Lecture I - Introduction

General topology, a language for communicating ideas of continuous geometry, provides us useful tools for studying local properties of space. Notions of compactness and connectedness though important, are quite inadequate for obtaining a reasonable understanding of the global geometry of space. For example, the sphere and the torus are not homeomorphic although they are both compact, path-connected, locally connected metric spaces.

Algebraic topology is a powerful tool in global analysis - the study involving the global geometry of space. It is difficult to define precisely at this point what global analysis is. Perhaps the few examples discussed in the following paragraphs may help in understanding this. The most basic example comes from advanced calculus in connection with Stokes' theorem where a student encounters the notion of orientability of a two dimensional surface in $\mathbb{R}^3$. A sphere is easily seen to be orientable inasmuch as it has "two sides". Small pieces of a surface obviously have "two sides" but the Möbius band "has only one side". How would one formulate a precise notion of an orientable surface and prove that the Möbius band is non-orientable? Is non-orientability an intrinsic property of the surface or does it depend on the way the surface is presented in $\mathbb{R}^3$?

Frequently one also sees an interplay between local and global analysis. The powerful algebraic techniques that we shall develop streamlines the process of piecing together local information (which is often trivial) to provide non-trivial information on the global geometry of space. A good example illustrating this "piecing of local information" is provided by the proof of the famous theorem in complex analysis asserting the impossibility of a continuous branch of the argument function on the punctured plane $\mathbb{C} - \{0\}$. Although formal use of algebraic topology can be avoided for this specific case, it is less obvious that the function $\sqrt{1-z^2}$ is holomorphic on $\mathbb{C} - [-1, 1]$. Analogous problems in several dimensions would be practically intractable without the use of algebraic topology or some other equally powerful tool in global analysis.

The first example in our list is provided by the famous Jordan curve theorem which also arose in connections with complex analysis.

Theorem 1.1 (Jordan Curve Theorem): A simple closed curve separates the plane into two disjoint open connected sets precisely one of which is bounded.

The theorem was used by Jordan in his formulation of Cauchy's theorem. Though the Jordan curve theorem no longer plays a central rôle in complex function theory it is nevertheless indispensable in many other branches such as ordinary differential equations. Let us consider the (non-trivial) problem of locating periodic solutions of systems of differential equations. In planar domains, a useful criterion is given by the following

Theorem 1.2 (Poincare Bendixon): Suppose given a planar system of differential equations

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{1.1}$$

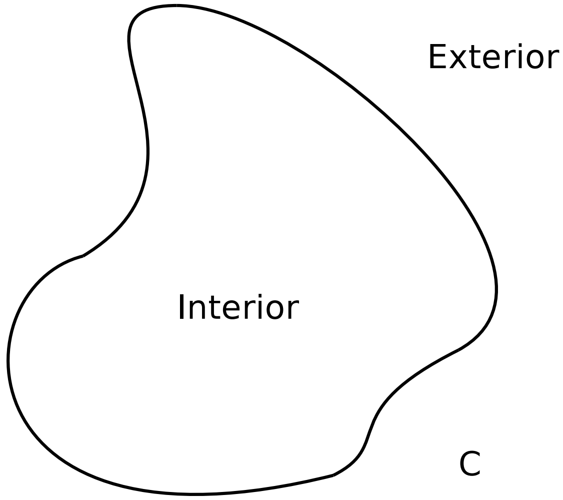

Figure 1: Simple closed curve

where $P(x, y)$ and $Q(x, y)$ are smooth functions in the plane. Assume that there is an annulus Ω not containing rest points[1] and invariant under the flow of the differential equation[2]. Then Ω must contain periodic orbits.

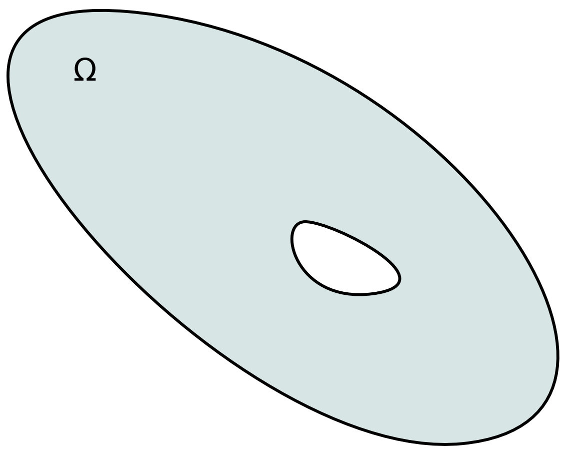

Figure 2: Invariant Annulus

The proof of this important result requires the Jordan curve theorem ([8], pp. 52-54). The analogue of theorem (1.2) is true for differential equations on the sphere but is false for differential equations on the torus. The Poincaré Bendixon theorem may be used to prove the existence of limit cycles for the Van der Pol oscillator

$$\dot{x} = -y, \quad \dot{y} = x + \epsilon(x^2 - 1)y$$

by finding an invariant annulus for the flow ([8], pp. 60-61). Another result from the theory of ordinary differential equations is the following result stated for planar systems (1.1) but holds in higher dimensions also. A proof may be given using Stokes' theorem or the Brouwer's fixed point theorem (see [8], p. 49).

[1]These are the common zeros of the pair $P(x, y)$ and $Q(x, y)$.
[2]This means a trajectory (solution curve) starting at a point of Ω stays in Ω for all times.

Theorem 1.3 Every closed trajectory of the system (1.1) contains a rest point in its interior.

Algebraic topology is a branch of geometry where properties of space are studied by assigning algebraic invariants (such as groups, rings etc.,) to space. Thus to each topological space X we attach an algebraic object such as a group $h(X)$ and to each continuous map $f : X \longrightarrow Y$ we attach a group homomorphism $h(f) : h(X) \longrightarrow h(Y)$ satisfying two basic properties:

1. If X, Y and Z are three topological spaces and $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous maps, then the corresponding group homomorphisms $h(f) : h(X) \longrightarrow h(Y)$, $h(g) : h(Y) \longrightarrow h(Z)$ and $h(g \circ f) : h(X) \longrightarrow h(Z)$ must satisfy the condition
$$h(g \circ f) = h(g) \circ h(f).$$
2. The identity map $i : X \longrightarrow X$ corresponds to the identity map $h(i) : h(X) \longrightarrow h(X)$

These properties are summarized by the statement that h is a (covariant) functor from the category of topological spaces to the category of groups. We shall provide formal definitions of a category and functor elucidating them through examples as we go along.

We shall introduce two important functors - the fundamental groups and the homology groups. We also indicate how these functors help in the understanding (under restrictive conditions) of two fundamental problems in topology - the extension problem and the lifting problem. The Tietze's extension theorem which provides a solution to the extension problem in certain special but important cases, is recalled in lecture 3 where we also place it against the general background of the extension problem. The extension problem reappears again in lecture 10 in connection with the Brouwer's fixed point theorem. Certain questions in complex analysis lead us naturally to the lifting problem as elaborated in lecture 18.

The course is organized as follows. Lectures 1 through 26 constitute the first part on fundamental groups and covering spaces. The second part on singular homology is covered in lectures 28 through 40. We begin with a review of general topology in the next four lectures. We shall touch upon some of the important results on compactness, connectedness, path-connectedness and their local analogues. This is followed by a longer chapter on quotient spaces with a large supply of examples that would occur frequently in the subsequent lectures. The exercises at the end of the lectures are designed as a warm up on these notions. The universal properties of the quotient is emphasized. We shall introduce the notion of a topological group in lecture 5 and discuss some important examples.

In the next lecture we introduce one of the principal thespians of the play - the fundamental group of a topological space. The theme will be developed in the subsequent lectures. The first non-trivial result is that the fundamental group of a circle is the group of integers which in turn implies several important results such as the Brouwer's fixed point theorem and the Perron-Frobenius theorem from matrix theory. The theory of covering spaces will be developed in lectures 13-17. The theory of covering spaces is important in many areas of mathematics but we shall study it here in close connection with the theory of the fundamental group. We introduce one of the fundamental problems in topology namely, the lifting problem for which an elegant solution is available in the context of covering spaces.

Many important spaces in mathematics such as the Klein's bottle, projective spaces and Riemann surfaces (the torus being an important example) occur as orbit spaces under the action of discrete groups. Lecture 18 is devoted to many of these examples. Unfortunately limitations in space and time prevent us from discussing the existence of a universal covering for a space.

Algebraic topology is certainly not a stand alone subject and we have tried (to the extent possible) to indicate connections with other areas of mathematics.