The Lecture Contains:

- Waiting time distribution
- Few important and interesting properties of Poisson process
- Derivation of correlation coefficient of Poisson process
- Interarrival times conditional distribution
- Few important and interesting properties of interarrival times conditional distribution

Previous Next

Waiting time distribution

Now let us consider the distribution function of $S_n = X_1 + X_2 + \cdots + X_n$, which is the waiting time for the occurrence of n number of events (which is deterministic) in a probabilistic time frame, say t.

Now the n^{th} events occurs prior or at time t iff the number of events occurring by time t is at least n, i.e., the following holds true, i.e.,

$$N(t) \geq n \Leftrightarrow S_n \leq t$$

Thus

$$P\{S_n \leq t\} = P\{N(t) \geq n\}$$

i.e., $F_{S_n}(t) = 1 - F_{N(t)}(n)$, where $P\{S_n \leq t\} = F_{S_n}(t)$ and $P\{N(t) \geq n\} = 1 - F_{N(t)}(n)$

Thus:

$$f_{S_n}(t) = \lim_{\delta t \to 0}\left\{\frac{F_{S_n}(t+\delta t) - F_{S_n}(t)}{(t+\delta t) - \delta t}\right\} = \frac{d}{dt}F_{S_n}(t)$$

$$\begin{aligned} P\{t \leq S_n \leq t + \delta t\} &= P\{N(t) - n - 1, 1\ event\ in\ the\ time\ interval\ (t, t+\delta t)\} + o(t) \\ &= P\{N(t) - n - 1\}P\{1\ event\ in\ the\ time\ interval\ (t, t+\delta t)\} + o(t) \\ &= P\{N(t) - n - 1\} \times P\{1\ event\ in\ the\ time\ interval\ (t, t+\delta t)\} + o(t) \\ &= e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!} \times e^{-\lambda \delta t}\frac{\lambda \delta t}{1!} + o(t) \end{aligned}$$

Thus:

$$\frac{d}{dt}F_{S_n}(t) = f_{S_n}(t)$$

which implies $\lim_{\delta t \to 0}\left|\frac{\delta P\{S_n\}}{\delta t}\right| = \lim_{\delta t \to 0}\left|\frac{P\{S_n \leq t + \delta t\} - P\{S_n \leq t\}}{\delta t}\right|$

$$f_{S_n}(t) = \lim_{\delta t \to 0}\left[\frac{1}{\delta t}e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!} \times e^{-\lambda \delta t}\frac{\lambda \delta t}{1!} + \frac{1}{\delta t}o(t)\right]$$

$$= \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

Using the fact that X_n, $n = 1,2,3 \ldots$ are *i.i.d* exponential random variables with mean λ^{-1} we show S_n has gamma distribution with n and λ as the parameters, hence the probability density function is of the form $\lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$.

Note

- Remember that the exponential distributed discussed here is also referred to as the gamma distribution wit h parameters λ and 1.

- From the reproduction property of gamma distribution one notes that if $X_i \sim G(\lambda, 1)$, then $S_n \sim G(\lambda, n)$.

Previous Next

Few important and interesting properties of Poisson process

Property 2.1

- Suppose $\{X(t), t \geq 0\}$ is a ***Poisson process*** with rate λ. Furthermore assume that the probability
  of an event being recorded is p and $M(t)$ is the number of recorded events in $(0, t]$. Then
  $M(t)$ is a ***Poisson process***, i.e., $\{M(t), t \geq 0\}$, with rate λp.

Proof of property 2.1

$$P\{M(t) = n\} = \sum_{r=0}^{\infty} P\{M(t) = n, X(t) = n + r\}$$

$$= \sum_{r=0}^{\infty} P\{X(t) = n + r\} P\{M(t) = n | X(t) = n + r\}$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!} \times \binom{n+r}{n} p^n (1-p)^r$$

$$= e^{-\lambda t} \frac{(\lambda p t)^n}{n!} \sum_{r=0}^{\infty} \frac{\{\lambda t(1-p)\}^r}{r!}$$

$$= e^{-\lambda t} \frac{(\lambda p t)^n}{n!} e^{\lambda t(1-p)}$$

$$= e^{-\lambda t p} \frac{(\lambda p t)^n}{n!}$$

■

Property 2.2

- Suppose $\{X(t), t \geq 0\}$ is a ***Poisson process*** with rate λ. Furthermore consider events are of two types which are say
  Type I (male) and Type II (female), with corresponding probabilities of p and $(1-p)$.
  Thus we have $X(t) = X_1(t) + X_2(t)$
  and here $X_1(t)$ is of Type I while $X_2(t)$ is of Type II.
  Then both $X_1(t)$ and $X_2(t)$ will be ***Poisson process*** with rates
  λp and $\lambda(1-p)$ respectively and they would be ***independent***.

Proof of property 2.2

$$P\{X_1(t) = m, X_2(t) = n\} = \sum_{k=0}^{\infty} P\{X_1(t) = m, X_2(t) = n, X(t) = k\}$$

$$= P\{X(t) = m + n\} P\{X_1(t) = m, X_2(t) = n | X(t) = m + n\}$$

$$= \frac{e^{-\lambda t}(\lambda t)^{m+n}}{(m+n)!} \times \binom{m+n}{n} p^m (1-p)^n$$

$$= \frac{e^{-\lambda t p}(\lambda t p)^m}{m!} \times \frac{e^{-\lambda t(1-p)}\{\lambda t(1-p)\}^n}{n!}$$

$$= P\{X_1(t) = m\} \times P\{X_2(t) = n\}$$

Property 2.3

Suppose we have two ***independent Poisson process*** denoted by $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ with respective rates of λ_1 and λ_2, then $\{X(t), t \geq 0\}$ is a ***Poisson process*** with rate $(\lambda_1 + \lambda_2)$.

Proof of property 2.3

$$P\{X(t) = n\} = P\{X_1(t) + X_2(t) = n\}$$

$$= \sum_{r=0}^{\infty} P\{X_1(t) = r, X_2(t) = n - r\} = \sum_{r=0}^{\infty} P\{X_1(t) = r\} P\{X_2(t) = n - r\}$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{r!} \times \frac{e^{-\lambda_2 t}(\lambda_2 t)^{(n-r)}}{(n-r)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}\{(\lambda_1+\lambda_2)t\}^n}{n!} \sum_{r=0}^{n} \frac{n!(\lambda_1 t)^r (\lambda_2 t)^{n-r}}{\{(\lambda_1+\lambda_2)t\}^n r!(n-r)!}$$

Let us consider

$$\sum_{r=0}^{n} \frac{n!(\lambda_1 t)^r (\lambda_2 t)^{(n-r)}}{\{(\lambda_1+\lambda_2)t\}^n r!(n-r)!} = \sum_{r=0}^{n} \binom{n}{r} \times \left\{\frac{\lambda_1 t}{(\lambda_1+\lambda_2)t}\right\}^r \times \left\{\frac{\lambda_2 t}{(\lambda_1+\lambda_2)t}\right\}^{n-r}$$

$$\sum_{r=0}^{n} \binom{n}{r} \times \left\{\frac{\lambda_1}{\lambda_1 + \lambda_2}\right\}^r \times \left\{\frac{\lambda_2}{\lambda_1 + \lambda_2}\right\}^{n-r}$$

$$= \left\{\frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\right\}^n = 1$$

Thus

$$P\{X(t) = n\} = \frac{e^{-(\lambda_1+\lambda_2)t}\{(\lambda_1+\lambda_2)t\}^n}{n!}$$

■

Previous Next

Property 2.4

Difference of two ***independent Poisson process*** will ***NOT*** be ***Poisson process***.

Derivation of correlation coefficient of Poisson process

Suppose $\{X(t), t \geq 0\}$ is a ***Poisson process*** with rate λ, then the correlation coefficient between $X(t)$ and $X(t+s)$ is given by $\left\{\frac{t}{(t+s)}\right\}^{\frac{1}{2}}$.

Proof

We know that $E\{N(t)\} = \lambda t$, $V\{N(t)\} = \lambda t$, $E\{N(t+s)\} = \lambda(t+s)$ and $V\{N(t+s)\} = \lambda(t+s)$.

Furthermore we know that:

$$\rho_{X(t),X(t+s)} = \frac{cov\{X(t), X(t+s)\}}{\sqrt{V\{X(t)\} \times V\{X(t+s)\}}}$$

$$= \frac{E\{X(t), X(t+s)\} - E\{X(t)\} \times E\{X(t+s)\}}{\sqrt{V\{X(t)\} \times V\{X(t+s)\}}}$$

$$= \frac{E[X(t)\{X(t+s) - X(t) + X(t)\}] - \lambda t \times \lambda(t+s)}{\sqrt{\lambda t \times \lambda(t+s)}}$$

$$= \frac{E[X(t)\{X(t+s) - X(t)\} + X^2(t)] - \lambda t \times \lambda(t+s)}{\sqrt{\lambda t \times \lambda(t+s)}}$$

$$= \frac{E\{X(t)\} \times E\{X(s)\} + E\{X^2(t)\} - \lambda t \times \lambda(t+s)}{\sqrt{\lambda t \times \lambda(t+s)}}$$

$$= \frac{\lambda^2 ts + \lambda t + (\lambda t)^2 - (\lambda t)^2 - \lambda^2 ts}{\sqrt{\lambda t \times \lambda(t+s)}}$$

$$= \sqrt{\frac{t}{(t+s)}}$$

■

Note

This correlation is known as auto-correlation.

Previous Next

Interarrival times conditional distribution

Suppose that you are the marketing executive of firm selling specialized engineering lubricants. To reconsider your company's marketing strategy you are keen to understand the sales figures in details and hence count the number of times of the occurrence of your sales figure crossing a certain value, say Rs. $1{,}00{,}00{,}000$. Furthermore you know that the distribution of the sales figures crossing this value is Poisson distributed and that exactly one such event has occurred within time t, say till the month of June for the financial year starting April. Now we know that a Poisson process has the property of being ***stationary*** with ***independent increments***, hence in each interval of a 3 month period, i.e., $[0, 3]$, which is of equal length, the probability of the sales figure crossing Rs. $1{,}00{,}00{,}000$ is equal. Hence it would mean that for $s \le 3$ we would have

$$P\{X_1 < s|N(t) = 1\} = \frac{P\{X_1 < s, N(t) = 1\}}{P\{N(t) = 1\}}$$

$$= \frac{P\{1\ event\ in[0,s)\} \times P\{1\ event\ in[s,3)\}}{P\{N(t) = 1\}}$$

$$= \frac{\lambda s e^{-\lambda s} e^{-\lambda(s-s)}}{3\lambda e^{-s\lambda}}$$

$$= \frac{s}{3}$$

Few important and interesting properties of interarrival times conditional distribution

Property 1

In general we have $P\{X_1 < s|N(t) = 1\} = \frac{s}{t}$

Previous Next

Property 2

Thus given $N(t) = n$ the n arrival times denoted by $S_1, S_2, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed in the interval $(0, t)$.

Proof of property 2

Let $0 < t_1 < t_2 < \cdots . < t_{n+1} = t$ and also assume h_i are small increments of time such that $t_i + h_i < t_{i+1}, \; i = 1, \ldots, n$. This concept is explained in the following Figure 2.7.

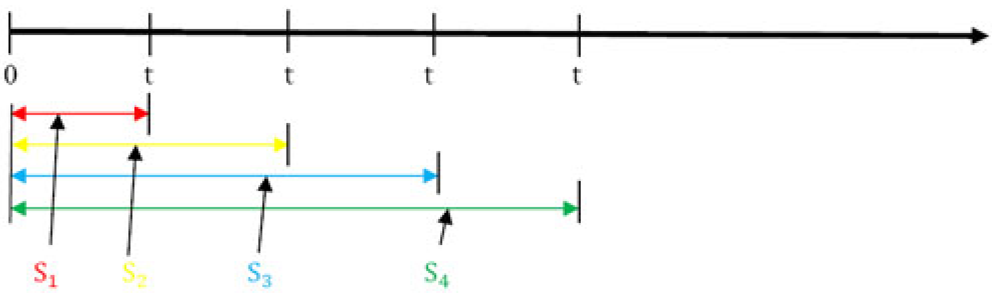

Figure 2.7: Concept of conditional distribution of arrival times

Hence:

$$P\{t_1 \le S_1 \le t_1 + h_1, t_2 \le S_2 \le t_2 + h_2, \ldots, t_n \le S_n \le t_n + h_n | N(t) = n\} =$$
$$P\{\text{exactly 1 event occurs in } [t_1, t_1 + h_1], \text{exactly 1 event occurs in } [t_2, t_2 + h_2], \ldots \ldots, \text{exactly 1 event occurs in } [t_n, t_n + h_n], 0 \text{ event occuts elsewhere in } [0, t]\}$$

$$P\{t_1 \le S_1 \le t_1 + h_1, t_2 \le S_2 \le t_2 + h_2, \ldots, t_n \le S_n \le t_n + h_n | N(t) = n\} = \frac{\lambda h_1 e^{-\lambda h_1} \times \lambda h_2 e^{-\lambda h_2} \times \ldots \times \lambda h_n e^{-\lambda h_n} \times e^{-\lambda(t - h_1 - h_2 - \cdots - h_n)}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

i.e.,

$$\frac{P\{t_1 \le S_1 \le t_1 + h_1, t_2 \le S_2 \le t_2 + h_2, \ldots, t_n \le S_n \le t_n + h_n | N(t) = n\}}{h_1 h_2 \ldots h_n} = \frac{n!}{t^n}$$
$$\lim_{h_1 \to 0, h_2 \to 0, \ldots, h_n \to 0} \left[\frac{P\{t_1 \le S_1 \le t_1 + h_1, t_2 \le S_2 \le t_2 + h_2, \ldots, t_n \le S_n \le t_n + h_n | N(t) = n\}}{h_1 h_2 \ldots h_n}\right] = \lim_{h_1 \to 0, h_2 \to 0, \ldots, h_n \to 0} \left\{\frac{n!}{t^n}\right\}$$

■

Previous Next