4.4 TRIDIAGONALIZATION OF A REAL SYMMETRIC MATRIX

Let $A = (a_{ij})$ be a real symmetric nxn matrix. Our aim is to get a real symmetric tridiagonal matrix T such that T is similar to A. The process of obtaining this T is called the Givens – Householder scheme. The idea is to first find a reduction process which annihilates the off – tridiagonal matrices in the first row and first column of A and repeatedly use this idea. We shall first see some preliminaries.

Let $U = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix}$ be a real nx1 nonzero vector.

Then, $H = UU^T$ is an nxn real symmetric matrix. Let α be a real number (which we shall suitably choose) and consider

$$P = I - \alpha H = I - \alpha UU^T \text{...........}(I)$$

We shall choose α such that P is its own inverse. (Note that $P^T = P$). So we need

$$P^2 = I$$

i.e.

$$(I - \alpha H)(I - \alpha H) = I$$

i.e.

$$(I - \alpha UU^T)(I - \alpha UU^T) = I$$

$$I - 2\alpha UU^T + \alpha^2 UU^T UU^T = I$$

So we choose α such that

$$\alpha^2\, UU^T\, UU^T = 2\,\alpha\, UU^T$$

Obviously, we choose $\alpha \neq 0$. Because otherwise we get P = I; and we don't get any new transformation.

Hence we need

$$\alpha\, UU^T\, UU^T = 2\; UU^T$$

But

$U^TU = U^2{}_1 + U^2{}_2 + \ldots.. + U^2{}_n = \|U\|_2$

is a real number $\neq 0$ (since U is a nonzero vector) and thus we have

$$\alpha\,(U^T U).\quad UU^T = 2\,UU^T$$

and hence

$$\alpha = \frac{2}{U^T U} \ldots\ldots\ldots\ \ldots(II)$$

Thus if U is an nx1 nonzero vector and α is as in (II) then P defined as

$$P = I - \alpha UU^T \ldots\ldots\ldots\ldots.(III)$$

is such that

$$P = P^T = P^{-1} \ldots\ldots\ldots\ldots.(IV)$$

Now we go back to our problem of tridiagonalization of A. Our first aim is to find a P of the form (IV) such that $P^t AP = PAP$ has off tridiagonal entries in 1st row and 1st column as zero. We can choose the P as follows:

Let $s^2 = a^2{}_{21} + a^2{}_{31} + \ldots.. + a^2{}_{n1} \ldots\ldots\ldots\ldots\ldots.(V)$

(the sum of the squares of the entries below the 1st diagonal in A)

Let s = nonnegative square root of s^2.

Let

$$U = \begin{pmatrix} 0 \\ a_{21} + s\,\text{sgn}\,.a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{pmatrix} \quad \text{..........} \ (VI)$$

Thus U is the same as the 1st column of A except that the 1st component is taken as 0 and second component is a variation of the second component in the 1st column of A. All other components of U are the same as the corresponding components of the1st column of A.

Then

$$\alpha = \left[\frac{U^T U}{2}\right]^{-1}$$

$$= \left[\frac{(a_{21} + s\,\text{sgn}.a_{21})^2 + a_{31}^2 + a_{41}^2 + \text{.....} + a_{n1}^2}{2}\right]^{-1}$$

$$= \left[\frac{a_{21}^2 + s^2 + 2|a_{21}|s + a_{31}^2 + \text{.....} + a_{n1}^2}{2}\right]^{-1}$$

$$== \left[\frac{(a_{21}^2 + a_{31}^2 + \text{.....} + a_{n1}^2) + s^2 + 2s|a_{21}|}{2}\right]^{-1}$$

$$\left[\frac{\left(a_{21}^2 + a_{31}^2 + \ldots\ldots + a_{n1}^2\right) + s^2 + 2s|a_{21}|}{2}\right]^{-1}$$

$$= \left\{\frac{2\left[s^2 + s|a_{21}|\right]}{2}\right\}^{-1} \text{ since } s^2 = a_{21}^2 + a_{31}^2 + \cdots + a_{n1}^2$$

$$= \frac{1}{s^2 + s|a_{21}|}$$

$$\therefore \alpha = \frac{1}{s^2 + s|a_{21}|} \qquad \text{(VII)}$$

Thus if α is as in (VII) and U is as in (VI) where s is as in (V) then

$$P = I - \alpha UU^T$$

is s.t. $P = P^T = P^{-1}$, and it can be shown that

$$A_2 = PA_1P = PAP \qquad (\text{ letting } A_1 = A)$$

is similar to A and has off tridiagonal entries in 1st row and 1st column as 0.

Now we apply this procedure to the matrix obtained by ignoring 1st column and 1st row of A_2.

Thus we now choose

$$s^2 = a^2{}_{32} + a^2{}_{42} + \ldots\ldots + a^2{}_{n2}$$

(where now a_{ij} denote entries of A_2)

(i.e. s^2 is sum of squares of the entries below second diagonal of A_2)

s = Positive square root of s^2

$$U = \begin{pmatrix} 0 \\ 0 \\ a_{32} + (sign.a_{32})s \\ a_{42} \\ \vdots \\ a_{n2} \end{pmatrix}$$

$$\alpha = \frac{1}{s^2 + s|a_{32}|}$$

$$P = I - \alpha UU^T$$

Then

$$A_3 = PA_2P$$

has off tridiagonal entries 1n 1^{st}, 2^{nd} rows and columns as zero. We proceed similarly and annihilate all off tridiagonal entries and get T, real symmetric tridiagonal and similar to A.

<u>Note:</u> For an nxn matrix we get tridiagonalization in n – 2 steps.

<u>Example:</u>

$$A = \begin{pmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{pmatrix}$$

A is a real symmetric matrix and is 4 x 4. Thus we get tridiagonalization after (4 – 2) i.e. 2 steps.

Step 1:

$$s^2 = 4^2 + 1^2 + 1^2 = 18$$
$$s = \sqrt{18} = 4.24264$$

$$\alpha = \frac{1}{s^2 + s|a_{21}|} = \frac{1}{18 + (4.24264)(4)} = \frac{1}{34.97056}$$

$$= 0.02860$$

$$U = \begin{pmatrix} 0 \\ a_{21} + s \operatorname{sgn} . a_{21} \\ a_{31} \\ a_{41} \end{pmatrix} = \begin{pmatrix} 0 \\ 4 + 4.24264 \\ 1 \\ 1 \end{pmatrix}$$

With this α, U, we get

$$P = I - \alpha U U^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.94281 & -0.23570 & -0.23570 \\ 0 & -0.23570 & 0.97140 & -0.02860 \\ 0 & -0.23570 & -0.02860 & 0.97140 \end{pmatrix}$$

$$A_2 = PAP = \begin{pmatrix} 5 & -4.24264 & 0 & 0 \\ -4.24264 & 6 & -1 & -1 \\ 0 & -1 & 3.5 & 1.5 \\ 0 & -1 & 1.5 & 3.5 \end{pmatrix}$$

<u>Step 2</u>

$$s^2 = (-1)^2 + (1)^2 = 2$$
$$s = \sqrt{2} = 1.41421$$

$$\alpha = \frac{1}{s^2 + s|a_{32}|} = \frac{1}{2 + (1.41421)(1)} = \frac{1}{3.41421} = 0.29289$$

$$U = \begin{pmatrix} 0 \\ 0 \\ a_{32} + s\,\mathrm{sgn}.a_{32} \\ a_{42} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 - 1.41421 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2.41421 \\ -1 \end{pmatrix}$$

$P = I - \alpha\, UU^T$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -0.70711 & -0.70711 \\ 0 & 0 & -0.70711 & 0.70711 \end{pmatrix}$$

$A_3 = PA_2P$

$$= \begin{pmatrix} 5 & -4.24264 & 0 & 0 \\ -4.24264 & 6 & 1.41421 & 0 \\ 0 & 1.41421 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

which is tridiagonal.

Thus the Givens – Householder scheme for finding the eigenvalues involves two steps, namely,

STEP 1: Find a tridiagonal T (real symmetric) similar to T (by the method described above)

STEP 2: Find the eigenvalues of T (by the method of sturm sequences and bisection described earlier)

However, it must be mentioned that this method is used mostly to calculate the eigenvalue of the largest modulus or to sharpen the calculations done by some other method.

If one wants to calculate all the eigenvalues at the same time then one uses the Jacobi iteration which we now describe.