Linear Dynamical Systems

Tutorial on State Feedback: Part I

Outline

1. State Feedback Design (Lecture slides $10 - 15$)
2. State Feedback Design (Lecture slides $10 - 15$)
3. BIBO and Asymptotic Stability with State Feedback
4. Limitations of Eigenvalue Placement
5. State Feedback design using Lyapunov method and `place` command (Lecture slides $18 - 23$)
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State Feedback Design

Problem 1

Given the system

$$\dot{x} = Ax + bu = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Find the stabilizing feedback matrix k and a Hurwitz closed loop matrix $A + bk$ using the controllability Gramian Q.

Solution to Problem 1

Assertion

The feedback matrix $K = -B'Q^{-1}$ produces a Hurwitz closed-loop matrix $A + BK = A - BB'Q^{-1}$, where Q is the controllability Gramian.

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The proof of this assertion is briefed next.

$$
\begin{aligned}
AQ + QA' &= \int_0^T \left[Ae^{-tA}BB'e^{-A'} + e^{-tA}BB'e^{-tA'}A' \right] dt \\
&= \int_0^T -\frac{d\left(e^{-tA}BB'e^{-tA'}\right)}{dt} dt \\
AQ + QA' &= BB' - e^{-TA}BB'e^{-TA'}
\end{aligned}
$$

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Set $K = -B'Q^{-1}$ and compute

$$(A + BK)\,Q + Q\,(A + BK)'$$

Solution to Problem 1

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Set $K = -B'Q^{-1}$ and compute

$$\begin{aligned} (A+BK)Q + Q(A+BK)' &= \left(A - BB'Q^{-1}\right)Q + Q\left(A - BB'Q^{-1}\right)' \\ &= AQ + QA' - 2BB' \\ &= -BB' - e^{-TA}BB'e^{-TA'} \end{aligned}$$

Solution to Problem 1

$$(A + BK)\,Q + Q\,(A + BK)' = -BB' - e^{-TA}BB'e^{-TA'}$$

Define $\hat{B} = \begin{bmatrix} B & e^{-TA}B \end{bmatrix}$, and note that the right hand side of the equation above may be written as

$$-\hat{B}\hat{B}' = -\begin{bmatrix} B & e^{-TA}B \end{bmatrix} \begin{bmatrix} B' \\ B'e^{-TA'} \end{bmatrix}$$

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$$-\hat{B}\hat{B}' = -\begin{bmatrix} B & e^{-TA}B \end{bmatrix} \begin{bmatrix} B' \\ B'e^{-TA'} \end{bmatrix}$$

which implies that the previously obtained equation is the Lyapunov equation

$$\left(A - BB'Q^{-1}\right) Q + Q\left(A - BB'Q^{-1}\right) + \hat{B}\hat{B}' = 0$$

Since (A, B) is controllable, $(A - BB'Q^{-1}, B)$ is also controllable. From the definition of $\hat{B}$ is follows that $(A - BB'Q^{-1}, \hat{B})$ is also controllable. Since Q and $\hat{B}\hat{B}'$ are positive definite matrices, we conclude that $A + BK = A - BB'Q^{-1}$ is Hurwitz.

Solution to Problem 1

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

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$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

and $A' = -A$. Thus, $e^{-A't} = e^{At}$.

Solution to Problem 1

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$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

and $A' = -A$. Thus, $e^{-A't} = e^{At}$. Choosing $T = 2\pi$, we compute the controllability Gramian as

$$\begin{aligned} Q_{2\pi} &= \int_0^{2\pi} \left(e^{-At} b b' e^{-A't} \right) dt \\ &= \int_0^{2\pi} \left(\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \right) dt \end{aligned}$$

Solution to Problem 1

$$= \int_0^{2\pi} \left(\begin{bmatrix} \sin^2 t & -\sin t \cos t \\ -\sin t \cos t & \cos^2 t \end{bmatrix} \right) dt$$
$$= \int_0^{2\pi} \left(\begin{bmatrix} \frac{1}{2} - \frac{1}{2}\cos 2t & -\sin t \cos t \\ -\sin t \cos t & \frac{1}{2} + \frac{1}{2}\cos 2t \end{bmatrix} \right) dt$$

On direct Integration,

$$Q_{2\pi} = \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix}$$

Calculating the stabilizing feedback matrix as

$$k = -b'Q^{-1} = -[0\ 1] \begin{bmatrix} 1/\pi & 0 \\ 0 & 1/\pi \end{bmatrix} = \begin{bmatrix} 0 \\ -1/\pi \end{bmatrix}$$

Solution to Problem 1

The closed-loop state matrix is given as

$$A + bk = \begin{bmatrix} 0 & 1 \\ -1 & -1/\pi \end{bmatrix}$$

which is Hurwitz.

State Feedback Design

Problem 2

Consider the discrete-time state equation

$$x[t+1] = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x[t] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u[t], \qquad y[t] = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} x[t]$$

a. Find the state feedback gain so that the resulting system has all eigenvalues at $x = 0$. Show that for any initial state the zero-input response of the feedback system becomes identically zero for $t \geq 3$.
b. Let $u = pr - kx$, where p is the feedforward gain and k is the *same* state feedback gain. Find a gain p so that the output will track "any" step reference input. Show also that $y(t) = r(t)$ for $t \geq 3$.

[1] Chen, Problem 8.8

Solution to Problem 2(a)

(A, b) is controllable.

$$\Delta(z) = z^3 - 3z^2 + 3z - 1$$
$$\Delta_f(z) = z^3$$
$$\bar{k} = \begin{bmatrix} 3 & -3 & 1 \end{bmatrix}$$

Calculating the gain k

$$k = \bar{k}\mathfrak{C}\mathfrak{C}^{-1} = \begin{bmatrix} 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -2 & -3 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 2 \end{bmatrix}$$

Thus, the state feedback equation becomes

$$x[t+1] = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \end{bmatrix} x[t] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} r[t]$$

$$= \begin{bmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{bmatrix} x[t] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} r[t]$$

The zero input response of the feedback system then becomes

$$y_{zi}[t] = c\bar{A}^t x[0]$$

where $\bar{A} = \begin{bmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{bmatrix}$ is the closed-loop state matrix. Calculating $\bar{A}^t$,

$$\bar{A} = Q \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} Q^{-1}$$

$$\bar{A}^t = Q \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^t Q^{-1} \qquad \text{where, } Q = \begin{bmatrix} 4 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Solution to Problem 2(a)

using the nilpotent property,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^t = \mathbf{0}, \text{ for } t \geq 3$$

Therefore,

$$y_{zi}[t] = c\mathbf{0}x[0] \; = 0, \text{ for } t \geq 3$$

Solution to Problem 2(b)

$$\Delta(z) = z^3 - 3z^2 + 3z - 1, \qquad \hat{g}(z) = \frac{2z^2 - 8z + 8}{z^3 - 3z^2 + 3z - 1}$$

$$\Delta_f(z) = z^3, \qquad u = pr - kx, \quad \hat{g}_f(z) = p\frac{2z^2 - 8z + 8}{z^3}$$

If the reference input is a step function with magnitude a, then at steady-state the output y is given by

$$y[t] = \hat{g}_f(1).a \qquad t \to \infty$$

thus in order for y to track any step reference input we need $\hat{g}_f(1) = 1$, i.e.

$$\hat{g}_f(1) = 2p = 1 \equiv p = 0.5$$

the resulting system can be described as

$$\begin{aligned} x[t+1] &= \begin{bmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{bmatrix} x[t] + \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix} r[t] \\ &= \bar{A}x[t] + \bar{b}r[t] \\ y[t] &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} x[t] \end{aligned}$$

the response excited by $r[t]$ is

$$y[t] = c\bar{A}^t x(0) + \sum_{m=0}^{t-1} c\ \bar{A}^{(t-1-m)}\bar{b}r(m)$$

Solution to Problem 2(b)

Since $\bar{A}^t = 0$ for $t \geq 3$, we have

$$\begin{aligned} y[t] &= c\bar{b}r[t-1] + c\bar{A}\bar{b}r[t-2] + c\bar{A}^2\bar{b}r[t-3] \\ &= r[t-1] - 4r[t-2] + 4r[t-3] \qquad \text{for } t \geq 3 \end{aligned}$$

For any step reference input $r[t] = a$ the response is

$$y[t] = (1 - 4 + 4)a = a = r[t], \qquad \text{for } t \geq 3$$

Observation

In the above problem, exact tracking is achieved in a finite number of sampling periods. This is possible if all poles of the resulting system are placed at $z = 0$. This is called the *dead-beat controller design*.

BIBO and Asymptotic Stability with State Feedback

Problem 3

Consider a system with transfer function

$$\hat{g}(s) = \frac{(s-1)(s+2)}{(s+1)(s-2)(s+3)}$$

Is it possible to change the transfer function to

$$\hat{g}_f(s) = \frac{(s-1)}{(s+2)(s+3)}$$

by state feedback? Is the resulting system BIBO stable? Asymptotically stable ?

[1] Chen, Problem 8.5

Solution to Problem 3

If we place the eigenvalues of the state feedback system at -2, -2, -3. Then the system has the transfer function

$$\hat{g}_f(s) = \frac{(s-1)(s+2)}{(s+2)^2(s+3)} = \frac{(s-1)}{(s+2)(s+3)}$$

The system is BIBO stable because $\hat{g}_f(s)$ has poles at poles at -2 and -3; it is asymptotically stable because the eigenvalues are -2, -2 and -3.

Limitations of Eigenvalue Placement

Problem 4

Consider the system

$$\dot{x} = Ax + bu = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Can a gain matrix k be computed, such that the eigenvalues of the system can be placed at any arbitrary position? Comment on the inference drawn from the result.

[1] Terrell, Example 6.3

Solution to Problem 4

Given,

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Since the rank of $\mathfrak{C} = 1$, the system is uncontrollable. If we try to find a state feedback gain k using $\det(sI - (A - bk)) = 0$, the characteristic equation becomes,

$$(s+1)(s-2+k_2) = 0$$

It is evident that no value of k_1 can be computed for the system and the eigenvalue at -1 cannot be changed by the state feedback.
For example, if we choose $k = \begin{bmatrix} 0 & -3 \end{bmatrix}$, the resulting system becomes

$$\dot{x} = (A + bk)x = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x$$

where the eigenvalue -1 is unchanged.
Inference : If the pair (A, B) is uncontrollable, then there are limitations on eigenvalue placement for the system.

State Feedback design using Lyapunov method and `place` command

Problem 5

Compute the feedback gain $k \in \mathbb{R}^{1\times n}$ for the system

$$\dot{x} = Ax + bu$$

with $A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ so that the closed loop eigenvalues are placed at -1, -2 and -3.

Repeat for $A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{bmatrix}$.

Solution to Problem 5

Recall!

Recall from the lecture slide 18, the method of computing the state feedback gains using the Lyapunov method.

The simplest choice for the matrix F which has the desired eigenvalues is

$$F = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Let $\bar{k} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$. It is easy to verify that the pair $(F', \bar{k}')$ is controllable. The Lyapunov equation which needs to be solved is then given as

$$AT - TF = b\bar{k}$$

Solution to Problem 5

Upon solving this equation, we compute the T matrix to be

$$T = \begin{bmatrix} -\frac{1}{3} & -\frac{6}{19} & -\frac{3}{43} \\ 1 & \frac{20}{19} & \frac{16}{43} \\ \frac{1}{3} & \frac{14}{19} & \frac{14}{43} \end{bmatrix}.$$

Finally, the required gain vector k is computed as $k = \bar{k}T^{-1}$, i.e.

$$k = \begin{bmatrix} \frac{63}{17} & \frac{26}{17} & \frac{36}{17} \end{bmatrix}.$$

Also, using the MATLAB '`place`' command, the gain vector is found to be $\begin{bmatrix} \frac{63}{17} & \frac{26}{17} & \frac{36}{17} \end{bmatrix}$.

Solution to Problem 5

For $A = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{bmatrix}$, let $F = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ and $\bar{K} = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{bmatrix}'$. Then, upon solving the Lyapunov equation $AT - TF = B\bar{K}$, we get

$$T = \begin{bmatrix} 13/3 & -4/5 & 0 \\ 23/3 & 12/5 & 1 \\ -11/3 & 2/5 & 0 \end{bmatrix}$$

Finally, the required state feedback gain is computed to be

$$K_{\mathsf{lyap}} = \begin{bmatrix} 31/9 & 1 & 53/9 \\ 25/9 & 1 & 41/9 \end{bmatrix}$$

Solution to Problem 5

While using the `place` command the state feedback gain is computed to be

$$K_{\texttt{place}} = \begin{bmatrix} -1.8710 & 0.6632 & -2.8196 \\ 6.0082 & 1.2348 & 10.0750 \end{bmatrix}$$

Note

Note that the designed K matrices are *different* using the two methods.

Solution to Problem 5

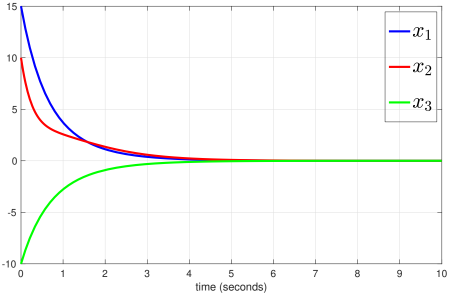


Figure: State trajectories with the feedback gain K_{lyap}

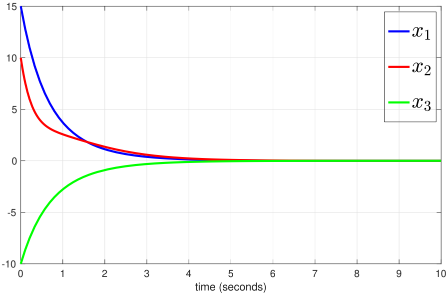


Figure: State trajectories with the feedback gain $K_{\texttt{place}}$

Solution to Problem 5

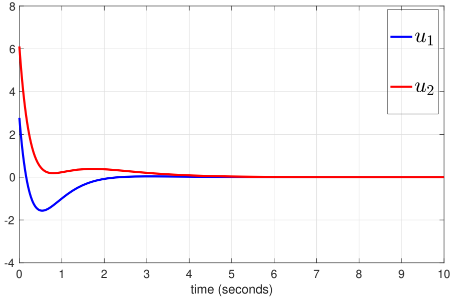


Figure: Input trajectories with the feedback gain K_{lyap}

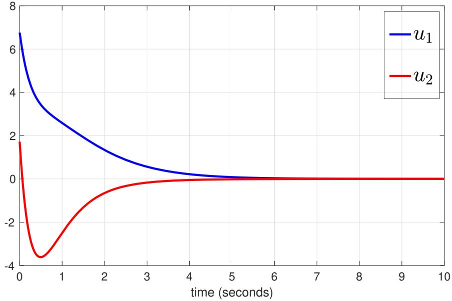


Figure: Input trajectories with the feedback gain $K_{\texttt{place}}$

Problem 6[1]:

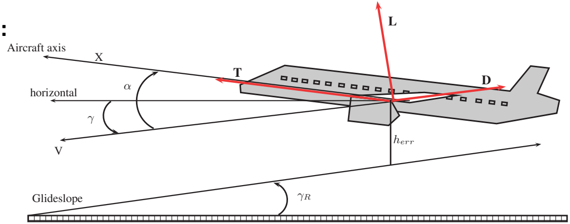


Figure: Aircraft during the landing phase

System description

1. The Instrument Landing System (ILS) on ground determines the difference between the actual trajectory of the aircraft and the reference trajectory imposed for the descent.
2. Lateral movement and rolling movements of the aircraft are ignored, but the longitudinal motion, assuming that these aspects are handled by another automated system.
3. Three outputs that are measured in real-time: the speed V , the angle γ of the flight path, and the distance from the center of mass of the aircraft relative to the glide-slope h_{err}.
4. The control inputs of the system are the aircraft thrust T and the elevator command δ.
5. The elevator is a movable aerodynamic surface located in the empennage that controls the pitch of the aircraft. We assume there are no dynamics between the elevator command and the angle of attack α of the wing. Thus, we view α as equivalent to δ, and consequently, for the sake of simplicity, we treat α as a control input. The thrust controls the speed V of the aircraft.

[1] Jain et al, International Journal of Applied Mathematics and Computer Science, 22(1), pp. 125-137, 2012

State Feedback and Tracking

Problem 6:
The non-linear model of the longitudinal dynamics of a large jet aircraft is given as:

$$\begin{bmatrix} m\frac{dV}{dt} \\ mV\frac{d\gamma}{dt} \\ \frac{dh_{err}}{dt} \end{bmatrix} = \begin{bmatrix} -D(\alpha, V) + T\cos\alpha - mg\sin\gamma \\ L(\alpha, V) + T\sin\alpha - mg\cos\gamma \\ V(\sin\gamma + \cos\gamma\tan\gamma_R) \end{bmatrix}$$

Control objective

The objective is that the aircraft follows along the glide–slope, making a desired flight path angle at 3 degrees clockwise (i.e., $\gamma_r = -3\,\mathrm{deg}$). Thus, it makes h_{err} zero.

[1] Jain et al, International Journal of Applied Mathematics and Computer Science, 22(1), pp. 125-137, 2012

Problem 6:
We use the following linearized model for designing a controller bank around the trim points, $\alpha = 2.686$ deg and $T = 4.23 \times 10^4 N$.

$$\dot{x} = Ax + Bu, \qquad z = Cx \qquad y = C_0 x$$

where A, B, C and C_0 are given as

$$A = \begin{bmatrix} -0.0180 & -9.7966 & 0 \\ 0.0029 & -0.0063 & 0 \\ 0 & 81.9123 & 0 \end{bmatrix}, \; B = \begin{bmatrix} -4.8374 & 5.2574 \times 10^{-6} \\ 0.5786 & 3.0149 \times 10^{-9} \\ 0 & 0 \end{bmatrix}$$

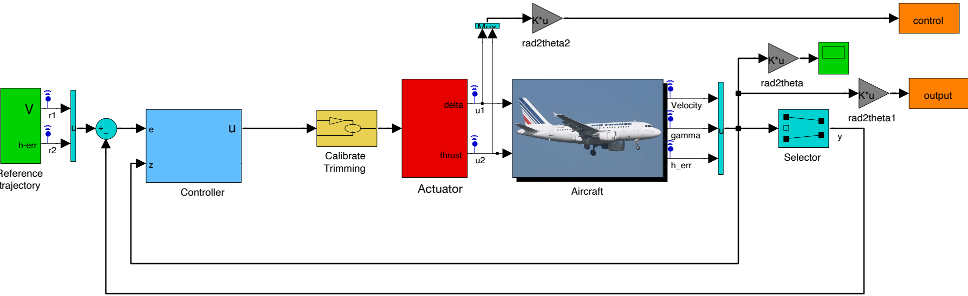
$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; C_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; x = \begin{bmatrix} V & \gamma & h_{err} \end{bmatrix}^T, u = \texttt{col}(\alpha, T)$$

Control problem

Design a controller such that the state of the closed-loop system is stable and tracks the output signal $\begin{bmatrix} 81.8m/s & 0m \end{bmatrix}^T$.

[1] Jain et al, International Journal of Applied Mathematics and Computer Science, 22(1), pp. 125-137, 2012

Solution to Problem 6



Solution to Problem 6

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