## The Lecture Contains:

## Theorem 1.3

A state $i$ is ***recurrent*** iff $\sum_{n=1}^{\infty} P_{ii}^{n} = \infty$

## Proof of Theorem 1.3

Assume state $i$ is ***recurrent*** then we ***must have must have*** $\sum_{n=1}^{\infty} f_{ii}^{n} = 1$, which is what we need to prove. Now pay close attention to the concept of *generating function* from where we see that, $A(s) = \sum_{k=0}^{\infty} a_k \times s^k$, $|s| < 1$ and this is the generic form,from which we have (i) $F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^{n} \times s^n$, $|s| < 1$, i.e., (ii) $P_{ij}(s) = \sum_{n=0}^{\infty} P_{ij}^{n} \times s^n$, $|s| < 1$ (*proof given above*),would also imply $\lim_{s \to 1^-}\left(\sum_{n=0}^{\infty} f_{ii}^{n} s^n\right) = \lim_{s \to 1^-} F_{ii}(s) = 1$, which would immediately prove that $\lim_{s \to 1^-} P_{ii}(s) = \lim_{s \to 1^-}\left(\sum_{n=0}^{\infty} P_{ii}^{n} s^n\right) = \infty$, as $P_{ii}(s) = \frac{1}{[1 - F_{ii}(s)]}$ for $|s| < 1$ (refer above prove). Now using the second proof which is: if $a_k \geq 0$ and $\lim_{s \to 1-}\left(\sum_{k=0}^{\infty} a_k \times s^k\right) = a \leq \infty$,

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This is the necessary condition. Now we need to concentrate on the sufficient condition prove.

Assume the $i^{th}$ state is transient, i.e., $\lim_{s\to 1-}\left(\sum_{k=0}^{\infty} a_k \times s^k\right) = a \leq \infty$. Using the two stated facts given below :

(i) $\sum_{k=0}^{\infty} a_k = \lim_{N\to\infty}\left(\sum_{k=0}^{N} a_k\right)$ if *converges* then $\lim_{s\to 1-}\left(\sum_{k=0}^{\infty} a_k \times s^k\right) = \sum_{k=0}^{\infty} a_k = a$

and

(ii) $P_{ii}(s) = \dfrac{1}{\left[1 - F_{ii}(s)\right]}$ for $|s| < 1$

we can infer $\lim_{s\to 1^-} P_{ii}(s) < \infty$.Again utilizing the fact that if $a_k \geq 0$ and $\lim_{s\to 1-}\left(\sum_{k=0}^{\infty} a_k \times s^k\right) = a \leq \infty$, then $\sum_{k=0}^{\infty} a_k = \lim_{N\to\infty}\left(\sum_{k=0}^{N} a_k\right)$ which leads us to $\sum_{n=1}^{\infty} P_{ii}^{n} < \infty$ being true. On seeing this we can immediately conclude that it contradicts our hypothesis based on which we started, i.e., $i^{th}$ state is transient. Hence $i^{th}$ state is ***not*** transient.

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## Corollary based on Theorem 1.3

Is the following holds true, i.e., $i \leftrightarrow j$, and if $i$ is ***recurrent***, then $j$ is also ***recurrent***.

## Proof of corollary based on Theorem 1.3

If $i \leftrightarrow j$, then there exists $m, n \geq 1$, such that we have $P_{ij}^{n} > 0$ and $P_{ij}^{m} > 0$. Now if $\nu > 0$, which is arbitrary, we can obtain $P_{jj}^{m+n+\nu} \geq P_{ji}^{m} \times P_{ii}^{\nu} \times P_{ij}^{n}$, which we utilize to sum up, which leads us to $\sum_{\nu=0}^{\infty} P_{jj}^{m+n+\nu} \geq \sum_{\nu=0}^{\infty} P_{ji}^{m} \times P_{ii}^{\nu} \times P_{ij}^{n} = P_{ji}^{m} \times P_{ij}^{n} \times \sum_{\nu=0}^{\infty} P_{ii}^{\nu}$. Now if $\sum_{\nu=0}^{\infty} P_{ii}^{\nu}$ diverges, so does $\sum_{\nu=0}^{\infty} P_{jj}^{\nu}$. Now we already know that a state $i$ is ***recurrent*** iff $\sum_{n=1}^{\infty} P_{ii}^{n} = \infty$, hence it would immediately lead us to the fact that $j$ is ***recurrent*** if $i$ is ***recurrent***.

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### Note/Remark

- ***Recurrence*** and ***periodicity*** are class property, which means that all states in an equivalence class are either recurrent or non-recurrent.
- The expected number of returns to state $i$, given that $X_0 = i$ is given by $\sum_{n=1}^{\infty} P_{ii}^{n}$, hence a state $i$ is recurrent iff the expected number of returns is infinite.

Let us define two power series

(a) $P_{i,j}(s) = \sum_{n=0}^{\infty} P_{i,j} s^n$, where $|s| < 1$

(b) $F(s) = \sum_{n=0}^{\infty} f_{i,j} s^n$, where $|s| < 1$

With these definitions we can now make the following claim which is

$$F_{ii}(s)P_{ii}(s) = P_{ii}(s) - 1$$

$$\Rightarrow \sum_{n=0}^{\infty}\left(P_{ii}^{0} f_{ii}^{n} + P_{ii}^{1} f_{ii}^{n-1} + \cdots + P_{ii}^{n} f_{ii}^{0}\right) s^n = P_{ii}^{0} f_{ii}^{0} + \sum_{n=1}^{\infty} \underbrace{\left(\sum_{k=0}^{\infty} f_{ii}^{k} P_{ii}^{n-k}\right)}_{P_{ii}^{n}} s^n$$

Now we also know that $P_{ii}(s) = 1 + \sum_{n=1}^{\infty} P_{ii}^{n} s^n$, i.e., $\sum_{n=1}^{\infty} P_{ii}^{n} s^n = P_{ii}(s) - 1$

Hence we have: $F_{ii}(s)P_{ii}(s) = P_{ii}(s) - 1$, i.e., $P_{ii}(s) = \left\{\frac{1}{1 - F_{ii}(s)}\right\}$

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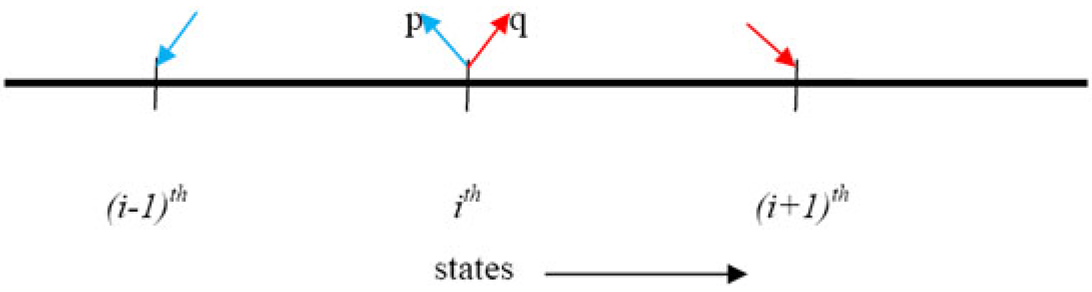
## Few more examples of recurrent Markov Chains

### Example 1.18

Consider again the simple case when a drunkard is moving one step right with probability $p$ , and one step to the left with probability $q$ , such that $p+q=1$




### Hence we have:

(i) $P_{00}^{2n+1}=0$, $n=0,1,2,\ldots$

(ii) $P_{00}^{2n}=\binom{2n}{n}p^n q^n=\left[\frac{(2n)!}{n!\times n!}\right]p^n q^n$

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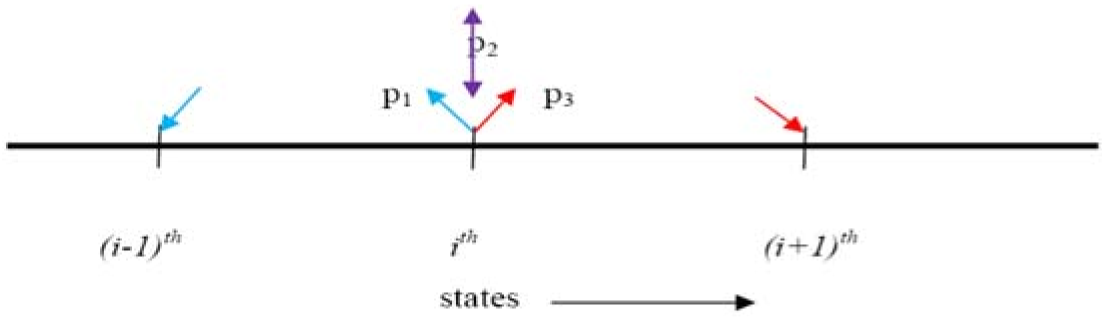
If one pays attention to (i), then this formulae would change the moment we have the following diagram ,which is given below




such that $(p_1 + p_2 + p_3) = 1$

Now going back to our original problem we solve using Stirling's formula or approximation (whichis $n! \sim n^{n+\frac{1}{2}} \times e^{-n} \times \sqrt{2\pi}$) we obta

(i) $P_{00}^{2n+1} = 0$, $n = 0,1,2,\ldots$

(ii) $P_{00}^{2n} = \binom{2n}{n} p^n q^n = \dfrac{(2n)^{2n+\frac{1}{2}} \times e^{-2n} \times \sqrt{2\pi}}{n^{n+\frac{1}{2}} \times n^{n+\frac{1}{2}} \times e^{-n} \times e^{-n} \times \sqrt{2\pi} \times \sqrt{2\pi}} p^n q^n \sim \dfrac{(4pq)^n}{\sqrt{n}} \times \dfrac{1}{\sqrt{\pi}}$

Now $pq \leq \frac{1}{4}$ and the value of $pq$ is maximum iff when $p = q = \frac{1}{2}$. Remember that this can be extended to
the case of $p_1 + \cdots. + p_n = 1$.

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Let us pay attention to the fact that $P_{00}^{2n} \sim \frac{(pq)^n 2^{2n}}{\sqrt{\pi n}} = \frac{(4pq)^n}{\sqrt{\pi n}}$ and this lead us to the fact that $P_{00}^{2n} \sim \frac{1}{\sqrt{\pi n}}$ for the case when $p = q = \frac{1}{2}$, else the rate of convergence of $P_{00}^{2n}$ is $0$. So now we have the sequence $P_{00}^{0}$, $P_{00}^{1}$, …., and the sum, i.e., $\sum_{n=0}^{\infty} P_{00}^{n} = \infty$ iff $p = q = \frac{1}{2}$. Thus the one dimension random walk is ***recurrent*** iff $p = q = \frac{1}{2}$, else it is ***transient***, i.e., we have convergence.

### Example 1.19

Can you say something of two dimensional random walk of the form, which is illustrated below, in the case when we have

(i) Probability of moving up is $p_1$

(ii) Probability of moving down $p_2$

(iii) Probability of moving right $q_1$

(iv) Probability of moving left $q_2$

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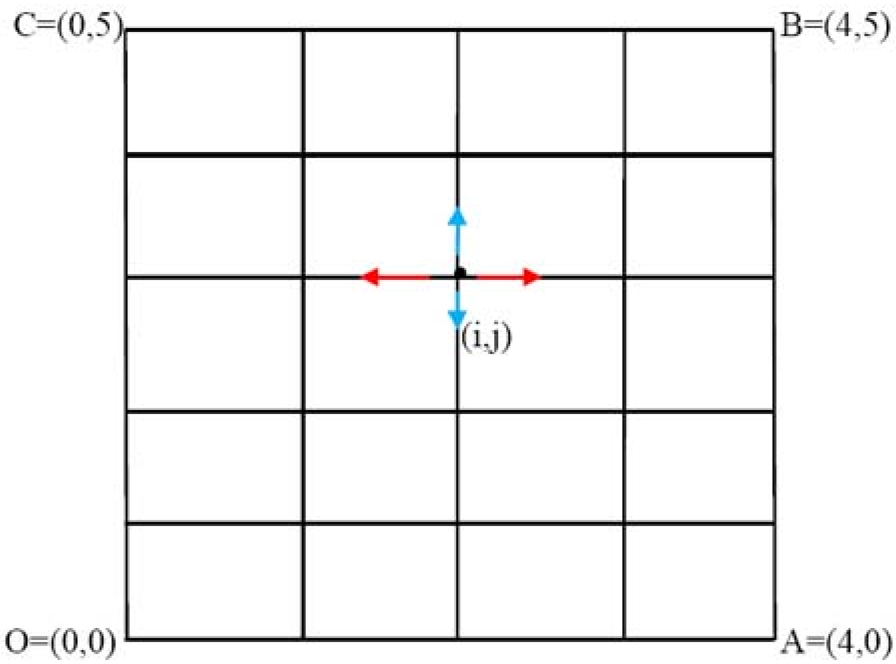
Such that $p_1 + p_2 + q_1 + q_2 = 1$, and $p_1 = p_2 = q_1 = q_2 = \frac{1}{4}$




Consider a battery operated car which can move randomly along the tracks in right, left, up, down with some fixed probability, where these probabilities do not change. Also consider the floor or plane to be infinite, i.e., there are infinite number of such states, or places the car can move. If $p_1 = p_2 = q_1 = q_2 = \frac{1}{4}$, we will see whether the origin from where the car starts is recurrent or not. Now if the car moves $i$ units to right, $i$ units to left, $j$ units to up and finally $j$ units to down, such that, , $2i + 2j = 2n$ then we have the following

(i) $P_{00}^{2n+1} = 0$, $n = 0,1,2,\ldots$ and

(ii) $P_{00}^{2n} = \sum_{\forall i,j:(i+j=n)} \left[ \frac{(2n)!}{i!\times i!\times j!\times j!} \right] \left( \frac{1}{4} \right)^{2n}$, $n = 1,2,3,\ldots$

Here we apply multinomial distribution to find the second term given above.

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## Note:

*Can you guess what happens in general when we have $p_1 \neq p_2 \neq q_1 \neq q_2$, and the movements are $i$ units to right, $j$ units to left, $k$ units to up and $l$ units to down, such that, $i+j+k+l=n$. Comment intelligently on this problem.*

Again let us continue with the problem which we were discussing. So we have

$$P_{00}^{2n} = \sum_{\forall i,j:(i+j=n)} \left[ \left\{ \frac{(2n)!}{i! \times i! \times j! \times j!} \right\} \times \left(\frac{1}{4}\right)^{2n} \times \left\{ \frac{(n!)^2}{(n!)^2} \right\} \right]$$

$$= \left(\frac{1}{4}\right)^{2n} \times \binom{2n}{n} \sum_{i=0}^{n} \left\{ \binom{n}{i} \times \binom{n}{n-1} \right\}$$

$$= \left(\frac{1}{4}\right)^{2n} \times \binom{2n}{n}^2 \text{ as } \sum_{i=0}^{n} \left\{ \binom{n}{i} \times \binom{n}{n-1} \right\} = \binom{2n}{n}$$

Using Stirling's formula or approximation, which is $n! = n^{\left(n+\frac{1}{2}\right)} \times \exp(-n) \times \sqrt{2\pi}$, we have $P_{00}^{2n} \sim \left(\frac{1}{\pi n}\right)$. Again let us pay attention to the fact that $P_{00}^{2n} \sim \left(\frac{1}{\pi n}\right)$ when $p_1 = p_2 = q_1 = q_2 = \frac{1}{4}$, hence the rate of convergence for $p_1 = p_2 = q_1 = q_2 = \frac{1}{4}$ is not zero, else the rate of convergence of $P_{00}^{2n+1}$ is 0. So now we have the sequence $P_{00}^{0}$, $P_{00}^{1}$, …., and the sum, i.e., $\sum_{n=0}^{\infty} P_{00}^{n} = \infty$ iff $p_1 = p_2 = q_1 = q_2 = \frac{1}{4}$. Thus the two dimension random walk is ***recurrent*** iff $p_1 = p_2 = q_1 = q_2 = \frac{1}{4}$.

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### Example 1.20

Can you say something of three dimensional random walk of the form, which is illustrated below, in the case when we have

(i) Probability of moving up $p_1$

(ii) Probability of moving down $p_2$

(iii) Probability of moving right $q_1$

(iv) Probability of moving left $q_2$

(v) Probability of moving front $r_1$

(vi) Probability of moving back $r_2$

Such that $p_1 + p_2 + q_1 + q_2 + r_1 + r_2 = 1$, and $p_1 = p_2 = q_1 = q_2 = r_1 = r_2 = \frac{1}{6}$

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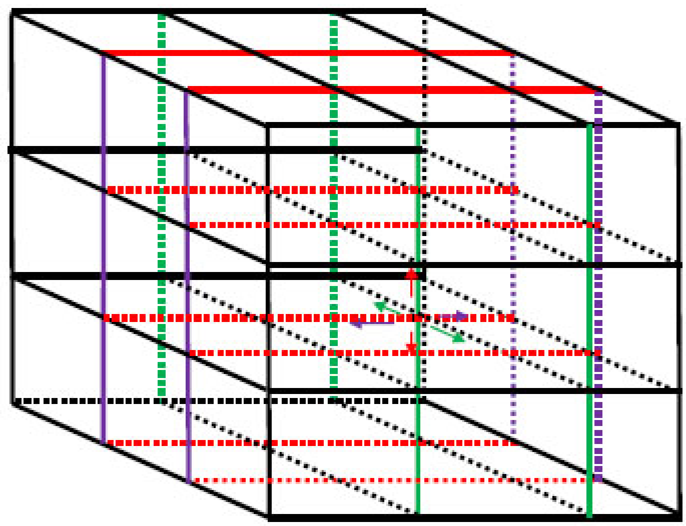

Consider again the simple case where a molecule (in a n adiabatic enclosure is randomly fluctuating all around the chamber or box, such that it rebounds from the walls and all other molecules without any loss of total energy. Consider the chamber has infinite dimension, such that there are infinite number of such states for the molecule. This problem of stating that the molecules have infinite states to visit is not impractical, as we can consider the chamber of finite size, but considering the size of the molecule it can take visit pr occue at infinite states. We can prove this formulae (which we will find out soon) for the case when $p_1 = p_2 = q_1 = q_2 = r_1 = r_2 = \left(\frac{1}{6}\right)$

(i) $P_{00}^{2n+1} = 0$, $n = 0,1,2,\ldots$

(ii) $P_{00}^{2n} = \sum_{\forall i,j,k(i+j+k=n)} \left[\frac{(2n)!}{i!\times i!\times j!\times j!\times k!\times k!}\right]\left(\frac{1}{6}\right)^{2n}$ $n = 1,2,3,\ldots$

Thus we have

$$P_{00}^{2n} = \sum_{\forall i,j,k(i+j+k=n)} \left[\frac{(2n)!}{i!\times i!\times j!\times j!\times k!\times k!}\right]\left(\frac{1}{6}\right)^{2n}$$

$$= \sum_{\forall i,j,k:(i+j+k=n)} \left\{\frac{(2n)}{i!\times i!\times j!\times j!\times k!\times k!}\right\}\left(\frac{1}{6}\right)^{2n} \times \left\{\frac{(n!)^2}{(n!)^2}\right\}$$

$$= \left(\frac{1}{2}\right)^{2n} \times \binom{2n}{n} \times \sum_{\forall i,j(i+j\le n)} \left\{\frac{n!}{i!\times j!\times (n-i-j)!}\right\}^2 \left(\frac{1}{3}\right)^{2n}$$

$$\le \max_{i,j:i+j\le n} \left\{\frac{n!}{i!\times j!\times (n-i-j)!}\right\} \times \left(\frac{1}{2}\right)^{2n} \times \binom{2n}{n} \times \left(\frac{1}{3^n}\right)$$

The above result can be obtained if we note that $\sum_{i,j:i+j\le n} \left\{\frac{n!}{i!\times j!\times (n-i-j)!}\right\} \times \left(\frac{1}{3}\right)^n = 1$. Now this fact that $\sum_{i,j:i+j\le n} \left\{\frac{n!}{i!\times j!\times (n-i-j)!}\right\} \times \left(\frac{1}{3}\right)^n = 1$ is true for this case as $p_1 = p_2 = q_1 = q_2 = r_1 = r_2 = \left(\frac{1}{6}\right)$, else we have to rework the whole problem.

Again going back to $P_{00}^{2n} \le \max_{i,j:i+j\le n} \left\{\frac{n!}{i!\times j!\times (n-i-j)!}\right\} \times \left(\frac{1}{2}\right)^{2n} \times \binom{2n}{n} \times \left(\frac{1}{3^n}\right)$, we see that for $n$ being a large value we have $\max_{i,j:i+j\le n} \left\{\frac{n!}{i!\times j!\times (n-i-j)!}\right\} \sim \left(\frac{n}{3}\right)$ when $i = j$, i.e., $i = j = k$. Thus we have

$P_{00}^{2n} \le \left\{\frac{n!}{\left(\frac{n}{3}\right)!\times \left(\frac{n}{3}\right)!\times \left(\frac{n}{3}\right)!\times 2^{2n}\times 3^n}\right\} \times \binom{2n}{n}$, and again using Stirling's formula or approximation, which is $n! = n^{\left(n+\frac{1}{2}\right)} \times \exp(-n) \times \sqrt{2n}$, we have the right hand side as $\left\{\frac{3\times\sqrt{3}}{2\pi^{\left(\frac{3}{2}\right)} \times n^{\left(\frac{3}{2}\right)}}\right\}$, when $n \to \infty$. But

if we have to find $P_{00}^0$, $P_{00}^1$, …., then the sum, i.e., , which is the

$$\sum_{n=1}^{\infty} P_{00}^{n} \leq \sum_{n=1}^{\infty}\left\{\frac{3\times\sqrt{3}}{2\pi^{\left(\frac{3}{2}\right)}\times n^{\left(\frac{3}{2}\right)}}\right\} < \infty$$

property of ***transient*** state and not ***recurrent*** state.

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Module 1:Concepts of Random walks, Markov Chains, Markov Processes

Lecture 3:Markov Chains

Now refer to the statement that ***recurrence*** and ***periodicity*** are class property, which means that all states in an equivalence class are ***either*** recurrent or non-recurrent. So for one dimension and two dimension all the states in that class are recurrent, but in three dimension it means that once the particle leaves the origin, it ***never*** returns to that state.

## Few classifications of the states and their corresponding limit theorems

The states of a Markov chain can be classified into distinct types depending on their respective limiting behavior. Suppose the Markov chain's initial state is $i$ and its final state is $j$. So if the ultimate return of the Markov chain to this $i^{th}$ is a certain event then that state, $i$, is called a ***recurrent*** state and the time of return for the first time, which is obviously a random variable is called the ***recurrent time***. In case the mean recurrence time for the first time return to the $i^{th}$ state, provided the Markov chain started from the $i^{th}$ state, is finite, then the state is called ***positive recurrent***, else if it is infinite then the state is ***non-recurrent***. Also we already know that in case the ultimate return to the $i^{th}$ state has a probability of less than one, then the state is called ***transient***.

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## Few definitions which are useful are

***Ephemeral state*** : A state $j^{th}$ is called ***ephemeral*** state if $p_{ij}=0$, $\forall\ i \in I$, i.e., this state cannot be reached from any other state. Now if we think rationally, the Markov chain can only be in the ephemeral state initially (because the process has not yet started) and pass out of the ephemeral state after the ***first*** transition, i.e., after $t=1$. Now if the characteristics of the ephemeral state are to be understood from the transition probability matrix point of view, then we have the ephemeral state as denoted by that state for which in the transition probability matrix all the probability values corresponding to that state (denoted by the corresponding column) are zeros as shown in the matrix P.

$$P=\begin{Vmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & q & 0 & p & \cdots & 0 & \cdots & 0 \\ 0 & 0 & q & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & q & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 \end{Vmatrix}$$

Let us suppose that the Markov chain is initially at state $i$, also let $f_{ii}^{(n)}$ be the probability that the next occurrence of state $i$ is at time $n$, i.e., $f_{ii}^{(1)}=p_{ii}$ and for $n=2,3,\ldots$, we have $f_{ii}^{(n)}=P\left[X_n=i; X_r\neq i: r=1,2,\ldots,n-1 \middle| X_0=i\right]$, which implies that the probability that based on the condition that the Markov chain started at $i^{th}$ state at time $t=0$, and would again be at $i^{th}$ state at time $t=n$, provided it did not ever come to the $i^{th}$ state at any of the times $t=1,2,\ldots,n-1$. This $f_{ii}^{(n)}$ is the ***first return probability*** for time $t=n$. Similarly first passage probability, $f_{ij}^{(n)}$, as the conditional probability that state $j$ is avoided at times, $t=1,2,\ldots,n-1$, and entered at time $t=n$, given that $i^{th}$ state is occupied initially. Thus we should have $f_{ij}^{(1)}=p_{ij}$ and for $n=2,3,\ldots$, we have $f_{ij}^{(n)}=P\left[X_n=i; X_r\neq i: r=1,2,\ldots,n-1 \middle| X_0=i\right]$, For the ***recurrent*** state, the ***mean recurrence time*** value is given by $\mu_i = \sum_{n=1}^{\infty}\left(n \times f_{ii}^{(n)}\right)$, and if $\mu_i$ is infinite then the state $i$ is ***null recurrent***, and in case $\mu_i$ is finite then the state $i$ is ***positive recurrent***. We must remember that $f_{ii}^{(n)}$, $n = 1,2,\ldots$ are the corresponding probabilities that state $i$ is revisited after the first, second, third, etc., transition times. In a similar line, $f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$, i.e., the sum of the probabilities that the state after starting from the $i^{th}$ state goes to $j^{th}$ state after $t = 1,2,\ldots$ time. So as $f_{ij}^{(n)}$ is the first passage probability, hence ***mean of the first passage time*** is given by $\sum_{n=1}^{\infty}\left(n \times f_{ij}^{(n)}\right)$.

Suppose a Markov chain starts at the $i^{th}$ state and comes back to the $i^{th}$ state again, but only after time periods of $t, 2t, \ldots$ and $t > 1$, then state $i$ is ***periodic***, with a periodicity of $t$ (where this $t$ is the largest integer with this property). This would imply that $p_{ii}^{(n)} = 0$ apart from when $n = k \times t$, $k \in Z^+$. A state which is not periodic is called ***aperiodic***. Just note that for a aperiodic state the periodicity is 1. An aperiodic state which is positive recurrent is called ***ergodic*** state. Below for our own convenience we summarize the definitions for a Markov chain

| SNo. | Type of state | Definition of state with $X_m = i$ | Definition in words |
|---|---|---|---|
| 1 | Periodic | $X_m = i$ and $X_{m+k\times t} = i$, $k \in Z^+$ | Return to state $i$ only possible at times $t, 2t, \ldots$ and $t > 1$ |
| 2 | Aperiodic | $X_m = i$ and $X_{m+k\times t} = i$, $k \notin Z^+$ | Not periodic |
| 3 | Recurrent | $X_m = i$ and $X_\infty = i$ | Eventual return to state $i$ is certain |
| 4 | Transient | $X_m = i$ and $X_\infty \neq i$ | Eventual return to state $i$ is uncertain |
| 5 | Positive recurrent | $X_m = i$ and $X_\infty = i$ and $\mu_i = \sum_{n=1}^{\infty}\left(n \times f_{ii}^{(n)}\right) < \infty$ | Recurrent with finite value of mean recurrence time |
| 6 | Null recurrent | $X_m = i$ and $X_\infty = i$ and $\mu_i = \sum_{n=1}^{\infty}\left(n \times f_{ii}^{(n)}\right) = \infty$ | Recurrent with infinite value of mean recurrence time |
| 7 | Ergodic | $X_m = i$ and $X_{m+k\times t} = i$, $k \notin Z^+$ and $\mu_i = \sum_{n=1}^{\infty}\left(n \times f_{ii}^{(n)}\right) < \infty$ | Aperiodic and positive recurrent (i.e., recurrent with finite value of mean recurrence time) |

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## Assignment 1.1

A psychological subject can make one of the two responses marked by $A_1$ and $A_2$, and associated with each response are a set of $N$ stimuli, i.e., $\{S_1, S_2, \ldots\ldots, S_N\}$. Each stimulus is conditioned to one of the responses. A single stimulus is sampled at random and all possibilities are equally likely and the subject responds according to the stimulus sampled. Reinforcement occurs at each trial with probability, $p\ (0 < P < 1)$ independence of the previous history of the process. When reinforcement occurs, the stimulus sampled does not alter its conditioning state. In the contrary event the stimulus becomes conditioned to the other response. Consider the Markov chain whose state variable is the number of stimuli conditioned to responseA1 . Determine the transition probability matrix for this Markov chain.

## Solution of Assignment 1.1

Let $X_n$ denote the number of stimuli conditioned to the response A1 at the nth trial. Clearly, $\{X_n : n = 1,2 \ldots \ldots\}$ represent the ***Markov chain*** with discrete state space $S = \{0,1,2, \ldots\ldots, N\}$ and the transition probabilities for this Markov chain are as followes.

$$P_{i,i} = P\{X_{n+1} = i | X_n = i\} = p$$
$$P_{i,i+1} = P\{X_{n+1} = i+1 | X_n = i\} = \frac{\{(N-i) \times (1-p)\}}{N}$$
$$P_{i,i-1} = P\{X_{n+1} = i-1 | X_n = i\} = \frac{\{i \times (1-p)\}}{N}$$

and

$$P_{0,0} = P\{X_{n+1} = 0 | X_n = 0\} = p$$
$$P_{0,1} = P\{X_{n+1} = 1 | X_n = 0\} = (1-p)$$

and

$$P_{N,N} = P\{X_{n+1} = N | X_n = N\} = p$$
$$P_{N,N-1} = P\{X_{n+1} = N-1 | X_n = N\} = (1-p)$$

Hence the transition probability for this Markov chain is given as shown below:

$$\begin{pmatrix} p & (1-p) & 0 & \cdots\cdots & 0 \\ \frac{(1-p)}{N} & p & \frac{(N-1)(1-p)}{N} & \cdots\cdots & 0 \\ \cdots\cdots\cdots & \ddots & \vdots & \cdots\cdots & 0 \\ 0 & 0 & 0 & (1-p) & p \end{pmatrix}$$

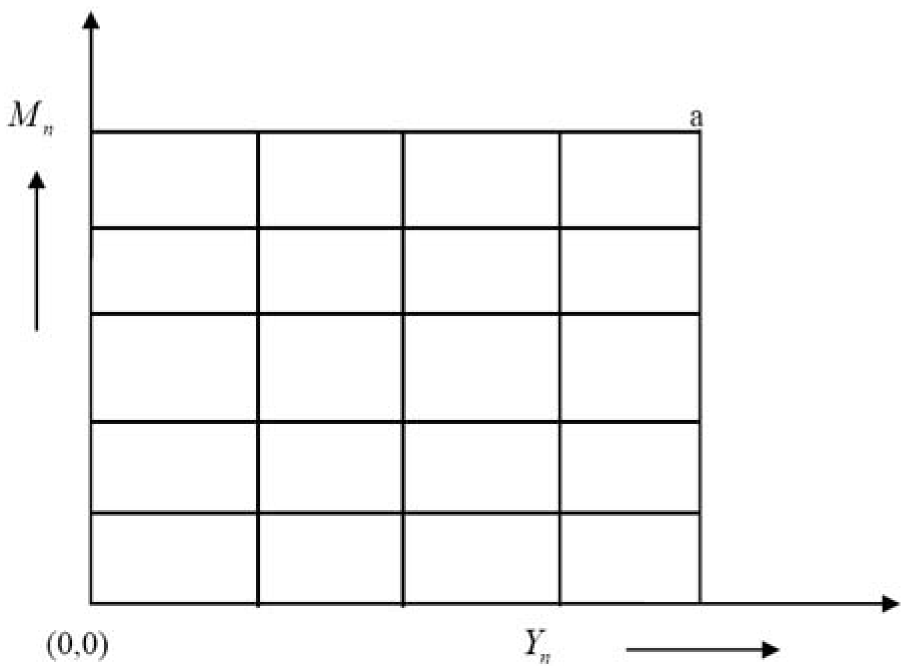
## Assignment 1.2

(a) Suppose $X_1, X_2, \ldots$ are independent with the following probabilities, i.e., $P[X_k = +1] = p$ and $P[X_k = -1] = q = (1-p)$, and $p \geq q$ .With $S_0 = 0$, set $S_n = (X_1 + \ldots + X_n)$, $M_n = Max\{S_k : 0 \leq k \leq n\}$ and $Y_n = (M_n - S_n)$. If $T(a) = \min\{n : S_n = a\}$, then show that

.

$$P\left[\max_{0 \leq k \leq T(a)} Y_k < y\right] = \begin{cases} \left(\dfrac{y}{1+y}\right)^a & if\ p = q = \dfrac{1}{2} \\ \left\{\dfrac{\dfrac{p}{q} - \left(\dfrac{p}{q}\right)^{y+1}}{1 - \left(\dfrac{p}{q}\right)^{y+1}}\right\}^a & if\ p \neq q \end{cases}$$

(b) Now if we consider the bivariate process $(M_n, Y_n)$ as a random walk on the positive two dimension lattice, then what is the probability that this random walk leaves the rectangle at the top?

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## Assignment 1.3

Determine and derive the generating function of the recurrent time from state 0 to state 0

## Solution of Assignment 1.3

Let $p_{ij}^{(n)}$ denotes the probability of transition from state i to state j in exactly n steps, and $f_{ij}^{(n)}$ denotes the probability of arriving at state j at time n for the first time, given that the process starts at state i.

Let:

$$P_{0,0}(s) = \sum_{n=0}^{\infty}\left\{p_{0,0}^{(n)} \times s(n)\right\} = 1 + \sum_{n=1}^{\infty}\left\{p_{0,0}^{(n)} \times s(n)\right\}, \quad |s|<1$$

$$F_{0,0}(s) = \sum_{n=0}^{\infty}\left\{f_{0,0}^{(n)} \times s(n)\right\} = \sum_{n=1}^{\infty}\left\{f_{0,0}^{(n)} \times s(n)\right\}, \quad |s|<1$$

be the generating function of the sequence $\{p_{00}^{(n)}\}$ and $\{f_{00}^{(n)}\}$ respectively

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## Assignments 1.4

Let $P = \begin{Vmatrix} 1-a & a \\ b & 1-b \end{Vmatrix}$, where $0 < a, b < 1$, then prove that

$$P^n = \frac{1}{(a+b)}\begin{Vmatrix} b & a \\ b & a \end{Vmatrix} + \frac{(1-a-b)^n}{(a+b)}\begin{Vmatrix} a & -a \\ -b & b \end{Vmatrix}$$

## Assignments 1.5

Consider a Markov chain and it has $r$ number of states, then prove

(a) If a state $k$ can be reached from state $j$, then it can be reached in $(r-1)$ steps or less.

(b) If $j$ is a recurrent state, then there exits $\alpha(0 < \alpha < 1)$ such that $n > r$, the probability that first return to state $j$ occurs after $n$ transition is $\leq \alpha^n$

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## Limit Theorem of Markov Chain and Applications

First let us consider few simple examples, which will motivate us about the idea why limiting theorems for Markov chains are important and their implications with respect to renewal process.

### Example 1.21

In order to obtain the pricing expressions for financial instruments, whose underlying asset may be described through a simple continuous-time random walk (CTRW) market model, one may use renewal equations pertinent to the renewal process to derive the expressions.

### Example 1.22

Suppose one is interested to find the software reliability and the costs are both deterministic as well as probabilistic. Then using the concepts of renewal process one can estimate the different metrics like mean error free time, number of errors remaining in the software product, etc.

### Example 1.23

Consider light bulbs are being replaced consecutively one after the another by a new one after the previous one fuses, then one may be interested to find the expected number of light bulbs replaced in some stipulated time, and for that one may use the concept of renewal process.

Having discussed these simple examples let us start with full earnestness the theorems necessary to under Markov chains

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### Theorem 1.4

Suppose $\{a_k\}$, $\{u_k\}$ and $\{b_k\}$ be *three* sequences indexed by $k = 0,\pm 1,\pm 2,\ldots$ . Also suppose the following are true: (i) $a_k \geq 0$, (ii) $\sum a_k = 1$, (iii) $\sum |k| \times a_k < \infty$, (iv) $\sum k \times a_k > 0$, (v) $\sum |b_k| < \infty$, and that the greatest common divisor of the integer $k$ for which $a_k > 0$ is 1.

### Proof of Theorem 1.4

(a) If the renewal equation $\left\{\left(u_n - \sum_{k=-\infty}^{\infty} a_{n-k} u_k\right) = b_n\right\}$ for $n = 0,\pm 1,\pm 2,\ldots$ is satisfied by a bounded sequence $\{u_n\}$ of real numbers, then (i) $\lim_{n\to\infty} u_n$ and (ii) $\lim_{n\to-\infty} u_n$ exist.

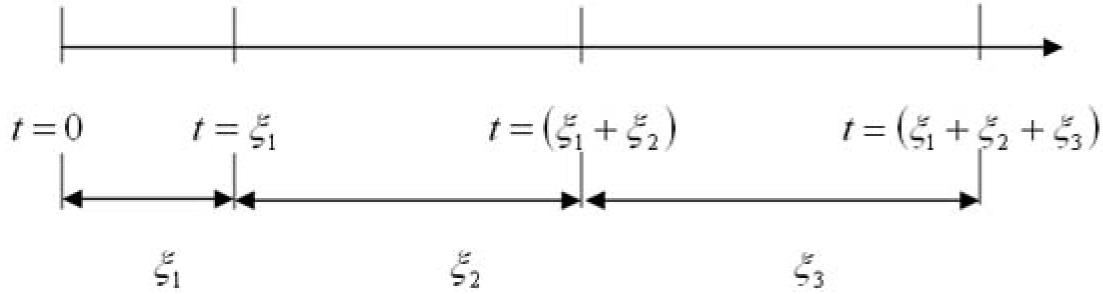
(b) Furthermore if $\lim_{n\to-\infty} u_n = 0$ (i.e., (ii)) then $\lim_{n\to\infty} u_n = \left\{\frac{\sum_{k=-\infty}^{\infty} b_k}{\sum_{k=-\infty}^{\infty}(k\times a_k)}\right\}$ [Note in case the denominator is equal to infinity, i.e., $\sum_{k=-\infty}^{\infty}(k\times a_k) = \infty$, then the limit is still valid provided we can interpret or say that

$$\left\{\frac{\sum_{k=-\infty}^{\infty} b_k}{\sum_{k=-\infty}^{\infty}(k\times a_k)}\right\} = 0$$

Now before going into the proof we will give the general definition of a renewal process where the equation would be of the form $\left\{\left(u_n - \sum_{k=0}^{n} a_{n-k} u_k\right) = b_n\right\}$ for $n = 0,1,2,\ldots$ . So consider a light bulb whose lifetime (obviously will be measured in discrete times) is a random variable, $\xi$, where $P[\xi = k] = a_k$ for $k = 0,1,2,\ldots$ , $a_k > 0$, $\sum_{k=0}^{\infty} a_k = 1$. Now if each bulb is replaced by a new one the moment the old bulb fails (fuses), such that first bulb lasts until $\xi_1$ time, the second bulb lasts until $(\xi_1 + \xi_2)$ time and so on and we must remember that $\xi_i$ are each *i.i.d*. Let $u_n$ denote the expected number of renewals (replacements) up to time $t = n$. So the first replacement occurs at time $k$, then the expected number of replacements in the remaining time upto to $n$ is $u_{n-k}$ and summing over all possible values of $k$ we have

$$u_n = \sum_{k=0}^{n}(1 + u_{n-k})\times a_k + 0\times \sum_{k=n+1}^{\infty} a_k$$

$$= \sum_{k=0}^{n} u_{n-k}\times a_k + \sum_{k=0}^{n} a_k$$




Now remember the first term, $(1+u_{n-k})$ is the expected number of bulbs replaced in time $n$ if the ***first*** bulb fails at time $k(0\le k\le n)$, and the probability of this event is $a_k$, while the second sum is the sum of the probability that the first bulb lasts a duration exceeding $n$ time units.

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## Few more example of Markov chain

### Example 1.24

Suppose customers arrive at a service station/centre in a $(M/G/1)$ queue system in accordance with Poisson process with an average rate of $\lambda$ . As this is a single server, thus when customers arrive, if they find the server in not being used they immediately go to the server and their respective job is processes, else if the server is busy the customers wait. We are to plan out scheduling system in such a way that we optimize on the metric which we consider as important to analyze how the queueing system works. Some of the metric may be average waiting time for the jobs, average idle time of the server, average processing time of the jobs, etc.

### Example 1.25

In this second example consider you have $m$ number of serves, such that that the queueing system is now denoted by $(M/G/m)$. Further more, consider that all the servers are machines which are particular types of machines like shaper, planner, grinder, etc., such that each is capable of performing only one operation. All the jobs are required to be machined by all of these $m$ servers/machines, but in any order. The arrival rate of the jobs is a Poisson process with an average rate of $\lambda$, and the throughput of the servers/machines are $\mu_i$ , $i=1,\ldots,m$ . As evident from the first example stated above, our plan for scheduling this queueing system can be based in a way that we optimize on the metric which we consider as important to analyze how the queueing system works. Some of the metric may be average waiting time for all the jobs, average idle time of all the servers/machines, average processing time of all the jobs, ratio of average utilization between the most used server/machine with respect to least used server/machine, etc.

Without any loss of generality let us discuss again the limit theorems for Markov chains

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## Theorem 1.5

Let $k$ be an arbitrary but fixed state, then

(i) $k$ is ***transient*** *iff* the series $\sum_{\forall n} p_{kk}^{(n)}$ is ***convergent*** (i.e., $P_{kk}(1) < \infty$) and in this case $\sum_{\forall n} p_{jk}^{(n)}$ is convergent for each $j$.

(ii) $k$ is ***recurrent*** *iff* the series $\sum_{\forall n} p_{kk}^{(n)}$ is ***divergent*** (i.e., $P_{kk}(1) = \infty$) and in this case $\sum_{\forall n} p_{jk}^{(n)}$ is convergent for each $j$ which communicates with $k$.

## Proof of Theorem 1.5

Let $k$ be a ***recurrent*** state and let $\mu_k = \sum_{\forall n} n f_{kk}(n)$ be its mean recurrence time, also define $\frac{1}{\mu_k} = 0$ if $\mu_k = \infty$

(i) If $k$ is periodic, then $\lim_{n\to\infty} p_{kk}^{(n)} = \frac{1}{\mu_k}$ and $\lim_{n\to\infty} p_{jk}^{(n)} = \left\{\frac{F_{jk}(1)}{\mu_k}\right\}$, where we already know that $P_{jk}(s) = F_{jk}(s) + F_{jk}(s)P_{kk}(s)$

(ii) If $k$ has a period $t$, then $\lim_{n\to\infty} p_{kk}^{(nt)} = \frac{t}{\mu_k}$ and for each state $j$ which communicates with $k$ $\lim_{n\to\infty} p_{jk}^{(r_{jk}+nt)} = \left\{\frac{tF_{jk}(1)}{\mu_k}\right\}$, where $r_{jk}$ is the smallest value of $r$ for which $p_{jk}^{(r)} > 0$

## Absorbing Markov Chains

Let us assume a hypothetical example where we have a Markov chain such that all the persistent states ($S$) of this Markov chain are absorbing, while $T$ set of states of this same Markov chain are transient. We rearrange the states in such a way (no one stops us from doing this as there is no set pattern in which the states will be reached in the stochastic process), such that we have the transition probability matrix as give: $P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix}$, and here $Q$ is a sub-matrix which corresponds to the transition among states, $i$, such that $i \in T$, while $I$ the unit matrix corresponds to the transition among state, $j$, such that $j \in S$ and $R$ is any matrix. Then calculate $P^n$.

Example for better illustration: consider we have the transition matrix as given $P = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & 0 \\ 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$

### Theorem 1.6

If $a_{ik}$ denotes the probability that the chain starting with a transient state, $i$ eventually gets absorbed in an absorbing state, $k$. Let us denote the absorption probability matrix by $A=(a_{ik})$, $\{i \in T, k \in (S-T)\}$, then $A=(I-Q)^{-1}R$

### Proof of Theorem 1.6

Now we have $a_{ik}=F_{ik}=\sum_{\forall} f_{ik}^{(n)}$ since transition between absorbing states are impossible. Moreover $F_{ik}=P\left[\bigcup_{n\geq 1}\{X_n=k|X_0=i\}\right]$. Since $k$ is absorbing so once the chain reaches an absorbing state $k$ after steps $(n+1),(n+2),\Lambda$, thus $\{X_n=k\}\subset\{X_{n-1}=k\}\subset \mathrm{K}$

This is true as $k$ can be any state in the Markov chain.

Note: Now we utilize the concept of ascending order of a sequence and its results, i.e., $P\left\{\bigcup_{i\geq 1} A_i\right\}=\lim_{n\to\infty} P(A_n)$ holds true for the case when $A_1\subset A_2\subset \mathrm{K}\subset A_n$. In case of descending order of a sequence we have $P\left\{\bigcap_{i\geq 1} A_i\right\}=\lim_{n\to\infty} P(A_n)$ holds true for the case when $A_1\supset A_2\supset \mathrm{K}\supset A_n$.

Hence utilizing this above fact we have $a_{ik}=F_{ik}=P\left\{\bigcup_{n\geq 1} p_{ik}^{(n)}\right\}=\lim_{n\to\infty} p_{ik}^{(n)}$

Also the Chapman Kolmorogrov equation can be written as $p_{ik}^{(n+1)}=\sum_{j\in S} p_{ij}^{(1)}p_{jk}^{(n)}$, remember the summation is being done for only those $j$ which are elements/members of $S$.

Now (i) $p_{kk}^{(n)}=1$ and (ii) $p_{jk}^{(n)}\neq 0$ iff $j\in T$ and $k\in S-T$

Hence we have: $p_{ik}^{(n+1)}=p_{ik}^{(1)}+\sum_{j\in T} p_{ij}^{(1)}p_{jk}^{(n)}$

$\lim_{n\to\infty} p_{ik}^{(n+1)}=\lim_{n\to\infty} p_{ik}^{(1)}+\lim_{n\to\infty}\sum_{j\in T} p_{ij}^{(1)}p_{jk}^{(n)}$

$a_{ik}=p_{ik}^{(1)}+\sum_{j\in T} p_{ij}^{(1)}a_{jk}$, which in matrix notation is $A=R+QA$, i.e., $A=(I-Q)^{-1}R$

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### Theorem 1.7

Let a finite Markov chain with state space $S=\{0,1,\ldots,l\}$ be also martingle. Then

(i) $\lim_{n\to\infty} p_{ij}^{(n)}=0$ for $j=1,2,\ldots(l-1)$

(ii) $\lim_{n\to\infty} p_{il}^{(n)}=\left(\frac{i}{l}\right)$ for $i=1,2,\ldots(l-1)$

(iii) $\lim_{n\to\infty} p_{i0}^{(n)}=1-\left(\frac{i}{l}\right)$ for $i=1,2,\ldots(l-1)$

### Proof of Theorem 1.7

Before we go through this simple proof we illustrate the concept of a ***martingle***. Now a stochastic process $\{X_n, n\geq 1\}$ is said to be martingle process if (i) $E\{|X_n|\}<\infty$ for all $n$ and (ii) $E\{X_{n+1}|X_1,X_2,\ldots,X_n\}=X_n$ .Now taking expectation we have , $E\{X_{n+1}\}=E\{X_n\}$ i.e., $E\{X_{n+1}\}=E\{X_1\}\ \forall n$

(i) Let $X_1,X_2,\ldots$ be independent random variables with mean 0, and let $S_n=\sum_{i=1}^{n}X_i$ , then $\{S_n, n\geq 1\}$ is a martingle

(ii) Let $X_1,X_2,\ldots$ be independent random variables with mean 1, and let $S_n=\prod_{i=1}^{n}X_i$ , then $\{S_n, n\geq 1\}$ is a martingle

Now as this is a martingle as well as a Markov chain with transition probability matrix P, then we would definitely have $E\{X_n|X_{n-1}=i\}=i$ , $\forall i$ , then it means that

$$\sum_{\forall j} j\times P(X_n=j|X_{n-1}=i)=i\text{, i.e., }\sum_{\forall j} j\times p_{ij}^{(1)}=i$$

Now $\sum_{\forall j} j\times p_{ij}^{(1)}=i$ is satisfied for $i=0$ iff $p_{00}^{(1)}=1$ and for $i=1$ iff $p_{ll}^{(1)}=1$. Thus if a finite Markov chain is also martingle, then its terminal states are absorbing.

Assuming that there are no further closed sets we get that the interior states, 1,2,…, ($l$-1) are all transient, hence $\lim_{n\to\infty} p_{ij}^{(n)}=0$ for $j=1,2,\ldots(l-1)$ and similarly we have $\lim_{n\to\infty} p_{il}^{(n)}=\left(\frac{i}{l}\right)$ for $i=1,2,\ldots(l-1)$ and $\lim_{n\to\infty} p_{i0}^{(n)}=1-\left(\frac{i}{l}\right)$ for $i=1,2,\ldots(l-1)$

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### Example 1.26

Let us consider a single counter at the railway booking counter at IIT Kanpur gate, where people arrive in order to buy/cancel railway tickets. Assume the time of arrivals are such that the server or the person at the counter can serve at time of 0, 1, 2,…., and for simplicity assume that in the time interval $(n, n+1)$ the number of customers is random which is denoted by $Y_n$ with $n = 0, 1, \ldots$., which are *i.i.d*. random variables with probability mass function of $P[Y_n = n] = p_n$. Furthermore consider that due to space limitation only $m$ number of persons can be accommodated in the small railway booking counter at the IIT Kanpur gate, where this $m$ includes the person at the counter who is booking/cancelling his/her ticket. In case if a passenger enters the booking counter and see that it is already full, then he/she leaves without booking/cancelling his/her ticket. Consider $X_n$ as the number of customer in the booking counter at time $n$., then $\{X_n, n > 0\}$ can be defined as a Markov chain which has the state space denoted by $\{0, 1, 2, \ldots, m\}$.

It is clear that we would have

$$X_{n+1} = \begin{cases} Y_n & \text{if } X_n = 0 \text{ and } 0 \le Y_n \le (m-1) \\ X_n - 1 + Y_n & \text{if } 1 \le X_n \le m \text{ and } 0 \le Y_n \le (m+1-X_n) \\ m & \text{otherwise} \end{cases}$$

and the probability transition matrix as $\begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0m} \\ p_{10} & p_{11} & \cdots & p_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m0} & p_{m1} & \cdots & p_{mm} \end{bmatrix}$

Now the probability distribution can easily be found out using the transition probability values, $p_{jk}$, so that we can easily write

$$\begin{aligned} &P\{X_0 = a, X_1 = b, \ldots, X_{n-1} = j, X_n = k\} \\ &= P\{X_n = k | X_{n-1} = j\} \times P\{X_{n-1} = k | X_{n-2} = l\} \times \ldots \times P\{X_1 = b | X_0 = a\} \times P\{X_0 = a\} \\ &= p_{jk} \times p_{lk} \times \ldots \times p_{ab} \times P\{X_0 = a\} \end{aligned}$$

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### Example 1.27

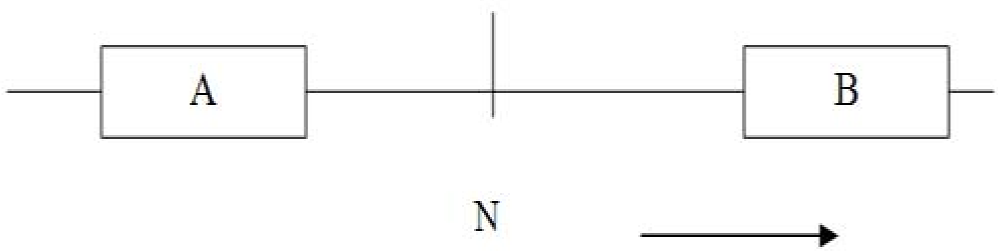
Let us consider the transition probability matrix as follows $\begin{bmatrix} ? & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & ? \\ 0 & ? & \frac{1}{4} \end{bmatrix}$. So we first find the missing values and then find the different probabilities of transition from one state to another.

### Note

The transition probability matrix along with the initial distribution (initial conditions) completely specifies the Markov chain.

### Property

1. ***Strong Markov property*** : In case $N$ is the stopping time for a Markov chain, and consider two different events A and B, such that we have




then if we have $P\{B|X_N = 1, A\} = P\{B|X_N = 1\}$, thus technically the evolution of the Markov chain starts afresh and repeats itself after it has reached the state $X_N$. Remember all discrete Markov chain have this strong Markov property.

2. ***Markov chain of order*** $s$: Consider a Markov chain, $\{X_n, n = 0,1,2,K\}$ and in case if we have s, then it is a Markov chain of order $s$. In general stock prices will be considered of order 1.

3. ***Markov chain of order 0***: For a Markov chain if we have $p_{jk} = p_k$ for $\forall j$, then it is a Markov chain of order 0.

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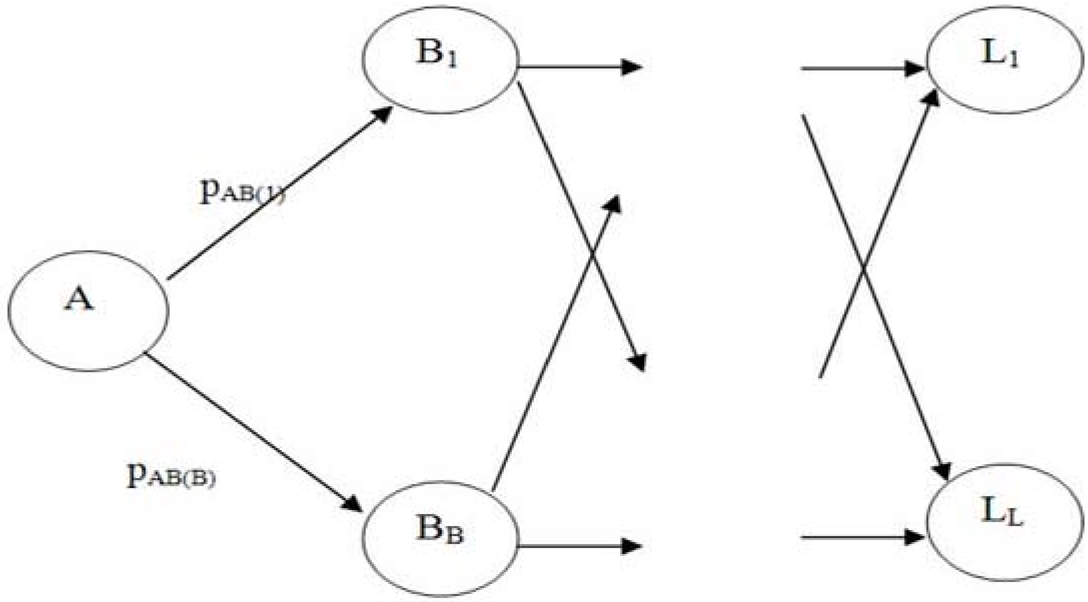
## Graph representation of Markov chain




Steady state graphical representation of Markov chains (depending on any order of the Markov chain)

Examples of (i) Graph colouring problem, (ii) stochastic network flows (oil flow, gas flow, information/data flow, etc.), (iii) network flows (maximum flow, minimum cut etc.), etc.