The homology groups $H_n(X)$ we have hitherto been studying are called the *absolute* homology groups. The relative homology groups $H_n(X, A)$ that we define here provide us a tool for understanding the geometry of a space X in relation with its subspace A. This is facilitated by a long exact sequence in homology for the pair (X, A). For instance if A is a retract of X, this sequence breaks off into a bunch of short exact sequences each of which splits. The groups $H_n(X, A)$ are related to the absolute homology groups $H_{n+1}(X, A)$ for sufficiently well behaved pairs (X, A) but we shall not get into this discussion here (see [16], p. 50).

Recall that if A is a subspace of X and z is a non-trivial $n-$cycle in A then it may be a boundary when viewed as a cycle in X. In other words, the inclusion map $i : A \longrightarrow X$ need not induce an injective map in homology. The relative homology group measures $H_n(i)$ the deviation from injectivity of the map .

Definition 38.1:

(i) Given a topological space X and a subspace A, $S_n(A)$may be regarded as a subgroup of $S_n(X)$ and the group $S_n(X, A)$ of relative $n-$chains is the quotient group $S_n(X)/S_n(A)$.

(ii) For each $n = 1, 2, \dots,$. we define the boundary maps $\partial_n : S_n(X, A) \longrightarrow S_{n-1}(X, A)$ as $\partial_n c = \partial_n c (37.1)$

It is readily verified that $\partial_{n-1} \circ \partial_n = 0$ leading to the quotient complex $S(X)/S(A)$ consisting of the sequence of groups $\{S_n(X, A)\}$ and the boundary maps (37.1).

$\{S_n(X, A)\}$

(iii) The homology groups of the quotient complex $S(X)/S(A)$ are called the relative homology groups and are denoted by the symbol $H_n(X, A)$.

For a slightly more explicit description of these groups we introduce the group $Z_n(X, A)$ of relative cycles and the group of relative boundaries. The group

$n-$ $B_n(X,A)$ $c \in S_n(X)$

$Z_n(X,A)$ is the subgroup of $S_n(X)$ consisting of chains $c \in S_n(X)$ such that the boundary $\partial_n c$ is a chain in A.

That is, $Z_n(X,A) = \{c \in S_n(X)/\partial_n c \in S_{n-1}(A)\}$.(37.2)

In keeping with the convention that $S_{-1}(A) = \{0\}$ (see definition (29.5))

$Z_0(X,A) = S_0(X)$. . We see that $c \in Z_n(X,A)$ if and only if c is in the kernel of ∂_n .

Likewise the group $c \in S_n(X)$ of relative boundaries is defined to be the subgroup of $S_n(X)$ consisting of chains $\partial_n c$ such that

$$c = \partial_{n+1} c' \text{ mod } (S_n(A)),$$

for some $c' \in S_{n+1}(X)$. In other words there exists $c' \in S_{n+1}(X)$ and $a \in S_n(A)$

$c - \partial_{n+1} c' = a.$

Obviously $c \in B_n(X,A)$ if and only if c belongs to the image of ∂_{n+1} whereby we conclude

Theorem 38.1:
$H_n(X,A) = Z_n(X,A)/B_n(X,A)$

We now consider the short exact sequence of complexes induced by the inclusion i and p denoting the projection onto the quotient:

$$0 \longrightarrow S(A) \xrightarrow{i_\sharp} S(X) \xrightarrow{p} S(X,A) \longrightarrow 0.$$

Equation (37.1) states that p is a chain map and exactness of this sequence is an easy exercise.

Theorem (29.6) now gives

Theorem 38.2:
For a pair (X,A) of topological spaces there is a long exact sequence in homology:

$$\longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(p)} H_n(X,A) \xrightarrow{\delta_n} H_{n-1}(A) \longrightarrow \quad (37.3)$$

We remark that the connecting homomorphism has a simple geometrical description in this case.
If we take a relative $n-$cycle namely an element $c \in Z_n(X,A)$ then $\partial_n c$ is an element of

and is simply viewed as a chain in A. We summarize this observation

$S_{n-1}(A) \quad i^{-1}(\partial_n c) \qquad \partial_n c$

as a lemma:

Lemma 38.3:
For a pair (X, A) of spaces the connecting homomorphism

$\delta_n : H_n(X, A) \longrightarrow H_{n-1}(A)$ is given by

$\delta_n c = \partial_n c, \quad c \in Z_n(X, A).(37.4)$

Despite the notation, $\partial_n c$ in (37.4) is not a boundary in $S_{n-1}(A)$ since c is not a chain in

$S_n(X, A)$ but a chain in $S_n(X)$. If ζ is a cycle in A then for sure, it is a cycle in X as well

but then it may be actually be a boundary X, in other words $H_n(i)\zeta = 0$. This happens

precisely when ζ is in the image of δ_{n+1} by exactness of (37.3). Figure below depicts a cycle in

A (annulus) which is a boundary in X (the polygonal region)

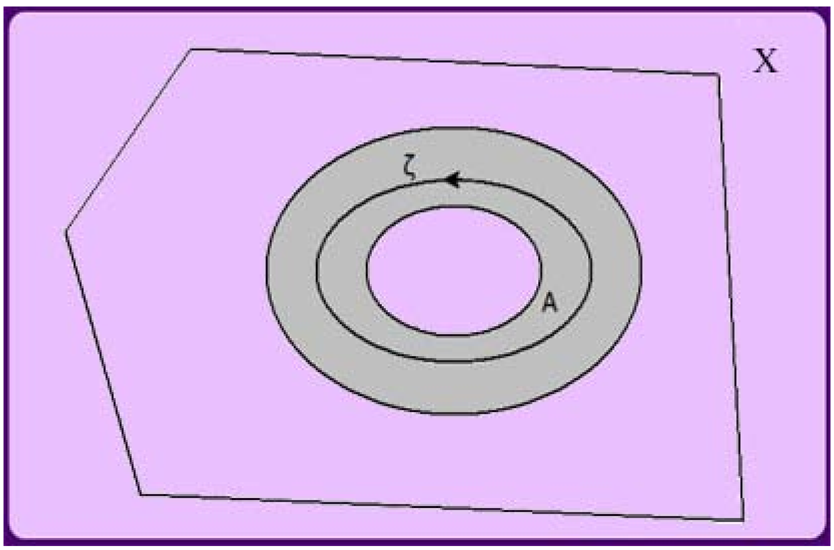

Figure 28

The long exact sequence in the preceding theorem is *natural* in the following sense.

Theorem 38.4
(Naturality): Given a map of pairs $f : (X, A) \longrightarrow (Y, B)$

the followig diagram commutes

$$\begin{array}{ccccccccc}
\longrightarrow & H_n(A) & \xrightarrow{i} & H_n(X) & \xrightarrow{p} & H_n(X,A) & \xrightarrow{\delta_n} & H_{n-1}(A) & \longrightarrow \\
& {\scriptstyle H_n(f)}\downarrow & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_{n-1}(f)} & \\
\longrightarrow & H_n(B) & \xrightarrow{i} & H_n(Y) & \xrightarrow{p} & H_n(Y,B) & \longrightarrow & H_{n-1}(B) & \longrightarrow
\end{array} \qquad (37.5)$$

Proof:

From theorem (29.6) or the specific description of δ'_n given above, (37.5) follows immediately.

Retraction:

We shall now define the notion of a split exact sequence and show that whenever A is a retract of X, the long exact sequence (37.3) breaks off into a bunch of short exact sequences each of which splits.

Definition 38.2:

A short exact sequence of abelian groups/chain complexes

$$0 \longrightarrow L \xrightarrow{f} G \xrightarrow{g} K \longrightarrow 0, (37.6)$$

splits on the right if there exists a group homomorphism (respectively a chain map) $\phi : K \longrightarrow G$ such that $g \circ \phi =_K$ idk. The short exact sequence (37.6) *splits on the left* if there exists group homomorphism (respectively a chain map) $\theta : G \longrightarrow L$ such that $\theta \circ f =$ id. $_L$

Lemma 38.5:

Given a short exact sequence (37.6), the following are equivalent:

(i) The sequence splits on the left.
(ii) The sequence splits on the right.
(iii) G is isomorphic to im. $f \oplus K$

Proof:

We begin by proving (ii) implies (iii). Note that ϕ is injective and so im ϕ is isomorphic to K.

Let $x \in G$ be arbitrary and observe that

$$x - \phi \circ g(x)$$

lies in the kernel of g and hence in the image of f. Thus,

$x = (x - \phi \circ g(x)) + \phi \circ g(x) \in$ im im $f +$ im ϕ.

We leave it to the reader to check that the sum $($im $f +$ im $\phi)$ is direct. It is easy to show that

(iii) implies (i). We now show that (i) implies (ii). Let $k \in K$ and choose any $x \in G$ such that $k = g(x)$. Define $\phi(k) = x - (f \circ \theta)(x)$. To check that this is well defined, suppose that

$k = g(x') = g(x'')$

for a pair of elements $x', x'' \in G$. There exists $y \in L$ such that $x' - x'' = f(y)$. Applying $f \circ \theta$ to this equation we get

$(f \circ \theta)(x') - (f \circ \theta)(x'') = f \circ \theta \circ f(y) = f(y) = x' - x'',$

from which we see that $(f \circ \theta)(x') - x' = (f \circ \theta)(x'') - x''$ It is trivial to see that the map ϕ that we have defined is a group homomorphism and satisfies the requirementg g o ϕ=id $_K$

Theorem 38.6:

A retraction $r : X \longrightarrow A$ gives for each $n = 0, 1, 2, \ldots$ a short exact sequence

$0 \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(p)} H_n(X, A) \longrightarrow 0.(37.7).$

Each of these short exact sequences splits. Thus

$$H_n(X) = H_n(A) \oplus H_n(X, A), \quad n = 0, 1, 2, \ldots$$

Proof:
We show that $\delta_n = 0$ for every n which would give us the sequences (37.7). For $c \in Z_n(X, A)$ we have the chains $r_\sharp(c) \in S_n(A)$ and $\partial_n c \in S_{n-1}(A)$.Now,

$$\partial_n r_\sharp(c) = r_\sharp(\partial_n c) = (r_\sharp \circ i_\sharp)(\partial_n c) = (r \circ i)_\sharp(\partial_n c) = \partial_n c$$

\Hence $\partial_n c$ is the boundary of the chain and so represents the zero element in $H_{n-1}(A)$. From lemma (37.3) we conclude that δ'_n is the zero map. The short exact sequence (37.7) splits on the left since $H_n(r) \circ H_n(i)$ is the identity map on $H_n(A)$.

Example 38.1

Let us calculate the relative homology groups $H_n(X, A)$ where X is the Möbius band and A is its boundary. Since the central circle is a deformation retract of X, we see that

$H_n(X, A) = 0$ when $n \geq 2$ and we infer from (37.3) that $H_n(X, A) = 0$ when $n \geq 3$.

We now recall that the map $i_* : \pi_1(A) \longrightarrow \pi_1(X)$ induced by inclusion is the group homomorphism of $\mathbb{Z}$ into itself given by $x \mapsto 2x$. Since the fundamental groups are abelian the map $H_1(i) = i_*$ and so the kernel of $H_1(i)$ is trivial. The portion of the exact sequence (37.3) with $n = 2$ gives $H_2(M, A) = 0$. Finally since $H_0(i) : H_0(A) \longrightarrow H_0(M)$ is an isomorphism (why?), we conclude from (37.3) (with $n = 0$) that the map $H_1(X) \longrightarrow H_1(X, A)$ is surjective with kernel $2\mathbb{Z}$. Hence $H_1(X, A) = \mathbb{Z}_2$.

Exercises

1. Verify that the diagram (37.5) commutes.
2. Determine $H_n(X, A)$ when $A = \emptyset$, and when A is a singleton and $n \geq 1$. What happens if $n = 0$?
3. Compute $H_n((S^1 \times S^1)/(S^1 \vee S^1))$ and compare it with the absolute homology $H_n((S^1 \times S^1)/(S^1 \vee S^1))$.
4. Compute $H_k(E^n/S^{n-1})$ and compare it with $H_k(E^n, S^{n-1})$.
5. In example (35.1), prove that X/A is homeomorphic to $\mathbb{R}P^2$. Compare the groups $H_n(X, A)$ with the groups $H_{n+1}(X, A)$. Hint: To set up the homeomorphism note that $(x, y) \mapsto (x\sqrt{1 - y^2}, y)$ maps each $[-1, 1] \times \{y\}$ homeomorphically onto the chord at height y.