Module 12:Application of stochastic processes in areas of engineering and management science

Lecture 40:Use of Markov Chain Monte Carlo Method

The Lecture Contains :

- Solution Methodology
- Reliability Based Optimization Method
    - Performance Measure Approach (PMA) and Reliability Index Approach (RIA) methods
- Sequential Optimization and Reliability Assessment (SORA)
- Use of MCMC
    - Case 1: (Generalized Exponential Distribution (GED)) non-Bayesian case
- Example of GED with Complete data
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Solution Methodology

Reliability search algorithms are characterized by the use of analytical techniques to find a particular point (Most Probable Point (MPP) of failure) in the feasible space which can be related to the probability of the system failure, defined by the limit state. The solution methodology proceeds as follows. We first transform the input vector $X=(X_1,\ldots,X_n)$ into the standard normal space $U=(U_1,\ldots,U_n)$, where $u_i=\Phi^{-1}\left[F_{X_i}(x_i)\right]$, $\forall\, i=1,\ldots,n$, and F^{-1} is the inverse of the normal distribution function. This transformation maintains the distribution functions being identical in both X and U space respectively. Thus the MPP, now in U space, is the minimum distance point from the constraint boundary $g_j(x,d,p)=g_j(u_x,d,u_p)=0$ to the origin, and this minimum distance is α_j.

Now in an inverse reliability problem, the required reliability α_j is given, and the percentile performance corresponding to α_j is to be evaluated, such that one needs to defined ß$_j$ (reliability index), which is given by $P\left[g_j(u_x,d,u_p)>0\right]=\alpha_j=\Phi(\beta_j)$. Thus the MPP becomes the tangent point of a hyper sphere in the U space with the radius ß$_j$ and the contour of g_j(ux, d, up), such that at MPP one achieves optimality. The type of optimization, i.e., maximization (minimization) achieved at MPP depends on which part of the tail of the distribution MPP corresponds to, and as such the right (left) position of MPP signifies that one achieves maximization (minimization) at those points respectively. So for (12.5) we consider the MPP corresponding to the left tail. Likewise for the maximization problem, when we have the objective function as maximum of $\mathrm{Re}\,turn(\Omega,V,M)$, we formulated it accordingly and solve it to obtain the MPP.

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Reliability Based Optimization Method

Performance Measure Approach (PMA) and Reliability Index Approach (RIA) methods

One of the most challenging issues of implementing the probabilistic design is associated with intense computational demand of uncertainty analysis. To capture the probabilistic characteristic of a system performance from a design perspective, we need to perform a number of deterministic analyses around the nominal point. One of the existing reliability based optimization methods is the decoupled method, where in, there are two loops, namely the (i) optimization synthesis or the outer loop that optimizes the original objective function based on the fact that the reliability constraints are formulated as deterministic constraints that approximate MPP and (ii) reliability assessment or the inner loop (there are two approaches used for solving this inner loop which are (i) Performance Measure Approach (PMA) method (Figure 12.7) and (ii) Reliability Index Approach (RIA) method (Figure 12.8) about we will discuss briefly), that finds the equivalent deterministic version of each probabilistic constraint by formulating and solving an optimization problem. It must be remember that these two loops, which are decoupled from one another, are applied one after another in a sequence. Since this decoupled loop method does not conduct the expensive MPP search at each step, its time efficiency is very high, but as it performs an approximation at each step, hence this may not guarantee that the results would always be optimal. But this decoupled method does generates a solution, even if sub-optimal, for maximum of the complex optimization problem formulations.

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In PMA we calculate the MPP by formulating a minimization problem of the form: Minimize g_j(ux, d, up), s.t. $(U_x^T U_x)^{1/2} = \beta_j$, and this MPP is identified as the percentile performance level, calculated by $g_j^{\alpha} = g_j(U_{X,MPP}) = g_j(X_{MPP})$, i.e., the g_j function evaluated at MPP. There are several methods to solve the MPP problem using PMA method, and that includes optimization techniques, traditional MPP search algorithm based on steepest ascent direction, the diagonal direction method and gradient based method, etc. On the other hand using RIA the MPP is calculated by finding a point which is on the constraint curve in the U-space and is nearest to the origin. The problem formulation corresponding to RIA approach is as follows: Minimize $(U_x^T U_x)^{1/2}$ s.t. g_j(ux, d, up) = 0. Here we ignore the desired reliability index ß$_j$ and the minimum U vector on the constraint boundary is found. Afterwards this point thus found is compared with ßj to get the optimal point.

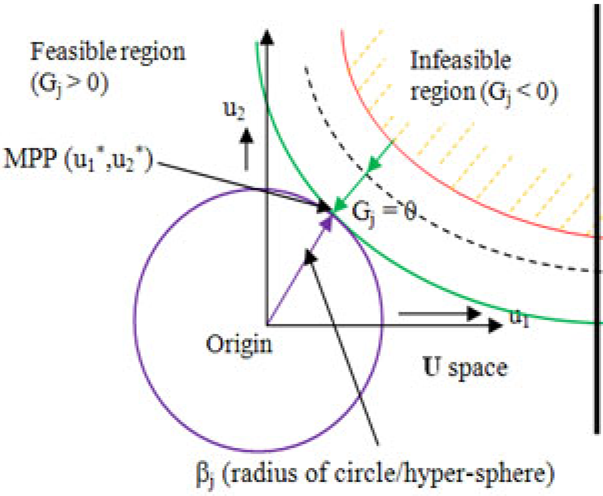

Figure 12.7: The PMA approach

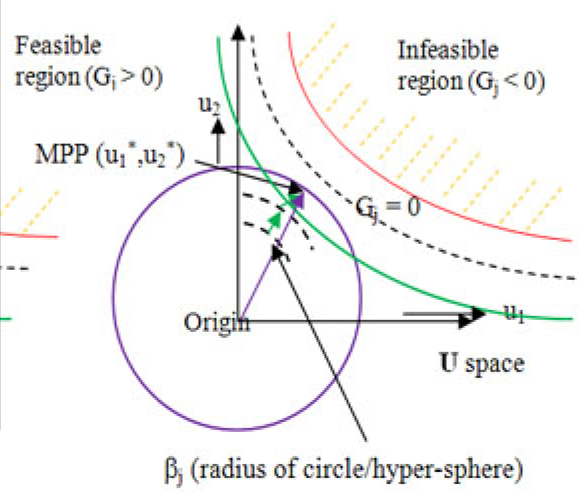

Figure 12.8: The RIA approach

Figure 12.7 and Figure 12.8, give a pictorial illustration of both PMA and RIA methods, where the interesting fact to note is how the minimization is achieved. In PMA the constraint boundary (or one side of the hyper plane in *n* dimension space) at each iteration moves, which is shown by the red lines, while in RIA it is the boundary of the circle (or hyper sphere) which moves, that is again shown by the red lines. Hence reliability optimization problem involves two steps, which are (i) the requisite optimization technique to solve the problem and (ii) the reliability assessment method which is required to incorporate the uncertainty in the variables. In recent times the Sequential Optimization Reliability Assessment (SORA), a form of decoupled solution method has gained importance for reliability based optimization.

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Sequential Optimization and Reliability Assessment (SORA)

Sequential Optimization and Reliability Assessment (SORA) works in a way in which a series of cycles of optimization task and reliability assessment are employed and in each such cycle, the optimization and reliability assessments are decoupled from one another i.e., no reliability assessment is required within optimization and the reliability assessment is conducted only after optimization process is complete. The key concept of the method is to shift the boundaries of the violated deterministic constraints to the feasible direction based on the information gained in the previous cycle such that both the optimization and reliability loops are repeated until convergence is achieved. Thus the three important steps for SORA are, (i) use α_j-percentile formulation to evaluate the design feasibility only at the desired reliability level α_j, (ii) employ equivalent deterministic optimization (first loop) to reduce the number of reliability assessments and (iii) use efficient MPP search algorithm for the reliability assessments loop. Note should made of the fact that the use of α_j-percentile formulation instead of the original reliability assessment is based on the fact that, closer the reliability is to 1.0, more is the computational effort needed. Thus for using MPP based methods, higher reliability would mean higher search regions in the standard normal space to locate the MPP and it is more likely that more functional evaluations are required. Our concern for a probabilistic constraint is not to find the actual reliability of the limit state function, but to determine whether it is probabilistically feasible, as some probabilistic constraint(s) may never be active whose reliability is close to one, and if these constraint(s) is/are least critical, then the evaluation of this/these reliability/reliabilities may dominate the computational effort. Thus use of α_j-percentile is performed to improve the efficiency of the overall process. Thus based on the relevant equations obtained after the MPP search is performed, the model given by (12.2) can be rewritten as shown in (12.6):

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the model given by (12.2) can be rewritten as shown in (12.6) :

SORA has been repeated twice. The second one is error free

$$\left.\begin{array}{lll} \underset{\forall(x,d)}{Optimize} & \sum_{i=1}^{I} f_i(\mu_x, d, \mu_p) & \\ s.t.: & g_j(\mathrm{x}_{\mathrm{MPP}}, \mathrm{d}, \mathrm{p}_{\mathrm{MPP}}) \geq 0 & j = 1, \ldots, J \\ & h_k(x, d, p) = 0 & k = 1, \ldots, K \\ & \mu_x \in \Re^n & n = 1, \ldots, N \\ & d \in \Re^m & m = 1, \ldots, M \\ & p \in \Re^l & l = 1, \ldots, L \end{array}\right\} \qquad (12.6)$$

where $\mu_x = (\mu_{x,1}, \ldots, \mu_{x,N})$, $x_{MPP} = (x_{MPP,1}, \ldots, x_{MPP,N})$, $\mu_p = (\mu_{p,1}, \ldots, \mu_{p,L})$, and $p_{MPP} = (p_{MPP,1}, \ldots, p_{MPP,L})$ are the corresponding mean value vectors and the MPP vectors for the decision and parameter variables respectively. Thus, (12.6) establishes the relation between a probabilistic optimization and a deterministic optimization since the original constraint functions $g_j(\mathrm{x}_{\mathrm{MPP}}, \mathrm{d}, \mathrm{p}_{\mathrm{MPP}})$ are used to evaluate the design feasibility using the inverse MPPs corresponding to desired reliability α_j. One can refer to Figure 12.9 to understand how a probabilistic constraint is converted to equivalent deterministic constraint.

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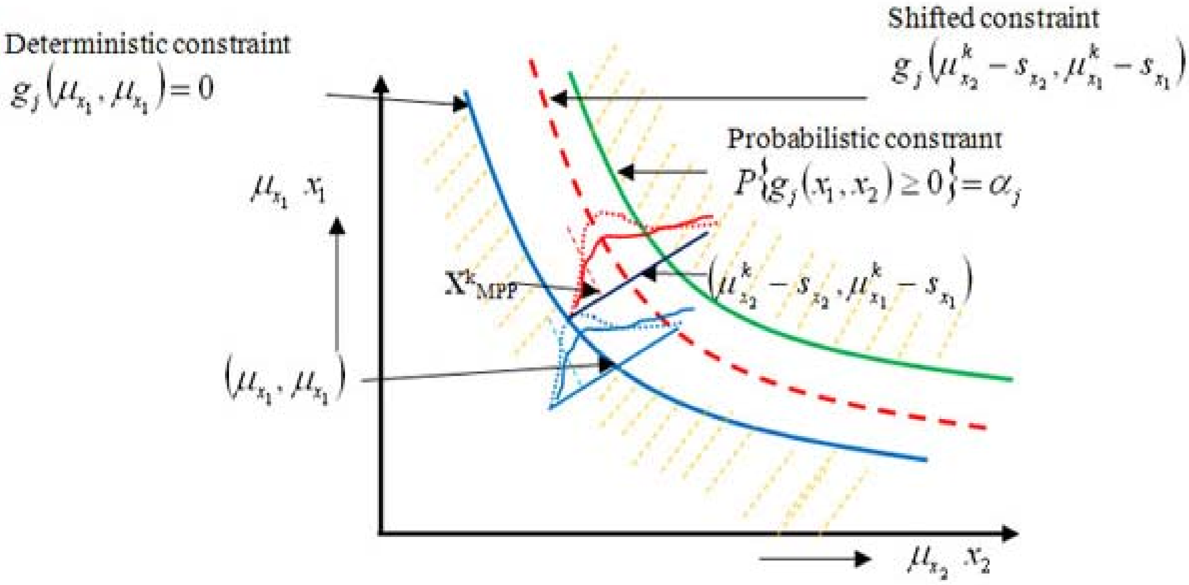
Deterministic constraint
$g_j(\mu_{x_1},\mu_{x_1})=0$
Shifted constraint
$g_j(\mu^k_{x_2}-s_{x_2},\mu^k_{x_1}-s_{x_1})$
Probabilistic constraint
$P\{g_j(x_1,x_2)\ge 0\}=\alpha_j$
μ_{x_1} x_1
X^k_{MPP}
$(\mu^k_{x_2}-s_{x_2},\mu^k_{x_1}-s_{x_1})$
(μ_{x_1},μ_{x_1})
μ_{x_2} x_2

Figure 12.9: Shifting of the constraint boundary

Figure 12.9, corresponds to two different co-ordinate systems, viz (i) x_1 vs x_2 (random space) and (ii) μ_1 vs μ_2 (design variables) and if we do not consider any uncertainty, then $g(\mu_1, \mu_2) = 0$ is the constraint boundary in the deterministic design, while for the uncertainty case, the constraint boundary is $P\{g_j(x_1,x_2)\ge 0\}=\alpha_j$. The probabilistic constraint feasible region is a reduced region in comparison to the deterministic constraint as the reliability of probabilistic constraint is much higher than that achieved for the deterministic constraint. Determining probabilistic constraint boundary requires reliability analysis, since $P\{g_j(x_1,x_2)\ge 0\}\ge\alpha_j$ is equivalent to $g_j(x_{MPP}, d, p_{MPP}) = 0$, where (x_{MPP}, p_{MPP}) is the inverse MPP point and evaluating a probabilistic constraint at (μ_1,μ_2) is equivalent to evaluating the deterministic constraint at the inverse MPP point. The employment of the equivalent deterministic optimization formulation allows us to use an effective single loop strategy and with this strategy, deterministic optimization and reliability assessment are conducted in sequential series. In Figure 12.9, what is also interesting is to note the red and blue (dotted and the continuous) joint distribution functions, $F_{x,d,p}\{g_j(x, d, p)\}$, where the bold (dotted) curves signify the non-normal (normal) distributions, such that the corresponding probability of failure and hence the MPP points would definitely dependent on the joint distributions.

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The flow chart of SORA is given in Figure 12.10, where in each cycle, the deterministic optimization is performed first, followed by reliability checking of $gj(x_{MPP}, d, p_{MPP}) = 0$. If all of the probability constraints simultaneously fail to fulfill the reliability requirements, then the MPP information obtained in the current cycle will be used to formulate the deterministic optimization in the next cycle. This procedure is repeated until convergence is achieved. The number of function evaluations would be reduced, as the reliability assessment is now equal to the number of optimization cycles. To explain the strategy of separating deterministic optimization and reliability assessment while ensuring that both the segments work together to bring the design solution quickly to a feasible and optimal point, we consider Figure 12.9. As shown, Figure 12.9, if at the result of the deterministic optimization, the reliability constraint is not feasible, then its boundary, $g_j(x_{MPP}, d, p_{MPP}) = 0$, in the optimization model is shifted towards the feasible region by a small distance, $s = (s_1,\ldots, s_J)$, where $s_j^{k+1} = \mu_x^k - x_{j,MPP}^k$, based on the MPP recently found, so that this shift is performed in a manner that the MPP is moved onto the deterministic boundary, and the constraint in the deterministic optimization model are reformulated as $g_j(\mu_x - s_j^{k+1}, d, p_{j,MPP}) \geq 0$, $\forall\ j = 1,\ldots,J$. The procedure is repeated until the objective converges and the reliability requirement is achieved when the shifting distances, s_j^k, $\forall\ j = 1,\ldots,J$ become zero. In order to improve the efficiency of SORA method, one may consider the starting point of inverse MPP search in any iteration as the MPP point obtained in previous iterations, and moreover similar starting point concept can be applied to the optimization cycle, where the optimal point of previous cycle is taken as the starting point.

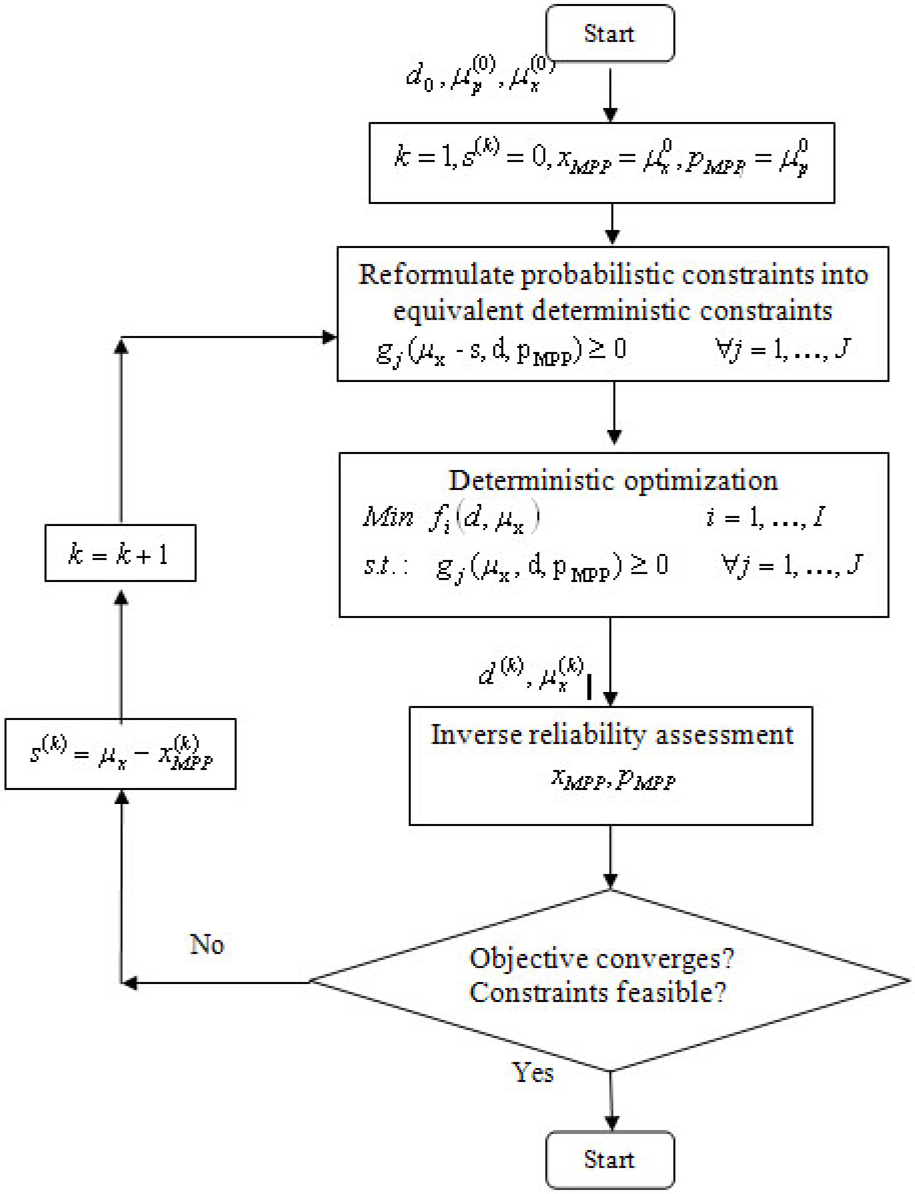
Start
$d_0, \mu_p^{(0)}, \mu_x^{(0)}$
$k=1, s^{(k)}=0, x_{MPP}=\mu_x^0, p_{MPP}=\mu_p^0$
Reformulate probabilistic constraints into equivalent deterministic constraints
$g_j(\mu_x - s, d, p_{MPP}) \geq 0 \quad \forall j=1,\dots,J$
Deterministic optimization
$Min\ f_i(d,\mu_x) \quad i=1,\dots,I$
$s.t.:\ g_j(\mu_x, d, p_{MPP}) \geq 0 \quad \forall j=1,\dots,J$
$d^{(k)}, \mu_x^{(k)}$
Inverse reliability assessment
x_{MPP}, p_{MPP}
Objective converges?
Constraints feasible?
No
Yes
$k=k+1$
$s^{(k)}=\mu_x - x_{MPP}^{(k)}$
Start

Figure 12.10: Flow Chart for SORA

So the stopping criteria for SORA method are based on the following two criteria, which are, (i) difference in the objective function between two consecutive cycles is less than a small e value, specified by the designer and (ii) all reliability requirements are satisfied. From the procedure of the SORA method we see that the reliability loop is completely decoupled from the optimization loop and in the optimization formulation, equivalent deterministic form of constraints, corresponding to the specified reliabilities, are used. As a result the computation requirements are reduced as compared to other methods.

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Use of MCMC

We already know that when an item is put to use it will definitely fail after some time, and then it will be replaced by a new item of the same type and the time concept for finding the failure rate for the item will start again. The question we need to answer in such cases is what happens then when we need to find the best estimate of the time to failure and also the parameter values of the distribution based on which the items function. In case if we have bulbs, each of which have a time to failure given by the exponential distribution, then our main concern is to calculate the parameter, λ or $\theta = \lambda, \alpha$ of that distribution and also T which would be the time to failure. This time to failure denotes when the first item (say A in Figure 12.11) fails such that the whole component fails. It may also be the case when we need to find the failure times for *two* items say E and F (Figure 12.11) such that the system stops functioning.

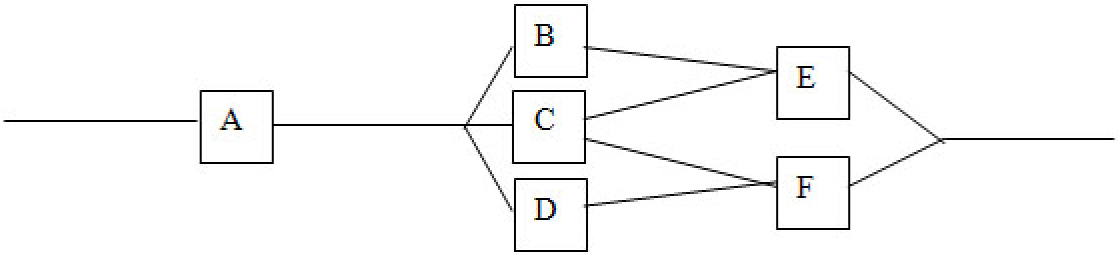

Figure 12.11: General set up for a system with series and parallel components

Based on this concept we have the competing-risk model which refers to a situation where a system (or organism) is exposed to two or more causes of failure (or death) but its eventual failure (or death) can be attributed to exactly one of the causes of failure. The basic information available in the competing-risks situation is the time to failure (T) of the system, and the corresponding cause of failure (δ).

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Assume a (replaceable) component or unit has '*k*' different ways it can fail. Then these are called failure modes and underlying each failure mode is a failure mechanism. The competing risk model evaluates component reliability by "building up" from the reliability models for each failure mode. Typically the following three assumptions hold for such models:

- Each failure mechanism leading to a particular type of failure (i.e., failure mode) proceeds independently (a very restrictive assumption in the practical sense) of every other mode, at least until a failure occurs. If we assume X_1, X_2,….,X_k as the latent lifetimes due to cause $1,2,\ldots,k$ respectively when the system is exposed to these risks independently, then $T=\min\left(X_1,X_2,\ldots,X_k\right)$ is the time to failure of the system. Obviously in a simplistic sense it means we consider a series circuit, such that any one component failing mean the whole system stops functioning and as an illustration one should refer to Figure 12.12.

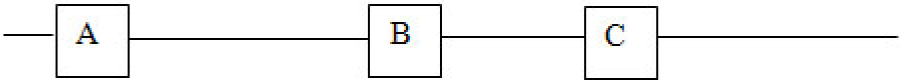

Figure 12.12: Series system

- The component fails when the first of all the competing failure mechanisms reaches a failure state and the system is said to fail due to cause, $j=1,2,\ldots,J$. The cause is denoted by $\delta_i=j$, j = 1, 2, …., J. In case if we do not have the information about the cause of failure of the system then the data is said to be masked/censored.

- Each of the *k* failure modes has a known life distribution model given by $F_k(t)$.

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If $S_c(t)$, $F_c(t)$, and $h_c(t)$, are the hazard rate function or failure rate function, then $h(t)=\frac{f(t)}{1-F(t)}=\frac{-S'(t)}{S(t)}=\frac{-d[\log S(t)]}{S(t)}$ denote the reliability, cumulative distribution function and failure rate for the component, respectively. On the other hand $S_i(t)$, $F_i(t)$ and $h_i(t)$ are the reliability, distribution function and failure rate for the *i*th failure mode, respectively. Thus the competing risk model formulas for the component is given by $S_c(t)=\prod_{i=1}^{k} S_i(t)$, $F_c(t)=\prod_{i=1}^{k}\{1-F_i(t)\}$ and $h_c(t)=\sum_{i=1}^{k} h_i(t)$. Now suppose n identical units are subject to failure by any one of the k competing risks, then the likelihood function for the observations $(\mathrm{T}_i, \delta_{ij})$ where $T_i=\min\{X_{i1}, X_{i2}, \ldots, X_{ij}\}$ and $\delta_{ij}=I(\mathrm{T}_i=\mathrm{X}_{ij})$, i =1, 2,….., n is given by $L=\prod_{j=1}^{k}\prod_{i=1}^{n}(f_j(t_i))^{\delta_{ij}}(\overline{S}_j(t_i))^{1-\delta_{ij}}$.

Note : We would like to draw the attendtion of the readers on the first and second part of the likelihood function. Each of them signify two different things. For e.g., let the distribution be exponential, then $f_j(t_i)=\lambda_j e^{-\lambda_j t_i}$ and $\overline{S}_j(t_i)=e^{-\lambda_j t_i}$, then the likelihood function is, $L=\prod_{j=1}^{k}\prod_{i=1}^{n}(\lambda_j e^{-\lambda_j t_i})^{\delta_{ij}}(e^{-\lambda_j t_i})^{1-\delta_{ij}}$. Now consider there are several mechanisms due to which a system can fail. Since the mechanisms are assumed to be independent to each other, one cannot know beforehand that the system will fail because of which cause, until the system has really failed and the component which fails first causes the system to fail. Under these conditions the component reliability is the product of the failure mode reliabilities and the component failure rate is the sum of failure mode rates. This holds true for any model as long as the condition of 'independence' and the concept of first mechanism failure causes the system to fail is true.

Here for our discussion we analyze both (i) Generalized Exponential distribution (GED) and and Lognormal distribution (LN) cases.

Let the number of causes be J (for our study $J=2$). Let $T_i^{(j)}$ denote the lifetime of ith subject due to cause j (j = 1, 2), where i = 1, 2…..,n. It is assumed that $T_i^{(j)}$ are independent for all i and j, and they are identically distributed for all i given a value of j. The corresponding distribution function, density function, survival function and hazard function are given as $F^{(j)}(.\,|\,\theta^{(j)})$, $f^{(j)}(.\,|\,\theta^{(j)})$, $S^{(j)}(.\,|\,\theta^{(j)})$ and $h^{(j)}(.\,|\,\theta^{(j)})$ respectively, where $\theta^{(j)}$ are the real valued parameters for each j. The observed lifetime of i^{th} subject is given by $T_i=\min\{T_i^1, T_i^2\}$, i.e., the system will fail due to the cause which *happens earlier* or which *occurs first*. Our task is to estimate the parameters for these causes assuming they follow a particular density function and find the confidence intervals for these parameter estimates. In a real time situation we can then analyze these causes and their behaviour and try to improve the system.

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Case 1: (Generalized Exponential Distribution (GED)) non-Bayesian case

Generalized Exponential (GE) distribution is a three parameter distribution in general. Out of the three parameters, the two parameters for GE distribution are the same as in Gamma and Weibull distribution and they are the scale (λ) and the shape (α) parameters. Gamma distribution has been extensively used for analyzing the lifetime data both for its increasing as well as decreasing failure rates which depends upon the shape parameter (α), but the non existence of any closed form expression for its distribution function resulted in Weibull distribution being used more frequently, since the latter has a closed form expression for its corresponding distribution function. Furthermore GE distribution was given as an option for the Gamma and Weibull distributions and the interesting fact is that this distribution function has closed form expression. GE distribution has likelihood ratio ordering with respect to the shape parameter when the scale parameter is constant, this property is same as that for the Gamma distribution.

The the three parameter GE along with its survival and hazard function formulae are given as follows

- $G(t)=(1-\rho e^{-\lambda x})^{\alpha}$ for $t>\frac{1}{\lambda}\ln\rho$, where ρ, α and λ are the respective parameters for the distribution. When ρ = 1 it is defined as Generalized Exponential (GE) distribution which has the following distribution function $F(x:\alpha,\lambda)=(1-e^{-\lambda x})^{\alpha};\ \alpha,\lambda,x>0$
- The density function is $f(x:\alpha,\lambda)=\alpha\lambda(1-e^{-\lambda x})^{\alpha}e^{-\lambda x}$, where α is the shape parameter, λ is the scale parameter and finally ρ is the location parameter.
- The corresponding survival function for the GE distribution is given by $S(x:\alpha,\lambda)=1-(1-e^{-\lambda x})^{\alpha}$
- The hazard function is given as $h(x:\alpha,\lambda)=\frac{\alpha\lambda(1-e^{-\lambda x})^{\alpha}e^{-\lambda x}}{1-(1-e^{-\lambda x})^{\alpha}}$

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Example of GED with Complete data

Suppose we have a two component system connected in series. The failure distribution for component 1 and component 2 are given by $f_1(t)$ and $f_2(t)$ respectively. We assume that the scale parameter (λ) for both the components as same and the shape parameters are α_1 for component 1 and α_2 for component 2 respectively. Consider we have *n* such systems which are identical to each other. We assume that X_{ij}'s are independent random variables for i=1, 2,…,n and j=1,2 and they are distributed according to Generalized Exponential distribution with parameters (α_j, λ). X_{i1} and X_{i2} are the time to failure for the ith system due to component 1 and component 2 respectively and $\delta_1, \ldots, \delta_n$ denote the cause of failure for the ith system where δ_i =1, 2. Then, the time to failure for the ith system will be the minimum of the two, i.e., $X_i = \min(X_{i1}, X_{i2})$. Then the expected lifetime of the system due to cause 1 is $E[X_1] = \int_0^\infty t f_1(t)dt = \int_0^\infty t\alpha_1\lambda(1-e^{-\lambda t})^{\alpha_1 - 1}e^{-\lambda t}dt$, where $E[X_1]$ is evaluated using the method of moments and is given by $E[X_1]$ = $\frac{1}{\lambda}(\psi(\alpha_1+1)-\psi(1))$, where ψ is the digamma function. The relative risk rate (π) due to cause 1 is $\pi_1 = \left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)$ an similarly $\pi_2 = \left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)$.Suppose that out of the n systems n_1 fail due to cause 1 and n_2 fail due to cause 2. Then the likelihood function for the observed data $(x_1,\delta_1),(x_2,\delta_2)\ldots\ldots,(x_n,\delta_n)$ is given by

$$L = \prod_{i=1}^{n}\left[f_1(x_i)\bar{F}_2(x_i)\right]^{I(\delta_i=1)}\left[f_2(x_i)\bar{F}_1(x_i)\right]^{I(\delta_i=2)}$$

$$= \prod_{i=1}^{n_1}\left[\alpha_1\lambda(1-e^{-\lambda x_i})^{\alpha_1-1}e^{-\lambda x_i}(1-(1-e^{-\lambda x_i})^{\alpha_2}\right]^{I(\delta_i=1)} *$$

$$\prod_{i=1}^{n_2}\left[\alpha_2\lambda(1-e^{-\lambda x_i})^{\alpha_2-1}e^{-\lambda x_i}(1-(1-e^{-\lambda x_i})^{\alpha_1}\right]^{I(\delta_i=2)} \qquad (12.7)$$

and the log Likelihood Function (ℓ) is given by

$$\ell = n_1\ln\alpha_1 + n_2\ln\alpha_2 + n\ln\lambda + \lambda\sum_{i=1}^{n}X_i + (\alpha_1-1)\sum_{i=1}^{n_1}\ln\left[(1-e^{-\lambda x_i})\right] + \sum_{i=1}^{n_1}\ln\left[(1-(1-e^{-\lambda x_i})^{\alpha_2}\right] + (\alpha_2-1)\sum_{i=1}^{n_2}\ln\left[(1-e^{-\lambda x_i})\right] + \sum_{i=1}^{n_2}\ln\left[(1-(1-e^{-\lambda x_i})^{\alpha 1}\right] \qquad (12.8)$$

The corresponding likelihood equations can be obtained by taking the partial derivatives with respect to the parameters α_1, α_1 and λ and remembering the fact that for evaluating the MLEs of α_1, α_2, and λ we have to simultaneously solve the set of equations (12.7) and (12.8). As they do not have a closed form solution, hence we use the multidimensional Newton Raphson Method to numerically solve the

Likelihood equations to get the MLE estimates $\hat{\alpha}_1, \hat{\alpha}_2$, and $\hat{\lambda}$.

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Case 1: (Generalized Exponential Distribution (GED)) Bayesian case

In our model we utilize certain priors to obtain the Bayesian estimates for the GE distribution in the competing risks setup for complete data. Suppose we have a two component system connected in series. The failure distribution for component 1 and component 2 are given by $f_1(t)$ and $f_2(t)$ respectively.We assume that the scale parameter (λ) for both the components are same and the shape parameters are α_1 and α_2 for component 1 and component 2 respectively.Let us consider n such systems which are identical to each other.
As usual we assume that X_{ij}'s are distributed according to GE distribution for i=1,2,…n.

We already know that X_{i1} and X_{i2} are the time to failure for the ith system due to component 1 and component 2 respectively and $\delta_1,\ldots,\delta_n$ denote the cause of failure for the i^th^ system where δ_i =1, 2. Then, the time to failure for the ith system will be the minimum of the two,i.e. $X_i = \min(X_{i1}, X_{i2})$
The likelihood function for the observed data $(x_1,\delta_1),(x_2,\delta_2),\ldots\ldots,(x_n,\delta_n)$

$$L((X,\delta)\,|\,\alpha_1,\alpha_2,\lambda)) = \alpha_1^{m_1}\prod_{i\in I_1}\left[\left(1-e^{-\lambda x_i}\right)^{\alpha_1-1}\right]e^{-\lambda\sum_{i\in I_1} x_i} * \lambda^{m_1}\prod_{i\in I_1}\left[\left(1-\left(1-e^{-\lambda x_i}\right)^{\alpha_2}\right)\right] *$$
$$\alpha_2^{m_2}\prod_{i\in I_2}\left[\left(1-e^{-\lambda x_i}\right)^{\alpha_2-1}\right]e^{-\lambda\sum_{i\in I_2} x_i} * \lambda^{m_2}\prod_{i\in I_2}\left[\left(1-\left(1-e^{-\lambda x_i}\right)^{\alpha_1}\right)\right]$$

Since both α and λ are non negative, we assume priors for α_1, α_2 and λ to be gamma priors.In many situations, the information about the shape and scale of the sampling distribution is available in an independent manner, and in the similar line we assume that the parameters of α and λ are independent a priori.

Hence the joint density function is given by

$$L(\alpha_1,\alpha_2,\lambda\,|\,X,\delta) \propto \frac{\pi(\alpha_1)\pi(\alpha_2)\pi(\lambda) * L((X,\delta)\,|\,\alpha_1,\alpha_2,\lambda))}{\int_0^\infty\int_0^\infty L((X,\delta)\,|\,\alpha_1,\alpha_2,\lambda))\pi(\alpha_1)\pi(\alpha_2)\pi(\lambda)\,d\alpha_1\,d\alpha_2\,d\lambda}$$

Therefore the Bayes estimator of any function of α_1, α_2 and λ, say g($\alpha_1, \alpha_2, \lambda$) is given by

$$= \frac{\int_0^\infty\int_0^\infty g(\alpha_1,\alpha_2,\lambda)L(data\,|\,\alpha_1,\alpha_2,\lambda))\pi(\alpha_1)\pi(\alpha_2)\pi(\lambda)\,d\alpha_1\,d\alpha_2\,d\lambda}{\int_0^\infty\int_0^\infty L((X,\delta)\,|\,\alpha_1,\alpha_2,\lambda))\pi(\alpha_1)\pi(\alpha_2)\pi(\lambda)\,d\alpha_1\,d\alpha_2\,d\lambda}$$

Since it is not possible to compute the function analytically, we use the Markov Chain Monte Carlo (MCMC) method to approximate this function.

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Results using MCMC

Using MCMC we obtain the average values of $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\lambda}$, variance and bias for the GE distribution. We also report the bootstrap values of the confidence intervals. For calculating the average values of $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\lambda}$ we consider a sample size *n* of 200 and conduct 200 runs for all the models. Thus in Table 12.1, 12.2 we report our simulation runs results using MCMC method.

Table 12.1: Average values, variance and bias of $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\lambda}$

GED complete data without EM and using MCMC			
	α_1=1	α_2 =2	λ =1.5
Average	1.009	2.001	1.501
Variance	0.009	0.027	0.016
Bias	0.073	0.134	0.097
GED incomplete data without EM and using MCMC			
	α_1=1	α_2 =2	λ =1.5
Average	1.010	1.999	1.521
Variance	0.008	0.023	0.009
Bias	0.074	0.126	0.080

Table 12.2: Bootstrap-1 and bootstrap-1 confidence intervals for α_1, α_2 and λ for both complete and masked data using EM algorithm and without EM algorithm and using MCMC

GED incomplete data without using EM and using MCMC			
	α_1	α_2	λ
Bootstrap-1	1.053(0.842)	2.164(0.898)	1.894(0.869)
Bootstrap-2	0.910 (0.827)	2.053(0.853)	1.667(0.784)

GED incomplete data using EM and using MCMC			
	α_1	α_2	λ
Bootstrap-1	1.063(0.835)	2.413(0.872)	2.149(0.838)
Bootstrap-2	0.861(0.827)	2.218(0.847)	2.051(0.816)

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