Lecture 30:Examples of application of stochastic processes in Queueing Theory

The Lecture Contains:

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Example 7.9

(a) Consider a single queue model where only one customer is allowed in the system. If another customer comes and finds the queue busy then he/she goes away and never returns, while is the queue is empty he/she approaches the server and is served. Assume the arrival distribution is Poisson with mean value of λ per unit time and the service time is Exponential with a mean value of $\left(\frac{1}{\mu}\right)$ time units. With this information

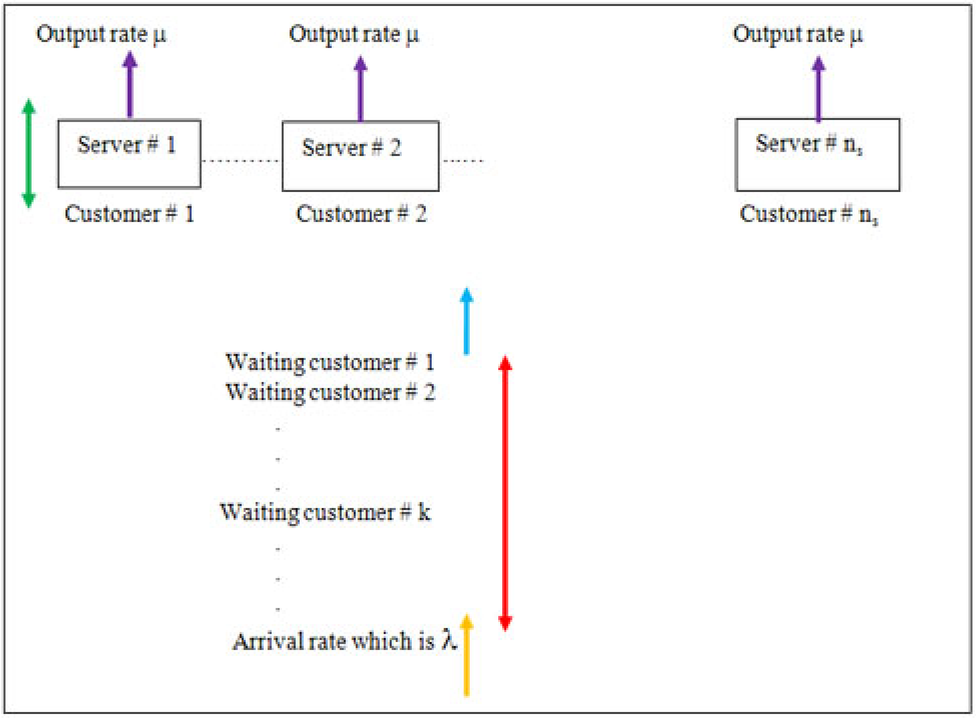
(i) Set up the transition diagram and determine the balance equations.
(ii) Determine the steady state probabilities.
(iii) Determine the average number in the system.

(b) Can you solve the same set of three (3) question as asked for Assignment # 3 (above) for the case when only two (2) customers are allowed in the sytem.

Specialized Poisson Queues

Here we have one (1) queue and n_s number of servers, i.e., a good look at the diagram below will illustrate the set up very clearly.

Thus we have the $\underline{n_s}$ number of server system, but only with ***one (1)*** queue having $\underline{n_q}$ number of people in the queue at any point of time. The arrival rate λ is the number of customers per unit time which is constant for the ***whole system***, while the service rate for the different servers are μ_i, $i = 1,2,3,\ldots,n_s$. For simplicity we consider they are the same, i.e., $\mu_1 = \mu_2 = \cdots \mu_n = \mu$. One should be aware that we make a distinction between the system and the queue, where the former implies the whole set up as such, while queue is the part of the system from where people/machines are being sent to the servers for the processing operation



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The green arrow depicts the ***server system*** or ***server facility***, which has ns number of servers and equals the number of ns customers who are being processed by the respective servers. While on the other hand the red arrow depicted portion signifies the ***queue system*** and there are some number of people standing in the queue, which we consider as nq. So in general if we consider the whole ***system*** as inside the box, then we have at any point of time (ns + nq) number of people in the system.
We must note that we will always consider the steady state condition, i.e., it is not a death or birth process, hence under this condition we have to define few metric or measure which are important to understand the efficiency of the system, i.e., the system in general. The following are the important measure and they are as given below

$L_s = \sum_{n=1}^{\infty}(n * p_n)$: Expected number of customers in the system

$L_{qs} = \sum_{n=c+1}^{\infty}\{(n - n_{ss}) * p_n\}$: Expected number of customers in the queue system, and remember here ns is not constant at any given point

$W_s = \frac{L_s}{\lambda_{eff}} = \left[\frac{\sum_{n=1}^{\infty}(n*p_n)}{\lambda_{eff}}\right]$: Expected waiting time in the system

$W_{qs} = \frac{L_{qs}}{\lambda_{eff}} = \left[\frac{\sum_{n=c+1}^{\infty}\{(n - n_{ss}) * p_n\}}{\lambda_{eff}}\right]$: Expected waiting time in the queue system

$\bar{n}_s$: Expected number of busy servers

Without being repetitive we would still like to mention that the system consists of the ***queue*** and the ***server*** facility combined together and this is an important which has to be noted by the reader. Moreover
λ_{eff}
is the effective arrival rate such that all the customers coming can join the queue. In case if the rate of arrival of customers is such that some customers cannot join and have to leave then $\lambda_{eff} < \lambda$, else if $\lambda_{eff} > \lambda$ would have no impact on any of the customers coming as they can come and immediately join the queue.
One can easily make out the relationship between $W_s = \frac{L_s}{\lambda_{eff}} = \left[\frac{\sum_{n=1}^{\infty}(n*p_n)}{\lambda_{eff}}\right]$ and $W_{qs} = \frac{L_{qs}}{\lambda_{eff}} = \left[\frac{\sum_{n=c+1}^{\infty}\{(n - n_{ss}) * p_n\}}{\lambda_{eff}}\right]$, which is $W_s = \left(W_{qs} + \frac{1}{\mu}\right)$, which is obtained from the fact that

$$\begin{bmatrix}\text{Expected waiting}\\ \text{time in system}\end{bmatrix} = \begin{bmatrix}\text{Expected waiting}\\ \text{time in queue}\end{bmatrix} + \begin{bmatrix}\text{Expected Service}\\ \text{time}\end{bmatrix}$$

, thus we will also have $L_s = \left(L_{qs} + \frac{\lambda_{eff}}{\mu}\right) = \left(L_{qs} + \bar{n}_{ss}\right)$

Example 7.10

Consider in IIT Kanpur computer centre (CC) where you have a cluster of computer servers, 6 in number, each of which processes the programs independently as and when the jobs/programs are input into the computer server system. By computer server system we mean the set of these 6 servers. The jobs are processed as per the queue, i.e., they are processed as and when their respective time comes. The arrival of jobs is Poisson distributed with an average rate of 10 per 30 minutes. The processing time is exponentially distributed with a mean of 45 minutes. Once jobs arrive they are made to wait in a queue and the temporary server (which holds the jobs in a queue) can hold maximum of 5 jobs considering its RAM and CPU capacity. From this set of information find

1. The probability, p_n, of n jobs in the system.
2. The effective arrival rate of programs/jobs that are actually held in the temporary server.
3. The average number of jobs in the temporary server.
4. The average time a job waits for being processed in the temporary server.
5. The average number of cluster servers which are busy.
6. The average utilization of the computer cluster server

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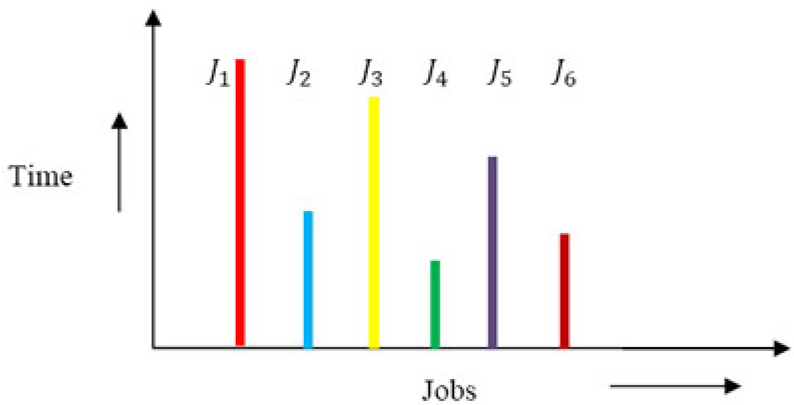
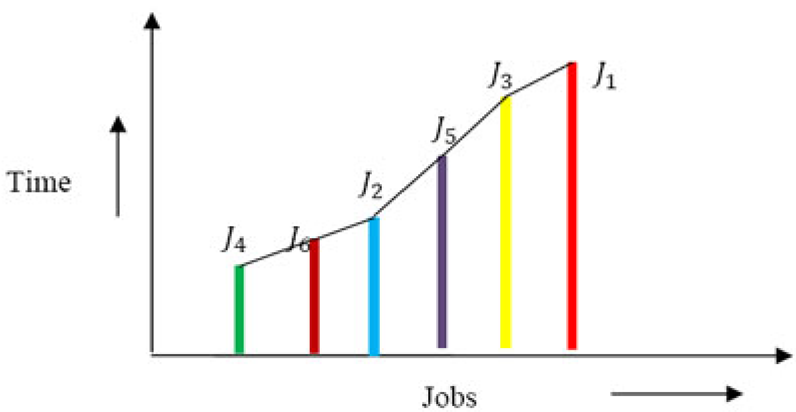
Module 7:Application of stochastic processes in queueing theory

Lecture 30:Examples of application of stochastic processes in Queueing Theory

Example 7.1



Solution 7.1



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Module 7:Application of stochastic processes in queueing theory

Lecture 30:Examples of application of stochastic processes in Queueing Theory

Example 7.2

Consider there is one machine and four (4) jobs of the same type to be processed one after the other in the machine. Also consider the following information is available and remember all the jobs are there standing in the queue, so how would you process the jobs?

Job	Servicing time
A	6
B	5
C	7
D	4

Solution 7.2

In case when the assignment is simple we have the sequence as D-B-A-C, which would minimine the average waiting of the jobs. Please note that in case there are priorities/priority, then one should assign the weights pertaining to the deadline of each job.

Exponential Distribution

We denote the random variable ***X*** *as Exponential, if,* $X \sim E(\lambda, \theta)$ $\forall x = [\theta, +\infty)$ *, i.e.,* $f(x) = \lambda \exp(-\lambda x)$ *, considering* $\theta = 0$*. It is a* ***memory loss distribution****, i.e.,* $P[t > T + S \mid t > S] = P[t > T]$

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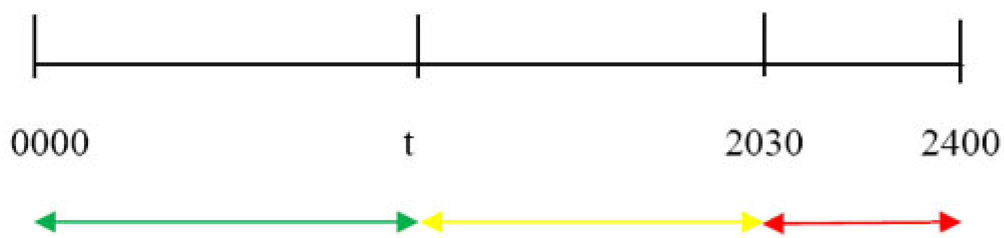
Example 7.3

A sophisticated CNC machine is being used in the shop floor and there is a critical component in the machine which regularly fails, hence you always keep few spare parts of that critical component. The average failure rate of the critical component is 5 hours. You as the shop floor manager ask you main operator to investigate the problem and he says that the break down occurs every night at a particular time which is 8.30 PM. Verify the claim made by the main operator is true or false.

Solution 7.3

From the information given the average failure rate of the machine is $\lambda = \frac{1}{5} = 0.2$ failures per hour, thus $f(x) = 0.2\exp(-0.2t)$, hence we will always have: $P\left[t < \frac{x}{24}\right] = k$, where k is constant.



Nomenclature

Machine functioning (green colour code)
Machine may or may not function (yellow colour code)
Machine failed (for this example it fails at 2030 hours) (red colour code)

What we are doing is to evaluate the probability on a time scale of 24 hours and check whether the value is equal or not, but as this is exponential, hence the statement that the value of the probability that the machine fails from where ever you measure your time is untrue, hence the claim of the operator is false.

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Example 7.4

Finished goods come out of the production process on an average every 12 minutes, then find the average production in a week, considering 24 hours utilization every day and 7 days a week.

Solution 7.4

Pure birth process

In general a pure birth process occurs in which ONLY arrivals are allowed. Remember that the exponential distribution will be used to describe the inter-arrival time in a pure birth process.
In a pure birth process we start with 0 number of customers at $t = 0$ and no queue is considered in the system and we consider arrivals happen at a particular rate which is λ.

So let us define the following

$p_0(t)$: Probability of ***no arrivals*** during a time period of t
λ: Average arrival rate of number of customer per unit time

Then we have:

$$p_0(t) = P[Inter\ arrival\ time \geq t] = 1 - P[Inter\ arrival\ time < t]$$

Now for a sufficient small interval time, h > 0, we would have

$p_0(h) = e^{-\lambda h} = 1 - \lambda h + \frac{(\lambda h)^2}{2!} - \ldots$,further more we can prove that the following also holds and is true which is $p_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$, which is Poisson and the expected value is λt .

Pure death process

In general a pure death process occurs in which ONLY departures are allowed. Remember that the exponential distribution will be used to describe the inter departure time in a pure death process.
In a pure death process we start with N number of customers at $t = 0$ and no new arrivals are considered in the system and we consider departure occurs at a particular rate which is μ .

So let us define the following
$p_0(t)$: Probability of ***no departures*** during a time period of t
μ: Average departure rate of number of customer per unit time

Using similarly we can prove that $p_0(t) = 1 - \sum_{n=1}^{N} p_0(t)$, where $p_n(t) = \frac{(\mu t)^{N-n} e^{-\mu t}}{(N-n)!}$ $\forall n = 1, 2, \ldots, N$.

Now remember that the following equations we have, is the truncated Possion distribution.

Example 7.5

Consider a white goods shop, which sells varieties of different capacities of refrigerators, $C_1, C_2, \ldots, C_n$. On an average the Ms.Naina Sabarwal, the owner of the shop, has the data that for the best selling brand/capacity, C_5, the average number sold is 3 per day and also that the actual demand follows a Poisson distribution, so she always stocks up 18 number of that product, i.e., C_5. When the stock level for that particular product, C_5, reaches a value below or equal to 5, a new order of 18 numbers are placed at the beginning of the following week.

With this set of information find the following

(i) The probability of placing an order in any one day of the week
(ii) The average number of that particular product which will be left unsold at the end of the week.

Solution 7.5

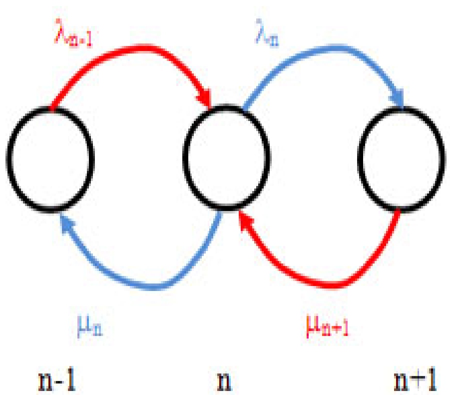
(i) The probability of placing an order is given by

$$p_{n\le 5}(t) = p_{n=0}(t) + p_{n=1}(t) + p_{n=2}(t) + p_{n=3}(t) + p_{n=4}(t) + p_{n=5}(t)$$
$$= p_{n=0}(t) + \sum_{i=1}^{5} \frac{(3t)^{18-i} e^{-3t}}{(18-i)!}$$

, here $t = 1,2, \ldots, 7$ as there are 7 days in a week

(ii) The average number of product C_5 , which will be left unsold at the end of the week $(t = 7)$ is actually E[n|t=7]. In order to calculate that we need $p_n(7)$, $n = 0,1,2,\ldots,18$, hence the value is $E[n|t=7] = \sum_{n=0}^{18} np_n(7) \cong 0.7$, i.e., about 1.

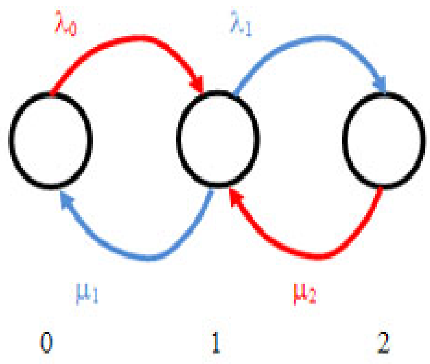
Consider the following set of information
We have a steady state system where
(i) n: Number of customers/jobs/items in the queue
(ii) λ_n: Arrival rate provided there are *n* customers/jobs/items in the queue
(iii) μ_n : Departure rate provided there are *n* customers/jobs/items in the queue
(iv)) p_n: Steady state probability that there are *n* customers/jobs/items in the queue



Now for the ***equilibrium*** position we must always have I/P is equal to O/P, i.e.,

$$\left[\lambda_{n-1}p_{n-1}+\mu_{n+1}p_{n+1}\right]=\left[\lambda_n p_n+\mu_n p_n\right]$$

Now remember the diagram for $n = 0$ is as given below



Hence the balance equation is: $\lambda_0 p_0 = \mu_1 p_1$, i.e., $p_1 = \left(\frac{\lambda_0}{\mu_1}\right) p_0$

For n = 1, the balance equation is: $\lambda_0 p_0 + \mu_2 p_2 = (\lambda_1 + \mu_1) p_1$, hence in general we have

$p_n = \left(\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\right)p_0$ and we know that $\sum_{n=0}^{\infty} p_n = 1$

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Example 7.6

A super market say BigBazar has three check our counters and the store manager knows the following from past data

# of customers	# of counters
1 to 3	1
4 to 6	2
> 6	3

If the manager also knows that the mean value of arrival of customers in the store is 10 per hour and the average processing time for the customer is 12 minutes, then find p_n. Why do you think as the store manager this value of p_n is important for you.

Solution 7.6

Now we already have the following set of information. Also remember that the outflow rate is 60/12 =5 customers from the system and the outflow doubles or increases by three the moment you have 2 counters or 3 counters respectively.

$$\lambda_n = \lambda = 10$$

$$\mu_n = \begin{cases} 5 & \forall n = 0,1,2,3 \\ 10 & \forall n = 4,5,6 \\ 15 & \forall n = 7,8 \end{cases}$$

Now utilizing $p_1 = \left(\frac{\lambda_0}{\mu_1}\right) p_0$, $p_2 = \left(\frac{\lambda_0\lambda_1}{\mu_1\mu_2}\right) p_0$,…., $p_n = \left(\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\right) p_0$ we have

$$p_1 = \left(\frac{10}{5}\right) p_0,\ p_1 = \left(\frac{10}{5}\right)^2 p_0,\ p_1 = \left(\frac{10}{5}\right)^3 p_0$$

$$p_4 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right) p_0,\ p_5 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^2 p_0,\ p_6 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^3 p_0$$

$$p_7 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^3 \left(\frac{10}{15}\right) p_0,\ p_8 = \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^3 \left(\frac{10}{15}\right)^2 p_0, \ldots,\ p_{n\geq 7} \left(\frac{10}{5}\right)^3 \left(\frac{10}{10}\right)^3 \left(\frac{10}{15}\right)^{n-6} p_0$$

Also $p_0 + p_1 + \cdots + p_\infty = 1$, i.e., $p_0\left[1+2+4+8+8+8+8+8\left(\frac{2}{3}\right)+8\left(\frac{2}{3}\right)^2+\ldots\right]=1$, i.e., using GP series summation we have $p_0\left[31 + 8\left(\frac{1}{1-\frac{2}{3}}\right)\right] = 1$, thus $p_0 = 1/55$

1. So n = 0,1,2,3 we find $P[\#\text{ of counter is } 1] = p_0 + p_1 + p_2 + p_3$
Thus $P[\#\text{ of counters is } 1] = p_0 + p_1 + p_2 + p_3 = (1 + 2 + 4 + 8) * (1/55) \approx 0.2727$

2. So for n = 4,5,6 we find $P[\#\ of\ counters\ is\ 2] = p_4 + p_5 + p_6$

3. So for n = 7,8,... we find $P[\#\ of\ counters\ is\ 3] = p_7 + p_8 + p_9 + \ldots\ldots$

How about finding the expected number of idle counters

If all 3 counters are idle which means the corresponding probability is p_0
If 2 counters are idle which means the corresponding probability is $(p_1 + p_2 + p_3)$
If 1 counter is idle which means the corresponding probability is $(p_4 + p_5 + p_6)$
If 0 counters are idle which means the corresponding probability is $(p_7 + p_8 + \ldots..)$

Hence expected value is

$$E[\#of\ idle\ counters] = \sum_{\forall n} nP[n] = \{3p_0 + 2(p_1 + p_2 + p_3) + 1(p_4 + p_5 + p_6) + 0(p_7 + p_8 + \cdots)\}$$

Similarly we can find the E[# of working counters], which would be given by

$$0(p_0) + 1(p_1 + p_2 + p_3) + 2(p_4 + p_5 + p_6) + 3(p_7 + p_8 + \cdots)$$

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Example 7.7

Billu has a hair cutting saloon and he is only able to serve one customer at a time and provides three (3) seats for his waiting customers. If his shop is full, i.e., all the three (3) seats are occupied and he is also busy attending another fourth (4th) customer, then all the rest customers who arrive after that go to Shot Gun Murugun's swanky hair cutting saloon. Billus is from IIM Calcutta so he has been able to understand that the arrival rate of customers, are Poisson distribution with mean value of 2 customers per 30 minutes. The average time Billu takes to do a haircut is 12 minutes. Billu has employed you as his strategic consultant and from this information which he has provided, please help Billu to find out the following:

(i) The steady state probability

(ii) The expected number of customers in Billu's hair cutting saloon.

(iii) The probability that customers from his shop will go to Shot Gun Murugun's swanky hair cutting saloon is?

Solution 7.7

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Example 7.8

You are the shop floor manager and jobs arrive with Poisson distribution at the rate of 18 jobs per hour. The average time needed to process and finish a job is 5 minutes. Remember your shop floor can accommodate a maximum of 30 jobs. If the shop floor is full then jobs are either send to the other shop floor or kept in the bay. With this information determine the following

(i) The probability that a arriving job will not be processes because as the shop floor is filled up with unfinished jobs?

(ii) The probability that any job arriving has to wait (or will have a waiting time) as the previous job(s) are being processed.

(iii) The average number of jobs in the shop floor.

Solution 7.8

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