Lecture IX - Functorial Property of the Fundamental Group

We now turn to the most basic functor in algebraic topology namely, the π_1 functor. Recall that the fundamental group of a space involves a base point and according to theorem (7.8) the fundamental group of a path connected space is unique upto isomorphism. However, this isomorphism is not canonical as theorem 7.9 shows and isomorphism classes of groups do not form a category. To get around this difficulty and to obtain a well-defined functor, we introduce the category of pointed topological spaces.

Definition 9.1 (The category of pointed topological spaces): This category will be denoted by $\mathbf{Top_0}$ and its objects consists of all pairs (X, x_0) where X is a topological space and x_0 is a point of X. Given two pairs of pointed spaces (X, x_0) and (Y, y_0), the morphisms between them consists of all continuous functions $f : X \longrightarrow Y$ such that $f(x_0) = y_0$.

Suppose that X, Y are path connected spaces and $f : X \longrightarrow Y$ is a continuous map such that $f(x_0) = y_0$ then f clearly defines a morphism, denoted by same letter, between pointed spaces

$$f : (X, x_0) \longrightarrow (Y, y_0).$$

The map $f_* : \pi_1(X.x_0) \longrightarrow \pi_1(Y, y_0)$ given by $f_*([\gamma]) = [f \circ \gamma]$ where γ in X based at x_0, is well defined since $f \circ \gamma$ is a loop in Y based at y_0 Therefore f_* is well defined because if γ_1, γ_2 are homotopic loops in X based at x_0 and F is the homotopy then $f \circ F$ is a homotopy between $f \circ \gamma_1$ and $f \circ \gamma_2$ in Y. It is immediately checked that $f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$ thereby giving a group homomorphism:

$$f_*([\gamma_1][\gamma_2]) = f_*([\gamma_1]) f_*([\gamma_2]).$$

The group homomorphism f_* is called the map induced by f on the fundamental groups. In other words we obtain a functor π_1 from $\mathbf{Top}_0$ to $\mathbf{Gr}$.

Lemma 9.1: Suppose that $(X, x_0), (Y, y_0)$ and (Z, z_0) are pointed topological spaces. Let $f : (X, x_0) \longrightarrow (Y, y_0)$ and $g : (Y, y_0) \longrightarrow (Z, z_0)$ be continuous maps of pairs, that is continuous maps satisfying $f(x_0) = y_0;\ g(y_0) = z_0$, then the induced homomorphisms on the respective fundamental groups satisfies

$$(g \circ f)_* = g_* \circ f_*.$$

If $\mathrm{id}_x : X \longrightarrow X$ is the identity map then $(\mathrm{id}_x)_* = \mathrm{id}_{\pi_1(X, x_0)}$. That is to say, the identity map on X induces the identity homomorphism on $\pi_1(X, x_0)$.

Proof: The second part is obvious. To prove the first part, for any loop γ in X based at x_0,

$$(g \circ f) \circ \gamma = g \circ (f \circ \gamma)$$

so we get upon passing to equivalence classes,

$$(g \circ f)_*[\gamma] = g_*[f \circ \gamma] = g_*(f_*([\gamma]))$$

In particular if $f : X \longrightarrow Y$ is a homeomorphism then $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is an isomorphism of groups. $\square$

Definition 9.2 (Retraction): Given a topological space X, a subset $A \subseteq X$ is said to be retract of X if there exits a continuous function $r : X \longrightarrow A$ such that $r(a) = a$ for all $a \in A$.

It is immediate that a retract of a Hausdorff space must be closed. The condition that A be a retract of X is quite a strong condition. For example if X is compact and connected then so must A. Thus $\{0, 1\}$ cannot be a retract of $[0, 1]$. The boundary $\dot{I}^2$ of I^2 is not a retract of I^2 but this is highly non-trivial.

Example 9.1:

(i) $S^1 \times \{1\}$ is a retract of $S^1 \times S^1$. A retraction is given by $r(z, w) = (z, 1)$.

(ii) $(S^1 \times \{1\}) \cup (\{1\} \times S^1)$ is not a retract of $S^1 \times S^1$ as we shall see later.

(iii) S^1 is a retract of $\mathbb{R}^2 - \{(0, 0)\}$ and the retraction is given by the map $\mathbf{x} \mapsto \mathbf{x}/\|\mathbf{x}\|$.

(iv) Suppose A is a retract of X then every continuous map $f : A \longrightarrow Y$ extends continuously to a map $\tilde{f} : X \longrightarrow Y$.

We shall show later (lectures 12-13) that $\pi_1(S^1, 1) = \mathbb{Z}$ is non-trivial but we present it here as a theorem for immediate use in the next lecture on the Brouwer's fixed point theorem.

Theorem 9.2: $\pi_1(S^1, 1) = \mathbb{Z}$ and the generator is given by the homotopy class of the loop

$$t \mapsto \exp(2\pi i t), \quad 0 \leq t \leq 1.$$

Lemma 9.3: Suppose $r : X \longrightarrow A$ is a retraction, $j : A \longrightarrow X$ is the inclusion, then for $a \in A$

$$r_* : \pi_1(X, a) \longrightarrow \pi_1(A, a)$$

is surjective and

$$j_* : \pi_1(A, a) \longrightarrow \pi_1(X, a)$$

is injective.

Proof: Since $r \circ j = \mathrm{id}_A$ we see that $r_* \circ j_* = \mathrm{id}_{\pi_1(A,a)}$. Hence r_* is surjective and j_* is injective. $\square$

Corollary 9.4 (No retraction theorem): S^1 is not a retract of $E^2 = \{\mathbf{x} \in \mathbb{R}^2 / \|\mathbf{x}\| \leq 1\}$

Proof: Suppose we have a retraction $r : E^2 \longrightarrow S^1$ then the induced map

$$r_* : \pi_1(E^2, 1) \longrightarrow \pi_1(S^1, 1)$$

would be surjective which means we have a surjective group homomorphism

$$r_* : \{1\} \longrightarrow \mathbb{Z}$$

which is impossible. $\square$

Corollary 9.5 (Brouwer's fixed point theorem): Every continuous function $f : E^2 \longrightarrow E^2$ has a fixed point where $E^2 = \{\mathbf{x} \in \mathbb{R}^2 / \|\mathbf{x}\| \leq 1\}$.

Proof: Will be done in the next lecture.

Fundamental group of a Product: The fundamental group functor has the pleasant property that it respects products. The following theorem summarizes the matter for finite products.

Theorem 9.6: Suppose that X and Y are two topological spaces and $x_0 \in X$ and $y_0 \in Y$. Then

$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Proof: Let p_1 and p_2 be the usual projection maps $X \times Y \longrightarrow X$ and $X \times Y \longrightarrow Y$ respectively and γ be a loop in $X \times Y$ based at (x_0, y_0). Then $p_1 \circ \gamma$ and $p_2 \circ \gamma$ are loops in X and Y based at x_0 and y_0 respectively. The map

$$\begin{aligned} \phi : \pi_1(X \times Y, (x_0, y_0)) &\longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0) \\ [\gamma] &\mapsto ([p_1 \circ \gamma], [p_2 \circ \gamma]) \end{aligned}$$

is well-defined and easily seen to be a surjective group homomorphism. Injectivity is also easy to check. Well, suppose that $[\gamma]$ is in the kernel of ϕ then $p_1 \circ \gamma$ and $p_2 \circ \gamma$ are homotopic to the constant loops ε_{x_0} and ε_{y_0} respectively via homotopies F_1 and F_2. That is to say there exists continuous maps $F_1 : I^2 \longrightarrow X$ and $F_2 : I^2 \longrightarrow Y$ such that

$$F_1(0, t) = p_1 \circ \gamma, F_1(1, t) = \varepsilon_{x_0}, F_2(0, t) = p_2 \circ \gamma, F_2(1, t) = \varepsilon_{y_0}.$$

and $F_1(s, 0) = F_1(s, 1) = x_0$, $F_2(s, 0) = F_2(s, 1) = y_0$ for all $s \in [0, 1]$. Putting these together we get a continuous map $F_1 \times F_2 : I^2 \longrightarrow X \times Y$ namely

$$(s, t) \mapsto (F_1(s, t), F_2(s, t))$$

which is a homotopy between γ and the constant loop at (x_0, y_0) proving that the kernel is trivial.

Corollary 9.7: $\pi_1(S^1 \times S^1, (1, 1)) = \mathbb{Z} \times \mathbb{Z}$

Exercises

1. Show that the sphere S^2 retracts onto one of its longitudes. If X is the space obtained from S^2 by taking its union with a diameter, there is a surjective group homomorphism $\pi_1(X) \longrightarrow \mathbb{Z}$.

2. Prove that A is a retract of X if and only if every space Y, every continuous map $f : A \longrightarrow Y$ has a continuous extension $\tilde{f} : X \longrightarrow Y$.

3. Show that the fundamental group respects arbitrary products.

4. Construct a retraction from $\{(x, y) : x \text{ or } y \text{ is an integer}\}$ onto the boundary of I^2.

5. Show that every homeomorphism of E^2 onto itself must map the boundary to the boundary.

6. Given that there exists a functor T from the category **Top** to the category **AbGr** such that $T(X)$ is the trivial group for every convex subset X of a Euclidean space and $T(S^n)$ is a non-trivial group, prove that S^n is not a retract of the closed unit ball in $\mathbb{R}^{n+1}$.