Lecture 1 : Overview

Objectives

(1) A historical sketch of main discoveries about formulas for roots of polynomials.
(2) Problems of classical Greek geometry.
(3) Discussion about the main themes of the course.

Key words and phrases: Quadratic, cubic and quartic equations, duplication of cube, trisection of an angle, construction of regular polygons, Galois group.

In this course we will study fields, Galois theory of field extensions and applications to geometry and theory of equations. We outline the main topics that we will study.

The formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the roots of the quadratic equation $ax^2 + bx + c = 0$ was known to Babylonians. During the reign of King Hammurabi (1750 B. C.), Babylonian mathematicians found methods of solving linear and quadratic equations in one and two variables. They described algorithms to solve specific examples. From these examples it is clear that they knew the formula for the roots of quadratic equations.

In 1494, an Italian mathematician Franciscan Luca Pacioli published the book *Summa de Arithmetica, Geometria, Proportioni et Proportionalita* containing all that was known in that period in arithmetic, algebra, geometry and trigonometry. Paciolo ended his book with a remark that solutions of cubic equations seemed impossible.

Generations of mathematicians at the University of Bologna in Italy tried to find solutions of cubic equations. This was the largest and one of the most famous universities at the turn of sixteenth century in Europe. Scipio Del Ferro at this univerisity solved the cubic but never published his findings.

In 1535, Niccolo Tartaglia, a mathematician from Venice proved in a public demonstration that he could solve cubic equations. But He kept his formula a secret. But a doctor from Milan Gerolamo Cardano obtained these formulas from Tartaglia under an oath that he will keep them a secret. Cardano wrote his textbook *Ars Magna* in 1545 which described Tartaglia's method and extended it to all cubic equations. It is easy to see that the equation $x^3+ax^2+bx+c=0$ is transformed into the equation $x^3+px+q=0$ by replacing x by $x-a/3$. Let x_1, x_2, x_3 denote the roots of $x^3+px+q=0$, $\delta = -(4p^3+27q^2)$ and $w = -1/2 - 1/2\sqrt{-3}$. Let

$$\begin{aligned} y_1 &= x_1 + w^2 x_2 + w x_3 \\ y_2 &= x_1 + w x_2 + w^2 x_3 \end{aligned}$$

Then Cardano's formulas are

$$\begin{aligned} y_1 &= \sqrt[3]{\frac{-27}{2} + \frac{3}{2}\sqrt{-3\delta}} \\ y_2 &= \sqrt[3]{\frac{-27}{2} - \frac{3}{2}\sqrt{-3\delta}} \end{aligned}$$

We also have $x_1+x_2+x_3=0$. These three linear equations determine the roots x_1, x_2, x_3.

In 1536 Lodovico Ferrari entered Cardano's house as a servant. Due to his extraordinary mathematical abilities he became a mathematician under Cardano's guidance. Ferrari showed that a quartic equation can be reduced to a cubic equation and therefore it can be solved by means of four arithmetic operations and extraction of square and cube roots. We will derive the formulas of Cardano and Ferrari later.

Some of the greatest mathematicians, e.g., Euler and Lagrange attempted to find similar formula for the roots of quintic equations. Lagrange gave a general method to solve equations of degree atmost four. But this method did not work for quintic equations.

Mathematicians became skeptical about existence of such formulas for equations of degree five and higher. Paolo Ruffini, born 1765 was a student of Lagrange. He published several papers(1802, 1813) about insolvability of general quintic equation. His proof was not complete. The first complete proof was given by Neils Henrik Abel (1802-1829) in 1824. Abel also proved that if the Galois group of the polynomial is commutative then the polynomial is solvable by radicals. Commutative groups are called Abelian to honour Abel for his deep work in many branches of mathematics.

Gauss made two fundamental contributions to the theory of equations. He provided complete solution by means of radicals of the cyclotomic equation

$$x^n - 1 = 0.$$

The roots of this equation are the complex numbers represented by the vertices of a regular polygon of n sides centered at the origin. Gauss' analysis of the roots of cyclotomic equation led him to find a criterion of the constructibility of regular polygons of n sides. We will discuss this later.

The second contribution was first rigorous proof of of the *Fundamental Theorem of Algebra*: Every polynomial with complex coefficients is a product of linear factors with complex coefficients.

The most decisive results in the theory of equations were found by Evariste Galois (1811-1832). Modern algebra began with the work of Galois. He introduced the Galois group of a polynomial which connected field theory with group theory. In 1829, Galois presented two papers to the Paris Academy of Sciences. These were sent to Cauchy who lost them. In 1830 he sent another paper to the Academy whose secretary was Fourier who died before he could examine this paper. The manuscript has never been found. In 1830, Galois published a summary of his results. The first theorem in this account is: An equation of prime degree is solvable by radicals if and only if if two of its roots are known then the others are rational functions of them. This implies that a general equation of degree five cannot be solved by means of radicals. The most decisive result is the solvability criterion: A polynomial is solvable by radicals if and only if its Galois group is a solvable.

We will also study the solutions of several problems in Greek Geometry using rudiments of field theory. In Euclidean Geometry, we carry out several geometric constructions with a ruler (unmarked) and compass such as bisection of line segments and angles, constructions of certain angles, triangles, quadrilaterals and circles. Ancient Greeks posed the following four problems:

(1) **The Delian Problem :** Construct the side of a cube of volume 2.

(2) **The angle trisection problem :** Divide a given angle in three equal parts.

(3) **Squaring a circle :** Construct a square having same area as that of a given circle.

(4) **Constructible regular polygons :** Find n for which regular polygon of n sides can be constructed by ruler and compass and describe their constructions.

The above problems remained open for almost 2200 years. The final solution employed techniques from abstract algebra and analysis. We will show that it is impossible to construct side of a cube whose volume is 2 by ruler and compass. The word Delian is derived from Delos which was a city in in ancient Greece. It is said that almost a quarter of population of Delos died of plague in 428 B.C. A delegation was sent to the oracle of Apollo at Delos to enquire how the plague could be arrested. The oracle replied that the cubical altar to the Sun God Apollo should be doubled. Instead of doubling the volume the faithfuls doubled the sides of the cube thereby increasing the volume eightfold.The second and the third problems also circulated among Greek geometers around the same time. It is not known who solved the Delian problem first. The angle trisection problem was solved by Gauss as a special case of his remarkable solution of the fourth problem. Gauss, barely 19, provided a construction of the 17-sided regular polygon. He also characterized n for which regular $n-$gons are constructible by ruler and compass. Recall that a prime of the form $2^{2^m}+1$ is called a Fermat prime. Gauss proved that a regular $n-$gon is constructible if and only if $n = 2^r p_1 p_2 \dots p_g$ where $n \geq 0$ and $p_1, p_2, \dots, p_g$ are distinct Fermat primes. Gauss's Theorem solves the angle trisection problem. If 20^o was constructible, then we can construct a regular 18-gon. Since $3^2|18$, we have a contradiction.

The values of n for which regular $n-$gons were known to be constructible upto the time of Gauss were $n = 2^m, 2^m.3, 2^m.5$ and $2^m.15$. No one was able to construct a heptagon or a regular $17-$gon.

In March, 1796 Gauss made his first mathematical discovery : construction of a 17-sided regular polygon by ruler and compass. He began noting down his mathematical discoveries in a diary which he maintained for the next 19 years. Gauss published "*Disquisitiones Arithmaticae*" in 1801 which has become a classic in mathematical literature. The last result of this is his solution to the fourth problem. Gauss was very proud of this discovery. He desired that a regular polygon of 17 sides be engraved on his tombstone. This wish was not fulfilled. It was fulfilled when a monument to Gauss was built in his birth place Braunschweig. Explicit construction of 17−sided regular polygon was given by Erchinger in 1800. In 1892 Richelot and Schwendenwein constructed a regular 257−gon. Around 1900 Hermes constructed a regular 65537-gon. The manuscript fills a box and it is found in Góttingen. The construction has now been computerized. See an article by Bishop in American Math. Monthly (1978).

Lindemann proved in 1882 that π is not a root of any polynomial with rational coefficients. This proved the impossibility of squaring a circle.