Theorem 38.2 (Excision):

Let (X, A) be a pair and U be a subset of A such that the closure of U is contained in the interior of A. Then, the homomorphism

$$H_n(i) \;:\; H_n(X-U,\; A-U) \longrightarrow H_n(X, A)$$

induced by inclusion $i : (X-U, A-U) \longrightarrow (X, A)$ is an isomorphism for every $n = 0, 1, 2, \ldots$. In other words the set U may be excised from the pair (X, A) without affecting the homology groups of the pair.

Proof:

The hypothesis implies that the pair $\mathcal{U} = \{\text{int } A,\; X - U\}$ is an open cover of X, where U denotes the closure of U. Likewise $\mathcal{V} = \{\text{ int } A,\; A - U\}$ is an open cover of A and $S^{\mathcal{V}}(A)$ is a subcomplex of $S^{\mathcal{U}}(X)$. By theorem (29.6) the short exact sequence of complexes

$$0 \longrightarrow S^{\mathcal{V}}(A) \longrightarrow S^{\mathcal{U}}(X) \longrightarrow S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A) \longrightarrow 0$$

gives rise to a long exact sequence in homology:

$$\longrightarrow H_n^{\mathcal{V}}(A) \longrightarrow H_n^{\mathcal{U}}(X) \longrightarrow H_n(S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A)) \longrightarrow H_{n-1}^{\mathcal{V}}(A) \longrightarrow$$

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On the other hand there is an obvious map of complexes induced by the inclusion maps namely

$$j\ :\ S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A) \longrightarrow S(X)/S(A),$$

resulting in a commutative diagram of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & S^{\mathcal{V}}(A) \arrow[r] \arrow[d, "i"] & S^{\mathcal{U}}(X) \arrow[r] \arrow[d, "i"] & S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A) \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & S(A) \arrow[r] & S(X) \arrow[r] & S(X)/S(A) \arrow[r] & 0
\end{tikzcd}$$

Since the long exact sequence in homology is natural (exercise 6 of lecture 29), we get the commutative diagram:

$$\begin{tikzcd}
\arrow[r] & H_n^{\mathcal{V}}(A) \arrow[r, "i"] \arrow[d, "i_*"'] & H_n^{\mathcal{U}}(X) \arrow[r, "p"] \arrow[d, "i_*"] & H_n(S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A)) \arrow[r] \arrow[d, "j_*"] & H_{n-1}^{\mathcal{V}}(A) \arrow[r] \arrow[d, "i_*"] & H_{n-1}^{\mathcal{U}}(X) \arrow[r] \arrow[d, "i_*"] & {} \\
\arrow[r] & H_n(A) \arrow[r, "i"] & H_n(X) \arrow[r, "p"] & H_n(S(X)/S(A)) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & {}
\end{tikzcd}$$

where we the subscript star indicates the map induced in homology. The five lemma enables us to conclude that

$$j_*\ :\ H_n(S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A)) \longrightarrow H_n(S(X)/S(A)) = H_n(X, A).$$

is an isomorphism. Note the inclusion

$$k\ :\ S(X - U) \longrightarrow S^{\mathcal{U}}(X)$$

maps $S(A - U)$ into whereby we get an isomorphism (exercise 2)

$\overline{k}\ :\ S(X - U)/S(A - U) = S^{\mathcal{U}}(X)/S^{\mathcal{V}}(A).(38.2)$ The composite $j \circ k$ is also

induced by the inclusion map $(X - U,\ A - U) \longrightarrow (X, A)$

and we have the desired isomorphism

$$(j \circ \overline{k})_* \; : \; H_n(X-U, A-U) \longrightarrow H_n(X, A), \quad n = 0, 1, 2, \ldots$$

Example 39.1:

Letand $X = S^n$ and $A = S^n - \mathbf{e}_{n+1}$. We take U to be the complement of the *polar ice cap* namely
the set of all $A \circ f$ such that $x_{n+1} \leq 2/3$ (reader is invited to draw a picture). Applying the excision theorem,
and denoting the polar ice cap by D,

$$H_n(S^n, S^n - \mathbf{e}_{n+1}) \cong H_n(S^n - U, \; D - \mathbf{e}_{n+1}) = H_n(D, \; D - \mathbf{e}_{n+1}).$$

Theorem (39.2) gives $H_n(S^n, S^n - \mathbf{e}_{n+1}) \cong H_n(S^n)$

and $H_n(D, D - \mathbf{e}_{n+1}) \cong H_{n-1}(D - \mathbf{e}_{n+1})$

Since the polar ice cap is homeomorphic to an open ball, $H_n(S^n) \cong H_{n-1}(S^{n-1}), \quad n \geq 2.$

.

Using theorem (39.1) we conclude that

$$H_n(S^n) \cong \mathbb{Z} \text{ for } n \geq 1.$$

Example 39.2:
In general, given a pair of open sets $\{U, V\}$ of a topological space X, let
$W = (U \cup V) - V$ Then, the closure of W in $U \cap V$ is contained in U and since

$$(X - W, \; U - W) = (V, U \cap V),$$

the excision theorem gives

$$H_n(U \cup V, \; U) \cong H_n(V, \; U \cap V), \quad n = 0, 1, 2, \ldots$$

Lemma 39.3 (Barrett and Whitehead):

Given a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
\longrightarrow & A_n & \xrightarrow{p_n} & B_n & \xrightarrow{q_n} & C_n & \xrightarrow{r_n} & A_{n-1} & \longrightarrow \\
& \alpha_n \downarrow & & \downarrow \beta_n & & \downarrow \gamma_n & & \downarrow \alpha_{n-1} & \\
\longrightarrow & A'_n & \xrightarrow{p'_n} & B'_n & \xrightarrow{q'_n} & C''_n & \xrightarrow{r'_n} & A'_{n-1} & \longrightarrow
\end{array}$$

If each of the maps $\gamma_n : C_n \longrightarrow C''_n$ is an isomorphism, then the sequence

$$\longrightarrow A_n \xrightarrow{\lambda_n} B_n \oplus A'_n \xrightarrow{\mu_n} B'_n \xrightarrow{\delta_n} A_{n-1} \longrightarrow$$

is exact where, the maps are given by

$$\lambda_n = (p_n, -\alpha_n), \quad \mu_n = \beta_n + p'_n, \quad \delta_n = r_n \circ \gamma_n^{-1} \circ q'_n.$$

Corollary 39.4 (Mayer Vietoris): If $\{U, V\}$ are open subsets of a topological space,

$$\kappa' : H_k(U \cap V) \longrightarrow H_k(U), \quad \kappa'' : H_k(U \cap V) \longrightarrow H_k(V)$$

is an exact sequence.

Proof:

The long exact sequence for a pair and its naturality gives a commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
\longrightarrow & H_n(U \cap U) & \longrightarrow & H_n(V) & \longrightarrow & H_n(V,\, U \cap V) & \longrightarrow & H_{n-1}(U \cap V) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_n(U) & \longrightarrow & H_n(U \cup V) & \longrightarrow & H_n(U \cap V,\, U) & \longrightarrow & H_{n-1}(U) & \longrightarrow
\end{array}$$

Applying the excision theorem to the inclusion $(U \cap V,\ V) \longrightarrow (U \cup V,\ U)$, we see that

the third arrow is an isomorphism in the displayed diagram. The result now follows from lemma (38.3).

Definition 39.2 (Local homology groups):

Given a topological space X and $p \in X$, the local homology groups of X at p are the

groups $H_n(X, X - \{p\})\ (n = 0, 1, 2, \dots)$.

Theorem 39.5:

$H_n(X, X-\{p\}) = H_n(V, V-\{p\})$ for any open neighborhood of p.

Proof:

This follows immediately from the excision theorem by taking $U = X - V$ Some applications of the local homology groups are indicated in the exercises below..

Exercises

1. Prove that the map η in the five lemma is surjective.
2. Show that the map (38.2) is indeed an isomorphism. To prove that it is surjective use the decompositions $S^{\mathcal{U}}(X) = S(X-U) + S(\text{ int } A)$ and

   $S^{\mathcal{V}}(A) = S(A-U) + S(\text{ int } A)$
3. Prove the Barrett-Whitehead lemma.
4. Calculate the local homology groups $H_2(X, X-\{p\})$ in the following cases:

   (i)The space X is the cylinder $S^1 \times [0,1]$ and p a point on its boundary.

   (ii)The space X is the Möbius band and p is a point on its boundary.

   Deduce that the cylinder and the Möbius band are not homeomorphic.
5. A topological manifold is a Hausdorff space in which each point has a neighborhood homeomorphic to an open ball in $\mathbb{R}^n$. Show that if p is a point on a topological manifold M,
$$H_n(M,\ M-\{p\}) \cong \mathbb{Z}.$$