# Lecture - XXXIII Homotopy invariance of homology

Homotopy of maps is one of the most important notions in topology and it is of interest to know what is its effect on the induced maps in homology. The result is simple and direct namely, if $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are a pair of homotopic maps then they induce the *same* maps in homology in every dimension. The further advantage here is that no base points are involved unlike the situation encountered in lecture 11 with the fundamental group. However the proof is not direct as one must algebraize the notion of homotopy in the context of chain maps. This leads to the notion of *chain homotopy* that we first define. We establish the purely algebraic result that a pair of chain homotopic maps induce equal maps in homology. We then proceed to relate the topological notion of homotopy of a pair of continuous maps $f, g$ as above with the chain homotopy between the induced chain maps $f_\sharp : S_n(X) \longrightarrow S_n(Y)$ and $g_\sharp : S_n(X) \longrightarrow S_n(Y)$. Some of these ideas have been implicitly used in the last lecture in the construction of the singular two chain $\sigma$ in lemma (32.2). We shall follow the treatment in the book by T. Dieck[7] defining first the notion of the cross product which seems more transparent. The student who is familiar with differential forms may notice some similarities with wedge products and the exterior derivative. As in the theory of differential forms where the construction of the exterior derivative $d$ is forced upon us through some of its properties, the cross product is determined by its properties described in theorem (33.1), as soon as one chooses for each pair $(p, q)$ a *model chain* namely, the $p + q$ chain $z$ in (33.4).

**The cross product:** This construction lies at the heart of the proof of Kunneth formula which relates the homology groups of $X \times Y$ in terms of the homologies of $X$ and $Y$. The first step would be to relate the singular chain complex of $X \times Y$ with those of $X$ and $Y$. This construction will be carried out naturally. Given a zero simplex $x \in X$ and a $q$ simplex $\sigma : \Delta_q \longrightarrow Y$ in $Y$, $x \times \sigma$ denotes the singular $q$ simplex in $X \times Y$ given by

$$\begin{aligned} x \times \sigma : \Delta_q &\longrightarrow X \times Y \\ t &\mapsto (x, \sigma(t)). \end{aligned}$$

Likewise given a $q$ simplex $\tau$ in $X$ and a zero simplex $y$ in $Y$, one defines a $q$ simplex $\tau \times y$ in $X \times Y$. For a pair of singular simplices $\sigma \in \Delta_p(X)$ and $\tau \in \Delta_q(Y)$ we call $p + q$ the total degree of the pair $(\sigma, \tau)$.

**Theorem 33.1:** There exists a bilinear map

$$\begin{aligned} S_p(X) \times S_q(Y) &\longrightarrow S_{p+q}(X \times Y) \\ (\sigma, \tau) &\mapsto \sigma \times \tau, \end{aligned}$$

[7]See also R. Stöcher and H. Zeischang, Algebraische Topologie, B. G. Teubner, Stuttgart (1988) 306-325.

with the following properties

(i) For zero simplices $x \in X$, $y \in Y$ and singular simplices $\sigma : \Delta_p \longrightarrow X$ and $\tau : \Delta_q \longrightarrow Y$ the products $x \times \tau$, $\sigma \times y$ are already defined above.

(ii) Naturality: Suppose that $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are two continuous maps and $f \times g : X \times Y \longrightarrow X' \times Y'$ denotes the product map $(f \times g)(x, y) = (f(x), g(y))$, then

$$(f \times g)_\sharp(\sigma \times \tau) = f_\sharp(\sigma) \times g_\sharp(\tau) \tag{33.1}$$

(iii) Generalized Leibnitz' rule: If $\sigma \in S_p(X)$ and $\tau \in S_q(Y)$ then

$$\partial(\sigma \times \tau) = \partial\sigma \times \tau + (-1)^p(\sigma \times \partial\tau) \tag{33.2}$$

**Proof:** The construction proceeds by induction on the total degree $p + q$ on pairs $(\sigma, \tau)$. It has already been carried out for the case when one of $\sigma$ or $\tau$ is a zero simplex and in particular when the total degree $p + q$ is zero. Further, and for this case, conditions (ii) and (iii) hold trivially. Assume that the cross product

$$S_p(X) \times S_q(Y) \longrightarrow S_{p+q}(X \times Y) \tag{33.3}$$

has been defined for all pairs $(p, q)$ such that $p + q < k$ satisfying (ii) and (iii). Now if $\sigma \in S_p(X)$ and $\tau \in S_q(Y)$ are such that $p + q = k$ then the right hand side of the formula in (iii) already makes sense and in particular this is so with the pair $\iota_p$ and $\iota_q$. Thus we need a singular $p + q$ chain $z$ such that

$$\partial z = \partial\iota_p \times \iota_q + (-1)^p(\iota_p \times \partial\iota_q). \tag{33.4}$$

Applying Leibnitz rule again to the right hand side one checks that it is a cycle. Since $\Delta_p \times \Delta_q$ is convex this cycle is also a boundary and so (33.4) has a (non-unique) solution $z \in S_{p+q}(\Delta_p \times \Delta_q)$. Once this choice is made the construction proceeds further as follows. Each $\sigma \in S_p(X)$ can be realized as $\sigma_\sharp(\iota_p)$ where $\sigma : \Delta_p \longrightarrow X$ and likewise for a singular $q$ simplex $\tau$ in $Y$. But now equation (33.1) forces upon us the definition

$$\sigma \times \tau = \sigma_\sharp(\iota_p) \times \tau_\sharp(\iota_q) = (\sigma \times \tau)_\sharp(\iota_p \times \iota_q) = (\sigma \times \tau) \circ z, \tag{33.5}$$

where the $\sigma \times \tau$ appearing on the extreme left of (33.5) is the object we are defining whereas the $\sigma$ and $\tau$ appearing in the middle and on the extreme right of (33.5) denote the functions $\sigma : \Delta_p \longrightarrow X$ and $\tau : \Delta_q \longrightarrow Y$. The easy verification of (33.1) is left for the reader. Proof of (33.2) runs as follows:

$$\begin{aligned}
\partial(\sigma \times \tau) &= \partial\big((\sigma \times \tau)_\sharp(\iota_p \times \iota_q)\big) \\
&= (\sigma \times \tau)_\sharp\partial(\iota_p \times \iota_q) \\
&= (\sigma \times \tau)_\sharp\partial z \\
&= (\sigma \times \tau)_\sharp\Big(\partial\iota_p \times \iota_q + (-1)^p(\iota_p \times \partial\iota_q)\Big)
\end{aligned}$$

Applying (33.1), which holds by induction hypothesis, and using the pair of equations $\sigma_\sharp\partial = \partial\sigma_\sharp$, $\tau_\sharp\partial = \partial\tau_\sharp$ we continue with our calculation:

$$\begin{aligned}
\partial(\sigma \times \tau) &= \sigma_\sharp(\partial\iota_p) \times \tau_\sharp(\iota_q) + (-1)^p(\sigma_\sharp(\iota_p) \times \tau_\sharp(\partial\iota_q)) \\
&= \partial\sigma \times \tau + (-1)^p(\sigma \times \partial\tau).
\end{aligned}$$

Having defined $\sigma \times \tau$ for singular simplices $\sigma$ and $\tau$, we can extend it as a bilinear map $S_p(X) \times S_q(Y) \longrightarrow S_{p+q}(X \times Y)$ since $S_p(X)$ and $S_q(Y)$ are free abelian groups.

**Homotopy and chain homotopy:** Chain homotopy is the algebraization of the topological notion of homotopic maps. Let $F : I \times X \longrightarrow Y$ be a homotopy between two continuous functions $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$. We use this map to define a sequence of maps

$$L_n : S_n(X) \longrightarrow S_{n+1}(Y) \tag{33.6}$$

satisfying the condition

$$\partial \circ L_n + L_{n-1} \circ \partial = f_\sharp - g_\sharp. \tag{33.7}$$

Let $u : \Delta_1 \longrightarrow I$ be the unique one simplex. For a singular $n$ simplex $\sigma$ in $X$, define

$$L_n(\sigma) = F_\sharp(u \times \sigma).$$

Then we compute using (33.1)-(33.2),

$$\begin{aligned}
\partial(L_n(\sigma)) &= F_\sharp(\partial u \times \sigma) - F_\sharp(u \times \partial\sigma) \\
&= F_\sharp(\partial u \times \sigma) - L_{n-1}(\partial\sigma) \\
\therefore\ \partial(L_n(\sigma)) + L_{n-1}(\partial\sigma) &= F_\sharp(\{1\} \times \sigma) - F_\sharp(\{0\} \times \sigma) \\
\therefore\ \partial(L_n(\sigma)) + L_{n-1}(\partial\sigma) &= F(1, \sigma(\cdot)) - F(0, \sigma(\cdot)).
\end{aligned}$$

So we have the important equation

$$\partial(L_n(\sigma)) + L_{n-1}(\partial\sigma) = g_\sharp(\sigma) - f_\sharp(\sigma), \quad \sigma \in S_n(X). \tag{33.8}$$

completing the proof of (33.7). The reader must go back to lemma (32.2) to observe some analogies. After these preparations we are ready to prove the following important result. Unlike theorems (11.2) - (11.5) we do not have to worry here about base points which makes life a lot easier.

**Theorem 33.2:** Homotopic maps $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ induce equal maps in homology. That is to say for each $n$ we have

$$H_n(f) = H_n(g). \tag{33.9}$$

**Proof:** Taking $\sigma \in Z_n(X)$ in (33.8), the term $L_{n-1}(\partial\sigma)$ drops out and we immediately see that the cycles $f_\sharp(\sigma)$ and $g_\sharp(\sigma)$ differ by a boundary. The proof is complete.

We see that equation (33.7) is the algebraic analogue of homotopy of continuous maps. As this phenomenon would recur often, we give a formal definition and a name for it.

**Definition 33.1:** Given chain maps $\phi_n : C_n \longrightarrow D_n$ and $\psi_n : C_n \longrightarrow D_n$ $(n = 1, 2, \dots)$ between chain complexes $C$ and $D$, a chain homotopy between $\phi$ and $\psi$ is a sequence $L_n : C_n \longrightarrow D_{n+1}$ of group homomorphisms such that

$$\partial \circ L_n + L_{n-1} \circ \partial = \phi_n - \psi_n \tag{33.10}$$

It is easy to see that that *chain homotopy* is an equivalence relation on the family of chain maps. Recalling now the definition of homotopy equivalence (see lecture 11, definition 11.2) we state the very useful result which follows immediately from theorem (33.2).

**Corollary 33.3:** If $X$ and $Y$ have the same homotopy type, then $H_n(X) = H_n(Y)$ for $n = 0, 1, 2, \ldots$.

# Exercises

1. Show that the $p + q - 1$ chain on the right hand side of (33.4) is a cycle.

2. Check that $\sigma \times \tau$ as defined by equation (33.5) satisfies (33.1).

3. Show that the product in theorem (33.1) defines a bilinear map $H_p(X) \times H_q(Y) \longrightarrow H_{p+q}(X \times Y)$.

4. Determine explicitly the two/three chain $z$ satisfying (33.4) when

   (i) $p = 1$ and $q = 1$.

   (ii) $p = 1$ and $q = 2$.

   Hint: In the proof of lemma (32.2), we chopped the square into two triangles. When $q = 2$ we need to chop a prism into three pieces and map $\Delta_3$ affinely onto each of them.

5. Use the map $\Pi_X$ of the previous lecture to calculate the generators of $H_1(S^1 \times S^1)$.

6. Use equation (33.1) to determine the image of the pair of generating one cycles of the previous exercise under the map $H_1(S^1) \times H_1(S^1) \longrightarrow H_2(S^1 \times S^1)$.