Lecture XX - Orbit Spaces

Many interesting spaces in geometry arise as the space of orbits under the action of groups. We have seen examples of this already in lecture 4. An important special case is when the group action is discrete such as the case of the multiplicative group $\{\pm 1\}$ on the sphere S^n resulting in the real projective space $\mathbb{R}P^n$.

Fundamental groups of orbit spaces: Recall that a group is said to act freely if there are no fixed points of the action. That is to say, if G acts on S such that if $g \cdot s = s$ for all $s \in S$ then $g = 1$. We now define a stronger notion when the group acts on a topological space.

Definition 20.1: Let Y be a topological space on which a group G acts. We say that the action is properly discontinuous if each point $y \in Y$ has a neighborhood U such that for any pair of *distinct* elements $g', g'' \in G$,

$$g'U \cap g''U = \emptyset.$$

Theorem 20.1: If a group G acts properly discontinuously on a topological space then the group action must then be free.

Proof: We shall show that if $g \cdot y = y$ for some $y \in Y$ and $g \in G$ then $g = 1$. If $g \neq 1$, choose a neighborhood U of y as in definition (20.1) which in particular implies $g \cdot U \cap U = \emptyset$. But $y \in g \cdot U \cap U$ and we get a contradiction.

The set of all orbits of the action with its quotient topology is denoted by Y/G and the following theorem expresses the covering properties of the quotient map

$$\eta : Y \longrightarrow Y/G.$$

Note that for each $g \in G$, the map $y \mapsto g \cdot y$ is a bijective map. If each of these maps is a homeomorphism of Y onto itself, we say that G acts as a group of homeomorphisms on Y.

Theorem 20.2: Let Y be a Hausdorff space and G be a group of acting properly discontinuously on Y as a group of homeomorphisms. Then,

(i) The orbit space Y/G is Hausdorff.

(ii) The quotient map $\eta : Y \longrightarrow Y/G$ is a covering projection.

(iii) G is the group of deck-transformations for the covering projection $\eta : Y \longrightarrow Y/G$.

(iv) In case Y is simply connected, $\pi_1(Y/G)$ is isomorphic to G.

Proof: Pick distinct points $\bar{y}, \bar{z} \in Y/G$ and let U and V be disjoint neighborhoods of y and z such that for every pair of distinct elements $g', g'' \in G$,

$$g'U \cap g''U = \emptyset, \quad g'V \cap g''V = \emptyset.$$

Then

$$\eta^{-1}(\eta(U)) = \bigcup_{g \in G} gU, \quad \eta^{-1}(\eta(V)) = \bigcup_{g \in G} gV.$$

Since G is a group of homeomorphisms, it follows from the definition of quotient topology that $\eta(U)$ and $\eta(V)$ are open sets containing $\bar{y}$ and $\bar{z}$. It is easy to see that $\eta(U)$ and $\eta(V)$ are disjoint and (i) follows and also that η is an open mapping. Now η restricted to each gU is a continuous, open bijection, that is a homeomorphism onto $\eta(U)$ and so (ii) follows. Conclusion (iv) follows from (iii). To prove (iii) first observe that the map

$$\phi_g : y \mapsto g \cdot y, \quad y \in Y$$

is a deck transformation for each $g \in G$. The map

$$\psi : g \mapsto \phi_g$$

is easily seen to be a group homomorphism. To see that it is surjective, let ϕ be a deck transformation and y_1 be a given point in Y and $\phi(y_1) = y_2$. Since y_1 and y_2 are in the same fiber, there is a unique element g of the group such that $g \cdot y_1 = y_2$. Then the deck transformations ϕ and ϕ_g agree at y_1 and so are identical which means $\psi(g) = \phi$ proving surjectivity. If $g \in \ker \psi$ then

$$\phi_g(y) = g \cdot y = y, \quad \forall y \in Y.$$

Since the action is properly discontinuous (and hence fixed point free) this forces $g = 1$.

Definition 20.2 (Lens spaces): Let $Y = S^3 = \{(z, w) \in \mathbb{C}^2 / |z|^2 + |w|^2 = 1\}$ and p be a prime, q be an integer relatively prime to p. The action of $\mathbb{Z}_p$ on S^3 given by

$$\exp(2\pi ik/p) \cdot (z_1, z_2) = (\exp(2\pi ik/p)z_1, \exp(2\pi ikq/p)z_2)$$

is fixed point free and hence properly discontinuous. The orbit space is called the lens space denoted by $L(p, q)$. Theorem (20.2) now implies

$$\pi_1(L(p, q)) = \mathbb{Z}_p.$$

Definition 20.3 (Generalized lens spaces): Let $q_1, q_2, \ldots, q_n$ be relatively prime to p. Define the action of the cyclic group $\mathbb{Z}_p$ on S^{2n+1} by

$$(z_0, z_1, \ldots, z_n) \mapsto \Big(z_0 \exp\Big(\frac{2\pi i}{p}\Big), z_1 \exp\Big(\frac{2\pi i q_1}{p}\Big), \ldots, \exp\Big(\frac{2\pi i q_n}{p}\Big)\Big),$$

where $S^{2n+1} = \{(z, w_1, w_2, \ldots, w_n) \in \mathbb{C}^{n+1} / |z|^2 + |w_1|^2 + \cdots + |w_n|^2 = 1\}$. The resulting orbit space is denoted by $L(p, q_1, q_2, \ldots, q_n)$ and its fundamental group is $\mathbb{Z}_p$ since the action is properly discontinuous.

The Mobius band: Consider the strip $Y = [0, 1] \times \mathbb{R}$ and let $S : Y \longrightarrow Y$ be the homeomorphism

$$S(x, y) = (1 - x, y + 1)$$

of Y. Then S generates an infinite cyclic group of homeomorphisms of Y acting properly discontinuously on Y. The resulting orbit space is the Möbius band. It is an exercise to show that the cylinder is a double cover of the Möbius band.

Klein's bottle: Let $Y = \mathbb{R}^2$ and G be the group generated by the affine maps T and S given by

$$T(x, y) = (x + 1, y), \quad S(x, y) = (1 - x, y + 1).$$

Note that T and S are isometies and the group generated by these acts properly discontinuously on $\mathbb{R}^2$. The orbit space is the Klein's bottle K. Thus $\pi_1(K) = G$. Now

$$TS(x, y) = (2 - x, y + 1), \quad ST(x, y) = (-x, y + 1)$$

The fundamental group of the Klein's bottle is non-abelian. Note that $TST = S$. There are no other independent relations and the fundamental group of the Klein's bottle is the group on two generators T and S with one relation $TST = S$. Summarizing we have

Theorem 20.3: The fundamental group of the Klein's bottle is the non-abelian group with two generators S and T with the relation $TST = S$.

Exercises:

1. Suppose that G is a finite group acting freely on a Hausdorff space then the action is properly discontinuous and hence deduce that the group action in the example of the generalized Lens space is properly discontinuous.

2. Suppose that $p : \tilde{X} \longrightarrow X$ is a covering projection and $\tilde{X}$ is locally path connected and simply connected. Show that if U is an evenly covered open set in X and $\tilde{U}$ is a sheet lying above it then $\phi(\tilde{U}) \cap \tilde{U} = \emptyset$ for every $\phi \in \mathrm{Deck}(\tilde{X}, X)$ and $\phi \neq \mathrm{id}_{\tilde{X}}$. Deduce that the group of deck transformations acts properly discontinuously on $\tilde{X}$. How does this relate to theorem 17.2?

3. Does the fundamental group of Klein's bottle have elements of finite order? Identify this group with a familiar group that we have already encountered in lecture 7. What is its abelianization?

4. Show that the torus is obtained as the orbit space of a group of homeomorphisms acting properly discontinuously on $\mathbb{R}^2$. Write out these homeomorphisms explicitly.

5. Show that the torus is a double cover of the Klein's bottle. Hence the fundamental group of the Klein's bottle must contain a subgroup of index two. Determine this subgroup.

6. Show that the cylinder is a two-sheeted cover of the Möbius band.

7. Suppose that G is a topological group, H is a discrete subgroup of G. Show that there exists a neighborhood U of the identity such that $U = U^{-1}$, $U \cap H = \{1\}$ and that $\{hU/h \in H\}$ is a family of disjoint open sets. Deduce that the quotient map $\eta : G \longrightarrow G/H$ is a covering projection. Also show that G/h is Hausdorff.