Historical origins of Modal Logic

A. V. Ravishankar Sarma

Department of Humanities and Social Sciences
IIT Kanpur
Email: avrs@iitk.ac.in

HSS, IIT Kanpur, June 2016

Outline

1 Historical Overview

2 Strict Implication

Quotations

> Ask three modal logician what modal logic is, and you are likely to get at least three different answers".
> form the Preface of the book Modal Logic by P. Blackburn, M. de Rijke and I. Venema

Two main notions of Modal Logic

1. the logic of modal sentences and modal operators,
2. the logic of relational structures (meant as Kripke frames).

The two notions do not coincide!!

Historical Overview

1. Pre-history
2. The syntactic era (1918-1959)
3. The classical era (1959-1972)
4. The modern era (1972-present)

Pre-history: Aristotles De Interpretatione

1. the problem of future contingents: a logical paradox by Diodorus Cronus, Megarian school of philosophy
2. what happens was necessarily going to happen
3. what does not happen was necessarily going to not happen
4. Aristotle: statements about the future are neither true nor false.

De Interpretatione

.....if a thing is white now, it was true before to say that it would be white, so that of anything that has taken place, it was always true to say it is or it will be. But if it was always true to say that a thing is or will be, it is not possible that it should not be or not come to be, and when a thing cannot not come to be, it is impossible that it should not come to be, and when it is impossible that it should not come to be, it must come to be. All then, that is about to be must of necessity take place. It results from this that nothing is uncertain or fortuitous, for if it were fortuitous it would not be necessary. [Translation, Ross, 1928]

Square of Opposition

1. $A : \Box p$, $E : \Box \neg p$, $I : \Diamond p$, $O : \Diamond \neg p$
2. contrary: $\neg(\Box p \wedge \Box \neg p)$; subaltern: $\Box p \rightarrow \Diamond p$, $\Box \neg p \rightarrow \Diamond \neg p$.
3. subcontrary: $\Diamond p \vee \Diamond \neg p$
4. contradictory:$\Box p \equiv \neg \Diamond p$, $\Box \neg p \equiv \neg \Diamond p$.

G.W. Leibniz (1646 –1716):

1. a possible world is made up of individuals that are compossible
2. that is, individuals that can exist together.
3. possible worlds exist as possibilities in the mind of God.
4. one world among them is realized as the actual world, and this is the most perfect one

modal status of a proposition:

1. truth: true in the actual world
2. falsity: false in the actual world
3. possibility: true in at least one possible world
4. impossibility: true in no possible world
5. necessity: true in all possible worlds
6. contingency: true in some possible worlds and false in others

The Syntactic Era: paradox of material implication

1. $\neg\phi \vdash (\phi \rightarrow \psi)$
2. $\psi \vdash (\phi \rightarrow \psi)$.
3. $(\phi \wedge \psi) \rightarrow \chi \rightarrow (\phi \rightarrow \chi) \vee (\psi \rightarrow \chi)$.
4. $(\phi \rightarrow \psi) \vdash (\phi \wedge \psi) \rightarrow \psi$.
5. $\phi \rightarrow \psi \vdash \neg\psi \rightarrow \neg\phi$.
6. $\phi \rightarrow \phi) \vee \psi \rightarrow \phi$.

Syntactic Era

1. axiomatic systems: S1 to S5 by Lewis proving distinctness theorems
2. lack of natural semantics
3. three lines of work to next stage:
   1. Carnap's state description (close to possible world semantics)
   2. Prior's tense logic: with semantic ideas and insights (the model
   3. Jonsson and Tarski: representation theorem of modal algebra (an algebraic analog of the canonical model techniques)

Results

1. Study of Lewis' systems from a syntactical point of view.
2. Mainly results of distinctness between Lewis' systems.
3. Godel (1933) de nes Lewis' systems using the modern notation, proves that Intuitionistic Propositional Logic (IPL) can be translated into S4.
4. McKinsey and Tarski (1944-1948) de ne some kinds of algebraic semantics algebras with operators" for Lewis' systems and show decidability for S2 and S4.

Classic Era

1. Kripke semantics: accessibility relation (Kripke, 1959; Hintikka, 1957; Kanger, 1957)
2. canonical model, completeness, filtration (Lemmon and Scott, 1977).
3. relational structures: as analytic tools, not really to be described
4. many applications in modeling of agents

Classical Era

1. Period : 1959-1972.
2. Systems: modal logics as semantically defined systems. Modalities: aletic, temporal, doxastic, deontic.
3. Problem: completeness, model theory.
4. Method: relational structures.
5. Strengths: clear insight of several modal systems, clear relations to propositional and firrst order logic.
6. Weaknesses: excessive trust on relational structures.

Modern Era 1972-Present

1. frame incompleteness (Thomason, 1972, 1974)
2. Sahlqvist correspondence theorem (1973)
3. universal algebra: algebraic semantics
4. classical model theory: correspondence theory, bisimulation (van Benthem)
5. connections with other fields.

Modern Era

1. computer science and AI: dynamic logic, description logic,
2. temporal logic, epistemic logic, complexity.
3. economics: game logic, (dynamic) epistemic logic
4. mathematics: co-algebra, non-well-founded set theory, geometry, topology
5. linguistics: feature logic

Modern Era- some results

1. Period : 1972-present.
2. Systems: modal logics in a wide sense.
3. Modalities: every kind of modality, introduction of the computational modalities.
4. Problem: algebraic completeness, model theory, expressivity, complexity.
5. Method: algebraic and relational structures, computational methods (automata, Turing machines).
6. Strengths: application of modal logics to other fields, in particular Theoretical Computer Science Weaknesses: ??

Dissatisfaction with the Material Implication

Paradox of Material Implication

1. $p \rightarrow (q \rightarrow p)$: A true proposition is implied by any strange kind of proposition
2. $\neg p \rightarrow (p \rightarrow q)$.: A false proposition implies any proposition.
3. $\neg(p \rightarrow q) \rightarrow (p \rightarrow \neg q)$ If a given proposition does not imply any other, then it implies the negative of that other.
4. $\neg(p \rightarrow q) \rightarrow (q \rightarrow p)$: Of any two propositions, if one does not imply the other, then the other implies the one.
5. $(p \wedge q) \rightarrow (p \rightarrow q)$: Of any two propositions, each implies the other.
6. $(\neg p \wedge \neg q) \rightarrow (p \rightarrow q)$: Of any two false propositions, each implies the other,
7. $\neg p \wedge q \rightarrow (p \rightarrow q)$. Of any two propositions, one of which is false and the other true, the false one implies the true one.

Other issues:

1. **Irrelevance/non-causality:** If the Sun is hot, then 2+2=4.
2. **Ex falsum quodlibet:** If 2+2=5 then the Moon is made of cheese.
3. **Monotonicity:** If I put sugar in my tea, then it will taste good.Therefore, If I put sugar and I put petrol in my tea then it will taste good.

Dorothy Edgington's Proof of the Existence of God

there are counter intuitive results to formulate if. . . then as $\neg p \vee q$.

Argument

If God does not exist($\neg G$), then it is not the case that if I ($P \rightarrow A$)). pray (P), my prayers will be answered(A): ($\neg(G \rightarrow \neg A)$
I dont Pray ($\neg P$).
Therefore, God Exist (G).

Contd. . .

Formal Analysis

1. $(\neg G \rightarrow \neg(P \rightarrow A))$
2. $(\neg G \rightarrow (P \wedge \neg A))$
3. $(G \vee (P \wedge \neg A))$
4. I don't pray: P=0
5. $G \vee (0 \wedge \neg A)$
6. $G \vee 0$
7. **therefore,**G (God Exist).

Intensional Disjunction and Implication

Disjunction $\vee_i$

1. Either Rani does not love me or I am beloved.
2. Either an emperor Ashoka died or the moon is made of greeen cheese.

Dissatisfaction with algebraic material implication

Why Dissatisfaction?

1. *The theorems are absurd only in the sense that they are utterly inapplicable to our modes of inference and proof. Properly, they are not rules for drawing inferences at all, but only propositions, but only propositions about nature of any world to which this system of material implication, would apply*
2. *In such a world, the all-possible must be the real, the true must be necessary, the contingent cannot exist, the false must be absurd and impossible, and the contrary to fact suppositions must be meaningless.*

Lewis on Strict implication

Lewis's Observations

1. For Lewis the ordinary meaning of *p* implies *q* is that *q* can be validly inferred from *p*, or is deducible from *p*,
2. This interpretation that he considered was not subject to these paradoxes.
3. Taking *p* implies *q* as synonymous with either not p or q, he distinguished extensional and intensional meanings of disjunction, providing two meanings for implies.
4. Extensional disjunction is the usual truth functional $\vee_e$ (or), which gives the material (algebraic) implication synonymous with it is false that p is true and q is false.
5. Intensional disjunction is such that at least one of the disjoined propositions is necessarily true.

Intensional and extensional disjunction

Intensional Disjunction

Intensional disjunction either p or q means it is impossible that *p* and *q* be both false. If either were false, the other would necessarily be true. The negation of either implies the other.

Extensional Disjunction

it happens to be the fact that atleast one of the propositions *p*, *q* , is true. It is not true that both are false.

Extensional vs Intensional disjunction:

Let $\vee_i$ be intensional disjunction and $\vee$ is extensional disjunction. $\rightarrow$ represent material implication.

For Material Implication

1. $(p \rightarrow q) = (\neg p \vee_i q) = (\neg p \vee q) = \neg(p \wedge \neg q)$.
2. $(p \wedge \neg q) = \neg(\neg p \vee q) = \neg(\neg p \vee_i q) = \neg(p \rightarrow q)$.

For strict implication:

1. $(p \rightarrow q) = (\neg p \vee_i q) \neq (\neg p \vee q) = \neg(p \wedge \neg q)$.
2. $(p \wedge \neg q) = \neg(\neg p \vee q) \neq \neg(\neg p \vee_i q) = \neg(p \rightarrow q)$.

Lewis on Strict Implication:

Lewis's Observations

1. For analyzing statement 1, we require non-boolean operators.
2. Extensional operators($\vee$, $\rightarrow$) have intensional counterparts.
3. recommends us to retain both extensional and intensional disjunction or implication, symbolize them differently, and use strict implication.
4. Adding new non-boolean operator, intensional disjunction: $\vee_e$, bearing the intuitive reading of Either A or B or, pehaps, necessarily either A or B.
5. $A \dashv B$ stands for the intuitive meaning A strictly implies B. to refer to intensional disjunction.

C I Lewis on Implication

. . . . . the expositors of the algebra of logic have not always taken pains to indicate that there is a difference between the algebraic and ordinary meanings of implication[p.522].[1]

[1] C. I. Lewis. Implication and the algebra of logic. Mind (New Series), 21:522531, 1912.

C. I Lewis on Material Implication

As indicated in the preceding chapter, we have here the further purpose to develop a calculus based upon a meaning of implies such that p implies q will be synonymous with *q is deducible from p.* The relation of material implication, which figures in most logistic calculus of propositions, does not accord with this usual meaning of *implies*. It leads to those paradoxes such as A false proposition implies every proposition and A true proposition is implied by any which have been set forth in Chapter **IVAs** we shall see, it is entirely possible so to develop the calculus of propositions that it accords with the usual meaning of *implies*. . . . . [a](p. 122-3)

[a]Lewis and Langford, Symbolic Logic, Chapter VI, The Logistic Calculus of Unanalyzed Propositions

Strict Implication(1932)

$\neg$, $\wedge$

X represents It is impossible that A is true. It is written as $\neg\Diamond X$ here.

Some Definitions

1. $A \dashv B = \neg A \vee_i B = \neg\Diamond(A \wedge \neg B)$
2. $(A = B) = (A \dashv B) \wedge (B \dashv A)$
3. $A \vee_i B =_{\text{Def}} \neg\Diamond(\neg A \wedge \neg B)$.
4. $A \dashv B =_{\text{Def}} \neg\Diamond(A \wedge \neg B)$.
5. $A = B =_{Def} \neg\Diamond(A \wedge \neg B) \wedge \neg\Diamond(B \wedge \neg A)$
6. $A \circ B$ represents intensional conjunction.
   $A \circ B = \neg\neg\Diamond(A \wedge B)$. A and B are consistent. It is possible that both A and B are true,
7. $A \circ B =_{\text{Def}} \neg(A \dashv \neg B)$

Material vs Strict Implication

Material Implication

1. 1.a. $(p \rightarrow q) \vee (q \rightarrow p)$
2. 2.a $(p \wedge q) \rightarrow (p \rightarrow q)$

Strict Implication:

1. 1.b. $(p \dashv\!\!\exists\ q) \vee (q \dashv\!\!\exists\ p)$
2. 2.b $(p \wedge q) \rightarrow (p \dashv\!\!\exists\ q)$

System S1

Lewis System: S1

1. $(p \wedge q) \dashv (q \wedge p)$
2. $(p \wedge q) \dashv p$
3. $p \dashv (p \wedge p)$
4. $((p \wedge q) \wedge r) \dashv (p \wedge (q \wedge r))$
5. $((p \dashv q) \wedge (q \dashv r) \dashv (p \dashv r)$
6. $(p \wedge (p \dashv q)) \dashv q$.

Rules of Inference

Rules

1. Uniform substitution of formulas for propositional variables.
2. Substitution of strict equivalents: from $(\alpha = \beta)$ and γ infer any formula obtained from γ by substituting β for some occurrence(s) of α.
3. Adjunction: from α and β infer $\alpha \wedge \beta$
4. Strict detachment: fromα and $\alpha \dashv \beta$ infer β

Lewis's systems

S2 to S5

1. **S2:** S1+$\Diamond(p \wedge q) \dashv p$
2. **S3:** S1+$(p \dashv q) \dashv (\neg\Diamond q \dashv \neg\Diamond p)$
3. **S4:** S1+$\Diamond\Diamond p \dashv \Diamond p$ or S1+$\Box p \dashv \Box\Box p$
4. **S5:** S1+$\Diamond p \dashv \Box\Diamond p$

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