Module 4:Renewal Processes and Theory, Limit theorems in renewal theory

Lecture 18:Renewal Process Continued

The Lecture Contains:

Previous Next

Theorem 4.7

In case for a renewal process $\frac{m(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$

Proof 4.7

Step # 1

Suppose that $\mu_X < \infty$ holds and also let us refer to the illustration given in **Figure 4.6**.

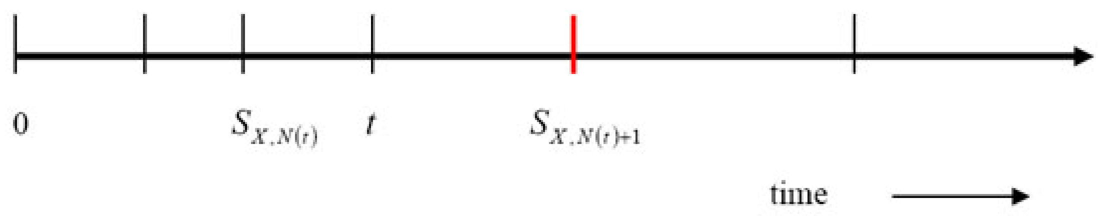


Figure 4.6: Schematic diagram to illustrate the concept asymptotic rate of progress of and its effect on average rate of occurrence

If you see Figure 4.6 carefully then we know that $S_{X,N(t)+1} > t$. So let us take the expectations for which we have

$$E\left[S_{X,N(t)+1}\right] > E[t]$$

$$E\left[S_{X,N(t)+1}\right] = \mu_X \times \left[m_X(t)+1\right] > E[t]$$

$$\text{i.e., } \liminf_{t\to\infty} \left[\frac{m_X(t)}{t}\right] \geq \frac{1}{\mu_X}$$

Now we need to show that $\lim_{t\to\infty}\left[\frac{m_X(t)}{t}\right] = \frac{1}{\mu_X}$, so let us prove the less than inequality, i.e., $\leq$.

Previous Next

Step # 2

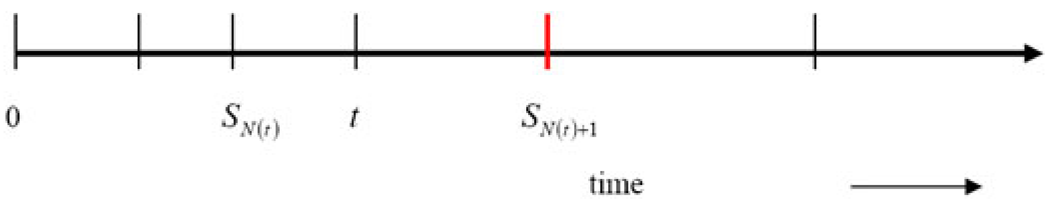
Define another renewal process Y_n , $i = 1,2\ldots$ and fix a constant M , such that we have

$$Y_n = \begin{cases} X_n & if\ X_n \le M \quad n = 1,2,\ldots \\ M & if\ X_n > M \end{cases}$$

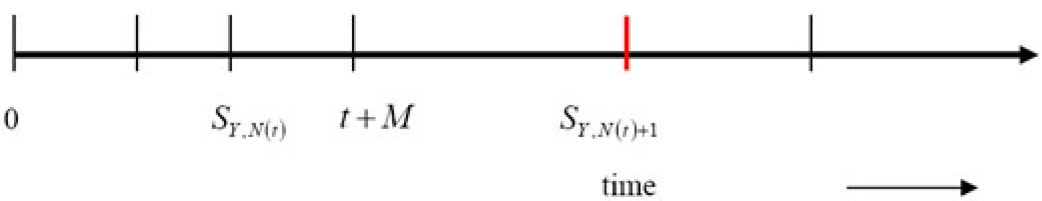
Let $S_{Y,n} = \sum_{i=1}^{n} Y_i$ and $N_Y(t) = \sup\{n : S_{Y,n} \le t\}$. Now if X_n is the inter arrival time, then by definition Y_n is also inter arrival time, but which has been truncated by the value of M, such that $S_{Y,N(t)+1} \le t + M$, i.e.,

$S_{Y,N(t)+1} = \sum_{i=1}^{N(t)+1} Y_i \le t + M$.

For X_i and $S_{X,n}$



For Y_i and $S_{Y,n}$



Thus from $S_{Y,N(t)+1} = \sum_{i=1}^{N(t)+1} Y_i \le t + M$ we have $[m_Y(t)+1]\mu_Y \le t + M$, hence $\lim_{t\to\infty} \sup\left[\frac{m_Y(t)}{t}\right] = \frac{1}{\mu_Y}$

.

Now as $S_{Y,n} = \sum_{i=1}^{n} Y_i \le S_{X,n} = \sum_{i=1}^{n} X_i$ and it would immediately follow that $N_Y(t) \ge N_X(t)$ and

$m_Y(t) \geq m_X(t)$, hence $\lim_{t\to\infty} \sup\left[\frac{m_X(t)}{t}\right] \leq \frac{1}{\mu_Y}$ and as $M \to \infty$, then $\mu_Y = \mu_X$, thus we have $\lim_{t\to\infty} \sup\left[\frac{m_X(t)}{t}\right] \leq \frac{1}{\mu_X}$.

Hence using $\lim_{t\to\infty} \inf\left[\frac{m_X(t)}{t}\right] \geq \frac{1}{\mu_X}$ and $\lim_{t\to\infty} \sup\left[\frac{m_X(t)}{t}\right] \leq \frac{1}{\mu_X}$ we immediately have $\frac{m(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$ ■

Remember in the sequential problem we are always interested to find the average value of N , i.e., as per the first order asymptotic property we should have $\lim_{f(N)\to 0} E(N) = D$ or else more generally and strongly we should have the second order asymptotic property as $\lim_{f(N)\to 0}\left[\frac{E(N)}{D}\right] = 1$.

Previous Next

Delayed Renewal Process

In many practical situation one may face a situation where the first ***interarrival time*** has a different distribution from subsequent values. Thus as per the definition given $\{X_n, n = 1,2\ldots\}$ is a sequence of independent non-negative random variables with X_1 having the distribution F_1, while X_n, $n > 1$ has the distributionF_2. Let $S_0 = 0$, $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$, then we define $N_D(t) = sup\{n: S_n \leq t\}$, such that the ***stochastic process*** $\{N_D(t), t \geq 0\}$ is called a ***general*** or a ***delayed renewal process***.

Properties of delayed renewal process

1. $P\left[\lim_{t\to\infty}\left\{\frac{N_D(t)}{t}\right\} \to \frac{1}{\mu}\right] = 1$
2. $P\left[\lim_{t\to\infty}\left\{\frac{E[N_D(t)]}{t}\right\} \to \frac{1}{\mu}\right] = 1$

Note

When $F_1 = F_2$, then we have the normal ***renewal process***.

Previous Next