Lectures XII - XIII The fundamental group of the circle.

We have already stated the fact that the fundamental group of the circle is the group of integers and derived some important consequences form it. The importance of this result is attested by the fact that the Brouwer's fixed point theorem for a disc follows immediately from it. In this lecture will provide a detailed proof that $\pi_1(S^1, 1) = \mathbb{Z}$. Some of the ideas of the proof would appear again later in a general context of covering spaces. Though this result is a special one from the theory of covering spaces it is worthwhile looking at this important special case without reference to the general theory but rather as a preview to it. This topic will be covered in two lectures but the numbering will be as that of lecture 12. We begin with an algebraic lemma [14] (p. [//]).

Lemma 12.1: Suppose S is a set on which two binary operations $*$ and $*'$ are defined such that

(a) Both $*$ and $*'$ have a common two sided unit.

(b) The binary operations $*$ and $*'$ are mutually distributive. That is,

$$(f_1 * g_1) *' (f_2 * g_2) = (f_1 *' f_2) * (g_1 *' g_2), \quad f_1,\ f_2,\ g_1,\ g_2 \in S.$$

Then,

(i) both $*$ and $*'$ are associative and commutative.

(ii) $f * g = f *' g$ for all $f, g \in S$.

Proof: Denoting the common two sided identity by 1,

$$(f * g) = (f *' 1) * (1 *' g) = (f * 1) *' (1 * g) = f *' g$$

which proves (ii). Next we prove commutativity:

$$g * f = (1 *' g) * (f *' 1) = (1 * f) *' (g * 1) = f *' g = f * g.$$

Finally, using (ii) we prove associativity:

$$(f * g) * h = (f * g) *' (1 * h) = (f *' 1) * (g *' h) = f * (g *' h) = f * (g * h)$$

Corollary 12.2: If X is a topological group with unit element e then $\pi_1(X, e)$ is abelian. Moreover, if $\gamma_1,\ \gamma_2$ are two loops based at e define the binary operation $\circ$ on $\pi_1(X, e)$ by[3]

$$[\gamma_1] \circ [\gamma_2] = [\gamma_1(t) \cdot \gamma_2(t)]$$

where $\gamma_1(t) \cdot \gamma_2(t)$ denotes the group multiplication in X. Then

$$[\gamma_1] \circ [\gamma_2] = [\gamma_1][\gamma_2],$$

the right hand side being the product in $\pi_1(X, e)$. In other words, $\gamma_1(t) \cdot \gamma_2(t) \sim \gamma_1 * \gamma_2$.

Proof: Let $\bar{e}$ denote the homotopy class of the constant loop based at e. We first show that the operation $\circ$ is well defined. If $\gamma_1' \sim \gamma_1''$ and $\gamma_2' \sim \gamma_2''$ via the respective homotopies $F, G : I \times I \longrightarrow X$, it is easily checked that the map $F \cdot G : [0,1] \times [0,1] \longrightarrow X$ given by

$$F \cdot G(s,t) = F(s,t) \cdot G(s,t),$$

the product on the right denoting with group multiplication in X, is a homotopy between $\gamma_1'(t)\gamma_2'(t)$ and $\gamma_1''(t)\gamma_2''(t)$. We conclude that $\circ$ is a well defined binary operation on $\pi_1(X, e)$ with a two sided unit $\bar{e}$. Clearly, $\bar{e}$ is a common two sided unit element for both binary operations on $\pi_1(X, e)$. To invoke the lemma we show that the two binary operations are mutually distributive. Let $\gamma_1', \gamma_2'\gamma_1'', \gamma_2''$ be loops based at e

$$(\ [\gamma_1'][\gamma_1'']\ ) \circ (\ [\gamma_2'][\gamma_2'']\ ) = [\ (\gamma_1' * \gamma_1'')(t) \cdot (\gamma_2' * \gamma_2'')(t)\ ]$$

We first verify through direct calculation that $(\gamma_1' * \gamma_1'') \cdot (\gamma_2' * \gamma_2'') = (\gamma_1' \cdot \gamma_2') * (\gamma_1'' \cdot \gamma_2'')$. Well,

$$\begin{aligned}
(\gamma_1' * \gamma_1'')(t) \cdot (\gamma_2' * \gamma_2'')(t) &= \gamma_1'(2t)\gamma_2'(2t), \qquad\qquad \text{if } 0 \le t \le \frac{1}{2} \\
&= \gamma_1''(2t-1)\gamma_2''(2t-1),\ \text{if } \frac{1}{2} \le t \le 1. \\
\therefore \quad [(\gamma_1' * \gamma_1'')(t) \cdot (\gamma_2' * \gamma_2'')(t)] &= [\gamma_1'(t) \cdot \gamma_2'(t)][\gamma_1''(t) \cdot \gamma_2''(t)]
\end{aligned}$$

So finally

$$([\gamma_1'][\gamma_1'']) \circ ([\gamma_2'][\gamma_2'']) = [\gamma_1'(t)\gamma_2'(t)][\gamma_1''(t)\gamma_2''(t)] = ([\gamma_1'] \circ [\gamma_2'])([\gamma_1''] \circ [\gamma_2''])$$

Thus lemma (12.1) is applicable for the binary operations $*$ and $\circ$ and the proof is complete.

Theorem 12.3: $\pi_1(S^1, 1) = \mathbb{Z}$ and the group is generated by homotopy class of the loop

$$t \mapsto \exp(2\pi i t), \quad 0 \le t \le 1$$

Proof: The proof is broken into several steps. We shall employ the exponential map ex: $\mathbb{R} \longrightarrow S^1$ given by

$$\text{ex}(t) = e^{2\pi i t}. \tag{12.1}$$

The function ex maps $(\frac{-1}{2},\ \frac{1}{2})$ homeomorphically onto $S^1 - \{-1\}$ and we denote its inverse by

$$\text{lg} : S^1 - \{-1\} \longrightarrow (\frac{-1}{2},\ \frac{1}{2}) \tag{12.2}$$

which is also a homeomorphism.

[3]To avoid introducing more notation we are being notationally imprecise. The expression $\gamma_1(t) \cdot \gamma_2(t)$ inside the brackets refers to the map $t \mapsto \gamma_1(t) \cdot \gamma_2(t)$.

Lemma 12.4 (The lifting lemma): Let X be a compact subset of $\mathbb{R}^n$ that is star shaped with respect to origin. Let $f : X \longrightarrow S^1$ be a continuous function such that $f(0) = \mathrm{ex}(t_0)$ for some $t_0 \in \mathbb{R}$. Then, there exists a continuous function $\tilde{f} : X \longrightarrow \mathbb{R}$ such that

$$\mathrm{ex}\tilde{f}(x) = f(x), \quad \tilde{f}(0) = t_0 \tag{12.3}$$

Moreover the function $\tilde{f}$ satisfying (12.3) is unique and is called the lift of f with respect to ex.

Proof: Invoking the uniform continuity of f with $\varepsilon = 2$, there exists $\delta > 0$ such that

$$\|x - y\| < \delta \Rightarrow |f(x) - f(y)| < 2$$

which in turn implies that $f(x) \neq -f(y)$. Now choose $n \in \mathbb{N}$ such that $n^{-1}\|x\| < \delta$ for all $x \in X$ which is possible since X is compact. This n is now fixed for the rest of the discussion. For $x \in X$ and $j = 0, 1, \ldots, n-1$

$$\Big\| \frac{j}{n}x - \frac{(j+1)}{n}x \Big\| < \delta,$$

whereby,

$$f(\frac{j+1}{n}x) \neq -f(\frac{j}{n}x).$$

From this we conclude that the function given by

$$\mathrm{lg}\Big(\frac{f(\frac{j+1}{n}x)}{f(\frac{j}{n}x)}\Big), \quad x \in X$$

is continuous with respect to x. We now claim that

$$\tilde{f}(x) = t_0 + \sum_{j=0}^{n-1} \mathrm{lg}\Big(\frac{f(\frac{j+1}{n}x)}{f(\frac{j}{n}x)}\Big)$$

is the required continuous function. Observe that $\mathrm{lg}(1) = 0$, $\tilde{f}(0) = t_0$ and

$$\mathrm{ex}\tilde{f}(x) = (\mathrm{ex}(t_0)) \cdot \frac{f(\frac{1}{n}x)}{f(0)} \cdot \frac{f(\frac{2}{n}x)}{f(\frac{1}{n}x)} \cdots \frac{f(\frac{n}{n}x)}{f(\frac{n-1}{n}x)} = f(x).$$

Turning to the proof of uniqueness of the lift $\tilde{f}$, suppose $\tilde{f}_1, \tilde{f}_2 : X \longrightarrow \mathbb{R}$ are two continuous functions such that $\tilde{f}_1(0) = \tilde{f}_2(0) = t_0$ and $\mathrm{ex}\tilde{f}_1(x) = \mathrm{ex}\tilde{f}_2(x) = f(x)$. Then $\mathrm{ex}(\tilde{f}_1(x) - \tilde{f}_2(x)) = 1$, which implies $\tilde{f}_1(x) - \tilde{f}_2(x) \in \mathbb{Z}$ (see note below). Since both functions are continuous, agree at the origin and X is connected, we conclude that

$$\tilde{f}_1(x) \equiv \tilde{f}_2(x).$$

□

Note: The properties of the exponential function used here must be established using power series expansions. Specifically prove using power series the following:

(i) $\mathrm{ex}(z_1 + z_2) = \mathrm{ex}(z_1) \cdot \mathrm{ex}(z_2)$

(ii) There exists a unique positive real root of $\cos(x) = 0$ in $[0, 2]$ (via the real power series for the cosine function) and we denote this root by $\pi/2$.

(iii) $\cos(2\pi + x) = \cos x$, $\sin(2\pi + x) = \sin x$ (using addition formula for sin and cos following (i))

(iv) If $\cos x = \cos y$, $\sin x = \sin y$ then there exists $k \in \mathbb{Z}$ such that $x - y = 2\pi i k$.

Definition 12.1: Let $\gamma : [0,1] \longrightarrow S^1$ be a loop based at 1. By the lifting lemma there exists unique lift $\tilde{\gamma} : [0,1] \longrightarrow \mathbb{R}$ such that $\tilde{\gamma}(0) = 0$, $\mathrm{ex}\tilde{\gamma}(1) = 1$. Thus, $\tilde{\gamma}(1) \in \mathbb{Z}$ and we call this integer the degree of the loop γ.

Lemma 12.5: If γ_1 and γ_2 are two homotopic loops based at 1. then $\deg\gamma_1 = \deg\gamma_2$. Thus the map $\phi : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}$ given by $[\gamma] \mapsto \deg \gamma$ is well-defined.

Proof: Let $F : I \times I \longrightarrow S^1$ be the homotopy between γ_1 and γ_2. Since $I \times I$ is star shaped with respect to $(0,0)$ and $F(0,0) = 1 = \mathrm{ex}(0)$, the lifting lemma gives a unique lift $\tilde{F} : I \times I \longrightarrow \mathbb{R}$ with $\tilde{F}(0,0) = 0$. The image $F(s,0)$ is a connected subset of $\mathbb{R}$ as s runs from 0 to 1 and $\exp\tilde{F}(s,0) = F(s,0) = 1$ for all $s \in [0,1]$. So $\tilde{F}(s,0)$ is integer valued and hence constant. From $\tilde{F}(0,0) = 0$ we conclude that $\tilde{F}(s,0) = 0$ for all $s \in [0,1]$. In particular the lifts $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ both start at the origin and so

$$\deg \gamma_1 = \tilde{\gamma}_1(1), \quad \deg \gamma_2 = \tilde{\gamma}_2(1).$$

Our job will be over if we show that $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$. Well, $\tilde{F}$ must map the connected set $\{(s,1)/0 \leq s \leq 1\}$ onto a connected subset J of $\mathbb{R}$ and since

$$\mathrm{ex}\tilde{F}(s,1) = F(s,1) = 1,$$

this connected subset J must be a subset of $\mathbb{Z}$ and hence reduces to a singleton which means

$$\tilde{F}(s,1) = \tilde{F}(0,1), \text{ for all } s \in [0,1]$$

Setting $s = 0$ and 1 we see that

$$\tilde{\gamma}_2(1) = \tilde{F}(1,1) = \tilde{F}(0,1) = \tilde{\gamma}_1(1),$$

thereby completing the proof that the map $\phi : [\gamma] \mapsto \deg\gamma$ is well defined.

Lemma 12.6: The map ϕ defined in lemma (12.5) is a group isomorphism.

Proof: Suppose γ_1 and γ_2 are two loops at 1 with lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$ starting at origin. Then the path $\tilde{\gamma}$ given by $\tilde{\gamma}(t) = \tilde{\gamma}_1(t) + \tilde{\gamma}_2(t)$ also starts at the origin and satisfies

$$\mathrm{ex}\,\tilde{\gamma}(t) = \mathrm{ex}\,\tilde{\gamma}_1(t) \cdot \mathrm{ex}\,\tilde{\gamma}_2(t) = \gamma_1(t) \cdot \gamma_2(t).$$

Hence $\tilde{\gamma}$ is the unique lift of $\gamma_1(t) \cdot \gamma_2(t)$ whereby,

$$\deg(\gamma_1(t)\gamma_2(t)) = \tilde{\gamma}(1) = \tilde{\gamma}_1(1) + \tilde{\gamma}_2(1) = \deg \gamma_1 + \deg \gamma_2.$$

Thus $\phi([\gamma_1 \cdot \gamma_2]) = \phi([\gamma_1]) + \phi([\gamma_2])$. From corollary (12.2), $[\gamma_1 \cdot \gamma_2] = [\gamma_1 * \gamma_2] = [\gamma_1][\gamma_2]$ whence $\phi([\gamma_1][\gamma_2]) = \phi([\gamma_1]) + \phi([\gamma_2])$ which means that ϕ is a group homomorphism.

Surjectivity of ϕ is easy to see. Let $n \in \mathbb{Z}$ be arbitrary and $\tilde{\gamma}(t) = nt$. Then $\tilde{\gamma}$ is the unique lift of $\gamma(t) = \mathrm{ex}\tilde{\gamma}(t)$ starting at the origin so that $\phi([\gamma]) = \tilde{\gamma}(1) = n$. We now show that the group homomorphism ϕ is injective. Suppose γ_1, γ_2 are two loops at 1 in S^1 such that $\deg \gamma_1 = \deg \gamma_2$. Then $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$, where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are the lifts of γ_1 and γ_2 starting at the origin. Since $\mathbb{R}$ is convex and the two curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have common end points, they are homotopic. That is to say, there exists a continuous function $\tilde{F} : I \times I \longrightarrow \mathbb{R}$ such that

$$\begin{aligned}\tilde{F}(0,t) &= \tilde{\gamma}_1(t), \quad \tilde{F}(1,t) = \tilde{\gamma}_2(t); \text{ for all } t \in [0,1] \\ \tilde{F}(s,0) &= 0, \quad \tilde{F}(s,1) = \tilde{\gamma}_1(1) = \tilde{\gamma}_2(1), \text{ for all } s \in [0,1].\end{aligned}$$

The function $F : [0,1] \times [0,1]$ given by

$$F(s,t) = \mathrm{ex}\tilde{F}(s,t)$$

is then a homotopy between γ_1 and γ_2 and we have shown that $\deg \gamma_1 = \deg \gamma_2$ implies $[\gamma_1] = [\gamma_2]$. This suffices for a proof.

Corollary 12.7 (Generators for $\pi_1(S^1, 1)$): (1) The generators for $\pi_1(S^1, 1)$ are given by the loops

$$\eta : t \mapsto \exp(\pm 2\pi i t) \tag{12.4}$$

(2) The loops (12.4) also generate the group $\pi_1(\mathbb{C} - \{0\}, 1)$.

Proof: The lifts of these starting at the origin are ± 1 so that these loops have degrees ± 1 respectively. The second conclusion follows from the fact that a deformation retraction induces an isomorphism of fundamental groups. $\square$

Definition 12.2 (Degree of a map): Suppose that $f : S^1 \longrightarrow S^1$ is a continuous map such that $f(1) = 1$, the degree of f is defined to be the degree of the loop

$$f \circ \eta : t \mapsto f(\exp(\pm 2\pi i t)), \quad 0 \leq t \leq 1. \tag{12.5}$$

Theorem 12.8: For a continuous map $f : S^1 \longrightarrow S^1$ with $f(1) = 1$, the degree satisfies the equation

$$f_*[\eta] = (\deg f)[\eta] \tag{12.6}$$

where the group operation on $\pi_1(S^1, 1)$ is viewed additively.

Proof: Since $[\eta]$ generates $\pi_1(S^1, 1)$, writing the group operation additively, we have

$$f_*[\eta] = c[\eta] \tag{12.6}$$

We have to show that $c = \deg f$. By definition, $f_*[\eta] = [f \circ \eta]$ which is mapped to $\deg f$ by the isomorphism ϕ of lemma (12.5). But this isomorphism maps $[\eta]$ to 1 and hence applying ϕ to (12.6) we get the result. $\square$

Theorem 12.9 (The Borsuk Ulam Theorem): Suppose $f : S^n \longrightarrow \mathbb{R}^n$ is a continuous map. Then there exists a pair of antipodal points $x, -x$ such that $f(x) = f(-x)$

Proof for the case $n = 2$: We follow the elegant proof given in [17] (p. 109). We first show that any continuous function $g : E^2 \longrightarrow S^1$ maps a pair of antipodal points on the *boundary* of E^2 to the same point. That is there exists $z \in E^2$ such that $|z| = 1$ and $g(z) = g(-z)$. Since E^2 is a compact convex set, by lemma (12.4) we see that any continuous map $g : E^2 \longrightarrow S^1$ has a continuous lift $\tilde{g} : E^2 \longrightarrow \mathbb{R}$. Since the real valued map

$$\theta \mapsto \tilde{g}(e^{2\pi i\theta}) - \tilde{g}(e^{-2\pi i\theta}), \quad 0 \leq \theta \leq 1,$$

changes sign we see that there is a pair of antipodal points $z, -z \in S^1$ such that $\tilde{g}(z) = \tilde{g}(-z)$ and hence $g(z) = g(-z)$. Turning now to a continuous map $f : S^2 \longrightarrow \mathbb{R}^2$, assume $f(\mathbf{x}) \neq f(-\mathbf{x})$ for every $\mathbf{x} \in S^2$. We construct the continuous function $g : E^2 \longrightarrow S^1$

$$g(z) = h(z)/|h(z)|$$

where

$$h(x_1, x_2) = f(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}) - f(-x_1, -x_2, -\sqrt{1 - x_1^2 - x_2^2}), \quad (x_1, x_2) \in E^2.$$

Since $|h(z)| = |h(-z)|$, we infer that there is no $z \in E^2$ satisfying $|z| = 1$ and $g(z) = g(-z)$ resulting in a contradiction.

Corollary 12.10: S^2 is not homeomorphic to any subset of $\mathbb{R}^2$

Proof: The Borsuk Ulam theorem shows that a continuous function $S^2 \longrightarrow \mathbb{R}^2$ cannot be injective.

Theorem 12.11 (Fundamental theorem of algebra): Every non-constant polynomial with complex coefficients has a complex root.

Proof: If the polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ has no zeros, then in particular, $p(1) \neq 0$. For $t \neq 0$, we define

$$p(z/t)t^n = \Big(z^n + a_1 z^{n-1} t + \cdots + a_n t^n\Big).$$

The right hand side makes sense even when $t = 0$ and we denote the right hand side by $g(z, t)$. Observe that $g(z, 0) = z^n$ and $g(z, 1) = p(z)$. However we need a homotopy of maps of S^1 preserving the base point 1. To this end we modify it consider instead the map $F : S^1 \times [0, 1] \longrightarrow S^1$ given by

$$F(z, t) = \frac{g(z, t)}{|g(z, t)|} \frac{|g(1, t)|}{g(1, t)}. \tag{12.7}$$

Clearly $g(z, 0) \neq 0$ for any $z \in S^1$ and if $0 < t \leq 1$ then again $g(z, t) = p(z/t)t^n \neq 0$. Thus (12.7) is a base point preserving homotopy between the function $f : S^1 \longrightarrow S^1$ given by

$$f(z) = \frac{p(z)}{|p(z)|} \frac{|p(1)|}{p(1)} \tag{12.8}$$

and the map $z \mapsto z^n$. We conclude that degree of f is n. However we have a base point preserving homotopy between (12.8) and the constant map namely, $G : S^1 \times [0, 1] \longrightarrow S^1$ given by

$$G(z, s) = \frac{p(sz)}{|p(sz)|} \frac{|p(s)|}{p(s)}.$$

We now conclude that degree of (12.8) is zero and we have a contradiction.

Exercises:

1. Formulate and prove the Borsuk Ulam theorem for continuous maps from S^1 to the real line.

2. Use the Borsuk Ulam theorem to prove that a pair of homogeneous polynomials of odd degree in three real variables have a common non-trivial zero.

3. For the following three maps $f : S^1 \longrightarrow S^1$ compute the induced map $f_* : \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$. All three maps preserve the base point 1.

   (i) $f(z) = z^n$

   (ii) $f(z) = \bar{z}$.

   (iii) $f(z) = \frac{z^2 - z + \frac{3}{2}}{|z^2 - z + \frac{3}{2}|}$. Hint: Is $(z^2 - z)t + 3/2 = 0$ for any $z \in S^1$ and $0 \leq t \leq 1$?

4. Let X be the union of the sphere S^2 and one of its diameters. Use exercise 1 of lecture 8 to determine a generator for $\pi_1(X, x_0)$, where x_0 is a point on the sphere.

5. Determine the generators of the group $\pi_1(S^1 \times S^1, (1, 1))$. Determine the generators for the fundamental group of the space X of example 11.3.

6. Compute $f_* : \pi_1(\mathbb{C} - \{0\}, 1) \longrightarrow \pi_1(\mathbb{C} - \{0\}, 1)$ for the function $f(z) = z^k$.