Lecture XVII - Action of $\pi_1(X, x_0)$ on the fibers $p^{-1}(x_0)$

Given a covering projection $p : \tilde{X} \longrightarrow X$, the lifting lemma would imply that the fundamental group of the base space X acts naturally on the fibers $p^{-1}(x_0)$ ($x_0 \in X$). We define this action and examine its basic properties such as its transitivity. The action provides a great deal of information about the fundamental group $\pi_1(X, x_0)$ and this is the primary application of the theory of covering spaces in this course.

Definition 17.1: Let $p : \tilde{X} \longrightarrow X$ be a covering projection and $x_0 \in X$ be a given point. For a loop γ in X based at x_0, define the right-action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ as follows. For $\tilde{x}_1 \in p^{-1}(x_0)$,

$$\tilde{x}_1 \cdot [\gamma] = \tilde{\gamma}(1), \tag{17.1}$$

where $\tilde{\gamma}$ is the unique lift of γ starting at $\tilde{x}_1$.

Theorem 17.1: The prescription (17.1) defines a right action of the fundamental group $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$.

Proof: We first show that the action is well-defined. That is to say if γ_1 and γ_2 are homotopic loops based at x_0 then for $\tilde{x} \in p^{-1}(x_0)$

$$\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1),$$

where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are lifts of γ_1 and γ_2 starting at $\tilde{x}$. Well, if F is the homotopy between γ_1 and γ_2 then F has a unique lift $\tilde{F}$ satisfying $\tilde{F}(0,0) = \tilde{x}$. In other words, $\tilde{F} : [0,1] \times [0,1] \longrightarrow \tilde{X}$ is the unique continuous map such that

$$p \circ \tilde{F} = F, \quad \tilde{F}(0,0) = \tilde{x}$$

In particular the image set $\{\tilde{F}(s,1)\}$ as s runs through $[0,1]$, must be a connected subset of $\tilde{X}$. But since F is a homotopy of loops based at x_0,

$$F(s,1) = p \circ \tilde{F}(s,1) = x_0, \quad \text{for all } s \in [0,1].$$

Hence $\{\tilde{F}(s,1)/s \in [0,1]\} \subset p^{-1}(x_0)$ which means $\{\tilde{F}(s,1)/s \in [0,1]\}$ is a singleton since $p^{-1}(x_0)$ is discrete. In particular,

$$\tilde{F}(0,1) = \tilde{F}(1,1), \quad \text{that is,} \quad \tilde{\gamma}_1(1) = \tilde{\gamma}_2(1).$$

Next, we show that (15.1) defines a right group action. First let us note that if $\tilde{x}_1, \tilde{x}_2$ and $\tilde{x}_3$ are three points in $p^{-1}(x_0)$ and $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ is a pair of paths joining $\tilde{x}_1$ to $\tilde{x}_2$ and $\tilde{x}_2$ to $\tilde{x}_3$ respectively then

$$p \circ (\tilde{\gamma}_1 * \tilde{\gamma}_2) = (p \circ \tilde{\gamma}_1) * (p \circ \tilde{\gamma}_2).$$

Now let γ_1 and γ_2 be two loops in X based at x_0. Assume that $\tilde{\gamma}_1$ is the unique lift of γ_1 starting at $\tilde{x}_1$ and $\tilde{\gamma}_2$ is the unique lift of γ_2 starting at the point $\tilde{x}_2 = \tilde{\gamma}_1(1)$ then the juxtaposition $\tilde{\gamma}_1 * \tilde{\gamma}_2$ is defined and is the unique lift of $\gamma_1 * \gamma_2$ starting at $\tilde{x}_1$. Thus,

$$\tilde{x}_1 \cdot ([\gamma_1][\gamma_2]) = \tilde{x}_1 \cdot [\gamma_1 * \gamma_2] = \tilde{\gamma}_1 * \tilde{\gamma}_2(1) = \tilde{\gamma}_2(1)$$

On the other hand,

$$\tilde{\gamma}_2(1) = \tilde{x}_2 \cdot [\gamma_2] = (\tilde{x}_1 \cdot [\gamma_1]) \cdot [\gamma_2].$$

Note that if we had tried to operate from the left we would instead get an anti-action. This is one of the instances where it is important to have the book-keeping done correctly from the very outset.

Finally the constant loop ε_{x_0} at x_0 lifts as the constant loop starting at $\tilde{x}_1 \in p^{-1}(x_0)$ and so (17.1) implies

$$\tilde{x}_1 \cdot [\varepsilon_{x_0}] = \tilde{x}_1, \quad \tilde{x}_1 \in p^{-1}(x_0).$$

We now examine the issues related to this group action namely, its transitivity and the stabilizer subgroups of various points of $p^{-1}(x_0)$.

Theorem 17.2: (i) The group action defined in theorem (17.1) is transitive.

(ii) For $x_0 \in X$ and each $\tilde{x} \in p^{-1}(x_0)$, the stabilizer of $\tilde{x}$ is the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}))$.

(iii) The family $\{p_*(\pi_1(\tilde{X}, \tilde{x}))/\tilde{x} \in p^{-1}(x_0)\}$ forms a complete conjugacy class of subgroups of $\pi_1(X, x_0)$.

(iv) $|p^{-1}(x_0)| = [\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$

Proofs: Statement (iv) follows from (ii). Assertion (iii) is a general fact about transitive group actions. To prove that the group action is transitive, let $\tilde{x}_1$ and $\tilde{x}_2$ be two points in the fiber $p^{-1}(x_0)$ and $\tilde{\gamma}$ be a path in $\tilde{X}$ joining $\tilde{x}_1$ and $\tilde{x}_2$. The image path $\gamma = p \circ \tilde{\gamma}$ is then a loop in X based at x_0 and so represents an element of $\pi_1(X, x_0)$. Also $\tilde{\gamma}$ being the lift of γ starting at $\tilde{x}_1$, we see that

$$\tilde{x}_1 \cdot [\gamma] = \tilde{\gamma}(1) = \tilde{x}_2.$$

Turning now to the proof of (ii), let $\tilde{x}_0 \in p^{-1}(x_0)$ and γ be an arbitrary loop in X based at x_0. Then $[\gamma]$ belongs to the stabilizer of $\tilde{x}_0$ if and only if its lift starting at $\tilde{x}_0$ terminates at the same point $\tilde{x}_0$. That is if and only if γ lifts as a loop based at $\tilde{x}_0$. But this is equivalent to saying $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ and $\gamma = p_*[\tilde{\gamma}]$. Conversely if $[\gamma] \in \pi_1(X, x_0)$ is the image under p_* of $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ then $\tilde{\gamma}$ is a loop homotopic to a lift of γ starting at $\tilde{x}_0$. But any two such lifts have the same terminal point which means

$$\tilde{x}_0 \cdot [\gamma] = \tilde{\gamma}(1) = \tilde{x}_0.$$

That is to say $[\gamma]$ belongs to the stabilizer of $\tilde{x}_0$ and that completes the proof.

Corollary 17.3: The fundamental group of $\mathbb{R}P^n$ $(n \geq 2)$ is the cyclic group of order two.

Proof: We know that the fundamental group of S^n is the trivial group and the standard quotient map $\eta : S^n \longrightarrow \mathbb{R}P^n$ is a covering projection. So from (iv) of the preceding theorem we get

$$|\eta^{-1}(x_0)| = 2 = [\pi_1(\mathbb{R}P^n, x_0) : \eta_*(\pi_1(S^n, \tilde{x}_0))]$$

From which follows that

$$|\pi_1(\mathbb{R}P^n, x_0)| = 2$$

and that completes the proof.

Regular coverings: These are coverings that exhibit symmetry. The precise meaning will be clarified in theorem (19.4) below and theorem (19.2) in lecture 19.

Theorem 17.4: For a covering projection $p : \tilde{X} \longrightarrow X$ with path connected X and $\tilde{X}$ the following are equivalent:

(i) The subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$, where $p(\tilde{x}_0) = x_0$.

(ii) For any loop in X based at x_0, either all its lifts are closed loops or none of the lifts is closed.

Proof: We begin with the observation that the condition spelled out in (i) is independent of the choice of x_0 and also independent of the choice of $\tilde{x}_0 \in p^{-1}(x_0)$. Well, changing the element $\tilde{x}_0$ in the fiber would give a conjugate subgroup but the normality hypothesis says that the conjugacy class of the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a singleton. Second, a group isomorphism must take a normal subgroup to a normal subgroup and so the condition (i) does not depend on the choice of the base point x_0.

Proof that (i) implies (ii). Suppose that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$ and γ is an arbitrary loop in X based at x_0. Let $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be two lifts of γ with initial points $\tilde{x}_1, \tilde{x}_2$ such that $\tilde{\gamma}_1$ a closed loop. Then

$$\tilde{x}_1 \cdot [\gamma] = \tilde{x}_1$$

which means $[\gamma]$ is in the stabilizer of $\tilde{x}_1$ and hence, by (iii) of theorem (17.2), $[\gamma]$ belongs to the stabilizer of $\tilde{x}_2$. Thus

$$\tilde{x}_2 \cdot [\gamma] = \tilde{\gamma}_2(1) = \tilde{x}_2$$

and we see that $\tilde{\gamma}_2$ is also closed.

Proof that (ii) implies (i). If $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is not a normal subgroup of $\pi_1(X, x_0)$ then it has at least two distinct conjugates which, by virtue of theorem (17.2), must be the stabilizers of say $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$. Thus there exists $[\gamma] \in p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = \text{Stab } \tilde{x}_1$ but $[\gamma] \notin p_*(\pi_1(\tilde{X}, \tilde{x}_2)) = \text{Stab } \tilde{x}_2$. In other words

$$\tilde{x}_1 \cdot [\gamma] = \tilde{\gamma}_1(1) = \tilde{x}_1, \quad \tilde{x}_2 \cdot [\gamma] = \tilde{\gamma}_2(1) \neq \tilde{x}_2,$$

where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are the lifts of γ starting at $\tilde{x}_1$ and $\tilde{x}_2$ respectively. Thus the lift $\tilde{\gamma}_1$ of γ is closed whereas the lift $\tilde{\gamma}_2$ is not closed.

Definition 17.2: A covering projection $p : \tilde{X} \longrightarrow X$ with path connected X and $\tilde{X}$ is said to be regular if it satisfies one of the equivalent conditions stated in theorem (15.4).

Corollary 17.5: If $\pi_1(X, x_0)$ is abelian then every covering of X is regular.

To construct an example of a non-regular covering we need a space with non-abelian fundamental group. We shall see an example in lecture 19 (exercise 3).

Exercises

1. Describe a path in S^n whose image under the standard map represents the generator of $\pi_1(\mathbb{R}P^n, x_0)$.

2. Let C_0 be the unit circle in the complex plane and $\omega_1, \omega_2, \dots, \omega_n$ denote the $n-$th roots of unity and at each of these a circle C_j of small radius touches the unit circle externally. Construct a continuous map p from the union of these $n+1$ circles onto the figure eight loop such that p is a regular covering. Hint: Take one lobe of the figure eight to be the unit circle C_0 and define $p(z) = z^n$ for $z \in C_0$. Let L be the other lobe of figure eight touching the lobe C_0 at say the point 1. For each j let $p_j : C_j \longrightarrow L$ be any homeomorphism such that $p_j(\omega_j) = 1$. Use gluing lemma to glue these maps to obtain the desired covering.

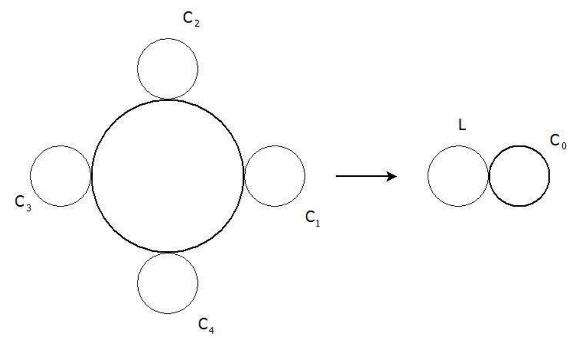

Figure 15: Covering of the figure eight loop

3. For the covering projection of the preceding exercise determine the action of the fundamental group of the base on a fiber assuming that the loops C_0 and L (based at 1) generate the fundamental group.

4. Consider the covering projection of exercise 6, lecture 15. Show by studying the lifts of various loops based at $(1, 1)$ that the covering is regular. We shall see another proof of regularity of this covering in lecture 19.

5. For the covering considered in the preceding exercise, determine the lifts of the loops

$$\gamma_1 : t \mapsto 1 - \exp(2\pi i t), \quad \gamma_2 : t \mapsto -1 + \exp(2\pi i t).$$

Find the lift of $\gamma_1 * \gamma_2 * \gamma_1^{-1} * \gamma_2^{-1}$ and deduce that the fundamental group of the figure eight space is non-abelian.

6. Show that the figure eight loop $(S^1 \times \{1\}) \cup (\{1\} \times S^1)$ is not a retract of the torus $S^1 \times S^1$. Show that the figure eight loop is a deformation retract of the torus minus a point.