2.2 JACOBI ITERATION

We write the system as in (4) of section 2.1 as

$$\left.\begin{array}{l} a_{11}x_1 = -a_{12}x_2 - a_{13}x_3 ..... - a_{1n}x_n + y_1 \\ a_{22}x_2 = -a_{21}x_1 - a_{23}x_3 ..... - a_{2n}x_n + y_2 \\ .... \qquad ...... \\ a_{nn}x_n = -a_{n1}x_1 - a_{n2}x_2 ..... - a_{nn-1}x_{n-1} + y_n \end{array}\right\} \qquad ....(11)$$

We start with an initial vector,

$$x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{pmatrix} ....(12)$$

and substitute this vector for x in the RHS of (11) and calculate x_1, x_2, ….., x_n and this vector is called $x^{(1)}$. We now substitute this vector in the RHS of (11) to calculate again x_1, x_2, ….., x_n and call this new vector as $x^{(2)}$ and continue this procedure to calculate the sequence $\{x^{(k)}\}$. Thus,

The equation (11) can be written as,

$$Dx = -(L + U)\,x + y \quad …………………. (13)$$

which we can write as

$$x = -D^{-1}(L+U)\,x + D^{-1}y,$$

giving

$$x = Jx + \hat{y} \quad ……………… (14)$$

where

$$J = -D^{-1}(L + U) \quad …………….(15)$$

and, we get

$$\left.\begin{array}{ll} x^{(0)} & \text{starting vector} \\ x^{(k)} = Jx^{(k-1)} + \hat{y} \text{ for } k = 1, 2, ... & \end{array}\right\} …………….(16)$$

as the iterative scheme. This is similar to (2 in section 2.1) with the iterating matrix M as $J = -D^{-1}(L + U)$; J is called the Jacobi Iteration Matrix. The scheme will converge to the solution x of our system if $\|J\|_{sp} < 1$. We shall see an easier condition below:

We have

$$D^{-1} = \begin{pmatrix} 1/a_{11} & & \bigcirc & \\ & 1/a_{22} & & \\ & & \ddots & \\ \bigcirc & & & 1/a_{nn} \end{pmatrix}$$

and therefore

$$J = -D^{-1}(L+U) = \begin{pmatrix} 0 & -a_{12}/a_{11} & -a_{13}/a_{11} & \cdots & -a_{1n}/a_{11} \\ -a_{21}/a_{22} & 0 & -a_{23}/a_{22} & \cdots & -a_{2n}/a_{22} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{n1}/a_{nn} & -a_{n2}/a_{nn} & \cdots & -a_{n,n-1}/a_{nn} & 0 \end{pmatrix}$$

Now therefore the i^{th} Absolute row sum for J is

$$R_i = \sum_{j \neq i} \left| \frac{a_{ij}}{a_{ii}} \right| = \left(|a_{i1}| + |a_{i2}| + \ldots + |a_{i,i-1}| + |a_{i,i+1}| + \ldots + |a_{in}| \right) \Big/ |a_{ii}|$$

$\therefore$ If $R_i < 1$ for every i =1,2,3,…..,n

then

$$\|J\|_\infty = \max\{R_1, \ldots, R_n\} < 1$$

and we have convergence.

Now $R_i < 1$ means

$$|a_{i1}|+|a_{i2}|+.....+|a_{i,i-1}|+|a_{i,i+1}|+.....+|a_{in}|<|a_{ii}|$$

i.e. in each row of A the sum of the absolute values of the non diagonal entries is dominated by the absolute value of the diagonal entry (in which case A is called 'strictly row diagonally dominant'). Thus the Jacobi iteration scheme for the system (3) converges if A is strictly row diagonally dominant (Of course, this condition may not be satisfied) and still Jacobi iteration scheme may converge if $\|J\|_{sp} < 1$.

<u>Example 1:</u>

Consider the system

$$\left.\begin{aligned} x_1 + 2x_2 - 2x_3 &= 1 \\ x_1 + x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + x_3 &= 0 \end{aligned}\right\} \qquad \text{............(I)}$$

Let us apply the Jacobi iteration scheme with the initial vector as

$$x^{(0)} = \theta = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{............(II)}$$

We have $A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$; $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$L + U = \begin{pmatrix} 0 & 2 & -2 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix} \quad ; \quad y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$J = -D^{-1}(L+U) = \begin{pmatrix} 0 & -2 & +2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} \quad ; \quad \hat{y} = D^{-1}y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Thus the Jacobi scheme (16) becomes

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x^{(k)} = Jx^{(k-1)} + \hat{y}, \qquad k = 1, 2, ...$$

$\therefore \mathrm{x}^{(1)} = J\mathrm{x}^{(0)} + \hat{y} = \hat{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ since $\mathrm{x}^{(0)}$ is the zero vector.

$$x^{(2)} = Jx^{(1)} + \hat{y} = \begin{pmatrix} 0 & -2 & +2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$x^{(3)} = Jx^{(2)} + \hat{y} = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$x^{(4)} = Jx^{(3)} + \hat{y} = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = x^{(3)}$$

$\therefore \quad \mathrm{x}^{(4)} = \mathrm{x}^{(5)} = \mathrm{x}^{(6)} = \dots\dots\dots. = \mathrm{x}^{(3)}$

$\therefore \quad \mathrm{x}^{(k)} = \mathrm{x}^{(3)}$ and $\mathrm{x}^{(k)}$ converges to $\mathrm{x}^{(3)}$

$\therefore$ The solution is

$$x = \lim_{k\to\infty} x^{(k)} = x^{(3)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

We can easily check that this is the exact solution.

Here, there is no convergence problem at all.

<u>Example 2:</u>

$$\left.\begin{aligned} 8x_1 + 2x_2 - 2x_3 &= 8 \\ x_1 - 8x_2 + 3x_3 &= 19 \\ 2x_1 + x_2 + 9x_3 &= 30 \end{aligned}\right\}$$

Let us apply Jacobi iteration scheme starting with $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

We have $\quad D = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 9 \end{pmatrix} \qquad \therefore D^{-1} = \begin{pmatrix} 1/8 & 0 & 0 \\ 0 & -1/8 & 0 \\ 0 & 0 & 1/9 \end{pmatrix}$

$$J = -D^{-1}(L+U) = \begin{pmatrix} 0 & -0.25 & +0.25 \\ +0.125 & 0 & 0.375 \\ -0.22222 & -0.11111 & 0 \end{pmatrix}$$

$$\hat{y} = D^{-1}y = \begin{pmatrix} 1 \\ -2.375 \\ 3.33333 \end{pmatrix}$$

Now the matrix is such that

$|a_{11}| = 8$ and $|a_{12}| + |a_{13}| = 2 + 2 = 4 \qquad \therefore |a_{11}| > |a_{12}| + |a_{13}|$

$|a_{22}| = 8$ and $|a_{21}| + |a_{23}| = 1 + 3 = 4; \qquad \therefore |a_{22}| > |a_{21}| + |a_{23}|$

$|a_{33}| = 9$ and $|a_{31}| + |a_{32}| = 2 + 1 = 3 \qquad \therefore |a_{33}| > |a_{31}| + |a_{32}|$

Thus we have strict row diagonally dominant matrix A. Hence the Jacobi iteration scheme will converge. The scheme is,

$$x^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x^{(k)} = Jx^{(k-1)} + \hat{y}$$

$$= \begin{pmatrix} 0 & -0.25 & 0.25 \\ 0.125 & 0 & 0.375 \\ -0.22222 & -0.11111 & 0 \end{pmatrix} x^{(k-1)} + \hat{y}$$

$$x^{(1)} = \hat{y} = \begin{pmatrix} 1 \\ -2.375 \\ 3.33333 \end{pmatrix}$$

We continue the iteration until the components of $x^{(k)}$ and $x^{(k+1)}$ differ by at most, say; $3x10^{-5}$, that is, $\left\| x^{(k+1)} - x^{(k)} \right\|_\infty \le 3x10^{-5}$, we get $\left\| x^{(1)} - x^{(0)} \right\|_\infty = 3.33333$. So we continue

$$x^{(2)} = Jx^{(1)} + \hat{y} = \begin{pmatrix} 2.42708 \\ -1.00000 \\ 3.37500 \end{pmatrix} \qquad \left\| x^{(2)} - x^{(1)} \right\|_\infty = 1.42708 \ \geq \in$$

$$x^{(3)} = Jx^{(2)} + \hat{y} = \begin{pmatrix} 2.09375 \\ -0.80599 \\ 2.90509 \end{pmatrix}; \quad \left\| x^{(3)} - x^{(2)} \right\|_\infty = 0.46991 \ \geq \in$$

$$x^{(4)} = Jx^{(3)} + \hat{y} = \begin{pmatrix} 1.92777 \\ -1.02387 \\ 2.95761 \end{pmatrix}; \quad \left\| x^{(4)} - x^{(3)} \right\|_\infty = 0.21788 \ \geq \in$$

$$x^{(5)} = Jx^{(4)} + \hat{y} = \begin{pmatrix} 1.99537 \\ -1.02492 \\ 3.01870 \end{pmatrix}; \quad \left\| x^{(5)} - x^{(4)} \right\|_{\infty} = 0.06760 \geq \in$$

$$x^{(6)} = Jx^{(5)} + \hat{y} = \begin{pmatrix} 2.01091 \\ -0.99356 \\ 3.00380 \end{pmatrix}; \quad \left\| x^{(6)} - x^{(5)} \right\|_{\infty} = 0.03136 \geq \in$$

$$x^{(7)} = Jx^{(6)} + \hat{y} = \begin{pmatrix} 1.99934 \\ -0.99721 \\ 2.99686 \end{pmatrix}; \quad \left\| x^{(7)} - x^{(6)} \right\|_{\infty} = 0.01157 \geq \in$$

$$x^{(8)} = Jx^{(7)} + \hat{y} = \begin{pmatrix} 1.99852 \\ -1.00126 \\ 2.99984 \end{pmatrix}; \quad \left\| x^{(8)} - x^{(7)} \right\|_{\infty} = 0.00405 \geq \in$$

$$x^{(9)} = Jx^{(8)} + \hat{y} = \begin{pmatrix} 2.00027 \\ -1.00025 \\ 3.00047 \end{pmatrix}; \quad \left\| x^{(9)} - x^{(8)} \right\|_{\infty} = 0.00176 \geq \in$$

$$x^{(10)} = Jx^{(9)} + \hat{y} = \begin{pmatrix} 2.00018 \\ -0.99979 \\ 2.99997 \end{pmatrix}; \quad \left\| x^{(10)} - x^{(9)} \right\|_{\infty} = 0.00050 \geq \in$$

$$x^{(11)} = Jx^{(10)} + \hat{y} = \begin{pmatrix} 1.99994 \\ -0.99999 \\ 2.99994 \end{pmatrix}; \quad \left\| x^{(11)} - x^{(10)} \right\|_{\infty} = 0.00024 \geq \in$$

$$x^{(12)} = Jx^{(11)} + \hat{y} = \begin{pmatrix} 1.99998 \\ -1.00003 \\ 3.00001 \end{pmatrix}; \quad \left\| x^{(12)} - x^{(11)} \right\|_{\infty} = 0.00008 \geq \in$$

$$x^{(13)} = Jx^{(12)} + \hat{y} = \begin{pmatrix} 2.00001 \\ -1.00000 \\ 3.00001 \end{pmatrix}; \quad \left\| x^{(13)} - x^{(12)} \right\|_{\infty} = 0.00003 = \in$$

Hence the solution is $x_1=2$; $x_2=-1$, $x_3=3.00001$
(The Exact solution is $x_1 = 1$, $x_2 = -2$, $x_3 = 3$).