Lecture XVI - Lifting of paths and homotopies

In the last lecture we discussed the lifting problem and proved that the lift if it exists is uniquely determined by its value at one point. In this lecture we shall prove the important result that covering projections enjoy the path lifting and covering homotopy properties. This theorem is fundamental in the the theory of covering projections and will be used in the next lecture to define an action of the fundamental group on the fibers.

Theorem 16.1 (path lifting lemma): Let $p : \tilde{X} \longrightarrow X$ be a covering projection and $\gamma : [0,1] \longrightarrow X$ be a path such that for some $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$,

$$\gamma(0) = x_0 = p(\tilde{x}_0). \tag{16.1}$$

Then there exists a unique path $\tilde{\gamma} : [0,1] \longrightarrow \tilde{X}$ such that

$$p \circ \tilde{\gamma} = \gamma, \quad \tilde{\gamma}(0) = \tilde{x}_0 \tag{16.2}$$

Thus each path in X lifts to a unique path in $\tilde{X}$ with a prescribed initial point in $p^{-1}(\gamma(0))$.

Proof: Let $\mathcal{O}$ be the open cover of X by evenly covered open sets and $\gamma^{-1}(\mathcal{O})$ be the family

$$\gamma^{-1}(\mathcal{O}) = \{\gamma^{-1}(G)/G \in \mathcal{O}\}$$

of open sets covering $[0,1]$. There is a Lebesgue number η for this cover and we choose n to be a natural number such that $1/n < \eta$. Consider the partition

$$\Big\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\Big\}.$$

For each $j = 1, 2, \dots, n$, the piece $\gamma([\frac{j-1}{n}, \frac{j}{n}])$ lies in an evenly covered open set in X. In particular if γ_0 denotes the restriction of γ to $[0, 1/n]$ then the image of γ_0 lies in an open set $G_0 \in \mathcal{O}$. The conditions

(16.1)-(16.2) say that there is a sheet $\tilde{G}_0$ lying over G_0 and containing the point $\tilde{x}_0$. Let p_0 denote the restriction of p to the sheet $\tilde{G}_0$ and q_0^{-1} be its inverse. On the sub-interval $[0, 1/n]$, we define

$$\tilde{\gamma}_0 = q_0 \circ \gamma_0$$

thereby obtaining an initial piece of the desired lift $\tilde{\gamma}$. We shall construct the lift $\tilde{\gamma}$ piece by piece defining it on each subinterval of the partition of $[0, 1]$. In what follows γ_j denotes the restriction of γ to the sub-interval $[\frac{j}{n}, \frac{j+1}{n}]$. Assume inductively that

$$\tilde{\gamma}_j : [\frac{j}{n}, \frac{j+1}{n}] \longrightarrow \tilde{X}$$

has been defined such that

$$\begin{aligned} p \circ \tilde{\gamma}_j &= \gamma_j \\ \tilde{\gamma}_j(j/n) &= \tilde{\gamma}_{j-1}(j/n), \qquad \text{in case } j \geq 1. \\ \tilde{\gamma}_0(0) &= \tilde{x}_0 \end{aligned}$$

For the inductive step we set up the notations for the endpoints of the lift $\tilde{\gamma}_j$ namely, let

$$\gamma_j\Big(\frac{j+1}{n}\Big) = x_{j+1}, \quad \tilde{\gamma}_j\Big(\frac{j+1}{n}\Big) = \tilde{x}_{j+1}, \quad p(\tilde{x}_{j+1}) = x_{j+1}.$$

Let $G_{j+1} \in \mathcal{O}$ be an evenly covered neighborhood containing x_{j+1} such that γ maps $[\frac{j+1}{n}, \frac{j+2}{n}]$ into G_{j+1} and $\tilde{G}_{j+1}$ be the sheet lying over G_{j+1} containing the point $\tilde{x}_{j+1}$. The restriction of p to $\tilde{G}_{j+1}$ is a homeomorphism with inverse q_{j+1} say, so that $q_{j+1}(x_{j+1}) = \tilde{x}_{j+1}$. We set

$$\tilde{\gamma}_{j+1} = q_{j+1} \circ \gamma_{j+1}$$

Then $\tilde{\gamma}_{j+1}$ is continuous, $p \circ \tilde{\gamma}_{j+1} = \gamma_{j+1}$ and

$$\tilde{\gamma}_{j+1}\Big(\frac{j+1}{n}\Big) = q_{j+1}(x_{j+1}) = \tilde{x}_{j+1} = \tilde{\gamma}_j\Big(\frac{j+1}{n}\Big)$$

By gluing lemma, the pieces $\tilde{\gamma}_j$ may be glued together to yield a continuous function $\tilde{\gamma} : [0, 1] \longrightarrow \tilde{X}$ such that

$$p \circ \tilde{\gamma} = \gamma, \quad \tilde{\gamma}(0) = \tilde{x}_0.$$

The proof is complete. The uniqueness has been already proved in general.

Lifting of homotopies: We now examine what happens when we lift homotopic paths with the lifts having the same initial points.

Theorem 16.2 (Covering homotopy property): Let $p : \tilde{X} \longrightarrow X$ be a covering projection and $\tilde{x}_0 \in \tilde{X}, x_0 \in X$ be chosen base points such that $p(\tilde{x}_0) = x_0$. Let γ_1, γ_2 be two curves in X starting at x_0 and having the same terminal points and $F : [0, 1] \times [0, 1] \longrightarrow X$ be a homotopy between γ_1 and γ_2. There is a unique lift $\tilde{F} : [0, 1] \times [0, 1] \longrightarrow \tilde{X}$ of F such that $\tilde{F}(0, 0) = \tilde{x}_0$. In particular the unique lifts of γ_1 and γ_2 starting at $\tilde{x}_0$ have the same terminal points.

Proof: The idea behind the proof is simple and parallels the proof of the previous theorem except that the book-keeping gets a bit more involved. Consider a covering $\mathcal{O}$ of X by evenly covered open neighborhoods and choose a Lebesgue number ϵ for the covering

$$\{F^{-1}(U)/U \in \mathcal{O}\}. \tag{16.3}$$

Choose n so large that any square in $[0,1]\times[0,1]$ of side $1/n$ is contained in one of the sets $F^{-1}(U)$ in (16.3). Partition $[0,1]\times[0,1]$ using the grid points

$$\Big\{\Big(\frac{j}{n},\frac{k}{n}\Big)/0 \le j \le n,\ 0 \le k \le n\Big\}$$

and $S_{j,k}$ be the square with vertices

$$\Big(\frac{j}{n},\frac{k}{n}\Big),\ \Big(\frac{j+1}{n},\frac{k}{n}\Big),\ \Big(\frac{j+1}{n},\frac{k+1}{n}\Big),\ \Big(\frac{j}{n},\frac{k+1}{n}\Big).$$

Let $U_{0,0}$ be an evenly covered neighborhood in X such that $F(S_{0,0}) \subset U_{0,0}$ and $\tilde{U}_{0,0}$ be the sheet in

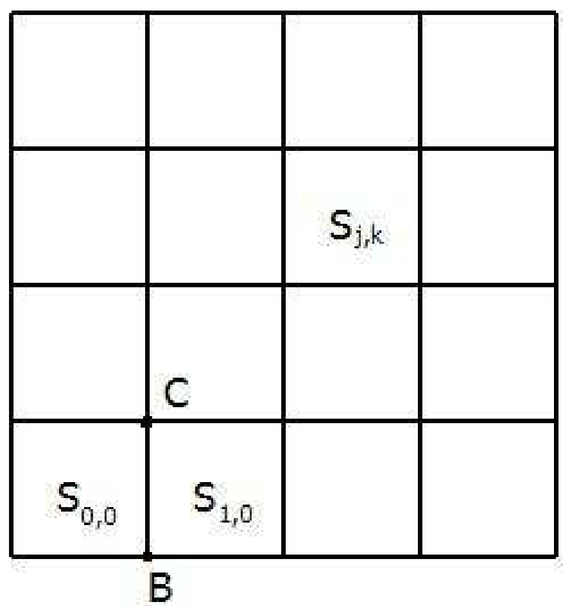

Figure 13: Homotopy lifting property

$\tilde{X}$ lying above $U_{0,0}$. Denoting by $p_{0,0}$ and $F_{0,0}$ the restrictions of p and F to $\tilde{U}_{0,0}$ and $S_{0,0}$ respectively, define

$$\tilde{F}_{0,0} = p_{0,0}^{-1} \circ F.$$

Thus $\tilde{F}_{0,0} : S_{0,0} \longrightarrow \tilde{X}$ is continuous, takes the value $\tilde{x}_0$ at the origin and is a part of the lift $\tilde{F}$ under construction. As in the previous theorem we shall construct the lift $\tilde{F}$ piece by piece and we now turn to the adjacent square $S_{1,0}$ which is mapped by F to an evenly covered neighborhood $U_{1,0}$ in the cover $\mathcal{O}$. In particular (referring to the figure) $F(B) \in U_{1,0}$. Choose a sheet $\tilde{U}_{1,0}$ lying above $U_{1,0}$ containing $\tilde{F}(B)$ and the restriction

$$p_{1,0} = p\Big|_{\tilde{U}_{1,0}}$$

maps $\tilde{U}_{1,0}$ homeomorphically onto $U_{1,0}$. Now we define the next piece of the lift $\tilde{F}_{1,0}$ as

$$\tilde{F}_{1,0} = p_{1,0}^{-1} \circ F$$

which is continuous on the square $S_{1,0}$ and

$$p \circ \tilde{F}_{1,0} = F\Big|_{S_{1,0}}$$

In order to glue together the pieces $\tilde{F}_{0,0}$ and $\tilde{F}_{1,0}$ we must ensure that they agree all along the common edge BC of the adjacent squares $S_{0,0}$ and $S_{1,0}$. Their restrictions along BC where $t = 0$ and $0 \leq s \leq 1/n$ agree at B namely

$$\tilde{F}_{0,0}(0, \frac{1}{n}) = \tilde{F}_{1,0}(0, \frac{1}{n})$$

and are both lifts of the map

$$s \mapsto F(s, \frac{1}{n}), \quad 0 \leq s \leq \frac{1}{n}$$

which implies, by uniqueness of lifts,

$$\tilde{F}_{0,0}(s, \frac{1}{n}) = \tilde{F}_{1,0}(s, \frac{1}{n}), \quad 0 \leq s \leq \frac{1}{n},$$

as desired. It is now clear how the construction ought to proceed and we get a lift $\tilde{F} : [0,1] \times [0,1] \longrightarrow \tilde{X}$ of F.

We now have to check that $\tilde{F}$ is indeed a homotopy of paths with fixed endpoints. Well,

$$p \circ \tilde{F}(s, 0) = F(s, 0) = x_0, \text{ for all } s \in [0, 1]$$

so that the connected set

$$\{\tilde{F}(s, 0)/0 \leq s \leq 1\}$$

is contained in the discrete set $p^{-1}(x_0)$ and so must reduce to a singleton. Likewise $\tilde{F}(s, 1)$ is constant as s varies over $[0, 1]$. Also $p \circ \tilde{F}(0, t) = F(0, t) = \gamma_1(t)$ and $p \circ \tilde{F}(1, t) = F(1, t) = \gamma_2(t)$ showing that $\tilde{F}$ is the desired homotopy between the lifts of γ_1 and γ_2 starting at $\tilde{x}_0$. □

Theorem 16.3: Given a covering projection $p : \tilde{X} \longrightarrow X$, for any $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ the induced group homomorphism

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$$

is injective.

Proof: Let $\tilde{\gamma}$ be a loop in $\tilde{X}$ based at $\tilde{x}_0$ that represents an element of ker p_*. This means the loop $\gamma = p \circ \tilde{\gamma}$ is homotopic to the constant loop in X based at x_0. But the constant loop ε_{x_0} at x_0 lifts as the constant loop $\varepsilon\tilde{x}_0$ at $\tilde{x}_0 \in \tilde{X}$. By the covering homotopy theorem we conclude that $\tilde{\gamma}$ and the constant loop $\varepsilon_{\tilde{x}_0}$ are homotopic. That is to say $[\tilde{\gamma}]$ is the trivial element in $\pi_1(\tilde{X}, \tilde{x}_0)$. □

Remark: The above theorem enables us to identify $\pi_1(\tilde{X}, \tilde{x}_0)$ as a subgroup of $\pi_1(X, x_0)$.

We shall now discuss another important consequence of the path lifting property.

Theorem 16.4: Given a covering projection $p : \tilde{X} \longrightarrow X$ where X and $\tilde{X}$ are path-connected, for any points $x_1, x_2 \in X$ the fibers $p^{-1}(x_1)$ and $p^{-1}(x_2)$ have the same cardinality.

Proof: We shall construct injective maps from $p^{-1}(x_1)$ into $p^{-1}(x_2)$ and vice versa. Fix a path γ in X joining x_1 and x_2. Pick $\tilde{x}_1 \in p^{-1}(x_1)$ and let $\tilde{\gamma}$ be the lift of γ starting at $\tilde{x}_1$ and define a map $T : p^{-1}(x_1) \longrightarrow p^{-1}(x_2)$ by the prescription

$$T : \tilde{x}_1 \mapsto \tilde{\gamma}(1).$$

Likewise let $S : p^{-1}(x_2) \longrightarrow p^{-1}(x_1)$ be the map in the reverse direction constructed using the path γ^{-1}. Since the inverse path $\tilde{\gamma}^{-1}$ is the unique lift of γ^{-1} starting at $\tilde{\gamma}(1)$, we see that

$$S(\tilde{\gamma}(1)) = \tilde{\gamma}(0) = \tilde{x}_1,$$

whereby we conclude $S \circ T$ is the identity map on $p^{-1}(x_1)$. By symmetry $T \circ S$ is the identity map on $p^{-1}(x_2)$ as desired. □

Exercises

1. Use the general results of this section to give an efficient and transparent proof that $\pi_1(S^1, 1) = \mathbb{Z}$. First show that for any loop γ based at 1, the map $\pi_1(S^1, 1) \longrightarrow \mathbb{Z}$ given by $[\gamma] \mapsto \tilde{\gamma}(1)$ is well defined by theorem 16.1, is a group homomorphism using uniqueness of lifts. Show that surjectivity follows from uniqueness of lifts and injectivity follows from theorem 16.1.

2. Let X be a topological spaces and $a, b \in X$. A simple chain connecting a and b is a finite sequence $U_1, U_2, \ldots, U_n$ of open sets such that $a \in U_1$, $b \in U_n$ and for $1 \leq i < j \leq n$, $U_i \cap U_j \neq \emptyset$ implies $j = i + 1$. Show that if X is a connected metric space and $\mathcal{U}$ is an open covering of X then any

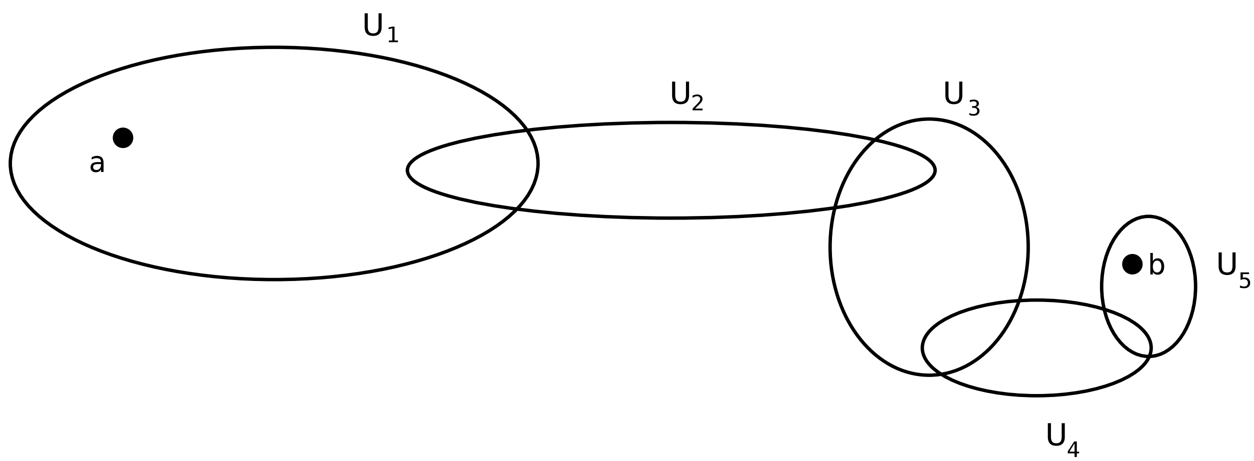

Figure 14: Chain connectedness

two points $a, b \in X$ can be connected by a simple chain. This property is referred to as chain connectedness. Is $\mathbb{Q}$ chain connected?

3. Use the above exercise to show that if X is a chain-connected space and $p : \tilde{X} \longrightarrow X$ is a covering projection then for any pair of points $x, y \in X$ the fibers $p^{-1}(x)$ and $p^{-1}(y)$ have the same cardinality. The point here is that X need not be path connected and the idea of using a path joining x and y as was done in the proof of theorem 14.4 is no longer available.

4. A toral knot is a group homomorphism $\kappa : S^1 \longrightarrow S^1 \times S^1$ given by $z \mapsto (z^m, z^n)$ where $m, n \in \mathbb{N}$. Regarding the toral knot as a loop on the torus determine its lifts with respect to the covering projection $\mathbb{R} \times \mathbb{R} \longrightarrow S^1 \times S^1$.

5. For the group homomorphism κ of the previous exercise describe the induced map κ_*.