Lecture 6 : Symmetric Polynomials I

Objectives

(1) Examples of symmetric polynomials.
(2) The fundamental theorem of symmetric polynomials.
(3) Newton's identities for power sum symmetric polynomials.

Key words and phrases: Symmetric polynomial, symmetrization of a monomial, power sum symmetric polynomials, Newton's identities.

Our next goal is to prove the Fundamental Theorem of Algebra : Every polynomial of positive degree with complex coefficients has a complex root. You must have seen its topological and complex analytic proofs. We will present a proof which uses symmetric polynomials and the construction of the splitting field of a polynomial. We will learn about symmetric polynomials in this section and splitting fields of polynomials in the next section.

Let R be a commutative ring with identity and $S = R[u_1, u_2, \dots, u_n]$ be the polynomial ring in n variables over R. Let $\phi \in S_n$, the symmetric group of all permutations of $\{1, 2, \dots, n\}$. A permutation $\phi \in S_n$ gives rise to an automorphism $g_\phi : S \to S$, defined as

$$g_\phi(f(u_1, \dots, u_n)) = f(u_{\phi(1)}, \dots, u_{\phi(n)}).$$

Definition 6.1. *A polynomial $f \in S$ is called a* **symmetric polynomial** *if for all $\phi \in S_n$*

$$f(u_1, \dots, u_n) = f(u_{\phi(1)}, \dots, u_{\phi(n)}).$$

Example 6.2. (1) Consider the general polynomial

$$\begin{aligned} f(x) &= (x - u_1)(x - u_2)\dots(x - u_n) \\ &= x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \dots + (-1)^n \sigma_n \end{aligned}$$

where

$$\sigma_1 = u_1 + \cdots + u_n,\ \sigma_2 = \sum_{i<j} u_i u_j, \ldots, \sigma_n = u_1 u_2 \cdots u_n.$$

It is easy to verify that $\sigma_1, \ldots, \sigma_n$ are symmetric. These are called the **elementary symmetric polynomials** in $u_1, u_2, \ldots, u_n$.

(2) The symmetrization of a monomial $u_1^{\alpha_1} \ldots u_n^{\alpha_n}$ is defined as

$$S(u_1^{\alpha_1} \cdots u_n^{\alpha_n}) = \sum_{\alpha \in S_n} u_{\sigma(1)}^{\alpha_1} u_{\sigma(2)}^{\alpha_2} \cdots u_{\sigma(n)}^{\alpha_n}.$$

It is clear that $S(u_1^{\alpha_1} \cdots u_n^{\alpha_n})$ is a symmetric polynomial. The symmetrization of $u_1^2 u_2$ is

$$S(u_1^2 u_2) = u_1^2 u_2 + u_1^2 u_3 + u_2^2 u_3 + u_3^2 u_1 + u_3^2 u_2 + u_2^2 u_1.$$

(3) For each k the polynomials $w_k = u_1^k + u_2^k + \cdots + u_n^k$ are symmetric polynomials.

(4) Let h_m denote the sum of all monomials of degree m in $u_1, u_2, \ldots, u_n$. It is called the complete homogeneous symmetric polynomial of degree m.

Fundamental Theorem for symmetric polynomials

Example 6.3. Consider the symmetric polynomial

$$f(u_1, u_2, u_3) = u_1^2 u_2 + u_1^2 u_3 + u_2^2 u_1 + u_2^2 u_3 + u_3^2 u_1 + u_3^2 u_2.$$

Then $f(u_1, u_2, 0) = u_1^2 u_2 + u_2^2 u_1 = u_1 u_2 (u_1 + u_2) = \sigma_1^0 \sigma_2^0$, where

$$\sigma_1^0 = \sigma_1(u_1, u_2, 0) = u_1 + u_2 \text{ and } \sigma_2^0 = \sigma_2(u_1, u_2, 0) = u_1 u_2.$$

Consider $f - \sigma_1 \sigma_2 = g$. Then $g|_{u_3=0} = 0$. Thus $u_3 \mid g$. Since g is symmetric $u_1 u_2 u_3 = \sigma_3 \mid g$. This gives $f - \sigma_1 \sigma_2 = -3 u_1 u_2 u_3 = -3\sigma_3$ and therefore $f = \sigma_1 \sigma_2 - 3\sigma_3$.

Theorem 6.4 (Newton). *Let R be a commutative ring. Then every symmetric polynomial in $R[u_1, u_2, \ldots, u_n]$ is a polynomial in the elementary symmetric polynomials in a unique way. In other words if $f(u_1, u_2, \ldots, u_n)$ is symmetric then there exists a unique polynomial $g \in R[x_1, \ldots, x_n]$ such that*

$$g(\sigma_1, \sigma_2, \ldots \sigma_n) = f(u_1, u_2, \ldots, u_n).$$

Proof. Apply induction on n. The $n = 1$ case is clear. Let the theorem be true for symmetric polynomials in $n-1$ variables. To prove the theorem in $R[u_1, u_2, \dots, u_n]$, apply induction on $\deg f$. If $\deg f = 0$ then f is a constant. It is clear in this case. Consider $f(u_1, u_2, \dots, u_{n-1}, 0) = f^0 \in R[u_1, u_2, \dots, u_{n-1}]$. Then f^0 is symmetric. By induction hypothesis we have $f^0 = g(\sigma_1^0, \sigma_2^0, \dots, \sigma_{n-1}^0)$. Then $f - g(\sigma_1, \sigma_2, \dots, \sigma_{n-1}) = f_1$ is symmetric and $f_1(u_1, \dots, u_{n-1}, 0) = 0$. Thus $u_n \mid f_1$ and hence $\sigma_n \mid f_1$, by symmetry. So $f_1 = \sigma_n h(u_1, \dots, u_n)$. Since σ_n is not a zerodivisor in $R[u_1, \dots, u_n]$, h is symmetric. Since $\deg h < \deg f$, by induction hypothesis h is a polynomial in $\sigma_1, \dots, \sigma_n$, hence f is so. Therefore f is a polynomial in $\sigma_1, \dots, \sigma_n$.

Uniqueness : Use induction on n. the $n = 1$ case is obvious. Let us first prove that the map

$$\begin{gathered}\phi : S = R[z_1, z_2, \dots, z_n] \to R[\sigma_1, \sigma_2, \dots, \sigma_n] \text{ such that} \\ \phi(z_i) = \sigma_i,\ i = 1, 2, \dots, n \quad \text{and } \phi|_R = id_R\end{gathered}$$

is an isomorphism. If it is not an isomorphism, we pick a nonzero polynomial $f(z_1, z_2, \dots, z_n) \in S$ of least degree such that

$$f(\sigma_1, \sigma_2, \dots, \sigma_n) = 0.$$

Write f as a polynomial in z_n with coefficients in $R[z_1, z_2, \dots, z_{n-1}]$:

$$f(z_1, z_2, \dots, z_n) = f_0(z_1, z_2, \dots, z_{n-1}) + \dots + f_d(z_1, z_2, \dots, z_{n-1}) z_n^d.$$

Then $f_0 \neq 0$. If so, then $f = z_n g$ where $g \in S$. Then $\sigma_n g(\sigma_1, \dots, \sigma_n) = 0$. Hence $g(\sigma_1, \dots, \sigma_n) = 0$. This contradicts the minimality of $\deg f$. Therefore we have

$$0 = f_0(\sigma_1, \dots, \sigma_{n-1}) + \dots + f_d(\sigma_1, \dots, \sigma_{n-1}).\sigma_n^d.$$

In this relation put $u_n = 0$ to get

$$f_0((\sigma_1)_0, (\sigma_2)_0, \dots, (\sigma_{n-1})_0) = 0.$$

This is a nontrivial relation among the elementary symmetric polynomials in $u_1, u_2, \dots, u_{n-1}$. This is a contradiction. $\square$

Newton's identities for power sum symmetric polynomials

By the Fundamental Theorem for symmetric polynomials the symmetric polynomials $w_k = u_1^k + \dots + u_n^k,\ k = 1, 2, 3, \dots$ are polynomials in the elementary symmetric polynomials. Isaac Newton found identities which express w_k in terms of $\sigma_1, \sigma_2, \ldots, \sigma_n$.

Theorem 6.5 (**Newton**).

$$\begin{aligned} w_k &= \sigma_1 w_{k-1} - \sigma_2 w_{k-2} + \cdots + (-1)^k \sigma_{k-1} w_1 + (-1)^{k+1} \sigma_k k \;\; \textit{if} \;\; k \leq n, \\ &= \sigma_1 w_{k-1} - \sigma_2 w_{k-2} + \cdots + (-1)^{n+1} \sigma_n w_{k-n} \;\; \textit{if} \;\; k \geq n. \end{aligned}$$

Proof. Let z, y be indeterminate. Then

$$(y-u_1)(y-u_2)\cdots(y-u_n) = y^n - \sigma_1 y^{n-1} + \sigma_2 y^{n-2} + \cdots + (-1)^n \sigma_n$$

Put $y = 1/z$ to get

$$(1-u_1 z)(1-u_2 z)\cdots(1-u_n z) = 1 - \sigma_1 z + \sigma_2 z^2 + \cdots + (-1)^n \sigma_n z^n := \sigma(z)$$

Consider the generating function of $w_1, w_2, \ldots$

$$\begin{aligned} w(z) &= w_1 z + w_2 z^2 + w_3 z^3 + \cdots = \sum_{k=1}^{\infty} w_k z^k \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^{n} u_i^k z^k = \sum_{i=1}^{n} \sum_{k=1}^{\infty} (u_i z)^k \\ &= \sum_{i=1}^{n} \frac{u_i z}{1 - u_i z} \end{aligned}$$

Since $\sigma(z) = (1-u_1 z)\cdots(1-u_n z)$,

$$\sigma'(z) = -\sum_{i=1}^{n} \frac{u_i \sigma(z)}{1-u_i z} \text{ and hence } w(z) = \sum_{i=1}^{n} \frac{u_i z}{1-u_i z} = \frac{-z\sigma'(z)}{\sigma(z)}$$

This implies that

$$\begin{aligned} w(z)\sigma(z) &= -z(-\sigma_1 + \sigma_2(2z) - \sigma_3(3z^2) + \cdots + (-1)^n n \sigma_n z^{n-1}) \\ &= \sigma_1 z - 2\sigma_2 z^2 + 3\sigma_3 z^3 + \cdots + (-1)^{n+1} n \sigma_n z^n \end{aligned}$$

if $k \leq n$, equating the coefficient of z^k we get

$$(-1)^{k+1} k \sigma_k = w_k - \sigma_1 w_{k-1} + w_k - 2\sigma_2 + \cdots + (-1)^k w_1 \sigma_{k-1}.$$

Hence
$$w_k = \sigma_1 w_{k-1} - \sigma_2 w_{k-2} + \cdots + (-1)^{k+1}\sigma_k k.$$
If $k > n$, equate coefficient of z^k to get
$$w_k - w_{k-1}\sigma_2 - \cdots + (-1)^n \sigma_n w_{k-n} = 0.$$
Therefore
$$w_k = \sigma_1 w_{k-1} - \sigma_2 w_{k-2} + \cdots + (-1)^{n+1}\sigma_n w_{k-n}.$$
□