Lecture XXII - Fundamental group of $SO(3,\mathbb{R})$ and $SO(4,\mathbb{R})$

For many applications, it is important to know The fundamental groups of the classical groups. We shall discuss in detail the orthogonal groups $SO(3,\mathbb{R})$ and $SO(4,\mathbb{R})$ since their underlying topological spaces are easily described. Indeed $SO(3,\mathbb{R})$ is the three dimensional real projective space and $SO(4,\mathbb{R})$, as a topological space, is the product of the three dimensional real projective space and the three dimensional sphere S^3. To unravel the structure of these spaces it is convenient to use quaternions. We shall assume some basic familiarity with quaternions (see [1]). We shall also use some basic facts from multi-variable calculus. The student who is unfamiliar with these parts of multi-variable calculus may omit these parts of the proof.

Theorem 22.1 The unit sphere S^3 is the double cover of the space $SO(3,\mathbb{R})$ and as a topological space is homeomorphic to $\mathbb{R}P^3$. In particular $\pi_1(SO(3,\mathbb{R}))$ is the cyclic group of order two.

The proof will be split into several lemmas. We begin by setting up a few notations which would remain in force throughout the lecture. We shall regard S^3 as the set of all unit quaternions forming a subgroup of the multiplicative group of non-zero quaternions.

Definition 22.1: A pure quaternion is one whose real part is zero. Thus a quaternion is pure q if and only if $\overline{q} = -q$, where the bar denotes the conjugate of q. We denote the set of all pure quaternions by Π. Thus Π is a three dimensional real vector space with the Euclidean norm inherited from $\mathbb{R}^4$.

We now list three lemmas whose proofs are left for the reader as easy exercises in linear algebra. It is useful to recall that a linear map of $\mathbb{R}^n$ to itself which preserves the Euclidean norm is an orthogonal transformation.

Lemma 22.2: If q is a pure quaternion then so is $x^{-1}qx$ for any non-zero quaternion x.

Thus, each non-zero quaternion x defines a non-singular linear map $T_x : \Pi \longrightarrow \Pi$ namely

$$T_x(q) = x^{-1}qx \tag{22.1}$$

Lemma 22.3: The linear map $T_x : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ given by equation (22.1) preserves the Euclidean norm and so defines an element of $O(4,\mathbb{R})$. Its restriction to Π still denoted by T_x maps Π onto itself and so may be identified as an element of $O(3,\mathbb{R})$. The map

$$\psi : x \mapsto T_x \tag{22.2}$$

is a group homomorphism from the multiplicative group of non-zero quaternions into $O(4,\mathbb{R})$.

Lemma 22.4: The kernel of ψ is the set of non-zero real numbers. In particular the kernel of the map

$$\psi : S^3 \longrightarrow O(3,\mathbb{R}) \tag{22.3}$$

obtained by restricting ψ to S^3 is the two element group ± 1.

Lemma 22.5: The image of $\psi : S^3 \longrightarrow O(3,\mathbb{R})$ is a compact connected subgroup of $SO(3,\mathbb{R})$.

Proof: Since S^3 is compact and connected, the image of the map $\psi : S^3 \longrightarrow O(3,\mathbb{R})$ is a compact and connected subgroup of $O(3,\mathbb{R})$. Now $O(3,\mathbb{R})$ is disconnected with two components and so the image must lie entirely in one of these components. Since $\psi(1)$ is the identity map this connected subgroup meets $SO(3,\mathbb{R})$ and so must be contained entirely in $SO(3,\mathbb{R})$.

Slightly more difficult is the proof that the image of S^3 under ψ is the whole of $SO(3,\mathbb{R})$. It is possible to give an argument which uses only linear algebra but we prefer to follow a slightly more sophisticated approach using the inverse function theorem. The student who is uncomfortable may merely skim through the argument and take the result on faith.

Lemma 22.6: The group homomorphism $\psi : S^3 \longrightarrow SO(3,\mathbb{R})$ is surjective and is a covering projection. As a topological space, $SO(3,\mathbb{R})$ is homeomorphic to $\mathbb{R}P^3$.

Proof: Once we show that $\psi : S^3 \longrightarrow SO(3,\mathbb{R})$ is surjective it follows from lemma (22.4) and the definition of real projective spaces that $SO(3,\mathbb{R})$ and $\mathbb{R}P^3$ are homeomorphic.

To prove the surjectivity of ψ, note that S^3 and $SO(3,\mathbb{R})$ are three dimensional manifolds and ψ is a smooth map. We show that the derivative $D\psi(1)$ is an invertible linear map and so by the inverse function theorem the image must contain a neighborhood of the identity. We merely have to recall from lecture 5 that if a subgroup H of a connected topological group G contains a neighborhood of the identity then $H = G$.

We now turn to the proof that $D\psi(1)$ is a surjective linear transformation. We shall regard ψ as a map from $\mathbb{R}^4$ to $SO(3,\mathbb{R}) \subset M(3,\mathbb{R})$ and compute its derivative at 1. For a quaternion h with sufficiently small norm,

$$\psi(1+h)v - \psi(1)v = \|1+h\|^{-2}(v + hv + v\overline{h}) - v + O(\|h\|^2) = -2h_0 v + hv + v\overline{h} + O(\|h\|^2),$$

where h_0 denotes the real part of h. We see that $D\psi(1)$ is the linear map $\mathbb{R}^4 \longrightarrow M(3,\mathbb{R})$ given by

$$h \mapsto -2h_0(\cdot) + h(\cdot) + (\cdot)\overline{h}. \tag{22.4}$$

The kernel of this linear map contains 1 and so is at-least one dimensional. It is exactly one dimensional since $D\psi(1)i$, $D\psi(1)j$ and $D\psi(1)k$ are linearly independent (skew-symmetric) matrices.

Remark: The curves σ_1, σ_2 and σ_3 given by

$$\sigma_1(t) = \cos t + i\sin t, \quad \sigma_2(t) = \cos t + j\sin t, \quad \sigma_3(t) = \cos t + k\sin t$$

lie on S^3 and pass through the point 1. Differentiating and setting $t = 0$ confirms that the vectors i, j, k span the tangent space to S^3 at 1. Thus $D\psi(1)i, D\psi(1)j$ and $D\psi(1)k$ span the image of $D\psi(1)$. We leave it to the reader to check, by calculating the derivatives of $\psi \circ \sigma_j$ $(j = 1, 2, 3)$ at $t = 0$, that $D\psi(1)i, D\psi(1)j$ and $D\psi(1)k$ are linearly independent.

Topological structure of $SO(4,\mathbb{R})$: Regard $L \in SO(4,\mathbb{R})$ as a linear transformation on the space $\mathbb{R}^4$ of all quaternions. In particular, $L(1)$ is a non-zero quaternion and we may define the linear map $L' : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ via the prescription

$$L'(x) = L(x)L(1)^{-1}, \quad x \in \mathbb{R}^4.$$

Lemma 22.7: The map L' preserves Euclidean distance and maps Π to itself.

Proof: The fact that it is distance preserving is clear so that it is an orthogonal transformation. Since L' also fixes the real axis by orthogonality it must map Π to itself.

Theorem 22.8: As a topological space, $SO(4,\mathbb{R})$ is homeomorphic to the product $S^3 \times SO(3,\mathbb{R})$.

Proof: We show that the map $\phi : SO(4,\mathbb{R}) \longrightarrow S^3 \times O(3,\mathbb{R})$ given by $\phi(L) = (L(1), L')$, where L' is defined as in the previous lemma, is a homeomorphism. The map L' is an element of $O(3,\mathbb{R})$ since it maps Π to itself and preserves Euclidean norm. Further, $L(1)$ is obviously a unit quaternion. The image of ϕ is a compact connected subspace of $S^3 \times O(3,\mathbb{R})$ and sends the identity element to the pair $(1, \mathrm{id})$ which means the image must be contained in $S^3 \times SO(3,\mathbb{R})$. It is an exercise that the map is bijective. Since the space $SO(4,\mathbb{R})$ is compact and $S^3 \times SO(3,\mathbb{R})$ is Hausdorff, it follows that ϕ is a homeomorphism.

Corollary 22.9: The fundamental group of $SO(4,\mathbb{R})$ is the cyclic group of order two. $\square$

Exercises

1. Show that the sphere S^3 is isomorphic (as a topological group) to $SU(2,\mathbb{C})$.
2. Show that the center of the group of non-zero quaternions is the set of non-zero real numbers. In the light of this explain why $\ker D\psi(1)$ in lemma (22.6) is non-trivial.
3. Explain why the map ϕ defined in theorem (22.8) is bijective.
4. Verify the properties of the map T_A in the proof of theorem (22.10). Also fill in the details concerning the properties of the map ϕ (except for the claims made concerning its derivative).
5. Use exercise 4 to find a generator of $\pi_1(SO(3,\mathbb{R}))$. Let $i : SO(2,\mathbb{R}) \longrightarrow SO(3,\mathbb{R})$ be given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad A \in SO(2,\mathbb{R}).$$

   Show that $i_* : \pi_1(SO(2,\mathbb{R})) \longrightarrow \pi_1(SO(3,\mathbb{R}))$ is surjective.