Module 4:Renewal Processes and Theory, Limit theorems in renewal theory

Lecture 16:Proof of Renewal Theory

The Lecture Contains:

Proposition 4.3

$m(t) < \infty$ for all $0 \le t < \infty$, where $m(t)$ is the renewal function.

Proof of Proposition 4.3

Since $0 \le P\{X_n = 0\} < 1$, then there must be some arbitrary α, such that $P\{X_n \ge \alpha\} > 0$. Define another renewal process $\{Y_n, n \ge 1\}$, such that $Y_n = \begin{cases} 0 & if\ X_n < \alpha \\ \alpha & if\ X_n \ge \alpha \end{cases}$ and also let $M_Y(t) = \sup\{n : Y_1 + \ldots Y_n \le t\}$

Assume we have the following illustration, Figure 4.3 for Proposition 4.3

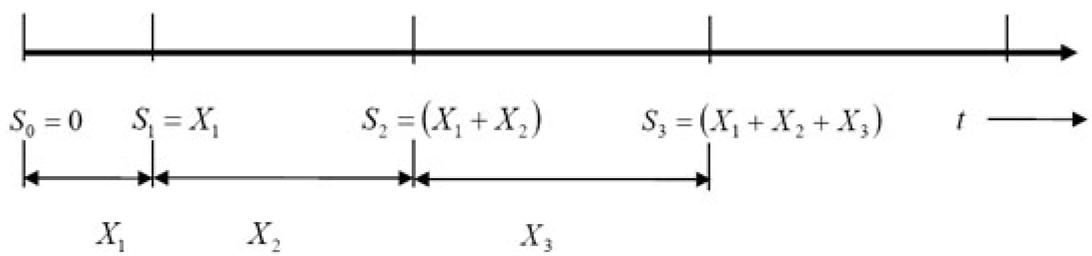

Figure 4.3: Schematic diagram to illustrate the concept of a renewal function and for the proof of Proposition 4.3

and suppose we can also write the following

For $n = 1$ if $X_1 < \alpha$, then $Y_1 = 0$, else $Y_1 = \alpha$

For $n = 2$ if $X_2 < \alpha$, then $Y_2 = 0$, else $Y_2 = \alpha$

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For $n = n$ if $X_n < \alpha$, then $Y_n = 0$, else $Y_n = \alpha$

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Thus Y_i is either 0 or α and this related renewal process which is $M_Y(t) = \sup\{n : Y_1 + \dots Y_n \le t\}$, can ***only*** take place at times $t = n\alpha$, where $n = 0,1,2,\dots$, Figure 4.4.

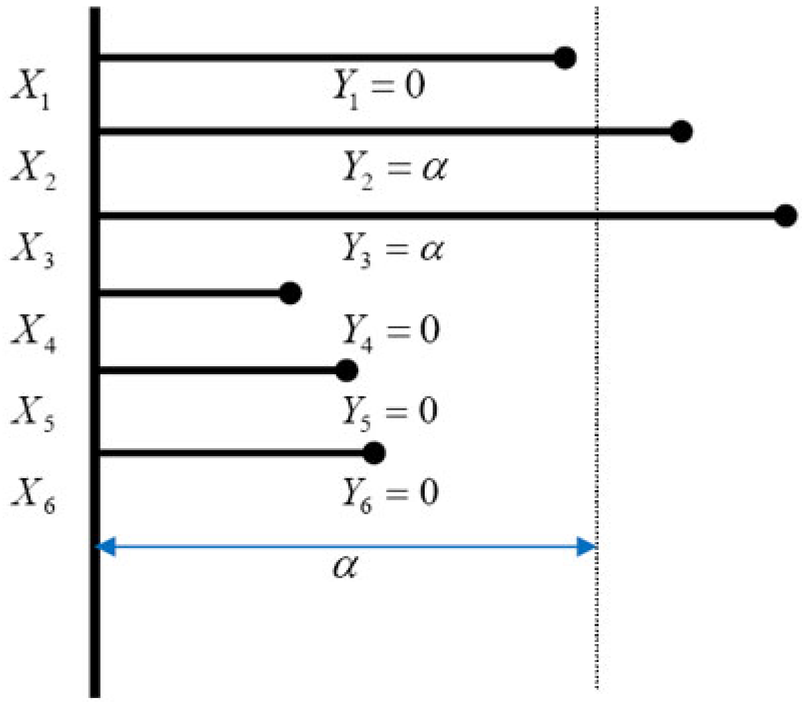

Figure 4.4: Schematic diagram to illustrate the concept of the new renewal process

Furthermore one can easily verify that $m(t) < \infty$ as $M(t) > N(t)$

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Let us denote $N(\infty)=\lim_{t\to\infty}N(t)$ as the total number of renewals that occur and then $N(\infty)=\infty$ with probability 1. It must be remembered that $N(t)$ is finite only when one of the inter arrival rates, i.e., X_i is infinite for any $i=1,2,\ldots$

Thus

$$P[N(\infty)<\infty]=P[X_n=\infty \ for\ some\ n]$$

$$=P\left\{\bigcup_{n=1}^{\infty}\{X_n=\infty\}\right\}$$

$$\leq\sum_{n=1}^{\infty}P\{X_n=\infty\}=0$$

Hence $N(t)$ goes to infinity as t also goes to infinity, but what is of more interest to us is the rate of increase of $N(t)$ with respect to t, i.e., we are interested to find $\lim_{t\to\infty}\left\{\frac{N(t)}{t}\right\}$.

So first consider the random variable $S_{N(t)}$ and the significance it has for determining our above answer. $S_{N(t)}$ represents the time for the $N(t)^{th}$ event and since there are only $N(t)$ number of events that have occurred before time t. $S_{N(t)}$ ***also*** represents the time for the last event, i.e., $N(t)^{th}$ event prior to or at time t. Thus one should clearly understand that $S_{N(t)}$ represents the time of the ***last renewal prior*** or at time t, while $S_{N(t)+1}$ represents the time of the ***first renewal after*** time t. Using a pictorial illustration we show this in Figure 4.5.

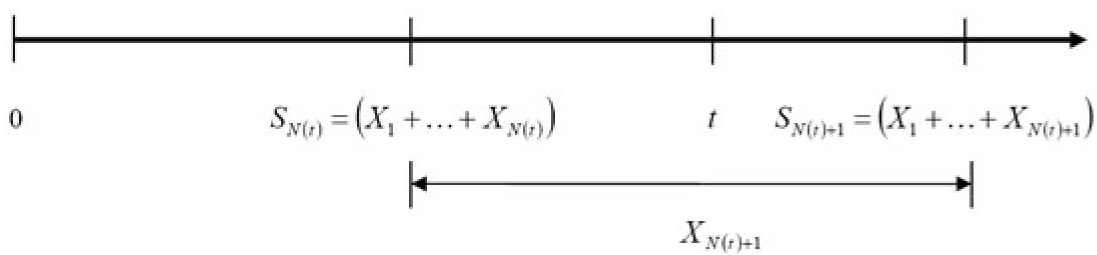

Figure 4.5: Schematic diagram to illustrate the concept of a last renewal and first renewal with respect to some time period t

Proposition 4.4

$$\lim_{t\to\infty}\left\{\frac{N(t)}{t}=\frac{1}{\mu}\right\}$$

Proof of Proposition 4.4

Now it is true that $S_{N(t)} \le t < S_{N(t)+1}$, such that we can also write it as $\frac{S_{N(t)}}{N(t)} \le \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$. Now watch carefully the first term which is $\frac{S_{N(t)}}{N(t)}$ and it is the average of first $N(t)$ inter arrival times and by the SLLN we have $\frac{S_{N(t)}}{N(t)} \to \mu$ as $N(t)\to\infty$. Now $N(t)\to\infty$ when $t\to\infty$, hence $\frac{S_{N(t)}}{N(t)} \to \mu$ as $t\to\infty$.

Also write $\frac{S_{N(t)+1}}{N(t)}=\left[\frac{S_{N(t)+1}}{N(t)+1}\right]\times\left[\frac{N(t)+1}{N(t)}\right]$. Again using similar logic we can deduce that $\frac{S_{N(t)+1}}{N(t)+1}\to\mu$ as $t\to\infty$ and separately we note that $\lim_{t\to\infty}=\left[\frac{N(t)+1}{N(t)}\right]=1$. Utilizing these we have $\frac{S_{N(t)+1}}{N(t)}\to\mu$ as $t\to\infty$.

Hence we have $\lim_{t\to\infty}\left\{\frac{N(t)}{t}=\frac{1}{\mu}\right\}$ as each of the left and right hand bounds separately converge to μ as $t\to\infty$.

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