Lectures - XXIX/XXX The singular chain complex and homology groups

The program of developing a calculus of chains is now formalized in this lecture. We introduce a new algebraic category of chain complexes and maps between them and prove the fundamental theorem about these algebraic gadgets. In particular, to each chain complex is associated a sequence of groups called the homology groups. Given a topological space X we associate a chain complex to it and obtain the homology functors from the category **Top** to the category **AbGr**. Thus we lay in this lecture the foundations for a systematic calculus of chains and cycles putting the heuristic ideas of the last lecture on a rigorous footing.

Definition 29.1 (The standard simplex): The standard $n-$simplex denoted by Δ_n is the convex hull of the $n+1$ the standard unit vectors in $\mathbb{R}^{n+1}$. Denoting the standard unit vectors by $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n+1}$, their convex hull is the set

$$\Delta_n = \{(t_1, t_2, \dots, t_{n+1}),\ \ t_1 \geq 0,\ t_2 \geq 0, \dots, t_{n+1} \geq 0,\ t_1 + t_2 + \cdots + t_{n+1} = 1\}.$$

We take the standard zero simplex Δ_0 to be the point $\mathbf{e}_1$.

Thus Δ_2 is the equilateral triangle with vertices $(1,0,0), (0,1,0)$ and $(0,0,1)$ and the one simplex Δ_1 is the line segment in $\mathbb{R}^2$ joining the points $(1,0)$ and $(0,1)$.

Note that Δ_2 contains three copies of Δ_1 namely the sides of the equilateral triangle. Likewise Δ_3 contains four copies of Δ_2, the four faces of the regular tetrahedron. To formalize this idea, we introduce $(n+1)$ affine maps $\Delta_{n-1} \longrightarrow \Delta_n$ called the face maps. For $i = 1, 2, 3$, the $i-$th *face* of Δ_2 is the face opposite to vertex $\mathbf{e}_i$ and consists of all points (t_1, t_2, t_3) with non-negative entries and $t_1 + t_2 + t_3 = 1$ such that the $i-$th coordinate t_i vanishes.

Now suppose that $(t_1, t_2, \dots, t_{n+1})$ denotes denotes a typical point on the last face of Δ_n. Then since t_{n+1} vanishes, we see that $(t_1, t_2, \dots, t_n)$ is a typical point on Δ_{n-1}. Turning the argument around we define the map

$$\begin{aligned} \Delta_{n-1} &\longrightarrow \Delta_n \\ (t_1, t_2, \dots, t_n) &\mapsto (t_1, t_2, \dots, t_n, 0), \end{aligned}$$

where the t_i are all non-negtive and $\sum t_i = 1$, and call it the standard n-th face map. The $i-$th face map $(0 \leq i \leq n)$ would be

$$\begin{aligned} \Phi^n_i : \Delta_{n-1} &\longrightarrow \Delta_n \\ (t_1, t_2, \dots, t_n) &\mapsto (t_1, t_2, \dots, t_{i-1}, 0, t_i, \dots, t_n), \end{aligned} \tag{29.1}$$

We leave it to the reader to write down explicitly the maps $\Phi^n_j \circ \Phi^{n-1}_i : \Delta_{n-2} \longrightarrow \Delta_n$ and prove the following result:

Lemma 29.1: Suppose that $0 \leq j < i \leq n$ then

$$\Phi_j^n \circ \Phi_{i-1}^{n-1} = \Phi_i^n \circ \Phi_j^{n-1} \tag{29.2}$$

Definition 29.2 (Singular chains): A singular $n-$simplex in a topological space X is a continuous map $\sigma : \Delta_n \longrightarrow X$. The free abelian group generated by the set of all singular $n-$simplices in X is called the group of singular $n-$chains in X. This group is denoted by $S_n(X)$ and a typical element of $S_n(X)$ is thus a formal sum

$$n_1\sigma_1 + n_2\sigma_2 + \cdots + n_k\sigma_k, \tag{29.3}$$

where the coefficients $n_1, n_2, \ldots, n_k$ are integers. For convenience we define $S_{-1}(X)$ to be the zero group.

The most important notion in homology theory is the algebraization of the notion of a boundary which applies to arbitrary singular simplices in an arbitrary topological space and not merely polyhedra in Euclidean spaces obtained by gluing together affine simplices. It is precisely this algebraization which provides considerable flexibility towards applications of homology theory.

Definition 29.3 (Boundary of a singular simplex): Given a singular $n-$simplex $\sigma : \Delta_n \longrightarrow X$, its $j-$th singular boundary is the singular $(n-1)$ simplex $\sigma \circ \Phi_j^n$ and the boundary $\partial_n\sigma$ of σ is then the $(n-1)$ chain given by the algebraic sum of its singular faces:

$$\partial_n\sigma = \sum_{j=0}^{n} (-1)^j (\sigma \circ \Phi_j^n). \tag{29.4}$$

The map ∂_n then extends as a group homomorphism $\sigma_n : S_n(X) \longrightarrow S_{n-1}(X)$. When $n = 0$ we define the boundary map ∂_0 to be the zero map.

The most important property of the maps ∂_n is the vanishing of $\partial_{n-1} \circ \partial_n$ which we now prove.

Theorem 29.2: For each $n \geq 1$, we have

$$\partial_{n-1} \circ \partial_n = 0. \tag{29.5}$$

Proof: It clearly suffices to check the result on the generators of $S_n(X)$. So let σ be an arbitrary singular n simplex. Using equation (29.4),

$$(\partial_{n-1} \circ \partial_n)\sigma = \partial_{n-1}\Big(\sum_{i=0}^{n} (-1)^i \sigma \circ \Phi_i^n \Big) = \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} \sigma \circ (\Phi_i^n \circ \Phi_j^{n-1}).$$

To use lemma (29.1) we break the double sum in two pieces and write

$$(\partial_{n-1} \circ \partial_n)\sigma = \sum_{i \leq j} (-1)^{i+j} \sigma \circ (\Phi_i^n \circ \Phi_j^{n-1}) + \sum_{j<i} (-1)^{i+j} \sigma \circ (\Phi_i^n \circ \Phi_j^{n-1})$$

Using (29.2) in the second piece we get

$$(\partial_{n-1} \circ \partial_n)\sigma = \sum_{i \leq j} (-1)^{i+j} \sigma \circ (\Phi_i^n \circ \Phi_j^{n-1}) + \sum_{j<i} (-1)^{i+j} \sigma \circ (\Phi_j^n \circ \Phi_{i-1}^{n-1})$$

It may be noted that each of the two pieces is a sum of $n(n+1)/2$ terms (why?). Renaming $i-1$ as k in the second sum gives

$$(\partial_{n-1} \circ \partial_n)\sigma = \sum_{i \le j \le n-1} (-1)^{i+j} \sigma \circ (\Phi_i^n \circ \Phi_j^{n-1}) + \sum_{j \le k \le n-1} (-1)^{k+j-1} \sigma \circ (\Phi_j^n \circ \Phi_k^{n-1}) = 0$$

as desired.

Now suppose that X and Y are two topological spaces and $f : X \longrightarrow Y$ is a continuous map then $f \circ \sigma$ is a singular $n-$simplex in Y whenever σ is a singular $n-$simplex in X.

Definition 29.4: Given a continuous map $f : X \longrightarrow Y$, the map $f_\sharp : S_n(X) \longrightarrow S_n(Y)$ is the group homomorphism which is defined on singular n simplices σ via the prescription

$$f_\sharp(\sigma) = f \circ \sigma, \tag{29.6}$$

and extended as a group homomorphism from $S_n(X)$ to $S_n(Y)$. We ought to denote this map by $f_\sharp^n$ but we shall suppress the superscript to enhance readability.

Theorem 29.3: (i) For a continuous map $f : X \longrightarrow Y$, the maps $f_\sharp : S_n(X) \longrightarrow S_n(Y)$ satisfy

$$\partial_n \circ f_\sharp = f_\sharp \circ \partial_n \tag{29.7}$$

The $f_\sharp$ on the right hand side obviously refers to the map $S_{n-1}(X) \longrightarrow S_{n-1}(Y)$ and ∂_n refers to the boundary operator on $S_n(Y)$ on the left hand side whereas it refers to the boundary operator on $S_n(X)$ on the right hand side.

(ii) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two continuous maps then the maps $f_\sharp : S_n(X) \longrightarrow S_n(Y)$ and $g_\sharp : S_n(Y) \longrightarrow S_n(Z)$ satisfy

$$(g \circ f)_\sharp = g_\sharp \circ f_\sharp \tag{29.8}$$

Proof: We shall only prove (29.7). It suffices to check these on singular simplices. So let $\sigma : \Delta_n \longrightarrow X$ be a singular n simplex in X. Using (29.4) we get

$$(f_\sharp \circ \partial_n)\sigma = f_\sharp\Big(\sum_{i=0}^{n}(-1)^i \sigma \circ \Phi_i^n\Big) = \sum_{i=0}^{n}(-1)^i((f \circ \sigma) \circ \Phi_i^n) = \partial_n(f \circ \sigma) = \partial_n(f_\sharp(\sigma)) = (\partial_n \circ f_\sharp)\sigma.$$

The category of chain complexes: We have associated to each topological space X a sequence $\{S_n(X)\}$ of *free* abelian groups and group homomorphisms $\partial_n : S_n(X) \longrightarrow S_{n-1}(X)$ satisfying (29.7). It is useful to describe these in an abstract setting where the groups in question need not be free abelian and prove some general results about them. This paragraph serves as an algebraic prerequisite for the study homology theory.

Definition 29.5: (i) A differential chain complex is a sequence $\{G_n/n = 0, 1, 2 \ldots\}$ of abelian groups together with a sequence of group homomorphism $\partial_n : G_n \longrightarrow G_{n-1}$ called the *boundary operator* satisfying the condition

$$\partial_n \circ \partial_{n+1} = 0, \quad n = 0, 1, 2, \ldots \tag{29.9}$$

with the convention $G_{-1} = \{0\}$ and $\partial_0 = 0$. We shall use the letter G to denote this chain complex.

(ii) For a chain complex G, we define the subgroup $Z_n(G)$ of n-cycles to be the kernel of ∂_n namely,

$$Z_n(G) = \{z \in G_n / \partial_n(z) = 0\} \tag{29.10}$$

and the subgroup of n-boundaries as the image of ∂_{n+1} namely

$$B_n(G) = \{\partial_{n+1}(x) / x \in G_{n+1}\}. \tag{29.11}$$

From (29.9) it is clear that $B_n(G) \subset Z_n(G)$ and also $Z_0(G) = G_0$.

(iii) The quotient group

$$H_n(G) = Z_n(G)/B_n(G) \tag{29.12}$$

is called the n-th homology of the chain complex G. If $z_n \in Z_n(G)$ is a cycle the symbol $\overline{z_n}$ refers to the coset of z_n in the quotient group $H_n(G)$, called the *homology class* of z_n. We shall simplify notations whenever feasible and write Z_n in place of $Z_n(G)$, B_n instead of $B_n(G)$ and sometimes ∂z in place of the cumbersome $\partial_n(z)$.

Given two chain complexes G and K one would like to study maps between them. These are the chain maps which we now define.

Definition 29.6: Given two chain complexes G and K with boundary maps $\partial' : G_n \longrightarrow G_{n-1}$ and $\partial'' : K_n \longrightarrow K_{n-1}$, a chain map $\phi : G \longrightarrow K$ is a sequence of group homomorphisms $\phi_n : G_n \longrightarrow K_n$ $(n = 0, 1, 2, \dots)$ such that

$$\partial''_n \circ \phi_n = \phi_{n-1} \circ \partial'_n \tag{29.13}$$

Equation (29.13) may be summarized by declaring that the following diagram commutes:

$$\begin{array}{ccc} G_n & \xrightarrow{\partial'_n} & G_{n-1} \\ {\scriptstyle \phi_n}\big\downarrow & & \big\downarrow{\scriptstyle \phi_{n-1}} \\ K_n & \xrightarrow{\partial''_n} & K_{n-1} \end{array} \tag{29.14}$$

Theorem 29.4: A chain map $\phi : G \longrightarrow K$ induces for each $n = 0, 1, 2, \dots$, a group homomorphism $H_n(\phi) : H_n(G) \longrightarrow H_n(K)$ given by

$$\overline{x} \;\mapsto\; \overline{\phi_n(x)}.$$

Proof: Thanks to (29.7), ϕ_n maps $Z_n(G)$ into $Z_n(K)$ and $B_n(G)$ into $B_n(K)$. Thus the map induced on the quotient groups is a well defined group homomorphism.

Theorem 29.5: Suppose given a pair of chain maps $\phi : L \longrightarrow G$ and $G \longrightarrow K$, then the composite $\psi \circ \phi : L \longrightarrow K$ is a chain map and

$$H_n(\psi \circ \phi) = H_n(\psi) \circ H_n(\phi), \quad n = 0, 1, 2, \dots \tag{29.15}$$

In other words for each n we get a covariant functor H_n from the category of chain complexes to the category **AbGr**.

Proof: For each $x \in Z_n(L)$,

$$H_n(\psi \circ \phi)(\overline{x}) = \overline{\psi_n \circ \phi_n(x)} = H_n(\psi)(\overline{\phi_n(x)}) = H_n(\psi) \circ H_n(\phi)(\overline{x}).$$

We shall at some point as we go along, drop the primes and denote both sets of boundary maps by ∂_n or even ∂. Observe that if $z \in \ker \phi_n$ then

$$\phi_{n-1}(\partial_n z) = \partial_{n-1}\phi_n(z) = 0,$$

whereby we conclude that ∂_n maps $\ker \phi_n$ into $\ker \phi_{n-1}$ and we get a chain complex

$$\longrightarrow \ker \phi_{n+1} \xrightarrow{\partial_{n+1}} \ker \phi_n \xrightarrow{\partial_n} \ker \phi_{n-1} \longrightarrow$$

which we denote by $\ker \phi$. Likewise we get the chain complex

$$\longrightarrow \mathrm{Im}\ \phi_{n+1} \xrightarrow{\partial_{n+1}} \mathrm{Im}\ \phi_n \xrightarrow{\partial_n} \mathrm{Im}\ \phi_{n-1} \longrightarrow$$

which we denote by Im ϕ. It is clear from (29.13) that ∂_n maps Im ϕ_n into Im ϕ_{n-1}.

The long exact homology sequence: We are now ready to prove the most basic result on chain complexes and their homologies. The symbol 0 in any diagram involving chain complexes refers to the zero chain complex in which all groups are zero and the boundary maps are all zero.

Definition 29.7: A short exact sequence of chain complexes consists of three chain complexes of abelian groups L, G and K and chain maps $f : L \longrightarrow G$ and $g : G \longrightarrow K$ such that

(i) For each n, the map f_n is injective.

(ii) For each n, the map g_n is surjective.

(iii) For each n, $\ker g_n = \mathrm{Im}\ f_n$.

Thus for each n we have the diagram

$$\{0\} \longrightarrow L_n \xrightarrow{f_n} G_n \xrightarrow{g_n} K_n \longrightarrow \{0\} \tag{29.16}$$

We now write out two more parallel rows with n replaced by $n-1$ and $n+1$ and the boundary maps going across the rows:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & L_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\partial_{n+1}"] & G_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\partial_{n+1}"] & K_{n+1} \arrow[r] \arrow[d, "\partial_{n+1}"] & 0 \\
0 \arrow[r] & L_n \arrow[r, "f_n"] \arrow[d, "\partial_n"] & G_n \arrow[r, "g_n"] \arrow[d, "\partial_n"] & K_n \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & L_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & G_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & K_{n-1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

We now state and prove the fundamental result.

Theorem 29.6: A short exact sequence of complexes (29.16) induces a long exact sequence in homology

$$\longrightarrow H_n(L) \xrightarrow{H_n(f)} H_n(G) \xrightarrow{H_n(g)} H_n(K) \xrightarrow{\delta_n} H_{n-1}(L) \longrightarrow \tag{29.17}$$

where the map $\delta_n : H_n(K) \longrightarrow H_{n-1}(L)$ known as the *connecting homomorphism* is given by the formula

$$\delta_n \overline{k_n} = \overline{f_{n-1}^{-1} \partial_n g_n^{-1}(k_n)}, \quad k_n \in Z_n(K) \tag{29.18}$$

Here $\overline{k_n}$ refers to the homology class of $k_n \in Z_n(K)$ and $g^{-1}(k_n)$ refers to any pre-image of k_n.

Proof: We must first show that displayed formula (29.18) gives a well-defined map since several choices are being made. First, for $k_n \in Z_n(K)$ surjectivity of g_n shows that there exists $x_n \in G_n$ such that $g_n(x_n) = k_n$. Applying the boundary map ∂_n we see that $g_{n-1}(\partial x_n) = \partial k_n = 0$ which, by virtue of exactness of (29.16) and injectivity of f_{n-1}, shows there is a unique $y_{n-1} \in L_{n-1}$ such that

$$f_{n-1}(y_{n-1}) = \partial x_n. \tag{29.19}$$

We have to now show that y_{n-1} is a cycle in L. This is clear if $n = 1$ and so we assume $n \geq 2$. Applying the boundary map to (29.19) gives $f_{n-2}(\partial y_{n-1}) = 0$ from which we conclude, since f_{n-2} is injective, that $y_{n-1} \in Z_{n-1}(L)$. Hence the assignment

$$k_n \mapsto \overline{y_{n-1}}, \quad k_n \in Z_n(K) \tag{29.20}$$

is well defined once we show that it is independent of the choice of $x_n \in g_n^{-1}(k_n)$.

Second, we suppose that for a given $k_n \in Z_n(K)$, x_n' and x_n'' are two members of $g_n^{-1}(k_n)$ then

$$x_n' - x_n'' \in \ker g_n = \operatorname{im} f_n.$$

So there is a $u_n \in L_n$ such that $x_n' - x_n'' = f_n(u_n)$. On the other hand, for these two choices there exist y_{n-1}' and y_{n-1}'' in L_{n-1} such that (29.19) holds and so

$$f_{n-1}(y_{n-1}') - f_{n-1}(y_{n-1}'') = \partial(x_n' - x_n'') = \partial f_n(u_n) = f_{n-1}(\partial u_n).$$

Injectivity of f_{n-1} implies y_{n-1}' and y_{n-1}'' differ by a boundary and so define the same homology class.

Third, we must show that the same homology class results if we begin with two homologous cycles k_n' and k_n''. In this there exists $v_{n+1} \in K_{n+1}$ and $x_{n+1} \in G_{n+1}$ such that

$$k_n' - k_n'' = \partial v_{n+1} = \partial g_{n+1}(x_{n+1}) = g_n(\partial x_{n+1}).$$

Let x_n' and x_n'' be chosen from $g_n^{-1}(k_n')$ and $g_n^{-1}(k_n'')$ respectively so that $g_n(x_n' - x_n'' - \partial x_{n+1}) = 0$. By exactness of (29.16) there is a $w_n \in L_n$ such that $x_n' - x_n'' - \partial x_{n+1} = f_n(w_n)$. Applying ∂ to this and and recalling (29.19) we see that the corresponding cycles y_{n-1}' and y_{n-1}'' satisfy

$$f_{n-1}(y_{n-1}' - y_{n-1}'') = \partial f_n(w_n) = f_{n-1}(\partial w_n).$$

Since f_{n-1} is injective we see that the cycles y_{n-1}' and y_{n-1}'' are homologous.

Exactness of (29.17): We first check the exactness at the junction $H_n(G)$. Since (29.15) implies $H_n(g) \circ H_n(f) = 0$, it suffices to prove ker $H_n(g) \subset$ im $H_n(f)$. So let $g_n(x_n) = \partial_{n+1} k_{n+1}$ for some $x_n \in Z_n(G)$ and $k_{n+1} \in K_{n+1}$. Since g_{n+1} is surjective we can find $x_{n+1} \in G_{n+1}$ such that $k_{n+1} = g_{n+1}(x_{n+1})$ and

$$g_n(x_n) = \partial_{n+1} g_{n+1}(x_{n+1}) = g_n(\partial_{n+1} x_{n+1}),$$

from which we conclude there exists $y_n \in L_n$ such that $x_n - \partial_{n+1} x_{n+1} = f_n(y_n)$. Applying the operator ∂_n and using injectivity of f_{n-1} we see that $y_n \in Z_n(L)$ and the result is established.

We now turn to the exactness at the junction $H_n(K)$. It is clear from (29.18) that $\delta_n(\overline{g_n x_n}) = 0$ for any $x_n \in Z_n(G)$ so that $\delta_n \circ H_n(g) = 0$. To prove ker $\delta_n(g) \subset$ im $H_n(g)$ let $k_n \in Z_n(K)$ such that $\delta_n(\overline{k_n}) = 0$. Equation (29.18) then implies, for any $x_n \in g_n^{-1}(k_n)$ there is $l_n \in L_n$ such that

$$f_{n-1}^{-1} \partial_n x_n = \partial_n l_n.$$

From this we get $x_n - f_n(l_n) = x_n' \in Z_n(G)$. Applying g_n we see that $k_n = g_n(x_n) = g_n(x_n')$ whereby we conclude $\overline{k_n} \in$ im $H_n(g)$.

Finally we come to the exactness at the junction $H_{n-1}(L)$. From (29.18) follows $H_{n-1}(f) \circ \delta_n = 0$. To show ker $H_{n-1}(f) \subset$ im δ_n, pick a cycle l_{n-1} such that $f_{n-1}(l_{n-1})$ is a boundary say $\partial_n x_n$ for some $x_n \in G_n$. Applying g_{n-1} to the equation

$$f_{n-1} l_{n-1} = \partial_n x_n$$

gives a cycle $k_n = g_n(x_n) \in Z_n(K)$. From (29.18) we infer $\delta_n(\overline{k_n}) = \overline{l_{n-1}}$ and this suffices for a proof.

Exercises

1. Sketch Δ_n for $n = 1, 2, 3$. Show that Δ_n is a compact and connected subspace of $\mathbb{R}^{n+1}$.

2. Discuss the continuity of the maps (29.1). Prove lemma (29.1). what about the cases $i \leq j$?

3. Verify equation (29.8).

4. Determine the values of n ($n = 1, 2, \ldots$) for which a constant function $\Delta_n \longrightarrow X$ an $n-$cycle.

5. Show that the family of all chain complexes forms a category in which the set of morphisms Mor(G, K) between any two chain complexes G and K is the set of all chain maps from G to K.

6. Naturality of (29.17)-(29.18). Assume given a commutative diagram of chain complexes with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\phi"] & G \arrow[r, "g"] \arrow[d, "\psi"] & K \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & L' \arrow[r, "f'"] & G' \arrow[r, "g'"] & K' \arrow[r] & 0
\end{tikzcd}$$

Denoting by δ_n and δ_n' the connecting homomorphisms, sketch relevant diagrams and prove that

$$\delta_n' \circ H_n(\eta) = H_n(\psi) \circ \delta_n$$