

## Lecture VII - Paths, homotopies and the fundamental group

In this lecture we shall introduce the most basic object in algebraic topology, the fundamental group. For this purpose we shall define the notion of homotopy of paths in a topological space  $X$  and show that this is an equivalence relation. We then fix a point  $x_0 \in X$  in the topological space and look at the set of all equivalence classes of loops starting and ending at  $x_0$ . This set is then endowed with a binary operation that turns it into a group known as the fundamental group  $\pi_1(X, x_0)$ . Besides being the most basic object in algebraic topology, it is of paramount importance in low dimensional topology. A detailed study of this group will occupy the rest of part I of this course. However in this lecture we shall focus only on the most elementary results.

All spaces considered here are path connected Hausdorff spaces.

**Definition 7.1 (homotopy of paths):** Two paths  $\gamma_0, \gamma_1$  in  $X$  with parameter interval  $[0, 1]$  such that

$\gamma_0(0) = \gamma_1(0), \gamma_0(1) = \gamma_1(1)$  (that is with the same end points) are said to be homotopic if there exists a continuous map  $F : [0, 1] \times [0, 1] \rightarrow X$  such that

$$\begin{aligned} F(0, t) &= \gamma_0(t) \\ F(1, t) &= \gamma_1(t) \\ F(s, 0) &= \gamma_0(0) = \gamma_1(0) \\ F(s, 1) &= \gamma_0(1) = \gamma_1(1) \end{aligned}$$

The definition says that the path  $\gamma_0(t)$  can be continuously deformed into  $\gamma_1(t)$  and  $F$  is the continuous

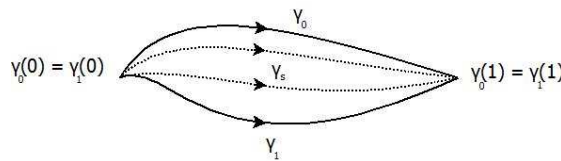


Figure 7: Homotopy of paths

function that does the deformation. The deformation takes place in unit time parametrized by  $s$ . For  $s \in [0, 1]$ , the function  $\gamma_s : t \rightarrow F(s, t)$  is the intermediate path. Finally, the conditions

$$F(s, 0) = \gamma_0(0) \quad \text{and} \quad F(s, 1) = \gamma_0(1)$$

imply that the ends  $\gamma_0(0), \gamma_0(1)$  do not move during the deformation. We shall now show that homotopy is an equivalence relation.

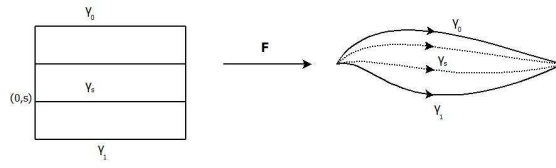


Figure 8: Homotopy of paths

**Theorem 7.1:** If  $\gamma_1, \gamma_2, \gamma_3$  are three paths in  $X$  such that

$$\gamma_1(0) = \gamma_2(0) = \gamma_3(0) \quad \text{and} \quad \gamma_1(1) = \gamma_2(1) = \gamma_3(1),$$

$\gamma_1$  and  $\gamma_2$  are homotopic;  $\gamma_2$  and  $\gamma_3$  are homotopic then  $\gamma_1$  and  $\gamma_3$  are homotopic.

**Proof:** It is clear that the homotopy is reflexive, and symmetry is left for student to verify. To prove transitivity let

$$F : [0, 1] \times [0, 1] \rightarrow X \quad \text{and} \quad G : [0, 1] \times [0, 1] \rightarrow X$$

be homotopies between the pairs  $\gamma_1, \gamma_2$  and  $\gamma_2, \gamma_3$  respectively. Define  $H : [0, 1] \times [0, 1] \rightarrow X$  by the prescription:

$$H(s, t) = \begin{cases} F(2s, t) & 0 \leq s \leq 1/2 \\ G(2s - 1, t) & 1/2 \leq s \leq 1 \end{cases}$$

Note that by gluing lemma  $H$  is continuous. We need to check the conditions at endpoints.

$$H(s, 0) = \begin{cases} F(2s, 0) = \gamma_1(0) = \gamma_3(0), & 0 \leq s \leq 1/2 \\ G(2s - 1, 0) = \gamma_2(0) = \gamma_3(0), & 1/2 \leq s \leq 1 \end{cases}$$

Likewise one verifies easily  $H(s, 1) = \gamma_1(1) = \gamma_3(1)$  for all  $s \in [0, 1]$ . Finally we see that  $H(0, t) = F(0, t) = \gamma_1(t)$  and  $H(1, t) = G(1, t) = \gamma_3(t)$ , which proves the result.

**Notation:** The equivalence class of  $\gamma$  will be denoted by  $[\gamma]$  and called the homotopy class of the path  $\gamma$ . When  $\gamma_1, \gamma_2$  are homotopic we write  $\gamma \sim \gamma_2$ .

**Theorem 7.2 (Reparametrization theorem):** Let  $X$  be a topological space. Suppose that  $\phi : [0, 1] \rightarrow [0, 1]$  is a continuous map such that  $\phi(0) = 0$  and  $\phi(1) = 1$ . Then for any given path  $\gamma$  in  $X$ , we have a homotopy

$$\gamma \sim \gamma \circ \phi$$

**Proof:** We must remark that we are not assuming anything about  $\phi$  besides continuity and the fact that it fixes 0 and 1. In particular  $\phi$  need not be monotone. The idea of proof is simple. The convexity of the unit square  $[0, 1] \times [0, 1]$  is used to tweak the graph of  $\phi$  onto the graph of the identity map of  $[0, 1]$ . Thus we define a continuous map  $F : [0, 1] \times [0, 1] \rightarrow X$  by the prescription

$$F(s, t) = \gamma(s\phi(t) + (1 - s)t)$$

Now  $F(0, t) = \gamma(t)$ ,  $F(1, t) = \gamma \circ \phi(t)$ . For verifying that the end points are fixed during deformation,

$$\begin{aligned} F(s, 0) &= \gamma(s\phi(0)) = \gamma(0) \\ F(s, 1) &= \gamma(s\phi(1) + (1 - s)) = \gamma(1), \quad 0 \leq s \leq 1. \end{aligned}$$

**Juxtaposition of paths:** Suppose that  $\gamma_1, \gamma_2$  are two paths such that  $\gamma_1(1) = \gamma_2(0)$ , that is to say, the end point of  $\gamma_1$  is the initial point of  $\gamma_2$ . The paths  $\gamma_1$  and  $\gamma_2$  can be juxtaposed to produce a path from  $\gamma_1(0)$  to  $\gamma_2(1)$  called the juxtaposition  $\gamma_1$  and  $\gamma_2$ , denoted by  $\gamma_1 * \gamma_2$  and defined as :

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

**Lemma 7.3:** If  $\gamma'_1$  and  $\gamma''_1$  are two homotopic paths starting at  $\gamma_0(1)$  then

$$\gamma_0 * \gamma'_1 \sim \gamma_0 * \gamma''_1$$

**Proof:** Let  $F : [0, 1] \times [0, 1] \rightarrow X$  be a homotopy between  $\gamma'_1$  and  $\gamma''_1$  so that  $F(0, t) = \gamma_0(1)$ ,  $F(1, t) = \gamma'_1(1) = \gamma''_1(1)$ . The homotopy we seek is the map  $H(s, t)$  given by

$$H(s, t) = \begin{cases} \gamma_0(2t), & 0 \leq t \leq 1/2 \\ F(s, 2t - 1), & 1/2 \leq t \leq 1 \end{cases}$$

It can be checked that the definition is meaningful along  $[0, 1] \times \{\frac{1}{2}\}$  and the continuity of  $H$  follows by the gluing lemma. The reader may complete the proof by verifying that

$$H(0, t) = \gamma_0 * \gamma'_1; \quad H(1, t) = \gamma_0 * \gamma''_1. \quad \square$$

**Corollary 7.4:** If  $\gamma'_1, \gamma''_1$  are homotopic paths starting at  $\gamma_0(1)$  then  $[\gamma_0 * \gamma'_1] = [\gamma_0 * \gamma''_1]$ . Likewise if  $\gamma_1$  is a path in  $X$  and  $\gamma'_0, \gamma''_0$  are homotopic paths whose terminal points are at  $\gamma_1(0)$  then  $\gamma'_0 * \gamma_1 \sim \gamma''_0 * \gamma_1$ .

**Definition 7.2:** If  $\gamma_1, \gamma_2$  are two paths in  $X$  such that initial point of  $\gamma_1$  is the terminal point of  $\gamma_2$ , then we define the product of the homotopy classes of paths as

$$[\gamma_1] \cdot [\gamma_2] = [\gamma_1 * \gamma_2]. \quad (7.1)$$

**The Inverse Path and the constant path:** Suppose  $\gamma : [0, 1] \rightarrow X$  is a path then the inverse path  $\gamma^{-1}(t)$  is the path traced in the reversed direction namely the map  $\gamma^{-1} : [0, 1] \rightarrow X$  given by

$$\gamma^{-1}(t) = \gamma(1 - t).$$

The initial point of  $\gamma$  is the terminal point of  $\gamma^{-1}$  and vice versa.

The constant path at  $x_0$  is the path  $\varepsilon_{x_0} : [0, 1] \rightarrow X$  given by

$$\varepsilon_{x_0}(t) = x_0 \quad \text{for all } t \in [0, 1].$$

The following lemma summarizes the main properties of the constant and the inverse paths in terms of the homotopy classes of paths. Theorem (7.6) spells out the associativity of multiplication of homotopy classes of paths. The reader would see analogies with the defining properties of a group.

**Lemma 7.5:**

- (i)  $\gamma * \gamma^{-1} \sim \varepsilon_{\gamma(0)}$ . Thus  $[\gamma] \cdot [\gamma^{-1}] = [\varepsilon_{\gamma(0)}]$ .
- (ii)  $\gamma * \varepsilon_{\gamma(1)} \sim \gamma$ . Thus  $[\gamma][\varepsilon_{\gamma(1)}] = [\gamma]$
- (iii)  $\varepsilon_{\gamma(0)} * \gamma \sim \gamma$ . Thus  $[\varepsilon_{\gamma(0)}][\gamma] = [\gamma]$ .

**Proofs:** One uses the reparametrization theorem to prove (ii) and (iii). Proof of (i) is more involved and we indicate two different methods by which this can be achieved. On the boundary  $\dot{I}^2$  of the unit square  $I^2$  we define a map  $\phi : \dot{I}^2 \longrightarrow [0, 1]$  as follows.

$$\phi(0, t) = 0, \quad \phi(s, 0) = 0, \quad \phi(s, 1) = 0$$

Along the part  $(1, t)$  of the boundary,

$$\phi(1, t) = \begin{cases} 2t & 0 \leq t \leq 1/2 \\ 2 - 2t & 1/2 \leq t \leq 1 \end{cases}$$

By Tietze's extension theorem  $\phi$  extends continuously to  $I^2$  taking values in  $[0, 1]$ . Consider now the map  $H : I^2 \longrightarrow X$  given by

$$H(s, t) = \gamma \circ \phi(s, t).$$

It is readily checked that  $H$  establishes a homotopy between  $\gamma * \gamma^{-1}$  and the constant path  $\varepsilon_{\gamma(0)}$ .  $\square$

**Theorem 7.6:** Suppose  $\gamma_1, \gamma_2, \gamma_3$  are three paths in  $X$  such that  $\gamma_1(1) = \gamma_2(0)$ ;  $\gamma_2(1) = \gamma_3(0)$  then

$$(\gamma_1 * \gamma_2) * \gamma_3 \sim \gamma_1 * (\gamma_2 * \gamma_3)$$

Hence

$$([\gamma_1][\gamma_2])[\gamma_3] = [\gamma_1]( [\gamma_2][\gamma_3] )$$

**Proof:** By direct calculation we get on the one hand

$$(\gamma_1 * \gamma_2) * \gamma_3 = \begin{cases} \gamma_1(4t) & 0 \leq t \leq 1/4 \\ \gamma_2(4t - 1) & 1/4 \leq t \leq 1/2 \\ \gamma_3(2t - 1) & 1/2 \leq t \leq 1. \end{cases}$$

On the other hand, for  $\gamma_1 * (\gamma_2 * \gamma_3)$  we find

$$\gamma_1 * (\gamma_2 * \gamma_3) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(4t - 2) & 1/2 \leq t \leq 3/4 \\ \gamma_3(4t - 3) & 3/4 \leq t \leq 1. \end{cases}$$

These two are homotopic by the reparametrization theorem. To see this define  $\phi : [0, 1] \rightarrow [0, 1]$  by

$$\phi(t) = \begin{cases} 2t & 0 \leq t \leq \frac{1}{4} \\ t + \frac{1}{4} & \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t}{2} + \frac{1}{2} & \frac{1}{2} \leq t \leq 1. \end{cases}$$

one verifies that  $\gamma \circ \phi = (\gamma_1 * \gamma_2) * \gamma_3$  where  $\gamma = \gamma_1 * (\gamma_2 * \gamma_3)$ . By theorem (7.2) the result follows.

We are now ready to define the fundamental group.

**Definition 7.3 (The fundamental group  $\pi_1(X, x_0)$ ):** Let  $X$  be a path connected topological space and  $x_0$  be a point of  $X$ . We define  $\pi_1(X, x_0)$  to be the set of all homotopy classes of paths beginning and ending at the given point  $x_0$  namely homotopy classes  $[\gamma]$  where  $\gamma : [0, 1] \rightarrow X$  is continuous and  $\gamma(0) = \gamma(1) = x_0$ :

$$\pi_1(X, x_0) = \{ [\gamma] / \gamma : [0, 1] \rightarrow X \text{ continuous and } \gamma(0) = \gamma(1) = x_0 \}.$$

**Terminology:** Paths in  $X$  starting and ending at  $x_0$  will be referred to as *loops* based at  $x_0$ . The distinguished point  $x_0 \in X$  is called the *base point* of  $X$ .

Note that if  $\gamma_1, \gamma_2$  are two loops based at  $x_0$ , their juxtaposition  $\gamma_1 * \gamma_2$  is defined whereby both the products  $[\gamma_1][\gamma_2]$  and  $[\gamma_2][\gamma_1]$  are defined. Also for  $[\gamma] \in \pi_1(X, x_0)$ ,  $[\gamma^{-1}]$  also belongs to  $\pi_1(X, x_0)$ .  $[\varepsilon_{x_0}] \in \pi_1(X, x_0)$  and lemma (7.5) and theorem (7.6) imply that  $\pi_1(X, x_0)$  is a group with unit element  $[\varepsilon_{x_0}]$ . This group is written multiplicatively and the unit element  $[\varepsilon_{x_0}]$  will be denoted by 1 when there is no danger of confusion. Summarizing,

**Theorem 7.7:** The set  $\pi_1(X, x_0)$  of homotopy classes of loops in  $X$  based at  $x_0$  is a group with respect to the binary operation defined by (7.1). The unit element of the group is the homotopy class of the constant loop at the base point  $x_0$  and the inverse of  $[\gamma]$  is the homotopy class of the loop  $\gamma^{-1}$ .

**Definition:** The group  $\pi_1(X, x_0)$  is called the fundamental group of the space  $X$  based at  $x_0$ . This group can be non-abelian although we need to do some work to produce an example. Indeed we need to do some work to produce such an example for which  $\pi_1(X, x_0)$  is non-trivial. All we shall do in the rest of this lecture is to show that it is trivial in case  $X$  is a convex subset of  $\mathbb{R}^n$ . First we shall see what happens when the base point is changed.

**Theorem 7.8:** Let  $X$  be a path connected topological space and  $x_1, x_2$  be two arbitrary points of  $X$ . Then  $\pi_1(X, x_1)$  and  $\pi_1(X, x_2)$  are isomorphic.

**Proof:** Let  $\sigma$  be a path joining  $x_1$  and  $x_2$ . Observe that if  $\gamma$  is a loop at the  $x_1$  then  $\sigma * \gamma * \sigma^{-1}$  is a loop at  $x_2$  thereby enabling us to define a map

$$\begin{aligned} h_{[\sigma]} : \pi_1(X, x_1) &\longrightarrow \pi_1(X, x_2) \\ [\gamma] &\mapsto [\sigma * \gamma * \sigma^{-1}]. \end{aligned}$$

Corollary (7.4) shows that the function is well defined and lemma (7.5) shows that it is a group

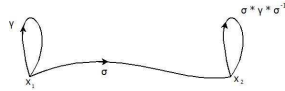


Figure 9: Change of base point

homomorphism. Let  $\Gamma$  be a loop at  $x_2$ . Then  $\sigma^{-1} * \Gamma * \sigma$  is a loop at  $x_1$  and  $h_{[\sigma]}([\sigma^{-1} * \Gamma * \sigma]) = [\Gamma]$  showing that  $h_{[\sigma]}$  is surjective. The map

$$h_{[\sigma^{-1}]} : [\Gamma] \longrightarrow [\sigma^{-1} * \Gamma * \sigma]$$

is the inverse of  $h_{[\sigma]}$ . □

**Remarks:** The isomorphism  $h$  depends on the homotopy class of the path  $\sigma$  joining  $x_1$  and  $x_2$  justifying the notation  $h_{[\sigma]}$ . The reason for the elaborate notation is that it will reappear in lecture 11. The next theorem tells us what happens when we choose various paths from  $x_1$  to  $x_2$ .

**Theorem 7.9:** Suppose  $\gamma'_0, \gamma''_0$  are two paths joining  $x_1$  and  $x_2$  and  $h', h''$  are the corresponding group isomorphisms from  $\pi_1(X, x_1) \longrightarrow \pi_1(X, x_2)$  given by the previous theorem. Then there exists an inner automorphism

$$\sigma : \pi_1(X, x_2) \longrightarrow \pi_1(X, x_2)$$

such that  $h' = \sigma \circ h''$ . In fact  $\sigma$  is the inner automorphism determined by  $[\gamma'_0][\gamma''_0]^{-1}$ .

If  $\pi_1(X, x_0)$  is abelian then  $\pi_1(X, x_0)$  and  $\pi_1(X, x_1)$  are naturally isomorphic. That is the isomorphism  $h_{[\sigma]}$  is canonical in this case.

**Proof:** Using lemma (7.5) we begin by writing

$$h'[\gamma] = [\gamma'_0 * \gamma * (\gamma'_0)^{-1}] = [\gamma'_0 * (\gamma''_0)^{-1} * \gamma''_0 * \gamma * (\gamma''_0)^{-1} * \gamma''_0 * \gamma_0^{-1}].$$

By definition (7.2), the right hand side equals

$$[\gamma'_0][\gamma''_0]^{-1}h''([\gamma])[\gamma''_0][\gamma'_0]^{-1} = (\sigma \circ h'' \circ \sigma^{-1})[\gamma]$$

□

**Definition:** A path connected space  $X$  is said to be simply connected if  $\pi_1(X, x_0) = \{1\}$ ,  $x_0 \in X$ .

**Definition 7.4 (Convex and star-shaped domains):** (i) A subset  $X$  of  $\mathbb{R}^n$  is said to be convex if for every pair of points  $a$  and  $b$  in  $X$ , the line segment  $ta + (1 - t)b$ ,  $0 \leq t \leq 1$  lies entirely in  $X$ .

(ii) A subset  $X$  of  $\mathbb{R}^n$  is said to be star shaped with respect to a point  $x_0$  if for every  $a \in X$ , the line segment  $ta + (1 - t)x_0$ ,  $0 \leq t \leq 1$  lies entirely in  $X$ .

So a convex domain is star shaped with respect to any of its points.

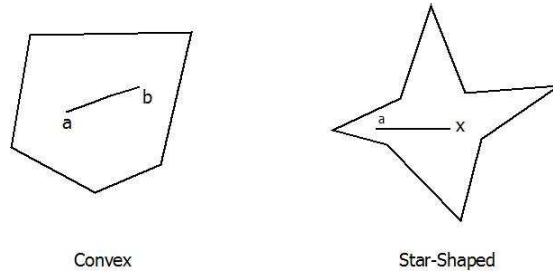


Figure 10: Convex and star-shaped domains

**Theorem 7.10:** If  $X$  is star shaped then  $\pi_1(X, x_0) = \{1\}$ . In particular the fundamental groups of the unit disc and  $\mathbb{R}^n$  are both equal to the trivial group.

**Proof:** By the previous result it is immaterial which point  $x_0$  is chosen as the base point. Assume that  $X$  is star shaped with respect to  $x_0$ . Let  $\gamma : [0, 1] \longrightarrow X$  be a loop in  $X$  based at  $x_0$ . We shall prove  $[\gamma] = [\varepsilon_{x_0}] = 1$  by constructing a homotopy  $F$  between  $\gamma$  and the constant loop  $\varepsilon_{x_0}$ , namely

$$F(s, t) = (1 - s)\gamma(t) + sx_0.$$

This makes sense because  $X$  is star shaped with respect to  $x_0$ . Turning to the boundary conditions,

$$\begin{aligned} F(0, t) &= \gamma(t), \quad F(1, t) = \varepsilon_{x_0} \\ F(s, 0) &= (1 - s)\gamma(0) + sx_0 = (1 - s)x_0 + sx_0 = x_0 \\ F(s, 1) &= (1 - s)\gamma(1) + sx_0 = (1 - s)x_0 + sx_0 = x_0. \end{aligned}$$

## Exercises:

1. Explicitly construct a homotopy between the loop  $\gamma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$  on the sphere  $S^2$  and the constant loop based at  $(1, 0, 0)$ . Note that an explicit formula is being demanded here.
2. Show that a loop in  $X$  based at a point  $x_0 \in X$  may be regarded as a continuous map  $f : S^1 \longrightarrow X$  such that  $f(1) = x_0$ . Show that if  $f$  is homotopic to the constant loop  $\varepsilon_{x_0}$  then  $f$  extends as a continuous map from the closed unit disc to  $X$ .
3. Show that if  $\gamma$  is a path starting at  $x_0$  and  $\gamma^{-1}$  is the inverse path then prove by imitating the proof of the reparametrization theorem (that is by taking convex combination of two functions) that  $\gamma * \gamma^{-1}$  is homotopic to the constant loop  $\varepsilon_{x_0}$ .
4. Prove theorems (7.2) and theorem (7.6) using Tietze's extension theorem.
5. Suppose  $\phi : [0, 1] \longrightarrow [0, 1]$  is a continuous function such that  $\phi(0) = \phi(1) = 0$  and  $\gamma$  is a closed loop in  $X$  based at  $x_0 \in X$ . Is it true that  $\gamma \circ \phi$  is homotopic to the constant loop  $\varepsilon_{x_0}$ ?
6. Show that the group isomorphism in theorem (7.8) is natural namely, if  $f : X \longrightarrow Y$  is continuous and  $x_1, x_2 \in X$  then

$$h_{[f \circ \sigma]} \circ f'_* = h_{[\sigma]} \circ f''_*$$

where,  $y_1 = f(x_1)$ ,  $y_2 = f(x_2)$  and  $\sigma$  is a path joining  $x_1$  and  $x_2$ . The maps  $f'_*$  and  $f''_*$  are the maps induced by  $f$  on the fundamental groups. This information is better described by saying that the following diagram *commutes*:

$$\begin{array}{ccc} \pi_1(X, x_1) & \xrightarrow{f'_*} & \pi_1(Y, y_1) \\ h_{[\sigma]} \downarrow & & h_{[f \circ \sigma]} \downarrow \\ \pi_1(X, x_2) & \xrightarrow{f''_*} & \pi_1(Y, y_2) \end{array}$$