Module 2:Poisson Process and Kolmorogov equations

Lecture 6:Derivation of Poisson Process

The Lecture Contains:

- Derivation of Poisson Process
- Interarrival time distribution

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Derivation of Poisson Process

Let us define $p_m(t) = P\{X(t) = m\}$, which means that m events occur in time t.

Now $P\{N(t+h) = 0\} = P\{N(t) = 0\} \times P\{N(h) = 0\}$, means the probability of the number of events in the time frame $(t+h)$ being zero is equal to the product of the probabilities that the number of events occurring in time frame t and h each is zero. ***This is true as the property of independent increments holds true for the counting process and the Poisson process we consider here***.

In the conceptual sense this time h is taken to be infinitesimally small, i.e.,$h \longrightarrow 0$ (Figure 2.4).

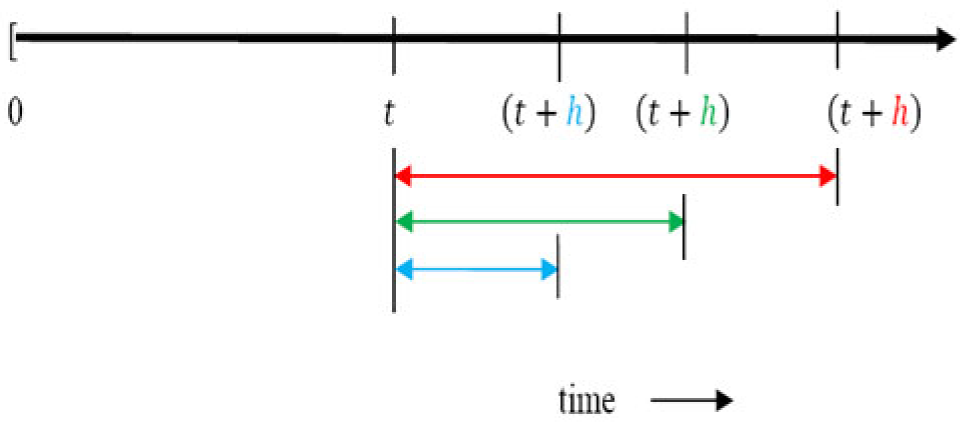

Figure 2.4: The concept of time h as it becomes zero

Hence

$$P\{N(t+h) = 0\} = P\{N(t) - N(0) = 0\} \times P\{N(t+h) - N(h) = 0\}$$

$$P\{N(t+h) = 0\} = P\{N(t) = 0\} \times [1 - \lambda h + o(h)]$$

$$\frac{P\{N(t+h) = 0\} - P\{N(t) = 0\}}{h} = -\lambda P\{N(t) = 0\} + \frac{o(h)}{h}$$

$$\lim_{h \to 0}\left\{\frac{P\{N(t+h) = 0\} - P\{N(t) = 0\}}{h}\right\} = \lim_{h \to 0}\left\{-\lambda P\{N(t) = 0\} + \frac{o(h)}{h}\right\}$$

$$\frac{\partial P\{N(t) = 0\}}{\partial t} = -\lambda P\{N(t) = 0\}$$

Solving this using simple integration furnishes us with $P\{N(t) = 0\} = e^c \times e^{-\lambda t}$, where c is the constant of integration

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Note

The boundary conditions gives us $P\{N(0) = 0\} = 1$, such that when we utilize this we have $e^c = 1$,

hence

$$P\{N(t) = 0\} = e^{-\lambda t} \qquad (2.1)$$

Now let us consider $n \geq 1$, such that the following one can easily visualize the following

Number of events in $(0, t+h]$	Number of events in $(0, t]$	Number of events in $(t, t+h]$
n	n	0
n	$(n-1)$	1
n	$(n-2)$	2
. .	. .	. .
n	1	$(n-1)$
n	0	n

Table 2.1: Breakup of occurrence of events in the time intervals $(0, t+h] = (0, t] + (t, t+h]$

Which implies the following holds where $\underline{i \geq 2}$

$$\begin{aligned} P\{N(t+h) = n\} &= P[\{N(t) - N(0)\} = n, \{N(t+h) - N(t) = 0\}] \\ &+ P[\{N(t) - N(0)\} = n-1, \{N(t+h) - N(t) = 1\}] \\ &+ P[\{N(t) - N(0)\} = n-i, \{N(t+h) - N(t) \geq i\}] \\ &= P\{N(t) = n\} \times P\{N(h) = 0\} + P\{N(t) = n-1\} \times P\{N(h) = 1\} \\ &+ \cdots + P\{N(t) = 0\} \times P\{N(h) = n\} \\ &= P\{N(t) = n\} \times P\{N(h) = 0\} + P\{N(t) = n-1\} \times P\{N(h) = 1\} \\ &+ \underbrace{\sum_{i=2}^{n} P\{N(t) = n-i\} \times P\{N(h) = i\}}_{o(h)} \end{aligned}$$

Using the assumptions under definition # 1 and definition # 2 we have:

$$P\{N(t+h) = n\} = (1 - \lambda h) P\{N(t) = n\} + \lambda h P\{N(t) = n-1\} + o(h)$$

Hence:

$$\lim_{h \to 0} \left[\frac{P\{N(t+h) = n\} - P\{N(t) = n\}}{h} \right] = \lim_{h \to 0} \left[-\lambda P\{N(t) = n\} + \lambda P\{N(t) = n-1\} + \frac{o(h)}{h} \right]$$

$$\frac{d}{dt} \left\{ e^{\lambda t} P\{N(t) = n\} \right\} = \lambda e^{\lambda t} P\{N(t) = n-1\}$$

This is a differential equation in time, but a difference equation in state. As usual we are interested to know about the initial conditions and they are $P\{N(t)=n\}=\begin{cases}1 & if\ n=0\\ 0 & if\ n>0\end{cases}$

For $n = 1$ we have

$\frac{d}{dt}\left\{e^{\lambda t} P\{N(t) = 1\}\right\} = \lambda$ (using (2.1))

i.e., $P\{N(t) = 1\} = (\lambda t + c)e^{-\lambda t}$

But $c = 0$ as $P\{N(0) = 0\} = 0$, hence:

$$P\{N(t) = 1\} = \lambda t e^{-\lambda t} \qquad (2.2)$$

Using mathematical induction we have:

$$P\{N(t) = n\} = e^{-(\lambda t)} \frac{(\lambda t)^n}{n!}$$

In general $E\{X(t)\} = \lambda t, V\{X(t)\} = \lambda t$ and both are not independent of t. This type of process can be said to be an ***evolutionary process***, thus ***Poisson process*** is not a ***stationary process*** but rather an ***evolutionary process***.

It is important to understand that $P\{N(t+s) - N(s) = n\} = e^{-(\lambda t)} \times \frac{(\lambda t)^n}{n!}$, and it gives us the same meaning when we assume $s \to 0$. Another important fact worth mentioning is that a ***Poisson process*** is a ***Markov Process*** with the transition probability being ***independent*** of the past.

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Note

Let T denote the time from 0 until the ***first event occurs***, then we have $e^{-\lambda t} = P\{T > t\}$, i.e., .

$$1 - e^{-\lambda t} = P\{T \leq t\}$$

Thus we have

$$\lambda e^{-\lambda t} = P\{N(t) = 1\} + P\{N(t) = 0\}$$

$$\lambda e^{-\lambda t} = P\{N(t) = 1\} = f_T(t)$$

We are already aware that $N(t+h) = m$, which denotes that in the time interval $(0, t+h]$ the event occurs m number of times. Thus diagrammatically it is as follows which is illustrated in Figure 2.5.

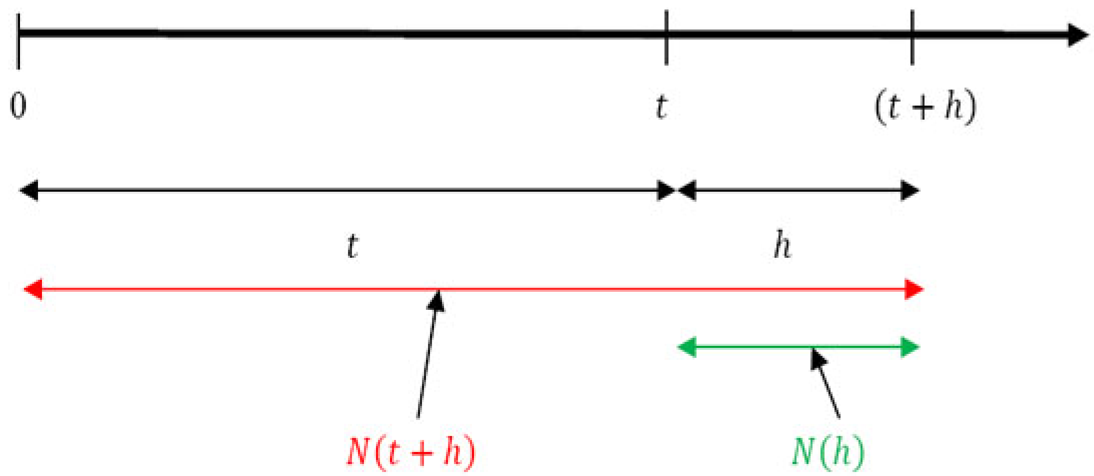

Figure 2.5 : The notational concept of time , and the number of events within each time intervals (which is denoted using colour scheme for ease of understanding)

Interarrival time distribution

For a Poisson process let us denote X_1 as the ***time*** for the occurrence of the 1st event, i.e., the ***time between the occurrence*** between the 1st and the 0^{th} event. Extending this notion we have X_n as the time between the occurrence the n^{th} and $(n-1)^{th}$ event. Then this sequence $\{X_n, n \geq 1\}$ is what we call as the the ***sequence of interarrival times***.

As an illustration one can refer to Figure 2.6, which points out the fact that the occurrence of the 1st, 2nd, 3rd,…. events (i.e., event number) are fixed, ***but when each of the individual event would occur is not known to us*** and hence the time interval between two consecutive occurrences of events say i^{th} and $(i+1)^{th}$ is stochastic.

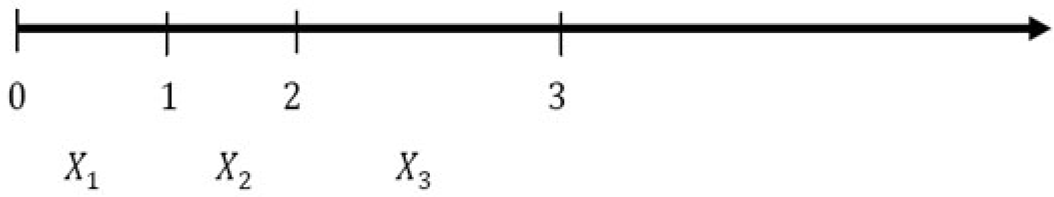

Figure 2.6: Illustration of interarrival time which is stochastic

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Note

- In Figure 2.6, the event number which is ***fixed*** is in increments of 1
- In Figure 2.6, the time between occurrence of events is ***non-deterministic***

Now what is of relevance to us is the distribution of X_n. In order to obtain the distribution of X_n we should note the following. In case $\{X_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$ it means that no events occur in the time interval $[0, t]$ and X_1 has exponential distribution with a mean value of λ^{-1}.

Now to find the distribution of X_n, $n = 2,3, \dots$ this gives us:

$$P\{X_2 > t|X_1 = s\} = P\{0\, events\, occur\, in\, (s, s+t]|X_1 = s\} = e^{-\lambda t}$$

$$P\{X_3 > t|X_2 = s\} = P\{0\, events\, occur\, in\, (s, s+t]|X_2 = s\} = e^{-\lambda t}$$

Hence X_n, $n = 1,2,3 \dots$ are *i.i.d* exponential random variables with mean λ^{-1}

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