

Lecture 1: Introduction to numbers

1.1 Introduction

"God gave us the natural numbers and rest is man's handiwork", so said the great German mathematician Leopold Kronecker, of the 19th century. Indeed numbers are fundamental to the very functioning of human society and of course are the bedrock for our sciences. Without numbers nothing would move. The numbers 1, 2, 3, 4, 5, 6, ..., the numbers with which we count have become so natural that we do not even think about them. But hidden in their simplicity lies an important fact, and that is called "abstraction." The same symbol 2, for example is used to denote two oranges and two elephants though those objects are distinctively different.

The language of modern mathematics is that of sets, an idea which was given to the mathematical world by George Cantor, a student of Kronecker. So the "obvious" natural numbers are now put in a set called the "Set of natural numbers" and is denoted by $\mathbb{N}$.

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$$

This set $\mathbb{N}$ might look slightly unfamiliar to those who have studied about sets in high school. This is probably your first encounter with an infinite set. Note that there is no end to the set of elements of $\mathbb{N}$. Apart from the number 1 every number in $\mathbb{N}$ has a predecessor and a successor.

Note that we can include the number zero and create the set of whole numbers, $\hat{\mathbb{N}}$

$$\hat{\mathbb{N}} = \{0\} \cup \mathbb{N}.$$

An important fact about $\mathbb{N}$ is that if one subtracts a natural number $a \in \mathbb{N}$ from $b \in \mathbb{N}$, to obtain $b - a$ and if $a > b$, then $b - a \notin \mathbb{N}$.

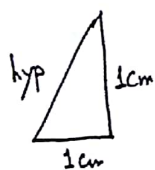
This allows us now to introduce the set of all integers which now includes negative integers. The set of all integers is called $\mathbb{Z}$ (In German 'Zahl' means number) and is given as

$$\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

Both $\mathbb{N}$ and $\mathbb{Z}$ have infinite elements. But $\mathbb{N}$ and $\mathbb{Z}$ are not the only possible type of numbers we need. We need for example fractions. Note that by dividing two integers we need not always get an integer (e.g. $\frac{1}{3}$ is not in $\mathbb{Z}$). But what we get is a fraction or a rational number. A rational number r is given as

$$r = \frac{p}{q}$$

where p and q are integers and $q \neq 0$. The set of all rational numbers is denoted by $\mathbb{Q}$. However the following question remains, can there be any number which is not rational. It might seem that $\mathbb{Q}$ is enough but unfortunately there can be numbers which are not in $\mathbb{Q}$. The ^{insight} answer is ~~surpr~~ surprisingly comes from geometry. Consider a right-angled triangle with ^{the} sides (i.e. base and height) as 1 cm and then applying ~~p~~ the Pythagoras theorem we can show that,



$$\text{length of the hypotenuse} = \sqrt{2} \text{ cm.}$$

In lecture 3 we will show that the number $\sqrt{2}$ cannot be a rational number. Numbers which are not rational are called "irrational numbers". So till now we have this fact

$$\mathbb{N} \subset \hat{\mathbb{N}} \subset \mathbb{Z} \subset \mathbb{Q}$$

The set of all ~~irrational~~ irrational numbers are denoted by $\mathbb{Q}^c$ where c denotes the complement

It is a big relief that there is nothing beyond irrational numbers and in fact there cannot be anything. This allows us to introduce the set of ^{real} rational numbers given as

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$$

We will speak more about $\mathbb{R}$ later and now speak about rational numbers.

1.2: The Properties of Rational Numbers

We shall begin by first talking about the set of natural numbers. Among the numbers in $\mathbb{N}$ some numbers are special and called prime numbers. These are the numbers which can be divided by only ^{by} itself and 1. The number 1 is not a prime number (by convention) but after that 2 is a prime and 3 is a prime and so on. We also divide the natural number into two important sets of numbers called odd numbers and even numbers

$$\text{odd numbers} = \{ 1, 3, 5, 7, \dots \}$$

$$\text{Even numbers} = \{ 2, 4, 6, 8, \dots \}$$

In fact 2 is the only even prime number. There is famous result which tells us that any $n \in \mathbb{N}$ can be uniquely represented as a product of the powers of its prime factors,

$$\text{i.e. } n = p_1^{q_1} p_2^{q_2} \dots p_r^{q_r}$$

$$\begin{aligned} \text{E.g. } 20 &= 2 \times 10 = 2 \times 2 \times 5 = 2^2 \times 5 \\ 28 &= 4 \times 7 = 2 \times 2 \times 7 = 2^2 \times 7 \end{aligned}$$

To keep this representation unique we have to avoid keeping 1 as a prime number. The important question is whether there are finite or infinite number of primes. We shall prove this after introducing an important aspect of natural numbers called the "Principle of Induction".

The principle of induction says the following.
A statement $P(n)$ is said to hold for all n if

- i) $P(1)$ is true
- ii) $P(n+1)$ is true whenever $P(n)$ is true.

As an example consider the following

$$P(n): 1 + 2 + 3 + 4 + \dots + n = \frac{n(n+1)}{2}$$

$$P(1) = 1, \text{ again } \frac{n(n+1)}{2} = 1 \text{ when } n=1.$$

Let $P(n)$ be true, then $P(n+1)$ is given as

$$\begin{aligned} & 1 + 2 + 3 + 4 + \dots + n + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{(n+1)}{1} \\ &= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2} \\ &= \frac{(n+1)(n+1+1)}{2} \end{aligned}$$

Hence $P(n+1)$ is also true and hence $P(n)$ holds for all n .

We shall now use this to show that ~~there are~~^{an} infinity of primes i.e. the set of prime numbers is not finite.

The proof we provide from the now celebrated book: "Proofs from the Book" by Aigner and Ziegler (Springer 2004, 3rd Edn).

Theorem 1.2.1 There are an infinity of primes.

Proof: Let us take a look at the "Fermat number" $F_n = 2^{2^n} + 1$. It is clear that F_n is an odd number. Of course as n varies over $\mathbb{N}$ it is clear that there are infinite number of Fermat numbers. We shall prove that any pair of Fermat numbers are relatively prime. This means that their only common divisor is 1 and there are no prime numbers which divide is a common divisor to the pair of Fermat numbers. Since each Fermat number is a common natural number

it has a ^{prime} common divisor and of course none of them has the same prime divisor and thus there must be an infinity of primes.
To show that any pair of Fermat numbers is relatively prime we use the fact

$$\prod_{k=0}^{n-1} F_k = F_n - 2 \quad (n \geq 1) \longrightarrow (A)$$

If q is a divisor of F_k and F_n ($k < n$), then q must divide 2 and thus q must be either 1 or 2. But q cannot be 2 as all Fermat numbers are odd. Thus F_k and F_n ($k < n$) are relatively prime.

We shall prove (A) by using the method of induction
For $n=1$, $F_0 = 3$, & $F_1 - 2 = 3$. Now observe that

$$\begin{aligned} \prod_{k=0}^n F_k &= \left(\prod_{k=0}^{n-1} F_k \right) F_n = (F_n - 2) F_n \\ &= (2^{2^n} + 1 - 2)(2^{2^n} + 1) \\ &= (2^{2^n} - 1)(2^{2^n} + 1) \\ &= (2^{2^n})^2 - 1 \\ &= 2^{2^{n+1}} + 1 - 2 \\ &= F_{n+1} - 2. \end{aligned}$$

This proves the result $\square$.

The most proof of this fact is the infinity of primes is due to Euclid and we skip that since this is available in many places.

We shall now discuss the important issue of representing rational numbers ~~as~~ through decimals. Decimal numbers or decimal representation of numbers is taught at school but we will see more formally what it means for rational numbers.

To begin with let us observe observe that if $\frac{p}{q}$ is a rational number then so is $\frac{rp}{rq}$ where $0 \neq r \in \mathbb{Z}$. The rational numbers are represented by either terminating or non-terminating decimals.

For example

$$\frac{1}{2} = 0.5, \quad \frac{1}{4} = 0.25, \quad \frac{1}{80} = 0.0125$$

are examples of terminating decimal representations of ^{the} rational numbers given above.

Some examples of non-terminating are

$$1 = 0.9999 \dots, \quad \frac{1}{3} = 0.33333 \dots, \quad 0.5 = 0.499999 \dots$$

So when does a rational number $\frac{p}{q}$ admits a terminating decimal expansion. To answer this question let us first take a look at the following example from Niven []

Consider the number $0.8625 = \frac{8625}{10000} = \frac{69}{80}$

Now we obtain the number 80 in the denominator is obtained by dividing 10,000 by 125 which is the greatest common divisor of 8625 and 10,000. The numerator is also obtained in the same way. Now observe the following

$$80 = 20 \times 4 = 5 \times 4 \times 4 = 5 \times 2^2 \times 2^2 = 5 \times 2^4$$

$$10,000 = 10^4 = (2 \times 5)^4 = 2^4 \times 5^4$$

i.e. both 80 and 10,000 have 2 and 5 in their unique prime factorization expression. Thus for a rational number $\frac{p}{q}$ in the lowest form, will always have only 2 and 5 as prime factors of q , since q is always a factor of some power of 10 and we know that $10 = 2 \cdot 5$.

Thus we can conclude that :

A rational ~~fraction~~ number $\frac{p}{q}$ in the lowest term admits a terminating decimal expansion if and only if q has no prime factors other than 2 and 5
(see Niven[] for a proof)

Now we will turn our attention to the fact the ~~non-period non-terminating~~ decimals (or infinite decimals) are periodic in nature. We shall follow Niven while discussing this. I am sure that the reader can observe the pattern in the following ~~numbers~~ expansions

$$\frac{5}{11} = 0.454545 \dots$$

$$\frac{3097}{9900} = 0.31282828 \dots$$

$$\cancel{0.3} = \frac{1}{3} = 0.333 \dots$$

So we see that a pattern ~~repeats~~ repeats itself. So we can write those expansions in short hand as

$$\frac{5}{11} = 0.\overline{45}, \quad \frac{1}{3} = 0.\overline{3}, \quad \frac{3097}{9900} = \cancel{0.3128} 0.\overline{3128}$$

Niven[] provides the following example of computing the decimal expansion of $\frac{2}{7}$

$$\begin{array}{r} 0.285714 \\ 7 \overline{) 2.000000 \dots} \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 30 \\ \underline{28} \\ 2 \end{array}$$

Observe that once we get to 2 the whole thing repeats itself and

$$\frac{2}{7} = 0.\overline{285714}$$

So here the first step recurs. But this need not happen always. The first step need not come but some other term can.

Niven provides the following example

$$\frac{209}{700} \text{ is}$$

$$\begin{array}{r} 0.29857142 \\ 700 \overline{) 209.00000000 \dots} \\ \underline{1400} \\ 6900 \\ \underline{6300} \\ 6000 \\ \underline{5600} \\ 4000 \\ \underline{3500} \\ 5000 \\ \underline{4900} \\ 1000 \\ \underline{700} \\ 3000 \\ \underline{2800} \\ 2000 \\ \underline{1400} \\ 600 \end{array}$$

So here we observe
the remainder 600,
the second hundred
recurs. This shows that

$$\frac{209}{700} = 0.29 \overline{857142}$$

I leave it to the reader
to think what happens
for a general rational number
 $\frac{p}{q}$.

Niven[] demonstrates ~~its~~ beautifully that any periodic decimal is
a rational number

Let

$$x = .a_1 a_2 \dots a_s \overline{b_1 b_2 \dots b_t} \quad (s > 0, t > 0, s, t \in \mathbb{N})$$

where $a_1 \dots a_s, b_1 \dots b_t$ are positive integers (natural numbers)

$$10^{t+s} x = a_1 a_2 a_3 \dots a_s b_1 b_2 \dots b_t + 0.\overline{b_1 b_2 \dots b_t}$$

$$10^s x = a_1 a_2 \dots a_s + 0.\overline{b_1 \dots b_t}$$

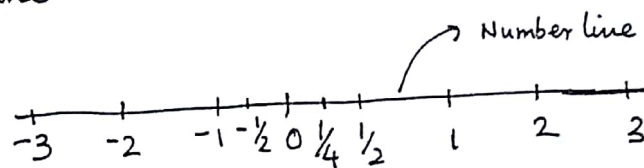
$$\therefore 10^{t+s} x - 10^s x = a_1 \dots a_s b_1 \dots b_t - a_1 \dots a_s$$

$$\therefore x = \frac{a_1 \dots a_s b_1 \dots b_t - a_1 \dots a_s}{10^{s+t} - 10^s}$$

Hence x is a rational number.

1.3 Real Numbers and their Rules

Real numbers are in some sense very geometric. It is usually represented as points of a straight line which is called the number line



The two major operations on real numbers are addition and multiplication. They are denoted as $+$ and $\cdot$. They follow (or are made to follow) certain rules. Let us write down these rules here. In this section we largely follow Zorich []

- Rules (or Axioms) for addition.

(The rule of addition $+$)

- a) If $a \in \mathbb{R}$, $b \in \mathbb{R}$, $a+b \in \mathbb{R}$ (closure property)
- b) If $(a+b)+c = a+(b+c)$ (Associative property)
- c) $a+0 = 0+a = a \quad \forall a \in \mathbb{R}$ (0 is the additive identity)
- d) If $a \in \mathbb{R}$, there exists $-a \in \mathbb{R}$, s.t

$$a+(-a) = (-a)+a = 0$$

(This is just taking the negative of a real number)

- e) $a+b = b+a, \quad \forall a, b \in \mathbb{R}$ (Commutative law)

- Rules for multiplication

- a') If $a \in \mathbb{R}$, $b \in \mathbb{R}$, $a \cdot b \in \mathbb{R}$ (closure property)
- b') $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in \mathbb{R}$ (Associative)
- c') $a \cdot 1 = 1 \cdot a = a$ (1 is the multiplicative identity)
- d') If $a \in \mathbb{R}$, $a \neq 0$, $\exists a^{-1} \in \mathbb{R}$, s.t $a \cdot a^{-1} = a^{-1} \cdot a = 1$ (multiplicative inverse)
- e') $a \cdot b = b \cdot a, \quad \forall a, b \in \mathbb{R}$ (Commutative law)

A very important law is the distribution of multiplication over addition.

$$i) a \cdot (b+c) = a \cdot b + a \cdot c$$

$$ii) (b+c) \cdot a = b \cdot a + c \cdot a = a \cdot b + a \cdot c$$

This rule is behind the multiplication rules for ~~two~~^a numbers of ~~2 or 2~~ two or more digits with another number of two or more digits taught at school.

The real number set is a completely ordered set since given any $x, y \in \mathbb{R}$, ~~the~~^{for} any of the following happens.

- ~~222~~ i) $x > y$
- ii) $x < y$
- iii) $y = x$

When x is on the right hand side of y then $x > y$, otherwise $x < y$, i.e. ~~as~~ if it is on the left. If x and y coincide then $x = y$.

$\Rightarrow$ If $x \leq y$, $x+z \leq y+z$, for any $z \in \mathbb{R}$. Further $0 \leq x$ & $0 \leq y$, $0 \leq x \cdot y$. These rules relate addition with order and multiplication with order.

— x —

Lec 2

Countability and Uncountability

2.1 Basics of Counting

The process of counting is very fundamental to human activity. We are now aware that the process of counting is done using the natural numbers $1, 2, 3, 4, \dots$. For this reason the natural numbers are often referred to as the counting numbers. Transport yourself to a time when no one knew about the natural numbers. If you are given a box of apples and a box of oranges you and you are asked the following "Which box has more fruits"? i.e. whether there are more oranges or more apples. I often ask this question when I give a more general talk on mathematics. Most of the people tell me they would weigh the boxes. This ~~does~~ really is not logical since the weight of the boxes may turn out same even if one has more fruits. However in some cases a few students would catch the basic principle. They say that they will pick an orange from the box containing oranges and pick up an apple from the other box and set them aside. As one does this if the box containing oranges become empty while the one with apples have still a few more left then we have more apples than oranges. Of course if both boxes are simultaneously empty, then there are equal number of them. Further if there are no apples left in the box and some oranges are still there then of course there are more oranges than apples.

This process of "one-to-one correspondence" is called counting. A set S is called countable if one can put its elements in one-to-one correspondence with the set of natural numbers.

$$\text{If } S = \{a_1, a_2, \dots, a_n, \dots\}$$

Then we can set up the correspondence as

$$\begin{array}{ccccccc} S = \{ & a_1 & a_2 & \dots & a_n & \dots & \} \\ & \updownarrow & \updownarrow & & \updownarrow & & \\ \mathbb{N} = \{ & 1 & 2 & \dots & n & \dots & \} \end{array}$$

Galelio in fact observed the following fact. Consider the set of natural numbers which are odd

$$O = \{1, 3, 5, 7, \dots\}$$

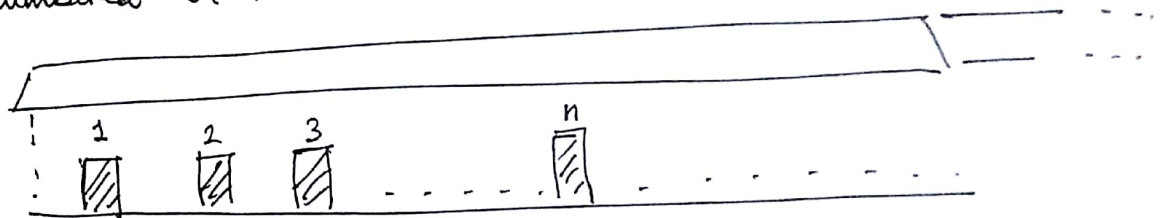
It may be quite natural to think that this set has much less amount of numbers than $\mathbb{N}$. However the following correspondence will show that it is not.

$$\begin{array}{ccccccc} O & = & \{ & 1 & , & 3 & , & 5 & , & 7 & , & \dots & \} \\ & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\ \mathbb{N} & = & \{ & 1 & , & 2 & , & 3 & , & 4 & , & \dots & \} \end{array}$$

So there are as many odd numbers as there are natural numbers. Strange are the ways of the infinite world. This fact is true for the set of even numbers.

In fact if the set A is a countable set, then $A \cup \{x\}$ is also countable. This simple looking fact is usually introduced by the story of an "Hilbert Hotel." This is a hotel which has rooms numbered $1, 2, 3, 4, 5, 6, 7, \dots$

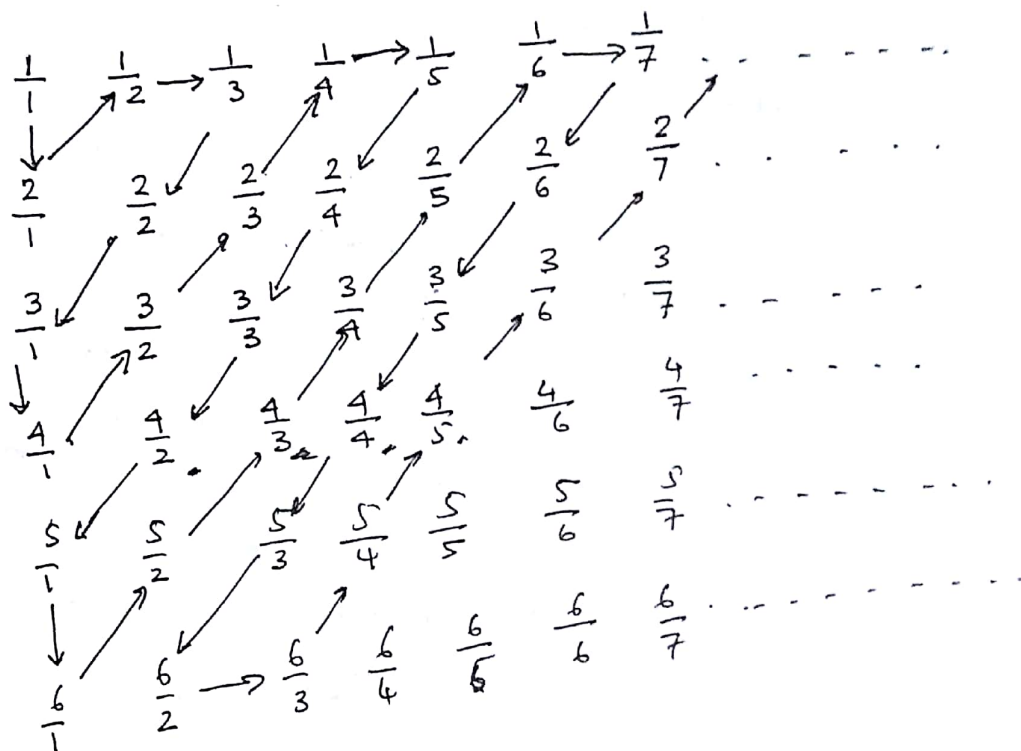
Hilbert's
Hotel



Suppose the hotel is completely filled with guests. A new guest who is also a mathematics teacher arrives at the hotel. The manager expresses his regret in not being able to give a room. The math teacher says no problem, just shift the occupant in room 1 to room 2 and that in room 2 to room 3 and room n to $n+1$ and so on. Thus the 1st room can be occupied by the math teacher. The manager was of course surprised.

Thus $\hat{\mathbb{N}} = \mathbb{N} \cup \{0\}$ is countable and thus $\hat{\mathbb{N}} \cup \{0\} \cup \{-1\}$ is countable and thus $\hat{\mathbb{N}} \cup \{0\} \cup \{-1\} \cup \{-2\}$ is countable and hence $\mathbb{Z}$ is countable.

What we can argue is the following : If M and N are countable sets then $M \cup N$ is countable. Further an infinite subset of a countable set is countable (Ponder over and think of examples). After the integers comes the chance of rationals. Are the rationals countable or uncountable. Surprisingly enough the rationals are countable. We shall show this in the following way. (See any good book on Mathematical Analysis or Proofs from book). Look at the following



Observe our technique for arrow path :

Consider $\frac{p}{q}$ with $p+q=n$, $n=2, 3, 4, 5, 6, \dots$

$\frac{p}{q}$: $p+q=2$, where $\frac{1}{1}$

$\frac{p}{q}$: $p+q=3$, $\frac{2}{1} \rightarrow \frac{1}{2}$

$\frac{p}{q}$: $p+q=4$, $\frac{3}{1} \rightarrow \frac{2}{2} \rightarrow \frac{1}{3}$

$\frac{p}{q}$: $p+q=5$, $\frac{4}{1} \rightarrow \frac{3}{2} \rightarrow \frac{2}{3} \rightarrow \frac{1}{4}$

So we are constructing sets of the form

$$M_\alpha = \left\{ \frac{p}{q}, \frac{2p}{2q}, \frac{3p}{3q}, \dots : p+q=\alpha \right\}$$

$$M_\alpha = \left\{ \frac{p}{q} : p+q=\alpha \right\}$$

Each M_α is countable and in fact finite. Thus $\bigcup_{\alpha=1}^{\infty} M_\alpha$ is countable. Note that the "Hilbert's Hotel principle" allows us to prove that countable union of countable sets is countable. Thus I leave it to the reader to rigorously show that $\mathbb{Q}$ is countable.

The approach we have taken above is due to Cantor George Cantor, a great German mathematician whose ideas were not well accepted when he was alive and never came to the centre of the German mathematical world. His own Supervisor Leopold Kronecker was displeased with him for taking on the idea of infinity head on. Cantor decided to define the notion of cardinality of the natural number set and wrote down the following

$$\text{Card}(\mathbb{N}) = \aleph_0 \text{ (Aleph-null)}$$

Aleph-null ~~was~~ ^{is} the first letter of Hebrew alphabet. So any ~~thin~~ set that is countable has cardinality $\aleph_0$. The question is the following. Is there a bigger infinity. In fact that is the question which was tackled by Cantor next. He showed that the set of real numbers is not countable. The argument he used to prove this is the "diagonal method" and is considered as one of the most elegant proof ever in the history of mathematics. Instead of considering the whole real line he simply showed the ~~uncom~~ uncountability of $(0,1)$. This stems from the fact that if $A \subset B$, and B is countable, then so is A . So if A is not countable then B cannot be so.

2.2 Uncountability of the Reals.

Cantor first proved the uncountability of the reals in ¹⁸⁷⁴~~1891~~. However his first approach is not the one which we will discuss. In 1891 he took a fresh look at the problem of proving the uncountability of the reals. In 1891 he presented the proof using the diagonal method.

Theorem 2.2.1 The interval $(0,1)$ of real numbers is uncountable.

Proof: Though this proof is now standard we will present it following the approach taken by William Dunham in his lovely book called "Journey through Genius".

The proof to begin with employs one of the greatest tools of the mathematician. This is "proof by contradiction". So we begin by assuming that under the given hypothesis the conclusion is false and then demonstrate that it will lead to contradiction.

So let us assume that $(0,1)$ is countable and let us show the correspondence with $\mathbb{N}$ as follows, ~~listing~~ noting that all real numbers will be given represented using decimal expansion. We shall represent a real number having two different representations with the representation having zeros not ~~as~~ nine, i.e. $0.5 = 0.50000000$ and not $0.5 = 0.4999999 \dots$. Hence we can now write

$\mathbb{N}$	$(0,1)$
1	$\leftrightarrow x_1 = 0.31424341 \dots$
2	$\leftrightarrow x_2 = 0.500000 \dots$
3	$\leftrightarrow x_3 = 0.009134256 \dots$
4	$\leftrightarrow x_4 = 0.3333333 \dots$
5	$\leftrightarrow x_5 = 0.986768 \dots$
$\vdots$	$\vdots$
n	$\leftrightarrow x_n = 0.a_1 a_2 a_3 a_4 a_5 \dots a_n \dots$
$\vdots$	$\vdots$

(5)

Our aim would be to construct a real number between (not), $0, c_1 c_2 c_3 c_4 \dots$ between 0 and 1, which is not included in the list.

We determine this number as follows.

- Choose c_1 to be any number between 1 and 8 and different from the first decimal digit of the x_1 , the number that corresponds to $1 \in \mathbb{N}$.
- Choose c_2 to be any number between 1 and 8 and different from the second decimal digit of x_2 , the number that corresponds to $2 \in \mathbb{N}$.
- Choose c_3 to be any number between 1 and 8 and different from the third decimal digit of x_3 , the number that corresponds to $3 \in \mathbb{N}$.

We shall continue this procedure to have, $0. c_1 c_2 c_3 c_4 \dots$.

Let us now just show how the number $0. c_1 c_2 c_3 c_4 \dots$ looks like.

First decimal^{digit} of x_1 is 3, so let $c_1 = 4$. The second decimal digit of x_2 is 5 so let $c_2 = 6$, the third decimal digit of x_3 is 9, so let $c_3 = 1$ and the fourth decimal digit of x_4 is 3 so $c_4 = 4$ and so on. Thus we have

$$0. c_1 c_2 c_3 c_4 \dots = 0. 4614. \dots$$

By excluding 0 and 9 as a choice, we strictly avoid the numbers $0. 00000 \dots = 0$ and $0. 9999999 \dots$ which is 1 and thus $0. c_1 c_2 c_3 c_4 \dots \in (0, 1)$.

i.e. it lies strictly between 0 and 1

(6)

(nine)
 Not choosing 9_k also has an advantage of ^{not} ending up with
 $0.499999\dots$ which is same as 0.5 which is represented as
 $0.50000\dots$. Now $0.c_1c_2c_3c_4\dots$ is clearly not among
 the numbers listed in the correspondence with $\mathbb{N}$. ~~Q~~

Of course $0.c_1c_2c_3c_4\dots$ is not equal to x_1 as their
 first decimal place do not match, it is not equal to x_2
 as their second decimal place do not match and so on.

Thus we have produced a number ~~$0.c_1c_2c_3$~~ . $0.c_1c_2c_3c_4\dots$
 which is in $(0,1)$ but not in the list. This is a contradiction
 and hence, $(0,1)$ is not countable and consequently $\mathbb{R}$ is
 not countable.

Chapter 3: Examples of Irrational Numbers Lecture 3:

Section 3.1: Some basic examples

In this lecture we will look into some basic examples of irrational numbers and how to construct an irrational number from a given one. The prototype example is $\sqrt{2}$.

This is taught even at the high school. We shall show that $\sqrt{2}$ is not rational. On the contrary assume that

$$\sqrt{2} = \frac{p}{q}$$

where p, q are in $\mathbb{Z}$ (~~pos~~) and $q \neq 0$ and let us assume $\frac{p}{q}$ in most reduced form; i.e. $\text{gcd}(p, q) = 1$

$$2 = \frac{p^2}{q^2}$$

$\therefore p^2 = 2q^2$, showing p^2 is an even number

Hence p^2 is an even number then so is p . Hence

$p = 2m$, thus showing that

$$4m^2 = 2q^2$$

which shows that $q^2 = 2m^2$, showing that q

is even and thus $\frac{p}{q}$ is not in the reduced form

Since they have a common factor 2. This is contradiction and hence $\sqrt{2}$ is not rational.

What about $\sqrt{3}$? This is usually not taught shown in a classroom setting. However Niven [] gives a very simple proof of this fact based on the fact that if 3 divides m^2 then 3 divides m . Consider now the contrary and

assume that $\sqrt{3} = \frac{p}{q}$

where $\frac{p}{q}$ is in the most reduced form, i.e. $\text{gcd}(p, q) = 1$.

(1)

Then

$$p^2 = 3q^2$$

Thus p^2 is divisible by 3 and hence p is divisible by 3.

So we have $p = 3r$, where ~~$r \in \mathbb{N}$~~ $r \in \mathbb{N}$.

Thus

$$9r^2 = 3q^2$$

$$\Rightarrow q^2 = 3r^2$$

Showing that q^2 is divisible by 3 and hence q is divisible by 3. Thus p and q have 3 as a common factor and contradicting the fact that $\frac{p}{q}$ is in the most reduced form.

Voila! we have thus showed that $\sqrt{3}$ is not a rational number.

Let us discuss another example from Niven [1]. Is $\sqrt{6}$ irrational? One might be tempted to prove that in the following way

$\sqrt{6} = \sqrt{2 \times 3} = \sqrt{2} \times \sqrt{3}$. Since $\sqrt{2}$ and $\sqrt{3}$ are irrational we may jump to the conclusion that $\sqrt{6}$ is irrational. But beware, product of two irrationals need not be irrational. As an example consider $\sqrt{2}$ and $\frac{1}{\sqrt{2}}$. The reader can easily show using the same approach as given for $\sqrt{2}$, that $\frac{1}{\sqrt{2}}$ is irrational. But $\sqrt{2} \times \frac{1}{\sqrt{2}} = 1$, and 1 is rational. So we should actually show that, $\sqrt{6}$ is indeed irrational. Again consider

$$\sqrt{6} = \frac{p}{q}$$

where as before $\frac{p}{q}$ is in the most reduced form.

(2)

Thus we have

$$6q^2 = p^2$$

Hence p^2 is even thus p is even and hence $p = 2m$ (say).

Therefore $p^2 = 4m^2$.

$$6q^2 = 4m^2$$

$$\underline{2q^2 = 3}$$

$$2m^2 = 3q^2$$

Thus $3q^2$ is an even number. Since 3 is odd it simply means that q^2 is even, i.e. q is even. Hence p and q have a common factor 2 and hence cannot be in the most reduced form. This is a contradiction.

Niven gives another interesting example of an irrational number, whose proof is different. Is $\sqrt{2} + \sqrt{3}$ irrational. Here we will use the fact that $\sqrt{6}$ is irrational. Let $\sqrt{2} + \sqrt{3}$ be rational.

$$\text{Let } \sqrt{2} + \sqrt{3} = r \quad (r: \text{rational})$$

Squaring both sides

$$2 + 2\sqrt{6} + 3 = r^2$$

$$\sqrt{6} = \frac{r^2 - 5}{2}$$

But $\frac{r^2 - 5}{2}$ is rational while $\sqrt{6}$ is irrational. Thus we have $\sqrt{2} + \sqrt{3}$ as an irrational number. Before we proceed further let us take a look at the following result from Niven []

Theorem 3.1.1: Let $\alpha \in \mathbb{Q}^c$, i.e. α irrational. Let $r \neq 0$ be a rational number. Then $\alpha + r$, $\alpha - r$, αr , $\frac{r}{\alpha}$, $-\alpha$, $\frac{1}{\alpha}$, $\frac{\alpha}{r}$ are all ~~rational number~~ irrational numbers.

Observe that all we have discussed so far goes under the name "surd".
 So the question is when is $\sqrt[n]{a} = a^{1/n}$ an irrational number.
 We will prove this amazing fact:

The number $\sqrt[n]{a}$ is either an integer or irrational.

We will take an approach, a beautiful one due to Niven.

Note that $\sqrt[n]{a}$ is a real-root of the polynomial equation
~~is~~ $x^n - a = 0$. So can the nature of the solutions of a
 polynomial equation give us any clue to the nature $\sqrt[n]{a}$.
 Let us consider a polynomial eqn in x given of degree n
 given as

$$c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0 \quad (c_n \neq 0)$$

A polynomial equation is an equation of the form.

$$c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0 = 0$$

We shall consider the coefficients $c_n, c_{n-1}, \dots, c_1, c_0$ are
 integers. Of course we can also consider $c_n, c_{n-1}, \dots, c_1, c_0$
 to be rational (give a little thought as to why we can do that).

To begin with let us look at the rational roots of a
 polynomial equation with integer coefficient. The key
 to ~~everything~~ that is the following result, which in turn is
 based on the prime factorization theorem.

Theorem 3.1.2: Let a, b, c , be integers. Let a be
 divisor of bc and a & b no common prime factors, i.e.
 $\gcd(a, b) = 1$. Then a is a divisor of c . In general
 we can under the given hypothesis also consider x to
 be a divisor of $b^n c$, where $n \in \mathbb{N}$, to get the same
 conclusion.