Lecture 4 : Ruler and Compass Constructions I

Objectives

(1) Describe standard ruler and compass constructions.
(2) The field of constructible numbers is closed under taking square roots of positive reals.
(3) Characterization of constructible real numbers via square root towers of fields.
(4) The degree of a constructible real number is a power of 2.
(5) Impossibility of squaring the circle, trisection of angles and duplication of cubes by ruler and compass.

Key words and phrases: Ruler and compass constructions, constructible real numbers, square root tower, trisection of an angle, duplication of cube, squaring a circle.

The four problems of classical Greek Geometry: duplication of cube, trisection of angles, squaring of circles and construction of regular polygons can now be solved using the rudiments of algebraic extensions of fields. A complete solution of the last problem about characterization of constructible regular polygons will use fundamental theorem of Galois theory. This will be discussed later. As we have remarked before, these problems remained open for more than 2000 years. We will see that the language of field extensions provides the right framework for discussion of these problems. Once translated into this language, the solutions are obtained quickly.

First we will precisely formulate constructibility by ruler and compass and the concept of constructible points, lines and constructible real numbers. A real number is called constructible if it is the length of a line segment connecting two constructible points. We will then see that a constructible real number is algebraic over $\mathbb{Q}$ and its degree over $\mathbb{Q}$ is a power of two. This criterion leads to solutions of the first three problems and a partial solution of the fourth problem.

Constructible points, lines, circles and real numbers:

Given a finite set $\{P_1, \ldots, P_n\}$ of points in the Cartesian plane $\mathbb{R}^2$, define the set S_m inductively. Put $S_0 = \{P_1, \ldots, P_n\}$. Suppose S_m has been defined. Put $S_{m+1} = S_m \cup T_m$ where T_m is the set of points of intersection of lines passing through points in S_m and circles with center at one point in S_m with radii equal to distance between points of S_m. Let $S = \cup_{m=0}^{\infty} S_m$. We say that $S = C(P_1, \ldots, P_n)$ is the set of points constructible from $P_1, P_2, \ldots, P_n$ by ruler and compass. A real number a is called constructible if $|a|$ is the distance between two constructible points. A line passing through two constructible points is called a constructible line. A circle is called constructible if its center is constructible and its radius is a constructible real number.

We can reformulate the problems according to the definition of constructible points. Let $P_1 = (0, 0)$ and $P_2 = (0, 1)$. Is $(\sqrt[3]{2}, 0) \in C(P_1, P_2)$? This is the Delian problem. For the squaring of the circle problem, if there exists a square with side a such that $a^2 = \pi$ then $a = \sqrt{\pi}$. So the problem is asking whether $(\sqrt{\pi}, 0) \in C(P_1, P_2)$. For the angle trisection problem, set $P_3 = (\cos\theta, \sin\theta)$. The problem asks whether $C(P_1, P_2, P_3)$ contains $(\cos\theta/3, \sin\theta/3)$. The problem of construction of regular n-gons asks for which values of n, $(\cos\frac{2\pi}{n}, \sin\frac{2\pi}{n}) \in C(P_1, P_2)$.

Trisection of an angle with a marked ruler and compass

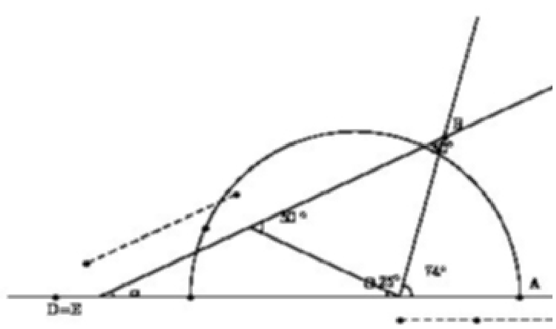

FIGURE 1. Trisection of an angle with a marked ruler and compass

Let $\angle AOB = \theta$. Draw a unit circle centered at O. Suppose one end of a ruler is E and point P is marked on the ruler such that $EP = 1$. Slide the ruler in such a way that E is on X-axis and P is on the circle and the edge passes through B. Then $\triangle DCO$ gives $\alpha + \alpha + \pi - \beta = \pi$. Hence $\beta = 2\alpha$. The $\triangle BOC$ gives $4\alpha + \pi - (\theta + \alpha) = \pi$. Hence $\alpha = \theta/3$

Duplication of a cube with a marked ruler and compass

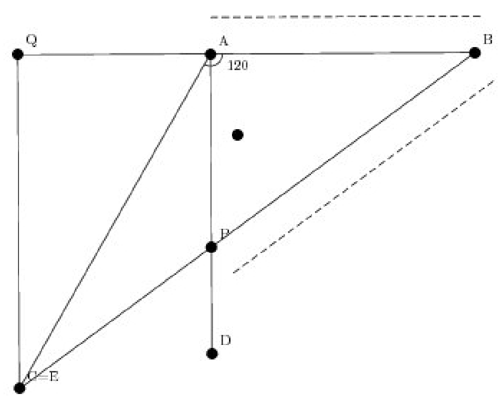

Figure 2. Duplication of a cube with a marked ruler

Use a ruler with one end point marked as E and a point marked as P with $EP = 1$. Let AB be a segment of unit length. Draw the angles $\angle BAD = 90^o$ and $\angle BAE = 120^o$. We show that $PB = \sqrt[3]{2}$. Let x be the length of PB and z be the length of AP. Then $x^2 = z^2 + 1$. Since $\triangle QEB \,||\, \triangle APB$, we get

$$\frac{x+1}{a+1} = \frac{x}{1} \text{ and } \frac{a\sqrt{3}}{z} = \frac{a+1}{1}.$$

Hence $a = 1/x$ and $\sqrt{3}/xz = x + 1/x$. We also have $\sqrt{3x} = x^2z + xz$ and hence $z = \sqrt{3x}/x(x+1) = \sqrt{3}/x + 1$. Since $x^2 = 3/(x+1)^2 + 1$, $x^4 + 2x^3 + x^2 = 3 + x^2 + 2x + 1$. Therefore $x^4 + 2x^3 - 2x - 4 = (x+2)(x^3 - 2) = 0$. Therefore $x = \sqrt[3]{2}$

Standard Constructions

(i) **Bisecting a line segment:** Suppose A and B are constructible points, we show that the mid point of the line segment AB is also constructible

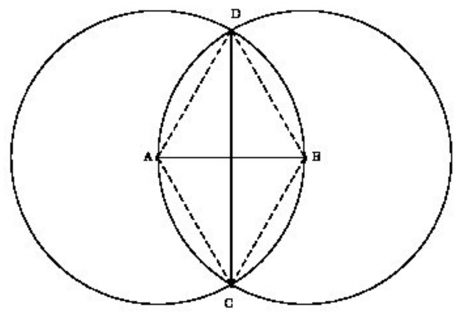

FIGURE 3. Bisection of a line segment

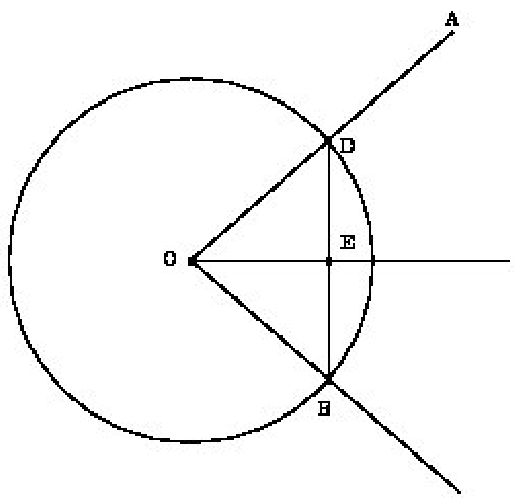

FIGURE 4. Bisection of an angle

Draw circles with centers A, B with radius AB. Then the intersection points of these circles C, D are constructible. The mid point of AB is the intersection of CD and AB. It is the mid point since it is the intersection of diagonals of the rhombus $ACBD$.

(ii) **Bisection of an angle:** Let A, O, B be three constructible points. They determine the angle AOB.

Draw a circle with center O and radius OB. It meets OA at D. Then D is constructible. Now bisect the segment BD at E. So E is also constructible. Then line OE bisects $\angle AOB$.

(iii) **Drawing a right angle:**

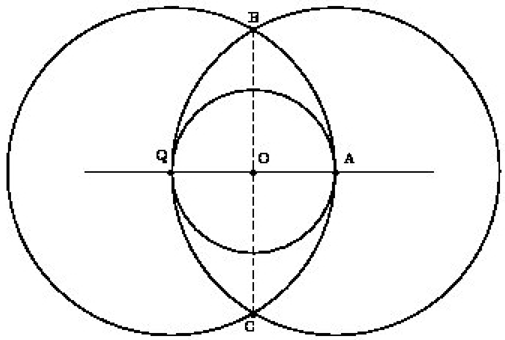

FIGURE 5. Drawing a right angle

Suppose O, A are constructible points. We wish to draw a perpendicular at O which is also a constructible line. Draw a circle $C(O, OA)$. It meets the extended line OA at Q. Draw circles $C(Q, QA)$ and $C(A, QA)$. These intersect at B and C. The triangle QAB is isosceles. Hence $\angle BOA = 90^o$

(iv) **Dropping a perpendicular:**

Suppose L is a constructible line and P a point outside this line which is constructible. Then we can draw a perpendicular onto L from P which is also constructible. Draw the circle $C(P, r)$ where r is a large constructible number so that $C(P, r)$ meets 2 points Q and R. Draw circles at centers Q and R of radius PQ. Join PS and take the intersection of QR and PS.

(v) **Drawing a parallel line.**

Suppose L is a constructible line and P is a constructible point outside L. Drop a perpendicular PO on L. Now draw 90^o on OP at P to get a parallel line.

Algebraic properties of Constructible Real Numbers

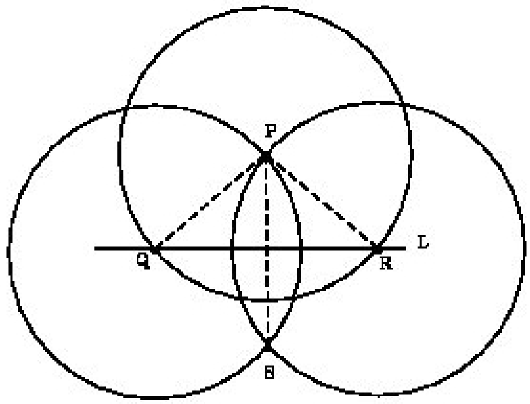

FIGURE 6. Construction of a perpendicular onto a line

Proposition 4.1. *A point $P = (a, b)$ is constructible if and only if a and b are constructible real numbers.*

Proof. Drop a perpendicular from P to X and Y axes to get constructible points A and B. So a and b are constructible real numbers. If a and b are constructible then we can draw circles $C(0, a)$ and $C(0, b)$ to get A and B. Now draw perpendicular at A and B to get P. $\square$

Proposition 4.2. *Constructible real numbers form a subfield of* $\mathbb{R}$.

Proof. It is easy to show that $a \pm b$ are constructible if a and b are so. To show ab and a/b for $b \neq 0$ are constructible, use the constructions in the figures below.

$\square$

Proposition 4.3. *If a is a positive constructible real number then so is* $\sqrt{a}$

Proof. Since $A = (a, 0)$ is constructible so is $B = (a + 1, 0)$. Hence the mid point $C = (a + 1/2, 0)$ is constructible. Draw the circle C with center $(0, \frac{a+1}{2})$ and radius $(a + 1)/2$. Draw a perpendicular at A which meets the circle at D. Since $\triangle ODA$ and $\triangle DBA$ are similar, $x/a = 1/x$ we have $x = \sqrt{a}$

$\square$

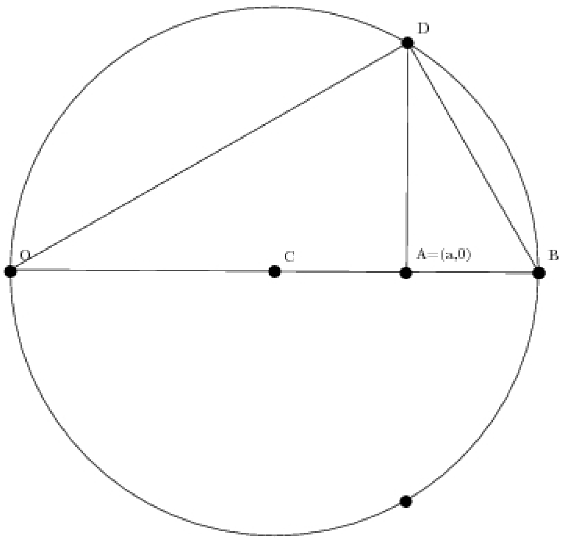

FIGURE 7. Construction of $\sqrt{a}$

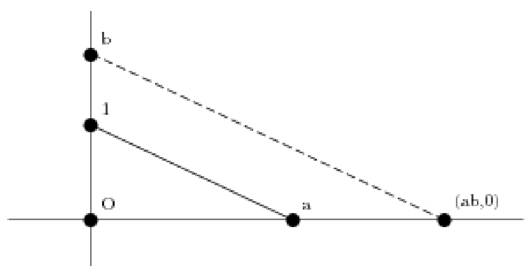

FIGURE 8. Construction of ab

Corollary 4.4. *Let $F \subseteq C$ be a subfield of the field C of constructible real numbers. Let $k > 0 \in F$. Then $F(\sqrt{k}) \subseteq C$.*

Proof. We need to show each number of $F(\sqrt{k})$ is constructible. Since an arbitrary element of $F(\sqrt{k})$ is of the form $a + b\sqrt{k}$ where $a, b \in C$, it is constructible since $\sqrt{k}$ is constructible. $\square$

Theorem 4.5. *Let $\mathbb{Q} \subset F_1 \subset F_2 \subset \cdots \subset F_n$ be a sequence of fields such that*

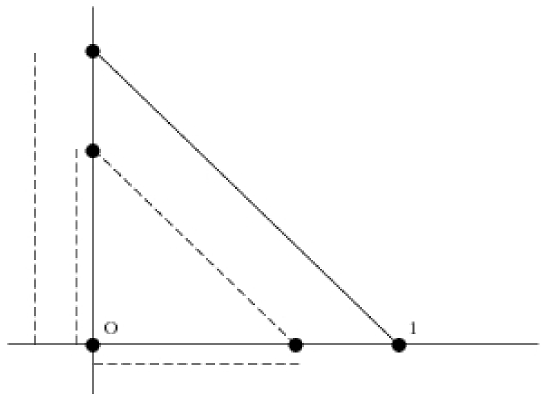

FIGURE 9. Construction of a/b

$$F_{j+1} = F_j(\sqrt{b_j}) \text{ and } 0 < b_j \in F_j \text{ for } j = 0, 1, \ldots, n-1.$$

Then all elements of F_n are constructible.

Proof. Apply induction on n.

□

Definition 4.6. *A tower of fields as in Theorem 4.5 is called a square root tower over $\mathbb{Q}$.*

Definition 4.7. *Let F be a field. Then F^2 is called the* **plane of F.** *Let $a, b, c \in F$ then the set $\{(x, y) \mid ax + by + c = 0\}$ is called* **a line in F^2** *and the set $\{(x, y) \mid x^2 + y^2 + ax + by + c = 0\}$ is called* **a circle in F^2.**

The proof of the next lemma is left as an exercise.

Lemma 4.8. *Let F be a subfield of $\mathbb{R}$.* (i) *The point of intersection, if any, of two lines in F^2 belongs to F^2.*

(ii) *The points of intersection of a line and a circle or two circles in F^2 lies in F^2 or $F(\sqrt{k})^2$ where $0 < k \in F$.*

Theorem 4.9. *A real number a is constructible if and only if there exists a square root tower $\mathbb{Q} \subset F_1 \subset \cdots \subset F_N$ such that $a \in F_N$.*

Proof. We have already proved that numbers in F_N are constructible. If a is constructible then $P = (a, 0)$ is a constructible point. We wish to show

$(a, 0) \in F_N^2$ where F_N is the last field in a square root over $\mathbb{Q}$. Beginning with $O = (0, 0)$ and $I = (1, 0)$, the point P is constructed in finite number of steps

$$S_0 = \{O, I\} \subset S_1 \subset \cdots S_m \subset \cdots$$

Let $P \in S_m$. Apply induction on m. If $m = 0$ then we are done. Let $m > 0$. By induction $S_{m-1} \subset F_N^2$, where F_N is the last field in a square root tower over $\mathbb{Q}$. The points in S_m are intersections of lines and circles in F_N^2. Hence they are in $F_N(\sqrt{k})^2$ for more $0 < k \in F_N$. Therefore P is in the plane of F_N. $\square$

Theorem 4.10. *Suppose a is a constructible real number, then*

$$[\mathbb{Q}(a) : \mathbb{Q}] = 2^m$$

for some $m \in \mathbb{N}$.

Proof. Let $\mathbb{Q} \subset F_1 \subset \cdots \subset F_N$ be a square root tower over $\mathbb{Q}$ and $a \in F_N$. Then

$$[\mathbb{Q}(a) : \mathbb{Q}][F_N : \mathbb{Q}(a)] = [F_N : \mathbb{Q}] = 2^N$$

Hence $[\mathbb{Q}(a) : \mathbb{Q}] = 2^m$ for some m. $\square$

Corollary 4.11. *It is impossible to duplicate a cube with ruler and compass.*

Proof. The number $\alpha = \sqrt[3]{2}$ is a root of the irreducible polynomial $x^3 - 2$ over $\mathbb{Q}$. Hence $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$ which is not a power of two. Therefore α is not constructible. $\square$

Corollary 4.12. *It is impossible to trisect an arbitrary angle θ with ruler and compass.*

Proof. Suppose the angle θ is given. We may assume $P = (\cos\theta, \sin\theta)$ is given along with $O = (0, 0)$ and $I = (1, 0)$. We wish to show that $(\cos\ \theta/3, \sin\ \theta/3)$ is not constructible. If so, then $\cos\ \theta/3$ and $\sin\ \theta/3$ are constructible real numbers. Using the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ we get $\cos\pi/3 = 4\cos^3\pi/9 - 3\cos\pi/9$. Therefore $u = \cos\pi/9$ satisfies $8u^3 - 6u - 1 = 0$. Hence $w^3 - 3w - 1 = 0$ where $w = 2u$. As $[\mathbb{Q}(w) : \mathbb{Q}] = 3$, u is not constructible. $\square$

Corollary 4.13. *It is impossible to square the unit circle by ruler and compass.*

Proof. Suppose it is possible to construct a segment a by ruler and compass such that $a^2 = \pi$. Then $a = \sqrt{\pi}$ is algebraic over $\mathbb{Q}$, hence so is π. But π is transcendental over $\mathbb{Q}$. Therefore $\sqrt{\pi}$ is not constructible by ruler and compass. $\square$