The Lecture Contains:

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Properties of Brownian Motion

One denotes the Brownian motion by $\{X(t): t\epsilon T\}$ where in general $T = \mathbb{R}$, and the following properties hold:

a. Given $t_0 < t_1 < \cdots . < t_n$, $\{X(t_1) - X(t_0)\}, \{X(t_2) - X(t_1)\}, \ldots.,$ $\{X(t_n) - X(t_{n-1})\}$ are mutually independent $\forall n$ and $\forall t$. Such a process has independent moments.

b. For any t, $X(t+h) - X(t)$, $h > 0$ depends only on h and not t. Which means the process has stationary increments, Figure 6.4.

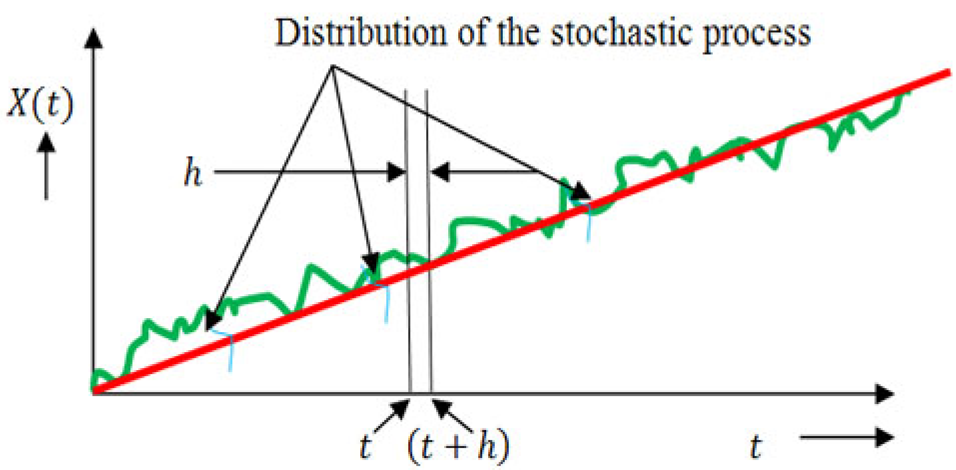

Figure 6.4: A hypothetical example of a stationary process with respect to variance

c. $P[\{X(t+h)-X(t)\} \leq x] = \frac{1}{\sqrt{2\pi\sigma^2 h}} \int_{u=-\infty}^{x} exp\left(\frac{-u^2}{2\sigma^2 h}\right) du$. If the mean is zero, and the variance is $\{X(t+h)-X(t)\} \sim N(0, \sigma^2 h)\, \sigma^2 h$, it would imply that . Thus the distribution looks as shown below, Figure 6.5.

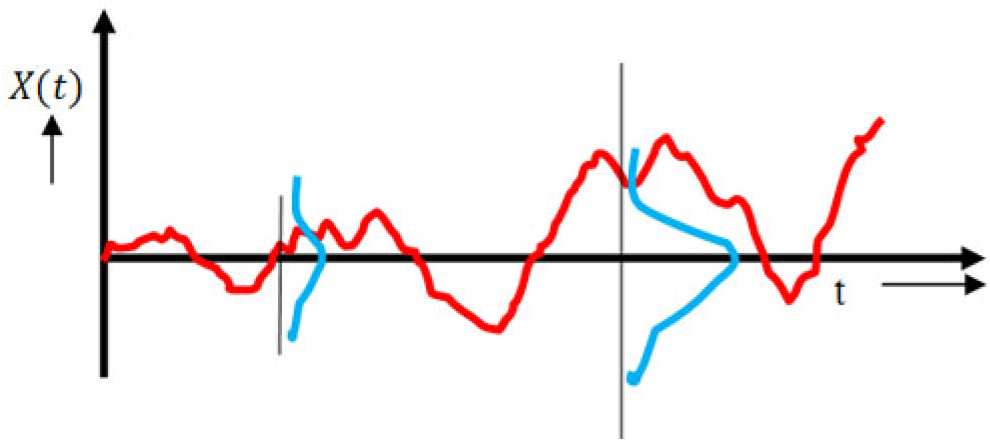

Figure 6.5: A hypothetical example of stationary process with respect to mean but not with respect to variance

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Furthermore it can be shown that given $t > t_n$ we have :

$P\{X(t) \leq x | X(t_1) = x_1, \ldots, X(t_n) = x_n\} = P\{X(t) \leq x | X(t_n) = x_n\}$, which when depicted using diagrams looks like that shown in Figure 6.6. In simple word it implies that it is the present value which only dictates the future and the values from the past have no bearing on the future. Thus it is only the present which effects the future.

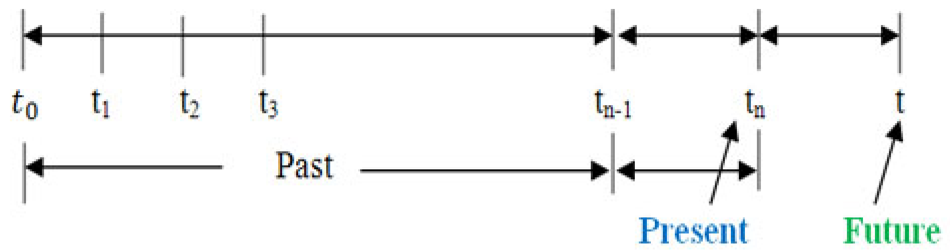

Figure 6.6: A hypothetical example of present affecting the future

Thus given what ever has happened in the past, only the present, i.e., t_n will effect the future, i.e., t, where $t > t_n$. Hence the distribution of the future, given the past and present is equal to the distribution of the future given the present only and this is what we mean by the no-memory process.

Few important facts about Brownian motion

1. Sample paths every where are continuous, but no where are they differentiable.
2. Any level may be attained with probability 1.
3. The expected time taken to reach any level is ∞.

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