Lecture - XXV Adjunction Spaces

The notion of push-outs in the category **Top** leads to an important class of spaces known as adjunction spaces. We shall see that most of the important spaces encountered are adjunction spaces. This lecture may be regarded as one on important examples of topological spaces.

Definition 25.1: Given a topological space X, a closed subset A and a continuous map $A \longrightarrow B$ we define an equivalence relation on the disjoint sum (coproduct) $X \sqcup B$ as follows

$$b \sim x \quad \text{if and only if} \quad x \in A \text{ and } \; f(x) = b.$$

Thus a point $x \in A$ is identified with its image $f(x) \in B$. There are no other identifications besides this. The quotient space under this equivalence relation is called the adjunction space or the space obtained by attaching X to B via the map f. The space is denoted by $X \sqcup_f B$. Thus

$$X \sqcup_f B = (X \sqcup B)/\sim$$

The situation may be pictorially described as

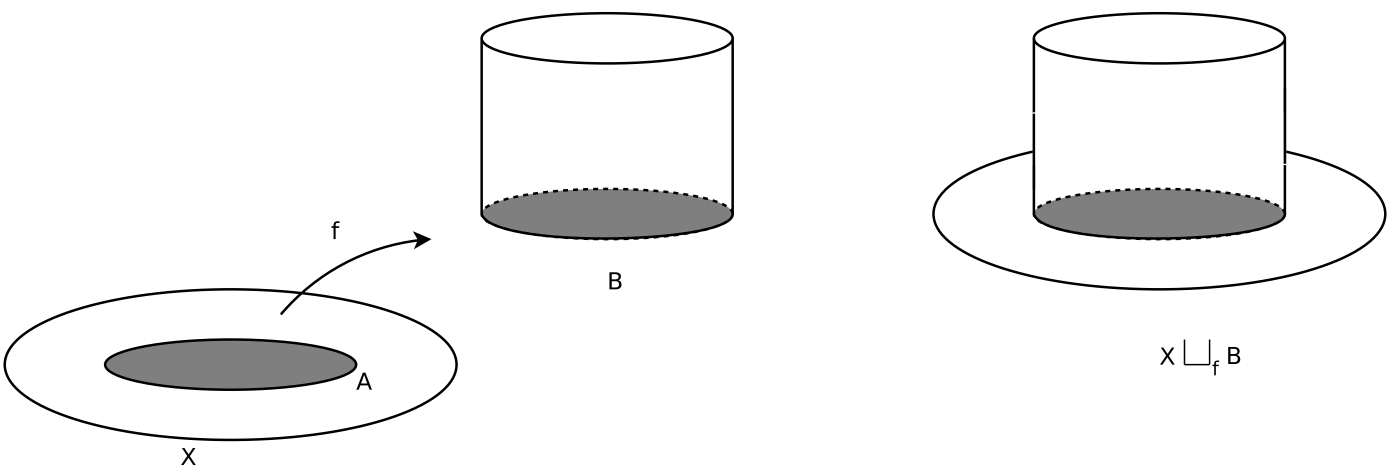

Figure 16: Adjunction Space

Example 25.1: Take $X = S^1$, $A = \{1\} \subset X$, $B = S^1$ and $f : A \longrightarrow B$ as $f(1) = 1$. The resulting space is the wedge of two circles $S^1 \vee S^1$.

Example 25.2 We now take $X = E^2$ the closed unit disc in the plane, $A = S^1$ the boundary of E^2, $B = \{1\}$ and f to be the constant map from S^1 to the singleton set B. The adjunction space is obtained by collapsing the boundary of E^2 to the single point B. The resulting space is S^2.

Before discussing further examples we relate this to the push-out construction.

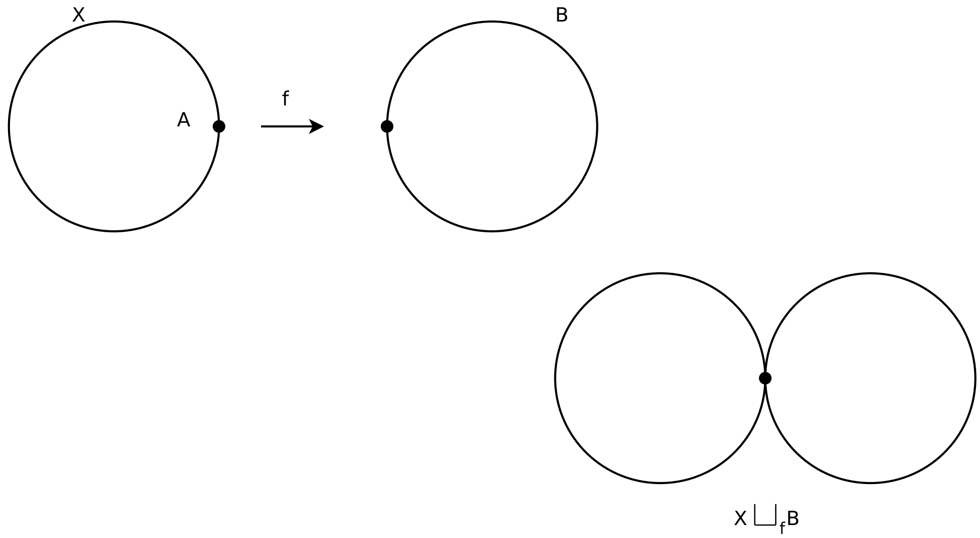

Figure 17: Wedge of two circles

Theorem 25.1: Let X and B be topological spaces, A be a closed subspace of X and $f : A \longrightarrow B$ be a continuous map. Then the space $X \sqcup_f B$ is the push-out for the following diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

where $i : A \longrightarrow X$ denotes the inclusion map.

Proof: We first define the associated maps $h_1 : X \longrightarrow X \sqcup_f B$ and $h_2 : X \longrightarrow X \sqcup_f B$. Let $\eta : X \sqcup B \longrightarrow X \sqcup_f B$ be the quotient map and $i_X : X \longrightarrow X \sqcup B$ and $i_B : B \longrightarrow X \sqcup B$ denote the inclusions. Then the associated maps h_1 and h_2 given by

$$h_1 = \eta \circ i_X, \quad h_2 = \eta \circ i_B. \tag{25.1}$$

For any $a \in A$ we have

$$h_1 \circ i(a) = \eta(i_X(a)) = \eta(a), \quad h_2 \circ f(a) = \eta(i_B(f(a))) = \eta(f(a))$$

Recalling the identifications we see that $h_1 \circ i = h_2 \circ f$. We now check the universal property. Suppose Z is a topological space and $g_1 : X \longrightarrow Z$, $g_2 : B \longrightarrow Z$ are continuous maps such that

$$g_1 \circ i = g_2 \circ f \tag{25.2}$$

Define the continuous map $\phi : X \sqcup B \longrightarrow Z$ as

$$\phi(x) = \begin{cases} g_1(x) & \text{if } x \in X \\ g_2(x) & \text{if } x \in B. \end{cases}$$

Condition (25.2) now shows that there is a unique map $\overline{\phi} : (X \sqcup_f B)/\sim \longrightarrow Z$ such that

$$\overline{\phi} \circ \eta = \phi. \tag{25.3}$$

The universal property of the quotient implies that $\overline{\phi}$ is continuous. Equations (25.1)-(25.3) immediately give

$$\overline{\phi} \circ h_1 = g_1, \quad \overline{\phi} \circ h_2 = g_2. \tag{25.4}$$

thereby completing the verification of the universal property.

Corollary 25.2: The square

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle h_1} \\ B & \xrightarrow{h_2} & X \sqcup_f B \end{array}$$

is a push-out where h_1 and h_2 are defined as in (25.1).

Proof: This is just a summary of the details of the maps involved.

Definition 25.2: An $n-$cell is any space that is homeomorphic to the closed unit ball E^n in $\mathbb{R}^n$.
Thus the square I^2 is an example of a two cell and the hemisphere

$$\{(x_1, x_2, \ldots, x_n) \in S^{n-1}/x_n \geq 0\}$$

is an $n-1$ cell.

Example 25.3 (The torus and the Klein's bottle): We now show that the Klein's bottle and the torus are obtained by attaching a two cell to the figure eight space $S^1 \vee S^1$. In both cases we take $X = I^2$ to be the two cell, $A = \dot{I}^2$ the boundary of I^2 and $B = S^1 \vee S^1$ regarded as a subset of $S^1 \times S^1$ namely $(S^1 \times \{1\}) \cup (\{1\} \times S^1)$. The distinguishing factor is that the attaching map $f : A \longrightarrow B$ is different in the two cases.

1. For the torus we define $f : A \longrightarrow B$ to be the continuous surjection

$$\begin{aligned} f(x,1) &= f(x,0) = (e^{2\pi i x}, 1), \quad x \in [0,1] \\ f(1,y) &= f(0,y) = (1, e^{2\pi i y}), \quad y \in [0,1] \end{aligned}$$

   It is geometrically clear that $X \sqcup_f B$ is a torus but we demonstrate this formally owing to the importance of the type of argument involved. Let $p : I^2 \longrightarrow S^1 \times S^1$ be the quotient map, $i_X : X \longrightarrow X \sqcup B$ the inclusion map and $\eta : X \sqcup B \longrightarrow X \sqcup_f B$ the quotient map. The map

$$\phi : S^1 \times S^1 \longrightarrow X \sqcup_f B$$

   given by $\phi(\exp(2\pi i x), \exp(2\pi i y)) = (\eta \circ i_X)(x,y)$ is well-defined, bijective and its continuity follows from the fact that $\phi \circ p = i_X \circ \eta$ and $i_X \circ \eta$ is continuous. Finally the compactness of $S^1 \times S^1$ and the fact that $X \sqcup_f B$ is Hausdorff shows that ϕ is a homeomorphism.

2. The argument for the Klein's bottle proceeds along similar lines and we merely indicate the attaching map $f : I^2 \longrightarrow S^1 \vee S^1$ namely,

$$\begin{aligned} f(x,1) &= f(x,0) = (e^{2\pi i x}, 1), \quad x \in [0,1] \\ f(1,y) &= f(0,1-y) = (1, e^{2\pi i y}), \quad y \in [0,1]. \end{aligned}$$

3. It is sometimes convenient to take the closed unit disc E^2 as the two cell. But the attaching map $f : S^1 \longrightarrow S^1 \vee S^1$ would be slightly more complicated to write down. For the Klein's bottle the attaching map is given by

$$f(z) = \begin{cases} (z^4, 1) & 0 \leq \arg z \leq \pi/2 \\ (1, z^4) & \pi/2 \leq \arg z \leq \pi \\ (-\overline{z}^4, 1) & \pi \leq \arg z \leq 3\pi/2 \\ (1, -\overline{z}^4) & 3\pi/2 \leq \arg z \leq 2\pi \end{cases} \tag{25.5}$$

For the torus the attaching map is obtained from (25.5) by suppressing the negative signs in the last two expressions. The student is invited to work out a similar construction for the double torus as well.

Example 25.4 (The projective plane): This is obtained by attaching a two cell to the circle. For the two cell we take the closed unit disc E^2 in the complex plane and its boundary as A. The attaching map is given by $f(z) = z^2$. We leave it to the reader to prove that the resulting adjunction space is indeed $\mathbb{R}P^2$.

Example 25.5 (Real projective spaces): We take the space X to be the closed unit disc E^n in $\mathbb{R}^n$ and A as its boundary. The space B is the lower dimensional projective space $\mathbb{R}P^{n-1}$. The attaching map is the quotient map $p : S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$. We leave the proof of the following result to the reader.

Theorem 25.3: The space $E^n \sqcup_p \mathbb{R}P^{n-1}$ is homeomorphic to the real projective space $\mathbb{R}P^n$. Thus $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell.

Definition 25.3 (The cone over a space): Let X be a topological space. The cone $C(X)$ over X is the quotient space

$$C(X) = (X \times [0,1])/(X \times \{0\})$$

We have an obvious inclusion map $i : X \longrightarrow C(X)$ given by $i(x) = [x, 1]$ where the square bracket

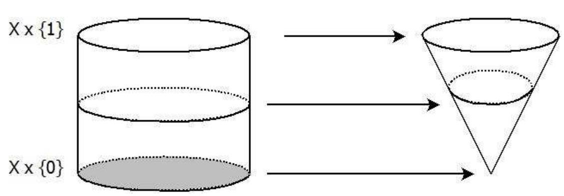

Figure 18: Cone over a space

refers to the image of $(x, 1) \in X \times [0,1]$ in the quotient $C(X)$.

Theorem 25.5: A continuous map $f : X \longrightarrow Y$ is homotopic to a constant map if and only if f extends continuously to a map $F : C(X) \longrightarrow Y$ namely $F \circ i = f$.

Proof: The proof writes itself out. Suppose that $G : X \times [0,1] \longrightarrow Y$ is a homotopy between f and the constant map taking the value y_0 say,

$$G(x,1) = f(x), \quad G(x,0) = f(y_0), \qquad \text{for all } \ x \in X.$$

The second equation in (25.12) says that G respects the identification made on $X \times [0,1]$ to yield $(X \times [0,1])/(X \times \{0\})$ whereby we conclude the existence of a map $F : C(X) \longrightarrow Y$ satisfying $F \circ \eta = G$. This map F is continuous by the universal property and the first equation in (25.12) gives $F[x,1] = G(x,1) = f(x)$. The proof of necessity is complete.

Conversely suppose given a continuous map $f : X \longrightarrow Y$ such that there is a $G : C(X) \longrightarrow Y$ with $F \circ i = G$. Denoting by η the quotient map $X \times [0,1] \longrightarrow C(X)$, the map $G \circ \eta$ provides a homotopy between f the constant map. $\square$

Exercises

1. We have obtained S^2 by attaching E^2 to a singleton with the attaching map as the constant map on the boundary of E^2. Discuss how would you obtain S^n analogously as an adjunction space.

2. Show that if X and B are connected/path-connected then $X \sqcup_f B$ is connected/path-connected.

3. Describe the push out resulting from the diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{i_1} & E^n \\ {\scriptstyle i_2}\big\downarrow & & \\ E^n & & \end{array}$$

4. Show that $S^m \times S^n$ results from attaching an $n+m$ cell to $S^n \vee S^m$. Hint: Let I denote $[0,1]$ and define a map $f : \partial(I^n \times I^m) \longrightarrow S^n \vee S^m$ as follows

$$f(z) = \begin{cases} (\eta_1(x), y_0) & \text{if } \ x \in \partial I^n \\ (x_0, \eta_2(y)) & \text{if } \ y \in \partial I^m \end{cases}$$

and $\eta_1 : I^n \longrightarrow S^n$ and $\eta_2 : I^m \longrightarrow S^m$ are the quotient maps of exercise 1.

5. Prove theorem (25.3).

6. Fill in the details in examples (25.4) and (25.5).