The Lecture Contains:

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Wald's equation

An integer valued random variable (r.v) is said to be a ***stopping time*** for a sequence $\{X_i : i = 1, 2, \ldots\}$ if the event $\{N = n\}$ is ***independent*** of X_{n+1}, $X_{n+2}, \ldots$ for all $n = 1, 2, \ldots$. Thus one would observe $X_1, \ldots, X_n$ in a sequential order where N (remember it is a random variable (r.v)) denotes the number observed before one stops. Thus if $N = n_1$, then we have ***observed*** $X_1, \ldots, X_{n_1}$ and ***not observed*** $X_{n_1+1}, X_{n_1+2}, \ldots$

Let us illustrate this concept of stopping time with an example.

Example 4.5

Consider X_n, $n = 1, 2, \ldots$ be independent such that $P\{X_n = 0\} = P\{X_n = 1\} = \frac{1}{2}$, $n = 1, 2, \ldots$.Assume that we have $N = \min\{n : X_1 + \ldots + X_n = 7\}$, then this N is a ***stopping time***. Example can be when we keeping flipping an unbiased coin and stop the experiment the moment the number of tails is 7.

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Theorem 4.5 (Wald's Equations)

Consider a sequence $\{X_i : i = 1, 2, \ldots\}$ which are *i.i.d*, have a finite expectation and also assume that $\{N = n\}$ is a stopping time for $\{X_i : i = 1, 2, \ldots\}$, such that $E[N] < \infty$, then

$$E\left[\sum_{i=1}^{N} X_n\right] = E[N] \times E[X].$$

Proof of Theorem 4.5

Before we venture in trying to proof the above theorem note the interesting thing when both $\{X_i : i = 1, 2, \ldots\}$ and $\{N = n\}$ are probabilistic.

Let us consider the indicator function, I_n, where $I_n = \begin{cases} 1 & \text{if } N \geq n \\ 0 & \text{if } N < n \end{cases}$, such that we will have

$$\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} (X_n \times I_n).$$

Hence

$$E\left[\sum_{n=1}^{N} X_n\right] = E\left[\sum_{n=1}^{\infty} (X_n \times I_n)\right] = \sum_{n=1}^{\infty} E(X_n \times I_n)$$

We will stop if we successfully observe $X_1, \ldots, X_{n-1}$. Thus we do not terminate our experiment as and when we like but continue to observe and stop ***ONLY*** at the last reading which is X_n. Therefore I_n is independent of X_n.

We thus obtain

$$E\left[\sum_{n=1}^{N} X_n\right] = \sum_{n=1}^{\infty} E(X_n) \times E(I_n)$$

$$= E(X) \sum_{n=1}^{\infty} E(I_n)$$ (remember here the suffix n is not relevant as it can be any n)

$$= E(X) \sum_{n=1}^{\infty} P\{N \geq n\}$$

$$= E(X) E(N)$$

■

Let denote the inter arrival time of a renewal process. In this experiment let us

$\{X_i : i = 1, 2, \ldots\}$

consider we stop at the ***first renewal*** after t, i.e., at the $N(t)+1$ renewal.

Note

$$N(t)+1 = n \Leftrightarrow N(t) = n-1$$

$$\Leftrightarrow X_1 + \ldots + X_{n-1} \le t \text{ and } X_1 + \ldots + X_{n-1} + X_n > t$$

Thus the event $\{N(t)+1 = n\}$ depends only on $X_1 + \ldots + X_n$ and is definitely independent of $X_{n+1}, X_{n+2}, \ldots$ such that $N(t)+1$ is the stopping time

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Example 4.6

Suppose a container contains an infinite collection of coins (may be biased or unbiased). Now whether the coin is biased or unbiased we still would have the probability lying between 0 and 1. Imagine you start flipping coins one after the other sequentially and our objective is to maximize the long run proportion of flips that land on heads.

Let us denote $N(n)$ as the number of tails in the first n flips of the coins, so if P_H denotes the proportions of heads in the long run, then we must have the following $P_H = \lim_{n\to\infty}\left[\frac{n-N(n)}{n}\right] = 1-\lim_{n\to\infty}\left[\frac{N(n)}{n}\right]$. Let us reformulate the experiment in a way such that we achieve the same thing but in a different way. Let us choose the first coin and keep flipping the coin and at the same time we also keep counting the number of heads we get. We stop the time we get the first tail. Hence the moment the tail is obtained for the first time we can discard the first coin and pick up the second coin and keep a note that $N(1)$ is the number of heads obtained using the first coin. On a similar line we continue flipping the second coin till we get the first tail for the first time, i.e., the moment the first tail is obtained from the second coin we discard that coin and note the number of heads obtained by this second coin as $N(2)$. We continue doing this experiment, such that intuitively we have $\lim_{n\to\infty}\left[\frac{N(n)}{n}\right] = \frac{1}{E[\#\,of\ \ flips\ between\ successive\ tails]}$.

Suppose $P(X=H)=p$ and $P(X=T)=q$ and remember we now concentrate on the simple concept of a geometric distribution of the form $G\left(\frac{1}{1-p}\right)$, where the mean is $\left(\frac{1}{1-p}\right)$.

It is true that

$$E[\#\,of\ \ flips\ between\ successive\ tails] = \int_{p=0}^{p=1}\frac{1}{(1-p)}dp = \infty\ ,$$

thus $\lim_{n\to\infty}\left[\frac{N(n)}{n}\right] = \frac{1}{E[\#\,of\ \ flips\ between\ successive\ tails]} = 0$ with probability 1.

Example 4.7

One can easy proof the renewal equation $m(t=)F(t)+\int_0^t m(t-xdF(x))$

Example 4.8

We can show that the renewal function $m(t)$, $0\le t<\infty$ uniquely determines the interarrival distribution F.

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Example 4.9

Suppose $\{N(t), t \geq 0\}$ be a renewal process and suppose that for all n and t, conditional on the event that $N(t)=n$, the events $S_1, \ldots, S_n$ are distributed as the order statistics of a set of independent uniform $(0,t)$ random variables. Then we can easily prove that $\{N(t), t \geq 0\}$ is a Poisson process.

For few more concepts of renewal theory and to dwell further into the concept of ***stopping time for the renewal process***, let us consider $X_1, X_2, \ldots$ as the inter arrival time of a renewal process. Now for this renewal process we should stop the first renewal after t, i.e., at the $[N(t)+1]^{th}$ renewal. What we need to verify is the fact that $N(t)+1$ is indeed ***a*** stopping time for this sequence $X_1, X_2, \ldots$, i.e., X_i, $i = 1, 2, \ldots$

Note

Notice I have made an emphasis on the word ***a***. This is due to the fact that depending on what is needed to be found, the stopping time will vary. Like if you are interested to find the mean for say a general distribution, $F(x)$, then the concept and the values of stopping time for find the mode will not be same as the former case when one is interested to find the mean.

Again continuing with our problem we have

$$N(t)+1 = n \Leftrightarrow N(t) = n-1$$

$$\Leftrightarrow X_1 + X_2 + \ldots + X_{n-1} \leq t \text{, i.e., } X_1 + X_2 + \ldots + X_n > t$$

Now see that the event $\{N(t)+1=n\}$ depends on $X_1, X_2, \ldots, X_n$ and ***definitely not on*** $X_{n+1}, X_{n+2}, \ldots$, hence obviously $N(t)+1$ is a stopping time which depends on $X_1, X_2, \ldots, X_n$ and ***definitely not*** on $X_{n+1}, X_{n+2}, \ldots$. From Wald's equation we know that when $E(X) < \infty$ we would have $E\left[X_1 + \ldots + X_{N(t)+1}\right] = E(X)E[N(t)+1]$

Thus the following corollary is true, which is

Corollary 4.6

If $\mu < \infty$, then $E\left[S_{N(t)+1}\right] = \mu \times [m(t)+1]$

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