Lecture - XL Inductive limits

We have frequently encountered situations where a certain space X is canonically embedded in a larger space Y. A familiar example the sequence of orthogonal groups and the canonical inclusions

$$SO(2,\mathbb{R}) \longrightarrow SO(3,\mathbb{R}) \longrightarrow SO(4,\mathbb{R}) \longrightarrow \dots \tag{40.1}$$

where, the inclusion map $SO(n,\mathbb{R}) \longrightarrow SO(n+1,\mathbb{R})$ is given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad A \in SO(n,\mathbb{R})$$

The inductive limit of a sequence such (40.1) is a space which contains each individual member of the sequence, and is the *smallest* such space. The precise meaning of the adjective *smallest* would be clear from the formal definition that we shall presently give.

Let us look at a situation in the category of abelian groups. For a fixed prime p let C_{p^k} denote the cyclic group of order p^k. Then for each $j \leq k$, the group C_{p^k} contains a (unique) cyclic group of order p^j giving us a sequence of groups

$$C_p \longrightarrow C_{p^2} \longrightarrow C_{p^3} \longrightarrow \dots, \tag{40.2}$$

in which the arrows inclusion maps. All these groups may be regarded as subgroups of $\mathbb{C} - \{0\}$ or as subgroups of the smaller group S^1. However there is a *smallest* group containing a copy of each the groups C_{p^k} namely, the group

$$\left\{ \exp\left(\frac{2\pi i l}{p^k}\right) \,/\, l,k \in \mathbb{Z} \right\} \tag{40.3}$$

consisting of all p^k-th roots of unity ($k = 1, 2, \dots$). This group (known as the Prüfer group) would then be the inductive limit of the family of cyclic groups C_{p^k} ($k = 1, 2, \dots$).

We now proceed to the formal definitions and prove the existence and uniqueness (upto isomorphism) of the inductive limit of a family of groups. We recall the notion of a directed set.

Definition 40.1 (Directed systems): (i) A directed set is a set Λ with a partial order $\leq$ such that for any pair $\alpha, \beta \in \Lambda$ there exists $\gamma \in \Lambda$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

(ii) A directed system of abelian groups is a family $\{G_\alpha \,/\, \alpha \in \Lambda\}$ of abelian groups indexed by a directed set Λ together with a family of group homomorphisms $\{f_{\alpha\beta} : G_\alpha \longrightarrow G_\beta \,/\, \alpha \leq \beta\}$ satisfying the two conditions

(a) $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$ for any three $\alpha, \beta, \gamma \in \Lambda$ such that $\alpha \leq \beta \leq \gamma$.

(b) $f_{\alpha\alpha} = \mathrm{id}_{G_\alpha}$ for each $\alpha \in \Lambda$.

(iii) By dropping the adjective *abelian* from (ii) we obtain a directed system of groups.
(iv) A directed system of topological spaces is a family $\{X_\alpha \ / \ \alpha \in \Lambda\}$ of topological spaces indexed by a directed set Λ together with a family of continuous maps $\{f_{\alpha\beta} : X_\alpha \longrightarrow X_\beta \ / \ \alpha \leq \beta\}$ satisfying the two conditions (a) and (b) in (ii).

So we shall speak of a directed system in the categories **Gr**, **AbGr** or **Top**.

Example 40.1: The most important example of a directed set is of course $\mathbb{N}$ with its usual order and (40.1)-(40.2) furnish examples of directed systems of topological spaces and abelian groups indexed by $\mathbb{N}$, where the maps $f_{\alpha\beta}$ are inclusions.

We record a lemma whose proof is left for the student to verify

Lemma 40.1: Suppose that $\{M_\alpha \ / \ \alpha \in \Lambda\}$ is directed system in one of the categories **Gr**, **AbGr** or **Top**, and for some pair $x_\alpha \in M_\alpha$ and $x_\beta \in M_\beta$ there exists $\gamma \in \Lambda$ such that $f_{\alpha\gamma}(x_\alpha) = f_{\beta\gamma}(x_\beta)$, then for every $\delta \geq \gamma$, $f_{\alpha\delta}(x_\alpha) = f_{\beta\delta}(x_\beta)$.

Definition 40.2 (Inductive limit): Given a directed system $\{M_\alpha \ / \ \alpha \in \Lambda\}$ in one of the categories **Gr**, **AbGr** or **Top** and a family of morphisms $\{f_{\alpha\beta} : M_\alpha \longrightarrow M_\beta \ / \ \alpha \leq \beta\}$ in the same category satisfying the conditions in definition (40.1), an inductive limit is an object M together with a family of morphisms $\{f_\alpha : M_\alpha \longrightarrow M\}$ such that the following two conditions hold:

(1) For every pair $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$, $f_\beta \circ f_{\alpha\beta} = f_\alpha$, summarized as a commutative diagram:

$$\begin{tikzcd} M_\alpha \arrow[rr, "f_{\alpha\beta}"] \arrow[dr, "f_\alpha"'] & & M_\beta \arrow[dl, "f_\beta"] \\ & M & \end{tikzcd}$$

(2) Universal property: Given an object L and a family of morphisms $g_\alpha : M_\alpha \longrightarrow L$ satisfying

$$g_\beta \circ f_{\alpha\beta} = g_\alpha, \quad \alpha, \beta \in \Lambda, \alpha \leq \beta,$$

there exists a unique morphism $\psi : M \longrightarrow L$ such that

$$\psi \circ f_\alpha = g_\alpha.$$

Notation: The inductive limit M of the system $\{M_\alpha \ /\alpha \in \Lambda\}$ will be denoted by $\varinjlim_\alpha M_\alpha$.

Theorem 40.2: (i) Every directed system of groups or abelian groups has an inductive limit which is unique upto isomorphism.

(ii) With the notations as in the definition (40.2), assume that $f_\alpha(x) = 0$ for some $x \in G_\alpha$. There exists $\beta \geq \alpha$ such that $f_{\alpha\beta}(x) = 0$.

Proof: (i) Let $\tilde{G}$ be the coproduct (direct sum) of the abelian groups $\{G_\alpha\}$ and we regard (for simplifying notations) the groups G_α as being subgroups of $\tilde{G}$ and $i_\alpha : G_\alpha \longrightarrow G$ the inclusion maps. Declare $x_\alpha \in G_\alpha$ and $x_\beta \in G_\beta$ as being equivalent if there exists $\gamma \in \Lambda$ such that $\gamma \geq \alpha$, $\gamma \geq \beta$ and

$$f_{\alpha\gamma}(x_\alpha) = f_{\beta\gamma}(x_\beta). \tag{40.4}$$

Lemma (40.1) states that this is a well defined equivalence relation. We denote by $\sim$ the equivalence relation just defined and define N to be the subgroup generated by

$$\{x_\alpha - x_\beta \;/\; x_\alpha \sim x_\beta\}.$$

Finally let $G = \tilde{G}/N$ and $\eta : \tilde{G} \longrightarrow G$ be the quotient map. We claim that G is the inductive limit with respect to the maps f_α given by the composition

$$G_\alpha \xrightarrow{\;i_\alpha\;} \tilde{G} \xrightarrow{\;\eta\;} \tilde{G}/N. \tag{40.5}$$

We now check the conditions (1) and (2) in definition (40.2). For $\alpha \leq \beta$ we derive from

$$(f_{\beta\beta} \circ f_{\alpha\beta})(x_\alpha) = f_{\alpha\beta}(x_\alpha).$$

the useful piece of information

$$f_{\alpha\beta}(x_\alpha) \sim x_\alpha, \quad x_\alpha \in G_\alpha. \tag{40.6}$$

Hence $f_{\alpha\beta}(x_\alpha) - x_\alpha \in N$ whereby we conclude

$$\eta(f_{\alpha\beta}(x_\alpha)) = \eta(x_\alpha),$$

which in turn implies $f_\beta \circ f_{\alpha\beta} = f_\alpha$. Turning to the universal property (2) assume given an abelian group H and a family of group homomorphisms $g_\alpha : G_\alpha \longrightarrow H$ such that

$$g_\beta \circ f_{\alpha\beta} = g_\alpha, \quad \alpha \leq \beta. \tag{40.7}$$

We first use the defining property of the coproduct to get a group homomorphism $\phi : \tilde{G} \longrightarrow H$ such that the following diagram commutes:

$$\begin{tikzcd}
G_\alpha \arrow[rr, "i_\alpha"] \arrow[dr, "g_\alpha"'] & & \tilde{G} \arrow[dl, "\phi"] \\
& H &
\end{tikzcd}$$

That is $\phi \circ i_\alpha = g_\alpha$. From (40.4) and (40.7) we get $g_\alpha(x_\alpha) = g_\beta(x_\beta)$, or in view of the fact that we have identified G_α as a subgroup of $\tilde{G}$, $\phi(x_\alpha) = \phi(x_\beta)$. Hence there is a group homomorphism $\psi : \tilde{G}/N \longrightarrow H$ such that

$$\psi \circ \eta = \phi. \tag{40.8}$$

Upon applying this to an arbitrary $x_\alpha \in G_\alpha$ we get using (40.5) that $\psi \circ f_\alpha = g_\alpha$ for every $\alpha \in \Lambda$. The homomorphism satisfying (40.8) is unique since the elements $\{f_\alpha(x_\alpha)/\alpha \in \Lambda \text{ and } x_\alpha \in G_\alpha\}$ generate the group $\tilde{G}/N$.

We now prove (ii) which we shall use in the next lecture. Since $x \in N$, there exists a finite set of indices $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k$ such that

$$x = \sum_{j=1}^{k} (x_{\alpha_j} - x_{\beta_j}) \tag{40.9}$$

where $x_{\alpha_j} \sim x_{\beta_j}$ for each j. Thus for each j there is a γ_j exceeding both α_j and β_j such that $f_{\alpha_j\gamma_j}(x_{\alpha_j}) = f_{\beta_j\gamma_j}(x_{\beta_j})$. Since (40.9) spells out a relation in the direct sum of the groups G_β, it decomposes into a bunch of equations namely

$$\begin{aligned} x &= \sum_{\alpha_i=\alpha} x_{\alpha_i} - \sum_{\beta_i=\alpha} x_{\beta_i} \\ 0 &= \sum_{\alpha_i=\lambda} x_{\alpha_i} - \sum_{\beta_i=\lambda} x_{\beta_i}, \quad \lambda \neq \alpha \end{aligned}$$

The index λ runs through a finite subset of $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$. Taking δ to be sufficiently large and applying $f_{\alpha\delta}$ to the first and $f_{\lambda\delta}$ to the second of the above displayed equations and adding we get

$$f_{\alpha\delta}(x) = \sum_{j=1}^{k} (f_{\alpha_i\delta}(x_{\alpha_j}) - f_{\beta_i\delta}(x_{\beta_j})) \tag{40.10}$$

Using lemma (40.1) we see that if δ is sufficiently large each of the summands on the right hand side of (40.10) is in N and so $f_{\alpha\delta}(x) = 0$ as asserted.

Remarks: (1) The construction can be carried out in exactly the same manner in the categories **Gr** and **Top**. In the Category **Gr**, the coproduct $\tilde{G}$ of the groups G_α is the free product with the group operation written multiplicatively and the candidate for N is the *normal subgroup* generated by

$$\{x_\alpha x_\beta^{-1} \,/ x_\alpha \sim x_\beta\},$$

where as before we regard each G_α to be a subgroup of $\tilde{G}$ to simplify notations.

(2) In the category **Top** we proceed analogously by taking the coproduct, the disjoint union of the spaces, and defining the equivalence relation (40.4) on it and passing on to the quotient space. In applications one uses the defining properties (1) and (2) of definition (40.2) and not these details involved in the actual construction.

Exercises:

1. Prove lemma (40.1)

2. Show that the Prüfer group (40.3) is the inductive limit of the sequence of multiplicative cyclic groups C_{p^k} of order p^k, where p is a prime number.

3. Discuss the existence of inductive limits of directed systems in the categories **Gr** and **Top**.

4. Suppose that $\{G_\alpha/\alpha \in \Lambda\}$ is a directed system of groups with inductive limit G and associated maps $f_\alpha : G_\alpha \longrightarrow G$, show that G is the set theoretic union of the images $f_\alpha(G_\alpha)$, $\alpha \in \Lambda$.