3.3 HERMITIAN MATRICES

Let A = (a_{ij}); be an nxn matrix. We define the Hermitian conjugate of A, denoted by A^* as ; $A^* = (a^*_{ij})$ where $a^*_{ij} = \overline{a}_{ji}$.

A^* is the conjugate of the transpose of A.

Example 1: $A = \begin{pmatrix} 1 & i \\ -i & i \end{pmatrix}$

Transpose of $A = \begin{pmatrix} 1 & -i \\ i & i \end{pmatrix}$

$$\therefore A^* = \begin{pmatrix} 1 & i \\ -i & -i \end{pmatrix}$$

Example 2: $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$

Transpose of $A = \begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix}$

$$\therefore A^* = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$$

Observe that in Example 1. $A^* \neq A$, whereas in Example 2, $A^* = A$.

DEFINITION: An nxn matrix A is said to be HERMITIAN if $A^* = A$.

We now state some properties of Hermitian matrices.

(1) If A = (a_{ij}) , A^* = (a^*_{ij}), and A = A^* , then $a_{ii} = a^*_{ii} = \overline{a_{ii}}$

Thus the DIAGONAL ENTRIES OF A HERMITIAN MATRIX ARE REAL.

(2) Let $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}; y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ be any two vectors in C^n and A a Hermitian matrix.

Let

$$Ax = \begin{pmatrix} (Ax)_1 \\ (Ax)_2 \\ \vdots \\ (Ax)_n \end{pmatrix}; \; Ay = \begin{pmatrix} (Ay)_1 \\ (Ay)_2 \\ \vdots \\ (Ay)_n \end{pmatrix}$$

We have

$$(Ax)_i = \sum_{j=1}^{n} a_{ij} x_j ; (Ay)_j = \sum_{i=1}^{n} a_{ji} y_i .$$

Now

$$(Ax, y) = \sum_{i=1}^{n} (Ax)_i \overline{y}_i$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \overline{y}_i$$

$$= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{n} a_{ij} \overline{y}_i \right)$$

$$= \sum_{j=1}^{n} x_j \overline{\left(\sum_{i=1}^{n} \overline{a_{ij}} y_i \right)}$$

$$= \sum_{j=1}^{n} x_j \overline{\left(\sum_{i=1}^{n} a_{ji} y_i \right)} \quad (\because \overline{a_{ij}} = a_{ji} \text{ since } A = A^*)$$

$$= \sum_{j=1}^{n} x_j \left(\overline{Ay} \right)_j$$

= (x, Ay)

Thus IF A IS HERMITIAN THEN

(Ax, y) = (x, Ay)

FOR ANY TWO VECTORS x, y.

(3) Let λ be any eigenvalue of a Hermitian matrix A. Then there is an $x \in C^n$, $x \neq \theta_n$ such that

$$Ax = \lambda x.$$

Now, since A is Hermitian we have,

$$\begin{aligned} \lambda(x, x) &= (\lambda x, x) = (Ax, x) \\ &= (x, Ax) \\ &= (x, \lambda x) \\ &= \overline{\lambda}(x, x) \end{aligned}$$

$$\therefore \ (\lambda - \overline{\lambda})(x, x) = 0 .\ But \ \ (x, x) \neq 0 \ \because \ x \neq \theta_n$$

$\therefore \ \lambda - \overline{\lambda} = 0 \ \therefore \ \lambda = \overline{\lambda}$ $\qquad \therefore \lambda$ is real.

THUS THE EIGENVALUES OF A HERMITIAN MATRIX ARE ALL REAL.

(4) Let λ, μ be two different eigenvalues of a Hermitian matrix A and x, y their corresponding eigenvectors. We have,

$$Ax = \lambda x \qquad \text{and} \qquad Ay = \mu y$$

and λ, μ are real by (3).

Now,

$$\begin{aligned} \lambda(x, y) &= (\lambda x, y) \\ &= (Ax, y) \\ &= (x, Ay) \text{ by } (2) \\ &= (x, \mu y) \\ &= \overline{\mu}(x, y) \\ &= \mu(x, y) \text{ since } \mu \text{ is real}. \end{aligned}$$

Hence we get

$$(\lambda - \mu)(x, y) = 0.$$

But $\lambda \neq \mu$

So we get (x,y) = 0 $\Rightarrow$ x and y are orthogonal.

THUS IF A IS A HERMITIAL MATRIX THEN THE EIGENVECTORS CORRESPONDING TO ITS DISTINCT EIGENVALUES ARE ORTHOGONAL.