\| \longrightarrow 0$ which is plainly false since $T\mathbf{u} \neq 0$.

More important examples are furnished by regarding S^2 as the one point compactification of the plane $\mathbb{C}$ and using the field structure on the plane. The proof of the following is an exercise.

Theorem 2.9: Any non-constant polynomial is a proper map of $\mathbb{C}$ onto itself and so may be viewed as a continuous map of S^2 to itself fixing the point at infinity.

Exercises

1. Prove that a topological space is compact if and only if it satisfies the following condition known as the *finite intersection property.* For every family $\{F_\alpha\}$ of closed sets with $\cap_\alpha F_\alpha = \emptyset$, there is a finite sub-collection whose intersection is empty

2. Show that $f : [0,1] \longrightarrow [0,1]$ is continuous if and only if its graph is a compact subset of I^2.

3. Examine whether the exponential map from $\mathbb{C}$ onto $\mathbb{C} - \{0\}$ is proper. What about the exponential map as a map from $\mathbb{R}$ onto $(0,\infty)$?

4. (Gluing Lemma) Suppose that $\Big\{U_\alpha\Big\}_{\alpha\in\Lambda}$ is a family of open subsets of a topological space and for each $\alpha \in \Lambda$ we are given a continuous function $f_\alpha : U_\alpha \longrightarrow Y$. Assume that whenever $f_\alpha(x) = f_\beta(x)$ whenever $x \in U_\alpha \cap U_\beta$. Show that there exists a unique continuous function $f : \bigcup\limits_{\alpha\in\Lambda} U_\alpha \longrightarrow Y$ such that $f(x) = f_\alpha(x)$ for all $x \in U_\alpha$ and for all $\alpha \in \Lambda$. Show that the result holds if all the U_α are closed sets and Λ is a finite set.

5. How would you show rigorously that the closed unit disc in the plane is homeomorphic to the closed triangular region determined by three non-collinear points? You are allowed to use results from complex analysis, provided you state them clearly.

6. Prove that any two closed triangular planar regions (as described in the previous exercise) are homeomorphic. Show that any such closed triangular region is homeomorphic to I^2.

7. Suppose that Z is a Hausdorff space and $f, g : Z \longrightarrow X$ are continuous functions then the set $\{z \in Z / f_1(z) = f_2(z)\}$ is closed in Z.

8. Show that the space obtained by rotating the circle $(x-2)^2 + y^2 = 1$ about the $y-$axis is homeomorphic to $S^1 \times S^1$.