Module 3:Branching process, Application of Markov chains, Markov Processes with discrete and continuous state space

Lecture 10:Application of Markov Chains

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Variance

Let Z be the sum of random number of random variables, i.e., $Z=\sum_{i=1}^{Y} X_i$, where Y is the random number and $X_i's$ are the random variables which are *i.i.d*.

Similarly we find the variance

$$V(X_n) = V\{E(X_n|X_{n-1})\} + E\{V(X_n|X_{n-1})\} \qquad (3.1)$$

Now $V(X_n|X_{n-1}) = V\left(\sum_{j=1}^{n-1} Y_j \middle| X_{n-1}\right) = X_{n-1} \times V(Y_i) = \sigma^2 \times X_{n-1}$, hence

$E\{V(X_n|X_{n-1})\} = \sigma^2 \times E(X_{n-1})$. Using these results and replacing them in (3.1) we have

$$\begin{aligned} V(X_n) &= V(X_{n-1} \times \mu) + \sigma^2 \times E(X_{n-1}) \\ &= \mu^2 V(X_{n-1}) + \sigma^2 \mu^{n-1} \\ &= \mu^2\{\sigma^2\mu^{n-2} + \mu^2 V(X_{n-2})\} + \sigma^2\mu^{n-1} \\ &= \sigma^2\mu^n + \sigma^2\mu^{n-1} + \mu^4 V(X_{n-2}) \\ &= \sigma^2\mu^{n-1} + \sigma^2\mu^n + \cdots . + \mu^{2n} V(X_0) \\ &= \sigma^2\mu^{n-1}(1 + \mu + \mu^2 + \cdots . + \mu^{n-1}) \text{ as } V(X_0) = 0 \end{aligned}$$

Thus

$$V(X_n) = \begin{cases} \sigma^2\mu^{n-1}\left(\frac{\mu^n - 1}{\mu - 1}\right) & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$$

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Note

In financial time series (specially stock prices) either the 1st, $\Delta X = (X_t - X_{t-1})$ or the 2nd, $\Delta X^2 = \{(X_t - X_{t-1}) - (X_{t-1} - X_{t-2})\}$, differences generally produce stationary time series, which in many cases lead to the fact that the changes are *i.i.d.* In time series we can utilize many of the techniques like GARCH, A-GARCH, E-GARCH, PC-GARCH, etc., to first find whether the series are in a sense *i.i.d*. In that case one should then utilize the fact that given the mean and variance of the first stage what are the subsequent mean and variances values.

Now let us take this example one step further, where π_0 denotes the probability that starting with only one individual the population dies out, i.e., $\pi_0 = Pr\{population\ dies\ out\}$, so that we have $\pi_0 = \sum_{j=0}^{\infty} Pr\{\text{population dies out}|X_i = j\} \times Pr\{X_1 = j\} = \sum_{j=0}^{\infty} Pr\{\text{population dies out}|X_i = j\} \times p_j$.

Logically the whole population will eventually die *iff* each of the families started by the members of the first generation eventually die (as we assume that generation of each member are being reproduced independently), hence we will obtain $\pi_0 = \sum_{\forall i} \pi_0^i \times \mathrm{p}_i$ (which is true as the concept of independence holds true).

Note

The probability of ultimate extinction of the branching process means $P\{X_n = 0\} = 1$, thus $X_n = 0$ implies that $X_m = 0\ \forall m > n$.

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Definition

Consider $X_0, X_1, \ldots$ are the random variables that denote the size of the 0^{th} generation, 1st generation, etc respectively, Figure 3.1. Also assume that an individual in any generation produce k number of off-springs and the probability be p_k, $k = 0,1,2\ldots$. It is quite simple to note that $\sum_{\forall k} p_k = 1$. Then the sequence, $X_0, X_1, \ldots$, i.e., $\{X_n, n = 0,1,2\ldots.\}$ is the Galton Watson branching process.

In the Galton Watson branching process let us assume $Z_n = X_0 + X_1 + \cdots + X_n$, where as per definition Z_n is the total number of progeny upto to the n^{th} generation.

Then:

- $E(Z_n) = E(X_0 + X_1 + \cdots + X_n) = 1 + \mu + \mu^2 + \cdots. + \mu^n = \frac{1-\mu^{n+1}}{1-\mu}$ if $\mu \neq 1$
- $E(Z_n) = E(X_0 + X_1 + \cdots + X_n) = 1 + \mu + \mu^2 + \cdots. + \mu^n = n + 1$ if $\mu = 1$
- $E(Z_n) = E(X_0 + X_1 + \cdots + X_n) = 1 + \mu + \mu^2 + \cdots. + \mu^n = \frac{1}{1-\mu}$ if $\mu < 1$ and $n \to \infty$

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Extinction probability

Let us now try to find the ultimate extinction probability.
Assume:

$$\pi_0 = \Pr\{\text{the population will ultimately die, assuming } X_0 = 1\}$$
$$= \lim_{n\to\infty} \Pr\{X_n = 0 | X_0 = 1\}$$

Now we know that $\mu^n = E(X_n) = \sum_{\forall i} i \times Pr\{X_n = i\} \geq \sum_{i=1}^{\infty} Pr\{X_n = i\} = Pr\{X_n \geq 1\}$

$$\Rightarrow Pr\{X_n \geq 1\} \leq \mu^n$$

So $Pr\{X_n \geq 1\} \to 0$ iff $\mu < 1$

$$\Rightarrow \lim_{n\to\infty} \Pr\{X_n = 0\} = 1$$

Thus $\Pr\{X_n = 0|X_0 = 1\} \to 1$ if $\mu < 1$ and it can also be shown that $\Pr\{X_n = 0|X_0 = 1\} \to 1$ if $\mu = 1$, i.e., $\pi_0 = 1$ if $\mu \leq 1$
Thus

$$\pi_0 = \Pr\{\text{ultimate extinction}\}$$
$$= \sum_{\forall i} \Pr\{\text{ultimate extinction}|X_i = i\} \times P\{X_i = i\}$$
$$= \sum_{\forall i} \pi_0^i \times p_i = \pi_0^0 \times p_0 + \pi_0^1 \times p_1 + \cdots . + \pi_0^n \times p_n$$

Thus when $\mu > 1$ then π_0 will be the smallest positive root of the equation $\pi_0 = \sum_{\forall i} \pi_0^i \times p_i$.

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