Lecture X - Brouwer's Theorem and its Applications.

In this lecture we shall prove the Brouwer's fixed point theorem and deduce some of its consequences such as the Perron-Frobenius' theorem. The one dimensional Brouwer's theorem follows from the intermediate value property as is indicated in the exercises of lecture 3. We also include a proof of the fact that the spheres S^n have trivial fundamental group when $n \geq 2$. This result has been included here to demonstrate why the fundamental group is insufficient to prove the Brouwer's fixed point theorem in dimension three or higher.

We begin by defining the fixed point property for a space. Here we require the fixed point property to hold for all continuous functions of the space into itself. Note that in analysis the spaces considered are somewhat special and so are the maps whose fixed point property are sought. A classic example of such a restricted fixed point theorem is the Banach's fixed point theorem.

Definition 10.1: A space X is said to have the fixed point property if every continuous map $f : X \longrightarrow X$ has a fixed point namely, there exists $p \in X$ such that $f(p) = p$.

Theorem 10.1: The fixed point property is a topological property. That is, if X and Y are homeomorphic and X has the fixed point property then so does Y.

Proof: Suppose that X has the fixed point property and $h : X \longrightarrow Y$ is a homeomorphism. Let $g : Y \longrightarrow Y$ be an arbitrary continuous map. Applying the fixed point property to the map $f = h^{-1} \circ g \circ h$ we get a point $p \in X$ such that $f(p) = p$. The fixed point of g is seen to be $h(p)$.

Examples 10.1: (i) The closed unit interval $[0, 1]$ has the fixed point property (exercise 1, lecture 3).

(ii) A non-trivial topological group does not have the fixed point property.

(iii) The space $\mathbb{R}P^{2n}$ has the fixed point property but we are not yet ready to prove this.

(iv) The open unit disc $U = \{z \in \mathbb{C}/|z| < 1\}$ does not have the fixed point property. For if a is a non-zero complex number with $|a| < 1$ then the map $f : U \longrightarrow U$ given by

$$f(z) = \frac{z - a}{1 - \overline{a}z}$$

has no fixed points in U. The reader must first check that f maps the open unit disc to itself and examine if it has any fixed points.

Theorem 10.2 (Brouwer's fixed point theorem): Every continuous function $f : E^2 \longrightarrow E^2$ has a fixed point where $E^2 = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}$.

Proof: We assume the contrary, that is to say a continuous function f of the closed unit disc into itself exists which has no fixed points. We produce a retraction from E^2 onto S^1 which would be a contradiction. The ray emanating from $f(x) \in E^2$ and passing through $x \in E^2$ namely

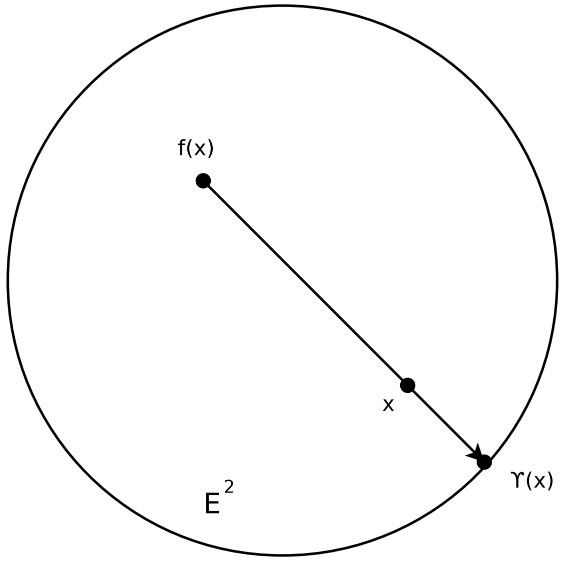

Figure 11: E^2 is not a retract of S^1

$$tx + (1-t)f(x), \quad t \geq 0,$$

meets the circle S^1 at a point denoted by $r(x) = t_0 x + (1-t_0)f(x)$ where, t_0 is a root of the quadratic

$$\langle tx + (1-t)f(x),\ tx + (1-t)f(x) \rangle = 1. \tag{10.1}$$

We recast this quadratic as

$$t^2(|f(\mathbf{x}) - \mathbf{x}|^2) - 2tf(\mathbf{x}) \cdot (f(\mathbf{x}) - \mathbf{x}) - (1 - |f(\mathbf{x})|^2) = 0. \tag{10.2}$$

Since the coefficient of t^2 is never zero, the roots are continuous functions of $\mathbf{x}$ and they are real. Moreover the roots differ in sign or one of the roots is zero. Take t_0 to be the larger root for constructing $r(x)$. From (10.1) we see that r maps E^2 to S^1. Note that if $|x| = 1$ then $t = 1$ satisfies the quadratic and so must be the larger root. Hence we conclude $r(x) = x$ if $|x| = 1$ and we get a retraction of E^2 onto S^1 which is a contradiction. $\square$

Remark: Note that the proof merely used the fact that π_1 functor is trivial on discs and nontrivial on circles. Any functor with this property may be used to prove the Brouwer's fixed point theorem.

Theorem 10.3 (Perron-Frobenius): A 3×3 matrix with strictly positive real entries has a positive eigen-value. The corresponding eigen-vector has non-negative entries.

Proof: Let A be a 3×3 matrix with strictly positive real entries and S be the part of the sphere

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, \quad x \geq 0, y \geq 0, z \geq 0\}$$

Then S is homeomorphic to the closed unit disc in the plane (why?) and so has the fixed point property. If $\mathbf{v}$ is any unit vector with non-negative entries then the entries of $A\mathbf{v}$ are non-negative and at-least one of the entries must be positive. Hence the map $f : S \longrightarrow S$ given by $f(\mathbf{v}) = A\mathbf{v}/\|A\mathbf{v}\|$ is continuous. By Brouwer's fixed point theorem, f has a fixed point $\mathbf{v}_0$ which means $A\mathbf{v}_0/\|A\mathbf{v}_0\| = \mathbf{v}_0$ from which we infer that $\|A\mathbf{v}_0\|$ is an eigen-value of A and this must be positive.

Fundamental groups of spheres: We close this lecture with a proof of the fact that $\pi_1(S^n) = \{1\}$ when $n \geq 2$. The student ought to try and figure out intuitively why is this so.

Theorem 10.4: If U and V are simply connected open subsets of X such that $X = U \cup V$ and $U \cap V$ is path connected then X is simply connected.

Proof: Let us choose a base point $x_0 \in U \cap V$ and γ be an arbitrary loop in X based at x_0. The open cover $\{\gamma^{-1}(U), \gamma^{-1}(V)\}$ of $[0, 1]$ has a Lebesgue number ϵ. Choose a partition

$$\{t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1\}.$$

such that the length of each sub-interval is less than ϵ. Then γ maps each $[t_j, t_{j+1}]$ into U or V. If γ maps two adjacent intervals into U or into V then drop the abutting point of the two intervals thereby coarsening the partition. Thus may arrange it such that for each $j = 1, 2, \ldots, n-1$, the point $\gamma(t_j)$ lies in $U \cap V$. We now choose a path σ_j joining x_0 and $\gamma(t_j)$ such that the image of σ_j lies entirely in $U \cap V$. This is possible since $U \cap V$ is path connected and $x_0 \in U \cap V$. Also let γ_j denote the restriction of γ to the sub-interval $[t_{j-1}, t_j]$ $(j = 1, 2, \ldots, n)$. We may reparametrize γ_j (retaining the name) so that its domain is $[0, 1]$. Now

$$\gamma \sim \gamma_1 * \sigma_1^{-1} * \sigma_1 * \gamma_2 * \sigma_2^{-1} * \sigma_2 * \gamma_3 * \cdots * \sigma_{n-1}^{-1} * \sigma_{n-1} * \gamma_n$$

Now each of the loops $\gamma_1 * \sigma_1^{-1}$, $\sigma_1 * \gamma_2 * \sigma_2^{-1}$,..., $\sigma_{n-1} * \gamma_n$ based at x_0 lies in one of the simply connected open sets U or V and so each of them is homotopic to the constant loop via a homotopy F_j. These homotopies F_j may be juxtaposed to provide a homotopy between

$$\gamma_1 * \sigma_1^{-1} * \sigma_1 * \gamma_2 * \sigma_2^{-1} * \sigma_2 * \gamma_3 * \cdots * \sigma_{n-1}^{-1} * \sigma_{n-1} * \gamma_n$$

and the constant loop. The proof is complete.

Theorem 10.5: For $n \geq 2$, the sphere S^n is simply connected.

Proof: Let U be the sphere minus the north pole and V be the sphere minus the south pole. Using the stereo-graphic projections, we see that U and V are simply connected open subsets of S^n and it is easily verified that $U \cap V$ is path connected. The result follows from the previous theorem.

Exercises

1. Suppose that a space X has the fixed point property, is it necessary that it be connected? Does it have to be path-connected?

2. Explain why a non-trivial topological group cannot have the fixed point property.

3. Prove the Brouwer's fixed point theorem for the closed unit ball in $\mathbb{R}^n$ given that that there exists a functor T from the category **Top** to the category **AbGr** such that $T(X)$ is the trivial group for every convex subset X of a Euclidean space and $T(S^{n-1})$ is a non-trivial group.
4. Show that the Brouwer's fixed point theorem implies the no retraction theorem.
5. Explain how the homotopies F_j in the proof of theorem 10.4 can be juxtaposed.
6. Show that the circle S^1 is not a retract of the sphere S^2.