

# Lecture IV - Further preliminaries from general topology:

We now begin with some preliminaries from general topology that is usually not covered or else is often perfunctorily treated in elementary courses on topology. Since many important examples in topology arise as quotient spaces, this lecture is completely devoted to this topic.

**Quotient Spaces:** Suppose that  $X$  is a topological space and  $f : X \longrightarrow Y$  is a surjective mapping, let us consider the various topologies on  $Y$  with respect to which  $f$  is continuous. Certainly the function  $f$  would be continuous if  $Y$  carries the trivial topology where the only open sets are  $\emptyset$  and  $Y$ . The quotient topology on  $Y$  is the strongest topology that makes  $f$  continuous. More explicitly consider the family

$$\mathcal{T} = \{A \subset Y : f^{-1}(A) \text{ is open in } X\}. \quad (4.1)$$

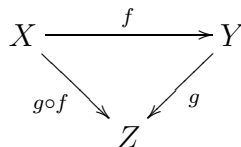
Since  $\mathcal{T}$  is closed under arbitrary unions, finite intersections and contains  $Y$  and the empty set, we conclude that  $\mathcal{T}$  is a topology on  $Y$  with respect to which  $f$  is continuous. It is also clear that any strictly larger topology would render  $f$  discontinuous.

**Definition 4.1:** (i) Given a topological space  $X$ , a set  $Y$  and a surjective map  $f : X \longrightarrow Y$ , the topology  $\mathcal{T}$  defined by (4.1) is called the *quotient topology* on  $Y$  induced by the function  $f$ . By construction  $f$  is continuous with this topology on  $Y$ .

(ii) Given a map  $f : X \longrightarrow Y$  between topological spaces  $X$  and  $Y$ , we say  $f$  is a *quotient map* if the given topology on  $Y$  agrees with the quotient topology on  $Y$  induced by  $f$ .

The quotient topology enjoys a universal property which is easy to prove but extremely useful.

**Theorem 4.1 (Universal property of quotients):** Suppose that  $X$  is a topological space,  $Y$  is a set,  $f : X \longrightarrow Y$  is a surjective map and  $Y$  is assigned the quotient topology induced by  $f$ . Then given any topological space  $Z$  and map  $g : Y \longrightarrow Z$ , the map  $g$  is continuous if and only if  $g \circ f : X \longrightarrow Z$  is continuous.



**Proof:** If  $g$  is continuous it is trivial that  $g \circ f$  is continuous. Conversely suppose that  $g \circ f$  is continuous. Let  $A$  be any open set in  $Z$  so that  $(g \circ f)^{-1}(A)$  is open in  $X$ . Thus  $f^{-1}(g^{-1}(A))$  is open in  $X$ . Invoking the definition of the quotient topology, we see that  $g^{-1}(A)$  must be open in  $Y$  which means  $g$  is continuous.

Before illustrating the use of the universal property of quotients we discuss the following issue. Suppose that  $X$  and  $Y$  are topological spaces and  $f : X \longrightarrow Y$  is a given continuous map, then the quotient topology on  $Y$  induced by  $f$  is weaker than the given topology on  $Y$ . When would the given topology on  $Y$  be equal to the quotient topology?

**Definition 4.2:** A (not necessarily continuous) map  $f : X \longrightarrow Y$  between topological spaces  $X$  and  $Y$  is said to be a closed map if the image of closed sets is closed. Likewise we say  $f$  is an open mapping if the image of open sets is open.

**Example 4.1:** (i) Suppose  $X$  is a compact space and  $Y$  is a Hausdorff space then any surjective continuous map  $f : X \longrightarrow Y$  is a closed map.

(ii) The reader may check that  $\phi : \mathbb{R} \longrightarrow S^1$  given by  $\phi(t) = \exp(2\pi it)$  is an open mapping.

(iii) The map  $\phi : [0, 1] \longrightarrow S^1$  given by  $\phi(t) = \exp(2\pi it)$  is closed but not open.

**Theorem 4.2:** Suppose that  $X$  and  $Y$  are topological spaces and  $f : X \longrightarrow Y$  is a surjective continuous closed/open map then the quotient topology on  $Y$  induced by  $f$  agrees with the given topology on  $Y$ .

**Proof:** The quotient topology on  $Y$  induced by  $f$  is stronger than the given topology. To obtain the reverse inclusion, suppose that  $f$  is a continuous open mapping and  $A \subset Y$  is open with respect to the quotient topology on  $Y$  induced by  $f$  which means  $f^{-1}(A)$  is open in  $X$  whereby  $f(f^{-1}(A))$  is open in  $Y$  in the given topology since  $f$  is an open mapping. But since  $f$  is surjective,  $f(f^{-1}(A)) = A$  and so we conclude  $A$  is open in the given topology as well.

Let us now turn to a continuous, closed surjective map  $f : X \longrightarrow Y$ . Again we merely have to show that the given topology on  $Y$  is stronger than the quotient topology since the reverse inclusion is trivial. So let  $A$  be an open set in  $Y$  with respect to the quotient topology induced by  $f$ . By definition  $f^{-1}(A)$  is open in  $X$ , or in other words  $X - f^{-1}(A)$  is closed in  $X$ . Since  $f$  is closed,  $f(X - f^{-1}(A)) = Y - A$  is closed in  $Y$  with respect to the given topology on  $Y$ . That is to say  $A$  is open with respect to the given topology on  $Y$ .

**Corollary 4.3:** Suppose  $X$  is a compact space and  $Y$  is a Hausdorff space then any continuous surjection from  $X$  onto  $Y$  is a quotient map.

**Identification spaces:** Suppose that  $X$  is a topological space and  $\sim$  is an equivalence relation on  $X$ . The set of all equivalence classes is denoted by  $X/\sim$  and  $\eta : X \longrightarrow X/\sim$  denotes the projection map

$$\eta(x) = \bar{x}, \quad x \in X,$$

where  $\bar{x}$  denotes the equivalence class of  $x$ . The space  $X/\sim$  with the quotient topology induced by  $\eta$  is called the identification space given by the equivalence relation. An important special case deserves mention as it is of frequent occurrence. Suppose that  $A$  is a subset of a topological space then we consider the equivalence relation for which all the points of  $A$  form one equivalence class and the equivalence class of any  $x \in X - A$  is a singleton. That is to say all the points of  $A$  are identified together as one point and no other identification is made. We shall refer to the resulting quotient space as **the space obtained from  $X$  by collapsing  $A$  to a singleton**.

**Theorem 4.4:** Let  $X$  and  $Y$  be topological spaces and  $f : X \longrightarrow Y$  be a surjective continuous map such that the given topology on  $Y$  agrees with the quotient topology on  $Y$  induced by  $f$ . Define an equivalence relation  $\sim$  on  $X$  as follows:

$$x_1 \sim x_2 \text{ if and only if } f(x_1) = f(x_2), \quad x_1, x_2 \in X$$

The identification space  $X/\sim$  is homeomorphic to  $Y$  via the map  $\phi : X/\sim \longrightarrow Y$  given by

$$\phi(\bar{x}) = f(x).$$

**Proof:** It is easy to see that  $\phi$  is well-defined, bijective and satisfies  $\phi \circ \eta = f$ . Since  $f$  is continuous and  $\eta$  is a quotient map we see by the universal property that  $\phi$  is continuous. Since  $f$  is a quotient map and  $\eta$  is continuous we may invoke the universal property again but this time to  $\phi^{-1} \circ f = \eta$  to conclude that  $\phi^{-1}$  is continuous as well.

**The real projective spaces  $\mathbb{R}P^n$ :** The projective space  $\mathbb{R}P^n$  is the identification space obtained from the sphere  $S^n$  by the equivalence relation  $\sim$  given by

$$\mathbf{x} \sim \mathbf{y} \text{ if and only if } \mathbf{x} = -\mathbf{y}, \quad \mathbf{x}, \mathbf{y} \in S^n. \quad (4.2)$$

That is to say, each pair of antipodal points are identified.

**Theorem 4.5:** (i) The projective spaces are compact and connected.

(ii) The projective space  $\mathbb{R}P^n$  is homeomorphic to the identification space  $(\mathbb{R}^{n+1} - \{0\})/\sim$  where

$$\mathbf{x} \sim \mathbf{y} \text{ if and only if for some } \lambda \in \mathbb{R}, \mathbf{x} = \lambda \mathbf{y}.$$

(iii) The projective space  $\mathbb{R}P^n$  is homeomorphic to the identification space  $E^n/\sim$  where  $\mathbf{x} \sim \mathbf{y}$  if and only if either  $\mathbf{x} = \mathbf{y}$  or else  $\mathbf{x}, \mathbf{y} \in S^{n-1}$  and  $\mathbf{x} = -\mathbf{y}$ .

**Proof:** (i) The sphere  $S^n$  is compact and connected and  $\mathbb{R}P^n$  is the continuous image of  $S^n$  under the projection map  $\eta$ .

(ii) Let  $\eta : S^n \rightarrow \mathbb{R}P^n$  and  $p : \mathbb{R}^{n+1} - \{0\} \rightarrow (\mathbb{R}^{n+1} - \{0\})/\sim$  be the projection maps. We have a continuous map  $\phi : \mathbb{R}^{n+1} - \{0\} \rightarrow \mathbb{R}P^n$  given by the prescription

$$\phi(x) = \bar{x}, \quad x \in \mathbb{R}^{n+1} - \{0\},$$

where  $\bar{x}$  is the equivalence class of  $x$  in  $\mathbb{R}^{n+1}/\sim$ . Denoting by  $[x]$  the equivalence class of  $x$  in  $(\mathbb{R}^{n+1} - \{0\})/\sim$ , the associated map  $\bar{\phi} : (\mathbb{R}^{n+1} - \{0\})/\sim \rightarrow \mathbb{R}P^n$  given by

$$\bar{\phi}([x]) = \bar{x}.$$

It is readily checked that  $\bar{\phi}$  is bijective and  $\bar{\phi} \circ \eta = \phi$ . The universal property now gives us the continuity of  $\bar{\phi}$ . Consider now the map

$$\psi : S^n \rightarrow (\mathbb{R}^{n+1} - \{0\})/\sim$$

given by  $\psi(\mathbf{x}) = [\mathbf{x}]$  which is evidently continuous map. There is a unique map

$$\bar{\psi} : \mathbb{R}P^n \rightarrow (\mathbb{R}^{n+1} - \{0\})/\sim$$

such that  $\bar{\psi} \circ \eta = \psi$ . By the universal property of the quotient we see that  $\bar{\psi}$  is continuous. It is left as an exercise to check that  $\bar{\psi}$  and  $\bar{\phi}$  are inverses of each other. Proof of (iii) is left as an exercise.

We shall see later that the spaces are Hausdorff as well. The space  $\mathbb{R}P^1$  is a familiar space and the proof of the following result will be left for the reader.

**Theorem 4.6:** The space  $\mathbb{R}P^1$  is homeomorphic to the circle  $S^1$ .

**The Möbius band and the Klein's bottle:** We describe the Möbius band and Klein's bottle as quotient spaces of  $I^2$  via identifications which are described as follows. Each point in the interior of  $I^2$  forms an equivalence class in itself. That is to say a point in the interior of  $I^2$  is not identified with any other point. Points on the boundary are identified according to the following scheme:

1. Möbius band: On the part of the boundary  $(\{0\} \times [0, 1]) \cup (\{1\} \times [0, 1])$ , the pair of points  $(0, y)$  and  $(1, 1 - y)$  are identified for each  $y$  with  $0 \leq y \leq 1$ . Points on the remaining part of the boundary namely

$$(0, 1) \times \{0\} \cup (0, 1) \times \{1\} \quad (4.3)$$

are left as they are. That is to say the equivalence class of each of the points (4.3) is a singleton.

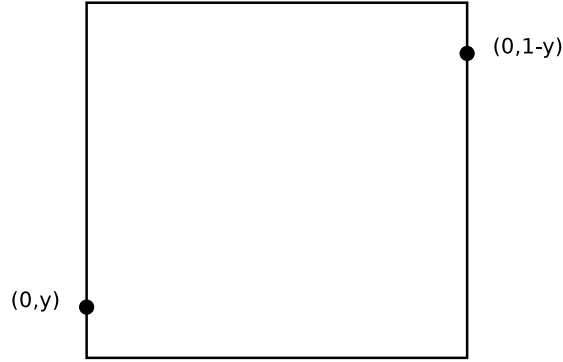


Figure 3: Möbius Band

2. Klein's bottle: As in the case of the Möbius band, for each  $0 \leq y \leq 1$ , the pair of points  $(0, y)$  and  $(1, 1 - y)$  are identified. However also for each  $0 \leq x \leq 1$ , the pair of points  $(x, 0)$  and  $(x, 1)$  are identified.

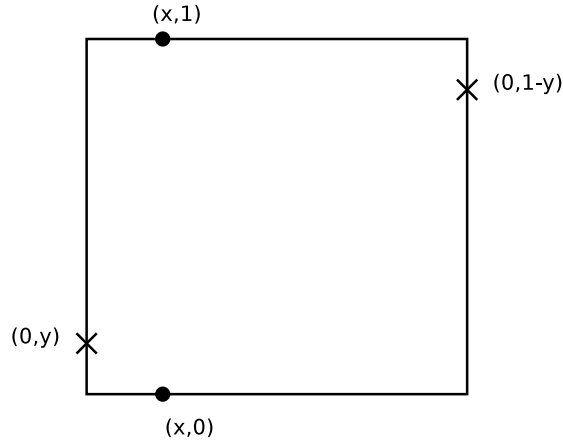


Figure 4: Klein's Bottle

**The torus:** This is obtained by identifying the opposite sides of the square  $I^2$  according to the following scheme. For each  $x \in [0, 1]$ , the pair of points  $(x, 0)$  and  $(x, 1)$  are identified. Likewise for

each  $y \in [0, 1]$  the pair of points  $(0, y)$  and  $(1, y)$  are identified. One first obtains a cylinder  $S^1 \times [0, 1]$  which is then “bent around” and the circular ends are glued together. One obtains a space which looks like the crust of a dough-nut (or medu vada).

**Example:** The torus defined above is homeomorphic to  $S^1 \times S^1$ . To see this, let  $T$  denote the torus and  $\eta : I^2 \longrightarrow T$  be the quotient map. Define the map  $f : I^2 \longrightarrow S^1 \times S^1$  as

$$f(x, y) = (\exp(2\pi ix), \exp(2\pi iy)).$$

There is a unique bijection  $\bar{f} : T \longrightarrow S^1 \times S^1$  such that  $\bar{f} \circ \eta = f$ . The universal property shows that  $\bar{f}$  is continuous and since  $T$  is compact and  $S^1 \times S^1$  is Hausdorff, the map  $\bar{f}$  is a closed map and so a homeomorphism.

**The wedge:** The *wedge* of two topological spaces  $X$  and  $Y$ , denote by  $X \vee Y$ , is the following subspace of  $X \times Y$

$$(X \times \{y_0\}) \cup (\{x_0\} \times Y),$$

where  $(x_0, y_0)$  is a chosen point of  $X \times Y$ .

**Theorem 4.7:** The quotient space  $(S^1 \times S^1)/(S^1 \vee S^1)$  is homeomorphic to the sphere  $S^2$ .

**Proof:** It is an exercise that the space obtained by collapsing the boundary of  $I^2$  to a singleton is homeomorphic to  $S^2$ . Let  $\eta_1$  denote the quotient map  $I^2 \longrightarrow S^2$  which collapses the boundary to a singleton and likewise let  $\eta_2 : S^1 \times S^1 \longrightarrow (S^1 \times S^1)/(S^1 \vee S^1)$  be the quotient map. The map  $\phi : S^1 \times S^1 \longrightarrow S^2$  given by the prescription

$$\phi(\exp(2\pi ix), \exp(2\pi iy)) = \eta_1(x, y)$$

is well-defined and surjective. Since  $\eta_1$  is continuous it follows that  $\phi$  is continuous (why?). There is a unique bijective map  $\bar{\phi} : (S^1 \times S^1)/(S^1 \vee S^1) \longrightarrow S^2$  such that  $\bar{\phi} \circ \eta_2 = \phi$ , from which follows that  $\bar{\phi}$  is continuous and a closed map since the domain is compact and the codomain is Hausdorff. Hence  $\bar{\phi}$  is a homeomorphism between  $(S^1 \times S^1)/(S^1 \vee S^1)$  and  $S^2$ .

**Surfaces:** The sphere  $S^2$ , torus, Klein’s bottle and projective plane are the four basic examples of a class of spaces called *surfaces*. We shall not formally define a surface but provide one more example namely, the double torus. Roughly the double torus is obtained by taking two copies of the torus and cutting out a little disc from each of them so as to obtain a pair of tori each with a boundary. One then glues these boundaries together to obtain a double torus. Analytically the double torus is the identification space obtained by identifying pairs of opposite sides of an octagon according to the following scheme. Obviously the process can be generalized and one can obtain for instance a triple torus by identifying pairs of opposite sides of a twelve sided polygon. The classification of surfaces forms an important chapter in topology and we refer to the book of [11].

**Hausdorff Quotients:** The quotient of a Hausdorff space need not be Hausdorff. Since quotient spaces occur in abundance we need easily verifiable sufficient conditions for a quotient space to be Hausdorff. We provide here one such condition which suffices for most applications [16]. Let  $X$  be a

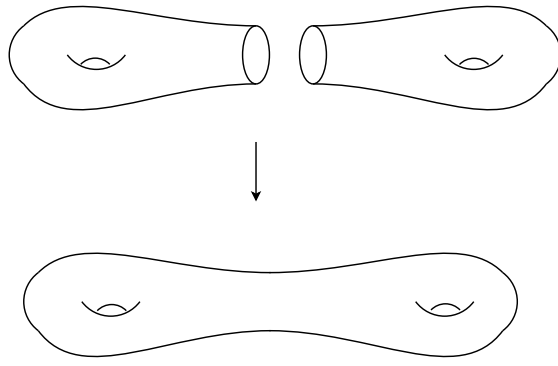


Figure 5: Double torus as a connected sum

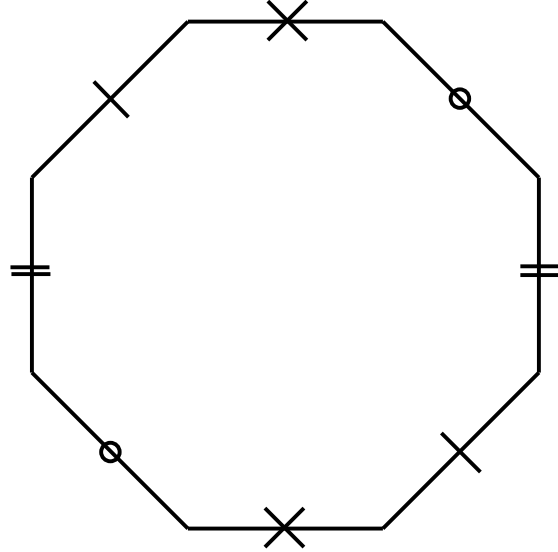


Figure 6: Double torus

space on which an equivalence relation  $\sim$  has been defined. Note that a relation on  $X$  is a subset  $\Gamma$  of the cartesian product  $X \times X$  on which we have the product topology. Thus,

$$\Gamma = \{(x, y) \in X \times X / x \sim y\}$$

**Definition 4.3:** The relation  $\sim$  is said to be closed if  $\Gamma$  is a closed subset of  $X \times X$ .

We shall employ the two projection maps

$$\begin{aligned} p : X \times X &\longrightarrow X, & q : X \times X &\longrightarrow X, \\ (x, y) &\mapsto x, & (x, y) &\mapsto y, \end{aligned}$$

and denote by  $\eta$  the quotient map  $\eta : X \longrightarrow X / \sim$ .

**Theorem 4.8:** Let  $X$  be a compact Hausdorff space and  $\sim$  be a closed relation on  $X$ . Then,

- (a) The map  $\eta$  is a closed map.
- (b) The quotient space  $X / \sim$  is Hausdorff.

**Proof:** Let  $C$  be a closed subset of  $X$ . Since  $p^{-1}(C)$  is closed in  $X \times X$  we note that  $p^{-1}(C) \cap \Gamma$  is closed in  $X \times X$  and hence is compact. Thus  $q(p^{-1}(C) \cap \Gamma)$  is compact and so is closed in  $X$ . Now,

$$\begin{aligned} q(p^{-1}(C) \cap \Gamma) &= \{y \in X / (x, y) \in p^{-1}(C) \cap \Gamma \text{ for some } x \in X\} \\ &= \{y \in X / y \sim x \text{ for some } x \in C\} \\ &= \eta^{-1}(\eta(C)) \end{aligned}$$

showing that  $\eta(C)$  is closed. This proves (a) and in particular we note that singleton sets in  $X/\sim$  are closed since they are images of singletons. Turning to the proof of (b), for an arbitrary pair of distinct elements  $\bar{x}$  and  $\bar{y}$  in  $X/\sim$ , the sets  $\eta^{-1}(\bar{x})$  and  $\eta^{-1}(\bar{y})$  are a pair of disjoint closed sets in  $X$ . Since  $X$  is normal there exist disjoint open sets  $U$  and  $V$  in  $X$  such that

$$\eta^{-1}(\bar{x}) \subset U \text{ and } \eta^{-1}(\bar{y}) \subset V.$$

The sets  $\eta(X - U)$  and  $\eta(X - V)$  are closed in  $X/\sim$  by (a). We leave it to the reader to verify that the complements

$$(X/\sim) - \eta(X - U) \text{ and } (X/\sim) - \eta(X - V)$$

are disjoint sets. Now  $\eta^{-1}(\bar{x}) \subset U$  implies  $\bar{x} \notin \eta(X - U)$  whereby  $\bar{x} \in (X/\sim) - \eta(X - U)$ . Likewise  $\bar{y} \in (X/\sim) - \eta(X - V)$  and the proof is complete.

**Corollary 4.9:** The projective spaces  $\mathbb{R}P^n$  are Hausdorff.

**Proof:** The relation  $\sim$  on  $S^n$  given by (4.2) defines a closed subset of  $S^n \times S^n$ .

## Exercises

1. What happens if we omit the surjectivity hypothesis on the function  $f : X \longrightarrow Y$  in the definition of quotient topology on  $Y$  induced by  $f$  ?
2. Show that the space obtained from the unit ball  $\{\mathbf{x} \in \mathbb{R}^n / \|\mathbf{x}\| \leq 1\}$  by collapsing its boundary to a singleton, is homeomorphic to the sphere  $S^n$ .
3. Show that  $\mathbb{R}P^1 \cong S^1$  by considering the map  $f : S^1 \longrightarrow S^1$  given by  $f(z) = z^2$ .
4. Try to show that  $S^2$  is not homeomorphic to  $\mathbb{R}P^2$ . Would the Jordan curve theorem help?
5. Show that the boundary of the Möbius band is homeomorphic to  $S^1$ .
6. Does a Möbius band result upon cutting the projective plane  $\mathbb{R}P^n$  along a closed curve on it?