## Lecture - XLI The Jordan-Brouwer separation theorem

We conclude the course with a proof of the Jordan Brouwer theorem, a far reaching generalization of the Jordan curve theorem (theorem 1.1). The most transparent and clear proof of the Jordan Brouwer theorem uses the notion of inductive limits developed in the previous lecture. We shall follow closely the treatment in [16] demonstrating the power of the Mayer Vietoris sequence.

**Theorem 41.1:** Let $X$ be a topological space and $\{X_\alpha \ / \ \alpha \in \Lambda\}$ be a directed system of open subsets of $X$ such that every compact subset of $X$ lies in some $X_\alpha$. For a pair of indices $\alpha \leq \beta$, the map $f_{\alpha\beta} : H_n(X_\alpha) \longrightarrow H_n(X_\beta)$ is the homomorphism induced by inclusion $X_\alpha \longrightarrow X_\beta$. Then, the family $\{H_n(X_\alpha) \ /\alpha \in \Lambda\}$ together with the maps $f_{\alpha\beta}$ forms an inductive system of abelian groups and

$$\varinjlim_{\alpha} H_n(X_\alpha) = H_n(X) = H_n\big(\varinjlim_{\alpha} X_\alpha\big) \tag{41.1}$$

**Proof:** The fact that $\{H_n(X_\alpha) \ /\alpha \in \Lambda\}$ is an inductive system is clear. Let $A$ denote the inductive limit of this system in **AbGr** and $f_\alpha : H_n(X_\alpha) \longrightarrow A$ denote the associated homomorphisms described in definition (40.2). The inclusion maps $X_\alpha \subset X$ induce homomorphisms $\iota_\alpha : H_n(X_\alpha) \longrightarrow H_n(X)$. To simplify notations, we shall suppress the bar and use the same symbol $\zeta$ to denote a cycle as well as the homology class it represents. The proof of (41.1) hinges on two simple facts:

(i) If $\zeta'$ is an $n-$chain in $X$ then there exists an $\alpha \in \Lambda$ such that the images of the constituent simplicies in $\zeta'$ are all contained in $X_\alpha$. We shall say that the chain $\zeta'$ *is supported in* $X_\alpha$. Thus $\zeta'$ may be viewed as a singular chain in $X_\alpha$ and the latter will be provisionally denoted by $\zeta$ in the proof. Further if $\zeta'$ is a cycle in $X$ then $\zeta$ is a cycle in $X_\alpha$ and $\zeta' = \iota_\alpha(\zeta)$.

(ii) If $\zeta'$ is a boundary of a chain $\omega'$ in $X$ then there exists a $\beta \in \Lambda$ such that $\zeta'$ and $\omega'$ are both supported in $X_\beta$ and the relation $\zeta = \partial\omega$ holds in $X_\beta$. In other words,

$$\iota_\alpha(\zeta) = 0 \quad \text{implies} \quad f_{\alpha\beta}(\zeta) = 0 \quad \text{for some} \quad \beta \geq \alpha. \tag{41.2}$$

To prove these note that the image of each singular simplex is a compact subset of $X$ and each chain is a finite linear combination of singular simplicies.

Property (2) of definition (40.2) may now be applied to the family of homomorphisms $\iota_\alpha$. There exists a group homomorphism $\phi : A \longrightarrow H_n(X)$ such that

$$\phi \circ f_\alpha = \iota_\alpha, \quad \alpha \in \Lambda \tag{41.3}$$

To show that $\phi$ is surjective, by (i) above, an arbitrary cycle $\zeta'$ in $X$ with support in $X_\alpha$ representing an element of $H_n(X)$ may be expressed as $\iota_\alpha(\zeta)$ where $\zeta$ is a cycle in $X_\alpha$. By (41.3) we see that

$\zeta' \in \text{im } \phi$. To show that $\phi$ is injective, let $\zeta' \in A$ be such that $\phi(\zeta') = 0$ in $X$. By exercise 4 of lecture 40, we can write

$$\zeta' = \sum f_\alpha(\zeta_\alpha) \tag{41.4}$$

where the sum is finite and each $\zeta_\alpha$ is a cycle in $X_\alpha$. Choose a $\beta$ exceeding all the indices in (41.4) and for each index $\alpha$ in (41.4), $f_\alpha(\zeta_\alpha) = f_\beta \circ f_{\alpha\beta}(\zeta_\alpha)$ and so using (41.3),

$$0 = \phi(\zeta') = (\phi \circ f_\beta)\big(\sum f_{\alpha\beta}(\zeta_\alpha)\big) = \iota_\beta\big(\sum f_{\alpha\beta}(\zeta_\alpha)\big)$$

Invoking (41.2) we arrive at $\sum f_{\alpha\beta}(\zeta_\alpha) = 0$ (perhaps with a larger $\beta$). Applying $f_\beta$ we see that $\zeta' = 0$ as desired. $\square$

**Theorem 41.2:** Let $K$ be a subset of $S^n$ that is homeomorphic to $I^k$ for some $k$ in the range $0 \leq k \leq n$. Then

$$H_j(S^n - K) = \begin{cases} \mathbb{Z} & \text{if } j = 0, \\ 0 & \text{if } j > 0. \end{cases}$$

**Proof:** If $k = 0$ then $K$ is a point and $S^n - K$ is homeomorphic to $\mathbb{R}^n$ and the result is true in this case. The proof now proceeds by induction on $k$. Assume that the result has been proved for $0 \leq k \leq m-1$ and let $h : K \longrightarrow I^m$ be a homeomorphism. Define the halves $I^+$ and $I^-$ as

$$I^+ = \{(x_1, x_2, \dots, x_m) \in I^m \ / \ x_n \geq 1/2\}, \quad I^- = \{(x_1, x_2, \dots, x_m) \in I^m \ / \ x_n \leq 1/2\}$$

and note that $I^+ \cap I^-$ is homeomorphic to the cube $I^{m-1}$. We construct the sets $K^+ = h^{-1}(I^+)$ and $K^- = h^{-1}(I^-)$, and use the Mayer Vietoris sequence to the following open cover of $S^n - (K^+ \cap K^{-1})$:

$$\{S^n - K^+,\ S^n - K^-\}.$$

Since $K^+ \cap K^-$ is homeomorphic to $I^{m-1}$, by induction hypothesis the end terms of the portion

$$H_{j+1}(S^n - K^+ \cap K^-) \longrightarrow H_j(S^n - K) \xrightarrow{(\kappa', -\kappa'')} H_j(S^n - K^+) \oplus H_j(S^n - K^-) \longrightarrow$$
$$\xrightarrow{q_j} H_j(S^n - K^+ \cap K^-)$$

are zero if $j > 0$ whereas the left most group is zero if $j = 0$. In any case $(\kappa', -\kappa'')$ is injective. Assume that for some $j > 0$, $H_j(S^n - K) \neq 0$. We choose $\zeta \in H_j(S^n - K)$, $\zeta \neq 0$ and it follows $\kappa'(\zeta) \neq 0$ or $\kappa''(\zeta) \neq 0$. Let us assume that $\kappa'(\zeta) \neq 0$. Since $K^+$ is homeomorphic to $I^m$, the process can be repeated subdividing $K^+$ into two pieces whose intersection is homeomorphic to $I^{m-1}$. Thus we construct a nested sequence of subsets

$$K = K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$$

such that for each $p$, the map $\kappa_p : H_j(S^n - K) \longrightarrow H_j(S^n - K_p)$ induced by inclusion, maps $\zeta$ to a non zero element. Composing with $f_p : H_j(S^n - K_p) \longrightarrow \varinjlim H_j(S^n - K_p)$ one checks that for $p, q \in \mathbb{N}$,

$$f_p \circ \kappa_p = f_q \circ \kappa_q,$$

thereby providing a map

$$f : H_j(S^n - K) \longrightarrow \varinjlim H_j(S^n - K_p).$$

Since the intersection $\bigcap K_i$ is homeomorphic to $I^{m-1}$, by induction hypothesis, $\varinjlim H_j(S^n - K_p) = \{0\}$. Hence $f_p(\kappa_p(\zeta)) = 0$ for every $p$ and hence by theorem (40.2) (ii), for some $q \in \mathbb{N}$, $\kappa_q(\zeta) = 0$ which is a contradiction.

Turning to the case $j = 0$, assume that rank of $H_0(S^n - K)$ is atleast two. If we select points $x$ and $y$ lying in distinct path components of $S^n - K$, the cycle $\zeta = x - y$ in $S^n - K$ is not a boundary. As before we construct a nested sequence of compact sets $\{K_p\}$ with $\kappa_p(\zeta) \neq 0$ for each $p \in \mathbb{N}$. But since $S^n - \bigcap K_p$ has only one path component, $\iota_p \circ \kappa_p(\zeta)$ is a boundary where $\iota_p$ is the map induced by the inclusion $S^n - K_p \longrightarrow S^n - \bigcap K_p$ whence

$$f_p(\kappa_p(\zeta)) = 0$$

by (41.3). This in turn forces $\kappa_p(\zeta) = 0$ by theorem (40.2) (ii) and we have a contradiction.

**Corollary 41.3:** Suppose $A$ is a subset of $S^n$ homeomorphic to $S^k$ for some $k$, $0 \leq k \leq n-1$, then

$$H_j(S^n - A) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } j = 0 \text{ and } k = n-1 \\ \mathbb{Z} & \text{if } j = 0 \text{ and } k \leq n-2 \\ \mathbb{Z} & \text{if } j = n-k-1 \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{41.5}$$

**Proof:** The result is clear if $k = 0$. We proceed by induction on $k$ and assume the result with $k-1$ in place of $k$. Let $A = A^+ \cup A^-$ where $A^+$ and $A^-$ are each homeomorphic to $S^{k-1}$ and $A^+ \cap A^-$ is homeomorphic to $S^{k-1}$. The Mayer Vietoris sequence may be applied to the open cover $\{S^n - A^+,\ S^n - A^-\}$ of $S^n - A$ and the reader ought to verify that

$$H_{j+1}(S^n - A^+ \cap A^-) \cong H_j(S^n - A), \quad j > 0.$$

By induction hypothesis we get (41.5) for the case $j > 0$. Let us now consider the case $j = 0$. The tail end of the Mayer Vietoris sequence gives

$$0 \longrightarrow H_1(S^n - S^{k-1}) \longrightarrow H_0(S^n - S^k) \xrightarrow{r} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{q} \text{img } q \longrightarrow 0$$

Since the image of $q$ is isomorphic to $\mathbb{Z}$, we see that the kernel of $q$ must also be isomorphic to $\mathbb{Z}$ giving a short exact sequence

$$0 \longrightarrow H_1(S^n - S^{k-1}) \longrightarrow H_0(S^n - S^k) \xrightarrow{r} \text{img } r \longrightarrow 0. \tag{41.6}$$

Since the image of $r$ is free of rank one, (41.6) splits and we have

$$H_0(S^n - S^k) = H_1(S^n - S^{k-1}) \oplus \mathbb{Z}.$$

If $k = n-1$ then $1 = n-(k-1)-1$ and so the induction hypothesis gives $H_1(S^n - S^{k-1}) = \mathbb{Z}$ whereas if $k \leq n-2$ then $H_1(S^n - S^{k-1}) = 0$. $\square$

**Corollary 41.4:** Suppose $A \subset S^n$ and $A$ is homeomorphic to $S^{n-1}$, then $S^n - A$ is disconnected and has precisely two components.

**Proof:** Equation (41.5) shows that $S^n - A$ has two path components. However since $S^n - A$ an open set, $S^n - A$ is locally path connected and so the path components are the same as components. Let these components be $C_1$ and $C_2$.

**Corollary 41.5 (Invariance of domain):** Suppose $U$ and $V$ are homeomorphic subsets of $\mathbb{R}^n$. Then $U$ is open if and only if $V$ is open. In particular if $h : A \longrightarrow B$ is a homeomorphism between subsets of $\mathbb{R}^n$ then $h$ maps interior points of $A$ to interior points of $B$.

**Proof:** Let $h$ be the homeomorphism between $U$ and $V$ and $p \in U$. We have to show that $h(p)$ is an interior point of $V$. Let $K$ be a closed ball centered at $p$ and contained in $U$ so that $K' = h(K)$ is a compact subset of $V$ containing $q = h(p)$. Let $B$ be the (topological) boundary of $K$ and $B' = h(B)$. We regard $U$ and $V$ as subsets of $S^n$. By theorem (41.2), $S^n - K'$ is path connected and $S^n - B'$ has two path components. However since the union

$$S^n - B' = (S^n - K') \cup (K' - B')$$

is a disjoint union of connected sets, the pieces $S^n - K'$ and $K' - B'$ are the components of $S^n - B'$. Hence they are both open in $S^n - B'$ (why?) and hence are open in $S^n$. The piece $K' - B'$ is then an open subset of $S^n$ containing $q$ and since $K' \subset V$ we see that $q$ is an interior point of $V$. $\square$

**Corollary 41.6 (Jordan Curve theorem):** The complement of a simple closed curve $C$ in $\mathbb{R}^2$ consists of two disjoint connected components precisely one of which is unbounded. $\square$

# Exercises

1. Prove the second equality in equation (41.1).
2. Prove corollary (41.6).
3. Prove that there is no injective continuous mapping from $S^n$ into $\mathbb{R}^n$. ([11], p. 217)
4. Show that no proper subset of $S^n$ can be homeomorphic to $S^n$. ([11], p. 217)
5. Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $f : \Omega \longrightarrow \mathbb{R}^n$ be an injective continuous map. Show that $f$ is a homeomorphism onto its image. ([11], p. 217)