Lecture - XXXI The homology groups and their functoriality

Having laid the algebraic foundations in the previous lecture we shall formally define the homology functors H_n, $n = 0, 1, 2, \dots$ from the category **Top** to the category **AbGr**. We shall discuss $H_0(X)$ completely and show that $H_0(X)$ is free abelian of rank equal to the number of path components of X. The groups $H_n(X)$ $(n \geq 1)$ vanish when X is a convex subset of $\mathbb{R}^n$. We shall prove this result using a technique that would be be considerably generalized in lecture 33. However the special case proved here for convex subsets would be needed in lecture 33. In the next lecture we shall see examples of topological spaces X for which $H_1(X)$ is non-trivial. However the reader would have to wait till lecture 34 to see more interesting examples.

The homology groups $H_n(X)$: Definitions (29.3)-(29.6) and theorems (29.2)-(29.5) from the previous lecture show that given a topological space X, the sequence of groups $S_n(X)$ and group homomorphisms $\partial_n : S_n(X) \longrightarrow S_{n-1}(X)$ provide an example of a chain complex called *the singular chain complex.* If $f : X \longrightarrow Y$ is a continuous function, the sequence $f_\sharp : S_n(X) \longrightarrow S_n(Y)$ $(n = 0, 1, 2, \dots)$ defines a chain map from the chain complex $S(X)$ to $S(Y)$. The general results on chain complexes when applied to this special case gives us the homology functors from **Top** to **AbGr**.

Definition 31.1: (i) The homology groups $H_n(X)$ of the space X are by definition the homology groups of the chain complex $S(X)$ namely

$$H_n(X) = Z_n(X)/B_n(X),$$

where $Z_n(X)$ is the kernel of the homomorphism $\partial_n : S_n(X) \longrightarrow S_{n-1}(X)$ and $B_n(X)$ is the image of the homomorphism $\partial_{n+1} : S_{n+1}(X) \longrightarrow S_n(X)$.

(ii) Given a continuous map $f : X \longrightarrow Y$, the induced maps $H_n(f) : H_n(X) \longrightarrow H_n(Y)$ in homology are the homomorphisms

$$H_n(f) : \overline{\sigma} \mapsto \overline{f_\sharp(\sigma)}, \quad \sigma \in Z_n(X).$$

Theorem (29.5) in this context is reproduced below:

Theorem 31.1: (i) Suppose $f : X \longrightarrow Y$ and $g : Y \longrightarrow W$ are continuous functions,

$$H_n(g \circ f) = H_n(g) \circ H_n(f), \quad n = 0, 1, 2, \dots.$$

The identity map on X induces the identity map on $H_n(X)$:

$$H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$$

In other words the $\{H_n / n = 0, 1, 2, \dots\}$ is a sequence of covariant functors from **Top** to **AbGr**. An immediate consequence is the following result.

Corollary 31.2: Suppose X and Y are homeomorphic, the groups $H_n(X)$ and $H_n(Y)$ are isomorphic for every $n = 0, 1, 2, \ldots$.

Another important consequence is the following result that parallels lemma (9.3).

Theorem 31.3: Suppose $r : X \longrightarrow A$ is a retraction, then for every $n = 0, 1, 2, \ldots$

$$H_n(r) : H_n(X) \longrightarrow H_n(A)$$

is surjective and

$$H_n(j) : H_n(A) \longrightarrow H_n(X)$$

is injective, where $j : A \longrightarrow X$ is the inclusion map.

The augmentation map $\epsilon : S_0(X) \longrightarrow \mathbb{Z}$: Since the standard Euclidean simplex Δ_0 is a singleton, each singular zero simplex $\Delta_0 \longrightarrow X$ can be identified with a point of X namely the image of the singular zero simplex. Thus we may think of a singular zero chain as an element of the free abelian group generated by the points of X, that is a formal expression

$$c_1p_1 + c_2p_2 + \cdots + c_kp_k, \tag{31.1}$$

where $p_1, p_2, \ldots, p_k$ are points of X and the coefficients $c_1, c_2, \ldots, c_k$ are integers.

Definition 31.2: The augmentation map $\epsilon : S_0(X) \longrightarrow \mathbb{Z}$ is the group homomorphism given by

$$c_1p_1 + c_2p_2 + \cdots + c_kp_k \mapsto c_1 + c_2 + \cdots + c_k.$$

If X is non-empty, the augmentation map is surjective. Since by definition, ∂_0 is the zero map and $Z_0(X) = S_0(X)$, we have to determine $B_0(X)$. The following theorem provides the answer.

Theorem 31.4: Suppose X is a path connected space then $B_0(X) = \ker \epsilon$. That is to say a singular zero chain (31.1) is a boundary if and only if the sum of its coefficients is zero. Thus, for a path connected space X,

$$H_0(X) \cong \mathbb{Z}.$$

Proof: We shall denote the ends of Δ_1 by a and b. If $\sigma : \Delta_1 \longrightarrow X$ is a singular one simplex then $\partial_1\sigma = \sigma(b) - \sigma(a)$ which is obviously in $\ker \epsilon$ and we conclude that $B_0(X) \subset \ker \epsilon$. To prove the reverse inclusion, let σ be an arbitrary element of $\ker \epsilon$ given by (31.1). That is, the coefficients satisfy $c_1 + c_2 + \cdots + c_k = 0$. Pick any point $p \in X$ and for each j let $\sigma_j : \Delta_1 \longrightarrow X$ be a path in X joining p and p_j. We claim that σ is the boundary of the one chain $\tau = c_1\sigma_1 + c_2\sigma_2 + \cdots + c_k\sigma_k$.

$$\begin{aligned} \partial_1\tau &= c_1(\sigma_1(b) - \sigma_1(a)) + c_2(\sigma_2(b) - \sigma_2(a)) + \cdots + c_k(\sigma_k(b) - \sigma_k(a)) \\ &= (c_1p_1 + c_2p_2 + \cdots + c_kp_k) - (c_1 + c_2 + \cdots + c_k)p = \sigma. \end{aligned}$$

The last part follows from the fundamental theorem on group homomorphisms.

Theorem 31.5: If $\{X_\alpha/\alpha \in \Lambda\}$ is the family of path components of a topological space, then for each $k = 0, 1, 2, \ldots$

$$H_k(X) = \bigoplus_{\alpha\in\Lambda} H_k(X_\alpha)$$

Proof: We shall only sketch the proof leaving the details as an exercise. Note that if σ is a singular $k-$simplex, the image of σ must be contained in one of the components X_α of X and so may be regarded as a singular $k-$simplex in X_α. This gives a natural decomposition of $S_k(X)$ as a direct sum of the family $S_k(X_\alpha)$. To see that the boundary map ∂_k respects the decomposition note that the boundary of a singular simplex σ is a sum of finitely many $k-1$ singular simplexes each of which must map into the same component as σ. It is easy to deduce from this the decompositions $Z_k(X) = \bigoplus Z_k(X_\alpha)$ and $B_k(X) = \bigoplus B_k(X_\alpha)$.

Convex sets and barycentric coordinates: Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ be given points. The convex hull of these points is the set consisting of all convex combinations $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k$, that is the coefficients $t_1, t_2, \dots, t_k$ are non-negative and $t_1 + t_2 + \cdots + t_k = 1$. The convex hull of these points is clearly a convex set. The points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are said to be *affinely independent* if the $k-1$ vectors $\mathbf{v}_1 - \mathbf{v}_k, \mathbf{v}_2 - \mathbf{v}_k, \dots, \mathbf{v}_{k-1} - \mathbf{v}_k$ are linearly independent (see exercise 4). The convex hull of a set of k affinely independent points is called the affine $k-$ simplex spanned by these points. The proof of the following result is left as an exercise.

Theorem 31.6: If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ are affinely independent then every point x in the convex hull of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ can be uniquely expressed as

$$x = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k, \tag{31.2}$$

where the coefficients t_j $(1 \leq j \leq k)$ are non-negative and $t_1 + t_2 + \cdots + t_k = 1$. These coefficients are called the *barycentric coordinates* of x.

We consider the standard n simplex Δ_n in $\mathbb{R}^{n+1}$ with summit $S = \mathbf{e}_{n+1}$. The figure below depicts a general point Q on the face Δ_{n-1} opposite to S and P an arbitrary point on the line segment joining Q and S. The reader may check that if $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$ are the barycentric coordinates of P then the coordinates of Q are given by the $n-$tuple

$$U(\lambda_1, \dots, \lambda_{n+1}) = \Big(\frac{\lambda_1}{1-\lambda_{n+1}}, \frac{\lambda_2}{1-\lambda_{n+1}}, \dots, \frac{\lambda_n}{1-\lambda_{n+1}}\Big) \tag{31.3}$$

Note that U is bounded but not continuous when $\lambda_{n+1} \longrightarrow 1$. As P approaches S the pyramid with

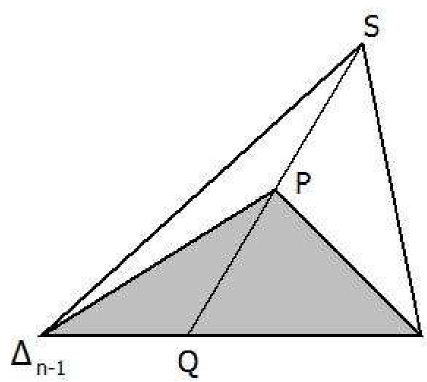

Figure 21:

base Δ_{n-1} and summit P fills up Δ_n. We are now in a position to prove the following theorem.

Theorem 31.7: Suppose X is a convex subset of a Euclidean space, $H_k(X) = 0$ for $k \geq 1$.

Proof: Choose a point $x_0 \in X$ and $F : X \times [0, 1] \longrightarrow X$ be the homotopy $F(x, t) = (1-t)x+tx_0$. We shall define a group homomorphism $T : S_{n-1}(X) \longrightarrow S_n(X)$ satisfying a certain property (31.6) below. This is a special case of a *chain homotopy* that we shall encounter later in a more general context. Since $S_{n-1}(X)$ is a free abelian group generated by singular $(n - 1)$ simplicies, it suffices to define T these. For a singular $(n - 1)$ simplex $\sigma : \Delta_{n-1} \longrightarrow X$, define the continuous map $T\sigma : \Delta_n \longrightarrow X$ in terms of the barycentric coordinates using the expression (31.3) namely

$$(T\sigma)(\lambda_1, \dots, \lambda_{n+1}) = F((\sigma \circ U)(\lambda_1, \dots, \lambda_{n+1}), \lambda_{n+1}). \tag{31.4}$$

The continuity of $T\sigma$ is left as an exercise. Let us calculate the boundary of $T\sigma$ using equations (29.1) and (29.4). Recalling the notations used in lecture 29, one checks that $(T\sigma) \circ \Phi^n_n = \sigma$.

For $0 \leq j \leq n - 1$, the $j-$th singular face is given by

$$(T\sigma) \circ \Phi^n_j(t_1, \dots, t_n) = F((\sigma \circ U)(t_1, \dots, t_{j-1}, 0, t_j, \dots, t_n), t_n). \tag{31.5}$$

On the other hand when $0 \leq j \leq n - 1$,

$$T(\sigma \circ \Phi^{n-1}_j)(t_1, \dots, t_n) = T\Big(\sigma\Big(\frac{t_1}{1 - t_n}, \dots, \frac{t_{j-1}}{1 - t_n}, 0, \frac{t_j}{1 - t_n}, \dots, \frac{t_{n-1}}{1 - t_n}\Big),\ t_n\Big),$$

which may be rewritten as $T((\sigma \circ U)(t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}), t_n)$, in agreement with the right hand side of (31.5). From equation (29.4) it follows that for $n \geq 1$,

$$\partial_n(T\sigma) - T(\partial_n\sigma) = \sigma, \quad \sigma \in S_n(X), \tag{31.6}$$

whereby we conclude that if $\sigma \in Z_n(X)$ then $\sigma = \partial_n(T\sigma) \in B_n(X)$. That is $Z_n(X) = B_n(X)$.

Exercises

1. Prove theorem (31.3).

2. Show that for a path connected space X, every singleton $\{p\}$ with $p \in X$ is a basis for $H_0(X)$.

3. Complete the proof of theorem (31.5).

4. Show that the set $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ is affinely independent if the vectors

$$\mathbf{v}_1 - \mathbf{v}_j, \dots, \mathbf{v}_{j-1} - \mathbf{v}_j, \mathbf{v}_{j+1} - \mathbf{v}_j, \dots, \mathbf{v}_{k-1} - \mathbf{v}_k$$

   are linearly independent for any j $(1 \leq j \leq k)$.

5. Prove theorem (31.6). Show that the barycentric coordinates are continuous functions of $\mathbf{x}$. All but the j-th barycentric coordinates of $\mathbf{v}_j$ vanish. The set of points in (31.2) obtained by setting $t_j = 0$ and varying the other coefficients is called the $j-$th face of the simplex spanned by the given points.

6. Check the continuity of the map $T\sigma$ in theorem (31.7).