Lecture - XXXIV Small simplicies

Recall that the Goursat lemma in complex analysis is proved by subdividing a triangle into four smaller triangles determined by the midpoints of the sides of the given triangle. The integral over the given triangle is then the sum of the integrals over the four little pieces. Likewise, in the proof of the classical Green's theorem (of which Cauchy's theorem is really a special case) one employs a subdivision into tiny squares. The contributions to the integral from an edge common to a pair of abutting triangles/squares cancel out.

A similar idea underlies the method of small simplicies where we perform a systematic subdivision operation known as *barycentric subdivision.* The barycentric subdivision enables us to replace a singular chain by a homotopic one in which the constituent singular simplicies are *small.* A small simplex is one whose image lies in an open set belonging to a prescribed open cover of the space. One achieves this through iterated barycentric subdivisions a process reminiscent of one used in the proof of the Goursat lemma. The fundamental theorem on small simplicies quickly leads us to the two fundamental results on algebraic topology - the excision theorem discussed in lecture 39 and the Mayer Vietoris sequence that we shall derive here and use in the next lecture.

Affine simplicies and barycentric subdivision: Given points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{p+1}$ in the standard $n-$simplex Δ_n, the continuous map $\sigma : \Delta_p \longrightarrow \Delta_n$ given in terms of the barycentric coordinates

$$\sum_{i=1}^{p+1} \lambda_i \mathbf{e}_i \mapsto \sum_{i=1}^{p+1} \lambda_i \mathbf{v}_i \tag{34.1}$$

is called an affine $p-$simplex and is denoted by $[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{p+1}]$. Note that the given need not be affinely independent. Each such σ is an element of $S_p(\Delta_n)$ and the subgroup generated by them is called the group of affine $p-$simplicies in Δ_n denoted by $A_p(\Delta_n)$. Thus $A_p(\Delta_n)$ is the set of all formal linear combinations with integer coefficients of affine simplicies. Since the face maps (29.1) are affine maps we conclude from exercise 2 that the boundary homomorphism $\partial_p : S_p(\Delta_n) \longrightarrow S_{p-1}(\Delta_n)$ maps $A_p(\Delta_n)$ into $A_{p-1}(\Delta_n)$ and so we get a subcomplex $\{A_p(\Delta_n)/p = 0, 1, 2, \dots\}$ with boundary maps as the restrictions of ∂_p to $A_p(\Delta_n)$.

If $\mathbf{b} \in \Delta_n$ is a given point the *cone* over the affine simplex $\sigma = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{p+1}]$ with vertex apex $\mathbf{b}$ is denoted by $K_{\mathbf{b}}\sigma$ and is defined as

$$K_{\mathbf{b}}\sigma = [\mathbf{b}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{p+1}] \tag{34.2}$$

The cone $K_{\mathbf{b}}\sigma$ is thus an affine $p+1$ simplex. If we start with a zero simplex namely, a point $\mathbf{v} \in S_n(\Delta_n)$, the cone over it is the line segment $[\mathbf{b}, \mathbf{v}]$. Since $A_p(\Delta_n)$ is a free abelian group generated by the affine p simplicies, we obtain by extension a group homomorphism $K_{\mathbf{b}} : A_p(\Delta_n) \longrightarrow A_{p+1}(\Delta_n)$. As in the proof of theorem (29.7) it is easy to compute the boundary of the cone $K_{\mathbf{b}}\sigma$ for any affine p simplex.

For a zero simplex $\sigma = [\mathbf{v}]$ we evidently have $\partial_1 K_{\mathbf{b}}(\sigma) = \sigma - [\mathbf{b}]$. We now calculate the faces of the affine $p+1$ simplex $K_{\mathbf{b}}(\sigma)$. If $j \geq 1$,

$$\begin{aligned}(K_{\mathbf{b}}\sigma \circ \Phi_j^p)(\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) &= K_{\mathbf{b}}\sigma(\lambda_1, \ldots, \lambda_j, 0, \lambda_{j+1}, \ldots, \lambda_{p+1}) \\ &= [\mathbf{b}, \mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{p+1}].\end{aligned}$$

This is the cone over the $j-$th face of $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1}]$. Turning to the case $j = 0$,

$$(K_{\mathbf{b}}\sigma \circ \Phi_0^p)(\lambda_1, \lambda_2, \ldots, \lambda_{p+1}) = K_{\mathbf{b}}\sigma(0, \lambda_1, \ldots, \lambda_{p+1}) = [\mathbf{v}_1, \ldots, \mathbf{v}_{p+1}].$$

Using equation (29.4) we immediately get the following result.

Theorem 34.1: The boundary of the cone $K_{\mathbf{b}}\sigma = [\mathbf{b}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1}]$ is given by

$$\partial K_{\mathbf{b}}\sigma = \sigma - K_{\mathbf{b}}\partial\sigma \tag{34.3}$$

Hitherto the choice of the apex $\mathbf{b}$ of the cone was arbitrary but now we shall specialize it to be the barycenter of $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1}]$ that we now define.

Definition 34.1 (Barycenter of an affine simplex): (i) The barycenter of a zero simplex, that is a point, is the zero simplex itself.

(ii) The barycenter of an affine p-simplex $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1}]$ is the point

$$\frac{1}{p+1}\Big(\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_{p+1}\Big) \tag{34.4}$$

The barycenter of a one simplex is its midpoint and the barycenter of a two simplex is the centroid of the triangle determined by the vertices. Roughly speaking, the barycentric subdivision of a one simplex is obtained by subdividing the segment at its midpoint, or equivalently constructing the cone of each of the two endpoints with apex as the barycenter. To subdivide a two simplex, we first subdivide each of its three sides resulting in six one simplicies and taking the cone of each of the six pieces with apex as the barycenter of the two simplex. Figure below depicts these subdivisions. More precisely

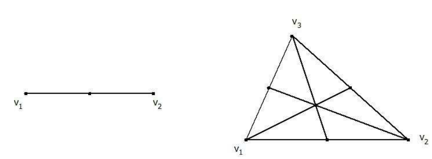

Figure 25:

the result of subdividing an affine $p-$simplex is a $p-$chain. The rough description above suggests an inductive definition.

Definition 34.2: The subdivision map $B : A_p(\Delta_n) \longrightarrow A_p(\Delta_n)$ is defined inductively as follows:

(i) For a zero simplex σ we define $B\sigma = \sigma$.

(ii) For $p \geq 1$, we assume that B is defined on $A_k(\Delta_n)$ for each $k \leq p-1$. For a $p-$simplex σ define

$$B\sigma = K_{\mathbf{b}}(B(\partial\sigma)), \tag{34.5}$$

the cone over the chain $B(\partial\sigma)$ with apex $\mathbf{b}$ as the barycenter of σ.

Theorem 34.2: The map $B : A_p(\Delta_n) \longrightarrow A_p(\Delta_n)$ is a chain map which is chain homotopic to the identity map.

Proof: If $p = 0$ and σ is a zero chain then $B\partial\sigma = 0$ whereas $\partial B\sigma = \partial\sigma = 0$. To handle the case $p > 0$ we assume inductively that for any $k-$chain σ with $k \leq p-1$, the equation $\partial B\sigma = B\partial\sigma$ holds. To prove it for p chains, let σ be an arbitrary affine $p-$simplex. Equations (34.5) and (34.3) combine to give

$$\partial B\sigma = \partial K_{\mathbf{b}}(B\partial\sigma) = B\partial\sigma - K_{\mathbf{b}}(\partial B\partial\sigma) = B\partial\sigma - K_{\mathbf{b}}(B\partial\partial\sigma) = B\partial\sigma.$$

Note that induction hypothesis justifies $\partial B\partial\sigma = B\partial\partial\sigma$. We have now shown that for every p chain σ,

$$B\partial\sigma = \partial B\sigma. \tag{34.6}$$

We now construct a chain homotopy $J : A_p(\Delta_n) \longrightarrow A_{p+1}(\Delta_n)$ between B and the identity map. Equation (34.3) suggests a formula of the type

$$J\sigma = K_{\mathbf{b}}f(\sigma),$$

where $\mathbf{b}$ is the barycenter of σ and $f : A_p(\Delta_n) \longrightarrow A_p(\Delta_n)$ is to be determined. The condition that J is a chain homotopy between B and the identity now forces

$$f(\sigma) - K_{\mathbf{b}}\partial f(\sigma) = B\sigma - \sigma - J(\partial\sigma). \tag{34.7}$$

Clearly $f(\sigma) = 0$ for a zero simplex σ. If we assume inductively that $J : A_k(\Delta_n) \longrightarrow A_{k+1}(\Delta_n)$ has already been defined for $k \leq p-1$, the right hand side of (34.7) is then a known function. Let us refer to the term $K_{\mathbf{b}}\partial f(\sigma)$ in equation (34.7) as *junk*. Exercise 3 invites the reader to check that retaining the junk term is unnecessary. We set it equal to zero and define formally for a $p-$simplex σ,

$$J\sigma = \begin{cases} 0 & \text{if } p = 0, \\ K_{\mathbf{b}}(B\sigma - \sigma - J(\partial\sigma)) & \text{if } p \geq 1. \end{cases}$$

Let us now verify that this formula does the job. The case $p = 0$ is trivial and let us assume

$$\partial J\sigma + J(\partial\sigma) = B\sigma - \sigma$$

for any k chain such that $k \leq p-1$. Using the formula of J we see that

$$\partial J\sigma = B\sigma - \sigma - J(\partial\sigma) - K_{\mathbf{b}}(\partial B\sigma - \partial\sigma - \partial J(\partial\sigma)). \tag{34.8}$$

By induction hypothesis $\partial J(\partial\sigma) = -J(\partial\partial\sigma) + B(\partial\sigma) - \partial\sigma$. Inserting this in (34.8) we get the desired result

$$\partial J\sigma = B\sigma - \sigma - J(\partial\sigma). \tag{34.9}$$

We shall now transfer the barycentric subdivision operator and the chain homotopy J to a chain map $\mathcal{B} : S_p(X) \longrightarrow S_p(X)$ and a chain homotopy $\mathcal{J} : S_p(X) \longrightarrow S_{p+1}(X)$. This will be unique subject to naturality.

Theorem 34.3: For each topological space X, there exists a unique chain map $\mathcal{B}_X : S_p(X) \longrightarrow S_p(X)$ and a chain homotopy $\mathcal{J}_X : S_p(X) \longrightarrow S_{p+1}(X)$ between $\mathcal{B}$ and the identity map, which satisfies the following two conditions.

(i) For a continuous map $f : X \longrightarrow Y$ between topological spaces X and Y,

$$\mathcal{B} \circ f_\sharp = f_\sharp \circ \mathcal{B}, \quad \mathcal{J} \circ f_\sharp = f_\sharp \circ \mathcal{J}.$$

(ii) $\mathcal{B}$ and $\mathcal{J}$ when restricted to the affine simplicies reduce to B and J respectively.

Proof: Let ι_p be the element of $A_p(\Delta_p)$ given by the identity map from Δ_p to itself. Since an arbitrary $\sigma \in S_p(X)$ can be written as $\sigma = \sigma_\sharp \iota_p$, condition (i) forces

$$\mathcal{B}\sigma = (\mathcal{B} \circ \sigma_\sharp)\iota_p = \sigma_\sharp(\mathcal{B}\iota_p) = \sigma_\sharp B\iota_p \tag{34.10}$$

since $\mathcal{B}\iota_p = B\iota_p$ by (ii). Thus the conditions (i) and (ii) determine $\mathcal{B}$ uniquely on the generators of the free abelian group $S_p(X)$. The same argument applies to $\mathcal{J}$.

We use (34.10) and (34.5) to show that B and $\mathcal{B}$ agree on any affine simplex $\sigma \in A_p(\Delta_n)$. Denoting by $\mathbf{g}$ the barycenter of ι_p and by $\mathbf{b}$ the barycenter of σ,

$$\mathcal{B}\sigma = \sigma_\sharp B\iota_p = \sigma_\sharp(K_{\mathbf{g}}(\partial_p \iota_p)).$$

Using exercise 2, this may be rewritten as

$$\mathcal{B}\sigma = K_{\sigma(\mathbf{g})}(\sigma_\sharp(\partial_p \iota_p)) = K_{\mathbf{b}}(\partial_p \sigma_\sharp \iota_p) = K_{\mathbf{b}}(\partial_p \sigma) = B\sigma.$$

The verification for $\mathcal{J}$ is similar. We now run through the proof that $\mathcal{B}$ is a chain map, which is now automatic. For an arbitrary $\sigma \in S_p(X)$, $\partial\mathcal{B}\sigma = \partial(\sigma_\sharp B\iota_p) = \sigma_\sharp(\partial B\iota_p)$. Since $\partial\iota_p$ is an affine chain and B is a chain map on the subcomplex of affine chains we get $\partial B\iota_p = B\partial\iota_p$. Applying $\sigma_\sharp$ to this gives $\partial\mathcal{B}\sigma = \sigma_\sharp(B\partial\iota_p)$. Working from the other end using the fact that $\sigma_\sharp$ is a chain map and $\mathcal{B}$ satisfies (i), we get

$$\mathcal{B}\partial\sigma = \mathcal{B}\partial(\sigma_\sharp\iota_p) = \mathcal{B}(\sigma_\sharp\partial\iota_p) = \sigma_\sharp\mathcal{B}(\partial\iota_p) = \sigma_\sharp B(\partial\iota_p).$$

Finally we show that $\mathcal{J}$ is a chain homotopy between $\mathcal{B}$ and the identity operator. For $\sigma \in S_p(X)$,

$$\begin{aligned} \mathcal{J}\partial\sigma &= \mathcal{J}\partial\sigma_\sharp\iota_p = \mathcal{J}\sigma_\sharp\partial\iota_p = \sigma_\sharp\mathcal{J}\partial\iota_p = \sigma_\sharp J\partial\iota_p \\ \partial\mathcal{J}\sigma &= \partial\mathcal{J}\sigma_\sharp\iota_p = \partial\sigma_\sharp\mathcal{J}\iota_p = \sigma_\sharp\partial\mathcal{J}\iota_p = \sigma_\sharp\partial J\iota_p. \end{aligned}$$

We have used (i) and (ii) and the fact that $\sigma_\sharp$ is a chain map. Subtracting and using (34.9) we get the desired result. $\square$

Theorem 34.4: (i) The diameter of an affine $p-$simplex $\sigma = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p+1}]$ is the length of its longest side namely,

$$\max_{i \neq j} \|\mathbf{v}_i - \mathbf{v}_j\|.$$

(ii) The diameter of any constituent simplex in the chain $B\sigma$ is $\left(\dfrac{p}{p+1}\right)\text{diam }(\sigma)$.

Proof: We leave (i) as an exercise for the reader. To prove (ii) we use induction on p setting aside the cases $p = 1, 2$ for the reader to investigate. Denoting by $\mathbf{b}$ the barycenter of σ, the reader may check that $\|\mathbf{b} - \mathbf{x}\| \leq p(p+1)^{-1}(\text{diam } \sigma)$ for any point $\mathbf{x}$ of σ. Let τ be one of the simplicies appearing in the chain $B\sigma$. Then the diameter of τ equals $\|w - z\|$ where w and z are two vertices of τ. If one of these is $\mathbf{b}$ then the result follows from the assertion in the previous sentence. If neither w nor z is $\mathbf{b}$ then they are both vertices of a face τ' of τ lying on a face σ' of σ. But τ' is then a constituent $(p-1)$ simplex of $B(\sigma')$ and by induction hypothesis, the result follows (how?). $\square$

Definition 34.3: Given an open covering $\mathcal{U}$ of X, $S_n^{\mathcal{U}}(X)$ denotes the subgroup of $S_n(X)$ generated by all the singular simplicies $\sigma : \Delta_n \longrightarrow X$ such that $\sigma(\Delta_n) \subset U_\sigma$ for some open set U_σ in the covering $\mathcal{U}$. That is to say, $S_n^{\mathcal{U}}(X)$ is the free abelian group generated by *small simplicies,* namely those with images contained in one of the open sets in the given covering. It is clear that that the boundary homomorphism ∂_n maps $S_n^{\mathcal{U}}(X)$ into $S_{n-1}^{\mathcal{U}}(X)$ and the resulting subcomplex is denoted by $S^{\mathcal{U}}(X)$. The homology groups of the complex $S^{\mathcal{U}}(X)$ will be denoted by $H_n^{\mathcal{U}}(X)$.

Lemma 34.5: (i) Given an open cover $\mathcal{U}$ of X and a singular simplex $\sigma \in S_p(X)$, there exists a $k \in \mathbb{N}$ such that $\mathcal{B}^k\sigma \in S_p^{\mathcal{U}}(X)$. In other words each of the simplicies occurring in $\mathcal{B}^k\sigma$ has its image in one of the open sets of the cover $\mathcal{U}$.

(ii) If σ is a singular p simplex whose image lies in an open set $U \in \mathcal{U}$ then $\mathcal{J}\sigma \in S_{p+1}^{\mathcal{U}}(X)$ where $\mathcal{J}$ is the chain homotopy constructed in theorem (34.3).

Proof: (i) Choose a Lebesgue number for the open cover $\{\sigma^{-1}(U) \ / U \in \mathcal{U}\}$. According to theorem (34.3), the images of the simplicies occurring in the chain $\mathcal{B}^k\sigma$ are the same as the images under σ of the affine simplicies occurring in $B^k\iota_p$, where ι_p is the identity map of Δ_p. However, theorem (34.4) states that the simplicies occurring in $B^k\iota_p$ have diameters less than $(p(p+1)^{-1})^k$. Thus, if we choose k sufficiently large the image of each of the simplicies in $B^k\sigma$ would lie in one of the open sets of $\mathcal{U}$.

To prove (ii) we use the naturality of $\mathcal{J}$ and proceed as in the proof of theorem (34.4). Let $\sigma : \Delta_p \longrightarrow X$ have its image in $U \in \mathcal{U}$. Then $\mathcal{J}\sigma = \sigma_\sharp(J\iota_p)$. But we see immediately from the definition of J in theorem (34.3) that $J\iota_p$ is a $\mathbb{Z}-$linear combination:

$$J\iota_p = \sum c_k\lambda_k$$

where each λ_k is a (degenerate) affine (p+1) simplex contained in Δ_p and hence $\sigma_\sharp(\lambda_k)$ is a singular $(p+1)$ simplex with image contained in U. $\square$

Theorem 34.6: The inclusion maps $S_n^{\mathcal{U}}(X) \longrightarrow S_n(X)$ $(n = 0, 1, 2, \ldots)$ define a chain map of complexes. Further, these inclusion maps induce isomorphisms in homology:

$$H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X), \quad n = 0, 1, 2, \ldots$$

Proof: The first assertion follows from the comments preceding lemma (34.5). To show that the inclusion maps induce an injective map on homologies, let $\sigma \in S_p^{\mathcal{U}}(X)$ be a singular chain such that $\sigma = \partial\eta$ for some $\eta \in S_{p+1}(X)$. Choose $k \in \mathbb{N}$ such that $\mathcal{B}^k\eta \in S_{p+1}^{\mathcal{U}}(X)$. We have to show that $\mathcal{B}^k\eta$ is a boundary in $S^{\mathcal{U}}$. By exercise 5, $\mathcal{B}^k$ is chain homotopic to the identity via a homotopy T_k say. Applying ∂ to

$$\mathcal{B}^k\eta - \eta = T_k\partial\eta + \partial T_k\eta,$$

we see that $\partial(\mathcal{B}^k\eta)-\sigma=\partial T_k\sigma$. By (ii) of lemma (34.5), $\partial T_k\sigma\in S_p^{\mathcal{U}}(X)$ which means σ is a boundary in $S_p^{\mathcal{U}}(X)$. To prove surjectivity, let σ be a cycle in $S(X)$ and $k\in\mathbb{N}$ be such that $\mathcal{B}^k\sigma\in S^{\mathcal{U}}(X)$. From $\mathcal{B}^k\sigma-\sigma=\partial T_k\sigma$ we conclude that σ is homologous to the cycle $\mathcal{B}^k\sigma$ in $S^{\mathcal{U}}(X)$. $\square$

Theorem 34.7 (Mayer Vietoris sequence): (i) Let $\{U,V\}$ be an open covering of X,

$$\kappa':H_k(U\cap V)\longrightarrow H_k(U),\quad \kappa'':H_k(U\cap V)\longrightarrow H_k(V)$$

be the maps induced by inclusions. Further, let $q_n:H_n(U)\oplus H_n(V)\longrightarrow H_n(U\cup V)$ be the map:

$$(a,b)\mapsto j_{1*}a+j_{2*}b,$$

where j_{1*} and j_{2*} are induced by the respective inclusions $j_1:U\longrightarrow U\cup V$ and $j_2:V\longrightarrow U\cup V$. Then, the following long exact sequence known as the *Mayer Vietoris sequence* holds:

$$\longrightarrow H_n(U\cap V)\xrightarrow{(\kappa',-\kappa'')}H_n(U)\oplus H_n(V)\xrightarrow{q_n}H_n(U\cup V)\xrightarrow{\delta_n}H_{n-1}(U\cap V)\longrightarrow$$

(ii) A cycle $\zeta\in Z_n(U\cup V)$ may be represented (modulo boundaries) as $\zeta=\zeta_1+\zeta_2$ for some $\zeta_1\in S_n(U)$ and $\zeta_2\in S_n(V)$ and the connecting homomorphism δ_n is given by

$$\delta_n\ :\ \zeta\mapsto\partial\zeta_1=-\partial\zeta_2.$$

Proof: In the diagrams below, the Left hand square depicts a push-out square of inclusions which goes over to a push-out square of complexes on the right:

$$\begin{tikzcd} U\cap V \arrow[r, "i_1"] \arrow[d, "i_2"'] & U \arrow[d, "j_1"] \\ V \arrow[r, "j_2"] & U\cup V \end{tikzcd} \qquad \begin{tikzcd} S(U\cap V) \arrow[r, "i_1"] \arrow[d, "i_2"'] & S(U) \arrow[d, "j_1"] \\ S(V) \arrow[r, "j_2"] & S^{\mathcal{U}}(U\cup V) \end{tikzcd}$$

The reader may check that the latter may be recast as a short exact sequence of chain complexes namely

$$0\longrightarrow S(U\cap V)\xrightarrow{(i_1,-i_2)}S(U)\oplus S(V)\xrightarrow{j_1+j_2}S^{\mathcal{U}}(U\cap V)\longrightarrow 0. \tag{34.11}$$

The corresponding long exact sequence in homology gives

$$\longrightarrow H_n(U\cap V)\xrightarrow{(\kappa',-\kappa'')}H_n(U)\oplus H_n(V)\xrightarrow{Q_n}H_n^{\mathcal{U}}(U\cup V)\xrightarrow{D_n}H_{n-1}(U\cap V)\longrightarrow$$

The definition of κ',κ'' and exercise 6 enables us to replace Q_n and D_n by the composites

$$q_n\ :\ H_n(U)\oplus H_n(V)\xrightarrow{Q_n}H_n^{\mathcal{U}}(U\cup V)\xrightarrow{\lambda}H_n(U\cup V)$$

$$\delta_n\ :\ H_n(U\cap V)\xrightarrow{\lambda^{-1}}H_n^{\mathcal{U}}(U\cup V)\xrightarrow{D_n}H_n(U\cap V) \tag{34.12}$$

where λ is the isomorphism given by theorem (34.6). The final result is the Mayer Vietoris sequence stated in the theorem. The second part is clear from (29.18). $\square$

Theorem 34.8 (Naturality of the Mayer Vietoris sequence): Given a continuous map of pairs $f : (U, V) \longrightarrow (A, B)$ where $\{U, V\}$ and $\{A, B\}$ are open coverings of topological spaces, the following diagram commutes where the vertical maps are induced by f.

$$\begin{tikzcd}
{} \arrow[r] & H_n(U \cap V) \arrow[r] \arrow[d] & H_n(U) \oplus H_n(V) \arrow[r] \arrow[d] & H_n(U \cup V) \arrow[r, "\delta_n"] \arrow[d] & H_{n-1}(U \cap V) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H_n(A \cap B) \arrow[r] & H_n(A) \oplus H_n(B) \arrow[r] & H_n(A \cup B)) \arrow[r] & H_{n-1}(A \cap B) \arrow[r] & {}
\end{tikzcd} \tag{34.13}$$

Proof: The proof is left for the reader. The non-trivial part concerns only the squares involving the connecting homomorphism for which (ii) of the previous theorem may be employed. ☐

Exercises

1. Show that the map defined by (34.1) is the restriction to Δ_p of an affine map. Note: An affine map is the composition of a linear map and a translation.

2. Suppose $T : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{m+1}$ is an affine map such that $T(\Delta_n) \subset \Delta_m$, then $T_\sharp$ maps the subgroup $A_p(\Delta_n)$ into $A_p(\Delta_m)$ and is a chain map from the complex $\{A_p(\Delta_n)\}$ to $\{A_p(\Delta_m)\}$. Further prove the following:

   (i) If $\mathbf{b} \in \Delta_n$ and $\sigma \in A_p(\Delta_n)$ then $T_\sharp(K_{\mathbf{b}}\sigma) = K_{T\mathbf{b}}(T_\sharp\sigma)$.

   (ii) If $\mathbf{b}$ is the barycenter of σ then $\mathbf{b}$ is the barycenter of $T_\sharp\sigma$.

   What happens if we consider a *degenerate* two simplex where the points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are not affinely independent? Discuss the case of the two simplex $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_2]$.

3. Examine what happens if the term referred to as junk in equation (34.7) is retained.

4. Complete the details of the proof of theorem (34.4).

5. Show that $\mathcal{B}^k$ is chain homotopic to the identity map. What is the chain homotopy?

6. Suppose that the maps g and h in the exact sequence

   $$A \longrightarrow B \xrightarrow{g} C \xrightarrow{h} D \longrightarrow E$$

   are replaced by the composites

   $$\tilde{g} : B \xrightarrow{g} C \xrightarrow{\lambda} X, \qquad \tilde{h} : X \xrightarrow{\lambda^{-1}} C \xrightarrow{h} D$$

   the result is again an exact sequence.

7. Fill in the details in the proof of theorem (34.8). See exercise 6 of lecture 29.