Lecture 21 : Galois Groups of Composite Extensions

Objectives

(1) Galois group of composite extensions
(2) Galois closure of a separable field extension.

Keywords and Phrases : Composite extensions, normal closure.

Let K be a field and $\overline{K}$ be an algebraic closure of K. Let E, F be intermediate subfields of $\overline{K}/K$. Recall that the **compositum** of E and F denoted by EF is the smallest subfield of $\overline{K}$ containing E and F. In this section we will discuss Galois groups of composite extensions and normal closure of an algebraic field extension.

Proposition 21.1. *If E/K is a Galois extension and F/K is a field extension, then EF/F is Galois. If F/K is Galois then EF/K and $E \cap F/K$ are Galois.*

Proof.

$$\begin{tikzcd}[arrows=dash]
 & \overline{K} \arrow[d] & \\
 & EF \arrow[dl] \arrow[dr] & \\
E \arrow[dr] & & F \arrow[dl] \\
 & E \cap F \arrow[d] & \\
 & K &
\end{tikzcd}$$

Consider the diagram above. As E/K is a separable and normal extension, it is a splitting field of a family $\{f_i(x)\}$ of separable polynomials over K. Then EF/F is the splitting field of the same family of polynomials. Hence EF/F is a Galois extension. If F/K is Galois then it is a splitting field of a family of polynomials $\{g_j(x)\}$ over K. Hence EF/K is a splitting field of the polynomials $\{f_i(x)\} \cup \{g_j(x)\}$. Hence EF/K is Galois.

Now we show that if E/K and F/K are Galois then $E \cap F/K$ are Galois. Let $\sigma : E \cap F \to \overline{K}$ be a K-embedding. Let $\tau : EF \to \overline{K}$ be an extension of σ. Then $\tau(E) = E$ and $\tau(F) = F$ since E/K and F/K are Galois. Therefore $\tau(E \cap F) \subseteq E \cap F$. Since $E \cap F/K$ is algebraic, $\tau(E \cap F) = E \cap F$. Hence $E \cap F/K$ is a Galois extension.

$\square$

Proposition 21.2. *Let E/K be a Galois extension and F/K be a field extension so that $E, F \subset \overline{K}$. Then the map $\psi : G(EF/F) \to G(E/K)$ defined by $\psi(\sigma) = \sigma|_E$ is injective and it induces an isomorphism:*

$$G(EF/F) \simeq G(E/E \cap F).$$

Proof. Since σ is an F-automorphism of EF, it is also a K-automorphism. Hence $\sigma|_E \in G(E/K)$. If $\sigma|_E = id_E$ then $\sigma = id_{EF}$. Hence ψ is an injective group homomorphism.

The image of ψ is a subgroup H of $G(E/K)$. By Artin's Theorem $G(E/E^H) = H$. Hence $E \cap F \subset E^H$. Let $a \in E \setminus (E \cap F)$. Then $a \in EF \setminus F$. Hence there is a $\sigma \in G(EF/F)$ so that $\sigma(a) \neq a$. Hence $a \notin E^H$. Therefore $E^H = E \cap F$ and we conclude that $G(E/E \cap F) = H \simeq G(EF/F)$. $\square$

Corollary 21.3. *Let E/K be a finite Galois extension and F as above. Then*

$$[EF : F] = [E : E \cap F].$$

In particular, $[EF : K] = [E : K][F : K]$ if and only if $E \cap F = K$.

Proof. Since $G(EF/F) \simeq G(E/E \cap F)$, we obtain

$$|G(EF/F)| = [EF : F] = |G(E/E \cap F)| = [E : E \cap F].$$

Therefore we have:

$$[EF : K] = [E : E \cap F][F : K] = \frac{[E : K][F : K]}{[E \cap F : K]}.$$

The conclusion follows from the equation above. $\square$

Theorem 21.4. *Let E/K and F/K be finite Galois extensions so that $E, F \subset \overline{K}$. Then the homomorphism*

$$\psi : G(EF/K) \longrightarrow G(E/K) \times G(F/K),\ \psi(\sigma) = (\sigma|_E, \sigma|_F)$$

is injective. If $E \cap F = K$ then ψ is an isomorphism.

Proof. It is clear that ψ is a group homomorphism. The kernel of ψ consists of $\sigma \in G(EF/K)$ so that $\sigma(a) = a$ for all $a \in E$ and for all $a \in F$. Hence such $\sigma = id_{EF}$. Thus ψ is injective.

Suppose that $E \cap F = K$. Then by Corollary 21.3,

$$|G(EF/K)| = [EF : E \cap F] = [F : K][E : K] = |G(E/K)||G(F/K)|.$$

This shows that ψ is an isomorphism. □

The Normal Closure of an Algebraic Extension

Let K/F be an algebraic extension and $K \subset \overline{F}$. The **normal closure of** K/F in $\overline{K}$ is the splitting field N over F of the polynomials $\{$ irr $(a, F) \mid a \in K\}$. It is clear that N is a normal extension of F containing K. Moreover any normal extension $N' \subset \overline{F}$ of F containing K must contain the splitting fields of $\{$ irr $(a, F) \mid a \in K\}$. Hence $N = N'$. If $K = F(a_1, a_2, \ldots, a_n)$ then N is the splitting field of the polynomials irr (a_i, F) for all $i = 1, 2, \ldots, n$.

If K/F is separable then N/F is a separable extension as it is obtained by adjoining roots of separable polynomials over F. Hence the normal closure of K/F when K/F is separable, is a Galois extension.

Let K/F be a separable extension that is not normal. Let N be a normal closure of K/F. Put $H = G(N/K)$. Then $K = N^H$. Let $H' < H$ be a normal subgroup of $G = G(N/F)$. Then $N^{H'} > N^H = K$ and $N^{H'}/F$ is a normal extension of K. Thus $N^{H'} = N$ by minimality of N. Hence $H' = (id)$.