Lectures - XXIII and XXIV Coproducts and Pushouts

We now discuss further categorical constructions that are essential for the formulation of the Seifert Van Kampen theorem. We first discuss the notion of coproducts which is a prerequisite for a proof of the existence of push-outs. The coproduct is popularly known as the free product in the context of groups but we shall also use the term coproduct which seems more appropriate from a categorical point of view ([11], p. 71). The notion of coproducts has already been introduced in the exercises to lecture 7 for the categories **Top** and **AbGr** where it is popularly known as the disjoint union and the direct sum respectively. However the construction is more complicated in the category **Gr**. The coproduct is defined in terms of a universal property.

Definition 23.1: Given two groups G_1 and G_2, their coproduct is a group G together with a pair of group homomorphisms $i_1 : G_1 \longrightarrow G$ and $i_2 : G_2 \longrightarrow G$ such that given any group H and group homomorphisms $f_1 : G_1 \longrightarrow H$ and $f_2 : G_2 \longrightarrow H$ there exists a **unique** homomorphism $\phi : G \longrightarrow H$ such that

$$\phi \circ i_1 = f_1, \quad \phi \circ i_2 = f_2 \tag{23.1}$$

summarized in the following diagram ($k = 1, 2.$):

$$\begin{tikzcd} G_k \arrow[rr, "i_k"] \arrow[dr, "f_k"'] & & G \arrow[dl, "\phi"] \\ & H & \end{tikzcd}$$

The definition immediately generalizes to any arbitrary (not necessarily finite) collection of groups. The uniqueness clause in the definition is important and the following theorem hinges upon it.

Theorem 23.1: If the coproduct (free product) exists then it is unique upto isomorphism. Denoting the coproduct by $G_1 * G_2$, the maps i_1 and i_2 are injective and so G_1 and G_2 may be regarded as subgroups of $G_1 * G_2$.

Proof: To establish uniqueness, suppose that G' is another candidate for the coproduct with the associated homomorphisms $j_1 : G_1 \longrightarrow G'$ and $j_2 : G_2 \longrightarrow G'$ satisfying the universal property. Taking $f_1 = j_1$ and $f_2 = j_2$ in the definition, there exists a homomorphism $\phi : G \longrightarrow G'$ such that

$$\phi \circ i_1 = j_1, \quad \phi \circ i_2 = j_2.$$

But since G' is also a coproduct we obtain reciprocally a group homomorphism $\psi : G' \longrightarrow G$ such that

$$\psi \circ j_1 = i_1, \quad \psi \circ j_2 = i_2.$$

Combining the two we get $(\psi \circ \phi) \circ i_1 = i_1$ and $(\psi \circ \phi) \circ i_2 = i_2$. We see that the identity map id_G as well as $\psi \circ \phi$ satisfy the universal property with $H = G$, $f_1 = i_1$ and $f_2 = i_2$. The uniqueness clause in the definition of the coproduct gives $\psi \circ \phi = \mathrm{id}_G$ Interchanging the roles of G and G' we get $\phi \circ \psi = \mathrm{id}_{G'}$. We leave it to the student to show that the maps i_1 and i_2 are injective.

Theorem 23.2: Coproducts exist in the category **Gr**.

Proof: We shall merely provide a sketch of the argument. Let G_1 and G_2 be two given groups. A word is by definition a finite sequence $(x_1, x_2, \ldots x_n)$ such that each x_i $(i = 1, 2 \ldots, n)$ belongs to one of the groups, no pair of adjacent terms of the sequence belong to the same group and none of the x_i is the identity element of either of the groups. We call the integer n the length of the word and also include the empty word of length zero. Denoting by W is the set of all words, the idea is to define a binary operation of *juxtaposition* of words. The empty word would serve as the identity and the inverse of a word $(x_1, x_2, \ldots, x_n)$ would be the word $(x_n^{-1}, x_{n-1}^{-1}, \ldots, x_1^{-1})$. One would hope that the operation of juxtaposition would make W a group. This however would not quite suffice. The juxtaposition of two words $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_m)$ may result in a sequence that does not qualify to be called a word for the simple reason that x_n and y_1 may belong to the same group. When this happens we may try to replace the juxtaposed string by the smaller string

$$(x_1, x_2, \ldots, x_{n-1}, z, y_2, \ldots, y_m)$$

where $z = x_n y_1$. If z is not the unit element we do get a legitimate word but if z is the unit element of one of the groups we must drop it altogether obtaining instead the still smaller string

$$(x_1, x_2, \ldots, x_{n-1}, y_2, \ldots, y_m)$$

If x_{n-1} and y_2 belong to the same group the above process must continue and thus in finitely many steps we obtain a legitimate word that ought to be the product of the two given words. To check that we do get a group that qualifies as the coproduct of the given groups can be tedious. The reader may consult [11], pp 72-73.

We now introduce the notion of a direct sum of abelian groups which will play a crucial role in the second part of the course.

Definition 23.2 (Coproduct of abelian groups or the direct sum): Given a family of abelian groups $\{G_\alpha \ / \ \alpha \in \Lambda\}$, their coproduct or direct sum is an abelian group G together with a family of group homomorphisms $\{\iota_\alpha : G_\alpha \longrightarrow G \ / \ \alpha \in \Lambda\}$ such that the following universal property holds.

Given any abelian group A and a family of group homomorphisms $f_\alpha : G_\alpha \longrightarrow A$, there exists a *unique* group homomorphism $\phi : G \longrightarrow A$ such that each of the diagrams commutes:

$$\begin{tikzcd} G_\alpha \arrow[rr, "\iota_\alpha"] \arrow[dr, "f_\alpha"'] & & G \arrow[dl, "\phi"] \\ & A & \end{tikzcd}$$

Theorem 23.3: Coproducts exist in the category **AbGr** and it is unique.

Proof: We use the additive notation and shall use the same symbol 0 to denote the identity element of all the groups. The cartesian product $\prod G_\alpha$ is a group with respect to component-wise addition and we consider the subgroup $\bigoplus G_\alpha$ given by

$$\bigoplus_{\alpha\in\Lambda} G_\alpha = \Big\{ (x_\alpha)_\alpha \in \prod_{\alpha\in\Lambda} G_\alpha \ / \ x_\alpha = 0 \text{ for all but finitely many indices } \alpha \Big\}.$$

For each $\beta \in \Lambda$ we define the standard inclusion map

$$\iota_\beta : G_\beta \longrightarrow \bigoplus_{\alpha\in\Lambda} G_\alpha$$

such that $\iota_\beta(x)$ has entry x in position β and all other coordinates are zero. We leave it to the reader to check that the group $\bigoplus_\alpha G_\alpha$ together with the family $\{\iota_\alpha : G_\alpha \longrightarrow G \ / \ \alpha \in \Lambda\}$ satisfies all the requirements.

Definition 23.3 (free groups): The coproduct in the category **Gr** (known as the free product) of k copies of $\mathbb{Z}$ is called the free group on k generators.

We shall denote a free group on k generators by F_k or if there is a need to specify the generators $a_1, a_2, \ldots, a_k$ we shall use the notation $F[a_1, a_2, \ldots, a_k]$.

Theorem 23.4: Any group H having k generators is a homomorphic image of F_k.

Proof: Let H be generated by $x_1, x_2, \ldots, x_k$ and for each $j = 1, 2, \ldots, k$ let G_j be the infinite cyclic group with generator a_j, regarded as a subgroup of F_k. Applying the definition of the coproduct to the collection of group homomorphisms $f_j : G_j \longrightarrow H$ defined by

$$f_j(a_j) = x_j, \quad j = 1, 2 \ldots, k,$$

we get a group homomorphism $\phi : F_k \longrightarrow H$ such that $\phi(a_j) = f_j(a_j) = x_j$. It is clear that ϕ is surjective and the proof is complete.

Generators and relations: Denoting by B the set of generators $a_1, a_2, \ldots, a_k$ of F_k, any collection S of words

$$a_{i_1}^{n_1} a_{i_2}^{n_2} \ldots, a_{i_p}^{n_p}, \quad a_{i_j} \in B,\ n_j \in \mathbb{Z},\ 1 \leq j \leq p. \tag{23.2}$$

gives rise to a group $F_k/\langle S\rangle$ where $\langle S\rangle$ denotes the normal subgroup generated by S. Conversely, let H be a finitely generated group and ϕ be as in the theorem. We take a set R of words (23.2) generating the kernel of ϕ and write

$$H = F_k/\langle R\rangle. \tag{23.3}$$

The elements of R are called *relators* and the set of equations

$$a_{i_1}^{n_1} a_{i_2}^{n_2} \ldots, a_{i_p}^{n_p} = 1 \tag{21.4}$$

obtained by setting each relator to 1 are called the relations for the group with respect to ϕ. The list of generators $\{a_1, a_2, \ldots, a_k\}$ and relations among them uniquely specifies H through equation (23.3). If a relation in the list (23.4) is a consequence of others, for example if one of them is the product of two others, we may clearly drop it from the list thereby shortening the list. In practice one would try to keep the list of relations down to a minimum. Such a description of H is called a presentation of the group H through generators and relations. A group in general has many presentations and it is usually very difficult to decide whether or not two presentations represent the same group.

Example 23.1 (Presentation of some groups): We describe some of the commonly occurring groups in terms of generators and relations. Some of these would appear as fundamental groups of spaces that we have already encountered or would do so in the next few lectures.

1. If we take the free group on two generators a, b and take $H = \mathbb{Z} \times \mathbb{Z}$ then every commutator $a^m b^n a^{-m} b^{-n}$ is a relator and hence each of the equations $a^m b^n a^{-m} b^{-n} = 1$ is a relation. However, all of them may be derived from the single relation $aba^{-1}b^{-1} = 1$. For example, we derive the relation $a^2ba^{-2}b^{-1} = 1$ as follows
$$a^2ba^{-2}b^{-1} = a(aba^{-1}b^{-1})ba^{-1}b^{-1} = aba^{-1}b^{-1} = 1.$$
Thus $\mathbb{Z} \times \mathbb{Z}$ has presentation
$$\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid ab = ba \rangle \tag{23.5}$$

2. The cyclic group of order n has presentation
$$\mathbb{Z}_n = \langle a \mid a^n = 1 \rangle \tag{23.6}$$

3. Recall from lecture 20 that the fundamental group of the Klein's bottle is given by the presentation
$$\mathbb{Z} \ltimes \mathbb{Z} = \langle a, b \mid aba = b \rangle \tag{23.7}$$

4. This example is from [15], p. [?]. Let us consider the group G given by the presentation
$$G = \langle a, b \mid a^2 = b^4 = 1,\ bab = a \rangle \tag{23.8}$$
To understand this group concretely, let us derive some consequences of the three displayed relations. Multiplying $bab = a$ on the left/right by a gives the relations $(ab)^2 = 1$ and $(ba)^2 = 1$. Further,
$$ab^3 = (ab)b^2 = b^3(bab)b^2 = b^3ab^2 = ba.$$
We conclude from this that G consists of the elements
$$\{1, a, b, b^2, b^3, ab, ba, ab^2\} \tag{23.9}$$
This however does not preclude further simplifications to a group of smaller order though it seems unlikely. The group has atleast three elements of order two and so if the elements listed in (21.9) are distinct then G must be the dihedral group D_4 of order eight if it is non-abelian or else must be an abelian group. In any case there must be atleast five elements of order two (why?). It is easy to see that ab^2 has order two. The map $f : a, b \longrightarrow D_4$ given by
$$f(a) = (13),\ f(b) = (1234)$$
respects the given relations since $(13)^2 = 1$, $(1234)^4 = 1$ and $(1234)(13)(1234) = (13)$. Hence f extends to a surjective group homomorphism $f : F_2 \longrightarrow D_4$. Since the kernel contains a^2, b^4 and bab we get a surjective group homomorphism $G \longrightarrow D_4$ and we conclude that G is indeed D_4.

Push-outs: The notion of push-outs is a convenient generalization of the coproduct and in the context of groups is also known as the free-product with amalgamation. In topology it is often referred to as the adjunction space though some authors in analogy with groups call it the amalgamated sum. We formulate this notion in general terms.

Definition 23.4: Suppose given a pair of morphisms $j_1 : C \longrightarrow A_1$ and $j_2 : C \longrightarrow A_2$ in a category $\mathcal{C}$, represented as a diagram:

$$\begin{tikzcd} C \arrow[r, "j_1"] \arrow[d, "j_2"'] & A_1 \\ A_2 & \end{tikzcd}$$

a push out is an object P in $\mathcal{C}$ together with a pair of morphisms $f_1 : A_1 \longrightarrow P$ and $f_2 : A_2 \longrightarrow P$ satisfying the following two conditions:

(i) $f_1 \circ j_1 = f_2 \circ j_2$

(ii) Universal property: Given any pair of morphisms $g_1 : A_1 \longrightarrow E$ and $g_2 : A_2 \longrightarrow E$ satisfying

$$g_1 \circ j_1 = g_2 \circ j_2$$

there exists a **unique** morphism $\phi : P \longrightarrow E$ such that

$$\phi \circ f_1 = g_1, \quad \phi \circ f_2 = g_2.$$

Remark: If P is a push-out for the pair $j_1 : C \longrightarrow A_1$ and $j_2 : C \longrightarrow A_2$ the commutative diagram

$$\begin{tikzcd} C \arrow[r, "j_1"] \arrow[d, "j_2"'] & A_1 \arrow[d, "f_1"] \\ A_2 \arrow[r, "f_2"] & P \end{tikzcd}$$

is also known as a cocartesian square.

Theorem 23.5: If the push out for the pair $j_1 : C \longrightarrow A_1$ and $j_2 : C \longrightarrow A_2$ exists in a given category, then it is unique.

Proof: If P' with morphisms $f_1' : A_1 \longrightarrow P'$ and $f_2' : A_2 \longrightarrow P'$ is another candidate we may apply the universal property to get a map $\phi : P \longrightarrow P'$ such that

$$\phi \circ f_1 = f_1', \quad \phi \circ f_2 = f_2'.$$

Reciprocally since P' is a push out, there is a map $\psi : P' \longrightarrow P$ such that

$$\psi \circ f_1' = f_1, \quad \psi \circ f_2' = f_2.$$

Combining we see that $(\psi \circ \phi) \circ f_1 = f_1$ and $(\psi \circ \phi) \circ f_2 = f_2$. We see that both $\psi \circ \phi$ and id_P satisfy the universal property with $E = P$, $g_1 = f_1$ and $g_2 = f_2$. The uniqueness clause in the definition gives $\psi \circ \phi = \mathrm{id}_P$. Likewise we get $\phi \circ \psi = \mathrm{id}_{P'}$ and the proof is complete.

Example: Let us now work in the category **Top** and recast the gluing lemma in terms of the push-out construction. Take a pair of open sets G_1, G_2 in a topological space X and the inclusions

$$j_1 : G_1 \cap G_2 \longrightarrow G_1, \quad j_2 : G_1 \cap G_2 \longrightarrow G_2.$$

The push out for this pair is the space $G_1 \cup G_2$ together with inclusion maps

$$i_1 : G_1 \longrightarrow G_1 \cup G_2, \quad i_2 : G_2 \longrightarrow G_1 \cup G_2$$

To see this suppose that Y is a topological space and $f_1 : G_1 \longrightarrow Y$ and $f_2 : G_2 \longrightarrow Y$ are a pair of continuous maps such that $f_1 \circ j_1 = f_2 \circ j_2$ then

$$f_1\Big|_{G_1 \cap G_2} = f_2\Big|_{G_1 \cap G_2}$$

The gluing lemma now says that there exists a unique map $\psi : G_1 \cup G_2 \longrightarrow Y$ such that

$$\psi\Big|_{G_1} = f_1, \quad \psi\Big|_{G_2} = f_2$$

which means $\psi \circ i_1 = f_1$ and $\psi \circ i_2 = f_2$ as desired. Instead of a pair of open subsets of a topological space one could choose a pair of closed sets.

Existence of push outs: We begin with the coproduct of A_1 and A_2 and perform some identifications. We examine the three categories **Gr**, **AbGr** and **Top** and show that the push-out exists in each of them. It may be noted that the popular term for the push out in the category of groups is *free product with amalgamation.*

Theorem 23.6: Push-outs exist in the categories **Gr**, **AbGr** and **Top**.

Proof: Let us begin with **Gr** and a given pair of morphisms $j_1 : C \longrightarrow A_1$ and $j_2 : C \longrightarrow A_2$. Let G be the coproduct of the groups A_1 and A_2. We regard A_1 and A_2 as subgroups of G. Let N be the normal subgroup of G generated by

$$\{j_1(c) j_2(c)^{-1} / c \in C\}$$

and $\eta : G \longrightarrow G/N$ be the quotient map. We claim that G/N qualifies as the push-out with the associated homomorphisms

$$f_1 = \eta \circ i_1, \quad f_2 = \eta \circ i_2$$

where i_1 and i_2 are the inclusions of A_1, A_2 in G. Since $\eta(j_1(c)) = \eta(j_2(c))$, we see that $f_1 \circ j_1 = f_2 \circ j_2$. To check the universal property, let $g_1 : A_1 \longrightarrow H$ and $g_2 : A_2 \longrightarrow H$ be a pair of morphisms such that

$$g_1 \circ j_1 = g_2 \circ j_2 \tag{23.10}$$

Aside from (23.10), by definition of coproduct, there exists a unique homomorphism $\psi : G \longrightarrow H$ such that $\psi \circ i_1 = g_1$ and $\psi \circ i_2 = g_2$ from which follows easily that the kernel of ψ contains N. Let $\overline{\psi} : G/N \longrightarrow H$ be the unique map such that $\overline{\psi} \circ \eta = \psi$. Then

$$\overline{\psi} \circ \eta \circ i_1 = \psi \circ i_1 = g_1, \quad \overline{\psi} \circ \eta \circ i_2 = g_2.$$

which means $\overline{\psi} \circ f_1 = g_1$ and $\overline{\psi} \circ f_2 = g_2$. That completes the job of verifying that G/N is indeed the push-out. Note that we have only used the definition of coproducts and the most basic property of quotients. As a result the proof goes through verbatim for the other two situations as we shall see. Leaving aside the case of abelian groups we pass on to the category **Top**.

Well, changing notations to suit the need, let $h_1 : Z \longrightarrow X$ and $h_2 : Z \longrightarrow Y$ be a pair of continuous functions and $X \sqcup Y$ be the disjoint union of X and Y, and $i_1 : X \longrightarrow X \sqcup Y$, $i_2 : Y \longrightarrow X \sqcup Y$ be the canonical inclusions. For each $z \in Z$ we identify the points $(i_1 \circ h_1)(z)$ and $(i_2 \circ h_2)(z)$ in $X \sqcup Y$ and W be the quotient space with the projection map

$$\eta : X \sqcup Y \longrightarrow W = (X \sqcup Y)/\sim$$

We claim that W qualifies to be the push-out with associated morphisms $f_1 = \eta \circ i_1 : X \longrightarrow W$ and $f_2 = \eta \circ i_2 : Y \longrightarrow W$. To check the first condition observe that since $(i_1 \circ h_1)(z)$ and $(i_2 \circ h_2)(z)$ are identified, $\eta(i_1(h_1(z))) = \eta(i_2(h_2(z)))$ which means $f_1 \circ h_1 = f_2 \circ h_2$. Turning now to the universal property let $g_1 : X \longrightarrow T$ and $g_2 : Y \longrightarrow T$ be two continuous maps such that

$$g_1 \circ h_1 = g_2 \circ h_2. \tag{23.11}$$

Aside from (23.11), since $X \sqcup Y$ is the coproduct in **Top**, there is a unique continuous map $\psi : X \sqcup Y \longrightarrow T$ such that $\psi \circ i_1 = g_1$ and $\psi \circ i_2 = g_2$. Now (23.11) implies that ψ respects the identification and so there is a unique $\overline{\psi} : (X \sqcup Y)/\sim \longrightarrow T$ such that $\overline{\psi} \circ \eta = \psi$. By the universal property of the quotient, $\overline{\psi}$ is continuous and

$$\overline{\psi} \circ f_1 = \overline{\psi} \circ \eta \circ i_1 = \psi \circ i_1 = g_1,$$

and likewise $\overline{\psi} \circ f_2 = g_2$. That suffices for a proof.

Exercises

1. Show that the maps i_1 and i_2 in definition (23.1) are injective and that the images of i_1 and i_2 generate $G_1 * G_2$. Hint: Use the universal property with $H = G_1$, $f_1 = i_1$ and $i_2 = 1$.

2. Show that abelianizing a free group on k generators results in a group isomorphic to the direct sum of k copies of $\mathbb{Z}$. Use the fact that the coproduct in **AbGr** is the direct sum.

3. Is there a surjective group homomorphism from the direct sum $\mathbb{Z} \times \mathbb{Z}$ onto $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$? Prove that if k and l are distinct positive integers, the free group on k generators is not isomorphic to the free group on l generators.

4. Show that $\langle a, c \mid a^2c^2 = 1\rangle$ is also a presentation of the fundamental group of the Klein's bottle.

5. Construct the push-out for the pair $j_1 : C \longrightarrow A_1$ and $j_2 : C \longrightarrow A_2$ in the category **AbGr**?

6. Suppose that C is the trivial group in the definition of push-out in the category **Gr**, show that the resulting group is the coproduct of the two given groups. What happens in the category **AbGr**? Describe explicitly the construction of the group specifying the subgroup that is being factored out.