Lecture XIX - Deck Transformations

Given a covering projection $p : \tilde{X} \longrightarrow X$, the deck transformations are, roughly speaking, the symmetries of the covering space. Thus it should not come as a surprise that they play a crucial part in the theory of covering spaces. In this lecture all spaces are assumed to be connected and locally path connected.

Definition 19.1 (Deck transformations): Let $p : \widetilde{X} \longrightarrow X$ be a covering projection. A deck transformation is a homeomorphism $\phi : \widetilde{X} \longrightarrow \widetilde{X}$ such that $p \circ \phi = p$, that is to say ϕ is a lift of p.

Examples 19.1: (i) For the covering space $\mathrm{ex} : \mathbb{R} \longrightarrow S^1$ given by $\mathrm{ex}(t) = \exp(2\pi i t)$ the deck transformations are the maps

$$T_n : \mathbb{R} \longrightarrow \mathbb{R}, \quad T_n(x) = x + n, \quad n \in \mathbb{Z}$$

(ii) For the two sheeted covering $p : S^n \longrightarrow \mathbb{R}P^n$ the deck transformations are the identity map and the antipodal map.

The following theorem summarizes the most basic properties of the group of deck transformations.

Theorem 19.1: Let $p : \widetilde{X} \longrightarrow X$ be a covering projection and ϕ be a deck transformation. Then

(i) ϕ is uniquely determined by its value at one point of $\widetilde{X}$

(ii) $\phi(\tilde{x}_0) \in p^{-1}(x_0)$ whenever $\tilde{x}_0 \in x_0$.

(iii) If $\phi(\tilde{x}_1) = \tilde{x}_2$, where $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ then

$$p_*\pi_1(\widetilde{X}, \tilde{x}_1) = p_*\pi_1(\widetilde{X}, \tilde{x}_2) \tag{19.1}$$

(iv) Conversely if (1) holds then there exists a unique deck transformation ϕ such that $\phi(\tilde{x}_1) = \tilde{x}_2$

Proof: Statement (i) follows from the uniqueness of lifts. Statement (ii) follows immediately from the definition. To prove (iii) apply the lifting criterion (necessity) to both ϕ and ϕ^{-1}. To prove (iv) apply lifting criterion (sufficiency) to get continuous functions $\phi : \widetilde{X} \longrightarrow \widetilde{X}$ and $\psi : \widetilde{X} \longrightarrow \widetilde{X}$ such that

$$p \circ \phi = p, \ \phi(\tilde{x}_1) = \tilde{x}_2; \quad p \circ \psi = p, \ \psi(\tilde{x}_2) = \tilde{x}_1.$$

Then $\phi \circ \psi$ and $\psi \circ \phi$ are both lifts of the map $p : \widetilde{X} \longrightarrow X$ such that

$$\phi \circ \psi(\tilde{x}_2) = \tilde{x}_2, \quad \psi \circ \phi(\tilde{x}_1) = \tilde{x}_1$$

The identity map on $\widetilde{X}$ is also a lift of p with these initial conditions. By uniqueness, we see that both $\phi \circ \psi$ and $\psi \circ \phi$ must be the identity map on $\widetilde{X}$ proving that ϕ and ψ are homeomorphisms. The uniqueness clause follows from the uniqueness of lifts. $\square$

Remark: If $\phi : \widetilde{X} \longrightarrow \widetilde{X}$ is a *continuous* map such that $p \circ \phi = p$, then prove that ϕ is a homeomorphism in the following cases:

(i) $\pi_1(\widetilde{X})$ is a finite group (ii) $p_*\pi_1(\widetilde{X}, \tilde{x}_0)$ has finite index in $\pi_1(X, x_0)$ (iii) $\widetilde{X}$ is a regular cover of X. Is this true in general? The point is that if H is a subgroup of G and $gHg^{-1} \subset H$ then it follows $gHg^{-1} = H$ in case H is finite or has finite index or is normal.

Definition 19.2: The set of deck transformations of a covering projection $p : \widetilde{X} \longrightarrow X$ forms a group under composition of maps denoted by $\mathrm{Deck}(\widetilde{X}, X)$.

Action of Deck($\widetilde{X}, X$) on the fibers $p^{-1}(x_0)$: We fix a base point $x_0 \in X$. Since each deck transformation is a bijection, it is a permutation of the fiber $p^{-1}(x_0)$ and so acts on $p^{-1}(x_0)$ as a group of permutations:

$$(\phi, \tilde{x}_0) \mapsto \phi(\tilde{x}_0)$$

We study this action closely and relate it to the action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$. We first look at the case of regular coverings

Theorem 19.2: The covering $p : \widetilde{X} \longrightarrow X$ is a regular covering if and only if the action of $\mathrm{Deck}(\widetilde{X}, X)$ is transitive on $p^{-1}(x_0)$.

Proof: Let $\tilde{x}_1$ and $\tilde{x}_2$ be two arbitrary points of $p^{-1}(x_0)$. The action of $\mathrm{Deck}(\widetilde{X}, X)$ is transitive on $p^{-1}(x_0)$ if and only if there is a $\phi \in \mathrm{Deck}(\widetilde{X}, X)$ carrying $\tilde{x}_1$ to $\tilde{x}_2$, which is the case if and only if (19.1) holds. This in turn implies that the conjugacy class

$$\Big\{p_*\pi_1(\widetilde{X}, \tilde{x}_0) : \tilde{x}_0 \in p^{-1}(x_0)\Big\}$$

reduces to a singleton and conversely, in other words, if and only if the covering is regular. $\square$

We now relate the (perhaps intransitive) action of $\mathrm{Deck}(\widetilde{X}, X)$ on $p^{-1}(x_0)$ with the transitive action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$. Pick $\phi \in \mathrm{Deck}(\widetilde{X}, X)$ and $\phi(\tilde{x}_1) = \tilde{x}_2$. Then on the one hand (19.1) must hold while since $p{*}\pi_1(\widetilde{X}, \tilde{x}_1) = \text{stab } \tilde{x}_1$ (for the action of $\pi_1(X, x_0)$), we have on the other hand

$$\text{stab } \tilde{x}_1 = \text{stab } \tilde{x}_2 = g(\text{stab } \tilde{x}_1)g^{-1}, \tag{19.2}$$

for some $g \in \pi_1(X, x_0)$. In fact (19.2) states that g belongs to the normalizer

$$N(\text{stab } \tilde{x}_1) = N(p{*}(\pi_1(\widetilde{X}, \tilde{x}_1)) \subset \pi_1(X, x_0).$$

This suggests that we must relate ϕ to the element $g \in N(p{*}(\pi_1(\widetilde{X}, \tilde{x}_1))$. However since there may be several such elements g it is expedient to define the map in the opposite direction.

Let $g \in N(p{*}(\pi_1(\widetilde{X}, \tilde{x}_1)) \subset \pi_1(X, x_0)$ and $\tilde{x}_1 \cdot g = \tilde{x}_2$. Then (19.1) holds since g is in the normalizer of stab $\tilde{x}_1$. There is a unique $\phi_g \in \mathrm{Deck}(\widetilde{X}, X)$ such that $\phi_g(\tilde{x}_1) = \tilde{x}_2 = \tilde{x}_1 \cdot g$. The map

$$\psi \ : \ N(p{*}(\pi_1(\widetilde{X}, \tilde{x}_1)) \longrightarrow \mathrm{Deck}(\widetilde{X}, X), \quad g \mapsto \phi_g \tag{19.3}$$

is a homomorphism. To see that it is surjective, let $\phi \in \mathrm{Deck}(\widetilde{X}, X)$. There is a $g \in \pi_1(X, x_0)$ such that

$$\tilde{x}_1 \cdot g = \phi(\tilde{x}_1)$$

then stab $\tilde{x}_1$ and stab $\phi(\tilde{x}_1)$ are conjugate by g but they are also equal by (iii) of Theorem (19.1), whereby we conclude g is in the normalizer $N(p*(\pi_1(\widetilde{X}, \tilde{x}_1))$ and $\phi = \phi_g$. To determine the kernel of ψ, observe that $\phi_g = \mathrm{id}$ if and only if

$$\phi_g(\tilde{x}_1) = \tilde{x}_1 \cdot g$$

that is, if and only if $g \in$ stab $\tilde{x}_1$. But stab $\tilde{x}_1 = p*(\pi_1(\widetilde{X}, \tilde{x}_1))$. Summarizing these observations,

Theorem 19.3: We the group isomorphism

$$\mathrm{Deck}(\widetilde{X}, X) \cong N(p*(\pi_1(\widetilde{X}, \tilde{x}_1))/p*(\pi_1(\widetilde{X}, \tilde{x}_1)). \tag{19.4}$$

Corollary 19.4: If $p : \widetilde{X} \longrightarrow X$ is a regular covering then

$$\mathrm{Deck}(\widetilde{X}, X) \cong \pi_1(X, x_0)/p*(\pi_1(\widetilde{X}, \tilde{x}_1)). \tag{19.5}$$

Corollary 19.5: If $\widetilde{X}$ is a simply connected covering of X then

$$\mathrm{Deck}(\widetilde{X}, X) \cong \pi_1(X, x_0). \tag{19.6}$$

Corollary 19.6: $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$

Existence of a simply connected covering space: Despite being an important theme, we shall not discuss this in any detail in this elementary course but make a few remarks about it. Most of the spaces that we shall encounter are reasonably well-behaved and indeed many of them such $SO(n, \mathbb{R})$, S^3 and the projective spaces are smooth manifolds. Given the existence of a simply connected covering - called a universal covering[4], one can develop a Galois correspondence for covering spaces which asserts the existence of a unique (upto isomorphism) covering corresponding to each conjugacy class of subgroups of $\pi_1(X, x_0)$.

Definition 19.3: Let us consider a fixed connected topological space X with a specified base point $x_0 \in X$. A homomorphism between two coverings $p : (Y, y_0) \longrightarrow (X, x_0)$ and $q : (Z, z_0) \longrightarrow (X, x))$ is a surjective continuous map $r : (Y, y_0) \longrightarrow (Z, z_0)$ such that $q \circ r = p$ or diagrammatically,

$$\begin{tikzcd}
(Y, y_0) \arrow[rr, "r"] \arrow[dr, "p"'] & & (Z, z_0) \arrow[dl, "q"] \\
& (X, x_0) &
\end{tikzcd}$$

The definition enables us to form a category of coverings of a given space X with a specified base point $x_0 \in X$. To obtain a satisfactory theory one must impose some additional assumption on X such as local connectedness. In other words r is a lift of p with respect to the covering map q. The universal covering is then defined in terms of a universal property.

[4]Actually the notion of a universal covering is more general than the notion of a simply connected coverings but the two notions coincide for all reasonable spaces and certainly for all spaces that we shall deal with.

Definition 19.4: The universal covering is a covering $e : (E, e_0) \longrightarrow (X, x_0)$ such that for every covering $p : (Y, y_0) \longrightarrow (X, x_0)$ there is a unique homomorphism $\psi : (E, e_0) \longrightarrow (Y, y_0)$, that is a continuous surjection ψ such that $p \circ \psi = e$.

The universal covering if it exists is unique and one can establish the existence of a universal covering for a reasonable nice class of topological spaces X.

Exercises

1. Suppose that G and $\tilde{G}$ are topological groups and $p : \tilde{G} \longrightarrow G$ is a covering projection that is also a group homomorphism then $\ker p = \text{Deck}(\tilde{G}, G)$.

2. Determine the deck transformations for the covering
$$\sin : \mathbb{C} - \left\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\} \longrightarrow \mathbb{C} - \{\pm 1\}$$

3. Determine the deck transformations for the covering
$$p : \mathbb{C} - \left\{\pm 1, \pm 2\right\} \longrightarrow \mathbb{C} - \{\pm 2\}$$
given by $p(z) = z^3 - 3z$. Show that this covering is not regular. Hint: Use Riemann's removable singularities theorem to show that a deck transformation must be analytic on the whole plane.

4. If p is a prime, what can you say about the group of deck transformations of a p-sheeted covering space?

5. Show using the universal property that the universal covering, if it exists is unique upto isomorphism of covering projections.