Lecture III - More preliminaries from general topology:

In this lecture we take up the second most important notion in point set topology, namely the notion of connectedness. This topic is usually covered in good detail in point set topology courses. Again we shall merely outline the theory emphasizing examples rather than proving standard results. We begin by recalling the definition of a connected subset of a topological space ([13], p. 42).

Definition 3.1: A subset Y of a topological space X is said to be disconnected if there are non-empty subsets A and B of X such that

$$Y = A \cup B, \quad \overline{A} \cap B = \emptyset, \quad A \cap \overline{B} = \emptyset.$$

If Y is not disconnected we say that Y is connected.

Examples 3.1: (i) The intervals $[0, 1]$ and $(0, 1)$ on the real line are connected. The only connected subsets of the real line are intervals (including the empty set). Hence the only connected subsets of $\mathbb{Z}$ are singletons and the empty set.

(ii) Product of connected spaces are connected. Thus the cube $[0, 1] \times [0, 1] \times [0, 1]$ is connected.

We now state the most basic theorem on connectedness whose proof ought to be done in standard courses on general topology and will not be repeated here.

Theorem 3.1: (i) If X and Y are topological spaces and $f : X \longrightarrow Y$ is a continuous map and A is a connected subset of X then $f(A)$ is a connected subset of Y.

(ii) A topological space X is connected if and only if every continuous function $f : X \longrightarrow \mathbb{Z}$ is constant.

(iii) If $\{A_n\}$ is a sequence of connected subsets of a topological space X and $A_n \cap A_{n+1}$ is non-empty for each $n = 1, 2, 3, \dots$ then $\cup_{n=1}^{\infty} A_n$ is connected. In particular, taking $A_2 = A_3 = \dots$ we get the result for two connected sets.

(iv) If A_α is a family of connected subsets of a topological space such that for some connected subset B, $A_\alpha \cap B \neq \emptyset$ for each α, then $\bigcup_\alpha A_\alpha$ is also connected.

(v) Suppose that A is a connected subset of a topological space and $A \subset B \subset \overline{A}$ then B is also connected.

(vi) A space X is connected if and only if the only subsets of X that are open and closed are X and $\emptyset$.

Example 3.2: The theorem can be used to prove that the sphere

$$S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} / x_1^2 + x_2^2 + \dots x_{n+1}^2 = 1\}$$

is connected. Define $S^n_\pm$ to be the closed upper and lower hemispheres. Then $S^n_\pm$ are connected. The intersection of these hemispheres is S^{n-1}. One can now apply induction observing first that the circle S^1 is connected since it is the continuous image of the real line under the map

$$t \mapsto \exp(2\pi i t).$$

Example 3.3: The set $GL(n, \mathbb{R})$ of all $n \times n$ invertible matrices with real entries is disconnected as a subspace of the space of all $n \times n$ matrices with real entries (the latter may be identified with $\mathbb{R}^{n^2}$).

If $GL(n, \mathbb{R})$ were connected then so would be its image under a continuous map. Well, the determinant map $d : GL(n, \mathbb{R}) \longrightarrow \mathbb{R}$ is continuous but the image is the real line minus the origin. The same argument shows that the set of all $n \times n$ orthogonal matrices $O(n, \mathbb{R})$ is disconnected.

Definition 3.2 (Path connectedness): A space X is said to be path connected if given any two points $x, y \in X$, there is a continuous function $f : [0, 1] \longrightarrow X$ such that $f(0) = x$ and $f(1) = y$.

Theorem 3.3: If X is path connected then it is connected.

Proof: Assume X is path connected but not connected. Then there is a non constant continuous function $g : X \longrightarrow \mathbb{Z}$ say $f(x) = m$ and $f(y) = n$ for a pair of distinct integers m and n. But there is also a continuous function $f : [0.1] \longrightarrow X$ such that $g(0) = x$ and $g(1) = y$. The composite function $f \circ g : [0, 1] \longrightarrow \mathbb{Z}$ is non constant which is a contradiction.

Corollary 3.4: A convex set in $\mathbb{R}^n$ and more generally a star shaped set in $\mathbb{R}^n$ is path connected and hence connected. In particular, the square $[0, 1] \times [0, 1]$ is path connected and hence connected.

Theorem 3.5: (i) If X and Y are topological spaces and $f : X \longrightarrow Y$ is a continuous map and A is a path connected subset of X then $f(A)$ is also a path connected subset of Y.

(ii) If $\{A_n\}$ is a sequence of path connected subsets of a topological space X and $A_n \cap A_{n+1}$ is non-empty for each $n = 1, 2, 3, \dots$ then $\cup_{n=1}^{\infty} A_n$ is path connected. In particular, taking $A_2 = A_3 = \dots$ we see get the result for two path connected sets.

Proof: This is usually done in point set topology courses and so the proof will not be repeated here.

Definition 3.3: A space X is said to be locally path connected if each point of X has a local base consisting of path connected neighborhoods.

Theorem 3.6: A connected, locally path-connected space is path connected. In particular, an open subset of $\mathbb{R}^n$ is path connected.

Proof: Let x and y be arbitrary points of X and let G be the set of all points of X that can be joined to x by a path. Clearly G is non-empty since it contains the point x. If we show that G is open and closed then by connectedness of X it would follow that G would equal the whole space X. In particular G contains y thereby proving that there is a path in X joining x and y. First we show that G is open. Well, let z be an arbitrary point of G and choose a path $\gamma : [0, 1] \longrightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = z$. Choose a path connected neighborhood N of z and $w \in N$ be arbitrary. Then there is a path σ lying in N joining z and w. We now juxtapose γ and σ by defining $\eta : [0, 1] \longrightarrow X$ as

$$\eta(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq 1/2 \\ \sigma(2t - 1), & 1/2 \leq t \leq 1 \end{cases}$$

By virtue of the gluing lemma η is continuous and defines a path joining x and w. Hence w belongs to G and so $N \subset G$. We now show that G is closed as well. Let $y \notin G$ and N be a path connected neighborhood of y. Then we show that $N \subset X - G$. Well, if not, pick $z \in G \cap N$ and there is a path γ in G joining x and z and a path σ in N joining z and y. Juxtaposing we would get a path in X joining x and y which would contradict the fact that $y \notin G$. So $X - G$ is also open in X and the proof is complete.

The Tietze's extension theorem: We shall make occasional use of this in the sequel. Since we need it for the special case of metric spaces, we shall state the theorem in this context.

Theorem 3.7: Suppose that X is a metric space, A is a closed subspace of X and $f : A \longrightarrow \mathbb{R}$ is a continuous function then f extends continuously to the whole of X. Furthermore if f is bounded from above/below then the extension may be so chosen that the bound(s) are preserved.

Remarks: Note that the Tietze's extension theorem is valid for maps taking values in $\mathbb{R}^n$ or a finite product of intervals such as $[0, 1]^n$.

Exercises:

1. Prove that any continuous function $f : [-1, 1] \longrightarrow [-1, 1]$ has a fixed point, that is to say, there exists a point $x \in [-1, 1]$ such that $f(x) = x$.

2. Prove that the unit interval $[0, 1]$ is connected. Is it true that if $f : [0, 1] \longrightarrow [0, 1]$ has connected graph then f is continuous? What if connectedness is replaced by path connectedness?

3. Suppose X is a locally compact, non-compact, connected Hausdorff space, is its one point compactification connected? What happens if X is already compact and Hausdorff?

4. Show that any connected metric space with more than one point must be uncountable. Hint: Use Tietze's extension theorem and the fact that the connected sets in the real line are intervals.

5. Show that the complement of a two dimensional linear subspace in $\mathbb{R}^4$ is connected. Hint: Denoting by V be the two dimensional vector space, show that $\Sigma = \{\mathbf{x}/\|\mathbf{x}\| \;/\; \mathbf{x} \in \mathbb{R}^4 - V\}$ is connected using stereographic projection or otherwise.

6. How many connected components are there in the complement of the cone
$$x_1^2 + x_2^2 + x_3^3 - x_4^2 = 0$$
in $\mathbb{R}^4$? Hint: The complement of this cone is filled up by families of hyperboloids. Examine if there is a connected set B meeting each member of a given family.

7. A map $f : X \longrightarrow Y$ is said to be a *local homeomorphism* if for $x \in X$ there exist neighborhoods U of x and V of $f(x)$ such that $f\Big|_U : U \longrightarrow V$ is a homeomorphism. If $f : X \longrightarrow Y$ is a local homeomorphism and a proper map, then for each $y \in Y$, $f^{-1}(y)$ is a finite set. Show that the map $f : \mathbb{C} - \{1, -1\} \longrightarrow \mathbb{C}$ given by $f(z) = z^3 - 3z$ is a local homeomorphism. Is it a proper map?