Module 6:Random walks and related areas

Lecture 24:Random woalk and other areas

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## Random Walk

As already discussed there is another type of Markov chain using which we model the behaviour of a particle experiencing a random nature of movement, such that if the particle is in state $i$ then we assume that it can either move up, i.e., go to state $(i+1)$ or stay at the same position, which is $(i)$ or move down to $(i-1)$ during the next period (which is one unit). We may denote the movements with the probabilities (i) $P(X_{n+1}=i+1|X_n=i)=p_i$; (ii) $P(X_{n+1}=i|X_n=i)=q_i$ and (iii) $P(X_{n+1}=i-1|X_n=i)=r_i$), where $p_i, q_i, r_i > 0$ and $p_i+q_i+r_i=1$. Thus the transition matrix would be denoted by $P=\begin{bmatrix} q_0 & p_0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ r_1 & q_1 & p_1 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & r_2 & q_2 & p_2 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & r_3 & q_3 & p_3 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots\ \cdots & & \cdots & \cdots\ \cdots & & \cdots & \cdots \end{bmatrix}$, which as we know is a sparse matrix or more precisely as the ***tri-diagonal matrix***. We can consider examples where (i) $p_0=p_1=p_2=\cdots$, $q_0=q_1=q_2=\cdots$ and $r_0=r_1=r_2=\cdots$ or (ii) $p_0=p_1=p_2=\cdots=q_0=q_1=q_2=\cdots=r_0=r_1=r_2=\cdots$ hold true such that the transition matrix will be modified accordingly.

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Consider the problem called the ***Gambler's ruin***, where the gambler starts with Rs. X and at each play of the game he/she wins Rs. 1 or losses Rs. 1. Once the gambler reaches the state where he/she losses all the money then he/she continues to stay at that level where his/her money is Rs. 0. Hence if we follow these assumptions then we should have $q_0 = 1$, $p_0 = 0$. Moreover $p_i = r_i = \frac{1}{2}$, while $q_i = 0$, $i > 0$. This provides us with the information such that one can easily draw the transition probability matrix as $P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$. Now holding the time tighter and tighter we can have this process converge to the ***Brownian moton***. As a remark we would like to mention that the relationship between convergence of the above model to the Brownian motion and that of binomial model to Black Schole's model (in finance) is quite interesting. Based on this we can work on a simple problem, where if we start with Rs. 100 then we can easily say that the answer to the question that we will eventually ruin with certainty is a yes and the answer to how long it will take to do that is infinity.

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## Ehrenfest Model

This is discrete model of diffusion through a membrane, something like osmosis. Imagine two cells have a total of $2a$ particles. A particle is selected at random and moved into the other container at each time period, $t$. For the explanation of this concept one can refer to the Figure 6.1 where we consider it to be a discrete time discrete state process:

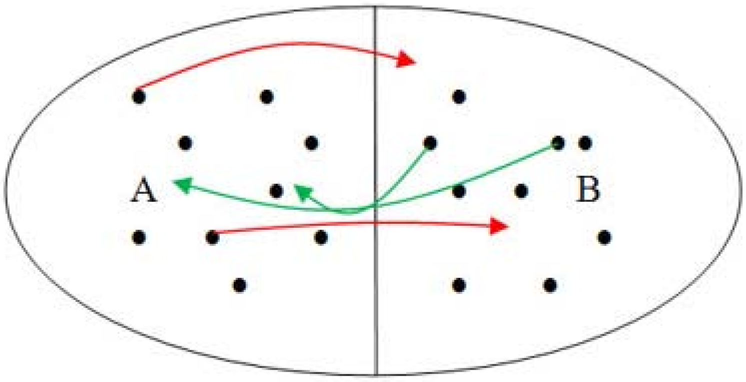


Figure 6.1: Ehrenfest model

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In this model we have $y_t = \rho y_{t-1} + \varepsilon_{normal}$, i.e., an AR(1) model. A close look at this AR(1) model immediately makes it clear that if $\rho = 1$, then the process blows up. Consider we have 6 particles in all and for simplicity each cell marked A and B has three particles in each. Hence from state $i$ we can go to state $(i+1)$ or $(i-1)$, which means that when we move from $i$ to $(i+1)$, then the corresponding probability is $\left(\frac{a-1}{2a}\right)$, while for the movement from state $i$ to $(i-1)$ it is $\left(\frac{a+1}{2a}\right)$. Thus the transition matrix is of the form as shown $P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 0 & 5/6 & 0 & 0 & 0 & 0 \\ 0 & 2/6 & 0 & 4/6 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 4/6 & 0 & 2/6 & 0 \\ 0 & 0 & 0 & 0 & 5/6 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ and if we draw the graph then we have the movement of the particle as given in Figure 6.2

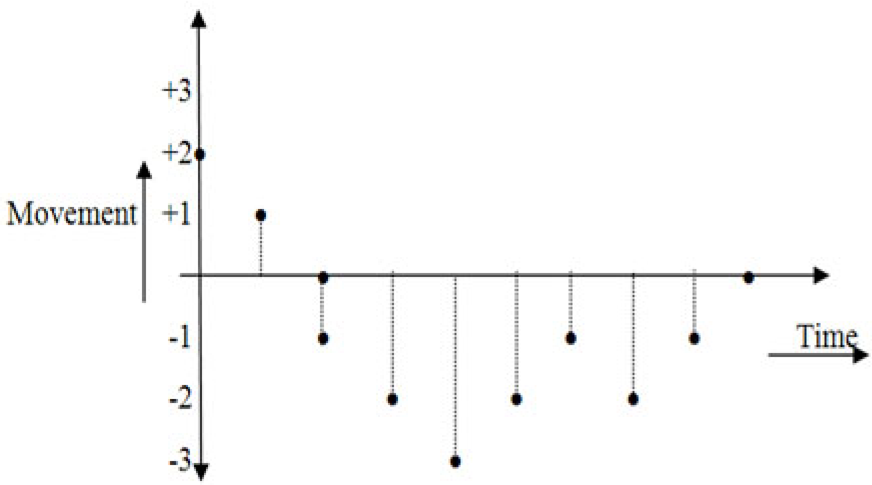


Figure 6.2: Diagrammatic representation of the movement of a particle

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This is an autoregressive process given by $\rho\gamma_{t-1}+\varepsilon_t$ and by stimulating it $N$ number of times and choosing $a$ we can model the process as an auto-regressive process. These types of model can be used to understand interest rate behavior in finance.

Consider stages are denotes by $n$, $n+r$ and $n+r+s$. Thus :

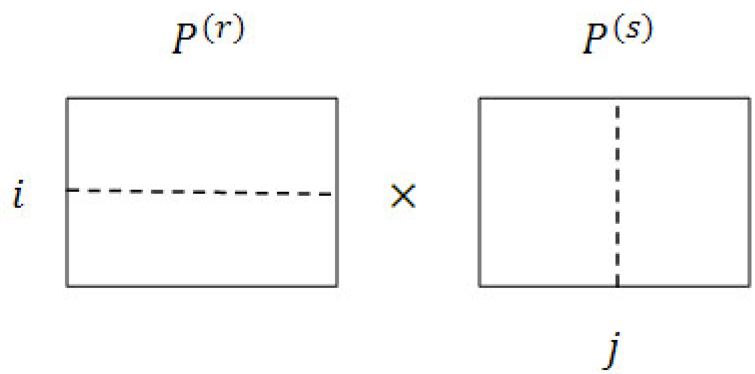
$$Pr(X_{n+k}=j|X_n=i)=\sum_{\forall l}Pr(X_{n+k}=j,X_{n+r}=l|X_n=i)$$

$$=\sum_{\forall l}Pr(X_{n+k}=j|X_{n+r}=l,X_n=i)\times Pr(X_{n+r}=l|X_n=i)$$

$$=\sum_{\forall l}Pr(X_{n+k}=j|X_{n+r}=l)\times Pr(X_{n+r}=l|X_n=i)$$

$$=\sum_{\forall l}p_{l,j}^{(k-r)}\times p_{i,l}^{(r)}=\sum_{\forall l}p_{l,j}^{(s)}\times p_{i,l}^{(r)}$$

This is the $(i,j)^{th}$ element of the $P^{(r)}\times P^{(s)}=P^{(k)}$




As an example consider $P=\begin{pmatrix}0.8 & 0.2\\ 0.5 & 0.5\end{pmatrix}$, then $P^{(2)}=P^{(1)}\times P^{(1)}=\begin{pmatrix}0.8 & 0.2\\ 0.5 & 0.5\end{pmatrix}\times\begin{pmatrix}0.8 & 0.2\\ 0.5 & 0.5\end{pmatrix}=\begin{pmatrix}0.74 & 0.26\\ 0.65 & 0.35\end{pmatrix}$. On similar lines one can find $P^{(4)}=P^{(2)}\times P^{(2)}=\begin{pmatrix}0.7166 & 0.2834\\ 0.7065 & 0.2915\end{pmatrix}$ and $P^{(10)}=\begin{pmatrix}0.7143 & 0.2857\\ 0.7143 & 0.2857\end{pmatrix}$.

In case $\lim_{n\to\infty}p_{i,j}^{(n)}=\pi_j$, i.e., the process depends on the final state and has a probability distribution denoted by $f_j(j)$, for different value of $j$, then we can deduce that each state, $j$ is a stationary state.

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