Lecture VIII - Categories and Functors

Note that one often works with several types of mathematical objects such as groups, abelian groups, vector spaces and topological spaces. Thus one talks of the family of all groups or the family of all topological spaces. These entities are huge and do not qualify to be sets. We shall call them families or classes and their individual members as objects. Between two objects of a family say between two topological spaces X and Y one is interested in the class of all continuous functions. Instead if we take two objects G and H from the class of all groups we are interested in the set of all group homomorphisms from G into H. Abstracting from these examples we say that a category consists of a family of objects and for each pair of objects X and Y we are given a family of maps $X \to Y$ called the set of morphisms $\mathrm{Mor}(X, Y)$ subject to the following properties:

(i) To each pair $\mathrm{Mor}(X, Y)$ and $\mathrm{Mor}(Y, Z)$ there is a map

$$\mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \longrightarrow \mathrm{Mor}(X, Z)$$
$$(f, g) \mapsto g \circ f$$

such that for $f \in \mathrm{Mor}(X, Y), g \in \mathrm{Mor}(Y, Z)$ and $h \in \mathrm{Mor}(Z, W)$,

$$(h \circ g) \circ f = h \circ (g \circ f)$$

(ii) To each object X there is a unique element $\mathrm{id}_X \in \mathrm{Mor}(X, X)$ such that for any $f \in \mathrm{Mor}(X, Y)$ and $g \in \mathrm{Mor}(Z, X)$

$$f \circ \mathrm{id}_X = f, \quad \mathrm{id}_X \circ g = g$$

Example 8.1: We see that the family of all groups **Gr** forms a category where $\mathrm{Mor}(G, H)$ consists of the set of all group homomorphisms from G to H.

(ii) Likewise we can look at the family **AbGr** of all abelian groups and as before $\mathrm{Mor}(G, H)$ consists of all group homomorphisms from G to H.

(iii) The class of all topological spaces **Top** forms a category if we take as morphisms between X and Y the set of all continuous functions from X to Y.

Definition 8.1 (Covariant functor): Given two categories $\mathcal{C}_1$ and $\mathcal{C}_2$, a covariant functor is a rule that assigns to each object $A \in \mathcal{C}_1$ an object $h(A) \in \mathcal{C}_2$ and to each morphism $f \in \mathrm{Mor}(A, B)$, where A, B are objects in $\mathcal{C}_1$, a unique morphism $h(f) \in \mathrm{Mor}(h(A), h(B))$ such that the following hold:

(i) Given objects A, B and C in $\mathcal{C}_1$ and a pair of morphisms $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$,

$$h(g \circ f) = h(g) \circ h(f)$$

(ii) $h(\mathrm{id}_A) = \mathrm{id}_{h(A)}$

Definition 8.2 (Contravariant functor): A contravariant functor between the two given categories $\mathcal{C}_1$ and $\mathcal{C}_2$ is a rule that assigns to each object $A \in \mathcal{C}_1$ an object $h(A) \in \mathcal{C}_2$ and to each morphism $f \in \mathrm{Mor}(A, B)$, where A, B are objects in $\mathcal{C}_1$, a unique morphism $h(f) \in \mathrm{Mor}(h(B), h(A))$ such that the following conditions hold:

(i) Given objects A, B and C in $\mathcal{C}_1$ and a pair of morphisms $f \in \mathrm{Mor}(A, B)$, $g \in \mathrm{Mor}(B, C)$,

$$h(g \circ f) = h(f) \circ h(g)$$

(ii) $h(\mathrm{id}_A) = \mathrm{id}_{h(A)}$

Example 8.2: To each group G we assign its commutator subgroup $[G, G]$. A group homomorphism $f : G \longrightarrow H$ maps the commutator subgroup into the commutator subgroup $[H, H]$ so that the restriction

$$f\Big|_{[G,G]} \;:\; [G, G] \longrightarrow [H, H]$$

is a meaningful group homomorphism enabling us to assign to the morphism f its restriction to $[G, G]$. The conditions of definition 8.1 are readily verified.

Example 8.3: Between the categories **Gr** and **AbGr** we define a map as follows. For $G \in \mathbf{Gr}$ we denote by A_G its abelianization namely the quotient group:

$$A_G = G/[G, G].$$

The quotient is an abelian group and so belongs to **AbGr**. For example if we take $G = S_n$ the symmetric group on n letters then its abelianization is the cyclic group of order two (why?). If G and H are two groups and $f : G \longrightarrow H$ is a group homomorphism then

$$\eta_H \circ f : G \longrightarrow H/[H, H]$$

is a group homomorphism into an abelian group where η_H is the quotient map $H \longrightarrow H/[H, H]$. The kernel of $\eta_H \circ f$ must contain all the commutators and so defines a group homomorphism

$$\begin{aligned} \tilde{f} : G/[G, G] &\longrightarrow H/[H, H] \\ \overline{x} &\mapsto \overline{f(x)}, \end{aligned}$$

where the bar over x denotes the residue class of x in the quotient. Thus to each object G of **Gr** we have assigned a unique object of **AbGr** namely the abelianization $G/[G, G]$ and to each morphism $f \in \mathrm{Mor}(G, H)$ we have associated a unique morphism $\tilde{f}$. The following properties are quite clear:

(i) If $f \in \mathrm{Mor}(G, H)$ and $g \in \mathrm{Mor}(H, K)$ then

$$\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$$

(ii) For any group G,

$$\widetilde{\mathrm{id}_G} = \mathrm{id}_{G/[G,G]}$$

This is an example of a covariant functor from one category to another.

Example 8.4: Here we give an example of a contra-variant functor. The family of all real vector spaces, denoted by **Vect** is a category and for a pair of real vector spaces V and W, the set $\mathrm{Mor}(V, W)$ consists of all linear transformations from V to W. We define a functor from **Vect** to itself by assigning to each V its dual V^* and to each $T \in \mathrm{Mor}(V, W)$ the adjoint map T^*. Again,

$$(\mathrm{id}_V)^* = \mathrm{id}_{V^*}$$

But if U, V and W are three vector spaces and $T \in \mathrm{Mor}(U, V)$ and $S \in \mathrm{Mor}(V, W)$ are two linear maps then

$$(S \circ T)^* = T^* \circ S^*$$

Let us look at an example of a functor from the category of topological spaces to the category **Rng** of commutative rings. We shall always assume that every ring that we shall deal with, has a unit element.

Example 8.5: Let X be a topological space and $C(X)$ be the set of all continuous functions from X to the real line (with its usual topology). Then $C(X)$ is a commutative ring with unity. Suppose that $f : X \longrightarrow Y$ is a continuous map between topological spaces then we define f^* to be the map

$$f^* : C(Y) \longrightarrow C(X)$$
$$\phi \mapsto \phi \circ f$$

It is obvious to see that f^* is a ring homomorphism and $\mathrm{id}_X^* = \mathrm{id}_{C(X)}$. Further, $(g \circ f)^* = f^* \circ g^*$ for $f \in \mathrm{Mor}(X, Y)$ and $g \in \mathrm{Mor}(Y, Z)$. We thus have a contravariant functor **Top** $\longrightarrow$ **Rng** sending the object $X \in$ **Top** to the object $C(X) \in$ **Rng** and assigning to $f \in \mathrm{Mor}(X, Y)$ the ring homomorphism $f^* \in \mathrm{Mor}(C(Y), C(X))$.

Category of pairs: Given topological spaces X and Y one is often not interested in the class of all continuous maps $f : X \longrightarrow Y$ but a restricted class of continuous functions satisfying some "side conditions" such as mapping a given subset A of X into a given subset B of Y.

Definition 8.3: The category $\mathbf{Top^2}$ of pairs has as its objects the class of all pairs of topological spaces (X, A) where X is a topological space and $A \subset X$. Given two pairs (X, A) and (Y, B) the set of morphisms between them is the class of all continuous functions $f : X \longrightarrow Y$ such that $f(A) \subset B$.

Exercises:

1. Recast the notion of homotopy of paths in terms of morphisms of the category $\mathbf{Top^2}$.

2. Define a binary operation on $\mathbb{Z} \times \mathbb{Z}$ as follows

$$(a, b) \cdot (c, d) = (a + c, b + (-1)^a d)$$

   Show that this defines a group operation on $\mathbb{Z} \times \mathbb{Z}$ and this group is called the semi-direct product of $\mathbb{Z}$ with itself. The standard notation for this is $\mathbb{Z} \ltimes \mathbb{Z}$. Compute the inverse of (a, b), compute the conjugate of (a, b) by (c, d) and the commutator of two elements. Determine the commutator subgroup and the the abelianization of $\mathbb{Z} \ltimes \mathbb{Z}$.

3. A morphism $\phi \in \text{Mor}(X, Y)$ in a category is said to be an equivalence if there exists $\psi \in \text{Mor}(Y, X)$ such that $\phi \circ \psi = \text{id}_Y$ and $\psi \circ \phi = \text{id}_X$. In a category whose objects are sets and morphisms are maps, show that if $g \circ f$ is an equivalence for $f \in \text{Mor}(X, Y)$ and $g \in \text{Mor}(Y, Z)$ then g is surjective and f is injective.

4. We say a category $\mathcal{C}$ admits finite products if for every pair of objects U, V in $\mathcal{C}$ there exists an object W and a pair of morphisms $p : W \longrightarrow U$, $q : W \longrightarrow V$ such that the following property holds. For every pair of morphisms $f : Z \longrightarrow U$, $g : Z \longrightarrow V$ there exists a unique morphism $f \times g \in \text{Mor}(Z, W)$ such that
$$p \circ (f \times g) = f, \quad q \circ (f \times g) = g.$$
Show that the categories **Top**, **Gr** and **AbGr** admit finite products and in fact the usual product of topological spaces/groups serve the purpose with p and q being the two projection maps.

5. Discuss arbitrary products in a category generalizing the preceding exercise and discuss the existence of arbitrary products in the categories **Top**, **Gr** and **AbGr**.

6. We say a category $\mathcal{C}$ admits finite coproducts if for every pair of objects U, V in $\mathcal{C}$ there exists an object W and a pair of morphisms $p : U \longrightarrow W$, $q : V \longrightarrow W$ such that the following property holds. For every pair of morphisms $f : U \longrightarrow Z$, $g : V \longrightarrow Z$ there exists a unique morphism $f \oplus g \in \text{Mor}(W, Z)$ such that
$$(f \oplus g) \circ p = f, \quad (f \oplus g) \circ q = g.$$
Show that the category **AbGr** admits finite coproducts and in fact the usual product of groups serves the purpose where the maps p and q are the canonical injections:
$$\begin{aligned} p : G \longrightarrow G \times H, &\quad q : H \longrightarrow G \times H \\ p(g) = (g, 1), &\quad q(h) = (1, h) \end{aligned}$$
What happens when this (naive construction) is tried out in the category **Gr** instead of **AbGr**? In the context of abelian groups the coproduct is referred to as the direct sum.

7. Discuss the coproduct of an arbitrary family of objects in the category **AbGr**. It is referred to as the direct sum of the family.

8. Suppose that X and Y are two topological spaces, form their disjoint union $X \sqcup Y$ which is the set theoretic union of their homeomorphic copies $X \times \{1\}$ and $Y \times \{2\}$. A subset G of $X \sqcup Y$ is declared open if $G \cap (X \times \{1\})$ and $G \cap (Y \times \{2\})$ are both open. Check that this defines a topology on $X \sqcup Y$ and the maps
$$\begin{aligned} p : X \longrightarrow X \sqcup Y, &\quad q : Y \longrightarrow X \sqcup Y \\ p(x) = (x, 1), &\quad q(y) = (y, 2) \end{aligned}$$
are both continuous. Show that the category **Top** admits finite coproducts.