Lecture - XXXII The abelianization of the fundamental group

In this lecture we shall establish a basic result relating the fundamental group $\pi_1(X, x_0)$ and the first homology group $H_1(X)$. The result is elegant and states that $H_1(X)$ is the abelianization of $\pi_1(X, x_0)$ when X is a path connected space. Further, the abelianization map is natural in the following sense. Suppose that $f : (X, x_0) \longrightarrow (Y, y_0)$ is a continuous map we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\ \Pi_X \Big\downarrow & & \Big\downarrow \Pi_Y \\ H_1(X) & \xrightarrow{H_1(f)} & H_1(Y) \end{array} \tag{32.1}$$

where $\Pi_X : \pi_1(X, x_0) \longrightarrow H_1(X)$ and $\Pi_Y : \pi_1(Y, y_0) \longrightarrow H_1(Y)$ are the quotient maps onto the respective abelianizations. We shall prove the main theorem (32.1) through a series of lemmas.

Theorem 32.1: Let X be a path connected topological space. There is a surjective group homomorphism

$$\Pi_X : \pi_1(X, x_0) \longrightarrow H_1(X) \tag{32.2}$$

whose kernel is the commutator subgroup $[\pi_1(X, x_0), \pi_1(X, x_0)]$. Thus

$$H_1(X) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \tag{32.3}$$

Before taking up the proof which will be completed in several steps, we set up the map Π_X. Note that if γ is a loop in X based at x_0 then γ is a one cycle, that is to say $\gamma \in Z_1(X)$ and we denote its homology class in the quotient $H_1(X)$ by $\overline{\gamma}$. This suggests that we define $\Pi_X : \pi_1(X, x_0) \longrightarrow H_1(X)$ as

$$\Pi_X : [\gamma] \mapsto \overline{\gamma} \tag{32.4}$$

We shall show that the map is a well-defined surjective group homomorphism and determine its kernel. We do each of these as a separate lemma. Since homotopy of loops is a map from the square $[0, 1] \times [0, 1]$ whereas a singular two simplex is a map from Δ_2 to X we must first set up some standard maps from Δ_2 to the square with specific properties. The usual proofs seem slightly tricky and we shall try an approach that would be useful in the next lecture.

Divide the square $[0, 1] \times [0, 1]$ into two triangles by drawing a diagonal from $(0, 0)$ to $(1, 1)$. Let T_i $(i = 1, 2)$ be two affine homeomorphisms mapping Δ_2 onto the these two triangles given by

$$\begin{aligned} T_1(\widehat{\mathbf{e}}_1) &= (0, 0), \quad T_2(\widehat{\mathbf{e}}_1) = (0, 0), \\ T_1(\widehat{\mathbf{e}}_2) &= (1, 0), \quad T_2(\widehat{\mathbf{e}}_2) = (0, 1), \\ T_1(\widehat{\mathbf{e}}_3) &= (1, 1), \quad T_2(\widehat{\mathbf{e}}_3) = (1, 1). \end{aligned}$$

We shall regard the maps T_i $(i = 1, 2)$ as maps from Δ_2 into I^2 and use them to prove the following:

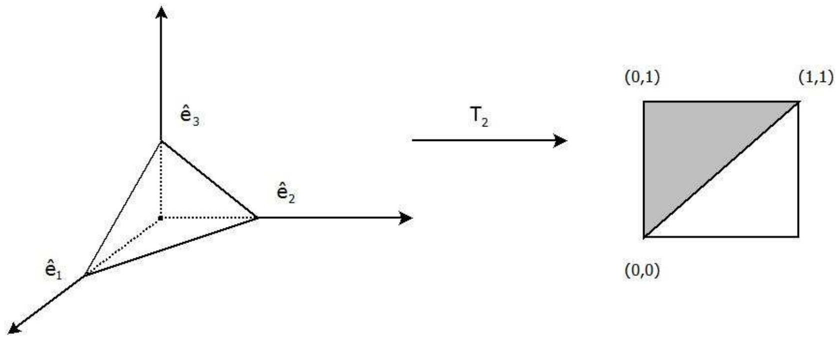

Figure 22:

Lemma 32.2: The map Π_X given by (32.4) is well defined.

Proof: Let γ_1 and γ_2 be two homotopic loops based at x_0 and let $F : I \times I \longrightarrow X$ be the homotopy fixing the base point x_0. Then $\sigma_i = F \circ T_i$ $(i = 1, 2)$ are two singular two simplicies. It is an exercise to compute the boundary of these two singular simplicies and we find easily

$$\partial(\sigma_1) = \partial(F \circ T_1) = \varepsilon_{x_0} + \gamma_1 - F(t, t)$$
$$\partial(\sigma_2) = \partial(F \circ T_2) = \varepsilon_{x_0} + \gamma_2 - F(t, t).$$

The one chain $\gamma_1 - \gamma_2$ is the boundary of the two chain $\sigma_1 - \sigma_2$ whence $\overline{\gamma}_1 = \overline{\gamma}_2$.

Lemma 32.3: The map Π_X given by (32.4) is a group homomorphism.

Proof: Let γ_1 and γ_2 be two loops in X based at x_0. We have to show that the one chain

$$\gamma_1 + \gamma_2 - \gamma_1 * \gamma_2$$

is a boundary of some singular two chain σ. The idea behind the construction is simple. We first define a map $\tilde{F} : I \times I \longrightarrow X$ whose restrictions to the four sides of the square are γ_1, γ_2, ε_{x_0} and $\gamma_1 * \gamma_2$. As in the previous lemma we shall employ the maps T_1, T_2 to construct our two chain σ.

We proceed as in lecture 7 by defining $\tilde{F}$ from the boundary of $I \times I$ to $[0, 1]$, using Tietze's theorem to extend it to the whole of $I \times I$ and then composing with $\gamma_1 * \gamma_2$. So we define

$$\begin{aligned} \tilde{F}(0, s) &= \frac{s}{2} \\ \tilde{F}(t, 1) &= \frac{t+1}{2} \\ \tilde{F}(t, 0) &= t \\ \tilde{F}(1, s) &= 1 \end{aligned}$$

and extend it continuously to $I \times I$. Let $F : I \times I \longrightarrow X$ be given by $F = (\gamma_1 * \gamma_2) \circ \tilde{F}$. The figure below depicts F along the boundary of I^2:

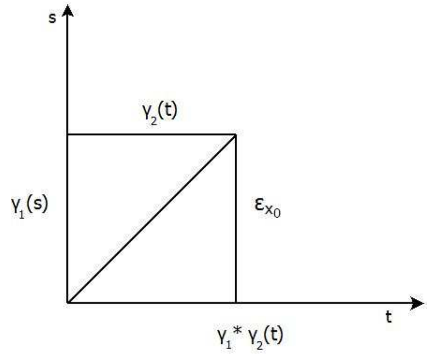

Figure 23:

It is now an easy matter to verify that the boundary of the two chain σ given by

$$\sigma = F \circ T_1 - F \circ T_2$$

is the one chain

$$\gamma_1 + \gamma_2 - \varepsilon_{x_0} - \gamma_1 * \gamma_2 \tag{32.5}$$

Now, if $\sigma' : \Delta_2 \longrightarrow X$ be the constant map taking value x_0 then $\partial\sigma' = \varepsilon_{x_0}$ whereby we conclude that

$$\gamma_1 + \gamma_2 - \gamma_1 * \gamma_2 = \partial\sigma + \partial\sigma', \tag{32.6}$$

which implies $\Pi_X([\gamma_1][\gamma_2]) = \Pi_X([\gamma_1 * \gamma_2]) = \Pi_X([\gamma_1]) + \Pi_X([\gamma_2])$. $\square$

Lemma 32.4: The map Π_X given by (32.4) is surjective.

Proof: Let λ be a singular one cycle say $\lambda = \sum n_j\gamma_j$, where $n_j \in \mathbb{Z}$ and $\gamma_j : [0,1] \longrightarrow X$. Since $\partial\lambda = 0$,

$$\sum_{j=1}^{k} n_j(\gamma_j(1) - \gamma_j(0)) = 0. \tag{32.7}$$

The idea is to complete each of the paths γ_j into a loop at x_0 by means of paths joining x_0 to the ends $\gamma_j(0)$ and $\gamma_j(1)$. The only non-trivial part is the book-keeping which has to be done carefully. Let S denote the set of endpoints

$$S = \{\gamma_j(1),\ \gamma_j(0)/\ j = 1, 2 \ldots, k\}.$$

For each $p \in S$, if m_p denotes the sum of the coefficients of p in (32.7) then m_p must be zero. Taking a path β_p in X joining x_0 and $p \in S$ we construct for each j a loop η_j in X based at x_0 namely,

$$\eta_j = \beta_{\gamma_j(0)} * \gamma_j * \beta^{-1}_{\gamma_j(1)}.$$

Finally

$$\Pi_X(\eta_1^{n_1} * \eta_2^{n_2} * \cdots * \eta_k^{n_k}) = \sum_{j=1}^{k} n_j\gamma_j - \sum_{j=1}^{k} n_j(\beta_{\gamma_j(1)} - \beta_{\gamma_j(0)}) = \lambda$$

since

$$\sum_{j=1}^{k} n_j(\beta_{\gamma_j(1)} - \beta_{\gamma_j(0)}) = \sum_{p \in S} m_p\beta_p = 0.$$

Lemma 32.5: Suppose G is a group and $x_1, x_2, \ldots, x_k$ are *distinct* elements of G such that $x_i \neq x_j^{-1}$ if $i \neq j$. Let w be a word involving integer powers of $x_1, x_2, \ldots, x_k$ such that the sum of the exponents of each x_i is zero. Then w lies in the commutator subgroup of G.

Proof: We leave the easy proof for the student to work out.

Lemma 32.6: The kernel of the map Π_X is the commutator subgroup $[\pi_1(X, x_0), \pi_1(X, x_0)]$.

Proof: Since the Π_X is a map into an abelian group, its kernel contains the commutator subgroup. To prove the converse suppose that γ is a loop based at x_0 such that $[\gamma] \in \operatorname{Ker} \Pi_X$. When considered as a singular one cycle it is a boundary of a singular two chain $\sum n_j \sigma_j$ where $\sigma_j : \Delta_2 \longrightarrow X$. Writing the boundary $\partial \sigma_j$ as a sum of its faces

$$\partial \sigma_j = \lambda_j + \mu_j + \nu_j$$

we see that

$$\sum_{j=1}^{k} n_j \partial \sigma_j = \sum_{j=1}^{k} n_j (\lambda_j + \mu_j + \nu_j) = \gamma. \tag{32.8}$$

We proceed as in lemma (32.4). Let S be the set distinct singular one simplicies in the list

$$\lambda_j, \mu_j, \nu_j \quad j = 1, 2, \ldots, k. \tag{32.9}$$

and choose auxiliary paths β_p joining x_0 and the endpoints p of each of the one simplicies in S. The loop γ also appears in the list (32.9) but since its ends are both x_0 there is no need to take the auxiliary paths β in this case. As in lemma (32.4), for each θ in the list (32.9), we denote by m_θ the sum of the

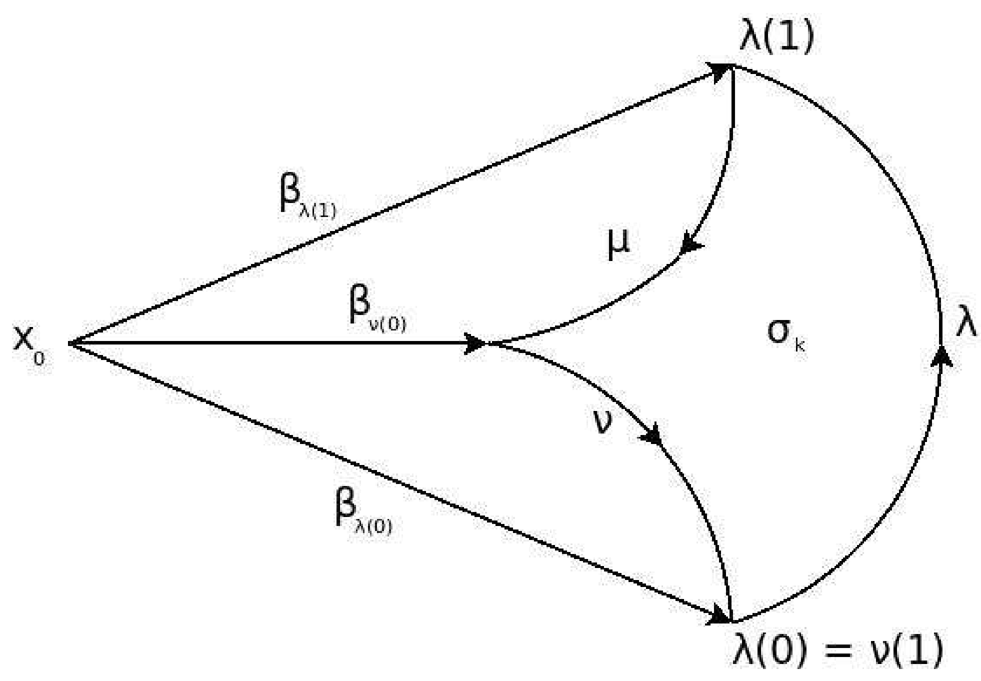

Figure 24:

coefficients of θ in (32.8) so that,

$$m_\theta = \begin{cases} 0 & \text{if } \theta \neq \gamma \\ 1 & \text{if } \theta = \gamma \end{cases} \tag{32.10}$$

For each two simplex σ_j we have the three loops (suppressing the subscript j)

$$\beta_{\lambda(0)} * \lambda * \beta^{-1}_{\lambda(1)},\ \beta_{\mu(0)} * \mu * \beta^{-1}_{\mu(1)},\ \beta_{\nu(0)} * \nu * \beta^{-1}_{\nu(1)},$$

whose juxtaposition η_j is easily seen to be homotopic to the trivial loop. For proving this one uses the equations $\lambda(1) = \mu(0)$, $\mu(1) = \nu(0)$ and $\nu(1) = \lambda(0)$. Corresponding to (32.8) we form the loop

$$\eta_1^{n_1} * \eta_2^{n_2} * \cdots * \eta_k^{n_k} * \gamma^{-1} \tag{32.11}$$

which is homotopic to γ^{-1} since the piece $\eta_1^{n_1} * \eta_2^{n_2} * \cdots * \eta_k^{n_k}$ is a juxtaposition of loops homotopic to the constant loop. On the other hand if we write out the expression (32.11) completely, we see that for each θ in the list (32.9), the factor $\beta * \theta * \beta^{-1}$ appears, probably in several positions, but the sum of its exponents is m_θ. In view of (32.10) and lemma (30.5) we see that the element of $\pi_1(X, x_0)$ represented by (32.11) lies in the commutator subgroup, that is to say, $[\gamma]^{-1}$ lies in the commutator subgroup of $\pi_1(X, x_0)$. The proof is complete.

Definition 32.1 (Natural transformation): Given a pair of functors $\pi : \mathcal{T} \longrightarrow \mathcal{G}$ and $H : \mathcal{T} \longrightarrow \mathcal{G}$, a natural transformation T between π and H is a function which assigns to each object X of $\mathcal{T}$ a morphism $\eta_X : \pi(X) \longrightarrow H(X)$ such that for each morphism $f : X \longrightarrow Y$ in $\mathcal{T}$, the following diagram commutes

$$\begin{array}{ccc} \pi(X) & \xrightarrow{\pi(f)} & \pi(Y) \\ \eta_X \big\downarrow & & \big\downarrow \eta_Y \\ H(X) & \xrightarrow{H(f)} & H(Y) \end{array} \tag{32.12}$$

The notation used in this definition is quite suggestive. The Poincaré-Hurewicz map provides a natural transformation between the functors π_1 and H_1.

Exercises

1. Verify the displayed results for $\partial\sigma_1$ and $\partial\sigma_2$ in lemma (32.2).
2. By writing out the boundary formula in detail verify equations (32.5) and (32.6).
3. Prove lemma (32.5).
4. Verify the naturality of Π_X by proving that the diagram (32.1) commutes.
5. Determine the first homology group of the Klein's bottle.
6. Determine the first homology groups of all the spaces described in the exercises to lecture 26.