Module 3:Branching process, Application of Markov chains, Markov Processes with discrete and continuous state space

Lecture 13:Differential Equation for Weiner Process

The Lecture Contains:

- Kolmogorov Equations
- Ornstein Uhlenbeck Process
- Note

Previous Next

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Kolmogorov Equations

As usual let us again consider $\{X(t), t \geq 0\}$ as the ***Markov process*** in continuous time and continuous state set up such that the following assumptions are considered to be true:

- $P\{|X(t) - X(s)| > \delta | X(t) = x\} = o(t-s)$, $s < t$
- $\lim_{\Delta t \to 0} \frac{E\{X(t+\Delta t) - X(t) | X(t) = x\}}{\Delta t} = \lim_{\Delta t \to 0} \int_{|y-x| \leq \delta} (y-x) p(x,t;y,t+\Delta t) dy = a(x,t)$, where $a(x,t)$ is the ***drift coefficient***.
- $\lim_{\Delta t \to 0} \frac{E\{[X(t+\Delta t) - X(t)]^2 | X(t) = x\}}{\Delta t} = \lim_{\Delta t \to 0} \int_{|y-x| \leq \delta} (y-x)^2 p(x,t;y,t+\Delta t) dy = b(x,t)$, where $b(x,t)$ is the ***diffusion coefficient***.

Then the corresponding ***forward Kolmorogov equation*** and ***backward Kolmorogov equation*** are given by:

$$\frac{\partial}{\partial t} p(x_0,t_0;x,t) = -\frac{\partial}{\partial x}\{a(x,t)p(x_0,t_0;x,t)\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\{b(x,t)p(x_0,t_0;x,t)\}$$

$$\left.\frac{\partial}{\partial t} p(x_0,t_0;x,t)\right|_{t=t_0} =$$

$$-a(x,t)|_{x=x_0,t=t_0} \times \left.\frac{\partial}{\partial x} p(x_0,t_0;x,t)\right|_{x=x_0} - \frac{1}{2} b(x_0,t_0)|_{x=x_0,t=t_0} \times \left.\frac{\partial^2}{\partial x^2} p(x_0,t_0;x,t)\right|_{x=x_0}$$

Note

- In case $p(x_0,t_0;x,t) = p(x_0,x;t-t_0)$ then the ***Markov process*** is ***homogeneous*** and we have $a(x,t) = a(x)$ and $b(x,t) = b(x)$, i.e., both are ***independent*** of time t.
- If the ***Markov process*** is ***additive***, then $\{X(t) - X(t_0)\}$ depends only on t and t_0 and not on x_0, hence $p(x_0,t_0;x,t) = p(x-x_0;t_0,t)$ and we have $a(x,t) = a(t)$ and $b(x,t) = b(t)$, i.e., both are ***independent*** of time x.

Previous Next

Ornstein Uhlenbeck Process

The Weiner process does not provide satisfactory result for the Brownian motion for small values of t. But for moderate and large values of t it does provide good results. As alternate model which does good results for small values of t is the Ornstein Uhlenbeck model. ***The main difference being instead of displacement, $X(t)$, we consider the velocity, $U(t) = \frac{dX(t)}{dt}$.***

Now the Brownian motion of a particle if expresses using the concept of velocity takes the form of $dU(t) = -\beta U(t)dt + dF(t)$.

The first term on the right hand side, i.e., $= -\beta U(t)$ represents the part due to the resistance of the medium in which the particle is, while $dF(t)$ is the random component and $F(t)$ is Weiner process with drift $\mu = 0$ and variance parameter σ^2. To make the derivation simply and simplistic we consider $\beta U(t)$ and $F(t)$ are independent.

Now for the Markov process $\{U(t), t \geq 0\}$ as $\Delta t \to 0$, $\{U(t+\Delta t) - U(t)\} = \Delta U(t) \to 0$, and we have

- $\lim_{\Delta t \to 0} \frac{E\{U(t+\Delta t) - U(t) | U(t) = u\}}{\Delta t} = -\beta u + \lim_{\Delta t \to 0} \frac{E\{\Delta F(t)\}}{\Delta t} = -\beta u$
- $\lim_{\Delta t \to 0} \frac{Var\{U(t+\Delta t) - U(t) | U(t) = u\}}{\Delta t} = \lim_{\Delta t \to 0} \frac{(\Delta t)^2}{\Delta t} + \lim_{\Delta t \to 0} \frac{Var\{\Delta F(t)\}}{\Delta t} = \sigma^2$

In other words the limit exists hence $\{U(t), t > 0\}$ is a diffusion process and it can be proved that its transition p.d.f, i.e., $p(u_0; u, t)$ satisfies the forward Kolmorogov equation, with $a(u,t) = -\beta u$ and $b(u,t) = \sigma^2$.

Hence $p(u_0; u, t)$ satisfies the differential equation $\frac{\partial p(u_0;u,t)}{\partial t} = \beta \frac{\partial u p(u_0;u,t)}{\partial u} + \frac{1}{2}\sigma^2 \frac{\partial^2 p(u_0;u,t)}{\partial u^2}$. It can be proves simply by considering the characteristic function that $p(u_0; u, t)$ is normal with a mean value of $u_0 e^{-\beta t}$ and variance of $\sigma^2 \left(1 - \frac{e^{-2\beta t}}{2\beta}\right)$.

Previous Next

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Note

- Thus the process $\{U(t), t \geq 0\}$ is a Gaussian process with mean $u_0 e^{-\beta t}$ and variance $\sigma^2 \left(1 - \frac{e^{-2\beta t}}{4\beta}\right)$.
- Thus the process $\{U(t), t \geq 0\}$ is a Markov process but it does not possess independent increment like the Weiner process.
- As $t \to \infty$, the mean value is 0 and variance is $\frac{\sigma^2}{2\beta}$. This implies the distribution of velocity is in statistical equilibrium.
- The Ornstein–Uhlenbeck process can be interpreted as a scaling limit of a discrete process, in the same way that Brownian motion is a scaling limit of random walks.

Previous Next