Lecture - XXXVI Maps of spheres

We are now in a position to prove the general Brouwer's fixed point theorem as well as a few other surprising results concerning maps of spheres. As demonstrated in lecture 10, these higher dimensional analogues were inaccessible via the theory of the fundamental group. We shall introduce the notion of the degree of a map of spheres generalizing the notion introduced in lectures (12-13).

Theorem 36.1 (No retraction theorem): The sphere S^{n-1} is not a retract of the closed unit ball E^n.

Proof: Assume $n \geq 2$. A retraction $r : E^n \longrightarrow S^{n-1}$ would imply that $H_{n-1}(r) : E^n \longrightarrow S^{n-1}$ is surjective which is plainly false since $H_{n-1}(E^n) = \{0\}$ whereas $H_{n-1}(S^{n-1}) = \mathbb{Z}$. The case $n = 1$ is left to the reader.

Corollary 36.2 (Brouwer's fixed point theorem): Every continuous map $f : E^n \longrightarrow E^n$ has a fixed point.

Proof: The proof is similar to the one given in lecture 10 for the case $n = 2$.

Degree of a map: We now generalize the notion of the degree of a map $f : S^1 \longrightarrow S^1$ that we have defined earlier in lectures 12-13. We shall show later that for each $n \in \mathbb{N}$, there is a continuous function having degree n.

Definition 36.1: For $n \geq 1$, the degree of a continuous map $f : S^n \longrightarrow S^n$ is defined to be the integer m such that

$$H_n(f)(\overline{\eta}) = m\,\overline{\eta} \tag{36.1}$$

where $\overline{\eta}$ is a generator for the infinite cyclic group $H_n(S^n)$. Since $H_n(f) : H_n(S^n) \longrightarrow H_n(S^n)$ is a group homomorphism the choice of either of the two generators would yield the same result.

Theorem 36.3: Suppose that $f : S^1 \longrightarrow S^1$ is a continuous map such that $f(1) = 1$ then the degree of f as defined above agrees with the notion of degree as defined in lectures (12-13). Moreover the generator for the group $H_1(S^1)$ is the homology class of the cycle

$$\eta : t \mapsto \exp(2\pi i t), \quad 0 \leq t \leq 1. \tag{36.2}$$

Note that we have tacitly identified the standard one simplex Δ_1 with $[0, 1]$.

Proof: Since $\pi_1(S^1, 1)$ is abelian, the abelianization $\Pi : \pi_1(S^1, 1) \longrightarrow H_1(S^1)$ is an isomorphism and hence maps a generator of $\pi_1(S^1, 1)$ to a generator of $H_1(S^1)$. Since (36.2) represents a generator for $\pi_1(S^1, 1)$ we infer that the cycle (36.2) is a generator for $H_1(S^1)$. We deduce from the diagram (32.1) that

$$f_* = \Pi^{-1} \circ H_1(f) \circ \Pi. \tag{36.3}$$

From (12.6) and (36.3) we see that

$$H_1(f)\Pi[\eta] = (\Pi \circ f_*)[\eta] = \Pi((\deg f)[\eta]) = (\deg f)\Pi[\eta].$$

Appealing to the definition (36.1) we see that $m = \deg f$.

Corollary 36.4: The map $f : S^1 \longrightarrow S^1$ given by $f(x, y) = (x, -y)$ has degree -1.

Proof: The induced map $f_* : \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$ is multiplication by -1.

Theorem 36.5: The degree satisfies the following properties.

(i) The degree of the identity map $S^n \longrightarrow S^n$ is $+1$.

(ii) If f and g are two continuous maps from S^n to itself then $\deg (g \circ f) = (\deg g)(\deg f)$.

(iii) Homotopic maps from S^n into itself have the same degree.

(iv) If $f : S^n \longrightarrow S^n$ is a homotopy equivalence then degree of f is ± 1.

(v) Any map homotopic to the constant map has degree zero.

(vi) Any two maps $S^n \longrightarrow S^n$ having the same degree are homotopic (Theorem of H. Hopf).

Proof: The first five are easy exercises for the reader. We shall not prove (vi).

The anti-podal map and its properties: Let us now calculate the degree of the anti-podal map $A : S^n \longrightarrow S^n$ given by $A(\mathbf{x}) = -\mathbf{x}$. The anti-podal map is the composite of reflections in the coordinate hyperplanes and so it suffices to compute the degree of one of them say

$$R_n : (x_1, x_2, \dots, x_{n+1}) \mapsto (-x_1, x_2, \dots, x_n, x_{n+1}). \tag{36.4}$$

Theorem 36.6: The degree of the map (36.4) is -1 and hence the degree of the antipodal map $A : S^n \longrightarrow S^n$ is $(-1)^{n+1}$.

Proof: From corollary (36.4) the case $n = 1$ follows (exercise 1). The general case is done by induction. Let us consider the covering $\{U, V\}$ where $U = S^n - \{\mathbf{e}_{n+1}\}$, and $V = S^n - \{-\mathbf{e}_{n+1}\}$. The map R_n fixes U and V but when restricted to the equator S^{n-1} gives R_{n-1}. The naturality of the Mayer Vietoris sequence gives us the commutative diagram

$$\begin{tikzcd}
H_n(U \cup V) \arrow[r, "\delta_n"] \arrow[d, "H_n(R_n)"'] & H_{n-1}(U \cap V) \arrow[r, "H_{n-1}(r)"] \arrow[d, "H_n(R_n)"] & H_{n-1}(S^{n-1}) \arrow[d, "H_{n-1}(R_{n-1})"] \\
H_n(U \cup V) \arrow[r, "\delta_n"] & H_{n-1}(U \cap V) \arrow[r, "H_{n-1}(r)"] & H_{n-1}(S^{n-1}).
\end{tikzcd}$$

The map $H_{n-1}(r)$ is isomorphism induced by the retraction of $U \cap V$ onto the equator S^{n-1}. The connecting homomorphisms δ_n are isomorphisms as we have seen in the last lecture. Since the map $H_{n-1}(R_{n-1}) : \mathbb{Z} \longrightarrow \mathbb{Z}$ on the extreme right is given by multiplication by -1, the same is the case with the map $H_n(R_n)$ on the extreme left whereby we conclude that R_n has degree -1.

Corollary 36.7: The antipodal map $A : S^n \longrightarrow S^n$ is homotopic to the identity map if and only if n is odd.

Proof: If n is even then the identity map and the anti-podal map have different degrees and so cannot be homotopic. The converse is done in exercise 3.

Theorem 36.8: If f and g are a pair of continuous maps from S^n to itself such that $f(\mathbf{x}) \neq g(\mathbf{x})$ for every $\mathbf{x} \in S^n$. Then g is homotopic to $A \circ f$.

Proof: Since $f(\mathbf{x}) \neq g(\mathbf{x})$ the reader may verify that $tAf(\mathbf{x}) + (1-t)g(\mathbf{x}) \neq 0$ for any $t \in [0,1]$. Normalizing we get the desired homotopy:

$$F : (t, \mathbf{x}) \mapsto \frac{tAf(\mathbf{x}) + (1-t)g(\mathbf{x})}{\|tAf(\mathbf{x}) + (1-t)g(\mathbf{x})\|}, \quad t \in [0,1], \ \ \mathbf{x} \in S^n.$$

Corollary 36.9: If n is odd then any continuous map $f : S^n \longrightarrow S^n$ has a fixed point or sends a point to its antipode. Hence the pair $\{f(\mathbf{x}), \mathbf{x}\}$ cannot be linearly independent for every $\mathbf{x} \in S^n$.

Proof: Suppose $f(\mathbf{x}) \neq \mathbf{x}$ for any $\mathbf{x} \in S^n$, we see by theorem (36.8) that f is homotopic to the antipodal map. Further, if also $f(\mathbf{x}) \neq -\mathbf{x}$ for every $\mathbf{x} \in S^n$, theorem (36.8) implies f is homotopic to the identity map. This contradicts corollary (36.7).

Corollary 36.10 (Hairy ball theorem): If n is even, any continuous tangent vector field on S^n must have a zero.

Proof: A continuous, non-vanishing tangent vector field upon normalization yields a continuous map $f : S^n \longrightarrow S^n$ such that the pair of vectors $\{f(\mathbf{x}), \ \mathbf{x}\}$ is every where orthonormal which contradicts corollary (36.9).

Suspension: Given a topological space X, the suspension of X denoted by ΣX, is obtained from $X \times [0,1]$ by passing to a quotient (see the figure that follows equation (36.6)):

$$\Sigma X = (X \times [0,1])/(X \times \{0\} \cup X \times \{1\})$$

Using polar coordinates we can see that $\Sigma S^{n-1} \cong S^n$ via the homeomorphism $\phi : S^{n-1} \times [0,1] \longrightarrow S^n$

$$(\omega, t) \mapsto ((\sin \pi t)\, \omega, \cos \pi t), \quad t \in [0,1], \ \ \omega \in S^{n-1} \subset \mathbb{R}^n. \tag{36.5}$$

With this identification, given $f : S^{n-1} \longrightarrow S^{n-1}$ continuous we define $\Sigma f : S^n \longrightarrow S^n$ by

$$(\Sigma f)((\sin \pi t)\, \omega, \cos \pi t) = ((\sin \pi t) f(\omega), \cos \pi t) \tag{36.6}$$

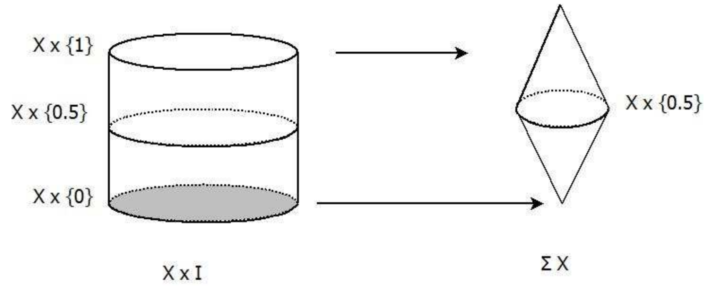

Figure 27: Suspension

Theorem 36.11: Given a continuous map $f : S^n \longrightarrow S^n$, the degree of Σf equals deg f. For every $m \in \mathbb{Z}$ there is a continuous map $f : S^n \longrightarrow S^n$ with degree m.

Proof: The argument parallels the one used to prove theorem (36.6) and is left for the reader.

Exercises

1. Show that if R' and R'' are two reflections (each with respect to a coordinate plane) then they are conjugate by a homeomorphism. Deduce that both R' and R'' have degree -1.

2. Show that if a continuous map $f : S^n \longrightarrow S^n$ misses a point of S^n then f is homotopic to the constant map and so has degree zero.

3. Show that if n is odd then the antipodal map of S^n is homotopic to the identity map. Hint: Do it first for the case $n = 1$ and show that the homotopy may be achieved via a continuous rotation. The general case follows along similar lines by working with pairs of coordinates.

4. Show that $\mathbb{R}P^{2n}$ has the fixed point property.

5. Let $\eta : S^{2n} \longrightarrow \mathbb{R}P^{2n}$ be the covering projection. Show that $H_{2n}(\eta)$ is the zero map.

6. Show that the map (36.5) is a homeomorphism and (36.6) defines a continuous map. More generally given a continuous map $f : X \longrightarrow Y$ show that the composite

$$X \times [0,1] \xrightarrow{f \times \mathrm{id}} Y \times [0,1] \longrightarrow \Sigma Y$$

   induces a map $\Sigma f : \Sigma X \longrightarrow \Sigma Y$. Imitate the computation in theorem [//] of lecture [//] to show that $H_{n+1}(\Sigma X) = H_n(X)$ when $n \geq 1$. What happens when $n = 0$?

7. Prove theorem (36.11). Note that the map $f : S^1 \longrightarrow S^1$ given by $f(z) = z^m$ has degree m.

8. Determine the degree of a polynomial as a map from S^2 to itself. Reprove the fundamental theorem of algebra.