Indian Institute of Technology Kanpur
भारतीय प्रौधोगिक संस्थान कानपुर

Crash Course on Propositional Logic

A. V. Ravishankar Sarma

Department of Humanities and Social Sciences
IIT Kanpur
Email: avrs@iitk.ac.in

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Outline

1 Syntax of Propositional Logic

2 Semantics of Propositional Logic

3 Semantic Tableaux Method: Knights and Knaves Puzzles

4 Examples:

What is Logic

1. Science or art of good reasoning.
2. Informal Logic is concerned with reasoning expressed in ordinary language, with a minimum symbolism.
3. Formal Logic is a more precise and systematic study of reasoning that employs symbolism for the purposes of brevity, clarity, generality, abstraction

Examples

Deduction

All Kanpurites are residents of Uttar Pradesh. All residents of Uttarpradesh are residents of India. Therefore all Kanpurites are the resident so India.

Induction

Crow 1 is black, Crow 2 is black.. . . N-crows are black. Most of the crows are black.

Informal Logic

All knowledge is power. All power corrupts. So, all knowledge corrupts.

Deduction, Induction, Abduction

Deduction

Rule: All the beans from this bag are white. Case: These beans are from this bag. Result: These beans are white.

Induction

Case: These beans are from this bag. Result: These beans are white. Rule: All the beans from this bag are white.

Abduction

Rule: All the beans from this bag are white. Result: These beans are white. Case: These beans are from this bag (CP 2.623).

Remark

Deduction infers a result (conclusion) that is certain. Induction produces a rule (conclusion) that is valid until a contrary instance is found. Abductive reasoning produces a case (conclusion) that is always uncertain (i.e. merely plausible). According to Peirce, deduction shows that something must be; induction shows that something in fact exists; and abduction shows that something may be.

Peirce, C.S. 193463. Collected Papers of Char les Sanders Peirce, Vols. 17, ed. Charles Hartshorne and Paul Weiss. Cambri dge, MA: Belknap Press of Harvard University.

Why Formal Logic?

1. Sentences in natural languages like English have very complex grammatical conventions, and it is not always easy to understand exactly what they mean.
2. it allows them isolate a claim they wish to defend, attack, or consider without any ambiguity or unclarity as to what they are getting at.
3. Russell thought that logic was so important was that he believed that all of mathematics could be derived from logic
4. Russell came to think that formal logic was at the core of how we should do metaphysics and epistemology.

Formal Language

1. Sentence Logic and Predicate Logic are formal languages.
2. A formal language is a set of sentences generated by rules of formation from a vocabluary.
3. The sentences of Sentence Logic and Predicate Logic are not part of natural language (though some may resemble natural-language sentences).
4. The formal languages Sentence Logic and Predicate Logic are the objects of our study, and as such they are called object languages.

Meta Language

1. If we are going to state anything about an object language, we must make use of a language.
2. We call a language used to study an object language a metalanguage.
3. In theory, the metalanguage may be identical or include the object language.
4. In some parts of linguistics, English is used to describe features of the English language.
5. We will strictly separate our metalanguage (English with some extra technical vocabulary) from our object languages.
6. Keeping the languages separate allows us to avoid paradoxes of self-reference, such as those described by Tarski.

Object and Metalanguage

1. Object language, which is the language being studied. The object language is the language about which we reason. The meta language is the language in which we reason about the object language
2. metalanguage, is the language in which the studying is being done. Sometimes these happen to be the same, as on the left, where Im making claims about English using English. But other times we might use English as a metalanguage to study some other object language.

Characteristics of Meta Language

1. Rich enough to construct a name (label) for every sentence of the object language,
2. . The language must contain the term true, which can be applied to sentences of the object language,
3. The language must contain the object language as part,
4. The language must be essentially richer than the object language,
5. The meta-language must contain logical terms, such as the expression if, if and only if.

Use vs Mention

1. When we employ the metalanguage to refer an item of language, we are said to mention that item of language.
2. To mention an item in the object language, one places it within single quotation marks.

Examples

1. 'Modi' has four letters and starts with an M.
2. 'Narendra Modi was born in Andhra Pradesh' is a false sentence.
3. 'This sentence is false' is true

In each of the examples, English as the metalanguage is used to mention words or sentences of English. In the sentence 'Barack Obama is president,' 'Barack Obama' is used to refer to the president

Language of Propositional Logic: Syntax



Definition

The alphabet of propositional logic consists of

1. Infinitely many propositional variables p0, p1, . . .,
2. The logical connectives: $\{\neg \vee, \wedge, \rightarrow, \leftrightarrow\}$
3. Parentheses $\{(.)\}$.
4. We usually write p, q, r, . . . for propositional variables. $\bot$ is pronounced bottom, and T as Top.

Definition

The formulas of propositional logic are given inductively by:

1. Every propositional variable is a formula. $\top$, $\bot$ are formulas,
2. If ϕ and ψ are formulas then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$.

Syntax

Definition

The set of formulas of propositional logic is given by the abstract syntax: Form
$\ni A, B, C ::= P \mid \bot \mid (\neg A) \mid (A \wedge B) \mid (A \vee B) \mid (A \to B)$
where P ranges over a countable set Prop , whose elements are called propositional symbols or propositional variables. (We also let Q, R range over Prop .)
Formulas of the form $\bot$, $\top$ or P are called atomic. $\top$ abbreviates $(\neg\bot)$ and $(A \leftrightarrow B)$ abbreviates $((A \to B) \wedge (B \to A))$

Conventions

Conventions to omit parentheses are:

1. outermost parentheses can be dropped;
2. the order of precedence (from the highest to the lowest) of connectives is: $\neg$, $\wedge$, $\vee$ and $\rightarrow$;
3. binary connectives are right-associative.

A is a subformula of B when A occurs in B.

Example

1. $((\neg p) \wedge ((\neg q) \wedge ((\neg r) \wedge ((\neg s) \wedge \top))))$
2. Indicate this reading by writing all parentheses:
   $\neg p \wedge q \rightarrow \neg r \vee \neg p \rightarrow r$
3. $(((\neg p) \wedge q) \rightarrow (((\neg r) \vee (\neg p)) \rightarrow r))$

Example

1. $((\neg p) \wedge ((\neg q) \wedge ((\neg r) \wedge ((\neg s) \wedge \top))))$
2. Indicate this reading by writing all parentheses:
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Semantics

1. T (true) and F (false) form the set of truth values.
2. A valuation is a function v: Prop $\Rightarrow\Rightarrow \{F, T\}$ that assigns truth values to propositional symbols.
3. Given a valuation v, the interpretation function $[[\ ]]v$: Form $\Rightarrow F, T$ is defined recursively as follows:
   1. $[[\bot]] = F$
   2. $[[P]]v = T$ iff $v(P) = T$.
   3. $[[\neg A]]v = T$ iff $[[A]]v = F$
   4. $[[A \wedge B]]v = T$ iff $[[A]]v = T$ and $[[B]]v = T$
   5. $[[A \vee B]]v = T$ iff $[[A]]v = T$ or $[[B]]v = T$
   6. $[[A \rightarrow B]]v = T$ iff $[[A]]v = F$ or $[[B]]v = T$

Definition Model in Propositional Logic

A propositional model $\mathcal{M}$ is a set of proposition symbols, i.e. $\mathcal{M} \subseteq$ Prop . The validity relation $\models$, $\subseteq$ P(Prop) x Form is defined inductively by:

1. $\mathcal{M} \models P$ iff $P \in \mathcal{M}$
2. $M \models \neg A$ iff $M \not\models A$.
3. $\mathcal{M} \models A \wedge B$ iff $M \models A$ and $M \models B$.
4. $M \models A \vee B$ iff $\mathcal{M} \models A$ or $\mathcal{M} \models B$
5. $\mathcal{M} \models A \rightarrow B$ iff $\mathcal{M} \not\models A$ or $\mathcal{M} \models B$.

Definition



1. A formula A is valid in a model $\mathcal{M}$ (or $\mathcal{M}$ satisfies A), iff $\mathcal{M} \models A$. When $\mathcal{M} \not\models A$, A is said refuted by $\mathcal{M}$.
2. A formula A is satisfiable iff there exists some model $\mathcal{M}$ such that $\mathcal{M} \models A$. It is refutable iff some model refutes A.
3. A formula A is valid (also called a tautology) iff every model satisfies A. A formula A is a contradiction iff every model refutes A.
4. A is logically equivalent to B, (denoted by $A \equiv B$) iff A and B are valid exactly in the same models.
5. Compactness: A (possibly infinite) set of formulas Γ is satisfiable if and only if every finite subset of Γ is satisfiable.

Examples: Logical equivalence

1. $\neg\neg A \equiv A$ (double negation)
2. $\neg(A \wedge B) \equiv \neg A \vee \neg B$ $\neg(A \vee B) \equiv \neg A \wedge \neg B$ (De Morgans laws)
3. $A \rightarrow B \equiv \neg A \vee B$ $\neg A \equiv A \rightarrow \bot$ (interdefinability)
4. $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$;
   $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ (distributivity)

Definitions

Notation

We let Γ, Γ' range over sets of formulas and use Γ, A to abbreviate $\Gamma \cup \{A\}$.

Definition

Let be a set of formulas:

1. Γ is valid in a model $\mathcal{M}$ (or M satisfies Γ), iff $\mathcal{M} \models A$ for every formula $A \in \Gamma$. .
2. We denote this by $\mathcal{M} \models$.
3. Γ is satisfiable iff there exists a model $\mathcal{M}$ such that $\mathcal{M} \models \Gamma$, and it is refutable iff there exists a model $\mathcal{M}$ such that $M\Gamma$.
4. Γ is valid, denoted by $\models \Gamma$, iff $\mathcal{M} \models \Gamma$ for every model $\mathcal{M}$, and it is unsatisfiable iff it is not satisfiable.

Logical Consequence

Let A be a formula and Γ a set of formulas. If every model that validates Γ also validates A, we say that entails A (or A is a logical consequence of Γ).
We denote this by $\Gamma \models A$ and call $\models \subseteq$ P(Form) x Form the semantic entailment or logical consequence relation.

Proposition

1. A is valid iff $\models A$, where $\models A$ abbreviates $\emptyset \models A$.
2. A is a contradiction iff $A \models \bot$.
3. $A \equiv B$ iff $A \models B$ and $B \models A$. (or equivalently, $A \leftrightarrow B$ is valid).

Soundness and Completeness

Soundness

A proof-theoretic consequence relation $\vdash$ is sound with respect to a semantic consequence relation $\models$ if and only if, for every possible set of premises $\{P1, P2, P3...\}$, and every possible conclusion Q, if $\{P1, P2, P3\} \vdash Q$ then $\{P1, P2, P3, ...\} \models Q$.

$\Gamma \vdash Q$ implies $\Gamma \models Q$.

If $\Gamma \vdash A$, then $\Gamma \models A$

Completeness

A proof-theoretic consequence relation $\vdash$ is complete with respect to a semantic consequence relation $\models$ if and only if, for every possible set of premises $\{P1, P2, P3...\}$, and every possible conclusion Q, if $\{P1, P2, P3\} \vdash Q$ then $\{P1, P2, P3, ...\} \models Q$.

Some Questions

The main questions about a formal logic are:

1. The soundness of the deductive system: no provable formula is invalid
2. The completeness of the deductive system: all valid formulae are provable

Maximally Consistent



Definition

Γ is maximally consistent iff it is consistent and furthermore, given any formula A, either A or $\neg A$ belongs to (but not both).

Proposition

Maximally consistent sets are closed for derivability, i.e. given a maximally consistent set Γ and given a formula A, $\Gamma \vdash A$ implies $A \in \Gamma$.

Lemma

If Γ is consistent, then there exists $\Gamma' \supset \Gamma$ s.t. Γ' is maximally consistent.

Satisfiability

Γ is consistent iff Γ is satisfiable.

Properties of Semantic Entailment

The semantic entailment relation satisfies the following properties (of an abstract consequence relation):

1. For all $A \in \Gamma$, $\Gamma \models A$. (inclusion)
2. If $\Gamma \models A$, then $\Gamma, B \models A$. (monotonicity)
3. If $\Gamma \models A$ and $\Gamma, A \models B$, then $\Gamma \models B$. (cut)
4. $\models A \wedge B$ iff $\Gamma \models A$ and $\Gamma \models B$
5. $\Gamma \models A \vee B$ iff $\Gamma \models A$ or $\Gamma \models B$
6. $\Gamma \models A \rightarrow B$ iff $, A \models B$
7. $\Gamma \models \neg A$ iff $\Gamma, A \models \bot$.
8. $\Gamma \models A$ iff $\Gamma, \neg A \models \bot$.
9. Γ is said inconsistent if $\Gamma \models \bot$ and otherwise is said consistent.
10. If Γ is consistent, then either $\Gamma \cup \{A\}$ or $\Gamma \cup \{\neg A\}$ is consistent (but not both).

Tableau Method

1. This method has been introduced for propositional logic and predicate logic by the Dutch philosopher and logician E W. Beth (1908-1964), Simplified by Hintikka(model sets), Raymond Smullyan.
2. A Semantic tree is a device for displaying all the valuations on which the formula or set of formulas is true.
3. Basic Idea: An inference is valid if and only if there exists no counter examples, i.e., there is no situation in which the premises hold and the conclusion is false.
4. This involves rule-based construction of a counter-example for a given inference. We start with negation of formula and see whether the tableau closes. Each step of the construction is given account of in a tree like structure called a *tableau.*
5. Tableau closes when there is a conflicting information.No counter example can be constructed (the branch is not

Tree Method

1. Truth trees are one of the most efficient ways of checking the semantic properties of propositional formulas.
2. In particular, it gives a very easy way of checking the validity of sequents.
3. The basic idea of truth trees is that they give a graphic way of displaying whether or not a set of formulas is inconsistent.

Tableau Rules: Alpha rules

$\neg p$

$\therefore \neg p$

$$\frac{(p \vee q)}{\begin{array}{c} \wedge \\ p \quad q \end{array}}$$

$$\frac{(p \wedge q)}{\begin{array}{c} p \\ q \end{array}}$$

$$\frac{(p \rightarrow q)}{\begin{array}{c} \wedge \\ \neg p \quad q \end{array}}$$

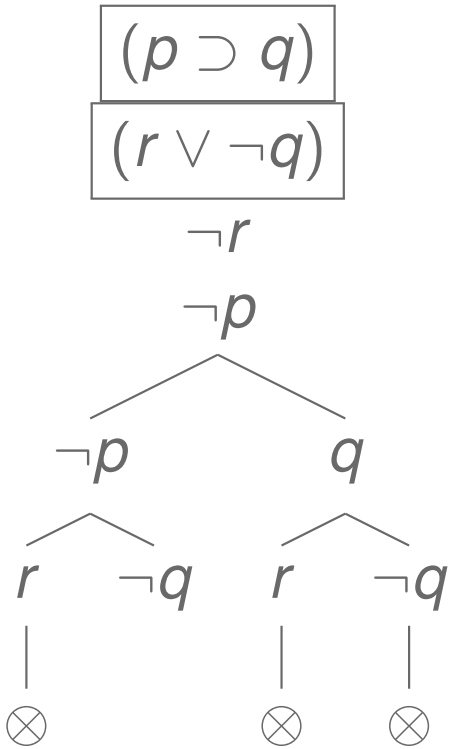
$$\frac{(p \leftrightarrow q)}{\begin{array}{c} \wedge \\ p,q \;//\neg p,\; \neg q \end{array}}$$

Tableau Rules: Beta rules

$\neg\neg p$
$\therefore p$

$$\frac{\neg(p \vee q)}{\neg p \quad \neg q} \qquad \frac{\neg(p \wedge q)}{\neg p, \neg q} \qquad \frac{\neg(p \rightarrow q)}{p \quad \neg q} \qquad \frac{\neg(p \leftrightarrow q)}{p, \neg q, \neg p, q}$$

Example



Why Semantic tableaux method?

1. Semantic tableaux method conducts a direct search for models. All the open paths of the tree corresponding to satisfiability of conjunction of formulas at the node.
2. Traditional approaches, such as constructing a truth table, can take 2^n steps for n. If n is large, truth table method is difficult o handle as propositional letters which are large.

Some Definitions

Path

A path of a tree (in any stage of construction) is a complete column of formulas from top to the bottom of the tree.
A route going from the initial set of formulas at the top down to one of the atomic formulas at the bottom, while choosing only one side of each branch as one goes, will be called here a path

Definitions

Finished Path

A path is finished if it is closed or if the only unchecked formula it contains are propositional variables or negations of prepositional variables so that no more rules apply to its formulas. A tree is closed or finished if all of its paths are closed.

Open and closed Path

An open path is a path that has not been ended with an X. A closed path is a path that has been ended with an X.

Some Definitions



Definition

A formula occurs on a path if it is on the path and is not merely a sub formula of some other formula on that path (2) it is unchecked.

Strategy

Using Tree Method

Apply non-branching rules before branching rules.

validity, Satisfiability



1. To determine whether a formula is valid, construct a tree starting with its premises and the negation of its conclusion. If all paths close, the formula is valid. If not, it is invalid and the open paths display the counter examples.
2. To show that $A \models B$ as valid, it suffices to show that $A \wedge \neg B$ is unsatisfiable.
3. To determine whether a formula or set of formulas is consistent, construct a tree starting with that formula (or set of formulas). If all paths close, that formula (set of formulas) is inconsistent. If not, it is consistent, and the open paths display the valuations that make the formula true.
4. A formula A is tautology iff $\neg A$ is unsatisfiable.
5. **Contingency:** Construct two different trees, one to test it for consistency and one to test for validity. If the formula is consistent but not valid, then it is contingent.

Theorems

1. **Main Theorem:**A completed semantic tableau for a formula A is closed if and only if A is unsatisfiable.
2. **Soundness**: If a tableau is closed, then A is unsatisfiable.
3. **Completeness:** If a wff *A* is unsatisfiable, then any tableau for A is closed.
4. **Corollary1:** A well formed formula *A* is a satisfiable formula if and only if any tableau for A is open.
5. **Corollary2:** A well formed formula *A* is a valid formula (tautology) if and only if a tableau for $\neg A$ is closed.

Some Theorems

soundness of the tableau method

If α is tableau provable, then α is a tautology, i.e.$\vdash \alpha$ implies $\models \alpha$

The tableau method is consistent. This means that there is no proposition α such that both $\vdash \alpha$ and $\vdash \neg\alpha$.

Soundness of the tableau method

If α is provable in the natural deduction system(ND), then α is a tautology, i.e. $\vdash_{ND} \alpha$ implies $\models \alpha$.

Knights and Knaves Puzzle using Truth Table:

On some island, there are knights (who always tell the truth) and knaves (who always lie).

Problem

You meet two islanders (call them A and B) and hear the first one say at least one of us is a knave. Can you tell whether the islanders are knights or knaves and which islander is which?

Knights and Knaves

P1

In this example, there are two people on the island: A and B. A says: I am a knave, but B isn't. (What are A and B?)

Explanaton

Think of this as the compund statement: I am a Knave and B is not a Knave.
There is no way the speaker can be a Knight, since the first part of the statement is false.
So A must be a KNAVE lying about B which makes B a KNAVE as well.

Knights and Knaves: Truth Table Method

Take p :A is a knight and
q : B is a knight.
Then the sentence At least one of us is a knave is translated as $(\neg p \vee \neg q)$, since being a knave is the negation of being a knight. $p \leftrightarrow (\neg p \vee \neg q)$

Knights and Knaves

p	q	$\neg p \vee \neg q$	$p \rightarrow (\neg p \vee \neg q)$
T	T	F	F
T	F	T	T
F	T	T	F
F	F	T	F

Knights and Knaves:

1. The island of Knights and Knaves has two types of inhabitants, Knights who always tell the truth, and Knaves who always lie (no knaves are knights).
2. Suppose A is the proposition person a is a knight and suppose a makes a statement S . Then, A is true if and only if S is true, since A is equivalent to S .
3. That is, $A \leftrightarrow S$. So, whenever an inhabitant x makes a claim S , we can infer that $X \leftrightarrow S$. That is, we can infer that x is a knight if and only if S is true.

Knights and Knaves Puzzles:

1. If a says *I am a Knight* then we can infer from this statement that $A \leftrightarrow A$. But, since this is logically true, we get no information from such a statement.
2. A native cannot say *I am a Knave*, since if this were true, then it would be false and if it were false, then it would be true (and, no Knights are Knaves).
3. If a says *I am the same type as b,* then we can infer $A \leftrightarrow (A \leftrightarrow B)$ which is equivalent to B (that is, $B \equiv A \leftrightarrow (A \leftrightarrow B)$. So, this statement allows us to infer that person b is a Knight!

Some Puzzles



1. You meet two people, A and B. A says: I'm a knave but B isn't. What are A and B?
2. Suppose A says: If I am a knight, then so is B. Can it be determined what A and B are?
3. Suppose you know that A and B are either both knights or both knaves. What do you make of A's statement If B is a knight, then I am a knave"?
4. Suppose A says: We are both knights" and B says Either A is a knight or I am a knight, but not both." What can you conclude?

Example

It is rumoured that there is gold buried on the Island. You ask one of the the natives, A, whether there is gold on the island. He makes the following response: *There is a gold on this island equivales I am a knight*. The problem is as follows:

1. Can it be determined whether A is knight or Knave?
2. Can it be determined whether there is gold on the island?

Gold in an Island

Let G denote the proposition: There is Gold in an Island
Statement made by A: $A \leftrightarrow G$.
The statement: $A \leftrightarrow G$

It is not possible to tell whether A is a knight or knave.

Tourist at the Fork

A tourist comes to a fork in the road, where one branch leads to Restaurant, and one doesn't. A native of the island is standing at the fork, Formulate a single yes/no question that the tourist can ask such that the answer will be yes if the left fork leads to the restaurant, and otherwise the answer will be no.

Tourist at the Fork

Let Q be the question.
Let A be the native is a knight.
Let L be the proposition, the left fork leads to the restaurant

We require that L equivalent to the response to the question is yes.
Response to the question Q is yes equivales QA.
We require $L \leftrightarrow (Q \leftrightarrow A)$.
$Q \leftrightarrow (L \leftrightarrow A)$.

Tourist at the Fork



Question

Is it the case that th statement that the left fork leads to the restaurant is equivalent to your being knight?

Knights and Knaves problem:

We have three inhabitants, A, B, and C, each of whom is a knight or a knave. Two people are said to be of the same type if they are both knights or both knaves. A and B make the following statements:

1. A: B is a knave.
2. B: A and C are of the same type.

What is C?

Knights and Knaves

Suppose, A and B say the following:

1. A: All of us are knaves.
2. B: Exactly one of us is a knave.

Can it be determined what B is? Can it be determined what C is?

Example



Problem

If today is Tuesday , I have a test in Mathematics or Economics . If my Economics professor is sick, I will not have a test in Economics. Today is Tuesday, and my Economics professor is sick. Therefore, I will have a test in Mathematics..

Converting to logical notation:
t = today is Tuesday
m= I have a test in Mathematics
e= I have a test in Economics
s = My Economics Professor is sick

Constraint Satisfaction problem

Planning a party

You are planning a party, but your friends are a bit touchy about who will be there.

1. If Jay comes, he will get very hostile if Sita is there.
2. Sita will only come if Kavita will be there also.
3. Kavita says she will not come unless Jay does.

Who can you invite without making someone unhappy?

Example

Translation to logic: let J;(S; K) represent Jay (Sita, Kavitha) comes to the party.
Then the constraints are:

1. $J \rightarrow \neg S$.
2. $S \rightarrow K$
3. $K \rightarrow J$.

Fir a Successful Party to be possible, we want the formula $\phi = (J \rightarrow \neg S) \wedge (S \rightarrow K) \wedge K \rightarrow J)$ to be satisfiable
Truth values for J, S, K making this true are satisfying assignments,

Example

Who is Guilty?

There was a robbery in which a lot of goods were stolen. The robber (s) left in a truck. It is known that :

1. Nobody else could have been involved other than A, B and C.
2. C never commits a crime without A's participation.
3. B does not know how to drive.

Is A innocent or guilty?

Example

There was a robbery in which a lot of goods were stolen. The robber (s) left in a truck. It is known that :

1. Nobody else could have been involved other than A, B and C. $(A \vee B \vee C)$.
2. C never commits a crime without A' s participation.$(C \rightarrow A)$
3. B does not know how to drive $(B \rightarrow [(B \wedge A) \vee (B \wedge C)]$.

Is A innocent or guilty?

A is Guilty


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Thank You!

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