Stochastic Differential Equation

Consider $H(t) = \int_{s=0}^{t} \mu(s)\,ds$. Moreover partition the interval $[0, t]$ into $0 = t_0 < t_1 < \cdots.. < t_n = t$ and define $H_n(t) = \sum_{i=1}^{n} \mu(t_i^*)[t_i - t_{i-1}]$, where $t_i^* \in [t_{i-1}, t_i]$.

Now by Rieman Integral we know that $H(t) = \lim_{\substack{n\to\infty \\ \delta_n \to 0}} H_n(t)$, where $\delta_n = \max_{1\le i\le n}|t_i - t_{i-1}|$. Moreover $I(t) = \int_{s=0}^{t} \sigma(s)dF(s)$.

Using the fundamentals mentioned above we can easily obtain the following which is $\frac{I_n(t+\varepsilon)-I_n(t)}{\varepsilon} = \sigma(t)\left\{\frac{F(t+\varepsilon)-F(\varepsilon)}{\varepsilon}\right\}$ and also define $\frac{dX(t)}{dt} = \mu(t)dt + \sigma(t)dF(t)$

Its Calculus

Without any loss of generality assume $X(0) = 0$ and define $X(t) = \int_0^t \mu\{X(s), s\}ds + \int_0^t \sigma\{X(s), s\}dZ(s)$ where $Z(.)$ is the standard Brownian motion.

Furthermore we may write $dX(t) = \mu\{X(t), t\}dt + \sigma\{X(t), t\}dZ(t)$.

Using some calculation we can show that the BS model differential equations is

$$\frac{1}{2}\sigma^2(X,t)\frac{\partial^2 p}{\partial x^2} + \mu(x,t)\frac{\partial p}{\partial x} = -\frac{\partial p}{\partial t}$$

Previous Next

Black-Scholes Model

Important : In Black-Scholes model the main assumption is the fact that $Ln(P)\sim$Brownian Motion process, i.e., $P(t)\sim$log normal distribution, where $P(t)$ is the price of stocks.

The main parameters for the BS model are:

1. S_0 = Stock price and this is known
2. K = strike price and this is known
3. r = interest rate and this is known
4. τ = time period and this is known
5. σ= volatility and this is unknown

One should remember that the main component in the model is $\sigma^2\tau$, but the bottle neck is the fact that σ which is the volatility is stochastic.

Two key assumptions is Black Scholes Model

1. Log normal prices
2. Volatility constant i.e., it is independent of time

Now recall that $\int_\alpha^\beta f(x)dx = F(\beta) - F(\alpha)$ and consider that $I = \int_0^a \frac{2}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}} dx$, if $W = \frac{x}{\sqrt{t}}$ which implies that $dw = dx/\sqrt{t}$

$$\therefore I = \int_0^{a/\sqrt{t}} \frac{2}{\sqrt{2\pi}} e^{-w^2/2} dw$$

$$\therefore \frac{\partial}{\partial t}\left[\int_0^{a/\sqrt{t}} \frac{2}{\sqrt{2\pi}} e^{-w^2/2} dx\right] = \frac{\partial}{\partial t}\left[H(a/\sqrt{t})\right] = H^{'}\left(\frac{a}{\sqrt{t}}\right)\frac{\partial}{\partial t}\left(a/\sqrt{t}\right)$$
$$= \left\{\frac{2}{\sqrt{2\pi}} e^{-a^2/2t}.\right\} a - \frac{1}{2}t^{-3/2}$$

$$\therefore g(t) = \begin{Bmatrix} \frac{a}{\sqrt{2\pi t^3}} e^{\frac{-a^2}{2t}} & t > 0 \\ 0 & otherwise \end{Bmatrix}$$, assuming $\sigma^2 = 1$

Now generally if σ^2 is there, then we have the following

$$g(t) = \begin{Bmatrix} \frac{a}{\sqrt{2\pi\sigma^2 t^3}} exp\left\{-\frac{(a-\mu t)^2}{2\sigma^2 t}\right\} & t > 0 \\ 0 & otherwise \end{Bmatrix}$$

NOTE

- If there is no drift then the expected time to reach a is ∞.
- If there is drift then the expected time to reach a is finite

Previous Next

Suppose at $t = 0$ the process is at the origin and let $p(k, n)$ denote the probability of being K step up after n trials

$\therefore P(k,n) = P_{0k}^{n} = P_{0,k-1}^{n-1} P_{k-1,k} + P_{0,k+1}^{n-1} P_{k+1,k}$ which is easily understood from Figure 10.13

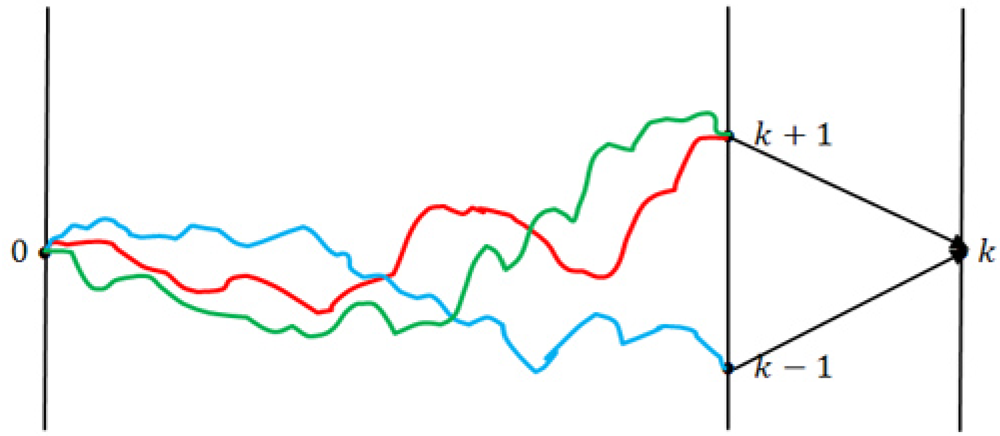

Figure 10.13: Movement of stock price according which can be considered as random

Thus:

$$p(K,n) = \frac{1}{2} p(K-1, n-1) + \frac{1}{2} p(K+1, n-1)$$

Let the time between step be Δ and let the jump size be η. Here $\frac{\eta^2}{\Delta}$ which is a constant is equal to σ^2. Hence we have $p \frac{(x,t+\Delta)-p(x,t)}{\Delta} = \frac{1}{2} \left[\frac{p(x+n,t)-2p(x,t)+p(x-n,t)}{\Delta}\right]$, i.e., $p \frac{(x,t+\Delta)-p(x,t)}{\Delta} = \frac{1}{2}\sigma^2 \left[\frac{p(x+n,t)-2p(x,t)+p(x-n,t)}{\eta^2}\right]$.

First set $x = k\eta$ and $n\Delta = t$ and let us also consider $\Delta \to 0$. Then we can easily prove that $\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2}$

Previous Next

Using first order expansion we have

$$C(S,t)\times(1+r\Delta)=pC(S+\eta S,t+\Delta)+(1-p)C(S-\eta S,t+\Delta)$$

$$=\frac{1}{2}\{C(S+\eta S,t+\Delta)+C(S-\eta S,t+\Delta)\}$$

$$+\frac{r\Delta}{2\eta}\{C(S+\eta S,t+\Delta)-C(S-\eta S,t+\Delta)\}$$

$$C(S,t)\times(1+r\Delta)-C(S,t\Delta)=\frac{1}{2}\{C(S+\eta S,t+\Delta)-2C(S,t+\Delta)+C(S-\eta S,t+\Delta)\}$$

$$+\frac{r\Delta}{2\eta}\{C(S+\eta S,t+\Delta)-C(S-\eta S,t+\Delta)\}$$

Let us now first divide both sides of the equations with Δ and take limits as $\Delta\rightarrow 0$. Furthermore we also divide and simultaneously multiply the first term and the second term on the right hand side of equality by $(\eta S)^2$ and $2\eta S$ respectively. This results in the following form which is as follows

$$\frac{1}{2}\frac{\partial^2 C(S,t)}{\partial S^2}\sigma^2 S^2+rS\frac{\partial C(S,t)}{\partial S}+\frac{\partial C(S,t)}{\partial t}-rC(S,t)=0$$

This is the fundamental ***partial differential equation*** (PDE) for pricing derivatives under the underlying assumption that logarithmic price of the underlying financial asset has ***Brownian motion***.

In case one is interested to understand how the theoretical relationship between and varies then Figure gives a fair idea how that behaves.

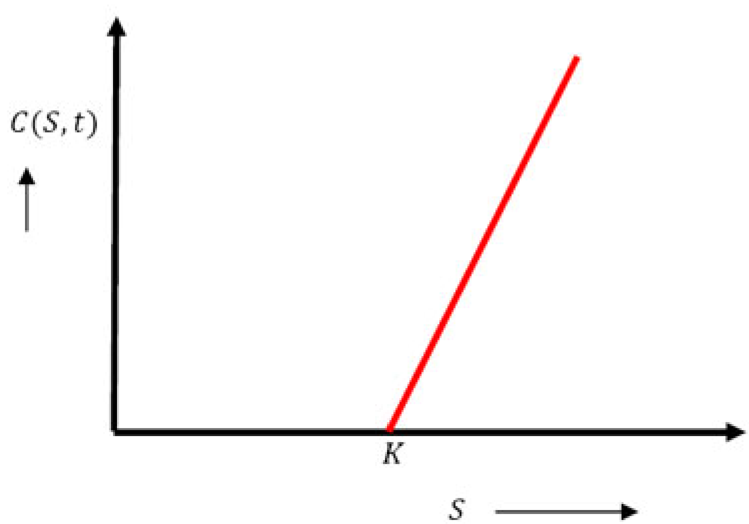

Figure 10.12: Illustration for PDE for pricing derivatives

Recall from Markov Chain Theory that $\rho_{ij}^n=\sum_{n=0}^{\infty}P_{ik}^{n-1}P_{kj}$. Now consider a simple random walk with equal probability of a step up and step down i.e., $\frac{1}{2}$

Module 10:Application of stochastic processes in areas like finance

Lecture 36:Black-Scholes Model

Now let us extend these concepts just discussed for the case when we have the figure given as

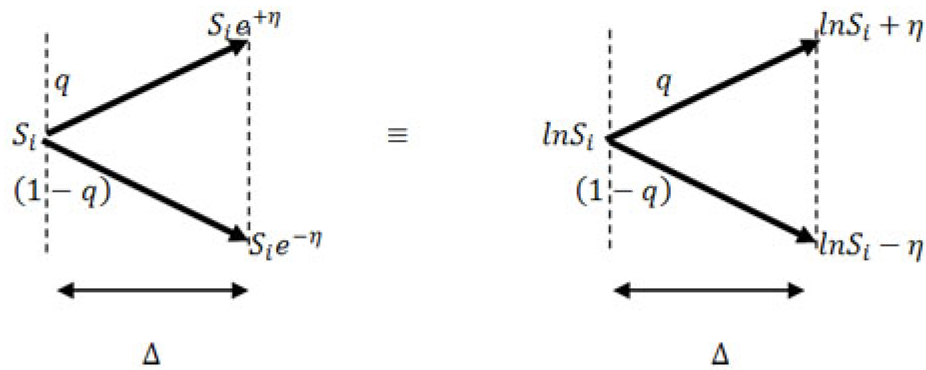

Figure 10.11: A stock and its increase and decrease of price

For the Figure we select Δ and η to be small, such that $\frac{\eta^2}{\Delta} = \sigma^2$ (a constant), and this σ^2 is variance per unit time. Hence it turns out that this ***random walk*** converges to a ***Brownian motion*** as $\Delta \rightarrow 0$.

Thus: $S_i e^{r\Delta} = S_i p e^{\eta} + S_i(1-p)e^{-\eta}$ where η and r have their usual meaning.

Using Taylor series expansion we get $r\Delta = p \times \eta + (1-p) \times (-\eta) = 2p\eta - \eta$.
Which results in

$$p = \frac{r\Delta}{2\eta} + \frac{1}{2}$$

Now when time is taken as 1 we have the following $C_i = e^{-r}\left[p \times C_{i+1}^{n+1} + (1-p) \times C_{i-1}^{n+1}\right]$, as true. In the general case when the time period is Δ, the equation takes the following form which is

$$C_i = e^{-r\Delta}\left[p \times C_{i+1}^{n+1} + (1-p) \times C_{i-1}^{n+1}\right].$$

Previous Next

Thus we have the following

$$(S_i + \xi^* C_i^n)e^r = S_i e^\eta + \xi^* C_{i+1}^{n+1} = S_i + \xi^* C_i^{n+1} = S_i e^{-\eta} + \xi^* C_{i-1}^{n+1}$$

Now if we have only $i+1$ and $i-1$ movement then the equation reduces to

$$(S_i + \xi^* C_i^n)e^r = S_i e^\eta + \xi^* C_{i+1}^{n+1} = S_i e^{-\eta} + \xi^* C_{i-1}^{n+1}$$

Thus we should have an unique value of p, say p^* such that the following holds true: $S_i e^r = S_i p e^\eta + S_i(1-p)e^{-\eta}$. In fact it can be easily proved for the case when we have only two sates, i.e., j and $i-1$, then $p^* = \frac{e^r - e^\eta}{e^r - e^{-\eta}}$.

Furthermore combining we can easily see that

$$\left(S_i e^\eta + \xi^* C_{i+1}^{n+1}\right) \times p^* + \left(S_i e^{-\eta} + \xi^* C_{i-1}^{n+1}\right) \times (1 - p^*)$$

Thus we have

$\xi^* C_i^n e^r = \xi^* \left\{p^* \times C_{i+1}^{n+1} + (1-p^*) \times C_{i-1}^{n+1}\right\}$ and as $\xi^* \neq 0$ one easily obtains

$$C_i^n = e^{-r}\left\{p^* \times C_{i+1}^{n+1} + (1-p^*) \times C_{i-1}^{n+1}\right\}$$

In general $p^* \neq q$ and it reflects the risk premium example. In case $p^* = q$, then we operate in the risk free environment and in that case p^* is termed as the ***risk-neutral probability measure***. In general using this value of p^* (whether risk free or not is immaterial) we can obtain the discounted expected value of the future claim.

Previous Next

The Lecture Contains:

Previous Next