Lecture - XXVI Seifert Van Kampen theorem and knots

This is one of the most famous theorem concerning the fundamental group which serves as a tool for computations and applications to combinatorial group theory. If U and V are path connected open subsets of a topological space such that $U \cap V$ is path connected, the theorem provides information on the geometry of $U \cup V$ in terms of the geometry of U, V and $U \cap V$. In precise terms it states that the π_1 functor maps the push-out diagram of pointed topological spaces with $x_0 \in U \cap V$,

$$\begin{array}{ccc} (U \cap V, x_0) & \xrightarrow{i_1} & (U, x_0) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ (V, x_0) & \xrightarrow{j_2} & (U \cup V, x_0) \end{array}$$

to the push-out diagram of groups:

$$\begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{i_{1*}} & \pi_1(U, x_0) \\ {\scriptstyle i_{2*}}\downarrow & & \downarrow{\scriptstyle j_{1*}} \\ \pi_1(V, x_0) & \xrightarrow{j_{2*}} & \pi_1(U \cup V, x_0) \end{array}$$

thereby giving a precise description of the group $\pi_1(U \cup V, x_0)$ in terms of the groups $\pi_1(U, x_0)$, $\pi_1(V, x_0)$ and $\pi_1(U \cap V, x_0)$. Thus $\pi_1(U \cup V, x_0)$ is the free product of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ amalgamated along $\pi_1(U \cap V, x_0)$. The theorem enables us to calculate quickly the fundamental groups of several important spaces.

Theorem 26.1 (Seifert and Van Kampen - version I): Let U, V be open path connected subsets of a topological space such that $U \cap V$ is path connected. Let $x_0 \in U \cap V$ and $i_1 : U \cap V \longrightarrow U$, $i_2 : U \cap V \longrightarrow V$ denote the inclusion maps. Then $\pi_1(U \cup V, x_0)$ is the free product (coproduct) of $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ amalgamated along $\pi_1(U \cap V, x_0)$ with respect to the maps i_{1*} and i_{2*}. That is to say if N is the normal subgroup

$$N = \langle i_{1*}[\gamma](i_{2*}[\gamma])^{-1} : [\gamma] \in \pi_1(U \cap V, x_0)\rangle \tag{26.1}$$

then the fundamental group of $U \cup V$ is given by

$$\pi_1(U \cup V, x_0) = \pi_1(U, x_0) * \pi_1(V, x_0)/N. \tag{26.2}$$

Considering $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ as subgroups of $\pi_1(U) * \pi_1(V)$, their images in the quotient group generate $\pi_1(U \cup V, x_0)$.

The result may be elegantly stated using a push-out diagram namely,

Theorem 26.2 (Seifert and Van Kampen - version II): Let U, V be open path connected subsets of a topological space such that $U \cap V$ is path connected. Let $x_0 \in U \cap V$ and $i_1 : U \cap V \longrightarrow U$, $i_2 : U \cap V \longrightarrow V$ denote the inclusion maps. Then the push-out data

$$\begin{tikzcd} \pi_1(U \cap V, x_0) \arrow[r, "i_{1*}"] \arrow[d, "i_{2*}"'] & \pi_1(U, x_0) \\ \pi_1(V, x_0) & \end{tikzcd}$$

may be completed to yield the push-out square

$$\begin{tikzcd} \pi_1(U \cap V, x_0) \arrow[r, "i_{1*}"] \arrow[d, "i_{2*}"'] & \pi_1(U, x_0) \arrow[d, "j_{1*}"] \\ \pi_1(V, x_0) \arrow[r, "j_{2*}"] & \pi_1(U \cup V, x_0) \end{tikzcd}$$

where the maps $j_1 : U \longrightarrow U \cup V$ and $j_2 : V \longrightarrow U \cup V$ are inclusions.

The proof is neatly presented on pages 110-113 of the book by J. Vick and need not be repeated here. Instead we move on to its applications to the computation of the fundamental groups of certain spaces.

Corollary 26.3: Suppose that U, V are open path-connected, simply connected subsets of a topological space such that $U \cap V$ is path connected then $U \cup V$ is simply connected.

Fundamental groups of spheres: An important example of this is the case $U = S^n - \{\mathbf{e}_n\}$ and $V = \{\mathbf{e}_n\}$. When $n \geq 2$, the spaces U and V are homeomorphic to $\mathbb{R}^n$ via the stereo-graphic projection and since $U \cap V$ is path connected we conclude that $U \cup V = S^n$ is simply connected.

Corollary 26.4: Suppose that U, V are open path-connected subsets of a topological space such that $U \cap V$ is simply connected then

$$\pi_1(U \cap V, x_0) = \pi_1(U, x_0) * \pi_1(V, x_0).$$

Wedge of two circles: Let us consider the space $S^1 \vee S^1$ given by the union of two circles of radius one in the plane touching each other externally at the origin. We take U and V to be the open sets obtained by deleting one of the points of each lobe (not the common point!). Then the circle is a deformation retract of both U and V and $U \cap V$ deformation retracts to the origin. Thus

$$\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}. \tag{26.3}$$

The last clause in theorem (26.1) also provides the generators of the fundamental group. Assuming the circles to centered at ± 1, the generators are given by the homotopy classes of the loops

$$\pm 1 + \exp(2\pi i t) \tag{26.4}$$

The generalization to a wedge of n circles is left as an exercise.

Corollary 26.5 Suppose that U, V are open path-connected subsets of a topological space such that $U \cap V$ and U are simply connected then with a base point $x_0 \in U \cap V$,

$$\pi_1(U \cup V, x_0) = \pi_1(V, x_0).$$

We turn to an important example to illustrate the use of this corollary. Regard $\mathbb{R}^3$ as a subset of S^3 via the stereo-graphic projection and K be a compact subset of $\mathbb{R}^3$ such that the complement $\mathbb{R}^3 - K$ is connected. We then have the following result.

Theorem 26.6: $\pi_1(S^3 - K) = \pi_1(\mathbb{R}^3 - K)$.

Proof: Let $p \in S^3$ denote the north-pole using which we project $S^3 - \{p\}$ stereo-graphically onto $\mathbb{R}^3$. Since K is compact there is a neighborhood U of p in S^3 homeomorphic to a ball which does not intersect K. Taking $V = S^3 - (K \cup \{p\}) = \mathbb{R}^3 - K$, we see that $U \cup V = S^3 - K$ and $U \cap V$ deformation retracts to S^2. The result now follows from the previous theorem.

Corollary 26.7: Suppose that U, V are open path-connected subsets of a topological space such that U is simply connected and $i : U \cap V \longrightarrow V$ is the inclusion map then, taking a base point $x_0 \in U \cap V$,

$$\pi_1(U \cup V, x_0) = \pi_1(V, x_0)/\langle \text{Im } i_* \rangle,$$

where $\langle \text{Im } i_* \rangle$ denotes the normal subgroup generated by the image of i_*.

Proof: The subgroup N in (26.1) reduces to $\langle \text{Im } i_* \rangle$.

The projective plane: We work this example out in meticulous detail. Such details will be progressively cut down and left for the students to fill in as we go along. The projective plane $\mathbb{R}P^2$ is obtained by attaching a two cell E^2 to S^1 using the map given in complex form as $f(z) = z^2$. Let p denote the center of E^2 and $\eta : E^2 \longrightarrow \mathbb{R}P^2$ be the quotient map. Taking U to be the interior of E^2 and $V = \mathbb{R}P^2 - \{p\}$ we apply corollary (26.7). For computing the image of i_* we take a generator for the infinite cyclic group $\pi_1(U \cap V, y_0)$ with base point $y_0 = 1/2$. The generator is the equivalence class of the loop

$$\gamma(t) = \frac{1}{2}\exp(2\pi i t), \quad 0 \le t \le 1. \tag{26.5}$$

We also need a base point x_0 sitting on the loop Γ given by

$$\Gamma(t) = \eta(\exp(i\pi t)), \quad 0 \le t \le 1, \tag{26.6}$$

which generates $\pi_1(\mathbb{R}P^2 - \{p\}, x_0)$. Taking a path β joining y_0 and x_0 we get a generator for the infinite cyclic group $\pi_1(\mathbb{R}P^2 - \{p\}, y_0)$ namely, the class of the loop $\beta * \Gamma * \beta^{-1}$. Having set the stage we are ready to compute $i_*[\gamma]$ namely, the homotopy class of the loop γ in $\mathbb{R}P^2 - \{p\}$. This loop γ based at y_0 is homotopic to the loop

$$\beta * \Gamma * \Gamma * \beta^{-1}. \tag{26.7}$$

The required homotopy is $\eta \circ F$ where F is a map of a rectangle onto a suitable annulus (see exercise (1)). Introducing a $\beta^{-1} * \beta$ we get

$$i_*[\gamma] = [\beta * \Gamma * \beta^{-1}][\beta * \Gamma * \beta^{-1}] \tag{26.8}$$

or in additive notation it is the map $\mathbb{Z} \longrightarrow \mathbb{Z}$ given by $n \mapsto 2n$. We conclude from corollary (26.7) that $\pi_1(\mathbb{R}P^2)$ is the cyclic group of order two.

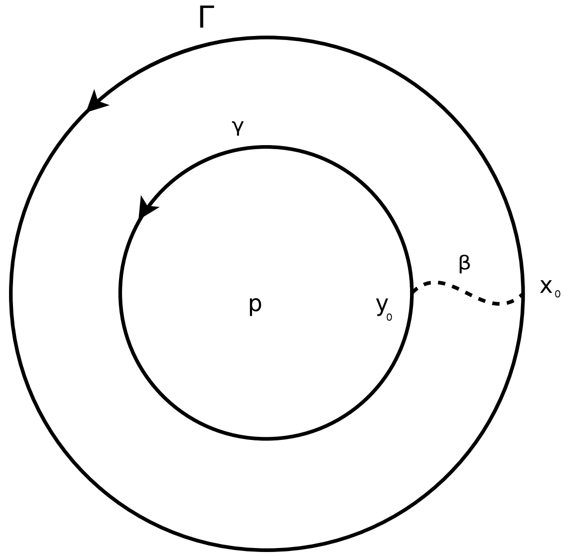

Figure 19: Computing $\pi_1(\mathbb{R}P^2)$

The torus and the Klein's bottle: We proceed along the same lines using the convenient form (25.5). Denoting by X either the torus or the Klein's bottle and p to be the origin, we see that $X - \{p\}$ deformation retracts to the figure eight loop and, in analogy with (26.6), the generators for the free group $\pi_1(X - \{p\})$ are given by

$$\Gamma_1(t) = \eta(\exp(i\pi t/2)), \quad \Gamma_2(t) = \eta(\exp(i\pi(t+1)/2)), \quad 0 \le t \le 1$$

We take U to be the open unit disc, V to be $X - \{p\}$ and the class of (26.5) as the generator for $\pi_1(U \cap V, y_0)$ where the base point y_0 is 1/2. Taking an auxiliary path β joining y_0 and the point $x_0 = 1$ common to both $\Gamma_1(t)$ and $\Gamma_2(t)$, we get the generators

$$[\beta * \Gamma_1 * \beta^{-1}] \quad \text{and} \quad [\beta * \Gamma_2 * \beta^{-1}] \tag{26.9}$$

for $\pi_1(X - \{p\}, y_0)$. The deformation of the previous example (exercise (1)) can be employed here again and this time we get

$$i_*[\gamma] = [\beta * \Gamma_1 * \beta^{-1}][\beta * \Gamma_2 * \beta^{-1}][\beta * \Gamma_1^{-1} * \beta^{-1}][\beta * \Gamma_2^{-1} * \beta^{-1}] \tag{26.10}$$

for the torus whereas for the Klein's bottle we get instead

$$i_*[\gamma] = [\beta * \Gamma_1 * \beta^{-1}][\beta * \Gamma_2 * \beta^{-1}][\beta * \Gamma_1 * \beta^{-1}][\beta * \Gamma_2^{-1} * \beta^{-1}] \tag{26.11}$$

One could also work with the other models described in example (25.3) where the spaces are obtained by identifying the opposite edges of a square. The homotopy $\eta \circ F$ of the last example would have to be modified to $\eta \circ G \circ F$ where G is a certain homeomorphism from the unit disc onto the square $[0,1] \times [0,1]$.

Denoting the generators (26.9) of $\pi_1(V, y_0)$ by S and T we are ready to apply corollary (26.7) since (26.10) gives us the image of the map i_*. The fundamental group of the torus is then

$$\langle S, T : ST = TS \rangle \cong \mathbb{Z} \times \mathbb{Z} \tag{26.12}$$

and the fundamental group of the Klein's bottle is

$$\langle S, T : TST = S \rangle \cong \mathbb{Z} \ltimes \mathbb{Z}. \tag{26.13}$$

The double torus: By writing out the attaching map $S^1 \longrightarrow S^1 \vee S^1 \vee S^1 \vee S^1$ akin to (25.5) or else using the identification of the sides of a regular octagon as described in lecture 4, the reader is invited to prove that the fundamental group of the double torus is

$$\langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} = 1\rangle \tag{26.14}$$

Fundamental groups of some adjunction spaces: The method used in the last few examples may be adapted to prove a general theorem about the fundamental group of the adjunction space $X \sqcup_f E^k$ obtained by attaching E^k to a given space X via a map $f : S^{k-1} \longrightarrow X$. As in the case of the projective plane, Klein's bottle and torus the crucial point is to obtain some specific information about the induced map f_*. We shall merely state the result and suppress the proof.

Theorem 26.8: Let $X \sqcup_f E^k$ be the space obtained by attaching a k cell to a path connected space X via a map $f : S^{k-1} \longrightarrow X$. Then for any choice of base point in $f(S^{k-1})$,

(i) $\pi_1(X \sqcup_f E^k, x_0) = \pi_1(X, x_0)$ if $k \geq 3$.

(i) $\pi_1(X \sqcup_f E^2, x_0) = \pi_1(X, x_0)/\langle \text{im } f_*\rangle$.

Exercises

1. Fill in the details in the computation of the fundamental group of the projective plane, Klein's bottle and the torus done in the lecture by providing a careful proof of equations (26.8), (26.10) and (26.11). Hint: Use polar coordinates. Continuously shrink the path β to the point x_0.

2. Show that the fundamental group of the wedge of n copies of S^1 is the free group on n generators. Calculate the fundamental group of the truncated grid
$$\{(x, y) \in \mathbb{R}^2/x \in \mathbb{Z} \text{ or } y \in \mathbb{Z},\ 0 \leq x \leq n,\ 0 \leq y \leq n\}.$$

3. Determine the generators of double torus by expressing it as a union of open sets each of which is a torus from which a tiny closed disc has been removed.

4. Let C be the union of the two *unlinked* circles
$$\begin{aligned}(x-2)^2 + y^2 &= 1,\ z = 0,\\ (x+2)^2 + y^2 &= 1,\ z = 0.\end{aligned}$$
in $\mathbb{R}^3$. Show that $\pi_1(\mathbb{R}^3 - C)$ is the free group on two generators.

5. Calculate the fundamental groups of the following spaces
   (i) $\mathbb{R}^4$ minus a line.
   (ii) $\mathbb{R}^4$ minus a two dimensional linear subspace.
   (iii) $\mathbb{R}^4$ minus two parallel lines.
   (iv) $\mathbb{R}^4$ minus two intersecting lines.
   (v) $\mathbb{R}^3$ minus the coordinate axes
   (vi) $\mathbb{C}^2 - \{(z_1, z_2)/z_1 z_2 = 0\}$
   (vii) $\mathbb{R}^3$ minus finitely many points.