Lecture XV - Covering Projections

The theory of covering projections sets a common stage for the development of diverse branches of mathematics. In this course we develop the theory of covering projections only to the extent that is relevant for the computation of the fundamental group. It may be useful for the student to review the proof that $\pi_1(S^1) = \mathbb{Z}$. In fact one of the paradigms for a covering projection is the map

$$t \mapsto \exp(2\pi i t)$$

wrapping the real line onto the circle.

Definition 15.1: A covering projection is a triple $(\tilde{X}, X, p)$ where $\tilde{X}$, X are connected topological spaces and a continuous map $p : \tilde{X} \longrightarrow X$ satisfying the following properties:

(i) The map p is surjective

(ii) Each $x \in X$ has a neighborhood U such that the inverse image $p^{-1}(U)$ is a disjoint union of a collection open subsets $\{U_\alpha\}$ of $\tilde{X}$.

(iii) Each U_α is mapped onto U homeomorphically by p.

The neighborhood U described in the definition above is called an *evenly covered neighborhood of x*, the open sets U_α are referred to as *sheets* lying above U and for $x \in X$, the subset $p^{-1}(x)$ of $\tilde{X}$ is called the *fiber* over x. This terminology will be used frequently. We shall also say that $\tilde{X}$ is a covering space of X when it is fairly clear what the map p is.

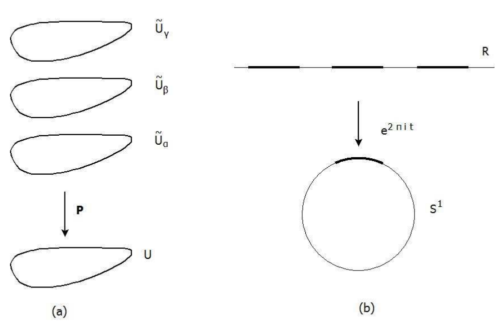

Figure 12: Covering projection

Remark: It is NOT sufficient that each U_α be homeomorphic to U but the homeomorphism must be given by the restricting p to U_α.

Examples 15.1: We now present four examples to illustrate the concept of a covering projection.

1. As indicated in the beginning the most basic example is the map $\mathrm{ex} : \mathbb{R} \longrightarrow S^1$ given by

$$\mathrm{ex}(t) = \exp(2\pi i t)$$

   For each point z on the circle take an arc U centered at z and of length say $\pi/2$. The reader may check that the inverse image of U under ex is a disjoint union of open intervals on the line.

2. Consider the map $T : \mathbb{C} - \{0\} \longrightarrow \mathbb{C} - \{0\}$ given by $T(z) = z^2$. If we pick a point $z \in \mathbb{C} - \{0\}$ and a small disc U centered at z not containing a pair of antipodal points then $T^{-1}(U)$ is a disjoint union of two open sets each of which is mapped bijectively onto U by T.

3. Consider the map $p : \mathbb{C} - \{\pm 1, \pm 2\} \longrightarrow \mathbb{C} - \{\pm 2\}$ given by

$$p(z) = z^3 - 3z$$

   The equation $p(z) = w$ has three distinct roots for each $w \in \mathbb{C} - \{\pm 2\}$ and the roots are continuous functions of w. For a sufficiently small neighborhood U of w, $p^{-1}(U)$ is a disjoint union of three open sets each of which is mapped onto U homeomorphically onto U by the open mapping theorem. Several examples of this type related to complex analysis are discussed in [6].

4. Consider the quotient map $\eta : S^n \longrightarrow \mathbb{R}P^n$. We show that η is a covering projection. Let U_1 be an open subset of S^n not containing a pair of anti-podal points and

$$U_2 = \{-x/x \in U_1\}.$$

   Then, $\eta(U_1) = \eta(U_2)$. Denoting these images by U, we see that $\eta^{-1}(U) = U_1 \cup U_2$ which is an open set in S^n and so U is open in $\mathbb{R}P^n$. Second, η maps each of U_1 and U_2 bijectively onto U. To see that η maps each of U_1 and U_2 homeomorphically onto U, we merely have to show that η is an open mapping. So let V_1 be an open subset of U_1 and $V_2 = \{-x/x \in V_1\}$. Then

$$\eta^{-1}(\eta(V_1)) = V_1 \cup V_2$$

   is open in S^n so that $\eta(V_1)$ is an open subset of $\mathbb{R}P^n$. Thus we have shown that η restricted to each U_j is an open mapping and that suffices for a proof.

We now summarize the most basic properties of covering projections.

Theorem 15.1: Suppose that $p : \tilde{X} \longrightarrow X$ is a covering projection. Then

(i) The map p is a local homeomorphism (see exercise 7, lecture 3).

(ii) The function p is an open mapping.

(iii) The fibers $p^{-1}(x)$ are discrete for each $x \in X$.

Proof: Let $\tilde{x} \in \tilde{X}$ be arbitrary and $x = p(\tilde{x})$. Choose an evenly covered neighborhood U of x and $\tilde{U}$ be a sheet lying over U and containing $\tilde{x}$. Then $\tilde{U}$ is an open set in $\tilde{X}$ containing $\tilde{x}$ that is mapped by p homeomorphically onto U. Thus p is a local homeomorphism and we have proved (i). Let $\tilde{G}$ be an arbitrary open set in $\tilde{X}$. Then $\tilde{G}$ can be covered by open sets $\tilde{U}$ such that p maps each $\tilde{U}$ homeomorphically onto an evenly covered open subset U of X (why?). Then $G = p(\tilde{G})$ is the union of such evenly covered neighborhoods U implying that G is an open set in X. Thus p is an open mapping. Finally to prove (iii) suppose that $\tilde{z} \in \tilde{X}$ is a limit point of $p^{-1}(x)$. Pick an arbitrary evenly covered neighborhood U of $z = p(\tilde{z})$ and a sheet $\tilde{U}$ lying over U containing $\tilde{z}$. In particular the restriction of p to the sheet $\tilde{U}$ is injective. But since $\tilde{z}$ is a limit point of $p^{-1}(x)$, this sheet must contain infinitely many points of $p^{-1}(x)$ which means p restricted to $\tilde{U}$ cannot be injective which is a contradiction.

The lifting problem: Suppose that $p : E \longrightarrow B$ is a surjective continuous map between topological spaces and $f : T \longrightarrow B$ is a given continuous map, a lift of f is by definition a continuous map $\tilde{f} : T \longrightarrow E$ such that

$$p \circ \tilde{f} = f$$

The lifting problem involves giving sufficient conditions for the existence of the lift $\tilde{f}$. The main point here is of course the continuity of the lift. The significance of the problem can be understood from complex analysis.

Example 15.2: Consider the exponential map $\exp : \mathbb{C} \longrightarrow \mathbb{C} - \{0\}$ and an open set $\Omega \subset \mathbb{C}$ and the inclusion map

$$j : \Omega \longrightarrow \mathbb{C} - \{0\}.$$

To say that the inclusion map j has a lift with respect to the exponential map means the existence of a continuous $\tilde{j} : \Omega \longrightarrow \mathbb{C}$ such that

$$\exp(\tilde{j}(z)) = z, \quad \text{for all } z \in \Omega.$$

In other words the existence of a lift of j is equivalent to the existence of a continuous branch of the logarithm on Ω. We know from complex variable theory that such a continuous branch need not exist in general such as for instance the case $\Omega = \mathbb{C} - \{0\}$.

In place of the exponential map we could consider the map $S : \mathbb{C} - \{0\} \longrightarrow \mathbb{C} - \{0\}$ given by $S(z) = z^2$. The problem of lifting the inclusion map of a domain $\Omega \subset \mathbb{C} - \{0\}$ is then equivalent to the existence of a continuous branch of the square root function on Ω. We also know from complex analysis that if the lift exists it need not be unique. Well, if a domain $\Omega \subset \mathbb{C} - \{0\}$ admits a continuous branch of the square root then it admits two branches. If it admits a continuous branch of the logarithm then it admits infinitely many any two of which differ by an integer multiple of $2\pi i$. On a connected domain, the branch however is uniquely specified by specifying a value at a point of the domain. The following theorem generalizes this in the context of covering spaces.

Theorem 15.2 (uniqueness of lifts): Suppose $p : \tilde{X} \longrightarrow X$ is a covering projection, T is a connected topological space and $f_1 : T \longrightarrow \tilde{X}$ and $f_2 : T \longrightarrow \tilde{X}$ are two lifts of a given continuous map $f : T \longrightarrow X$ such that $f_1(t_0) = f_2(t_0)$ for some $t_0 \in T$. Then the two lifts agree on T namely, $f_1(t) = f_2(t)$ for all $t \in T$.

Proof: Let G be the subset given by $G = \{t \in T / f_1(t) = f_2(t)\}$. The set G is non-empty since $t_0 \in G$. We shall show that G is both open and closed in T from which the result would follow since T is connected. For $t \in G$ pick an evenly covered neighborhood U of

$$x = p(f_1(t)) = p(f_2(t)).$$

and $\tilde{U}$ be the sheet lying over U and containing $f_1(t) = f_2(t)$. The set

$$N = f_1^{-1}(\tilde{U}) \cap f_2^{-1}(\tilde{U})$$

is open and contains t. If $z \in N$ then $f_1(z)$ and $f_2(z)$ both belong to $\tilde{U}$ and $p(f_1(z)) = p(f_2(z)) = f(z)$. But p restricted to $\tilde{U}$ is injective and so $f_1(z) = f_2(z)$ for all $z \in N$ and we conclude that $N \subset G$. The proof that G is closed is left as an exercise. The student may assume that the spaces involved are Hausdorff (see exercise 7 of lecture 2).

Exercises:

1. Explain why the map $\phi : \mathbb{C} - \{0, 1/2\} \longrightarrow \mathbb{C} - \{-1/4\}$ given by $\phi(z) = z(z-1)$ is not a covering projection?
2. Show that the map $f : S^1 \longrightarrow S^1$ given by $f(z) = z^k$ is a covering projection for every $k \in \mathbb{N}$.
3. Suppose $p : \tilde{X} \longrightarrow X$ is a covering projection and E is a closed subset of X. Is the map
$$p : \tilde{X} - p^{-1}(E) \longrightarrow X - E$$
a covering projection?
4. Find a discrete subset E of $\mathbb{C}$ such that $\sin : \mathbb{C} - E \longrightarrow \mathbb{C} - \{-1, 1\}$ is a covering projection.
5. Suppose that $p : \tilde{X} \longrightarrow X$ and $q : \tilde{Y} \longrightarrow Y$ are covering projections then the product map $(p, q) : \tilde{X} \times \tilde{Y} \longrightarrow X \times Y$ given by
$$(p, q)(z, w) = (p(z), q(w)), \quad z \in \tilde{X}, w \in \tilde{Y},$$
is a covering projection. In particular the plane $\mathbb{R}^2$ is a covering space of the torus $S^1 \times S^1$.
6. Let Y be the infinite grid
$$Y = \{(x, y) \in \mathbb{R}^2 / x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$$
is a covering projection of the figure eight loop. Draw the figure eight loop on the torus.
7. Show that the set G in theorem (15.2) is closed without using the Hausdorff assumption on T.