Lecture XVIII - The lifting criterion

We have already discussed the lifting problem and examined its significance in the light of complex analysis. We have seen in connection with the exponential map/squaring map that the existence of a lift of the inclusion map of a domain Ω into $\mathbb{C}-\{0\}$ is equivalent to the existence of a continuous branch of the logarithm/square-root function on Ω. Thus it is desirable to have a necessary and sufficient condition for the existence of lifts. We prove one such theorem in this lecture which provides an elegant necessary and sufficient condition.

Theorem 18.1: Let X and Y be connected locally path connected spaces, $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a covering projection and $f : (Y, y_0) \longrightarrow (X, x_0)$ is a continuous function. A lift $\tilde{f} : Y \longrightarrow \tilde{X}$ satisfying $\tilde{f}(y_0) = \tilde{x}_0$ exists if and only if

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0)). \tag{18.1}$$

In particular, if Y is simply connected, that is if $\pi_1(Y, y_0)$ is trivial, then (18.1) holds and the lift $\tilde{f} : Y \longrightarrow \tilde{X}$ satisfying $\tilde{f}(y_0) = \tilde{x}_0$ exists.

Proof: To prove that the condition (18.1) is necessary, let us assume that a the lift exists. Then $p \circ \tilde{f} = f$ and $p_* \circ \tilde{f}_* = f_*$ whereby,

$$f_*(\pi_1(Y, y_0)) = p_*\Big(\tilde{f}_*(\pi_1(Y, y_0))\Big) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

We now turn to the proof of sufficiency of (18.1). To construct the lift $\tilde{f}$ let $y \in Y$ and γ be a path in Y joining y_0 and y. Take the lift of $f \circ \gamma : [0,1] \longrightarrow X$ starting at $\tilde{x}_0$ and we declare

$$\tilde{f}(y) = \widetilde{f \circ \gamma}(1).$$

To show that the function $\tilde{f}$ is well-defined, take two paths γ_1 and γ_2 joining y_0 and y in Y and form the closed loop $\gamma_1 * \gamma_2^{-1}$ at y_0. Then $f \circ (\gamma_1 * \gamma_2^{-1})$ is a loop in X based at x_0 and so

$$[f \circ (\gamma_1 * \gamma_2^{-1})] \in f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

Choose a loop σ in $\tilde{X}$ based at $\tilde{x}_0$ such that $p_*([\sigma]) = [f \circ (\gamma_1 * \gamma_2^{-1})]$. In other words, the loop $(f \circ \gamma_1) * (f \circ \gamma_2^{-1})$ is homotopic to $p \circ \sigma$. By the covering homotopy lemma, The lift of $(f \circ \gamma_1) * (f \circ \gamma_2^{-1})$ starting at $\tilde{x}_0$ which will be denoted by τ, is homotopic to σ. As a result, τ is also closed loop at $\tilde{x}_0$. Let $\widetilde{f \circ \gamma_1}$ be the lift of $f \circ \gamma_1$ starting at $\tilde{x}_0$ and $\widetilde{f \circ \gamma_2^{-1}}$ be the lift of $f \circ \gamma_2^{-1}$ starting at the terminal point $\widetilde{f \circ \gamma_1}(1)$. Observe that

$$\tau(t) = \begin{cases} \widetilde{f \circ \gamma_1}(2t) & 0 \leq t \leq 1/2 \\ \widetilde{f \circ \gamma_2^{-1}}(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

We now look at the projection of the two paths $\tau(s/2)$ and $\tau\left(\frac{2-s}{2}\right)$ $(0 \leq s \leq 1)$:

$$p \circ \tau(s/2) = f \circ \gamma_1(s), \quad 0 \leq s \leq 1$$

and

$$p \circ \tau\left(\frac{2-s}{2}\right) = f \circ \gamma_2(s), \quad 0 \leq s \leq 1.$$

The paths $\tau(s/2)$ and $\tau\left(\frac{2-s}{2}\right)$ $(0 \leq s \leq 1)$ are thus lifts of $f \circ \gamma_1$ and $f \circ \gamma_2$, both starting at $\tilde{x}_0$ since τ is a closed loop. Hence

$$\widetilde{f \circ \gamma_1}(1) = \tau(1/2) = \widetilde{f \circ \gamma_2}(1)$$

proving that $\tilde{f}(y)$ is well-defined.

Continuity of the lift $\tilde{f}$: Let $y \in Y$ be arbitrary, and let $f(y) = x$ and $\tilde{f}(y) = \tilde{x}$. Choose an evenly covered neighborhood U of x and $\tilde{U}$ be the sheet containing $\tilde{x}$ lying above U. By continuity of f we obtain a neighborhood V of y in Y such that $f(V) \subset U$ and hence $\tilde{f}(V) \subset p^{-1}(U)$ (since $p \circ \tilde{f} = f$).

Now if we assume that $\tilde{f}$ maps the neighborhood V into $\tilde{U}$, then the following would be valid:

$$\tilde{f} = \left(p\Big|_{\tilde{U}}\right)^{-1} \circ f, \tag{18.2}$$

which would prove the continuity of $\tilde{f}$. To prove that $\tilde{f}(V) \subset \tilde{U}$, we shall assume that the neighborhoods U, V and $\tilde{U}$ are path connected and invoke the construction of $\tilde{f}$. Choose a path γ in Y joining y_0 and y and for each $z \in V$ pick a path η joining y and z and then we get the path $\gamma * \eta$ joining y_0 and z. Lift $f \circ \gamma$ and $f \circ \eta$ to paths in $\tilde{X}$ starting at $\tilde{x}_0$ and $\widetilde{f \circ \gamma}(1)$ respectively. Since $f \circ \eta$ lies in U, its lift must lie entirely in $\tilde{U}$ and hence

$$\tilde{f}(z) = f \circ \widetilde{(\gamma * \eta)}(1) = \widetilde{f \circ \eta}(1) \in \tilde{U}.$$

Theorem 18.2 (Uniqueness of simply connected covers): Suppose that $p_1 : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (X, x_0)$ and $p_2 : (\tilde{X}_2, \tilde{x}_2) \longrightarrow (X, x_0)$ are covering projections such that both $\tilde{X}_1$ and $\tilde{X}_2$ are simply connected and locally path connected. Then there is a homeomorphism $\psi : \tilde{X}_1 \longrightarrow \tilde{X}_2$ such that

$$p_2 \circ \psi = p_1. \quad INSERTDIAGRAM$$

Proof: Since $\tilde{X}_1$ is simply connected the map p_1 has a lift $\phi_1 : \tilde{X}_1 \longrightarrow \tilde{X}_2$ with respect to the covering projection $p_2 : \tilde{X}_2 \longrightarrow X$, such that $\phi_1(\tilde{x}_1) = \tilde{x}_2$. Likewise there exists a lift $\phi_2 : \tilde{X}_2 \longrightarrow \tilde{X}_1$ of the map p_2 with respect to the covering $p_1 : \tilde{X}_1 \longrightarrow X$, such that $\phi_2(\tilde{x}_2) = \tilde{x}_1$. From $p_1 \circ \phi_2 = p_2$ and $p_2 \circ \phi_1 = p_1$ follows $p_1 \circ (\phi_2 \circ \phi_1) = p_1$ and $(\phi_2 \circ \phi_1)(\tilde{x}_1) = \tilde{x}_1$. Thus, the identity map on $\tilde{X}_1$ and $\phi_2 \circ \phi_1 : \tilde{X}_1 \longrightarrow \tilde{X}_1$ are both lifts of $p_1 : \tilde{X}_1 \longrightarrow X$ with respect to itself. By uniqueness of lifts we conclude that $\phi_2 \circ \phi_1$ is the identity map on $\tilde{X}_1$. Likewise $\phi_1 \circ \phi_2$ is the identity map on $\tilde{X}_2$. $\square$

Example 18.1 (Some applications to complex analysis): (i) Let Ω be a *simply connected* open subset of $\mathbb{C} - \{0\}$ and $j : \Omega \longrightarrow \mathbb{C} - \{0\}$ be the inclusion and $\exp : \mathbb{C} \longrightarrow \mathbb{C} - \{0\}$ be the exponential map. Then p is a covering projection with respect to which j has a lift $\tilde{j} : \Omega \longrightarrow \mathbb{C}$ which means

$$\exp(\tilde{j}(z)) = z, \quad z \in \Omega \tag{18.3}$$

Thus there is a continuous branch of the logarithm on any simply connected open subset of $\mathbb{C} - \{0\}$. In the exercises the student is asked to show that any continuous lift is holomorphic.

(ii) Consider the map $S : \mathbb{C} - \{0\} \longrightarrow \mathbb{C} - \{0\}$ given by $S(z) = z^2$. Let $\Omega = \mathbb{C} - [0, 1/2]$ and $f : \Omega \longrightarrow \mathbb{C} - \{0\}$ be given by

$$f(z) = z(2z - 1). \tag{16.4}$$

Let us determine the induced map $f_* : \pi_1(\Omega, 1) \longrightarrow \pi_1(\mathbb{C} - \{0\}, 1)$. The group $\pi_1(\Omega, 1)$ is the infinite cyclic group generated by the homotopy class of the loop $\gamma(t) = \exp(2\pi i t)$. Since $\mathbb{C} - \{0\}$ is a topological group under multiplication of complex numbers, we may apply corollary (12.2) to get

$$[f \circ \gamma(t)] = [\gamma(t)] + [2\gamma(t) - 1]. \tag{18.5}$$

The additive notation is used for the infinite cyclic group. The last equation may be rewritten as

$$[f \circ \gamma(t)] = [\gamma(t)] + \Big[\gamma(t)\Big(2 - \frac{1}{\gamma(t)}\Big)\Big] = 2[\gamma(t)] + \Big[2 - \frac{1}{\gamma(t)}\Big] = 2, \tag{18.6}$$

since $|\gamma(t)| = 1$ and the loop $\Big(2 - \frac{1}{\gamma(t)}\Big)$ can be contracted to the constant loop in $\mathbb{C} - \{0\}$. Hence

$$f_*(\pi_1(\mathbb{C} - [0, 1/2], 1) = 2\mathbb{Z} = S_*(\mathbb{C} - \{0\}, 1). \tag{18.7}$$

The lifting criterion holds and f has a unique lift $\tilde{f}$ such that $\tilde{f}(1) = 1$. This lift is the continuous branch of $\sqrt{z(2z-1)}$ defined on Ω. In exercise 3, the student is asked to show that the lift $\tilde{f}$ is holomorphic. Note that the space Ω is not simply connected.

The next example is Picard's theorem which is a corollary of the following highly non-trivial result.

Theorem 16.3: The open unit disc is a covering space for the plane with two points removed.

Theorem 16.4 (The Little Picard Theorem): An entire function that misses two or more points is a constant.

Proof: Suppose an entire function f misses two points p and q. The map $f : \mathbb{C} \longrightarrow \mathbb{C} - \{p, q\}$ lifts to a map $\tilde{f} : \mathbb{C} \longrightarrow \{z \in \mathbb{C}/|z| < 1\}$. As before the lift is holomorphic and hence is an entire function taking its values in the unit disc. By Liouville's theorem, $\tilde{f}$ is constant and so must f.

Exercises:

1. For the map S in example (18.3) show that S_* is the map $\mathbb{Z} \longrightarrow \mathbb{Z}$ given by $x \mapsto 2x$.
2. Suppose G is a path connected topological group with unit element e and $p : \tilde{G} \longrightarrow G$ is a covering map. For any choice of $\tilde{e} \in p^{-1}(e)$ show that there is a group operation on $\tilde{G}$ with unit element $\tilde{e}$ that makes $\tilde{G}$ into a topological group and p is a continuous group homomorphism.
3. Show that if Ω is an open subset of $\mathbb{C} - \{0\}$ on which a continuous branch of the logarithm exists then this branch is automatically holomorphic. Likewise show that the continuous branch of $\sqrt{z(2z-1)}$ on $\mathbb{C} - [0, 1/2]$ obtained in the lecture is holomorphic.
4. Use the fact that S^{n-1} is not a retract of S^n to prove that $\mathbb{R}P^{n-1}$ is not a retract of $\mathbb{R}P^n$.
5. Show that any continuous map $S^n \longrightarrow S^1$ is homotopic to the constant map if $n \geq 2$. What about maps from the projective spaces $\mathbb{R}P^n \longrightarrow S^1$ $(n \geq 2)$?