Lecture XI - Homotopies of maps. Deformation retracts.

We generalize the notion of homotopy of paths to homotopy of a pair of continuous maps between topological spaces. This would be particularly useful in the second part of the course. It also leads to a powerful notion of deformation retracts which is often useful in deciding whether two spaces have the same fundamental group. Homotopy of maps is a useful coarsening of the notion of homeomorphism of two spaces leading to the notion of homotopy equivalence of spaces. Over the decades homotopy has proved to be the most important notion in topology, susceptible to considerable generalization with wide applicability.

Definition 11.1 (Homotopies of maps): (i) Given continuous maps $f, g : X \longrightarrow Y$ between topological spaces we say that f and g are homotopic if there exists a continuous map $F : X \times [0,1] \longrightarrow Y$ such that

$$F(x,0) = f(x), \quad F(x,1) = g(x), \quad \text{for all } x \in X \tag{11.1}$$

We shall occasionally use the notation $f \sim g$ to indicate that f and g are homotopic. One can formulate a notion for pairs of spaces:

(ii) Two continuous maps $f, g : (X, A) \longrightarrow (Y, B)$ between pairs of topological spaces are said to be homotopic if there exists $F : (X \times I, A \times I) \longrightarrow (Y, B)$ such that in addition to (11.1) the following condition holds:

$$F(a,t) \in B, \quad \text{for all } a \in A, t \in [0,1]. \tag{11.2}$$

Condition (11.2) is a boundary condition which states that the intermediate functions

$$F_t : x \mapsto F(x,t)$$

all map A into B. Note that when $A = \{x_0\}$ and $B = \{y_0\}$, the condition says that all the intermediate maps F_t are base point preserving. We leave it to the reader to prove the following two simple results.

Theorem 11.1: Homotopy is an equivalence relation.

Theorem 11.2: Suppose that f and g are homotopic maps of pairs (X, x_0) and (Y, y_0) then the induced group homomorphisms f_* and g_* from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$ are equal.

Now suppose that f and g are homotopic maps from X to Y such that for a base point $x_0 \in X$, $f(x_0) = g(x_0) = y_0$ say, but the intermediate maps do not respect these base points. Then it is not necessary that $f_* = g_*$ as maps from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$. The following theorem addresses this issue.

Theorem 11.3: Suppose that F is a homotopy between maps $f, g : X \longrightarrow Y$ and for a point $x_0 \in X$, $f(x_0) = g(x_0) = y_0$. Then the group homomorphisms f_* and g_* are conjugate by the inner-automorphism generated by the loop

$$\sigma : t \mapsto F(x_0, t) \tag{11.3}$$

Proof: The idea of proof is simple. Observe that (11.3) is the image of the base point x_0 under the deformation suggesting the use of theorem (7.8). If we fix an intermediate time $s \in [0,1]$ then the curve σ_s given by $\sigma_s(t) = t \mapsto \sigma(st)$ starts at y_0 and we could use it to construct a loop at y_0 namely

$$\sigma_s * F(\gamma(\,.\,), s) * \sigma_s^{-1}$$

In detail, for each loop $\gamma(t) \in X$ based at x_0, the homotopy $\phi : [0,1] \times [0,1] \longrightarrow Y$ given by

$$\begin{aligned}
\phi(s,t) &= \sigma(3st) \ \text{ if } 0 \le t \le 1/3 \\
&= F(\gamma(3t-1), s) \ \text{ if } 1/3 \le t \le 2/3 \\
&= \sigma(3s - 3st) \ \text{ if } 2/3 \le t \le 1.
\end{aligned}$$

establishes the equality of $f_*[\gamma]$ and $[\sigma](g_*[\gamma])[\sigma^{-1}]$.

Corollary 11.4: Suppose that F is a homotopy between maps $f, g : X \longrightarrow Y$ and for a point $x_0 \in X$, $f(x_0) = g(x_0) = y_0$. If $\pi_1(Y, y_0)$ is abelian then the group homomorphisms f_* and g_* are equal.

If we drop the hypothesis $f(x_0) = g(x_0)$ in theorem 11.3 the proof still goes through but since σ is no longer a loop we merely get that the induced maps f_* and g_* differ by a composition through the isomorphism $h_{[\sigma]}$ encountered in theorem (7.8). We record the result as a theorem and the reader may rework the proof of theorem 11.3 to fit it in the present context.

Theorem 11.5: Suppose that F is a homotopy between maps $f, g : X \longrightarrow Y$ then for $x_0 \in X$, the induced maps $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ and $g_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, g(x_0))$ satisfy the relation

$$h_{[\sigma]} \circ f_* = g_* \tag{11.4}$$

where $h_{[\sigma]}$ is the isomorphism

$$h_{[\sigma]} : [\gamma] \mapsto [\sigma * \gamma * \sigma^{-1}] \tag{11.5}$$

we have encountered earlier with σ being the curve $F(x_0, t)$ joining $f(x_0)$ and $g(x_0)$.

Definition 11.2 (Homotopy equivalence): (i) A map $f : X \longrightarrow Y$ is said to be a homotopy equivalence if there exists a map $g : Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are respectively homotopic to the identity maps id_Y and id_X respectively. Under this circumstance we say that the spaces X and Y are homotopically equivalent or have the same homotopy type.

(ii) A space that is homotopy equivalent to a point is said to be contractible. This is equivalent to the statement that the identity map on X is homotopic to a constant map.

The student may check that if X and Y are homotopy equivalent and Y and Z are homotopically equivalent then X and Z are homotopy equivalent.

Theorem 11.6: If $f : X \longrightarrow Y$ is a homotopy equivalence then the groups $\pi_1(X, x_0)$ and $\pi_1(Y, f(x_0))$ are isomorphic.

Proof: There exists $g : Y \longrightarrow X$ such that $f \circ g$ and $g \circ f$ are respectively homotopic to id_Y and id_X. By theorem 11.5 $f_* \circ g_*$ differs from the identity map on $\pi_1(Y, (f \circ g)(y_0))$ by a composition with the isomorphism $h_{[\sigma]}$ where σ is a path joining $f(g(y_0))$ and y_0. In particular $f_* \circ g_*$ is bijective and so f_* is surjective and g_* is injective. Likewise, working with $g \circ f$ one concludes that g_* is surjective and f_* is injective. Hence f_* is an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(Y, f(x_0))$.

Deformation retract: A subspace A of X is said to be a deformation retract if there exists a continuous map $r : X \longrightarrow A$ such that $r \circ j = \mathrm{id}_A$ and $j \circ r \sim \mathrm{id}_X$ where j denotes the inclusion of A into X. In particular, X and A have the same homotopy type.

Theorem 11.7: Suppose that A is a deformation retract of X via a map $r : X \longrightarrow A$. Then for $x_0 \in A$, the maps $r_* : \pi_1(X, x_0) \longrightarrow \pi_1(A, x_0)$ and $i_* : \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$ are isomorphisms.

Proof: Let $r : X \longrightarrow A$ be a retraction such that $j \circ r \sim \mathrm{id}_X$. By (the proof of) theorem 11.6, r_* is injective. But the composition $r \circ j = \mathrm{id}_A$ shows that r_* is surjective. Hence r_* establishes an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(A, x_0)$.

Example 11.1: The sphere S^{n-1} is a deformation retract of $\mathbb{R}^n - \{0\}$. A retraction $r : \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$ is given by the formula $r(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$. The homotopy between $j \circ r$ and the identity map on $\mathbb{R}^n - \{0\}$ is provided by the convex combination

$$F(\mathbf{x}, t) = t\mathbf{x} + (1 - t)\frac{\mathbf{x}}{\|\mathbf{x}\|} \tag{11.6}$$

The student must however check that $F(\mathbf{x}, t)$ omits the zero vector. From this we get the following important result.

Theorem 11.8: (i) The fundamental group of the punctured plane is the additive group $\mathbb{Z}$ and the homotopy class of the loop

$$t \mapsto \exp(2\pi i t), \quad 0 \leq t \leq 1 \tag{11.7}$$

provides a generator for the group.

(ii) The fundamental group of $\mathbb{R}^n - \{0\}$ is the trivial group.

Example 11.2: Let X be the union of the sphere S^2 and one of its diameters. Then X is homotopy equivalent to the space $S^2 \vee S^1$. While it is easy to construct the map $f : X \longrightarrow S^2 \vee S^1$, the map g in the opposite direction is not easy to write down. Exercise 6 shows how to get around the difficulty.

Example 11.3: Let L be the line $\{(0, 0, x_3)/x_3 \in \mathbb{R}\}$ in $\mathbb{R}^3$ and C be the circle

$$(x_1 - 1)^2 + x_2^2 = 1/4, \quad x_3 = 0.$$

We show that the torus is a deformation retract of the space $X = \mathbb{R}^3 - (L \cup C)$. The idea is simple but some details ought to be examined. Let us begin with the punctured half plane

$$H_0' = \{(x_1, 0, x_3)/x_1 > 0\} - \{(1, 0, 0)\}$$

which clearly deformation retracts to the circle C_0 given by

$$C_0 : (x_1 - 1)^2 + x_3^2 = 1/4, \quad x_2 = 0.$$

The homotopy $F : H_0' \times [0,1] \longrightarrow H_0'$ is simply given by the convex combination:

$$F(\mathbf{x}, t) = (1-t)\mathbf{x} + t\Big(\mathbf{e}_1 + \frac{\mathbf{x} - \mathbf{e}_1}{\|\mathbf{x} - \mathbf{e}_1\|}\Big), \quad \mathbf{e}_1 = (1, 0, 0).$$

The idea is to rotate the picture about the x_3-axis. It is expedient to use spherical polar coordinates given by

$$x_1 = \rho\cos\theta\sin\phi,\ x_2 = \rho\sin\theta\sin\phi,\ x_3 = \rho\cos\phi, \quad 0 < \phi < \pi,\ \theta \in \mathbb{R}.$$

Let H_θ' be the half plane bounded by the x_3-axis making angle θ with H_0' and R_θ denote the rotation about the x_3-axis mapping H_θ' onto H_0' namely,

$$R_\theta(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi) = (\rho\sin\phi, 0, \rho\cos\phi)$$

The homotopy we are looking for is then the map $G : X \times [0,1] \longrightarrow X$ given by

$$G(\mathbf{x}, t) = R_\theta^{-1} \circ F(R_\theta(\mathbf{x}), t). \tag{11.8}$$

It is easy to see using the properties of rotations, that

(i) G is well defined, that is the image of G avoids the circle C

(ii) Satisfies the requisite boundary conditions at $t = 0$ and $t = 1$.

However, the continuity of G is not automatic since the θ appearing in the definition of G depends also on $\mathbf{x}$ and we know that θ cannot be defined as a continuous function of $\mathbf{x}$ on X. One can either write a formula (which is easy) and see that θ occurs in (11.8) only as $\cos\theta$ and $\sin\theta$ which are continuous functions on X or better still use the property of quotients. We leave the amusing details to the reader.

Corollary 11.9: The fundamental group of the complement of $L \cup C$ in $\mathbb{R}^3$ is $\mathbb{Z} \times \mathbb{Z}$.

Exercises:

1. Check that the map ϕ constructed in the proof of theorem 11.3 is continuous and is indeed a homotopy. Work out the proof of theorem 11.5.

2. Show that the boundary ∂M of the Möbius band M is not a deformation retract of M by taking a base point x_0 on the boundary and computing explicitly the group homomorphism

$$i_* : \pi_1(\partial M, x_0) \longrightarrow \pi_1(M, x_0).$$

3. Show that the boundary of the Möbius band is not even a retract of the Möbius band.

4. Fill in the details on the continuity of the map G in the example preceding corollary 11.9.

5. Show that the space $\mathbb{R}^3 - \{(x, y, z)/x^2 + y^2 = 1,\ z = 0\}$ deformation retracts to a sphere with a diameter attached to it.

6. Let X be the union of S^2 and one of its diameters D, $Y = S^2 \vee S^1$ and Z be the union of S^2 with a punctured half disc contained in a half with edge along D. Show that X and Y are both deformation retracts of Z and so they have the same homotopy type.