Module 6:Random walks and related areas

Lecture 28:Random Walks in more than one dimension

The Lecture Contains:

- One dimension random walks
- Higher dimension random walks
- Wiener Process

Previous Next

Definition : the Markov chain is ***irreducible*** if the equivalence reduces only to one class i.e., all state ***communicate***.

One dimension random walks

In one dimension random walk our variable is defined as $X(t)$, where the realized values are scalars only and are defined as x_1, x_2,...., x_n,....... Let us illustrate this with a simple example. Consider you toss an unbiased coin repeatedly. Then Figure 6.8 shows how the random walk movement takes place in the one dimension case considering the actual occurrences are either a head (H) or a tail (T). Moreover one can define the outcomes occurring after each toss as a ***stage*** and the collection of all different occurrences at any stage as ***states***. Thus I, II, III, and so on represents the stages while the outcomes in each stage is the corresponding states of that stage,

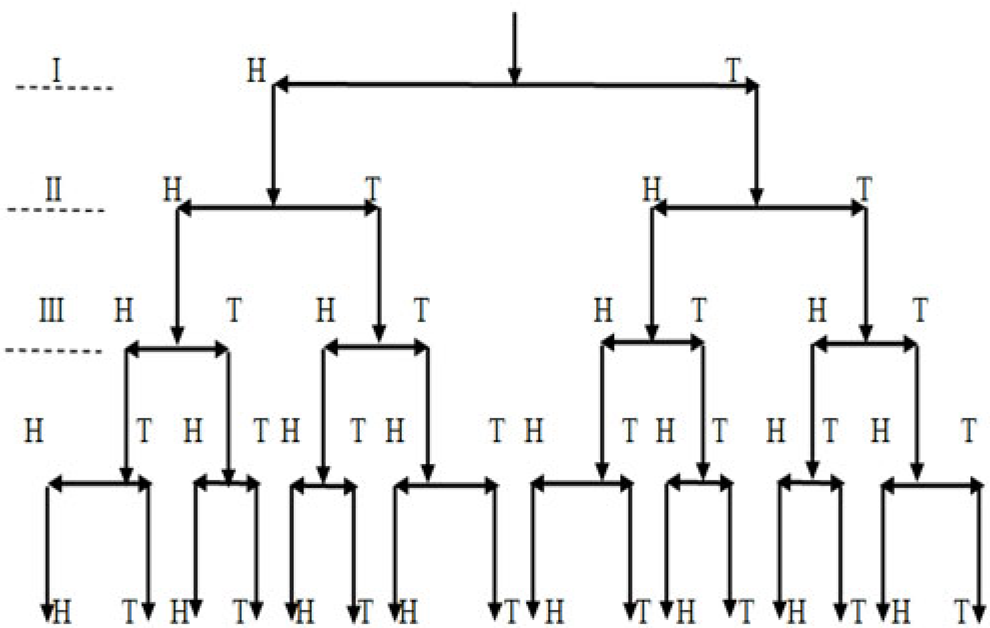

Figure 6.8: One dimension random walk with the example where you toss a coin

One should note that the ***Central Limit Theorem*** (CLT) describes the important aspects of the behavious of the simple random walk (in one dimension).

Previous Next

Higher dimension random walks

Let us now consider a random walk with identically distrubuted, independent steps on a periodic lattice of dimension d. It must be noted that our aim is to analyze the random time it takes the particle to return to its starting point.

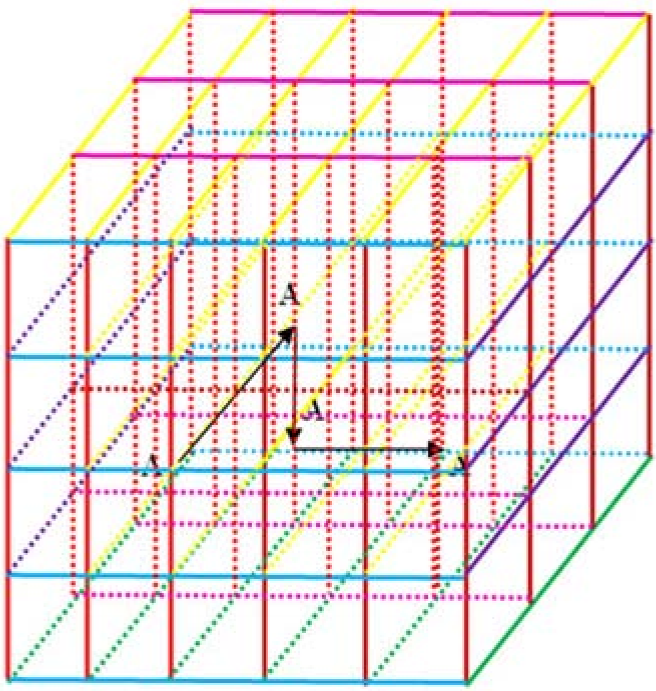

Figure 6.9: Three dimension random walk (an example of higher dimension random walk) with the example where a gas particle is moving inside an enclosed cube

Previous Next

For an ease of understanding let is refer to Figure 6.9, which illustrates the random movement of a particle inside a chamber. The movement of the particle A happens as shown by the arrow.

Let the partile start at x_0 and suppose it makes n steps. Let us also define the following
$p(\Delta x)$ = Probability that the particle moves a distance Δx which is independent of either x_0 or n
$P_n(x|x_0)$ = Probability that the particle is at x (which is a vector of dimension d)
$F_n(x|x_0)$ = Probability that the particle is at x for the first time

Thus we can show that $P_n(x|x_0) = \sum_{j=1}^{n} F_j(x|x_0)P_{n-j}(x|x) + \delta_{n,0}\delta_{x,x_0}$. Now to solve F_n we introduce the generating function and apply the discrete version of the convolution theorem.

Thus we have:

$$P(x|x_0;z) = F(x|x_0;z)P(x|x;z) + \delta_{x,x_0} \Rightarrow F(x|x_0;z) = \frac{P(x|x_0;z) - \delta_{x,x_0}}{P(x|x;z)}$$

Now inverting the trsansform and using Taylor series expansion we get $F_n(x|x_0) = \frac{1}{n!}\frac{d^n F(x|x_0;z)}{dz^n}\Big|_{z=0}$

We can also find the eventualprobability that the particle returns to its original place from where it started and that is already calculated by us.

Previous Next

Wiener Process

In mathematics, the Wiener process is a continuous-time stochastic process named in honor of Norbert Wiener, and this process is also called the standard Brownian motion, after Robert Brown. It is one of the best known Lévy processes (stochastic processes with stationary independent increments) and is quite frequently used in in pure and applied mathematics, economics, physics, finance, etc.

Let us characterize the Wiener process, W_t, as given below

- $W_t = 0$
- W_t has independent increments such that $W_t - W_s \sim N(0, t-s)$, where $0 \leq s < t$

The basic properties of the Wiener process are as follows :

The unconditional probability density function at a fixed time t is given by $f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$, such that we can easily prove that the following holds.

- $E(W_t) = 0$
- $Var(W_t) = t$
- $cov(W_s, W_t) = min(s, t)$
- $corr(W_s, W_t) = \sqrt{\frac{min(s,t)}{max(s,t)}}$

Previous Next