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Weiner process

Consider a particle which is undergoing ***Brownian motion*** performes a random walk such that as time changes from t to $t+\Delta t$, the position of the particle also changes from x to $x+\Delta x$. One should be aware that the total displacement of the particle in time t is x. Also suppose that the random variable Z_i denotes the length of the i^{th} step taken by the particle in the time interval of Δt, such that $P\{Z_i=\Delta x\}=p$ and $P\{Z_i=-\Delta x\}=(1-p)$ and p is independent of both x and t.

Now these $Z_i^{'}s$ are *i.i.d*. and assume you divide the interval length into n equal subintervals each of length Δt, such that $n\Delta t=t$. The total displacement $X(t)$is given by $X(t)=\sum_{i=1}^{n(t)} Z_i$

Hence:

$$E(Z_i)=(p-q)\Delta x \text{ such that } E\{X(t)\}=nE(Z_i)=\frac{t(p-q)\Delta x}{\Delta t}$$
$$V(Z_i)=4pq(\Delta x)^2 \text{ such that } V\{X(t)\}=nV(Z_i)=\frac{4pqt(\Delta x)^2}{\Delta t}$$

Now assume as $\Delta x\to 0$, $\Delta t\to 0$ then

$$\frac{(\Delta x)^2}{\Delta t}\to \text{constant and } (p-q)\to \text{a multiple of } (\Delta x) \quad (3.6)$$

Moreover consider

$$\lim_{\Delta x\to 0,\Delta t\to 0} E\{X(t)\}=\lim_{\Delta x\to 0,\Delta t\to 0}\frac{t(p-q)\Delta x}{\Delta t}=\mu t \quad (3.7)$$
$$\lim_{\Delta x\to 0,\Delta t\to 0} V\{X(t)\}=\lim_{\Delta x\to 0,\Delta t\to 0}\frac{4pqt(\Delta x)^2}{\Delta t}=\sigma^2 t \quad (3.8)$$

If (3.6), (3.7) and (3.8) are true it would mean that

$$\Delta x=\sigma(\Delta t)^{\frac{1}{2}},\ p=\frac{1}{2}\left\{1+\frac{\mu\sqrt{\Delta t}}{\sigma}\right\} \text{ and } q=\frac{1}{2}\left\{1-\frac{\mu\sqrt{\Delta t}}{\sigma}\right\} \quad (3.9)$$

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Module 3:Branching process, Application of Markov chains, Markov Processes with discrete and continuous state space

Lecture 12:Weiner Process

Note

- For large values of n, $X(t) \sim N(\mu t, \sigma^2 t)$
- $\{X(t)-X(s)\} \sim N\{\mu(t-s), \sigma^2(t-s)\}$
- $X(s)-X(0)$ and $X(s)-X(0)$ are mutually independent, which implies $X(t)$ is a ***Markov process***

Thus we say a stochastic process is ***Weiner-Einsten process*** (Figure 3.2) with ***drift parameter*** μ and ***variance parameter*** σ^2 if

1. For disjoint intervals (s,t) and (u,v) where $s \le t \le u \le v$, $X(t)-X(s)$ and $X(v)-X(u)$ are independent and this implies it is ***Markov process*** with independent increments (Figure 3.2).

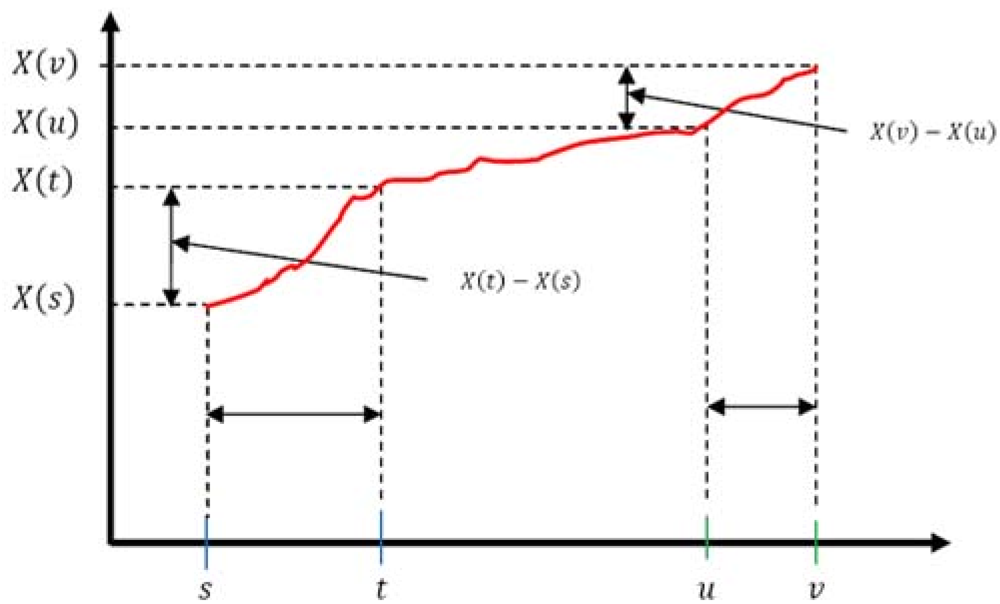

Figure 3.2: Illustration of Weiner-Einsten process to show that it has independent increments

2. $\{X(t)-X(s)\} \sim N\{\mu(t-s), \sigma^2(t-s)\}$ and this implies it is ***Gaussian***.

3. Since # 2 above is true hence the ***transition probability density function*** is given by

$$p(x_0, x; t) = P\{x \le X(t) \le x + dx | X(t) = x_0\} = \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{(x-x_0-\mu t)^2}{2\sigma^2 t}} dx$$

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Differential equations for Weiner process

Consider a ***Weiner process*** with $p(x_0, x; t) \times \Delta x$ as the transition probability considering it starts at x_0 when $t = 0$. We arbitrarily consider its state as x when time is $(t - \Delta t)$. Furthermore as time progresses to t, and then to $(t + \Delta t)$, the position of the particle changes to $x + \Delta x$ or $x - \Delta x$ or again back to x as illustrated in Figure 3.3.

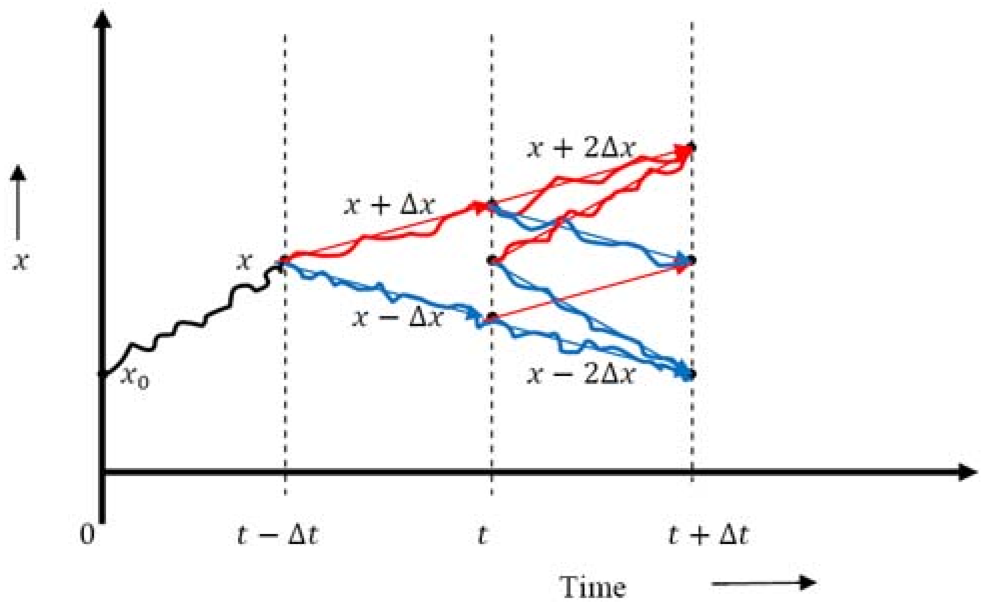

Figure 3.3: Illustration of Weiner-Einsten process in order to find the differential equations

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In order to find the differential equations of ***Weiner process*** we are interestd to find the infinitesimal shift in the particle's position as time progresses. Assume the probabilities of p and q are ***independent*** of x and t. Then using Taylor series expansion we have:

$$p(x_0,x\pm\Delta x;t-\Delta t) = p(x_0,x;t) + \frac{-\Delta t}{1!}\times\frac{\partial p(x_0,x;t)}{\partial t} + \frac{\pm\Delta x}{1!}\times\frac{\partial p(x_0,x;t)}{\partial x} + \frac{(-\Delta t)^2}{2!}\times\frac{\partial^2 p(x_0,x;t)}{\partial t^2} + \frac{(\pm\Delta x)^2}{2!}\times\frac{\partial^2 p(x_0,x;t)}{\partial x^2} + 2\frac{(-\Delta t)\times(\pm\Delta x)}{2!}\times\frac{\partial^2 p(x_0,x;t)}{\partial x\partial t} + \cdots.$$

Here we consider $\frac{\partial^2 p(x_0,x;t)}{\partial x\partial t} = \frac{\partial^2 p(x_0,x;t)}{\partial t\partial x}$

Thus:

$$p(x_0,x\pm\Delta x;t-\Delta t) = p(x_0,x;t) + \frac{-\Delta t}{1!}\times\frac{\partial p(x_0,x;t)}{\partial t} + \frac{\pm\Delta x}{1!}\times\frac{\partial p(x_0,x;t)}{\partial x} + \frac{(\pm\Delta x)^2}{2!}\times\frac{\partial^2 p(x_0,x;t)}{\partial x^2} + o(\Delta t) \qquad (3.10)$$

We also know that

$$p(x_0,x;t)\Delta x = p(x_0,x-\Delta x;t-\Delta t)\Delta x\times p + p(x_0,x+\Delta x;t-\Delta t)\Delta x\times q \qquad (3.11)$$

Using (3.9), (3.10) and (3.11) and after some simplification along with the fact that $\Delta x\to 0$, $\Delta t\to 0$ we obtain

$$\frac{\partial p(x_0,x;t)}{\partial t} = -\mu\frac{\partial p(x_0,x;t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 p(x_0,x;t)}{\partial x^2} \qquad (3.12)$$

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(3.12) is the ***forward diffusion equation*** of the Winer process and is of first order in t and of second order in x.

Similarly using

$$p(x_0,x \pm \Delta x;t+\Delta t) = p(x_0,x;t) + \frac{+\Delta t}{1!} \times \frac{\partial p(x_0,x;t)}{\partial t} + \frac{\pm \Delta x}{1!} \times \frac{\partial p(x_0,x;t)}{\partial x} + \frac{(+\Delta t)^2}{2!} \times \frac{\partial^2 p(x_0,x;t)}{\partial t^2}$$
$$+ \frac{(\pm\Delta x)^2}{2!} \times \frac{\partial^2 p(x_0,x;t)}{\partial x^2} + 2\frac{(+\Delta t)\times(\pm \Delta x)}{2!} \times \frac{\partial^2 p(x_0,x;t)}{\partial x \partial t} + \cdots.$$

Using which we have

$$\frac{\partial p(x_0,x;t)}{\partial t} = \mu \frac{\partial p(x_0,x;t)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 p(x_0,x;t)}{\partial x^2} \qquad (3.13)$$

(3.13) is the ***backward diffusion equation*** of the Winer process and is of first order in t and of second order in x.

Note

Till now we consider μ and σ^2 are ***independent*** of time.

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