Lecture V - Topological Groups

A topological group is a topological space which is also a group such that the group operations (multiplication and inversion) are continuous. They arise naturally as continuous groups of symmetries of topological spaces. A case in point is the group $SO(3, \mathbb{R})$ of rotations of $\mathbb{R}^3$ about the origin which is a group of symmetries of the sphere S^2. Many familiar examples of topological spaces are in fact topological groups. The most basic example of-course is the real line with the group structure given by addition. Other obvious examples are $\mathbb{R}^n$ under addition, the multiplicative group of unit complex numbers S^1 and the multiplicative group $\mathbb{C}^*$.

In the previous lectures we have seen that the group $SO(n, \mathbb{R})$ of orthogonal matrices with determinant one and the group $U(n)$ of unitary matrices are compact. In this lecture we initiate a systematic study of topological groups and take a closer look at some of the matrix groups such as $SO(n, \mathbb{R})$ and the unitary groups $U(n)$.

Definition 5.1: A topological group is a group which is also a topological space such that the singleton set containing the identity element is closed and the group operation

$$\begin{aligned} G \times G &\longrightarrow G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned}$$

and the inversion $j : G \longrightarrow G$ given by $j(g) = g^{-1}$ are continuous, where $G \times G$ is given the product topology.

We leave it to the reader to prove that a topological group is a Hausdorff space. It is immediate that the following maps of a topological group G are continuous:

1. Given $h \in G$ the maps $L_h : G \longrightarrow G$ and $R_h : G \longrightarrow G$ given by $L_h(g) = hg$ and $R_h(g) = gh$. These are the left and right translations by h.

2. The inner-automorphism given by $g \mapsto hgh^{-1}$ which is a homeomorphism.

Note that the determinant map is a continuous group homomorphism from $GL_n(\mathbb{R}) \longrightarrow R - \{0\}$. The image is surjective from which it follows that $GL_n(\mathbb{R})$ is disconnected as a topological space.

Theorem 5.1: The connected component of the identity in a topological group is a subgroup.

Proof: Let G_0 be the connected component of G containing the identity and $h, k \in G_0$ be arbitrary. The set $h^{-1}G_0$ is connected and contains the identity and so $G_0 \cup h^{-1}G_0$ is also connected. Since G_0 is a component, we have $G_0 \cup h^{-1}G_0 = G_0$ which implies $h^{-1}G_0 \subset G_0$. In particular $h^{-1}k$ belongs to G_0 from which we conclude that G_0 is a subgroup.

Interesting properties of topological groups arise in connection with quotients:

Theorem 5.2: Suppose that G is a topological group and K is a subgroup and the coset space G/K is given the quotient topology then

1. If K and G/K are connected then G is connected.

2. If K and G/K are compact then G is compact.

Proof: If G is connected then so is G/K since the quotient map $\eta : G \longrightarrow G/K$ is a continuous surjection. To prove the converse suppose that K and G/K are connected and $f : G \longrightarrow \{0,1\}$ be an arbitrary continuous map. We have to show that f is constant. The restriction of f to K must be constant and since each coset gK is connected, f must be constant on gK as well taking value $f(g)$. Thus we have a well defined map $\tilde{f} : G/K \longrightarrow \{0,1\}$ such that $\tilde{f} \circ \eta = f$. By the fundamental property of quotient spaces, it follows that $\tilde{f}$ is continuous and so must be constant since G/K is connected. Hence f is also constant and we conclude that G is connected. $\square$

Since we shall not need (2), we shall omit the proof. A proof is available in [12], p. 109.

Theorem 5.3: The groups $SO(n, \mathbb{R})$ are connected when $n \geq 2$.

Proof: It is clear that $SO(2, \mathbb{R})$ is connected (why?). Turning to the case $n \geq 3$, we consider the action of $SO(n, \mathbb{R})$ on the standard unit sphere S^{n-1} in $\mathbb{R}^n$ given by

$$(A, \mathbf{v}) \mapsto A\mathbf{v},$$

where $A \in SO(n, \mathbb{R})$ and $\mathbf{v} \in S^{n-1}$. It is an exercise for the student to check that this group action is transitive and that the stabilizer of the unit vector $\hat{\mathbf{e}}_n$ is the subgroup K consisting of all those matrices in $SO(n, \mathbb{R})$ whose last column is $\hat{\mathbf{e}}_n$. The subgroup K is homeomorphic to $SO(n-1, \mathbb{R})$ and so, by induction hypothesis, is connected. By exercise 3, the quotient space $SO(n, \mathbb{R})$ is homeomorphic to S^{n-1} which is connected. So the theorem can be applied with $G = SO(n, \mathbb{R})$, $H = SO(n-1, \mathbb{R})$ and G/H is the sphere S^{n-1} with $n \geq 2$. $\square$

Theorem 5.4: If G is a connected topological group and H is a subgroup which contains a neighborhood of the identity then $H = G$. In particular, an open subgroup of G equals G.

Proof: Let U be the open neighborhood of the identity that is contained in H and $h \in H$ be arbitrary. Since multiplication by h is a homeomorphism, the set $Uh = \{uh/u \in U\}$ is also open and also contained in H. Hence the set

$$L = \bigcup_{h \in H} Uh$$

is open and contained in H. Since U contains the identity element, $H \subset L$ and we conclude that H is open. Our job will be over if we can show that H is closed as well. Let $x \in \overline{H}$ be arbitrary. Since the neighborhood Ux of x contains a point $y \in H$, there exists $u \in U$ such that $y = ux$ which, in view of the fact that $U \subset H$, implies $x \in H$. Hence $\overline{H} = H$. $\square$

Theorem 5.5: Suppose G is a connected topological group and H is a discrete normal subgroup of G then H is contained in the center of G.

Proof: Since H is discrete, the identity element is not a limit point of H and so there is a neighborhood U of the identity such that $U \cap H = \{1\}$. We may assume U has the property that if u_1, u_2 are in U then the product $u_1^{-1}u_2$ is in U. This follows from the continuity of the group operation and a detailed verification is left as an exercise. It is easy to see that if h_1 and h_2 are two distinct elements of H then

$$Uh_1 \cap Uh_2 = \emptyset.$$

Fix $h \in H$ and consider now the set K given by

$$K = \{g \in G \ / \ gh = hg\}$$

We shall show that the subgroup K contains a neighborhood of the identity. Pick a neighborhood V of the identity such that $V = V^{-1}$ and $(hVh^{-1}V) \cap H = \{1\}$. Then for any $g \in V$, we have on the one hand

$$hgh^{-1}g^{-1} \in hVh^{-1}V$$

and on the other hand $hgh^{-1}g^{-1} \in H$ since H is normal. Hence $hgh^{-1}g^{-1} \in (hVh^{-1}V) \cap H = \{1\}$ which shows that g belongs to K and K contains a neighborhood of the unit element. We may now invoke the previous theorem. $\square$

Remark: The result is false if the hypothesis of normality of H is dropped. For example consider a cube in $\mathbb{R}^3$ with center at the origin and H be the subgroup of $G = SO(3, \mathbb{R})$ that map the cube to itself. Then H is the symmetric group on four letters (proof?). Clearly H is not in the center of G.

Exercises

1. Show that in a topological group, the connected component of the identity is a normal subgroup.

2. Show that the action of the group $SO(n, \mathbb{R})$ on the sphere S^{n-1} given by matrix multiplication is transitive. You need to employ the Gram-Schmidt theorem to complete a given unit vector to an orthonormal basis.

3. Suppose a group G acts transitively on a set S and x, y are a pair of points in S and $y = gx$. Then the subgroups stab x and stab y are conjugates and $g^{-1}(\text{stab } y)g = \text{stab } x$.

   (i) Show that the map $\overline{\phi} : G/\text{stab } x \longrightarrow S$ given by $\overline{\phi}(\overline{g}) = gx$ is well-defined, bijective and $\overline{\phi} \circ \eta = \phi$.

   (ii) Suppose that S is a topological space, G is a topological group and the action $G \times S \longrightarrow S$ is continuous. Show that the map $\overline{\phi}$ is continuous.

   (iii) Deduce that if G is compact and S is Hausdorff then $G/\text{stab } x$ and S are homeomorphic.

4. Examine whether the map $\phi : SU(n) \times S^1 \longrightarrow U(n)$ given by $\phi(A, z) = zA$ is a homeomorphism.

5. Show that the group of all unitary matrices $U(n)$ is compact and connected. Regarding $U(n-1)$ as a subgroup of $U(n)$ in a natural way, recognize the quotient space as a familiar space.

6. Show that the subgroups $SU(n)$ consisting of matrices in $U(n)$ with determinant one are connected for every n.

7. Suppose G is a topological group and H is a normal subgroup, prove that G/H is Hausdorff if and only if H is closed.