Lecture - XXVIII Introductory remarks on homology theory

In the first part of the course we focused on the fundamental group and its basic properties. We discussed an elegant solution of the lifting problem for covering projections in terms of the fundamental group. While the theory of fundamental groups and covering spaces is fairly adequate for many applications in low dimensional geometry and other parts of mathematics such as the theory of function of one complex variable, it is quite ineffective when higher dimensional objects are involved. For instance the ball B^n and the sphere S^{n-1} both have trivial fundamental group ($n \geq 3$) which renders it useless for proving the higher dimensional analogues of the Brouwer's fixed point theorem.

Homology theory provides a functor that is quite convenient for understanding the geometry of "higher dimensional objects" which has the added advantage of being easily computable (at least for a large class of interesting spaces). While the fundamental group functor respect products, the homology groups of $X \times Y$ are not so easily described in terms of the homology groups of X and Y. A covering projection is a very special case of a fiber bundle with discrete fibers. We have seen that in the case of a covering projection $p : \tilde{X} \longrightarrow X$ we have a relationship between $\pi_1(X)$ and $\pi_1(\tilde{X})$. The story is decidedly more complicated with homology groups. For instance some work is required to compute the homology groups of the real projective spaces $\mathbb{R}P^n$. Homotopy theory is better suited for studying fibrations where the use of homology would entail the formidable machinery of "spectral sequences". However, on the computational side there is a very useful substitute for the Seifert Van Kampen theorem in homology known as the Mayer Vietoris sequence. We shall use it to calculate efficiently the homology groups of a large number of spaces.

There are several approaches to the homology theory, the oldest being the simplicial theory. Homology theory evolved over several decades through the early part of the twentieth century becoming progressively abstract.

The theory we discuss in this course is known as the singular homology theory and would appear somewhat non-intuitive in the beginning but we hope that the examples and applications presented would enable the students to digest the material. Singular homology theory appeared rather late in the development of algebraic topology and is a culmination of efforts spanning a few decades by several eminent topologists. In the intervening years several seemingly different homology theories developed the oldest and most intuitive being simplicial homology theory that applies to the restricted class of simplicial complexes. However the topological invariance is highly non-trivial and beset with technical complications.

Some motivation for singular homology: Let us recall some of the notions in the theory of contour integrals in elementary complex analysis. Given a holomorphic function $f : \Omega \longrightarrow \mathbb{C}$ one defines a line integral

$$\int_\gamma f(z)dz \tag{28.1}$$

along a path[5] $\gamma : [a, b] \longrightarrow \Omega$ lying in the domain Ω. If the path γ is the juxtaposition of several paths $\gamma_1, \gamma_2, \dots, \gamma_k$ then one knows that

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \dots + \int_{\gamma_k} f(z)dz \tag{28.2}$$

[5] The path would have to satisfy some regularity condition such as being piecewise continuously differentiable. However since this is merely supposed to be a motivation we shall brush aside these technicalities.

Thus one can *break* the path γ into several pieces, compute the integral over the individual pieces and add the results. One can also reparametrize the pieces and regard all the pieces γ_j as being maps from $[0,1]$. In view of all these, it seems meaningful to write

$$\gamma_1 + \gamma_2 + \cdots + \gamma_k \tag{28.3}$$

in place of

$$\gamma_1 * \gamma_2 * \cdots * \gamma_k.$$

We see that the rigidity present in the theory of the fundamental group where one deals with homotopy classes of loops all of which are based at a given point, is now significantly relaxed.

Also, one checks that integration along the inverse path reverses the sign:

$$\int_{\gamma^{-1}} f(z)dz = -\int_{\gamma} f(z)dz \tag{28.4}$$

Taking a specific example with $f(z) = 1/z$ and integrating along two concentric circles γ_1, γ_2 traced counter clockwise, we see that

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz. \tag{28.5}$$

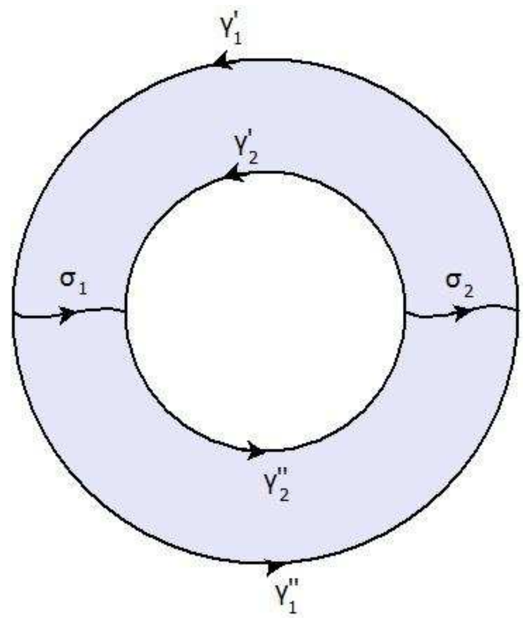

Figure 20:

Using (28.1) and (28.4) this may be rewritten as

$$\int_{\gamma_1 - \gamma_2} f(z)dz = 0, \tag{28.6}$$

where, in keeping with the additive notation (28.3) we have written $-\gamma_2$ in place of γ_2^{-1}. Equation (28.6) is interesting since γ_1 and γ_2 are the two pieces of the boundary of the annular region A bounded by them, where the function f is holomorphic. Equation (28.6) suggests that the two paths γ_1 and

γ_2 ought to be regarded as being *equivalent* with regard to f or more precisely with regard to A since nothing changes if f is replaced by any other function holomorphic in an neighborhood of A. However (28.6) fails for $f(z) = (z-p)^{-1}$, where p is any point in the interior of A. This is a reflection of the fact that the paths γ_1 and γ_2 do not constitute the full boundary of the punctured annulus $A - \{p\}$ which is where $(z-p)^{-1}$ is holomorphic.

In doing contour integrals one occasionally introduces auxiliary paths such as σ_j $(j = 1, 2)$ indicated in the figure below and writes the integral (28.6) over $\gamma_1 - \gamma_2$ as the sum

$$(\gamma_1' + \sigma_1 - \gamma_2' + \sigma_2) + (\gamma_1'' - \sigma_1 - \gamma_2'' - \sigma_2) \tag{28.7}$$

Each of the two parenthesis indicates a boundary of one of the halves of the annulus and so each ought to *equivalent* to a null path or in other words, the equivalence of γ_1 and γ_2 translates to $\gamma_1 - \gamma_2$ being equivalent to a *null* path. We write $\gamma_1 \sim \gamma_2$ to indicate the equivalence of $\gamma_1 - \gamma_2$ to a null-path.

These considerations suggest an underlying *calculus of paths* bounding regions in the plane. Indeed homology theory does develop such a calculus of paths as well as its higher dimensional analogues. Perhaps the student has encountered these higher dimensional analogues in connection with the Gauss' divergence theorem in vector calculus[6].

Note that the sum indicated in (28.3) is a formal sum we are lead to the free abelian group generated by the set of all piecewise smooth functions from $[0, 1]$ to Ω called the group of one chains. Thus $\gamma_1 - \gamma_2$ in (28.6) and $\gamma_1 + \gamma_2 + \cdots + \gamma_k$ displayed in (28.3) are examples of one chains. Note that the one chain appearing in (28.2) is *different* from γ though in the final stage of construction they would be identified. The Cauchy theory suggests that the chains whose pieces are all closed curves would play a distinguished role and these are examples of one cycles - a certain subgroup of the group of chains called the group of one cycles Z_1. If a chain such as $\gamma_1 - \gamma_2$ appearing in equation (28.6) is the oriented boundary of a sub-domain we would regard it as being equivalent to zero and we would call such chains as boundaries. These form a subgroup of Z_1 known as the group of boundaries B. The equivalence relation is thus $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1 - \gamma_2 \in B$. Passing to the quotient of Z via this equivalence relation or in algebraic terms, passing to the quotient group Z/B would give us the first homology group of the space Ω. All these heuristics are rigorously defined in the next couple of lectures. We shall of course have to dispense with the notion of piecewise smoothness and talk of continuous paths $\gamma : [0, 1] \longrightarrow X$ called *singular one simplexes* and and their formal linear combinations with integer coefficients called *singular one chains* . To develop a calculus of higher dimensional chains, one has the option of introducing *singular cubes* namely continuous maps $[0, 1]^n \longrightarrow X$, which is the approach taken by W. Massey. This however necessitates certain preliminary reductions but has some distinct advantages later particularly in applications of homology theory to the study of homotopy groups. We shall follow the traditional approach, as in J. Vick's book and use singular simplices instead.

[6]For a discussion along the lines of vector calculus see [11]