The Lecture Contains:

Non-homogenous Poisson Process

A counting process denoted by $\{N(t), t \geq 0\}$ is said to be a non-stationary or non-homogeneous Poisson process with intensity function $\lambda(t)$, $t \geq 0$ if it has the following four properties which are:

- $N(0) = 0$ **:** This means the number of occurrences at time $t = 0$, i.e., when the process has just started is zero.
- $\{N(t), t \geq 0\}$ has independent increments.
- $P\{N(t+h) - N(t) = 1\} = \lambda(t)h + o(h)$: Which means that the probability of the number of events in the time interval $(0, h]$ being exactly equal to 1 is equal to the product of the rate of the process (which is dependent on time, t) and the time interval plus some incremental function of time interval, i.e., h.
- $P\{N(t+h) - N(t) \geq 2\} = o(h)$: The fourth property denotes that in case we are interested to find the probability that the number of events in the time interval $(0, h]$ is equal to 2 or more, then that probability becomes zero as the time interval shrinks or is made smaller and smaller. For the benefit of the reader we like to mention that a function $g(x)$ is said to be $o(x)$ if we have $\lim_{x \to 0}\left\{\frac{g(x)}{x}\right\} = 0$.

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Chapman Kolmogorov Equation

First let us build the motivation for ***Chapman Kolmogorov equation***. One is already aware that $P_{ij}(t) = P\{X(t+s) = j|X(s) = i\}$. Thus we are given the probability of the values such as $P\{X_{m+1} = j|X_m = i\} = p_{ij}$, which in general notation is denoted by

$$P\{X_{m+1} = j|X_m = i\} = p_{ij}^{(m,m+1)} = p_{ij}^{(1)}.$$

Hence given $p_{ij}^{(1)}$ our main concern is to find $P\{X_{m+n} = j|X_m = i\} = p_{ij}^{(m,m+n)} = p_{ij}^{(n)}$ and we also want to comment intelligently whether this probability $P\{X_{m+n} = j|X_m = i\} = p_{ij}^{(m,m+n)} = p_{ij}^{(n)}$ is ***dependent*** or ***independent*** on n.

If we look carefully we will notice that in the equation $P\{X_{m+n} = j|X_m = i\} = p_{ij}^{(m,m+n)} = p_{ij}^{(n)}$ we do not have m on the right hand side of the equation and such a process is called ***homogeneous Markov chain***, ***otherwise a non-homogeneous Markov chain***.

Note

An example of a Markov chain may be when you play a gamble with a unbiased/biased coin or a dice and the transition probability values are generally independent of the state. Then you say that the Markov chain in ***homogeneous*** in nature. On the other hand if you have a drunken person who depending on his movement consumes an arbitrary amount of liquor as he/she moves to the right or left, then the transition probability values will be effected by the value of m, and this type of process would be termed as non-***homogenous*** Markov chain.

Now from $P\{X_{m+2} = j|X_m = i\} = p_{ij}^{(m,m+2)} = p_{ij}^{(2)} = \sum_{\forall r} p_{ir} \times p_{rj}$, we easily derive the fact that

$$P\{X_{m+2} = j|X_m = i\} = p_{ij}^{(m,m+2)} = p_{ij}^{(2)} = \sum_{\forall r} p_{ir} \times p_{rj}$$

$$P\{X_{m+3} = j|X_m = i\} = p_{ij}^{(m,m+3)} = p_{ij}^{(3)} = \sum_{\forall r,s} p_{ir} \times p_{rs} \times p_{sj}$$ and so on

Using the method of induction we can easily derive the general form as

$$p_{ij}^{(m+n)} = \sum_{\forall r} p_{ir}^{(m)} \times p_{rj}^{(n)} = \sum_{\forall r} p_{ir}^{(n)} \times p_{rj}^{(m)}.$$

The equation form is $p_{ij}^{(m+n)} \geq p_{ir}^{(m)} \times p_{rj}^{(n)}$ for any value of r. Can you comment why this is be true? One should remember that for the case of homogeneous Markov chain we have: $P^{(m+n)} = P^{(m)} \times P^{(n)}$

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Example 2.1

Consider a sequence of trials, each of which has two outcomes, but the outcomes are not independent of each other from trial to trail, as we would expect for the case when we toss a coin or do an experiment where the assumption of Bernoulli trail holds. Let us assume the transition probability matrix which dictates the relationship between the trials as: $\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$, also suppose that the initial condition as $P[X_0 = 1] = p_1 = 1 - P[X_0 = 0]$. What is of interest to us is to find (i) $P[X_n = 0]$ and (ii) $P[X_n = 1]$, so for which we should have:

(i) $P[X_n = 0] = P[X_n = 0, X_{n-1} = 0] + P[X_n = 0, X_{n-1} = 1]$ and

(ii) $P[X_n = 1] = P[X_n = 1, X_{n-1} = 0] + P[X_n = 1, X_{n-1} = 1]$

To find (i) and (ii) each of them can only come from either state 0 or 1, hence through method of induction we have: $P[X_n = 1] = \left(\frac{a}{a+b}\right) + \left(p_1 - \frac{a}{a+b}\right) \times (1-a-b)^n$.

Now $p_n = P[X_n = 1]$ and $q_n = P[X_n = 0] = 1 - p_n$, hence we have

$$p_n = P[X_n = 1] = P[X_n = 1, X_{n-1} = 1] + P[X_n = 1, X_{n-1} = 0] = p_{n-1} \times (1-b) + q_{n-1} \times a$$

Furthermore $p_{n-1} = P[X_{n-1} = 1]$ can be found in terms of p_{n-2} or q_{n-2} and so can $q_{n-1} = P[X_{n-1} = 0]$ be found in terms of p_{n-2} of q_{n-2}. If we extend this logic then one can find all the terms, and hence $p_n = P[X_n = 1]$ and $q_n = P[X_n = 0]$ in terms of the initial conditions.

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Example 2.2

Consider you have an experiment which consists of trials each of which has three (3) outcomes, and the trials are dependent and we have the transition probability matrix as $\begin{bmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{bmatrix}$. Assume initial condition as $P[X_0 = 0] = p_1$, $P[X_0 = 1] = q_1$ and $P[X_0 = 2] = r_1$, with the additional conditional that $p_1 + q_1 + r_1 = 1$. Then find $p_n = P[X_n = 0]$, $q_n = P[X_n = 1]$ and $r_n = P[X_n = 2]$ in terms of the transition probability values, $p_{00}, \ldots, p_{22}$ and p_1, q_1, r_1.

Example 2.3

Let us further extend example 2.5, where you have an experiment which consists of $N = 5, 10, \infty$, trials. For the case

(i) $N = 5$, each of the trial has 5 outcomes, and the trials are not independent but have the transition probability matrix such that $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$,

(ii) when $N = 10$, each has 10 outcomes, and the trials are not independent but have the transition probability matrix such that $\overbrace{\begin{bmatrix} \frac{1}{10} & \cdots & \frac{1}{10} \\ \vdots & \cdots & \frac{1}{10} \\ \frac{1}{10} & \cdots & \frac{1}{10} \end{bmatrix}}^{10}$,

(iii) finally for the case $N = \infty$, each has ∞ outcomes, and the trials are not independent but have the transition probability matrix given by Poisson distribution, $f(n;\lambda) = \frac{\lambda^n e^{-n}}{n!}$, with mean of 3 and remember that $n = 0, 1, \ldots$.

For the three different sub-problem let us consider the initial conditions as (i) $P[X_i = i] = \frac{1}{5}$,

$i=1,2,3,4,5$, (ii) $P[X_i = i] = k \times i$, $i=1,\ldots,10$ and (iii) Poisson distribution with mean of 3. With this set of information one can easily find (i) $P[X_{50}=i]$, for $i=1,2,3,4,5$, $E(X_{50})$, $V(X_{50})$, (ii) $P[X_{50}=i]$, for $i=1,\ldots,10$, $E(X_{50})$, $V(X_{50})$, (iii) $P[X_{50}=i]\, i=1,\ldots$, for, $E(X_{50})$, $V(X_{50})$.

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Example 2.4

Consider a communications system which has states 0 and 1 only, but through many channels of communication, say n in number. It is like you are calling your friend in Mumbai and you stay in Kanpur, and say the communications can be routed through many different cities phone lines. So as there are two states let us denote the following

q = P[digit which is entered is transmitted unaltered]
p = P[digit which is entered is transmitted altered]

Also assume that $P[X_0 = 0] = a$ and $P[X_0 = 1] = b = (1-a)$. Then the transition matrix is given as $p = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} q & p \\ p & q \end{pmatrix}$, where

$p_{00} = P[X_1 = 0|X_0 = 0]$; $p_{01} = P[X_1 = 1|X_0 = 0]$; $p_{10} = P[X_1 = 0|X_0 = 1]$; $p_{11} = P[X_1 = 1|X_0 = 1]$

Now $P[X_1 = 1, X_0 = 0] = P[X_0 = 0] \times P[X_1 = 1|X_0 = 0] = a \times p$

Utilizing this we can deduce $P[X_2 = 0, X_0 = 0]$, $P[X_3 = 0, X_0 = 0], \ldots,$ $P[X_n = 0, X_0 = 0]$

Also we know that $p^2 = p \times p$, i.e., $p^2 = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \times \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} q & p \\ p & q \end{pmatrix} \times \begin{pmatrix} q & p \\ p & q \end{pmatrix}$, i.e., $p^2 = \begin{pmatrix} q^2 + p^2 & 2pq \\ 2pq & q^2 + p^2 \end{pmatrix}$. One can easily verify that the row sum is 1 as $(p+q)^2 = 1^2$.

Furthermore we can show that $p^n = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(q-p)^n & \frac{1}{2} - \frac{1}{2}(q-p)^n \\ \frac{1}{2} - \frac{1}{2}(q-p)^n & \frac{1}{2} + \frac{1}{2}(q-p)^n \end{pmatrix} = \begin{pmatrix} p_{00}^{(n)} & p_{01}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} \end{pmatrix}$

Thus we can find

$$P[X_m = 0, X_0 = 0] = P[X_0 = 1] \times P[X_m = 0|X_0 = 1] = b \times p_{10}^{(m)}$$

$$P[X_0 = 0|X_m = 0] = P[X_0 = 0] \times P[X_m = 0|X_0 = 0] + P[X_0 = 1] \times P[X_m = 0|X_0 = 1]$$ and so on

Now as $n \to \infty$ and as we always have $0 < p, q < 1$, hence

$$p_{00}^{(n)} = p_{11}^{(n)} = \frac{1}{2} + \frac{1}{2}(q-p)^n = \frac{1}{2}$$

$$p_{01}^{(n)} = p_{10}^{(n)} = \frac{1}{2} - \frac{1}{2}(q-p)^n = \frac{1}{2}$$

By now we should be aware that the following sets of equations given below can be utilized for solving different transition matrices values and the set of equations are given as:

1. $p_{ij}^{(n)} = \sum_{\forall k} p_{ik} \times p_{kj}^{(n-1)}$

2. $p_{ij}^{(m+n)} = \sum_{\forall k} p_{ik}^{(m)} \times p_{kj}^{(n)} = \sum_{\forall k} p_{ik}^{(n)} \times p_{kj}^{(m)}$

3. $P^{n} = P \times P^{n-1}$

4. $P^{m+n} = P^{m} \times P^{n}$

These are known as the Chapman Kolmorogrov equations and they are widely used in the study of stochastic processes.

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