# Lecture - XXXV The Mayer Vietoris sequence and its applications

The proof of Mayer Vietoris sequence is reminiscent of the Seifert Van Kampen theorem. While the Seifert Van Kampen theorem enables us to relate the fundamental group of a union $U \cup V$ in terms of the fundamental groups of $U, V$ and $U \cap V$, the situation here is slightly more involved. The precise relationship between the homologies of $U, V, U \cap V$ and $U \cup V$ is described in terms of the long exact sequence of theorem (34.7).

As in the Seifert Van Kampen theorem we obtain from a push-out diagram of topological spaces a push out diagram of chain complexes which turns into a short exact sequence of complexes. The corresponding long-exact sequence gives, after an application of the excision theorem of the last lecture, the Mayer Vietoris sequence. It is one of the most efficient tools available for the computation of homology groups. We restate here the theorem for convenience.

**Theorem 35.1:** Suppose $U$ and $V$ are subsets of a topological space such that $\text{Int } U \cup \text{Int } V = X$. Then there is a long exact sequence

$$\longrightarrow H_n(U \cap V) \xrightarrow{(\kappa', -\kappa'')} H_n(U) \oplus H_n(V) \xrightarrow{q_n} H_n(U \cup V) \xrightarrow{\delta_n} H_{n-1}(U \cap V) \longrightarrow$$

**Interpretation of the connecting homomorphism:** We use equation (29.18) to describe explicitly the connecting homomorphism in the Mayer Vietoris sequence. Take a representative cycle $\zeta$ in $H_n(U \cup V)$. Theorem (34.6) implies that an arbitrary element of $H_n(U \cup V)$ can be represented as a sum of chains

$$\zeta = \zeta_1 + \zeta_2$$

where $\zeta_1 \in S_n(U)$ and $\zeta_2 \in S_n(V)$. Note that we are resorting to an abuse notation in writing $\zeta_1$ instead of $i_\sharp(\zeta_1)$. We conclude that $\partial\zeta_1 = -\partial\zeta_2$. Thus $\partial\zeta_1$ and $\partial\zeta_2$ are both cycles in $U \cap V$. According to (29.18), the homomorphism $\delta_n$ is given by

$$\delta_n(\overline{\zeta}) = \overline{\partial\zeta_1}$$

**Corollary 35.2:** The homology groups of the spheres $S^n$ $(n \geq 1)$ are given by

$$H_m(S^n) = \begin{cases} 0 & \text{if } m \neq 0,\ m \neq n \\ \mathbb{Z} & \text{if } m = 0,\ m = n \end{cases}$$

**Proof:** We take $U = S^n - \{\mathbf{e}_{n+1}\}$ and $V = S^n - \{-\mathbf{e}_{n+1}\}$ and note that $U \cap V$ deformation retracts to $S^{n-1}$. Consider the portion of the Mayer Vietoris sequence

$$\longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(U \cup V) \xrightarrow{\delta_n} H_{n-1}(U \cap V) \longrightarrow H_{n-1}(U) \oplus H_{n-1}(V) \longrightarrow$$

Since $U$ and $V$ are contractible spaces, we get for the case $n \geq 2$,

$$0 \longrightarrow H_n(U \cup V) \xrightarrow{\delta_n} H_{n-1}(U \cap V) \longrightarrow 0,$$

and hence $H_n(S^n) \cong H_{n-1}(S^{n-1})$ $(n \geq 2)$. By induction the result would follow as soon as we prove it for the case $n = 1$. For this case let us take a look at the end of the Mayer Vietoris sequence:

$$0 \longrightarrow H_1(S^1) \xrightarrow{\delta_1} H_0(U \cap V) \xrightarrow{(\kappa', -\kappa'')} H_0(U) \oplus H_0(V) \longrightarrow H_0(U \cup V) \longrightarrow$$

Since $\delta_1$ is injective,

$$H_1(S^1) \cong \text{im } \delta_1 = \ker\,(\kappa', -\kappa'').$$

To understand the map $(\kappa', -\kappa'')$ we take a basis of $H_0(U \cap V)$ consisting of a pair of points $a \in U$ and $b \in V$. The singleton $\{a\}$ generates $H_0(U)$ and

$$k'(ma + nb) = ma + nb = n(b - a) + (m + n)a$$

which is a boundary in $H_0(U)$ if and only if $m + n = 0$. Likewise $k''(ma + nb) = 0$ in $H_0(V)$ if and only if $m + n = 0$. Thus the kernel of $(\kappa', \kappa'')$ is the infinite cyclic group generated by the zero chain $a - b$. Hence we get

$$H_1(S^1) \cong \mathbb{Z}.$$

To calculate $H_n(S^1)$ for $n \geq 2$ we look at the portion of the Mayer Vietoris sequence

$$\longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(U \cup V) \xrightarrow{\delta_n} H_{n-1}(U \cap V) \longrightarrow$$

and observe that since $H_n(U) = H_n(V) = H_{n-1}(U \cap V) = 0$ when $n \geq 2$,

$$H_n(S^1) = \{0\}, \quad n \geq 2.$$

**Corollary 35.3:** For $m, n \in \mathbb{N}$ with $m < n$, the spheres $S^m$ and $S^n$ are non-homeomorphic. Also $\mathbb{R}^m$ and $\mathbb{R}^n$ are non-homeomorphic.

**Proof:** The first part follows from the fact that the homology groups $H_n(S^m)$ and $H_n(S^n)$ are non-isomorphic. If $\mathbb{R}^m$ and $\mathbb{R}^m$ were homeomorphic then their one-point compactifications would also be homeomorphic which means $S^n$ and $S^m$ would be homeomorphic leading to a contradiction.

**Homology groups of adjunction spaces:** We shall now consider the space $Y = X \sqcup_f E^k$ obtained by attaching a $k-$cell $E^n$ to $X$ via an attaching map

$$f : S^{k-1} \longrightarrow X.$$

We shall closely follow the method used in lecture 26 to compute the fundamental groups of the projective plane and Klein's bottle. We do not have to keep track of base points and use the Mayer Vietoris sequence instead of the Seifert Van Kampen theorem. We shall use the same notations and denote by $p$ the center of $E^k$, the interior of $E^k$ by $U$ and the space $Y - \{p\}$ by $V$. The space $U \cap V$ deformation retracts to a space homeomorphic to $S^{k-1}$. Since $V$ deformation retracts to $X$, the spaces $V$ and $X$ have the same homology groups and $H_n(U) = \{0\}$ when $n \geq 1$. We are ready to prove the following result:

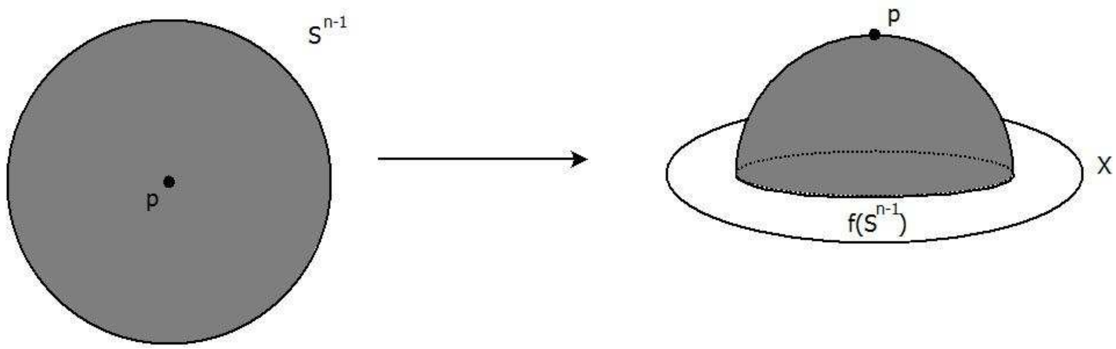


Figure 26: Adjucntion space

**Theorem 35.4** $H_n(X \sqcup_f E^k) = H_n(X)$ if $n \neq k, k-1$.

**Proof:** Looking at the portion of the Mayer Vietoris sequence

$$\longrightarrow H_n(U \cap V) \longrightarrow H_n(U) \oplus H_n(V) \longrightarrow H_n(Y) \longrightarrow H_{n-1}(U \cap V) \longrightarrow$$

we get the result directly when $n \geq 2$. If $n = 1$ then necessarily $k \geq 3$ and we look at the portion of the Mayer Vietoris sequence

$$\longrightarrow H_1(U \cap V) \longrightarrow H_1(U) \oplus H_1(V) \longrightarrow H_1(Y) \longrightarrow H_0(U \cap V) \xrightarrow{\cong} H_0(U) \oplus H_0(V).$$

Observe that $H_1(U \cap V) = \{0\}$ and we get the exact sequence

$$0 \longrightarrow H_1(V) \longrightarrow H_1(Y) \longrightarrow 0,$$

establishing the result when $n = 1$. $\square$

The cases $n = k, k-1$ are more technical and we shall merely state the relevant results.

**Theorem 35.5:** With notations as in theorem (35.4),

$$H_{k-1}(X \sqcup_f E^k) = H_{k-1}(X)/\text{im } H_{k-1}(f), \quad H_k(X \sqcup_f E^k) = H_k(X) \oplus \ker H_{k-1}(f)$$

**Corollary 35.6 (Homology groups of $\mathbb{R}P^2$):** $H_0(\mathbb{R}P^2) = \mathbb{Z}$, $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$. All other homology groups vanish.

**Proof:** Recall example (25.4) that $\mathbb{R}P^2$ arises from $S^1$ by attaching a two cell using the attaching map $f : S^1 \longrightarrow S^1$ given by $f(z) = z^2$. Since $H_1(f) : \mathbb{Z} \longrightarrow \mathbb{Z}$ is given by $n \mapsto 2n$ the result immediately follows from theorems (35.4)-(35.5).

# Exercises

1. Prove that a homeomorphism $E^n$ onto itself maps each boundary point of $E^n$ to a boundary point.

2. Determine the homology groups of the Klein's bottle.

3. Determine the homology groups of the double torus.
4. Establish the isomorphism $H_0(U \cap V) \longrightarrow H_0(U) \oplus H_0(V)$ in the proof of theorem (35.4)
5. Let $C_k$ be the disjoint union of $k$ copies of $S^1$ in $\mathbb{R}^3$. Determine the homology groups of the complement $\mathbb{R}^3 - C_k$.
6. Determine the homology groups of $\mathbb{R}P^3$. Try computing the homology groups of $\mathbb{R}P^4$.
7. Determine the homology groups of $S^n \vee S^m$. Use exercise 4 of lecture 25. to calculate the homology groups of $S^2 \times S^4$.