Module 4:Renewal Processes and Theory, Limit theorems in renewal theory

Lecture 15:Renewal Theory Continued

The Lecture Contains:

- Renewal Theory
- What is Convolution
- Proposition
- Proof of Proposition

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Renewal Theory

We know that for a Poisson process, the inter arrival time (a some sort of ***counting process*** which counts the inter arrival time in this case) are ***independent*** and ***identically distributed*** (*i.i.d*) random variables (r.v.'s) where the underlying distribution is exponential in nature. Consider the ***counting process*** has an underlying distribioution and let it be denoted by any arbitrary distribution say, $F_X(x)$. Then this general counting process is termed as a ***renewal process***.

Before going into details of a ***renewal process***, let us formalize the definition of a counting process. Given any arbitrary distribution, let $\{X_n : n = 1,2 \ldots\}$ be a sequence of non-negative independent random variables (r.v.'s) (example being the time between the $(n-1)^{th}$ and the n^{th} event) with a common distribution, $F_X(x)$ such that $F_X(0) = P\{X_n = 0\} < 1$. Let the ***mean time between successive events*** be denoted by $\mu = E(X_n) = \sum_{x=0}^{\infty}\{x \times P(x)\}$. Now $X_n \geq 0$ and $F_X(0) = P\{X_n = 0\} < 1$, will ensure that $0 < \mu \leq \infty$.

Furthermore let us denote $S_n = \sum_{i=1}^{n} X_i = (X_1 + X_2 + \ldots + X_n)$, where $n \geq 1$, as the sum of the ***inter arrival times*** of n number of such events or better still as the time of the n^{th} event, with the added condition that $S_0 = 0$, Figure 4.1.

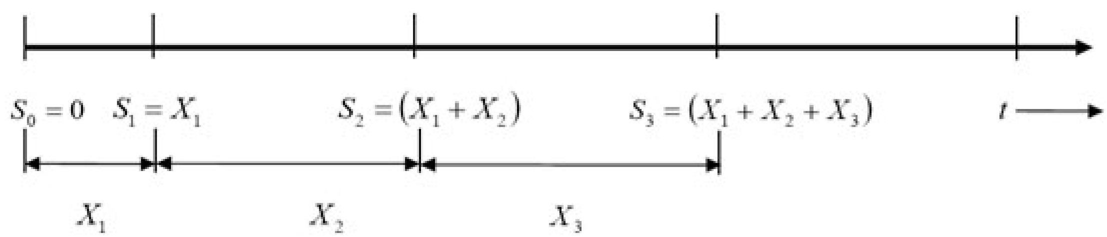

Figure 4.2: Schematic diagram to illustrate the concept of a renewal process using interarrival times and sum of interarrival times

Remember, the number of events by time t will equal the largest value of n for which the n^{th} event occurs before or at time t. We already have $N(t)$, as the number of events by time t which is given by $N(t) = \sup\{n : S_n \leq t\}$, and this $\{N(t), t \geq 0\}$ is what we call a ***renewal process***.

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Note

- Since the inter arrival times are *i.i.d*, hence at each renewal or event the process probability starts over again.
- There are finite numbers of renewals which can occur in a finite time, as by the ***strong law of large numbers (SLLN)*** we can show that $P\left\{\lim_{n\to\infty}\left(\frac{S_n}{n}\right)=\mu\right\}=P\left\{\lim_{n\to\infty}\left(\frac{X_1+\ldots+X_n}{n}\right)=\mu\right\}=1$. But since $0<\mu\leq\infty$, hence S_n must go towards infinity as n goes to infinity. Thus, S_n can be less than or equal to t for at most a finite number of values of n, hence $N(t)=\sup\{n: S_n\leq t\}$ must be finite and one can write $N(t)=\max\{n: S_n\leq t\}$

For a better understanding of the concept of ***renewal theory*** it is important that we find the distribution of $N(t)$, but before that one must note the important relationship that the number of renewals by time t is greater than or equal to n iff the n^{th} renewal occurs before or at time t. Hence we need to check the following theorem.

Theorem 4.1

$N(t)\geq n \Leftrightarrow S_n\leq t$

Proof 4.1

$$N(t)\geq n \Leftrightarrow S_n\leq t$$

i.e., $P\{N(t)=n\}=P\{N(t)\geq n\}-P\{N(t)\geq n+1\}$

$$=P\{S_n\leq t\}-P\{S_{n+1}\leq t\}$$

Now X_i, $i\geq 1$ are *i.i.d*, hence it follows that $S_n=\sum_{i=1}^{n}X_i$ is distributed F_n which is the n-fold convolution of F itself which is the distribution of X_i, $i\geq 1$.

Hence

$$P\{N(t)=n\}=F_n(t)-F_{n+1}(t)$$

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What is convolution

In general functional analysis convolution means that if we have two different functions, f and g producing a third function, then we denote it by $(f*g)(x)=\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau=\int_{-\infty}^{+\infty} f(t-\tau)g(\tau)d\tau$, where the operator denoted by $*$ is what is termed as convolution. In general the properties followed by convolution, considering there are two functions, f and g, are:

1. ***Commutative property***, i.e., $f*g=g*f$
2. ***Associative property***, i.e., $f*(g*h)=(f*g)*h$
3. ***Distributive property***, i.e., $f*(g+h)=(f*g)+(f*h)$
4. ***Associative property with scalar multiplication***, i.e., $\alpha(f*g)=(\alpha f)*g=f*(\alpha g)$

Let us now denote $m(t)=E[N(t)]=\sum_{n=1}^{\infty} n\times P\{N(t)=n\}$ as the ***renewal function*** and one of our main focus in ***renewal theory*** is to determine the properties of this ***renewal function***. A natural question which arises is how does one understand the importance of this renewal function.

Proposition 4.2

$$m(t)=E[N(t)]=\sum_{n=1}^{\infty} n\times P\{N(t)=n\}=\sum_{n=1}^{\infty} F_n(t)$$

Proof of Proposition 4.2

We can write

$N(t)=\sum_{n=1}^{\infty} I_n$, where $I_n=\begin{cases}1 & \text{if the } n^{th} \text{ renewal occurred in } [0,t]\\ 0 & \text{otherwise}\end{cases}$

Hence

$$\begin{aligned}E[N(t)]=E\left[\sum_{n=1}^{\infty} I_n\right]&=\sum_{n=1}^{\infty} E(I_n)\\ &=\sum_{n=1}^{\infty}\{1\times P(I_n=1)\}+\{0\times P(I_n=0)\}\\ &=\sum_{n=1}^{\infty} P(I_n=1)\\ &=\sum_{n=1}^{\infty} P(S_n\le t)=\sum_{n=1}^{\infty} F_n(t)\end{aligned}$$

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