Linear Dynamical Systems

Week 3 and 4 - Controllability and State Feedback

Outline of Week 3 and 4

1. Controllable and Reachable Subspaces
2. Fundamental Theorem of Linear Equations (review)
3. Various Tests for Controllability
4. Controllable Decompositions
5. Stabilizability
6. State feedback controller design
7. Regulation and Tracking control problems

Simple Illustration of Controllability

Controllability

Consider the continuous-time LTV system

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

Using variation of constants formula, a given input $u(\bullet)$ transfers the state $x(t_0) := x_0$ at time t_0 to the state $x(t_1) := x_1$ at time t_1,

$$x_1 = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

where $\phi(\bullet)$ denoted the system's state transition matrix.

Controllability

Consider the continuous-time LTV system

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

Using variation of constants formula, a given input $u(\bullet)$ transfers the state $x(t_0) := x_0$ at time t_0 to the state $x(t_1) := x_1$ at time t_1,

$$x_1 = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

where $\phi(\bullet)$ denoted the system's state transition matrix.

We want to express how powerful the input is in terms of transferring the state between two given states.

Controllability

Consider the continuous-time LTV system

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

Using variation of constants formula, a given input $u(\bullet)$ transfers the state $x(t_0) := x_0$ at time t_0 to the state $x(t_1) := x_1$ at time t_1,

$$x_1 = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

where $\phi(\bullet)$ denoted the system's state transition matrix.

We want to express how powerful the input is in terms of transferring the state between two given states.

Definition (Controllability)

The LTV[1] system is said to be *controllable* at t_0, whenever there exists a finite $t_1 > t_0$ such that for any x_0 and any x_1, there exists an input $u(\cdot)$ that transfers x_0 to x_1 at time t_1.
Otherwise, the system is uncontrollable at t_0.

[1] In the time-invariant case, if the state equation is controllable then it is controllable at every t_0 and for every $t_1 > t_0$; thus there is **no need to specify** t_0 **and** t_1. In the time-varying case, the specification of t_0 and t_1 is *crucial*.

Subspaces

Controllability from the origin

Definition (Reachable Subspace)

Controllability to the origin

Definition (Controllable Subspace)

Subspaces

Controllability from the origin

Definition (Reachable Subspace)

Given two times $t_1 > t_0 \geq 0$, the *reachable* or *controllable-from-the-origin on* $[t_0, t_1]$ subspace $\mathcal{R}[t_0, t_1]$ consists of all states x_1 for which there exists an input $u : [t_0, t_1] \to \mathbb{R}^k$ that transfers the state from $x(t_0) = 0$ to $x(t_1) = x_1$; i.e

$$\mathcal{R}[t_0, t_1] \triangleq \left\{ x_1 \in \mathbb{R}^n : \exists u(\bullet), \; x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau \right\}.$$

Controllability to the origin

Definition (Controllable Subspace)

Subspaces

Controllability from the origin

Definition (Reachable Subspace)

Given two times $t_1 > t_0 \geq 0$, the *reachable* or *controllable-from-the-origin on* $[t_0, t_1]$ subspace $\mathcal{R}[t_0, t_1]$ consists of all states x_1 for which there exists an input $u : [t_0, t_1] \to \mathbb{R}^k$ that transfers the state from $x(t_0) = 0$ to $x(t_1) = x_1$; i.e

$$\mathcal{R}[t_0, t_1] \triangleq \left\{ x_1 \in \mathbb{R}^n : \exists u(\bullet),\ x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau \right\}.$$

Controllability to the origin

Definition (Controllable Subspace)

Given two times $t_1 > t_0 \geq 0$, the *controllable* or *controllable-to-the-origin on* $[t_0, t_1]$ subspace $\mathcal{C}[t_0, t_1]$ consists of all states x_0 for which there exists an input $u : [t_0, t_1] \to \mathbb{R}^k$ that transfers the state from $x(t_0) = x_0$ to $x(t_1) = 0$; i.e.,

$$\mathcal{C}[t_0, t_1] \triangleq \left\{ x_0 \in \mathbb{R}^n : \exists u(\bullet),\ 0 = \phi(t_1, t_0) x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau \right\}.$$

Subspaces

The matrices $C(\cdot)$ and $D(\cdot)$ play no role in these definitions; therefore one often simply talks about the reachable or controllable subspaces of the system

$$\dot{x} = A(t)x + B(t)u, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{AB-CLTV}$$

or of the pair $(A(\cdot), B(\cdot))$.

Subspaces

The matrices $C(\cdot)$ and $D(\cdot)$ play no role in these definitions; therefore one often simply talks about the reachable or controllable subspaces of the system

$$\dot{x} = A(t)x + B(t)u, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{AB-CLTV}$$

or of the pair $(A(\cdot), B(\cdot))$.

Attention!

Determining the reachable subspace amounts to finding for which vectors $x_1 \in \mathbb{R}^n$, the equation

$$x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$$

has a solution $u(\cdot)$.

Subspaces

The matrices $C(\cdot)$ and $D(\cdot)$ play no role in these definitions; therefore one often simply talks about the reachable or controllable subspaces of the system

$$\dot{x} = A(t)x + B(t)u, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{AB-CLTV}$$

or of the pair $(A(\cdot), B(\cdot))$.

Attention!

Determining the reachable subspace amounts to finding for which vectors $x_1 \in \mathbb{R}^n$, the equation

$$x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

has a solution $u(\cdot)$.

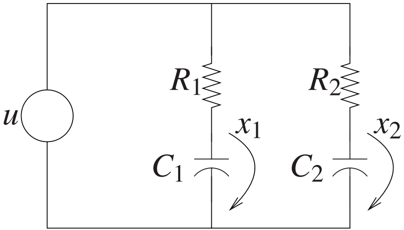
Similarly, determining the controllable subspace amounts to finding the vectors $x_0 \in \mathbb{R}^n$ for which the equation

$$0 = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau \iff x_0 = \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)v(\tau)d\tau$$

has a solution $v(\cdot) = -u(\cdot)$.

Examples and System Interconnections

Parallel RC network

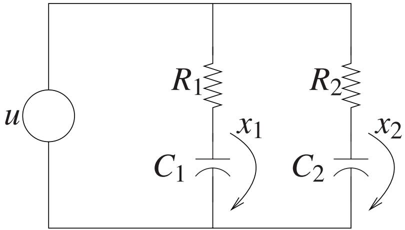


State space model of parallel electrical network is given as

$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} u$$

Examples and System Interconnections

Parallel RC network



State space model of parallel electrical network is given as

$$\dot{x} = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ 0 & -\frac{1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ \frac{1}{R_2C_2} \end{bmatrix} u$$

The solution to this system is given by

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-\frac{t}{R_1C_1}} x_1(0) \\ e^{-\frac{t}{R_2C_2}} x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{e^{-\frac{t-\tau}{R_1C_1}}}{R_1C_1} \\ \frac{e^{-\frac{t-\tau}{R_2C_2}}}{R_2C_2} \end{bmatrix} u(\tau)d\tau$$

Examples and System Interconnections

Parallel RC network

When the two branches have the same time constant, i.e. $\frac{1}{R_1C_1} = \frac{1}{R_2C_2} = \omega$, we have

$$x(t) = e^{-\omega t}x(0) + \omega \int_0^t e^{-\omega(t-\tau)}u(\tau)d\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Examples and System Interconnections

Parallel RC network

When the two branches have the same time constant, i.e. $\frac{1}{R_1 C_1} = \frac{1}{R_2 C_2} = \omega$, we have

$$x(t) = e^{-\omega t} x(0) + \omega \int_0^t e^{-\omega(t-\tau)} u(\tau) d\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If $x(0) = 0$, then $x(t)$ reduces to

$$x(t) = \alpha(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \alpha(t) = \omega \int_0^t e^{-\omega(t-\tau)} u(\tau) d\tau$$

Examples and System Interconnections

Parallel RC network

When the two branches have the same time constant, i.e. $\frac{1}{R_1C_1} = \frac{1}{R_2C_2} = \omega$, we have

$$x(t) = e^{-\omega t}x(0) + \omega \int_0^t e^{-\omega(t-\tau)}u(\tau)d\tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If $x(0) = 0$, then $x(t)$ reduces to

$$x(t) = \alpha(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \alpha(t) = \omega \int_0^t e^{-\omega(t-\tau)}u(\tau)d\tau$$

We can observe that transferring the system from the origin to any state with $x_1(t) = x_2(t)$ is permissible. Hence, the reachable subspace of this system is

$$\mathcal{R}[t_0, t_1] = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}, \quad \forall t_1 > t_0 \geq 0.$$

Examples and System Interconnections

Parallel RC network

Suppose now that we want to transfer $x(0)$ to the origin.

Examples and System Interconnections

Parallel RC network

Suppose now that we want to transfer $x(0)$ to the origin.Then we need

$$0 = e^{-\omega t}x(0) + \alpha(t)\begin{bmatrix}1\\1\end{bmatrix}, \qquad \alpha(t) = \omega\int_0^t e^{-\omega(t-\tau)}u(\tau)d\tau$$

Examples and System Interconnections

Parallel RC network

Suppose now that we want to transfer $x(0)$ to the origin.Then we need

$$0 = e^{-\omega t}x(0) + \alpha(t)\begin{bmatrix}1\\1\end{bmatrix}, \qquad \alpha(t) = \omega\int_0^t e^{-\omega(t-\tau)}u(\tau)d\tau$$

Clearly, this is possible only if $x(0)$ is aligned with $\begin{bmatrix}1 & 1\end{bmatrix}'$.

Examples and System Interconnections

Parallel RC network

Suppose now that we want to transfer $x(0)$ to the origin.Then we need

$$0 = e^{-\omega t} x(0) + \alpha(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \alpha(t) = \omega \int_0^t e^{-\omega(t-\tau)} u(\tau) d\tau$$

Clearly, this is possible only if $x(0)$ is aligned with $\begin{bmatrix} 1 & 1 \end{bmatrix}'$.The controllable subspace for this system is

$$\mathcal{C}[t_0, t_1] = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}, \quad \forall t_1 > t_0 \geq 0.$$

Examples and System Interconnections

Parallel RC network

Suppose now that we want to transfer $x(0)$ to the origin. Then we need

$$0 = e^{-\omega t}x(0) + \alpha(t)\begin{bmatrix}1\\1\end{bmatrix}, \qquad \alpha(t) = \omega\int_0^t e^{-\omega(t-\tau)}u(\tau)d\tau$$

Clearly, this is possible only if $x(0)$ is aligned with $\begin{bmatrix}1 & 1\end{bmatrix}'$. The controllable subspace for this system is

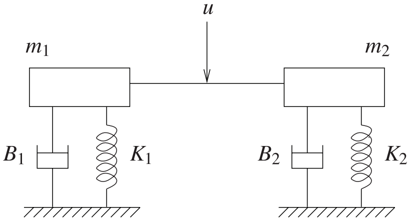
$$\mathcal{C}[t_0, t_1] = \left\{\alpha\begin{bmatrix}1\\1\end{bmatrix} : \alpha \in \mathbb{R}\right\}, \quad \forall t_1 > t_0 \geq 0.$$

However, we shall see shortly that when the time constants are different; i.e. $\frac{1}{R_1C_1} \neq \frac{1}{R_2C_2}$, *any* vector in $\mathbb{R}^2$ can be reached from the origin and the origin can be reached from *any* initial condition in $\mathbb{R}^2$; i.e.

$$\mathcal{R}[t_0, t_1] = \mathcal{C}[t_0, t_1] = \mathbb{R}^2$$

Examples and System Interconnections

Suspension System



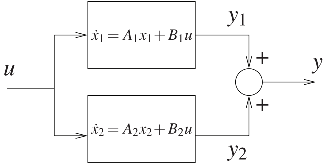
The steady state space model of the mechanical system is given by

$$\dot{x} = \begin{bmatrix} -\frac{b_1}{m_1} & -\frac{k_1}{m_1} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{b_2}{m_2} & -\frac{k_2}{m_2} \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{2m_1} \\ 0 \\ \frac{1}{2m_2} \\ 0 \end{bmatrix} u$$

where $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}'$, and x_1 and x_2 are the spring displacements with respect to the equilibrium position. We assumed that the bar has negligible mass and therefore the force u is equally distributed between the two spring systems.

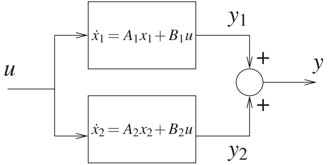
Examples and System Interconnections

Parallel Interconnection



Examples and System Interconnections

Parallel Interconnection



Consider the parallel interconnection of two systems with states $x_1, x_2 \in \mathbb{R}^n$. The overall system corresponds to the state space model:

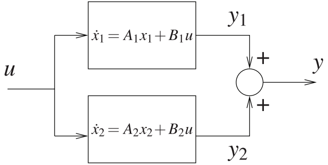
$$\dot{x} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

The solution to the system when $A_1 = A_2 = A$, $B_1 = B_2 = B$ is given by

$$x(t) = \begin{bmatrix} e^{At}x_1(0) \\ e^{At}x_2(0) \end{bmatrix} + \begin{bmatrix} I \\ I \end{bmatrix} \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau.$$

Examples and System Interconnections

Parallel Interconnection



Consider the parallel interconnection of two systems with states $x_1, x_2 \in \mathbb{R}^n$. The overall system corresponds to the state space model:

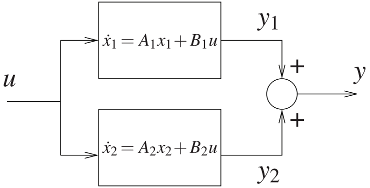
$$\dot{x} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

The solution to the system when $A_1 = A_2 = A$, $B_1 = B_2 = B$ is given by

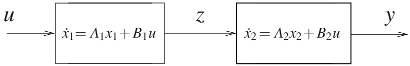
$$x(t) = \begin{bmatrix} e^{At}x_1(0) \\ e^{At}x_2(0) \end{bmatrix} + \begin{bmatrix} I \\ I \end{bmatrix} \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

This shows that if $x(0) = 0$, we cannot transfer the system from the origin to any state with $x_1(t) \neq x_2(t)$. Similarly, to transfer a state $x(t_0)$ to the origin, we must have $x_1(t_0) = x_2(t_0)$.

Examples and System Interconnections



(a) parallel

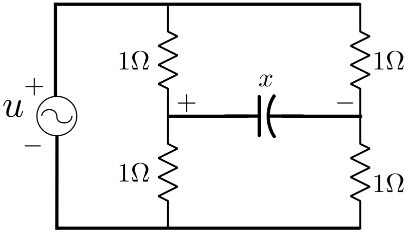


(b) cascade

Attention

- Parallel connections of similar systems are a common mechanism that leads to lack of reachability and controllability.
- Cascade connections, generally do not have this problem. However, they may lead to stability problems through resonance.

Examples and System Interconnections



State variable x is the voltage across the capacitor.

If $x(0) = 0$, then $x(t) = 0$ for all $t \geq 0$ no matter what input is applied.

This is due to the symmetry of the network, and the input has no effect on the voltage across the capacitor.

Fundamental theorem of linear equations

Given an $m \times n$ matrix W

Definition (Image and Rank of a matrix)

Definition (Kernel and Nullity of a matrix)

Fundamental theorem of linear equations

Given an $m \times n$ matrix W

Definition (Image and Rank of a matrix)

The *range* or *image* is the set of vectors $y \in \mathbb{R}^m$ for which $y = Wx$ has a solution $x \in \mathbb{R}^n$; i.e.,

$$\mathtt{Im}W \triangleq \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n, y = Wx\}$$

The image of W is a linear subspace of $\mathbb{R}^m$ and its dimension is called the *rank* of the matrix W.

Definition (Kernel and Nullity of a matrix)

Fundamental theorem of linear equations

Given an $m \times n$ matrix W

Definition (Image and Rank of a matrix)

The *range* or *image* is the set of vectors $y \in \mathbb{R}^m$ for which $y = Wx$ has a solution $x \in \mathbb{R}^n$; i.e.,

$$\mathtt{Im}W \triangleq \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n, y = Wx\}$$

The image of W is a linear subspace of $\mathbb{R}^m$ and its dimension is called the *rank* of the matrix W.

The rank of W is equal to the number of linearly independent columns of W, which is also equal to the number of linearly independent rows of W.

Definition (Kernel and Nullity of a matrix)

Fundamental theorem of linear equations

Given an $m \times n$ matrix W

Definition (Image and Rank of a matrix)

The *range* or *image* is the set of vectors $y \in \mathbb{R}^m$ for which $y = Wx$ has a solution $x \in \mathbb{R}^n$; i.e.,

$$\mathtt{Im}W \triangleq \{y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n, y = Wx\}$$

The image of W is a linear subspace of $\mathbb{R}^m$ and its dimension is called the *rank* of the matrix W.

The rank of W is equal to the number of linearly independent columns of W, which is also equal to the number of linearly independent rows of W.

Definition (Kernel and Nullity of a matrix)

The *kernel* or *null space* is the set

$$\mathtt{ker}W = \{x \in \mathbb{R}^n : Wx = 0\}$$

The kernel of W is a linear subspace of $\mathbb{R}^n$, and its dimension is called the *nullity* of the matrix W.

Fundamental theorem of linear equations

Theorem (Fundamental theorem of linear equations[1])

For every $m \times n$ matrix W,

$$\dim \mathtt{ker} W + \dim \mathtt{Im} W = n.$$

[1] Strang, Linear Algebra and its Applications, 1988

Fundamental theorem of linear equations

Theorem (Fundamental theorem of linear equations[1])

For every $m \times n$ *matrix* W*,*

$$\dim \mathtt{ker} W + \dim \mathtt{Im} W = n.$$

There exists a simple relationship between the kernel and image spaces.

[1] Strang, Linear Algebra and its Applications, 1988

Fundamental theorem of linear equations

Theorem (Fundamental theorem of linear equations[1])

For every $m \times n$ matrix W,

$$\dim \mathtt{ker} W + \dim \mathtt{Im} W = n.$$

There exists a simple relationship between the kernel and image spaces. The $orthogonal\ complement$ $\mathcal{V}^\perp$ of a linear subspace $\mathcal{V} \subset \mathbb{R}^n$ is the set of all vectors that are orthogonal to every vector in $\mathcal{V}$; i.e.,

$$\mathcal{V}^\perp = \left\{x \in \mathbb{R}^n : x'z = 0, \forall z \in \mathcal{V}\right\}.$$

[1] Strang, Linear Algebra and its Applications, 1988

Fundamental theorem of linear equations

Theorem (Fundamental theorem of linear equations[1])

For every $m \times n$ matrix W,

$$\dim \mathtt{ker} W + \dim \mathtt{Im} W = n.$$

There exists a simple relationship between the kernel and image spaces. The *orthogonal complement* $\mathcal{V}^{\perp}$ of a linear subspace $\mathcal{V} \subset \mathbb{R}^n$ is the set of all vectors that are orthogonal to every vector in $\mathcal{V}$; i.e.,

$$\mathcal{V}^{\perp} = \left\{ x \in \mathbb{R}^n : x'z = 0, \forall z \in \mathcal{V} \right\}.$$

Theorem (Range vs null space)

For every $m \times n$ matrix W,

$$\mathtt{Im} W = (\mathtt{ker} W')^{\perp}, \qquad \mathtt{ker} W = (\mathtt{Im} W')^{\perp}.$$

[1] Strang, Linear Algebra and its Applications, 1988

Reachability and Controllability Gramians

As the name suggests, the Gramians allow one to compute the aforesaid subspaces.

Reachability and Controllability Gramians

As the name suggests, the Gramians allow one to compute the aforesaid subspaces.

Definition (Gramians)

Given two times $t_1 > t_0 \geq 0$, the reachability and controllability Gramians of the system (AB-CLTV) are defined, respectively, by

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T d\tau$$

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \tau) B(\tau) B(\tau)^T \phi(t_0, \tau)^T d\tau$$

Reachability and Controllability Gramians

As the name suggests, the Gramians allow one to compute the aforesaid subspaces.

Definition (Gramians)

Given two times $t_1 > t_0 \geq 0$, the reachability and controllability Gramians of the system (AB-CLTV) are defined, respectively, by

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B(\tau)^T\phi(t_1, \tau)^T d\tau$$

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)B(\tau)^T\phi(t_0, \tau)^T d\tau$$

Note

Both Gramians are symmetric and positive-semidefinite $n \times n$ matrices.

Reachability and Controllability Gramians

Definition (Reachable Subspace, Gramian, Image)

$$\mathcal{R}[t_0, t_1] \triangleq \left\{ x_1 \in \mathbb{R}^n : \exists u(\bullet),\ x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau \right\}$$

$$W_R(t_0, t_1) \triangleq \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T d\tau, \quad \mathtt{Im} W \triangleq \{ y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n, y = Wx \}$$

Reachability and Controllability Gramians

Definition (Reachable Subspace, Gramian, Image)

$$\mathcal{R}[t_0, t_1] \triangleq \left\{ x_1 \in \mathbb{R}^n : \exists u(\bullet),\ x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau \right\}$$

$$W_R(t_0, t_1) \triangleq \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T d\tau, \quad \mathtt{Im} W \triangleq \{ y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n, y = Wx \}$$

Theorem (Reachable subspace)

Given two times $t_1 > t_0 \geq 0$*,*

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1).$$

Reachability and Controllability Gramians

Definition (Reachable Subspace, Gramian, Image)

$$\mathcal{R}[t_0, t_1] \triangleq \left\{ x_1 \in \mathbb{R}^n : \exists u(\bullet),\ x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau \right\}$$

$$W_R(t_0, t_1) \triangleq \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T d\tau, \quad \mathtt{Im} W \triangleq \{ y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n, y = Wx \}$$

Theorem (Reachable subspace)

Given two times $t_1 > t_0 \geq 0$,

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1).$$

Moreover, if $x_1 = W_R(t_0, t_1)\eta_1 \in \mathtt{Im} W_R(t_0, t_1)$, the control

$$u(t) = B(t)^T \phi(t_1, t)^T \eta_1, \qquad t \in [t_0, t_1] \tag{1}$$

can be used to transfer the state from $x(t_0) = 0$ to $x(t_1) = x_1$.

Reachability and Controllability Gramians

Definition (Reachable Subspace, Gramian, Image)

$$\mathcal{R}[t_0, t_1] \triangleq \left\{ x_1 \in \mathbb{R}^n : \exists u(\bullet),\ x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau \right\}$$

$$W_R(t_0, t_1) \triangleq \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T d\tau, \quad \mathtt{Im} W \triangleq \{ y \in \mathbb{R}^m : \exists x \in \mathbb{R}^n, y = Wx \}$$

Theorem (Reachable subspace)

Given two times $t_1 > t_0 \geq 0$,

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1).$$

Moreover, if $x_1 = W_R(t_0, t_1)\eta_1 \in \mathtt{Im} W_R(t_0, t_1)$, the control

$$u(t) = B(t)^T \phi(t_1, t)^T \eta_1, \qquad t \in [t_0, t_1] \tag{1}$$

can be used to transfer the state from $x(t_0) = 0$ to $x(t_1) = x_1$.

Logical idea of the proof.

The proof shall be done in two parts

- $x_1 \in \mathtt{Im} W_R(t_0, t_1) \implies x_1 \in \mathcal{R}[t_0, t_1]$
- $x_1 \in \mathtt{Im} W_R(t_0, t_1) \impliedby x_1 \in \mathcal{R}[t_0, t_1]$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \implies x_1 \in \mathcal{R}[t_0, t_1]$

Recall

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T d\tau$$

$$u(t) = B(t)^T \phi(t_1, t)^T \eta_1, \qquad t \in [t_0, t_1]$$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \implies x_1 \in \mathcal{R}[t_0, t_1]$

Recall

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T d\tau$$

$$u(t) = B(t)^T \phi(t_1, t)^T \eta_1, \qquad t \in [t_0, t_1]$$

When $x_1 \in \mathtt{Im} W_R(t_0, t_1)$, there exists a vector $\eta_1 \in \mathbb{R}^n$ such that

$$x_1 = W_R(t_0, t_1)\eta_1.$$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \implies x_1 \in \mathcal{R}[t_0, t_1]$

Recall

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T d\tau$$

$$u(t) = B(t)^T \phi(t_1, t)^T \eta_1, \qquad t \in [t_0, t_1]$$

When $x_1 \in \mathtt{Im} W_R(t_0, t_1)$, there exists a vector $\eta_1 \in \mathbb{R}^n$ such that

$$x_1 = W_R(t_0, t_1)\eta_1.$$

To prove that $x_1 \in \mathcal{R}[t_0, t_1]$, it suffices to show that the input $u(t)$ does indeed transfer the state from $x(t_0) = 0$ to $x(t_1) = x_1$.

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \Longrightarrow x_1 \in \mathcal{R}[t_0, t_1]$

Recall

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T d\tau$$
$$u(t) = B(t)^T \phi(t_1, t)^T \eta_1, \qquad t \in [t_0, t_1]$$

When $x_1 \in \mathtt{Im} W_R(t_0, t_1)$, there exists a vector $\eta_1 \in \mathbb{R}^n$ such that

$$x_1 = W_R(t_0, t_1)\eta_1.$$

To prove that $x_1 \in \mathcal{R}[t_0, t_1]$, it suffices to show that the input $u(t)$ does indeed transfer the state from $x(t_0) = 0$ to $x(t_1) = x_1$. To verify that this is so, we use the variations of constants formula for the input

$$x(t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \implies x_1 \in \mathcal{R}[t_0, t_1]$

Recall

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T d\tau$$

$$u(t) = B(t)^T \phi(t_1, t)^T \eta_1, \qquad t \in [t_0, t_1]$$

When $x_1 \in \mathtt{Im} W_R(t_0, t_1)$, there exists a vector $\eta_1 \in \mathbb{R}^n$ such that

$$x_1 = W_R(t_0, t_1) \eta_1.$$

To prove that $x_1 \in \mathcal{R}[t_0, t_1]$, it suffices to show that the input $u(t)$ does indeed transfer the state from $x(t_0) = 0$ to $x(t_1) = x_1$. To verify that this is so, we use the variations of constants formula for the input

$$x(t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$$

$$x(t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) \underbrace{B(\tau)^T \phi(t_1, \tau)^T \eta_1}_{u(\tau)} d\tau$$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \implies x_1 \in \mathcal{R}[t_0, t_1]$

Recall

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T d\tau$$

$$u(t) = B(t)^T \phi(t_1, t)^T \eta_1, \qquad t \in [t_0, t_1]$$

When $x_1 \in \mathtt{Im} W_R(t_0, t_1)$, there exists a vector $\eta_1 \in \mathbb{R}^n$ such that

$$x_1 = W_R(t_0, t_1) \eta_1.$$

To prove that $x_1 \in \mathcal{R}[t_0, t_1]$, it suffices to show that the input $u(t)$ does indeed transfer the state from $x(t_0) = 0$ to $x(t_1) = x_1$. To verify that this is so, we use the variations of constants formula for the input

$$x(t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$$

$$x(t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) \underbrace{B(\tau)^T \phi(t_1, \tau)^T \eta_1}_{u(\tau)} d\tau = W_R(t_0, t_1) \eta_1 = x_1.$$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \impliedby x_1 \in \mathcal{R}[t_0, t_1]$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \impliedby x_1 \in \mathcal{R}[t_0, t_1]$

When $x_1 \in \mathcal{R}[t_0, t_1]$ there exists an input $u(\cdot)$ for which $x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \impliedby x_1 \in \mathcal{R}[t_0, t_1]$

When $x_1 \in \mathcal{R}[t_0, t_1]$ there exists an input $u(\cdot)$ for which $x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$

We show next that this leads to $x_1 \in \mathtt{Im} W_R(t_0, t_1) = (\mathtt{ker} W_R(t_0, t_1))^\perp$, which is to say that

$$x_1^T \eta_1 = 0, \qquad \forall \eta_1 \in \mathtt{ker} W_R(t_0, t_1) \tag{2}$$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \impliedby x_1 \in \mathcal{R}[t_0, t_1]$

When $x_1 \in \mathcal{R}[t_0, t_1]$ there exists an input $u(\cdot)$ for which $x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$

We show next that this leads to $x_1 \in \mathtt{Im} W_R(t_0, t_1) = (\mathtt{ker} W_R(t_0, t_1))^{\perp}$, which is to say that

$$x_1^T \eta_1 = 0, \qquad \forall \eta_1 \in \mathtt{ker} W_R(t_0, t_1) \tag{2}$$

To verify that, we pick some arbitrary vector $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$ and compute

$$x_1^T \eta_1 = \int_{t_0}^{t_1} u(\tau)^T B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \tag{3}$$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \Longleftarrow x_1 \in \mathcal{R}[t_0, t_1]$

When $x_1 \in \mathcal{R}[t_0, t_1]$ there exists an input $u(\cdot)$ for which $x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$

We show next that this leads to $x_1 \in \mathtt{Im} W_R(t_0, t_1) = (\mathtt{ker} W_R(t_0, t_1))^{\perp}$, which is to say that

$$x_1^T \eta_1 = 0, \qquad \forall \eta_1 \in \mathtt{ker} W_R(t_0, t_1) \tag{2}$$

To verify that, we pick some arbitrary vector $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$ and compute

$$x_1^T \eta_1 = \int_{t_0}^{t_1} u(\tau)^T B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \tag{3}$$

Compute further

$$\eta_1^T W_R(t_0, t_1) \eta_1 = \int_{t_0}^{t_1} \eta_1^T \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau$$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \impliedby x_1 \in \mathcal{R}[t_0, t_1]$

When $x_1 \in \mathcal{R}[t_0, t_1]$ there exists an input $u(\cdot)$ for which $x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$

We show next that this leads to $x_1 \in \mathtt{Im} W_R(t_0, t_1) = (\mathtt{ker} W_R(t_0, t_1))^{\perp}$, which is to say that

$$x_1^T \eta_1 = 0, \qquad \forall \eta_1 \in \mathtt{ker} W_R(t_0, t_1) \tag{2}$$

To verify that, we pick some arbitrary vector $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$ and compute

$$x_1^T \eta_1 = \int_{t_0}^{t_1} u(\tau)^T B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \tag{3}$$

Compute further

$$\begin{aligned} \eta_1^T W_R(t_0, t_1) \eta_1 &= \int_{t_0}^{t_1} \eta_1^T \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \\ &= \int_{t_0}^{t_1} \| B(\tau)^T \phi(t_1, \tau)^T \eta_1 \|^2 d\tau \end{aligned}$$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \impliedby x_1 \in \mathcal{R}[t_0, t_1]$

When $x_1 \in \mathcal{R}[t_0, t_1]$ there exists an input $u(\cdot)$ for which $x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$

We show next that this leads to $x_1 \in \mathtt{Im} W_R(t_0, t_1) = (\mathtt{ker} W_R(t_0, t_1))^{\perp}$, which is to say that

$$x_1^T \eta_1 = 0, \qquad \forall \eta_1 \in \mathtt{ker} W_R(t_0, t_1) \tag{2}$$

To verify that, we pick some arbitrary vector $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$ and compute

$$x_1^T \eta_1 = \int_{t_0}^{t_1} u(\tau)^T B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \tag{3}$$

Compute further

$$\begin{aligned} \eta_1^T W_R(t_0, t_1) \eta_1 &= \int_{t_0}^{t_1} \eta_1^T \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \\ &= \int_{t_0}^{t_1} \|B(\tau)^T \phi(t_1, \tau)^T \eta_1\|^2 d\tau \end{aligned}$$

But since $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$, we have

$$\eta_1^T W_R(t_0, t_1) \eta_1 = \int_{t_0}^{t_1} \|B(\tau)^T \phi(t_1, \tau)^T \eta_1\|^2 d\tau = 0$$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \impliedby x_1 \in \mathfrak{R}[t_0, t_1]$

When $x_1 \in \mathfrak{R}[t_0, t_1]$ there exists an input $u(\cdot)$ for which $x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$

We show next that this leads to $x_1 \in \mathtt{Im} W_R(t_0, t_1) = (\mathtt{ker} W_R(t_0, t_1))^\perp$, which is to say that

$$x_1^T \eta_1 = 0, \qquad \forall \eta_1 \in \mathtt{ker} W_R(t_0, t_1) \tag{2}$$

To verify that, we pick some arbitrary vector $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$ and compute

$$x_1^T \eta_1 = \int_{t_0}^{t_1} u(\tau)^T B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \tag{3}$$

Compute further

$$\begin{aligned} \eta_1^T W_R(t_0, t_1) \eta_1 &= \int_{t_0}^{t_1} \eta_1^T \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \\ &= \int_{t_0}^{t_1} \|B(\tau)^T \phi(t_1, \tau)^T \eta_1\|^2 d\tau \end{aligned}$$

But since $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$, we have

$$\eta_1^T W_R(t_0, t_1) \eta_1 = \int_{t_0}^{t_1} \|B(\tau)^T \phi(t_1, \tau)^T \eta_1\|^2 d\tau = 0$$

which implies that

$$B(\tau)^T \phi(t_1, \tau)^T \eta_1 = 0, \qquad \forall \tau \in [t_0, t_1].$$

Proof: $x_1 \in \mathtt{Im} W_R(t_0, t_1) \impliedby x_1 \in \mathcal{R}[t_0, t_1]$

When $x_1 \in \mathcal{R}[t_0, t_1]$ there exists an input $u(\cdot)$ for which $x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$

We show next that this leads to $x_1 \in \mathtt{Im} W_R(t_0, t_1) = (\mathtt{ker} W_R(t_0, t_1))^{\perp}$, which is to say that

$$x_1^T \eta_1 = 0, \qquad \forall \eta_1 \in \mathtt{ker} W_R(t_0, t_1) \tag{2}$$

To verify that, we pick some arbitrary vector $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$ and compute

$$x_1^T \eta_1 = \int_{t_0}^{t_1} u(\tau)^T B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \tag{3}$$

Compute further

$$\begin{aligned} \eta_1^T W_R(t_0, t_1) \eta_1 &= \int_{t_0}^{t_1} \eta_1^T \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \\ &= \int_{t_0}^{t_1} \|B(\tau)^T \phi(t_1, \tau)^T \eta_1\|^2 d\tau \end{aligned}$$

But since $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$, we have

$$\eta_1^T W_R(t_0, t_1) \eta_1 = \int_{t_0}^{t_1} \|B(\tau)^T \phi(t_1, \tau)^T \eta_1\|^2 d\tau = 0$$

which implies that

$$B(\tau)^T \phi(t_1, \tau)^T \eta_1 = 0, \qquad \forall \tau \in [t_0, t_1].$$

From this and (3), we conclude that (2) indeed holds.

Reachability and Controllability Gramians

Theorem (Controllable subspace)

Given two times $t_1 > t_0 \geq 0$

$$\mathcal{C}[t_0, t_1] = \mathtt{Im} W_C(t_0, t_1)$$

Moreover, if $x_0 = W_C(t_0, t_1)\eta_0 \in \mathtt{Im} W_C(t_0, t_1)$*, the control*

$$u(t) = -B(t)^T \phi(t_0, t)^T \eta_0, \qquad t \in [t_0, t_1]$$

can be used to transfer the state $x(t_0) = x_0$ *to* $x(t_1) = 0$

Controllability Matrix (LTI)

Consider the continuous time LTI system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{AB-CLTI}$$

Controllability Matrix (LTI)

Consider the continuous time LTI system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{AB-CLTI}$$

For this system, the reachability and controllability Gramians are given, respectively, by

$$W_R[t_0, t_1] = \int_{t_0}^{t_1} e^{A(t_1-\tau)} BB^T e^{A^T(t_1-\tau)} d\tau = \int_0^{t_1-t_0} e^{At} BB^T e^{A^T t} dt$$
$$W_C[t_0, t_1] = \int_{t_0}^{t_1} e^{A(t_0-\tau)} BB^T e^{A^T(t_0-\tau)} d\tau = \int_0^{t_1-t_0} e^{-At} BB^T e^{-A^T t} dt$$

Controllability Matrix (LTI)

Consider the continuous time LTI system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \qquad \text{(AB-CLTI)}$$

For this system, the reachability and controllability Gramians are given, respectively, by

$$W_R[t_0, t_1] = \int_{t_0}^{t_1} e^{A(t_1-\tau)} BB^T e^{A^T(t_1-\tau)} d\tau = \int_0^{t_1-t_0} e^{At} BB^T e^{A^T t} dt$$
$$W_C[t_0, t_1] = \int_{t_0}^{t_1} e^{A(t_0-\tau)} BB^T e^{A^T(t_0-\tau)} d\tau = \int_0^{t_1-t_0} e^{-At} BB^T e^{-A^T t} dt$$

The *controllability matrix* of (AB-CLTI) is to be defined by

$$\mathfrak{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}_{n \times (kn)}$$

and provides a “simple method” to compute the reachable and controllable subspace.

Controllability Matrix (LTI)

Theorem

For any two times t_0 *and* t_1*, with* $t_1 > t_0 \geq 0$

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1) = \mathtt{Im} \mathfrak{C} = \mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1]$$

Controllability Matrix (LTI)

Theorem

For any two times t_0 and t_1, with $t_1 > t_0 \geq 0$

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1) = \mathtt{Im}\mathfrak{C} = \mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1]$$

Attention

The notion of reachability and controllability **coincide** for continuous time LTI system, which means if one can go from origin to some state x_1, then one can also go from x_1 to origin.
Because of this, one studies controllability for continuous time system, and neglect reachability.

Controllability Matrix (LTI)

Theorem

For any two times t_0 and t_1, with $t_1 > t_0 \geq 0$

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1) = \mathtt{Im}\mathfrak{C} = \mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1]$$

Attention

The notion of reachability and controllability **coincide** for continuous time LTI system, which means if one can go from origin to some state x_1, then one can also go from x_1 to origin.
Because of this, one studies controllability for continuous time system, and neglect reachability.

Logical idea of the proof.

$$\underbrace{\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1)}_{\text{Reachable subspace}} = \mathtt{Im}\mathfrak{C} = \overbrace{\mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1]}^{\text{Controllable subspace}}$$

Controllability Matrix (LTI)

Theorem

For any two times t_0 *and* t_1*, with* $t_1 > t_0 \geq 0$

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1) = \mathtt{Im}\mathfrak{C} = \mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1]$$

Attention

The notion of reachability and controllability **coincide** for continuous time LTI system, which means if one can go from origin to some state x_1, then one can also go from x_1 to origin.
Because of this, one studies controllability for continuous time system, and neglect reachability.

Logical idea of the proof.

$$\underbrace{\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1)}_{\text{Reachable subspace}} = \mathtt{Im}\mathfrak{C} = \overbrace{\mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1]}^{\text{Controllable subspace}}$$

The rest of the proof shall be done in two parts

- $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \implies x_1 \in \mathtt{Im}\mathfrak{C}$.
- $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \impliedby x_1 \in \mathtt{Im}\mathfrak{C}$.

Proof: $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \implies x_1 \in \mathtt{Im}\mathfrak{C}$

When $x_1 \in \mathcal{R}[t_0, t_1]$, there exists an input $u(\cdot)$ that transfers the state from $x(t_0) = 0$ to $x(t_1) = x_1$, and therefore

$$x_1 = \int_{t_0}^{t_1} e^{A(t_1-\tau)} B u(\tau) d\tau$$

Proof: $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \implies x_1 \in \mathtt{Im}\mathfrak{C}$

When $x_1 \in \mathcal{R}[t_0, t_1]$, there exists an input $u(\cdot)$ that transfers the state from $x(t_0) = 0$ to $x(t_1) = x_1$, and therefore

$$x_1 = \int_{t_0}^{t_1} e^{A(t_1-\tau)} Bu(\tau) d\tau$$

Recall

Theorem (Caley-Hamilton theorem)

For every $n \times n$ matrix A,

$$\Delta(A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I_{n\times n} = 0_{n\times n}$$

where

$$\Delta(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

is the characteristics polynomial of A.

Proof: $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \implies x_1 \in \mathtt{Im}\mathfrak{C}$

When $x_1 \in \mathcal{R}[t_0, t_1]$, there exists an input $u(\cdot)$ that transfers the state from $x(t_0) = 0$ to $x(t_1) = x_1$, and therefore

$$x_1 = \int_{t_0}^{t_1} e^{A(t_1-\tau)} B u(\tau) d\tau$$

Using Caley-Hamilton theorem, we can write

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i, \qquad \forall t \in \mathbb{R}$$

for appropriately defined scalar functions $\alpha_0(t), \alpha_1(t), \alpha_2(t), \ldots, \alpha_{n-1}(t)$.

Proof: $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \implies x_1 \in \mathtt{Im}\mathfrak{C}$

When $x_1 \in \mathcal{R}[t_0, t_1]$, there exists an input $u(\cdot)$ that transfers the state from $x(t_0) = 0$ to $x(t_1) = x_1$, and therefore

$$x_1 = \int_{t_0}^{t_1} e^{A(t_1 - \tau)} B u(\tau) d\tau$$

Using Caley-Hamilton theorem, we can write

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i, \qquad \forall t \in \mathbb{R}$$

for appropriately defined scalar functions $\alpha_0(t), \alpha_1(t), \alpha_2(t), \ldots, \alpha_{n-1}(t)$.
Therefore,

$$x_1 = \sum_{i=0}^{n-1} A^i B \left(\int_{t_0}^{t_1} \alpha_i(t_1 - \tau) u(\tau) d\tau \right)$$

Proof: $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \implies x_1 \in \mathtt{Im}\mathfrak{C}$

When $x_1 \in \mathcal{R}[t_0, t_1]$, there exists an input $u(\cdot)$ that transfers the state from $x(t_0) = 0$ to $x(t_1) = x_1$, and therefore

$$x_1 = \int_{t_0}^{t_1} e^{A(t_1 - \tau)} B u(\tau) d\tau$$

Using Caley-Hamilton theorem, we can write

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i, \qquad \forall t \in \mathbb{R}$$

for appropriately defined scalar functions $\alpha_0(t), \alpha_1(t), \alpha_2(t), \ldots, \alpha_{n-1}(t)$.
Therefore,

$$x_1 = \sum_{i=0}^{n-1} A^i B \left(\int_{t_0}^{t_1} \alpha_i(t_1 - \tau) u(\tau) d\tau \right) = \mathfrak{C} \begin{bmatrix} \int_{t_0}^{t_1} \alpha_0(t_1 - \tau) u(\tau) d\tau \\ \vdots \\ \int_{t_0}^{t_1} \alpha_{n-1}(t_1 - \tau) u(\tau) d\tau \end{bmatrix}$$

which shows that $x_1 \in \mathtt{Im}\mathfrak{C}$.

Proof: $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \impliedby x_1 \in \mathtt{Im}\mathfrak{C}$

Proof: $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \impliedby x_1 \in \mathtt{Im}\mathfrak{C}$

When $x_1 \in \mathtt{Im}\mathfrak{C}$, there exists a vector $v \in \mathbb{R}^{kn}$ for which

$$x_1 = \mathfrak{C}v.$$

Proof: $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \Longleftarrow x_1 \in \mathtt{Im}\mathfrak{C}$

When $x_1 \in \mathtt{Im}\mathfrak{C}$, there exists a vector $v \in \mathbb{R}^{kn}$ for which

$$x_1 = \mathfrak{C}v.$$

We show next that this leads to $x_1 \in \mathtt{Im}W_R(t_0, t_1) = (\mathtt{ker}W_R(t_0, t_1))^{\perp}$, which is to say that

$$\eta_1^T x_1 = \eta_1^T \mathfrak{C}v = 0, \qquad \forall \eta_1 \in \mathtt{ker}W_R(t_0, t_1). \tag{4}$$

Proof: $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \impliedby x_1 \in \mathtt{Im}\mathfrak{C}$

When $x_1 \in \mathtt{Im}\mathfrak{C}$, there exists a vector $v \in \mathbb{R}^{kn}$ for which

$$x_1 = \mathfrak{C}v.$$

We show next that this leads to $x_1 \in \mathtt{Im} W_R(t_0, t_1) = (\mathtt{ker} W_R(t_0, t_1))^\perp$, which is to say that

$$\eta_1^T x_1 = \eta_1^T \mathfrak{C} v = 0, \qquad \forall \eta_1 \in \mathtt{ker} W_R(t_0, t_1). \tag{4}$$

To verify that this is so, we pick an arbitrary vector $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$.
We saw in the proof of the reachable subspace that such vector η_1 has the property that

$$\eta_1^T e^{A(t_1 - \tau)} B = 0, \qquad \forall \tau \in [t_0, t_1].$$

To verify that, we pick some arbitrary vector $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$ and compute

$$x_1^T \eta_1 = \int_{t_0}^{t_1} u(\tau)^T B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \tag{3}$$

But since $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$, we have

$$\begin{aligned} \eta_1^T W_R(t_0, t_1) \eta_1 &= \int_{t_0}^{t_1} \eta_1^T \phi(t_1, \tau) B(\tau) B(\tau)^T \phi(t_1, \tau)^T \eta_1 d\tau \\ &= \int_{t_0}^{t_1} \|B(\tau)^T \phi(t_1, \tau)^T \eta_1\|^2 d\tau = 0 \end{aligned}$$

which implies that

$$B(\tau)^T \phi(t_1, \tau)^T \eta_1 = 0, \qquad \forall \tau \in [t_0, t_1].$$

Proof: $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \impliedby x_1 \in \mathtt{Im}\mathfrak{C}$

When $x_1 \in \mathtt{Im}\mathfrak{C}$, there exists a vector $v \in \mathbb{R}^{kn}$ for which

$$x_1 = \mathfrak{C}v.$$

We show next that this leads to $x_1 \in \mathtt{Im} W_R(t_0, t_1) = (\mathtt{ker} W_R(t_0, t_1))^\perp$, which is to say that

$$\eta_1^T x_1 = \eta_1^T \mathfrak{C} v = 0, \qquad \forall \eta_1 \in \mathtt{ker} W_R(t_0, t_1). \tag{4}$$

To verify that this is so, we pick an arbitrary vector $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$. We saw in the proof of the reachable subspace that such vector η_1 has the property that

$$\eta_1^T e^{A(t_1 - \tau)} B = 0, \qquad \forall \tau \in [t_0, t_1].$$

Taking k time derivatives with respect to τ, we further conclude that

$$(-1)^k \eta_1^T A^k e^{A(t_1 - \tau)} B = 0, \qquad \forall \tau \in [t_0, t_1], k \geq 0.$$

Proof: $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \impliedby x_1 \in \mathtt{Im}\mathfrak{C}$

When $x_1 \in \mathtt{Im}\mathfrak{C}$, there exists a vector $v \in \mathbb{R}^{kn}$ for which

$$x_1 = \mathfrak{C}v.$$

We show next that this leads to $x_1 \in \mathtt{Im} W_R(t_0, t_1) = (\mathtt{ker} W_R(t_0, t_1))^{\perp}$, which is to say that

$$\eta_1^T x_1 = \eta_1^T \mathfrak{C} v = 0, \qquad \forall \eta_1 \in \mathtt{ker} W_R(t_0, t_1). \tag{4}$$

To verify that this is so, we pick an arbitrary vector $\eta_1 \in \mathtt{ker} W_R(t_0, t_1)$. We saw in the proof of the reachable subspace that such vector η_1 has the property that

$$\eta_1^T e^{A(t_1 - \tau)} B = 0, \qquad \forall \tau \in [t_0, t_1].$$

Taking k time derivatives with respect to τ, we further conclude that

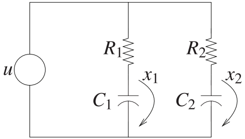
$$(-1)^k \eta_1^T A^k e^{A(t_1 - \tau)} B = 0, \qquad \forall \tau \in [t_0, t_1], k \geq 0.$$

and in particular for $\tau = t_1$, we obtain

$$\eta_1^T A^k B = 0, \qquad \forall k \geq 0$$

It follows that $\eta_1^T \mathfrak{C} = 0$ and therefore (4) indeed holds.

Parallel RC network example (continued)



The controllability matrix for the network is given by

$$\mathfrak{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1 C_1} & \frac{-1}{R_1^2 C_1^2} \\ \frac{1}{R_2 C_2} & \frac{-1}{R_2^2 C_2^2} \end{bmatrix}$$

When two branches have the same time constants, i.e., $\frac{1}{R_1 C_1} = \frac{1}{R_2 C_2} = \omega$, we have

$$\mathfrak{C} = \begin{bmatrix} \omega & -\omega^2 \\ \omega & -\omega^2 \end{bmatrix}$$

and therefore

$$\mathcal{R}[t_0, t_1] = \mathcal{C}[t_0, t_1] = \mathtt{Im}\mathfrak{C} = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} : \alpha \in \mathbb{R} \right\}, \quad \forall t_1 > t_0 \geq 0$$

However, when the time constants are different, i.e. $\frac{1}{R_1 C_1} \neq \frac{1}{R_2 C_2}$.

$$\det \mathfrak{C} = \frac{1}{R_1^2 C_1^2 R_2 C_2} - \frac{1}{R_1 C_1 R_2^2 C_2^2} = \frac{1}{R_1 C_1 R_2 C_2} \left(\frac{1}{R_1 C_1} - \frac{1}{R_2 C_2} \right) \neq 0$$

which means that $\mathfrak{C}$ is nonsingular, and therefore

$$\mathcal{R}[t_0, t_1] = \mathcal{C}[t_0, t_1] = \mathtt{Im}\mathfrak{C} = \mathbb{R}^2$$

Discrete-Time Case

Consider discrete-time LTV system

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k \quad \text{(AB-DLTV)}$$

A given input $u(\cdot)$ transfers of state $x(t_0) = x_0$ at time t_0 to the state $x(t_1) = x_1$ at time t_1 given by the variation of constants formula,

$$x_1 = \phi(t_1, t_0)x_0 + \sum_{\tau=t_0}^{t_1-1} \phi(t_1, \tau+1)B(\tau)u(\tau),$$

where $\phi(\cdot)$ denotes the system's state transition matrix.

We want to express how powerful the input is in terms of transferring the state between two given states.

Discrete-Time Case

Definition (Reachable subspace)

Given two times $t_1 > t_0 \geq 0$, the *reachable* or *controllable-from-the-origin on* $[t_0, t_1]$ subspace $\mathcal{R}[t_0, t_1]$ consists of all states x_1 for which there exists an input $u : \{t_0, t_0 + 1, \ldots, t_1 - 1\} \to \mathbb{R}^k$ that transfers the state from $x(t_0) = 0$ to $x(t_1) = x_1$; i.e.

$$\mathcal{R}[t_0, t_1] = \left\{ x_1 \in \mathbb{R}^n : \exists u(\cdot), x_1 = \sum_{\tau=t_0}^{t_1-1} \phi(t_1, \tau + 1) B(\tau) u(\tau) \right\}.$$

Definition (Controllable subspace)

Given two times $t_1 > t_0 \geq 0$, the *controllable* or *controller-to-the-origin on* $[t_0, t_1]$ subspace $\mathcal{C}[t_0, t_1]$ consists of all states x_0 for which there exists an input $u : \{t_0, t_0 + 1, \ldots, t_1 - 1\} \to \mathbb{R}^k$ that transfers the state from $x(t_0) = x_0$ to $x(t_1) = 0$; i.e.,

$$\mathcal{C}[t_0, t_1] = \left\{ x_0 \in \mathbb{R}^n : \exists u(\cdot), 0 = \phi(t_1, t_0) x_0 + \sum_{\tau=t_0}^{t_1-1} \phi(t_1, \tau + 1) B(\tau) u(\tau) \right\}.$$

Discrete-Time Case

Theorem (Reachability and controllability Gramians)

Given two times $t_1 > t_0 \geq 0$, the reachability and controllability Gramians of the system (AB-DLTV) are defined, respectively, by

$$W_R(t_0, t_1) := \sum_{\tau=t_0}^{t_1-1} \phi(t_1, \tau+1)B(\tau)B(\tau)'\phi(t_1, \tau+1)',$$

$$W_C(t_0, t_1) := \sum_{\tau=t_0}^{t_1-1} \phi(t_0, \tau+1)B(\tau)B(\tau)'\phi(t_0, \tau+1)'.$$

These Gramians allow us to determine exactly what the reachable and controllable spaces are.

Discrete-Time Case

Theorem (Reachability and controllability Gramians)

Given two times $t_1 > t_0 \geq 0$*, the reachability and controllability Gramians of the system (AB-DLTV) are defined, respectively, by*

$$W_R(t_0, t_1) := \sum_{\tau=t_0}^{t_1-1} \phi(t_1, \tau+1)B(\tau)B(\tau)'\phi(t_1, \tau+1)',$$

$$W_C(t_0, t_1) := \sum_{\tau=t_0}^{t_1-1} \phi(t_0, \tau+1)B(\tau)B(\tau)'\phi(t_0, \tau+1)'.$$

Attention!

The discrete-time controllability Gramian requires a backward-in-time state transition matrix $\phi(t_0, \tau+1)$ from time $\tau+1$ to time $t_0 \leq \tau < \tau+1$.

These Gramians allow us to determine exactly what the reachable and controllable spaces are.

Discrete-Time Case

Theorem (Reachability and controllability Gramians)

Given two times $t_1 > t_0 \geq 0$, the reachability and controllability Gramians of the system (AB-DLTV) are defined, respectively, by

$$W_R(t_0, t_1) := \sum_{\tau=t_0}^{t_1-1} \phi(t_1, \tau+1)B(\tau)B(\tau)'\phi(t_1, \tau+1)',$$

$$W_C(t_0, t_1) := \sum_{\tau=t_0}^{t_1-1} \phi(t_0, \tau+1)B(\tau)B(\tau)'\phi(t_0, \tau+1)'.$$

Attention!

The discrete-time controllability Gramian requires a backward-in-time state transition matrix $\phi(t_0, \tau+1)$ from time $\tau+1$ to time $t_0 \leq \tau < \tau+1$.
The matrix is well defined *only* when

$$x(\tau+1) = A(\tau)A(\tau-1)\cdots A(t_0)x(t_0), \qquad t_0 \leq \tau < t_1 - 1$$

can be solved for $x(t_0)$, i.e. when all the matrices $A(t_0), A(t_0+1)\ldots A(t_1-1)$ are **nonsingular**.

These Gramians allow us to determine exactly what the reachable and controllable spaces are.

Discrete-Time Case

Theorem (Reachability and controllability Gramians)

Given two times $t_1 > t_0 \geq 0$, the reachability and controllability Gramians of the system (AB-DLTV) are defined, respectively, by

$$W_R(t_0, t_1) := \sum_{\tau=t_0}^{t_1-1} \phi(t_1, \tau+1)B(\tau)B(\tau)'\phi(t_1, \tau+1)',$$

$$W_C(t_0, t_1) := \sum_{\tau=t_0}^{t_1-1} \phi(t_0, \tau+1)B(\tau)B(\tau)'\phi(t_0, \tau+1)'.$$

Attention!

The discrete-time controllability Gramian requires a backward-in-time state transition matrix $\phi(t_0, \tau+1)$ from time $\tau+1$ to time $t_0 \leq \tau < \tau+1$.
The matrix is well defined $only$ when

$$x(\tau+1) = A(\tau)A(\tau-1)\cdots A(t_0)x(t_0), \qquad t_0 \leq \tau < t_1 - 1$$

can be solved for $x(t_0)$, i.e. when all the matrices $A(t_0), A(t_0+1)\ldots A(t_1-1)$ are **nonsingular**.
When this does not happen, $the\ controllability\ Gramian\ cannot\ be\ defined.$

These Gramians allow us to determine exactly what the reachable and controllable spaces are.

Discrete-Time Case

Theorem (Reachable and controllable subspaces)

Given two times $t_1 > t_0 \geq 0$,

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1),$$

Moreover

1. *if* $x_1 = W_R(t_0, t_1)\eta_1 \in \mathtt{Im} W_R(t_0, t_1)$, *the control*

$$u(t) = B(t)^T \phi(t_1, t+1)^T \eta_1, \qquad t \in [t_0, t_1 - 1]$$

can be used to transfer the state from $x(t_0) = 0$ *to* $x(t_1) = x_1$,

Discrete-Time Case

Theorem (Reachable and controllable subspaces)

Given two times $t_1 > t_0 \geq 0$,

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1),$$
$$\mathcal{C}[t_0, t_1] = \mathtt{Im} W_C(t_0, t_1)$$

Moreover

1. *if* $x_1 = W_R(t_0, t_1)\eta_1 \in \mathtt{Im} W_R(t_0, t_1)$, *the control*
$$u(t) = B(t)^T \phi(t_1, t + 1)^T \eta_1, \qquad t \in [t_0, t_1 - 1]$$
can be used to transfer the state from $x(t_0) = 0$ *to* $x(t_1) = x_1$, *and*
2. *if* $x_0 = W_C(t_0, t_1)\eta_0 \in \mathtt{Im} W_C(t_0, t_1)$, *the control*
$$u(t) = -B(t)^T \phi(t_0, t + 1)^T \eta_0, \qquad t \in [t_0, t_1 - 1]$$
can be used to transfer the state from $x(t_0) = x_0$ *to* $x(t_1) = 0$.

Discrete-Time Case

Theorem (Reachable and controllable subspaces)

Given two times $t_1 > t_0 \geq 0$,

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1),$$
$$\mathcal{C}[t_0, t_1] = \mathtt{Im} W_C(t_0, t_1)$$

Moreover

1. *if* $x_1 = W_R(t_0, t_1)\eta_1 \in \mathtt{Im} W_R(t_0, t_1)$, *the control*

$$u(t) = B(t)^T \phi(t_1, t+1)^T \eta_1, \qquad t \in [t_0, t_1 - 1]$$

can be used to transfer the state from $x(t_0) = 0$ *to* $x(t_1) = x_1$, *and*

2. *if* $x_0 = W_C(t_0, t_1)\eta_0 \in \mathtt{Im} W_C(t_0, t_1)$, *the control*

$$u(t) = -B(t)^T \phi(t_0, t+1)^T \eta_0, \qquad t \in [t_0, t_1 - 1]$$

can be used to transfer the state from $x(t_0) = x_0$ *to* $x(t_1) = 0$.

Logical idea of the proof.

The proof can be done in two parts

- $x_1 \in \mathtt{Im} W_R(t_0, t_1) \implies x_1 \in \mathcal{R}[t_0, t_1]$
- $x_1 \in \mathtt{Im} W_R(t_0, t_1) \impliedby x_1 \in \mathcal{R}[t_0, t_1]$

Discrete-Time Case: LTI

Consider now the discrete-time LTI ststem

$$x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{AB-DLTI}$$

For this system, the reachability and controllability Gramians are given, respectively, by

$$W_R(t_0, t_1) = \sum_{\tau=t_0}^{t_1-1} A^{t_1-1-\tau} BB^T (A^T)^{t_1-1-\tau},$$
$$W_C(t_0, t_1)^1 = \sum_{\tau=t_0}^{t_1-1} A^{t_0-1-\tau} BB^T (A^T)^{t_0-1-\tau}$$

and the *controllability matrix* is given by

$$\boxed{\mathfrak{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}_{n\times(kn)}.}$$

[1]The controllability Gramian can be defined only when A is nonsingular.

Discrete-Time Case: LTI

Theorem

For any two times $t_1 > t_0 \geq 0$, *with* $t_1 \geq t_0 + n$, *we have*[1]

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1) = \mathtt{Im}\mathfrak{C} = \mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1].$$

[1] The results regarding the controllability Gramian implicitly assume that A is nonsingular.

Discrete-Time Case: LTI

Theorem

For any two times $t_1 > t_0 \geq 0$, with $t_1 \geq t_0 + n$, we have[1]

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1) = \mathtt{Im}\mathfrak{C} = \mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1].$$

Attention

In discrete time, the notions of controllable and reachable subspaces **coincide** only when the matrix A is nonsingular.

[1] The results regarding the controllability Gramian implicitly assume that A is nonsingular.

Discrete-Time Case: LTI

Theorem

For any two times $t_1 > t_0 \geq 0$, with $t_1 \geq t_0 + n$, we have[1]

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1) = \mathtt{Im}\mathfrak{C} = \mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1].$$

Attention

In discrete time, the notions of controllable and reachable subspaces **coincide** only when the matrix A is nonsingular.
Otherwise, we have

$$\mathcal{R}[t_0, t_1] = \mathtt{Im}\mathfrak{C} \subset \mathcal{C}[t_0, t_1],$$

[1] The results regarding the controllability Gramian implicitly assume that A is nonsingular.

Discrete-Time Case: LTI

Theorem

For any two times $t_1 > t_0 \geq 0$, *with* $t_1 \geq t_0 + n$, *we have*[1]

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1) = \mathtt{Im}\mathfrak{C} = \mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1].$$

Attention

In discrete time, the notions of controllable and reachable subspaces **coincide** only when the matrix A is nonsingular.
Otherwise, we have

$$\mathcal{R}[t_0, t_1] = \mathtt{Im}\mathfrak{C} \subset \mathcal{C}[t_0, t_1],$$

but the reverse inclusion *does not* hold, i.e., there are states x_1 that can be transferred to the origin, but it is not possible to find an input to transfer the origin to these states.

[1] The results regarding the controllability Gramian implicitly assume that A is nonsingular.

Discrete-Time Case: LTI

Theorem

For any two times $t_1 > t_0 \geq 0$, *with* $t_1 \geq t_0 + n$, *we have*[1]

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1) = \mathtt{Im}\mathfrak{C} = \mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1].$$

Attention

In discrete time, the notions of controllable and reachable subspaces **coincide** only when the matrix A is nonsingular.
Otherwise, we have

$$\mathcal{R}[t_0, t_1] = \mathtt{Im}\mathfrak{C} \subset \mathcal{C}[t_0, t_1],$$

but the reverse inclusion *does not* hold, i.e., there are states x_1 that can be transferred to the origin, but it is not possible to find an input to transfer the origin to these states.

Because of this, one *must* study reachability and controllability of discrete time systems separately.

[1] The results regarding the controllability Gramian implicitly assume that A is nonsingular.

Discrete-Time Case: LTI

Theorem

For any two times $t_1 > t_0 \geq 0$*, with* $t_1 \geq t_0 + n$*, we have*[1]

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1) = \mathtt{Im}\mathfrak{C} = \mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1].$$

Logical idea of the proof.

$$\underbrace{\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1)}_{\text{Reachable subspace}} = \mathtt{Im}\mathfrak{C} = \overbrace{\mathtt{Im} W_C(t_0, t_1) = \mathcal{C}[t_0, t_1]}^{\text{Controllable subspace}}$$

The proof can be done in two parts

- $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \implies x_1 \in \mathtt{Im}\mathfrak{C}$.
- $x_1 \in \mathcal{R}[t_0, t_1] = \mathtt{Im} W_R[t_0, t_1] \impliedby x_1 \in \mathtt{Im}\mathfrak{C}$.

$\square$

[1] The results regarding the controllability Gramian implicitly assume that A is nonsingular.

Outline of Controllable Systems

1. Matrix test
2. Eigenvector test
3. Lyapunov test

Matrix Test

Consider the following continuous and discrete-time LTV systems

$$\dot{x} = A(t)x + B(t)u \quad / \quad x(t+1) = A(t)x(t) + B(t)u(t), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{AB-LTV}$$

Definition (Reachable system)

Given two times $t_1 > t_0 \geq 0$, the system (AB-LTV), or simply the pair $(A(.), B(.))$, is *(completely state-) reachable on* $[t_0, t_1]$ whenever $\mathcal{R}[t_0, t_1] = \mathbb{R}^n$, i.e., whenever the origin can be transferred to every state.

Definition (Controllable system)

Given two times $t_1 > t_0 \geq 0$, the system (AB-LTV), or simply the pair $(A(.), B(.))$, is *(completely state-) controllable on* $[t_0, t_1]$ whenever $\mathcal{C}[t_0, t_1] = R^n$, i.e., whenever every state can be transferred to the origin.

[1] Here, we jointly present the results for continuous and discrete time and use a slash to separate the two cases.

Matrix Test

Theorem

The following two statements are equivalent.

1. *The $n-$dimensional pair $(A(t), B(t))$ is controllable at time t_0.*
2. *there exists a finite $t_1 > t_0$ such that the $n \times n$ matrix*
$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B(\tau)'\phi(t_1, \tau)'d\tau$$
is nonsingular.

Matrix Test

Theorem

The following two statements are equivalent.

1. *The* $n-$*dimensional pair* $(A(t), B(t))$ *is controllable at time* t_0*.*
2. *there exists a finite* $t_1 > t_0$ *such that the* $n \times n$ *matrix*

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B(\tau)'\phi(t_1, \tau)'d\tau$$

is nonsingular.

Proof: $2 \implies 1.$

Matrix Test

Theorem

The following two statements are equivalent.

1. *The* $n-$*dimensional pair* $(A(t), B(t))$ *is controllable at time* t_0.
2. *there exists a finite* $t_1 > t_0$ *such that the* $n \times n$ *matrix*

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B(\tau)'\phi(t_1, \tau)'d\tau$$

is nonsingular.

Proof: $2 \implies 1.$

The response of (AB-LTV) at t_1 is given as

$$x(t_1) = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

Matrix Test

Theorem

The following two statements are equivalent.

1. *The* $n-$*dimensional pair* $(A(t), B(t))$ *is controllable at time* t_0.
2. *there exists a finite* $t_1 > t_0$ *such that the* $n \times n$ *matrix*

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B(\tau)'\phi(t_1, \tau)'d\tau$$

   is nonsingular.

Proof: $2 \implies 1.$

The response of (AB-LTV) at t_1 is given as

$$x(t_1) = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

We claim that the input

$$u(t) = -B'(t)\phi'(t_1, t)\eta, \qquad \eta = W_C^{-1}(t_0, t_1)[\phi(t_1, t_0)(x_0 - x_1)]$$

will transfer x_0 at time t_0 to x_1 at time t_1.

Matrix Test

Theorem

The following two statements are equivalent.

1. *The $n-$dimensional pair $(A(t), B(t))$ is controllable at time t_0.*
2. *there exists a finite $t_1 > t_0$ such that the $n \times n$ matrix*

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B(\tau)'\phi(t_1, \tau)'d\tau$$

is nonsingular.

Proof: $2 \implies 1.$

The response of (AB-LTV) at t_1 is given as

$$x(t_1) = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

We claim that the input

$$u(t) = -B'(t)\phi'(t_1, t)\eta, \qquad \eta = W_C^{-1}(t_0, t_1)[\phi(t_1, t_0)(x_0 - x_1)]$$

will transfer x_0 at time t_0 to x_1 at time t_1.From the above, we have

$$x(t_1) = \phi(t_1, t_0)x_0 - \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B'(\tau)\phi'(t_1, \tau)d\tau \times \eta$$

Matrix Test

Theorem

The following two statements are equivalent.

1. *The* $n-$*dimensional pair* $(A(t), B(t))$ *is controllable at time* t_0.
2. *there exists a finite* $t_1 > t_0$ *such that the* $n \times n$ *matrix*

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B(\tau)'\phi(t_1, \tau)'d\tau$$

is nonsingular.

Proof: $2 \implies 1.$

The response of (AB-LTV) at t_1 is given as

$$x(t_1) = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

We claim that the input

$$u(t) = -B'(t)\phi'(t_1, t)\eta, \qquad \eta = W_C^{-1}(t_0, t_1)[\phi(t_1, t_0)(x_0 - x_1)]$$

will transfer x_0 at time t_0 to x_1 at time t_1.From the above, we have

$$\begin{aligned} x(t_1) &= \phi(t_1, t_0)x_0 - \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B'(\tau)\phi'(t_1, \tau)d\tau \times \eta \\ &= \phi(t_1, t_0)x_0 - W_c(t_0, t_1)\underbrace{W_c^{-1}(t_0, t_1)[\phi(t_1, t_0)x_0 - x_1]}_{\eta} \end{aligned}$$

Matrix Test

Theorem

The following two statements are equivalent.

1. *The $n-$dimensional pair $(A(t), B(t))$ is controllable at time t_0.*
2. *there exists a finite $t_1 > t_0$ such that the $n \times n$ matrix*
$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B(\tau)'\phi(t_1, \tau)'d\tau$$
is nonsingular.

Proof: $2 \implies 1.$

The response of (AB-LTV) at t_1 is given as

$$x(t_1) = \phi(t_1, t_0)x_0 + \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)u(\tau)d\tau$$

We claim that the input

$$u(t) = -B'(t)\phi'(t_1, t)\eta, \qquad \eta = W_C^{-1}(t_0, t_1)[\phi(t_1, t_0)(x_0 - x_1)]$$

will transfer x_0 at time t_0 to x_1 at time t_1.From the above, we have

$$x(t_1) = \phi(t_1, t_0)x_0 - \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)B'(\tau)\phi'(t_1, \tau)d\tau \times \eta$$
$$= \phi(t_1, t_0)x_0 - W_c(t_0, t_1)\underbrace{W_c^{-1}(t_0, t_1)[\phi(t_1, t_0)x_0 - x_1]}_{\eta} = x_1$$

Proof: $2 \Longleftarrow 1$ OR $\neg 2 \Longrightarrow \neg 1$

Proof: $2 \Longleftarrow 1$ OR $\neg 2 \Longrightarrow \neg 1$

We prove this by contradiction.
Suppose $W_c(t_0, t_1)$ is singular or positive-semidefinite, for all $t_1 > t_0$.

Proof: $2 \impliedby 1$ OR $\neg 2 \implies \neg 1$

We prove this by contradiction.
Suppose $W_c(t_0, t_1)$ is singular or positive-semidefinite, for all $t_1 > t_0$. Then there exists a nonzero constant vector $v \in \mathbb{R}^n$ such that

$$v' W_c(t_0, t_1) v = 0$$

Proof: $2 \Longleftarrow 1$ OR $\neg 2 \Longrightarrow \neg 1$

We prove this by contradiction.
Suppose $W_c(t_0, t_1)$ is singular or positive-semidefinite, for all $t_1 > t_0$.Then there exists a nonzero constant vector $v \in \mathbb{R}^n$ such that

$$v' W_c(t_0, t_1) v = 0 = \int_{t_0}^{t_1} v' \phi(t_1, \tau) B(\tau) B'(\tau) \phi'(t_1, \tau) v d\tau$$

Proof: $2 \Longleftarrow 1$ OR $\neg 2 \Longrightarrow \neg 1$

We prove this by contradiction.
Suppose $W_c(t_0, t_1)$ is singular or positive-semidefinite, for all $t_1 > t_0$.Then there exists a nonzero constant vector $v \in \mathbb{R}^n$ such that

$$\begin{aligned} v' W_c(t_0, t_1) v = 0 &= \int_{t_0}^{t_1} v' \phi(t_1, \tau) B(\tau) B'(\tau) \phi'(t_1, \tau) v d\tau \\ &= \int_{t_0}^{t_1} \left\| B'(\tau) \phi'(t_1, \tau) v \right\|^2 d\tau \end{aligned}$$

Proof: $2 \Longleftarrow 1$ OR $\neg 2 \Longrightarrow \neg 1$

We prove this by contradiction.
Suppose $W_c(t_0, t_1)$ is singular or positive-semidefinite, for all $t_1 > t_0$.Then there exists a nonzero constant vector $v \in \mathbb{R}^n$ such that

$$\begin{aligned} v' W_c(t_0, t_1) v = 0 &= \int_{t_0}^{t_1} v' \phi(t_1, \tau) B(\tau) B'(\tau) \phi'(t_1, \tau) v d\tau \\ &= \int_{t_0}^{t_1} \left\| B'(\tau) \phi'(t_1, \tau) v \right\|^2 d\tau \end{aligned}$$

which implies

$$B'(\tau)\phi'(t_1, \tau) v = 0 \quad \text{or} \quad v'\phi(t_1, \tau) B(\tau) = 0 \qquad \forall \tau \in [t_0, t_1]$$

Proof: $2 \Longleftarrow 1$ OR $\neg 2 \implies \neg 1$

We prove this by contradiction.
Suppose $W_c(t_0, t_1)$ is singular or positive-semidefinite, for all $t_1 > t_0$.Then there exists a nonzero constant vector $v \in \mathbb{R}^n$ such that

$$\begin{aligned} v'W_c(t_0,t_1)v = 0 &= \int_{t_0}^{t_1} v'\phi(t_1,\tau)B(\tau)B'(\tau)\phi'(t_1,\tau)v d\tau \\ &= \int_{t_0}^{t_1} \left\|B'(\tau)\phi'(t_1,\tau)v\right\|^2 d\tau \end{aligned}$$

which implies

$$B'(\tau)\phi'(t_1,\tau)v = 0 \quad \text{or} \quad v'\phi(t_1,\tau)B(\tau) = 0 \qquad \forall \tau \in [t_0, t_1]$$

If (AB-LTV) is controllable, there exists an input that transfers the initial state $x_0 = \phi(t_0, t_1)v$ at t_0 to $x(t_1) = 0$.

Proof: $2 \Longleftarrow 1$ OR $\neg 2 \implies \neg 1$

We prove this by contradiction.
Suppose $W_c(t_0, t_1)$ is singular or positive-semidefinite, for all $t_1 > t_0$.Then there exists a nonzero constant vector $v \in \mathbb{R}^n$ such that

$$v' W_c(t_0, t_1) v = 0 = \int_{t_0}^{t_1} v' \phi(t_1, \tau) B(\tau) B'(\tau) \phi'(t_1, \tau) v d\tau$$
$$= \int_{t_0}^{t_1} \left\| B'(\tau) \phi'(t_1, \tau) v \right\|^2 d\tau$$

which implies

$$B'(\tau)\phi'(t_1,\tau)v = 0 \quad \text{or} \quad v'\phi(t_1,\tau)B(\tau) = 0 \qquad \forall \tau \in [t_0, t_1]$$

If (AB-LTV) is controllable, there exists an input that transfers the initial state $x_0 = \phi(t_0, t_1)v$ at t_0 to $x(t_1) = 0$. Then the solution of the state equations is given by

$$0 = \overbrace{\phi(t_1, t_0)\,\underbrace{\phi(t_0, t_1)}_{}}^{I}\!\!\underbrace{\vphantom{\phi}v}_{x_0} + \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$$

Proof: $2 \Longleftarrow 1$ OR $\neg 2 \Longrightarrow \neg 1$

We prove this by contradiction.
Suppose $W_c(t_0, t_1)$ is singular or positive-semidefinite, for all $t_1 > t_0$.Then there exists a nonzero constant vector $v \in \mathbb{R}^n$ such that

$$\begin{aligned} v' W_c(t_0, t_1) v = 0 &= \int_{t_0}^{t_1} v' \phi(t_1, \tau) B(\tau) B'(\tau) \phi'(t_1, \tau) v d\tau \\ &= \int_{t_0}^{t_1} \left\| B'(\tau) \phi'(t_1, \tau) v \right\|^2 d\tau \end{aligned}$$

which implies

$$B'(\tau)\phi'(t_1, \tau) v = 0 \quad \text{or} \quad v'\phi(t_1, \tau) B(\tau) = 0 \qquad \forall \tau \in [t_0, t_1]$$

If (AB-LTV) is controllable, there exists an input that transfers the initial state $x_0 = \phi(t_0, t_1) v$ at t_0 to $x(t_1) = 0$. Then the solution of the state equations is given by

$$0 = \overbrace{\phi(t_1, t_0)\, \phi(t_0, t_1)}^{I} \underbrace{ v}_{x_0} + \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau$$

Premultiplication with v' yields

$$0 = v'v + v' \int \phi(t_1, \tau) B(\tau) u(\tau) d\tau$$

Proof: $2 \Longleftarrow 1$ OR $\neg 2 \Longrightarrow \neg 1$

We prove this by contradiction.
Suppose $W_c(t_0, t_1)$ is singular or positive-semidefinite, for all $t_1 > t_0$.Then there exists a nonzero constant vector $v \in \mathbb{R}^n$ such that

$$
\begin{aligned}
v' W_c(t_0, t_1) v = 0 &= \int_{t_0}^{t_1} v' \phi(t_1, \tau) B(\tau) B'(\tau) \phi'(t_1, \tau) v d\tau \\
&= \int_{t_0}^{t_1} \left\| B'(\tau) \phi'(t_1, \tau) v \right\|^2 d\tau
\end{aligned}
$$

which implies

$$
B'(\tau)\phi'(t_1, \tau) v = 0 \quad \text{or} \quad v'\phi(t_1, \tau) B(\tau) = 0 \qquad \forall \tau \in [t_0, t_1]
$$

If (AB-LTV) is controllable, there exists an input that transfers the initial state $x_0 = \phi(t_0, t_1) v$ at t_0 to $x(t_1) = 0$. Then the solution of the state equations is given by

$$
0 = \overbrace{\phi(t_1, t_0)\, \phi(t_0, t_1)}^{I} \!\underbrace{ v}_{x_0} + \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau
$$

Premultiplication with v' yields

$$
0 = v'v + v' \int \phi(t_1, \tau) B(\tau) u(\tau) d\tau = \|v\|^2 + 0.
$$

This contradicts the hypothesis $v \neq 0$.

Matrix Test

Attention!

1. We need the knowledge of the state transition matrix which may not be available

Matrix Test

Attention!

1. We need the knowledge of the state transition matrix which may not be available
2. Therefore, it is desirable to develop a controllability condition without involving $\phi(t, \tau)$

Matrix Test

Attention!

1. We need the knowledge of the state transition matrix which may not be available
2. Therefore, it is desirable to develop a controllability condition without involving $\phi(t, \tau)$
3. This is possible if we have additional conditions on $A(t)$ and $B(t)$

Matrix Test

Attention!

1. We need the knowledge of the state transition matrix which may not be available
2. Therefore, it is desirable to develop a controllability condition without involving $\phi(t, \tau)$
3. This is possible if we have additional conditions on $A(t)$ and $B(t)$
4. Recall that we have assumed $A(t)$ and $B(t)$ to be continuous. Now we require them to be $(n-1)$ times continuously differentiable.

Matrix Test

Define $M_0(t) \triangleq B(t)$, define using recursion:

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t);\ m = 0, 1, \ldots, (n-1)$$

Matrix Test

Define $M_0(t) \triangleq B(t)$, define using recursion:

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t);\ m = 0, 1, \ldots, (n-1)$$

Clearly, we have

$$\phi(t_2, t)B(t) = \phi(t_2, t)M_0(t) \text{ for any fixed } t_2$$

Matrix Test

Define $M_0(t) \triangleq B(t)$, define using recursion:

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t);\ m = 0, 1, \ldots, (n-1)$$

Clearly, we have

$$\phi(t_2, t)B(t) = \phi(t_2, t)M_0(t) \text{ for any fixed } t_2$$

Using

$$\boxed{\frac{d}{dt}\phi(t_2, t) = -\phi(t_2, t)A(t),}$$

Matrix Test

Define $M_0(t) \triangleq B(t)$, define using recursion:

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t);\ m = 0, 1, \ldots, (n-1)$$

Clearly, we have

$$\phi(t_2, t)B(t) = \phi(t_2, t)M_0(t) \text{ for any fixed } t_2$$

Using

$$\boxed{\frac{d}{dt}\phi(t_2, t) = -\phi(t_2, t)A(t),}$$

compute

$$\frac{d}{dt}\left[\phi(t_2, t)B(t)\right]$$

Matrix Test

Define $M_0(t) \triangleq B(t)$, define using recursion:

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t);\ m = 0, 1, \ldots, (n-1)$$

Clearly, we have

$$\phi(t_2, t)B(t) = \phi(t_2, t)M_0(t) \text{ for any fixed } t_2$$

Using

$$\boxed{\frac{d}{dt}\phi(t_2, t) = -\phi(t_2, t)A(t),}$$

compute

$$\frac{d}{dt}\left[\phi(t_2, t)B(t)\right] = \frac{d}{dt}\left[\phi(t_2, t)\right]B(t) + \phi(t_2, t)\frac{d}{dt}B(t)$$

Matrix Test

Define $M_0(t) \triangleq B(t)$, define using recursion:

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t);\ m = 0, 1, \ldots, (n-1)$$

Clearly, we have

$$\phi(t_2, t)B(t) = \phi(t_2, t)M_0(t) \text{ for any fixed } t_2$$

Using

$$\boxed{\frac{d}{dt}\phi(t_2, t) = -\phi(t_2, t)A(t),}$$

compute

$$\begin{aligned} \frac{d}{dt}\left[\phi(t_2, t)B(t)\right] &= \frac{d}{dt}\left[\phi(t_2, t)\right]B(t) + \phi(t_2, t)\frac{d}{dt}B(t) \\ &= \phi(t_2, t)\left[-A(t)M_0(t) + \frac{d}{dt}M_0(t)\right] \end{aligned}$$

Matrix Test

Define $M_0(t) \triangleq B(t)$, define using recursion:

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t); \; m = 0, 1, \ldots, (n-1)$$

Clearly, we have

$$\phi(t_2, t)B(t) = \phi(t_2, t)M_0(t) \text{ for any fixed } t_2$$

Using

$$\boxed{\frac{d}{dt}\phi(t_2, t) = -\phi(t_2, t)A(t),}$$

compute

$$\begin{aligned}
\frac{d}{dt}\left[\phi(t_2, t)B(t)\right] &= \frac{d}{dt}\left[\phi(t_2, t)\right]B(t) + \phi(t_2, t)\frac{d}{dt}B(t) \\
&= \phi(t_2, t)\left[-A(t)M_0(t) + \frac{d}{dt}M_0(t)\right] = \phi(t_t, t)M_1(t)
\end{aligned}$$

Matrix Test

Define $M_0(t) \triangleq B(t)$, define using recursion:

$$M_{m+1}(t) = -A(t)M_m(t) + \frac{d}{dt}M_m(t);\;\; m = 0, 1, \ldots, (n-1)$$

Clearly, we have

$$\phi(t_2, t)B(t) = \phi(t_2, t)M_0(t) \text{ for any fixed } t_2$$

Using

$$\boxed{\frac{d}{dt}\phi(t_2, t) = -\phi(t_2, t)A(t),}$$

compute

$$\begin{aligned}\frac{d}{dt}\left[\phi(t_2, t)B(t)\right] &= \frac{d}{dt}\left[\phi(t_2, t)\right]B(t) + \phi(t_2, t)\frac{d}{dt}B(t)\\ &= \phi(t_2, t)\left[-A(t)M_0(t) + \frac{d}{dt}M_0(t)\right] = \phi(t_t, t)M_1(t)\end{aligned}$$

Proceeding forward, we have

$$\frac{d^m}{dt^m}\phi(t_2, t)B(t) = \phi(t_2, t)M_m(t); m = 0, 1, 2, \ldots$$

Matrix Test

Theorem

Let $A(t)$ and $B(t)$ be $(n-1)$ times continuously differentiable, then the $n-$dimensional pair $(A(t), B(t))$ is controllable at t_0 if there exists a finite $t_1 > t_0$ such that

$$\texttt{rank}\begin{bmatrix} M_0(t_1) & M_1(t_1) & \cdots & M_{n-1}(t_1) \end{bmatrix} = n$$

$$\boxed{\begin{aligned} M_0(t) &= B(t) \\ M_{m+1}(t) &= -A(t)M_m(t) + \frac{d}{dt}M_m(t);\ m = 0, 1, \ldots, (n-1) \end{aligned}}$$

Note

The above theorem is sufficient but not necessary.

Proof

We show that if the rank condition holds, then $W_c(t_0, t_1)$ is non-singular for all $t \geq t_1$.

Proof

We show that if the rank condition holds, then $W_c(t_0, t_1)$ is non-singular for all $t \geq t_1$.
Suppose not, i.e. $W_c(t_0, t_1)$ is singular or positive semidefinite for some $t_2 \geq t_1$. Then there exists a nonzero constant vector v such that

$$v' W_c(t_0, t_2) v = 0 = \int_{t_0}^{t_2} \|B'(\tau)\phi'(t_2, \tau)v\|^2 d\tau$$

which implies

$$B'(\tau)\phi'(t_2, \tau)v = 0 \quad \text{or} \quad v'\phi(t_2, \tau)B(\tau) = 0$$

for all τ in $[t_0, t_2]$.

Proof

We show that if the rank condition holds, then $W_c(t_0, t_1)$ is non-singular for all $t \geq t_1$.
Suppose not, i.e. $W_c(t_0, t_1)$ is singular or positive semidefinite for some $t_2 \geq t_1$. Then there exists a nonzero constant vector v such that

$$v' W_c(t_0, t_2) v = 0 = \int_{t_0}^{t_2} \|B'(\tau)\phi'(t_2, \tau)v\|^2 d\tau$$

which implies

$$B'(\tau)\phi'(t_2, \tau)v = 0 \quad \text{or} \quad v'\phi(t_2, \tau)B(\tau) = 0$$

for all τ in $[t_0, t_2]$. Its differentiation with respect to τ yield as derived previously

$$v'\phi(t_2, \tau)M_m(\tau) = 0$$

for $m = 0, 1, 2, \ldots, n-1$, and all $\tau \in [t_0, t_2]$, in particular, at t_1.

Proof

We show that if the rank condition holds, then $W_c(t_0, t_1)$ is non-singular for all $t \geq t_1$.
Suppose not, i.e. $W_c(t_0, t_1)$ is singular or positive semidefinite for some $t_2 \geq t_1$. Then there exists a nonzero constant vector v such that

$$v' W_c(t_0, t_2) v = 0 = \int_{t_0}^{t_2} \|B'(\tau)\phi'(t_2, \tau)v\|^2 d\tau$$

which implies

$$B'(\tau)\phi'(t_2, \tau)v = 0 \quad \text{or} \quad v'\phi(t_2, \tau)B(\tau) = 0$$

for all τ in $[t_0, t_2]$. Its differentiation with respect to τ yield as derived previously

$$v'\phi(t_2, \tau)M_m(\tau) = 0$$

for $m = 0, 1, 2, \ldots, n-1$, and all $\tau \in [t_0, t_2]$, in particular, at t_1. They can be arranged as

$$v'\phi(t_2, t_1) \begin{bmatrix} M_0(t_1) & M_1(t_1) & \cdots & M_{n-1}(t_1) \end{bmatrix} = 0$$

Proof

We show that if the rank condition holds, then $W_c(t_0, t_1)$ is non-singular for all $t \geq t_1$.
Suppose not, i.e. $W_c(t_0, t_1)$ is singular or positive semidefinite for some $t_2 \geq t_1$. Then there exists a nonzero constant vector v such that

$$v' W_c(t_0, t_2) v = 0 = \int_{t_0}^{t_2} \|B'(\tau)\phi'(t_2, \tau)v\|^2 d\tau$$

which implies

$$B'(\tau)\phi'(t_2, \tau)v = 0 \quad \text{or} \quad v'\phi(t_2, \tau)B(\tau) = 0$$

for all τ in $[t_0, t_2]$. Its differentiation with respect to τ yield as derived previously

$$v'\phi(t_2, \tau)M_m(\tau) = 0$$

for $m = 0, 1, 2, \ldots, n-1$, and all $\tau \in [t_0, t_2]$, in particular, at t_1. They can be arranged as

$$v'\phi(t_2, t_1)\begin{bmatrix} M_0(t_1) & M_1(t_1) & \cdots & M_{n-1}(t_1) \end{bmatrix} = 0$$

Because $\phi(t_2, t_1)$ is nonsingular, $v'\phi(t_2, t_1)$ is nonzero. This contradicts the rank condition.

Example

Consider

$$\dot{x} = \begin{bmatrix} t & -1 & 0 \\ 0 & -t & t \\ 0 & 0 & t \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

Example

Consider

$$\dot{x} = \begin{bmatrix} t & -1 & 0 \\ 0 & -t & t \\ 0 & 0 & t \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

We have $M_0 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}'$ and compute

$$M_1 = -A(t)M_0 + \frac{d}{dt}M_0 = \begin{bmatrix} 1 \\ 0 \\ -t \end{bmatrix}$$

$$M_2 = -A(t)M_1 + \frac{d}{dt}M_1 = \begin{bmatrix} -t \\ t^2 \\ t^2 - 1 \end{bmatrix}$$

Example

Consider

$$\dot{x} = \begin{bmatrix} t & -1 & 0 \\ 0 & -t & t \\ 0 & 0 & t \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

We have $M_0 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}'$ and compute

$$M_1 = -A(t)M_0 + \frac{d}{dt}M_0 = \begin{bmatrix} 1 \\ 0 \\ -t \end{bmatrix}$$

$$M_2 = -A(t)M_1 + \frac{d}{dt}M_1 = \begin{bmatrix} -t \\ t^2 \\ t^2 - 1 \end{bmatrix}$$

The determinant of the matrix

$$\begin{bmatrix} M_0 & M_1 & M_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -t \\ 1 & 0 & t^2 \\ 1 & -t & t^2 - 1 \end{bmatrix}$$

is $t^2 + 1$, which is nonzero for all t.

Matrix Test

Consider now the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x(t+1) = Ax(t) + Bu(t), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \quad \text{(AB-LTI)}$$

Notes

- For continuous-time LTI systems $\mathcal{R}[t_0, t_1] = \mathcal{C}[t_0, t_1]$, and therefore one often talks about only controllability.
- A system that is not controllable is called *uncontrollable*.
- In discrete time, this holds for $t_1 - t_0 \geq n$, and nonsingular A.

Matrix Test

Earlier, we saw that

$$\mathtt{Im}\mathfrak{C} = \mathcal{R}[t_0, t_1] = \mathcal{C}[t_0, t_1].$$

Matrix Test

Earlier, we saw that

$$\mathtt{Im}\mathfrak{C} = \mathcal{R}[t_0, t_1] = \mathcal{C}[t_0, t_1].$$

Since $\mathfrak{C}$ has n rows, $\mathtt{Im}\mathfrak{C}$ is a subspace of $\mathbb{R}^n$, so its dimension can be at most n.

Matrix Test

Earlier, we saw that

$$\mathtt{Im}\mathfrak{C} = \mathcal{R}[t_0, t_1] = \mathcal{C}[t_0, t_1].$$

Since $\mathfrak{C}$ has n rows, $\mathtt{Im}\mathfrak{C}$ is a subspace of $\mathbb{R}^n$, so its dimension can be at most n.

For controllability, $\mathtt{Im}\mathfrak{C} = \mathbb{R}^n$, and therefore the dimension of $\mathtt{Im}\mathfrak{C}$ must be exactly n.

Matrix Test

Earlier, we saw that

$$\mathtt{Im}\mathfrak{C} = \mathcal{R}[t_0, t_1] = \mathcal{C}[t_0, t_1].$$

Since $\mathfrak{C}$ has n rows, $\mathtt{Im}\mathfrak{C}$ is a subspace of $\mathbb{R}^n$, so its dimension can be at most n.

For controllability, $\mathtt{Im}\mathfrak{C} = \mathbb{R}^n$, and therefore the dimension of $\mathtt{Im}\mathfrak{C}$ must be exactly n.

Theorem (Controllabilty matrix test)

The LTI system (AB-LTI) *is controllable if and only if*

$$\mathtt{rank}\mathfrak{C} = n$$

Matrix Test

Earlier, we saw that

$$\mathtt{Im}\mathfrak{C} = \mathcal{R}[t_0, t_1] = \mathcal{C}[t_0, t_1].$$

Since $\mathfrak{C}$ has n rows, $\mathtt{Im}\mathfrak{C}$ is a subspace of $\mathbb{R}^n$, so its dimension can be at most n.

For controllability, $\mathtt{Im}\mathfrak{C} = \mathbb{R}^n$, and therefore the dimension of $\mathtt{Im}\mathfrak{C}$ must be exactly n.

Theorem (Controllabilty matrix test)

The LTI system (AB-LTI) *is controllable if and only if*

$$\mathtt{rank}\mathfrak{C} = n$$

Notes

In discrete time, when A is singular, we simply have

$$\mathtt{Im}\mathfrak{C} = \mathcal{R}[t_0, t_1] \subset \mathcal{C}[t_0, t_1]$$

and

- $\mathtt{rank}\mathfrak{C} = n \implies \mathcal{R}[t_0, t_1] = \mathcal{C}[t_0, t_1] = \mathbb{R}^n$
- $\mathtt{rank}\mathfrak{C} < n \implies \mathtt{Im}\mathfrak{C} = \mathcal{R}[t_0, t_1] \subset \mathcal{C}[t_0, t_1] = \mathbb{R}^n$

Eigenvector Test

Eigenvector Test

Definition (A-invariant)

Given an $n \times n$ matrix A, a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$ is said to be A-*invariant* whenever for every vector $v \in \mathcal{V}$ we have $Av \in \mathcal{V}$.

Eigenvector Test

Definition (A-invariant)

Given an $n \times n$ matrix A, a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$ is said to be A-*invariant* whenever for every vector $v \in \mathcal{V}$ we have $Av \in \mathcal{V}$.

Properties: Given an $n \times n$ matrix A, a linear subspace $\mathcal{V} \subset \mathbb{R}^n$, the following statements are true.

Lemma (Property 1)

If one constructs an $n \times k$ matrix V whose columns form a basis for $\mathcal{V}$, there exists a $k \times k$ matrix Γ such that

$$AV = V\Gamma.$$

Eigenvector Test

Definition (A-invariant)

Given an $n \times n$ matrix A, a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$ is said to be A-*invariant* whenever for every vector $v \in \mathcal{V}$ we have $Av \in \mathcal{V}$.

Properties: Given an $n \times n$ matrix A, a linear subspace $\mathcal{V} \subset \mathbb{R}^n$, the following statements are true.

Lemma (Property 1)

If one constructs an $n \times k$ matrix V whose columns form a basis for $\mathcal{V}$, there exists a $k \times k$ matrix Γ such that

$$AV = V\Gamma.$$

Proof.

Since the ith column v_i of the matrix V belongs to $\mathcal{V}$ and $\mathcal{V}$ is A-invariant, $Av_i \in \mathcal{V}$.

Eigenvector Test

Definition (A-invariant)

Given an $n \times n$ matrix A, a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$ is said to be A-*invariant* whenever for every vector $v \in \mathcal{V}$ we have $Av \in \mathcal{V}$.

Properties: Given an $n \times n$ matrix A, a linear subspace $\mathcal{V} \subset \mathbb{R}^n$, the following statements are true.

Lemma (Property 1)

If one constructs an $n \times k$ matrix V whose columns form a basis for $\mathcal{V}$, there exists a $k \times k$ matrix Γ such that

$$AV = V\Gamma.$$

Proof.

Since the ith column v_i of the matrix V belongs to $\mathcal{V}$ and $\mathcal{V}$ is A-invariant, $Av_i \in \mathcal{V}$. This means that Av_i can be written as linear combination of columns of V; i.e., there exists a column vector γ_i such that

$$Av_i = V\gamma_i \qquad \forall i \in \{1, 2, \dots, k\}.$$

Eigenvector Test

Definition (A-invariant)

Given an $n \times n$ matrix A, a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$ is said to be A-*invariant* whenever for every vector $v \in \mathcal{V}$ we have $Av \in \mathcal{V}$.

Properties: Given an $n \times n$ matrix A, a linear subspace $\mathcal{V} \subset \mathbb{R}^n$, the following statements are true.

Lemma (Property 1)

If one constructs an $n \times k$ matrix V whose columns form a basis for $\mathcal{V}$, there exists a $k \times k$ matrix Γ such that

$$AV = V\Gamma.$$

Proof.

Since the ith column v_i of the matrix V belongs to $\mathcal{V}$ and $\mathcal{V}$ is A-invariant, $Av_i \in \mathcal{V}$. This means that Av_i can be written as linear combination of columns of V; i.e., there exists a column vector γ_i such that

$$Av_i = V\gamma_i \qquad \forall i \in \{1, 2, \ldots, k\}.$$

Putting all these equations together, we conclude that

$$\begin{bmatrix} Av_1 & Av_2 & \cdots & Av_k \end{bmatrix} = \begin{bmatrix} V\gamma_1 & V\gamma_2 & \cdots & V\gamma_k \end{bmatrix} \iff AV = V\Gamma$$

where all the γ_i are used as columns for Γ. □

Eigenvector Test

Lemma (Property 2)

$\mathcal{V}$ contains at least one eigenvector of A.

Eigenvector Test

Lemma (Property 2)

$\mathcal{V}$ contains at least one eigenvector of A.

Proof.

Let $\bar{v}$ is an eigenvector of the matrix Γ corresponding to the eigenvalue λ. Then

$$AV\bar{v} = V\Gamma\bar{v} = \lambda V\bar{v}$$

Eigenvector Test

Lemma (Property 2)

$\mathcal{V}$ contains at least one eigenvector of A.

Proof.

Let $\bar{v}$ is an eigenvector of the matrix Γ corresponding to the eigenvalue λ. Then

$$AV\bar{v} = V\Gamma\bar{v} = \lambda V\bar{v}$$

and therefore, $v := V\bar{v}$ is an eigenvector of the matrix A. Moreover, since v is a linear combination of the columns of V, it must belong to $\mathcal{V}$. $\square$

Eigenvector Test

Theorem (Eigenvector test for controllability)

The following two statements are equivalent.

1. *The LTI system (AB-LTI) is controllable.*
2. *There is no eigenvector of A^T in the kernel of B^T.*

Eigenvector Test

Theorem (Eigenvector test for controllability)

The following two statements are equivalent.

1. *The LTI system (AB-LTI) is controllable.*
2. *There is no eigenvector of A^T in the kernel of B^T.*

$1 \implies 2$ OR $\neg 1 \impliedby \neg 2.$

Eigenvector Test

Theorem (Eigenvector test for controllability)

The following two statements are equivalent.

1. *The LTI system (AB-LTI) is controllable.*
2. *There is no eigenvector of A^T in the kernel of B^T.*

$1 \implies 2$ OR $\neg 1 \impliedby \neg 2$.

Suppose there exists an eigenvalue $A^T x = \lambda x$ with $x \neq 0$, for which $B^T x = 0$.

Eigenvector Test

Theorem (Eigenvector test for controllability)

The following two statements are equivalent.

1. *The LTI system (AB-LTI) is controllable.*
2. *There is no eigenvector of A^T in the kernel of B^T.*

$1 \implies 2$ OR $\neg 1 \impliedby \neg 2$.

Suppose there exists an eigenvalue $A^T x = \lambda x$ with $x \neq 0$, for which $B^T x = 0$.Then

$$\mathfrak{C}^T x = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} x = \begin{bmatrix} B^T x \\ \lambda B^T x \\ \vdots \\ \lambda^{n-1} B^T x \end{bmatrix} = 0. \tag{7}$$

Eigenvector Test

Theorem (Eigenvector test for controllability)

The following two statements are equivalent.

1. *The LTI system (AB-LTI) is controllable.*
2. *There is no eigenvector of A^T in the kernel of B^T.*

$1 \implies 2$ OR $\neg 1 \impliedby \neg 2$.

Suppose there exists an eigenvalue $A^T x = \lambda x$ with $x \neq 0$, for which $B^T x = 0$.Then

$$\mathfrak{C}^T x = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} x = \begin{bmatrix} B^T x \\ \lambda B^T x \\ \vdots \\ \lambda^{n-1} B^T x \end{bmatrix} = 0. \tag{7}$$

This means that the null space of $\mathfrak{C}$ has at least one nonzero vector

$$\dim \mathtt{ker} \mathfrak{C}^T \geq 1$$

□

Eigenvector Test

Theorem (Eigenvector test for controllability)

The following two statements are equivalent.

1. *The LTI system (AB-LTI) is controllable.*
2. *There is no eigenvector of A^T in the kernel of B^T.*

$1 \implies 2 \quad$ OR $\quad \neg 1 \impliedby \neg 2.$

Suppose there exists an eigenvalue $A^T x = \lambda x$ with $x \neq 0$, for which $B^T x = 0$.Then

$$\mathfrak{C}^T x = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} x = \begin{bmatrix} B^T x \\ \lambda B^T x \\ \vdots \\ \lambda^{n-1} B^T x \end{bmatrix} = 0. \tag{7}$$

This means that the null space of $\mathfrak{C}$ has at least one nonzero vector, and therefore, from the fundamental theorem of linear equations, we conclude that

$$\dim \texttt{ker} \mathfrak{C}^T \geq 1 \implies \qquad \texttt{rank} \mathfrak{C}^T = n - \dim \texttt{ker} \mathfrak{C}^T < n$$

$\square$

Eigenvector Test

Theorem (Eigenvector test for controllability)

The following two statements are equivalent.

1. *The LTI system (AB-LTI) is controllable.*
2. *There is no eigenvector of A^T in the kernel of B^T.*

$1 \implies 2$ OR $\neg 1 \impliedby \neg 2.$

Suppose there exists an eigenvalue $A^T x = \lambda x$ with $x \neq 0$, for which $B^T x = 0$.Then

$$\mathfrak{C}^T x = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} x = \begin{bmatrix} B^T x \\ \lambda B^T x \\ \vdots \\ \lambda^{n-1} B^T x \end{bmatrix} = 0. \tag{7}$$

This means that the null space of $\mathfrak{C}$ has at least one nonzero vector, and therefore, from the fundamental theorem of linear equations, we conclude that

$$\dim \mathtt{ker} \mathfrak{C}^T \geq 1 \implies \mathtt{rank} \mathfrak{C} = \mathtt{rank} \mathfrak{C}^T = n - \dim \mathtt{ker} \mathfrak{C}^T < n$$

which contradicts the controllability of (AB-LTI). □

Proof: $2 \implies 1$ OR $\neg 2 \impliedby \neg 1$

Proof: $2 \implies 1$ OR $\neg 2 \impliedby \neg 1$

To prove this part, firstly we shall show that the subspace $\mathtt{ker}\mathfrak{C}^T$ is A^T-invariant.

Subsequently, we use the property 2 to conclude that if (AB-LTI) is *not* controllable then there exists an eigenvector of A^T in the kernel of B^T.

Property 2

$\mathcal{V}$ contains at least one eigenvector of A.

Proof: $2 \implies 1$ OR $\neg 2 \impliedby \neg 1$

$\mathtt{ker}\mathfrak{C}^T$ is A^T-invariant

Since (AB-LTI) is not controllable, we have

$$\mathtt{rank}\mathfrak{C} = \mathtt{rank}\mathfrak{C}^T < n \implies \dim \mathtt{ker}\mathfrak{C}^T = n - \mathtt{rank}\mathfrak{C}^T \geq 1$$

Proof: $2 \implies 1$ OR $\neg 2 \impliedby \neg 1$

$\mathtt{ker}\mathfrak{C}^T$ is A^T-invariant

Since (AB-LTI) is not controllable, we have

$$\mathtt{rank}\mathfrak{C} = \mathtt{rank}\mathfrak{C}^T < n \implies \dim \mathtt{ker}\mathfrak{C}^T = n - \mathtt{rank}\mathfrak{C}^T \geq 1$$

Indeed, if $x \in \mathtt{ker}\mathfrak{C}^T$, then (7) holds, and therefore

$$x \in \mathtt{ker}\mathfrak{C}^T \implies \mathfrak{C}^T A^T x = \begin{bmatrix} B^T A^T \\ B^T (A^T)^2 \\ \vdots \\ B^T (A^T)^n \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ B^T (A^T)^n x \end{bmatrix}$$

Proof: $2 \implies 1$ OR $\neg 2 \impliedby \neg 1$

$\mathtt{ker}\mathfrak{C}^T$ is A^T-invariant

Since (AB-LTI) is not controllable, we have

$$\mathtt{rank}\mathfrak{C} = \mathtt{rank}\mathfrak{C}^T < n \implies \dim \mathtt{ker}\mathfrak{C}^T = n - \mathtt{rank}\mathfrak{C}^T \geq 1$$

Indeed, if $x \in \mathtt{ker}\mathfrak{C}^T$, then (7) holds, and therefore

$$x \in \mathtt{ker}\mathfrak{C}^T \implies \mathfrak{C}^T A^T x = \begin{bmatrix} B^T A^T \\ B^T(A^T)^2 \\ \vdots \\ B^T(A^T)^n \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ B^T(A^T)^n x \end{bmatrix}$$

But by the Caley-Hamilton theorem, $(A^T)^n$ can be written as a linear combination of the lower powers of A^T, and therefore $B^T(A^T)^n x$ can be written as a linear combination of the terms

$$B^T x, B^T A^T x, \ldots, B^T (A^T)^{n-1} x,$$

Proof: $2 \implies 1$ OR $\neg 2 \impliedby \neg 1$

$\mathtt{ker}\mathfrak{C}^T$ is A^T-invariant

Since (AB-LTI) is not controllable, we have

$$\mathtt{rank}\mathfrak{C} = \mathtt{rank}\mathfrak{C}^T < n \implies \dim \mathtt{ker}\mathfrak{C}^T = n - \mathtt{rank}\mathfrak{C}^T \geq 1$$

Indeed, if $x \in \mathtt{ker}\mathfrak{C}^T$, then (7) holds, and therefore

$$x \in \mathtt{ker}\mathfrak{C}^T \implies \mathfrak{C}^T A^T x = \begin{bmatrix} B^T A^T \\ B^T (A^T)^2 \\ \vdots \\ B^T (A^T)^n \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ B^T (A^T)^n x \end{bmatrix}$$

But by the Caley-Hamilton theorem, $(A^T)^n$ can be written as a linear combination of the lower powers of A^T, and therefore $B^T (A^T)^n x$ can be written as a linear combination of the terms

$$B^T x, B^T A^T x, \ldots, B^T (A^T)^{n-1} x,$$

which are all zero because of (7). We therefore conclude that

$$x \in \mathtt{ker}\mathfrak{C}^T \implies \mathfrak{C}^T A^T x = 0 \implies A^T x \in \mathtt{ker}\mathfrak{C}^T.$$

Proof: $2 \implies 1$ OR $\neg 2 \impliedby \neg 1$

Property 2

$\mathcal{V}$ contains at least one eigenvector of A.

Use Property 2

From Property 2, we then conclude that $\mathtt{ker}\mathfrak{C}^T$ must contain at least one eigenvector x of A^T. But since $\mathfrak{C}^T x = 0$, we necessarily have $B^T x = 0$.

$$\mathfrak{C}^T x = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} x = \begin{bmatrix} B^T x \\ \lambda B^T x \\ \vdots \\ \lambda^{n-1} B^T x \end{bmatrix} = 0.$$

Eigenvector Test (Elegant restatement)

Theorem (Popov-Belevitch-Hautus (PBH) test for controllability)

The LTI system (AB-LTI) is controllable if and only if

$$\texttt{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \qquad \forall \lambda \in \mathbb{C} \tag{8}$$

Eigenvector Test (Elegant restatement)

Theorem (Popov-Belevitch-Hautus (PBH) test for controllability)

The LTI system (AB-LTI) is controllable if and only if

$$\mathbf{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \qquad \forall \lambda \in \mathbb{C} \tag{8}$$

Proof: Equivalence of rank and eigenvector condition.

Eigenvector Test (Elegant restatement)

Theorem (Popov-Belevitch-Hautus (PBH) test for controllability)

The LTI system (AB-LTI) is controllable if and only if

$$\mathtt{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \qquad \forall \lambda \in \mathbb{C} \tag{8}$$

Proof: Equivalence of rank and eigenvector condition.

From the fundamental theorems of linear equations, we conclude that

$$\dim \mathtt{ker} \begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} = n - \mathtt{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix}, \qquad \forall \lambda \in \mathbb{C}$$

Eigenvector Test (Elegant *re*statement)

Theorem (Popov-Belevitch-Hautus (PBH) test for controllability)

The LTI system (AB-LTI) is controllable if and only if

$$\texttt{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n \qquad \forall \lambda \in \mathbb{C} \tag{8}$$

Proof: Equivalence of rank and eigenvector condition.

From the fundamental theorems of linear equations, we conclude that

$$\dim \texttt{ker}\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix} = n - \texttt{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix}, \qquad \forall \lambda \in \mathbb{C}$$

and therefore the above condition can also be written as

$$\dim \texttt{ker}\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix} = 0 \qquad \forall \lambda \in \mathbb{C},$$

Eigenvector Test (Elegant *re*statement)

Theorem (Popov-Belevitch-Hautus (PBH) test for controllability)

The LTI system (AB-LTI) is controllable if and only if

$$\texttt{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n \qquad \forall \lambda \in \mathbb{C} \tag{8}$$

Proof: Equivalence of rank and eigenvector condition.

From the fundamental theorems of linear equations, we conclude that

$$\dim \texttt{ker}\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix} = n - \texttt{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix}, \qquad \forall \lambda \in \mathbb{C}$$

and therefore the above condition can also be written as

$$\dim \texttt{ker}\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix} = 0 \qquad \forall \lambda \in \mathbb{C},$$

which means that the kernel of $\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix}$ can contain only the zero vector.

Eigenvector Test (Elegant restatement)

Theorem (Popov-Belevitch-Hautus (PBH) test for controllability)

The LTI system (AB-LTI) is controllable if and only if

$$\texttt{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n \qquad \forall \lambda \in \mathbb{C} \tag{8}$$

Proof: Equivalence of rank and eigenvector condition.

From the fundamental theorems of linear equations, we conclude that

$$\dim \texttt{ker}\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix} = n - \texttt{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix}, \qquad \forall \lambda \in \mathbb{C}$$

and therefore the above condition can also be written as

$$\dim \texttt{ker}\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix} = 0 \qquad \forall \lambda \in \mathbb{C},$$

which means that the kernel of $\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix}$ can contain only the zero vector. This means the said condition is also equivalent to

$$\texttt{ker}\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix} = \left\{x \in \mathbb{R}^n : A^T x = \lambda x, B^T x = 0\right\}$$

Eigenvector Test (Elegant *re*statement)

Theorem (Popov-Belevitch-Hautus (PBH) test for controllability)

The LTI system (AB-LTI) is controllable if and only if

$$\texttt{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \qquad \forall \lambda \in \mathbb{C} \tag{8}$$

Proof: Equivalence of rank and eigenvector condition.

From the fundamental theorems of linear equations, we conclude that

$$\dim \texttt{ker}\begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} = n - \texttt{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix}, \qquad \forall \lambda \in \mathbb{C}$$

and therefore the above condition can also be written as

$$\dim \texttt{ker}\begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} = 0 \qquad \forall \lambda \in \mathbb{C},$$

which means that the kernel of $\begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix}$ can contain only the zero vector. This means the said condition is also equivalent to

$$\texttt{ker}\begin{bmatrix} A^T - \lambda I \\ B^T \end{bmatrix} = \left\{ x \in \mathbb{R}^n : A^T x = \lambda x, B^T x = 0 \right\} = \{0\}, \qquad \lambda \in \mathbb{C}$$

Eigenvector Test (Elegant *re*statement)

Theorem (Popov-Belevitch-Hautus (PBH) test for controllability)

The LTI system (AB-LTI) is controllable if and only if

$$\texttt{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n \qquad \forall\lambda\in\mathbb{C} \tag{8}$$

Proof: Equivalence of rank and eigenvector condition.

From the fundamental theorems of linear equations, we conclude that

$$\dim\texttt{ker}\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix} = n - \texttt{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix}, \qquad \forall\lambda\in\mathbb{C}$$

and therefore the above condition can also be written as

$$\dim\texttt{ker}\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix} = 0 \qquad \forall\lambda\in\mathbb{C},$$

which means that the kernel of $\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix}$ can contain only the zero vector. This means the said condition is also equivalent to

$$\texttt{ker}\begin{bmatrix} A^T-\lambda I \\ B^T \end{bmatrix} = \left\{x\in\mathbb{R}^n : A^T x = \lambda x, B^T x = 0\right\} = \{0\}, \qquad \lambda\in\mathbb{C}$$

which is precisely equivalent to the statement that there can be no eigenvector of A^T in the kernel of B^T. $\square$

Lyapunov test for controllability

Consider now the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x(t+1) = Ax(t) + Bu(t), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

Theorem (Lyapunov test for controllability)

Assume that A is a stability matrix. The LTI system (AB-LTI) is controllable if and only if there is a unique positive-definite solution W to the following Lyapunov equation

$$AW + WA^T = -BB^T \quad / \quad AWA^T - W = -BB^T \quad (\mathfrak{C}\text{-Lyapunov Eq.})$$

Moreover, the unique solution to ($\mathfrak{C}$-Lyapunov Eq.) *is equal to*

$$W = \int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau = \lim_{t_1 - t_0 \to \infty} W_R(t_0, t_1)$$

$$/ \quad W = \sum_{\tau=0}^{\infty} A^\tau BB^T (A^T)^\tau = \lim_{t_1 - t_0 \to \infty} W_R(t_0, t_1) \quad (9)$$

$2 \implies 1$ (C-LTI systems)

We use the eigenvector test to prove this implication.

$2 \implies 1$ (C-LTI systems)

We use the eigenvector test to prove this implication.

Assume that ($\mathfrak{C}$-Lyapunov Eq.) holds, and $x \neq 0$ be an eigenvector of A^T associated with the eigenvalue λ, i.e., $A^T x = \lambda x$.

$2 \implies 1$ (C-LTI systems)

We use the eigenvector test to prove this implication.

Assume that ($\mathfrak{C}$-Lyapunov Eq.) holds, and $x \neq 0$ be an eigenvector of A^T associated with the eigenvalue λ, i.e., $A^T x = \lambda x$. Then

$$x^*(AW + WA^T)x = -x^* BB^T x = -\|B^T x\|^2, \tag{10}$$

where $(\cdot)^*$ denotes the complex conjugate transpose.

$2 \implies 1$ (C-LTI systems)

We use the eigenvector test to prove this implication.

Assume that ($\mathfrak{C}$-Lyapunov Eq.) holds, and $x \neq 0$ be an eigenvector of A^T associated with the eigenvalue λ, i.e., $A^T x = \lambda x$. Then

$$x^*(AW + WA^T)x = -x^* BB^T x = -\|B^T x\|^2, \quad (10)$$

where $(\cdot)^*$ denotes the complex conjugate transpose. But the left-hand side of this equation is equal to

$$(A^T x^{*T})^T W x + x^* W A^T x = \lambda^* x^* W x + \lambda x^* W x = 2\mathtt{Re}[\lambda] x^* W x. \quad (11)$$

$2 \implies 1$ (C-LTI systems)

We use the eigenvector test to prove this implication.

Assume that ($\mathfrak{C}$-Lyapunov Eq.) holds, and $x \neq 0$ be an eigenvector of A^T associated with the eigenvalue λ, i.e., $A^T x = \lambda x$. Then

$$x^*(AW + WA^T)x = -x^* BB^T x = -\|B^T x\|^2, \tag{10}$$

where $(\cdot)^*$ denotes the complex conjugate transpose. But the left-hand side of this equation is equal to

$$(A^T x^{*T})^T W x + x^* W A^T x = \lambda^* x^* W x + \lambda x^* W x = 2\mathtt{Re}[\lambda] x^* W x. \tag{11}$$

Since W is positive-definite, this expression must be strictly negative (note that $\mathtt{Re}[\lambda] < 0$ because A is a stability matrix), and therefore $B^T x \neq 0$.

$2 \implies 1$ (C-LTI systems)

We use the eigenvector test to prove this implication.

Assume that ($\mathfrak{C}$-Lyapunov Eq.) holds, and $x \neq 0$ be an eigenvector of A^T associated with the eigenvalue λ, i.e., $A^T x = \lambda x$. Then

$$x^*(AW + WA^T)x = -x^* BB^T x = -\|B^T x\|^2, \tag{10}$$

where $(\cdot)^*$ denotes the complex conjugate transpose. But the left-hand side of this equation is equal to

$$(A^T x^{*T})^T W x + x^* W A^T x = \lambda^* x^* W x + \lambda x^* W x = 2\texttt{Re}[\lambda] x^* W x. \tag{11}$$

Since W is positive-definite, this expression must be strictly negative (note that $\texttt{Re}[\lambda] < 0$ because A is a stability matrix), and therefore $B^T x \neq 0$.

We conclude that every eigenvalue of A^T is not in the kernel of B^T, which implies controllability by the eigenvector test.

$1 \implies 2$ (C-LTI systems)

Assume that (AB-LTI) is controllable. The previously studied Lyapunov equation can be written as

$$\bar{A}^T W + W\bar{A} = -Q, \qquad \bar{A} := A^T, \qquad Q := BB^T.$$

Since A is a stability matrix, $\bar{A} := A^T$ is also a stability matrix, and therefore we can reuse the proof of the Lyapunov stability theorem to conclude that (9) is a unique solution to ($\mathfrak{C}$-Lyapunov Eq.).

$1 \implies 2$ (C-LTI systems)

Assume that (AB-LTI) is controllable. The previously studied Lyapunov equation can be written as

$$\bar{A}^T W + W\bar{A} = -Q, \qquad \bar{A} := A^T, \qquad Q := BB^T.$$

Since A is a stability matrix, $\bar{A} := A^T$ is also a stability matrix, and therefore we can reuse the proof of the Lyapunov stability theorem to conclude that (9) is a unique solution to ($\mathfrak{C}$-Lyapunov Eq.).

The only issue that needs special attention is that in Lyapunov stability theorem we used the fact that $Q = BB^T$ was positive-definite to show that the solution W was also positive-definite. Here, $Q = BB^T$ may not be positive-definite,

$1 \implies 2$ (C-LTI systems)

Assume that (AB-LTI) is controllable. The previously studied Lyapunov equation can be written as

$$\bar{A}^T W + W\bar{A} = -Q, \qquad \bar{A} := A^T, \qquad Q := BB^T.$$

Since A is a stability matrix, $\bar{A} := A^T$ is also a stability matrix, and therefore we can reuse the proof of the Lyapunov stability theorem to conclude that (9) is a unique solution to ($\mathfrak{C}$-Lyapunov Eq.).

The only issue that needs special attention is that in Lyapunov stability theorem we used the fact that $Q = BB^T$ was positive-definite to show that the solution W was also positive-definite. Here, $Q = BB^T$ may not be positive-definite, but it turns out that controllability of the pair (A,B) *suffices* to establish that W is positive-definite, even if Q is not.

$1 \implies 2$ (C-LTI systems)

Assume that (AB-LTI) is controllable. The previously studied Lyapunov equation can be written as

$$\bar{A}^T W + W\bar{A} = -Q, \qquad \bar{A} := A^T, \qquad Q := BB^T.$$

Since A is a stability matrix, $\bar{A} := A^T$ is also a stability matrix, and therefore we can reuse the proof of the Lyapunov stability theorem to conclude that (9) is a unique solution to ($\mathfrak{C}$-Lyapunov Eq.).

The only issue that needs special attention is that in Lyapunov stability theorem we used the fact that $Q = BB^T$ was positive-definite to show that the solution W was also positive-definite. Here, $Q = BB^T$ may not be positive-definite, but it turns out that controllability of the pair (A,B) *suffices* to establish that W is positive-definite, even if Q is not. Indeed, given any arbitrary vector $x \neq 0$,

$$x^T W x = x^T \left(\int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau \right) x$$

$1 \implies 2$ (C-LTI systems)

Assume that (AB-LTI) is controllable. The previously studied Lyapunov equation can be written as

$$\bar{A}^T W + W \bar{A} = -Q, \qquad \bar{A} := A^T, \qquad Q := BB^T.$$

Since A is a stability matrix, $\bar{A} := A^T$ is also a stability matrix, and therefore we can reuse the proof of the Lyapunov stability theorem to conclude that (9) is a unique solution to ($\mathfrak{C}$-Lyapunov Eq.).

The only issue that needs special attention is that in Lyapunov stability theorem we used the fact that $Q = BB^T$ was positive-definite to show that the solution W was also positive-definite. Here, $Q = BB^T$ may not be positive-definite, but it turns out that controllability of the pair (A,B) *suffices* to establish that W is positive-definite, even if Q is not. Indeed, given any arbitrary vector $x \neq 0$,

$$x^T W x = x^T \left(\int_0^\infty e^{A\tau} BB^T e^{A^T \tau} d\tau \right) x \geq x^T \left(\int_0^1 e^{A\tau} BB^T e^{A^T \tau} d\tau \right) x$$

$1 \implies 2$ (C-LTI systems)

Assume that (AB-LTI) is controllable. The previously studied Lyapunov equation can be written as

$$\bar{A}^T W + W\bar{A} = -Q, \qquad \bar{A} := A^T, \qquad Q := BB^T.$$

Since A is a stability matrix, $\bar{A} := A^T$ is also a stability matrix, and therefore we can reuse the proof of the Lyapunov stability theorem to conclude that (9) is a unique solution to ($\mathfrak{C}$-Lyapunov Eq.).

The only issue that needs special attention is that in Lyapunov stability theorem we used the fact that $Q = BB^T$ was positive-definite to show that the solution W was also positive-definite. Here, $Q = BB^T$ may not be positive-definite, but it turns out that controllability of the pair (A,B) *suffices* to establish that W is positive-definite, even if Q is not. Indeed, given any arbitrary vector $x \neq 0$,

$$\begin{aligned} x^T W x &= x^T \left(\int_0^\infty e^{A\tau} BB^T e^{A^T \tau} d\tau \right) x \geq x^T \left(\int_0^1 e^{A\tau} BB^T e^{A^T \tau} d\tau \right) x \\ &= x^T W_R(0,1) x \end{aligned}$$

$1 \implies 2$ (C-LTI systems)

Assume that (AB-LTI) is controllable. The previously studied Lyapunov equation can be written as

$$\bar{A}^T W + W\bar{A} = -Q, \qquad \bar{A} := A^T, \qquad Q := BB^T.$$

Since A is a stability matrix, $\bar{A} := A^T$ is also a stability matrix, and therefore we can reuse the proof of the Lyapunov stability theorem to conclude that (9) is a unique solution to ($\mathfrak{C}$-Lyapunov Eq.).

The only issue that needs special attention is that in Lyapunov stability theorem we used the fact that $Q = BB^T$ was positive-definite to show that the solution W was also positive-definite. Here, $Q = BB^T$ may not be positive-definite, but it turns out that controllability of the pair (A,B) *suffices* to establish that W is positive-definite, even if Q is not. Indeed, given any arbitrary vector $x \neq 0$,

$$\begin{aligned} x^T W x = x^T \left(\int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau \right) x \geq x^T \left(\int_0^1 e^{A\tau} BB^T e^{A^T\tau} d\tau \right) x \\ = x^T W_R(0,1) x > 0, \end{aligned}$$

because $W_R(0,1) > 0$, due to controllability.

Lyapunov test for controllability

The result of the above theorem allow us to add a six equivalent conditions to the Lyapunov stability theorem.

Theorem (Lyapunov stability, *updated*)

The following six conditions are equivalent.

1. *The system (H-CLTI) is asymptotically stable.*
2. *The system (H-CLTI) is exponentially stable.*
3. *All the eigenvalues of A have strictly negative real parts.*
4. *For every symmetric positive-definite matrix Q, there exists a unique solution P to the Lyapunov equation*

$$A^T P + PA = -Q \tag{Lyapunov Eq.}$$

Moreover, P is symmetric, positive-definite, and equal to $P := \int_0^\infty e^{A^T t} Q e^{At} dt$.

5. *There exists symmetric, positive-definite matrix P for which the following Lyapunov matrix inequality holds*

$$A^T P + PA < 0 \tag{LMI}$$

6. *For every matrix B for which the pair (A, B) is controllable, there exists a unique solution P to the Lyapunov equation*

$$AP + PA^T = -BB^T \tag{$\mathfrak{C}$-Lyapunov Eq.}$$

Moreover, P is symmetric, positive-definite, and equal to $P = \int_0^\infty e^{A\tau} BB^T e^{A^T \tau} d\tau$.

Summary of Controllability Tests

The following statements are equivalent.

1. The n-dimensional pair (A, B) is controllable.
2. The $n \times n$ Gramian

$$W_c(t) = \int_0^t e^{At} BB' e^{A'\tau} d\tau = \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)} d\tau$$

(Gramian - Matrix Test)

is non-singular for any $t > 0$.

3. The $n \times kn$ controllability matrix

$$\mathfrak{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

(Matrix Test)

has rank n (full row rank).

4. The matrix

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix}$$

(PBH Test)

has full row rank at every eigenvalue λ of A.

5. If, in addition, all eigenvalues of A have negative real parts, then the unique solution of

$$AW_c + W_c A = -BB'$$

($\mathfrak{C}$-Lyapunov Eq.)

is positive definite and can be expressed as $W_c = \int_0^\infty e^{A\tau} BB' exp^{A'\tau} d\tau$.

Outline of this section

1. Invariance with respect to similarity transformations
2. Controllable decomposition
3. Block diagram interpretation
4. Transfer function

Invariance with Respect to Similarity Transformations

Consider the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

and a similarity transformation $\bar{x} = T^{-1}x$, leading to

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad \bar{A} = T^{-1}AT \quad \bar{B} = T^{-1}B \tag{$\bar{A}\bar{B}$-LTI}$$

Invariance with Respect to Similarity Transformations

Consider the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

and a similarity transformation $\bar{x} = T^{-1}x$, leading to

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad \bar{A} = T^{-1}AT \quad \bar{B} = T^{-1}B \tag{$\bar{A}\bar{B}$-LTI}$$

The controllability matrices $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ of the systems (AB-LTI) and ($\bar{A}\bar{B}$-LTI), respectively, are related by

$$\bar{\mathfrak{C}} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \end{bmatrix}$$

Invariance with Respect to Similarity Transformations

Consider the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

and a similarity transformation $\bar{x} = T^{-1}x$, leading to

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad \bar{A} = T^{-1}AT \quad \bar{B} = T^{-1}B \qquad (\bar{A}\bar{B}\text{-LTI})$$

The controllability matrices $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ of the systems (AB-LTI) and ($\bar{A}\bar{B}$-LTI), respectively, are related by

$$\bar{\mathfrak{C}} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \end{bmatrix}$$
$$= \begin{bmatrix} T^{-1}B & T^{-1}AB & \cdots & T^{-1}A^{n-1}B \end{bmatrix}$$

Invariance with Respect to Similarity Transformations

Consider the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

and a similarity transformation $\bar{x} = T^{-1}x$, leading to

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad \bar{A} = T^{-1}AT \quad \bar{B} = T^{-1}B \tag{$\bar{A}\bar{B}$-LTI}$$

The controllability matrices $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ of the systems (AB-LTI) and ($\bar{A}\bar{B}$-LTI), respectively, are related by

$$\begin{aligned}\bar{\mathfrak{C}} = \begin{bmatrix}\bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B}\end{bmatrix} \\ = \begin{bmatrix}T^{-1}B & T^{-1}AB & \cdots & T^{-1}A^{n-1}B\end{bmatrix} = T^{-1}\mathfrak{C}\end{aligned}$$

Invariance with Respect to Similarity Transformations

Consider the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

and a similarity transformation $\bar{x} = T^{-1}x$, leading to

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad \bar{A} = T^{-1}AT \quad \bar{B} = T^{-1}B \qquad (\bar{A}\bar{B}\text{-LTI})$$

The controllability matrices $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ of the systems (AB-LTI) and ($\bar{A}\bar{B}$-LTI), respectively, are related by

$$\begin{aligned}\bar{\mathfrak{C}} &= \begin{bmatrix}\bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B}\end{bmatrix} \\ &= \begin{bmatrix}T^{-1}B & T^{-1}AB & \cdots & T^{-1}A^{n-1}B\end{bmatrix} = T^{-1}\mathfrak{C}\end{aligned}$$

Therefore,

$$\texttt{rank}\bar{\mathfrak{C}} = \texttt{rank}T^{-1}\mathfrak{C} = \texttt{rank}\mathfrak{C}$$

because T^{-1} is nonsingular.

Invariance with Respect to Similarity Transformations

Consider the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

and a similarity transformation $\bar{x} = T^{-1}x$, leading to

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad \bar{A} = T^{-1}AT \quad \bar{B} = T^{-1}B \qquad (\bar{A}\bar{B}\text{-LTI})$$

The controllability matrices $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ of the systems (AB-LTI) and ($\bar{A}\bar{B}$-LTI), respectively, are related by

$$\begin{aligned}\bar{\mathfrak{C}} = \begin{bmatrix}\bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B}\end{bmatrix} \\ = \begin{bmatrix}T^{-1}B & T^{-1}AB & \cdots & T^{-1}A^{n-1}B\end{bmatrix} = T^{-1}\mathfrak{C}\end{aligned}$$

Therefore,

$$\texttt{rank}\bar{\mathfrak{C}} = \texttt{rank}T^{-1}\mathfrak{C} = \texttt{rank}\mathfrak{C}$$

because T^{-1} is nonsingular. Since the controllability of a system is determined by the rank of its controllability matrix, we conclude that controllability is *preserved* through similarity transformations.

Invariance with Respect to Similarity Transformations

Consider the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k$$

and a similarity transformation $\bar{x} = T^{-1}x$, leading to

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad \bar{A} = T^{-1}AT \quad \bar{B} = T^{-1}B \qquad (\bar{A}\bar{B}\text{-LTI})$$

The controllability matrices $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ of the systems (AB-LTI) and ($\bar{A}\bar{B}$-LTI), respectively, are related by

$$\bar{\mathfrak{C}} = \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \cdots & \bar{A}^{n-1}\bar{B} \end{bmatrix}$$
$$= \begin{bmatrix} T^{-1}B & T^{-1}AB & \cdots & T^{-1}A^{n-1}B \end{bmatrix} = T^{-1}\mathfrak{C}$$

Therefore,

$$\texttt{rank}\bar{\mathfrak{C}} = \texttt{rank}T^{-1}\mathfrak{C} = \texttt{rank}\mathfrak{C}$$

because T^{-1} is nonsingular. Since the controllability of a system is determined by the rank of its controllability matrix, we conclude that controllability is *preserved* through similarity transformations.

Theorem (Invariance with respect to Similarity transformation)

The pair (A, B) is controllable if and only if $(\bar{A}, \bar{B}) = (T^{-1}AT, T^{-1}B)$ is controllable.

Controllable Decomposition

Consider again the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \qquad \text{(AB-LTI)}$$

Note

The controllable subspace $\mathcal{C}$ of the system (AB-LTI) is A-invariant and contains $\mathtt{Im}B$.

Controllable Decomposition

Consider again the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \qquad \text{(AB-LTI)}$$

Note

The controllable subspace $\mathcal{C}$ of the system (AB-LTI) is A-invariant and contains $\mathtt{Im}B$.

Because of A-invariance, by constructing an $n \times \bar{n}$ matrix V[2] whose columns form a basis for $\mathcal{C}$, there exists an $\bar{n} \times \bar{n}$ matrix A_c such that

$$AV = VA_c$$

[2]The number of columns of V is $\bar{n}$, and therefore $\bar{n}$ is also the dimension of the controllable subspace.

Controllable Decomposition

Consider again the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \qquad \text{(AB-LTI)}$$

Note

The controllable subspace $\mathcal{C}$ of the system (AB-LTI) is A-invariant and contains $\mathtt{Im}B$.

Because of A-invariance, by constructing an $n \times \bar{n}$ matrix V[2] whose columns form a basis for $\mathcal{C}$, there exists an $\bar{n} \times \bar{n}$ matrix A_c such that

$$AV = VA_c$$

Moreover, since $\mathtt{Im}B \subset \mathcal{C}$, the columns of B can be written as a linear combination of the columns of V,

[2] The number of columns of V is $\bar{n}$, and therefore $\bar{n}$ is also the dimension of the controllable subspace.

Controllable Decomposition

Consider again the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \qquad \text{(AB-LTI)}$$

Note

The controllable subspace $\mathcal{C}$ of the system (AB-LTI) is A-invariant and contains $\mathtt{Im}B$.

Because of A-invariance, by constructing an $n \times \bar{n}$ matrix V[2] whose columns form a basis for $\mathcal{C}$, there exists an $\bar{n} \times \bar{n}$ matrix A_c such that

$$AV = VA_c$$

Moreover, since $\mathtt{Im}B \subset \mathcal{C}$, the columns of B can be written as a linear combination of the columns of V, and therefore there exists an $\bar{n} \times k$ matrix B_c such that

$$B = VB_c.$$

[2] The number of columns of V is $\bar{n}$, and therefore $\bar{n}$ is also the dimension of the controllable subspace.

Controllable Decomposition

Consider again the LTI systems

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{AB-LTI}$$

Note

The controllable subspace $\mathcal{C}$ of the system (AB-LTI) is A-invariant and contains $\mathtt{Im}B$.

Because of A-invariance, by constructing an $n \times \bar{n}$ matrix V[2] whose columns form a basis for $\mathcal{C}$, there exists an $\bar{n} \times \bar{n}$ matrix A_c such that

$$AV = VA_c$$

Moreover, since $\mathtt{Im}B \subset \mathcal{C}$, the columns of B can be written as a linear combination of the columns of V, and therefore there exists an $\bar{n} \times k$ matrix B_c such that

$$B = VB_c.$$

When the system (AB-LTI) is controllable, $\bar{n} = \dim \mathcal{C} = n$, and the matrix V is square and nonsingular.

[2] The number of columns of V is $\bar{n}$, and therefore $\bar{n}$ is also the dimension of the controllable subspace.

Controllable Decomposition

Otherwise, let U be an $n \times (n - \bar{n})$ matrix whose columns are linearly independent of each other and also linearly independent of the columns of V.

Controllable Decomposition

Otherwise, let U be an $n \times (n - \bar{n})$ matrix whose columns are linearly independent of each other and also linearly independent of the columns of V. Suppose that we define a nonsingular matrix T by combining V and U side by side:

$$T := \begin{bmatrix} V_{n\times\bar{n}} & U_{n\times(n-\bar{n})} \end{bmatrix}_{n\times n}$$

Controllable Decomposition

Otherwise, let U be an $n \times (n - \bar{n})$ matrix whose columns are linearly independent of each other and also linearly independent of the columns of V. Suppose that we define a nonsingular matrix T by combining V and U side by side:

$$T := \begin{bmatrix} V_{n\times\bar{n}} & U_{n\times(n-\bar{n})} \end{bmatrix}_{n\times n}$$

We then conclude that

$$AT = A\begin{bmatrix} V & U \end{bmatrix} = \begin{bmatrix} AV & AU \end{bmatrix} = \begin{bmatrix} VA_c & TT^{-1}AU \end{bmatrix} = \begin{bmatrix} T\begin{bmatrix} A_c \\ 0 \end{bmatrix} & TT^{-1}AU \end{bmatrix}$$

Controllable Decomposition

Otherwise, let U be an $n \times (n - \bar{n})$ matrix whose columns are linearly independent of each other and also linearly independent of the columns of V. Suppose that we define a nonsingular matrix T by combining V and U side by side:

$$T := \begin{bmatrix} V_{n\times\bar{n}} & U_{n\times(n-\bar{n})} \end{bmatrix}_{n\times n}$$

We then conclude that

$$AT = A\begin{bmatrix} V & U \end{bmatrix} = \begin{bmatrix} AV & AU \end{bmatrix} = \begin{bmatrix} VA_c & TT^{-1}AU \end{bmatrix} = \begin{bmatrix} T\begin{bmatrix} A_c \\ 0 \end{bmatrix} & TT^{-1}AU \end{bmatrix}$$

By partitioning the $n \times (n - \bar{n})$ matrix $T^{-1}AU$ as

$$T^{-1}AU = \begin{bmatrix} A_{12} \\ A_u \end{bmatrix}$$

Controllable Decomposition

Otherwise, let U be an $n \times (n - \bar{n})$ matrix whose columns are linearly independent of each other and also linearly independent of the columns of V. Suppose that we define a nonsingular matrix T by combining V and U side by side:

$$T := \begin{bmatrix} V_{n\times\bar{n}} & U_{n\times(n-\bar{n})} \end{bmatrix}_{n\times n}$$

We then conclude that

$$AT = A\begin{bmatrix} V & U \end{bmatrix} = \begin{bmatrix} AV & AU \end{bmatrix} = \begin{bmatrix} VA_c & TT^{-1}AU \end{bmatrix} = \begin{bmatrix} T\begin{bmatrix} A_c \\ 0 \end{bmatrix} & TT^{-1}AU \end{bmatrix}$$

By partitioning the $n \times (n - \bar{n})$ matrix $T^{-1}AU$ as

$$T^{-1}AU = \begin{bmatrix} A_{12} \\ A_u \end{bmatrix}$$

we further obtain

$$AT = T\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}, \qquad B = VB_c = T\begin{bmatrix} B_c \\ 0 \end{bmatrix},$$

Controllable Decomposition

Otherwise, let U be an $n \times (n - \bar{n})$ matrix whose columns are linearly independent of each other and also linearly independent of the columns of V. Suppose that we define a nonsingular matrix T by combining V and U side by side:

$$T := \begin{bmatrix} V_{n\times\bar{n}} & U_{n\times(n-\bar{n})} \end{bmatrix}_{n\times n}$$

We then conclude that

$$AT = A\begin{bmatrix} V & U \end{bmatrix} = \begin{bmatrix} AV & AU \end{bmatrix} = \begin{bmatrix} VA_c & TT^{-1}AU \end{bmatrix} = \begin{bmatrix} T\begin{bmatrix} A_c \\ 0 \end{bmatrix} & TT^{-1}AU \end{bmatrix}$$

By partitioning the $n \times (n - \bar{n})$ matrix $T^{-1}AU$ as

$$T^{-1}AU = \begin{bmatrix} A_{12} \\ A_u \end{bmatrix}$$

we further obtain

$$AT = T\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}, \qquad B = VB_c = T\begin{bmatrix} B_c \\ 0 \end{bmatrix},$$

which can be rewritten as

$$\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} = T^{-1}AT, \qquad \begin{bmatrix} B_c \\ 0 \end{bmatrix} = T^{-1}B.$$

The similarity transformation constructed using this procedure is called a *controllable decomposition*.

Controllable Decomposition

Recall[3]

$$\begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} = T^{-1}AT, \qquad \begin{bmatrix} B_c \\ 0 \end{bmatrix} = T^{-1}B. \tag{12}$$

Theorem (Controllable decomposition)

For every LTI system (AB-LTI)*, there is a similarity transformation that takes the system to the form* (12)[3] *for which*

1. *the controllable subspace of the transformed system* (12) *is given by*

$$\bar{\mathcal{C}} = \mathtt{Im} \begin{bmatrix} I_{\bar{n}\times\bar{n}} \\ 0 \end{bmatrix},$$

   and

2. *the pair* (A_c, B_c) *is controllable.*

---

[3] This form is often called the *standard form for uncontrollable systems*.

[3] MATLAB: `[Abar,Bbar,Cbar,T] = ctrbf(A,B,C)` computes the controllable decomposition of the system with realization A, B, C.

Proof (Controllable decomposition)

To compute the controllability subspace of the transformed system, we compute its controllability matrix

$$\bar{\mathfrak{C}} = \begin{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} & \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} & \dots & \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}^{n-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} B_c & A_c B_c & \dots & A_c^{n-1} B_c \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Proof (Controllable decomposition)

To compute the controllability subspace of the transformed system, we compute its controllability matrix

$$\begin{aligned}\bar{\mathfrak{C}} &= \left[\begin{bmatrix} B_c \\ 0 \end{bmatrix} \quad \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}\begin{bmatrix} B_c \\ 0 \end{bmatrix} \quad \dots \quad \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}^{n-1}\begin{bmatrix} B_c \\ 0 \end{bmatrix}\right] \\ &= \begin{bmatrix} B_c & A_c B_c & \dots & A_c^{n-1} B_c \\ 0 & 0 & \dots & 0 \end{bmatrix}.\end{aligned}$$

Since, similarity transformation preserve the dimension of the controllability subspace, which was $\bar{n}$ for the original system,

$$\texttt{rank}\bar{\mathfrak{C}} = \bar{n}.$$

Proof (Controllable decomposition)

To compute the controllability subspace of the transformed system, we compute its controllability matrix

$$\bar{\mathfrak{C}} = \begin{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} & \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} & \dots & \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}^{n-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} B_c & A_c B_c & \dots & A_c^{n-1} B_c \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Since, similarity transformation preserve the dimension of the controllability subspace, which was $\bar{n}$ for the original system,

$$\mathtt{rank}\bar{\mathfrak{C}} = \bar{n}.$$

Since the number of nonzero rows of $\mathfrak{C}$ is exactly $\bar{n}$, all these rows must be linearly independent. Therefore

$$\mathtt{Im}\bar{\mathfrak{C}} = \mathtt{Im} \begin{bmatrix} I_{\bar{n} \times \bar{n}} \\ 0 \end{bmatrix}.$$

Proof (Controllable decomposition)

To compute the controllability subspace of the transformed system, we compute its controllability matrix

$$\bar{\mathfrak{C}} = \left[\begin{bmatrix} B_c \\ 0 \end{bmatrix} \quad \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}^{n-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} \right]$$
$$= \begin{bmatrix} B_c & A_c B_c & \ldots & A_c^{n-1} B_c \\ 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Since, similarity transformation preserve the dimension of the controllability subspace, which was $\bar{n}$ for the original system,

$$\texttt{rank}\bar{\mathfrak{C}} = \bar{n}.$$

Since the number of nonzero rows of $\mathfrak{C}$ is exactly $\bar{n}$, all these rows must be linearly independent. Therefore

$$\texttt{Im}\bar{\mathfrak{C}} = \texttt{Im} \begin{bmatrix} I_{\bar{n}\times\bar{n}} \\ 0 \end{bmatrix}.$$

Moreover,

$$\texttt{rank} \begin{bmatrix} B_c & A_c B_c & \cdots & A_c^{n-1} B_c \end{bmatrix} = \bar{n}.$$

Proof (Controllable decomposition)

To compute the controllability subspace of the transformed system, we compute its controllability matrix

$$\bar{\mathfrak{C}} = \left[\begin{bmatrix} B_c \\ 0 \end{bmatrix} \quad \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} \quad \dots \quad \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix}^{n-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} \right]$$
$$= \begin{bmatrix} B_c & A_c B_c & \dots & A_c^{n-1} B_c \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Since, similarity transformation preserve the dimension of the controllability subspace, which was $\bar{n}$ for the original system,

$$\mathtt{rank}\bar{\mathfrak{C}} = \bar{n}.$$

Since the number of nonzero rows of $\mathfrak{C}$ is exactly $\bar{n}$, all these rows must be linearly independent. Therefore

$$\mathtt{Im}\bar{\mathfrak{C}} = \mathtt{Im} \begin{bmatrix} I_{\bar{n} \times \bar{n}} \\ 0 \end{bmatrix}.$$

Moreover,

$$\mathtt{rank} \begin{bmatrix} B_c & A_c B_c & \cdots & A_c^{n-1} B_c \end{bmatrix} = \bar{n}.$$

But since A_c is $\bar{n} \times \bar{n}$, by the Cayley-Hamilton theorem,

$$\mathtt{rank} \begin{bmatrix} B_c & A_c B_c & \cdots & A_c^{n-1} B_c \end{bmatrix} = \mathtt{rank} \begin{bmatrix} B_c & A_c B_c & \cdots & A_c^{\bar{n}-1} B_c \end{bmatrix} = \bar{n},$$

which proves the pair (A_c, B_c) is controllable.

Block Diagram Interpretation

Consider now the LTI systems with outputs

$$\dot{x}/x^{+} = Ax + Bu, \quad y = Cx + Du, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k, y \in \mathbb{R}^m \tag{AB-LTI}$$

and let T be the similarity transformation that leads to the controllable decomposition

$$\begin{bmatrix} A_c & A_{12} \\ 0 & Au \end{bmatrix} = T^{-1}AT, \quad \begin{bmatrix} B_c \\ 0 \end{bmatrix} = T^{-1}B, \quad \begin{bmatrix} C_c & C_u \end{bmatrix} = CT.$$

Partitioning the state of the transformed system as

$$\bar{x} = T^{-1}x = \begin{bmatrix} x_c \\ x_u \end{bmatrix} \qquad x_c \in \mathbb{R}^{\bar{n}}, x_u \in \mathbb{R}^{n-\bar{n}}$$

its state space model can be written as

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u \quad y = \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + Du.$$

Block Diagram Interpretation

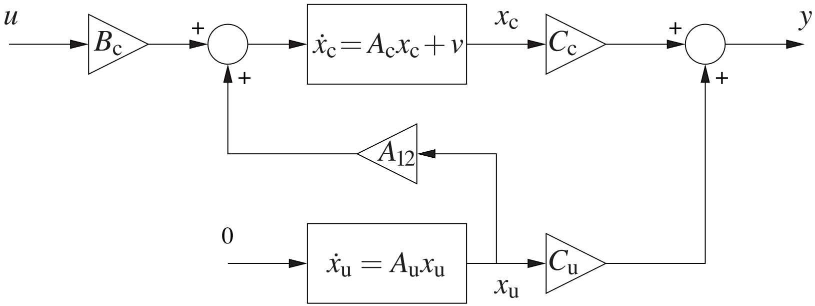


Figure: Controllable decomposition. The direct feed-through term D was eliminated to simplify the diagram

1. This figure highlights the fact that the input u *cannot affect* the x_u component of the state.
2. The controllability of the pair (A_c, B_c) means that the x_c component of the state can *always* be taken to the origin by the appropriate choice of $u(\cdot)$.

Transfer function

Since similarity transformations do not change the system's transfer function, we can use the state-space model for the transformed system to compute the transfer function $T(s)$ of the original system

$$T(s) = \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} sI - A & -A_{12} \\ 0 & sI - A_u \end{bmatrix}^{-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D$$

Transfer function

Since similarity transformations do not change the system's transfer function, we can use the state-space model for the transformed system to compute the transfer function $T(s)$ of the original system

$$T(s) = \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} sI - A & -A_{12} \\ 0 & sI - A_u \end{bmatrix}^{-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D$$

Since the matrix that needs to be inverted is upper triangular, its inverse is also upper triangular, and the diagonal blocks of the inverse are the inverses of the diagonal block of the matrix. Therefore,

$$\begin{aligned} T(s) &= \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} (sI - A)^{-1} & * \\ 0 & (sI - A_u)^{-1} \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D \\ &= C_c(sI - A_c)^{-1} B_c + D. \end{aligned}$$

Transfer function

Since similarity transformations do not change the system's transfer function, we can use the state-space model for the transformed system to compute the transfer function $T(s)$ of the original system

$$T(s) = \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} sI - A & -A_{12} \\ 0 & sI - A_u \end{bmatrix}^{-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D$$

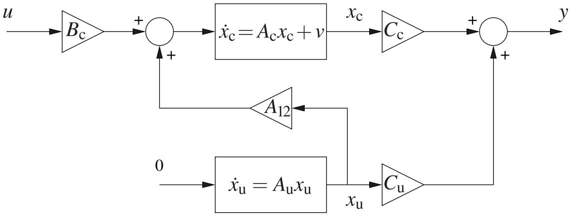
Since the matrix that needs to be inverted is upper triangular, its inverse is also upper triangular, and the diagonal blocks of the inverse are the inverses of the diagonal block of the matrix. Therefore,

$$\begin{aligned} T(s) &= \begin{bmatrix} C_c & C_u \end{bmatrix} \begin{bmatrix} (sI - A)^{-1} & * \\ 0 & (sI - A_u)^{-1} \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D \\ &= C_c(sI - A_c)^{-1} B_c + D. \end{aligned}$$

This shows that the transfer function of the system (LTI) is equal to the transfer function of its controllable part.

Outline of this section

1. Stabilizable system
2. Tests for stabilizability
   1. Eigenvector test
   2. PBH test
   3. Lyapunov test



Stabilizable system

Earlier we saw that any LTI system is *algebraically equivalent* to a system in the following standard form for uncontrollable systems:

$$\begin{bmatrix} \dot{x}_c/x_c^+ \\ \dot{x}_u/x_u^+ \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u, \quad x_c \in \mathbb{R}^{\bar{n}}, x_u \in \mathbb{R}^{n-\bar{n}} \tag{13}$$

Stabilizable system

Earlier we saw that any LTI system is *algebraically equivalent* to a system in the following standard form for uncontrollable systems:

$$\begin{bmatrix} \dot{x}_c / x_c^+ \\ \dot{x}_u / x_u^+ \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u, \quad x_c \in \mathbb{R}^{\bar{n}}, x_u \in \mathbb{R}^{n-\bar{n}} \tag{13}$$

Definition (Stabilizable systems)

The pair (A, B) is *stabilizable* whenever it is algebraically equivalent to a system in the standard form for uncontrollable systems (13) with $n = \bar{n}$ (i.e., A_u nonexistent) or with A_u a stability matrix.

Stabilizable System

Since for stabilizable systems we have

$$\dot{x}_u / x_u^+ = A_u x_u,$$

with A_u a stability matrix, x_u converges to zero exponentially fast, and therefore we have

$$\dot{x}_c / x_c^+ = A_c x_c + B_c u + d, \quad y = C_c x_c + Du + n,$$

where

$$d(t) := A_{12} x_u(t), \quad n(t) := C_u x_u(t), \qquad \forall t \geq 0$$

can be viewed as disturbance and noise terms, respectively, that convergence to zero exponentially fast.

Stabilizable System

Since for stabilizable systems we have

$$\dot{x}_u/x_u^+ = A_u x_u,$$

with A_u a stability matrix, x_u converges to zero exponentially fast, and therefore we have

$$\dot{x}_c/x_c^+ = A_c x_c + B_c u + d, \quad y = C_c x_c + Du + n,$$

where

$$d(t) := A_{12} x_u(t), \quad n(t) := C_u x_u(t), \qquad \forall t \geq 0$$

can be viewed as disturbance and noise terms, respectively, that convergence to zero exponentially fast.

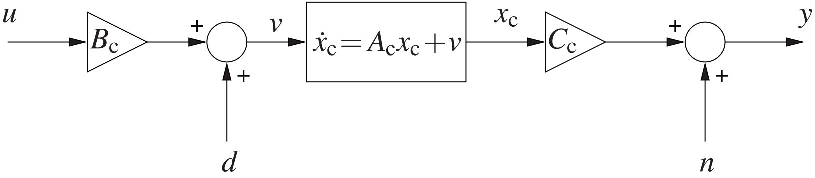


Figure: Controllable part of a stabilizable system. The direct feed-through term D was omitted to simplify the diagram

Eigenvector test for stabilizability

Investigating the stabilizability of an LTI system

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{AB-LTI}$$

from the definition *requires* the computation of its controllable decomposition.

Eigenvector test for stabilizability

Investigating the stabilizability of an LTI system

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \qquad \text{(AB-LTI)}$$

from the definition *requires* the computation of its controllable decomposition. However, there are alternative tests that avoid this intermediate step.

Theorem (Eigenvector test for stabilizability)

1. *The continuous-time system* (AB-LTI) *is stabilizable if and only if every eigenvector of A' corresponding to an eigenvalue with a positive or zero real part is not in kernel of B'.*
2. *The discrete-time system* (AB-LTI) *is stabilizable if and only if every eigenvector of A' corresponding to an eigenvalue with magnitude larger or equal to 1 is not in the kernel of B'.*

Eigenvector test for stabilizability

Investigating the stabilizability of an LTI system

$$\dot{x} = Ax + Bu \quad / \quad x^+ = Ax + Bu, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{AB-LTI}$$

from the definition *requires* the computation of its controllable decomposition. However, there are alternative tests that avoid this intermediate step.

Theorem (Eigenvector test for stabilizability)

1. *The continuous-time system* (AB-LTI) *is stabilizable if and only if every eigenvector of A' corresponding to an eigenvalue with a positive or zero real part is not in kernel of B'.*
2. *The discrete-time system* (AB-LTI) *is stabilizable if and only if every eigenvector of A' corresponding to an eigenvalue with magnitude larger or equal to 1 is not in the kernel of B'.*

Before seeing the proof, let us recall a couple of things.
Let T be the similarity transformation that leads the system (AB-LTI) to the controllable decomposition; i.e.,

$$\bar{A} := \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} = T^{-1}AT, \qquad \bar{B} := \begin{bmatrix} B_c \\ 0 \end{bmatrix} = T^{-1}B$$

$$\bar{x} = T^{-1}x.$$

Proof ((AB-LTI) is stabilizable $\implies$ every "unstable" eigenvector of $A' \notin \texttt{ker} B'$)

Proof ((AB-LTI) is stabilizable $\implies$ every "unstable" eigenvector of $A' \notin \texttt{ker} B'$)

We prove this by contradiction!

Proof ((AB-LTI) is stabilizable $\implies$ every "unstable" eigenvector of $A' \notin \mathtt{ker} B'$)

We prove this by contradiction!
Assume that there exists an "unstable" eigenvalue-eigenvector[4] pair (λ, x) for which

$$
\begin{aligned}
A'x = \lambda x, \quad B'x = 0 \iff & \left(T\bar{A}T^{-1}\right)' x = \lambda x, & (T\bar{B})'x = 0 \\
\iff & \begin{bmatrix} A_c' & 0 \\ A_{12}' & A_u' \end{bmatrix} T'x = \lambda T'x, & \begin{bmatrix} B_c' & 0 \end{bmatrix} T'x = 0 \\
\iff & \begin{bmatrix} A_c' & 0 \\ A_{12}' & A_u' \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} = \lambda \begin{bmatrix} x_c \\ x_u \end{bmatrix}, & \begin{bmatrix} B_c' & 0 \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} = 0
\end{aligned}
$$

where $\begin{bmatrix} x_c' & x_u' \end{bmatrix}' := T'x \neq 0$.

[4] for the purposes of stabilizability, eigenvalues on the "boundary" are considered unstable.

Proof ((AB-LTI) is stabilizable $\implies$ every "unstable" eigenvector of $A' \notin \mathtt{ker} B'$)

We prove this by contradiction!
Assume that there exists an "unstable" eigenvalue-eigenvector[4] pair (λ, x) for which

$$A'x = \lambda x, \quad B'x = 0 \iff \left(T\bar{A}T^{-1}\right)' x = \lambda x, \qquad (T\bar{B})'x = 0$$

$$\iff \begin{bmatrix} A_c' & 0 \\ A_{12}' & A_u' \end{bmatrix} T'x = \lambda T'x, \qquad \begin{bmatrix} B_c' & 0 \end{bmatrix} T'x = 0$$

$$\iff \begin{bmatrix} A_c' & 0 \\ A_{12}' & A_u' \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} = \lambda \begin{bmatrix} x_c \\ x_u \end{bmatrix}, \quad \begin{bmatrix} B_c' & 0 \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} = 0$$

where $\begin{bmatrix} x_c' & x_u' \end{bmatrix}' := T'x \neq 0$. Since the pair (A_c, B_c) is controllable and

$$A_c' x_c = \lambda x_c, \qquad B_c' x_c = 0,$$

we must have $x_c = 0$ (and consequently $x_u \neq 0$), since otherwise this would violate the eigenvector test for controllability.

[4] for the purposes of stabilizability, eigenvalues on the "boundary" are considered unstable.

Proof ((AB-LTI) is stabilizable $\implies$ every "unstable" eigenvector of $A' \notin \ker B'$)

We prove this by contradiction!
Assume that there exists an "unstable" eigenvalue-eigenvector[4] pair (λ, x) for which

$$A'x = \lambda x, \quad B'x = 0 \iff \left(T\bar{A}T^{-1}\right)' x = \lambda x, \qquad (T\bar{B})'x = 0$$

$$\iff \begin{bmatrix} A_c' & 0 \\ A_{12}' & A_u' \end{bmatrix} T'x = \lambda T'x, \qquad \begin{bmatrix} B_c' & 0 \end{bmatrix} T'x = 0$$

$$\iff \begin{bmatrix} A_c' & 0 \\ A_{12}' & A_u' \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} = \lambda \begin{bmatrix} x_c \\ x_u \end{bmatrix}, \quad \begin{bmatrix} B_c' & 0 \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} = 0$$

where $\begin{bmatrix} x_c' & x_u' \end{bmatrix}' := T'x \neq 0$. Since the pair (A_c, B_c) is controllable and

$$A_c' x_c = \lambda x_c, \qquad B_c' x_c = 0,$$

we must have $x_c = 0$ (and consequently $x_u \neq 0$), since otherwise this would violate the eigenvector test for controllability. This means that λ must be an eigenvalue of A_u because

$$A_u' x_u = \lambda x_u$$

which contradicts the stabilizability of the system (AB-LTI) because λ is "unstable".

[4] for the purposes of stabilizability, eigenvalues on the "boundary" are considered unstable.

Proof (every "unstable" eigenvector of $A' \notin \texttt{ker} B' \implies$ (AB-LTI) is stabilizable.)

Proof (every "unstable" eigenvector of $A' \notin \mathtt{ker} B' \implies$ (AB-LTI) is stabilizable.)

Suppose now that the system (AB-LTI) is not stabilizable.

Proof (every "unstable" eigenvector of $A' \notin \texttt{ker} B' \implies$ (AB-LTI) is stabilizable.)

Suppose now that the system (AB-LTI) is not stabilizable.
Therefore A'_u has an "unstable" eigenvalue-eigenvector pair

$$A'_u x_u = \lambda x_u, \qquad x_u \neq 0.$$

Proof (every "unstable" eigenvector of $A' \notin \texttt{ker} B' \implies$ (AB-LTI) is stabilizable.)

Suppose now that the system (AB-LTI) is *not* stabilizable.
Therefore A'_u has an "unstable" eigenvalue-eigenvector pair

$$A'_u x_u = \lambda x_u, \qquad x_u \neq 0.$$

Then,

$$\bar{A}' \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \begin{bmatrix} A'_c & 0 \\ A'_{12} & A'_u \end{bmatrix} \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ x_u \end{bmatrix},$$

$$\bar{B}' \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \begin{bmatrix} B'_c & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_u \end{bmatrix} = 0, \qquad \begin{bmatrix} 0 \\ x_u \end{bmatrix} \neq 0.$$

Proof (every "unstable" eigenvector of $A' \notin \texttt{ker} B' \implies$ (AB-LTI) is stabilizable.)

Suppose now that the system (AB-LTI) is not stabilizable.
Therefore A'_u has an "unstable" eigenvalue-eigenvector pair

$$A'_u x_u = \lambda x_u, \qquad x_u \neq 0.$$

Then,

$$\bar{A}' \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \begin{bmatrix} A'_c & 0 \\ A'_{12} & A'_u \end{bmatrix} \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ x_u \end{bmatrix},$$
$$\bar{B}' \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \begin{bmatrix} B'_c & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_u \end{bmatrix} = 0, \qquad \begin{bmatrix} 0 \\ x_u \end{bmatrix} \neq 0.$$

We have thus far found an "unstable" eigenvector of $\bar{A}'$ in the kernel of $\bar{B}'$, so $(\bar{A}, \bar{B})$ cannot be stabilizable.

Proof (every "unstable" eigenvector of $A' \notin \texttt{ker} B' \implies$ (AB-LTI) is stabilizable.)

Suppose now that the system (AB-LTI) is *not* stabilizable.
Therefore A'_u has an "unstable" eigenvalue-eigenvector pair

$$A'_u x_u = \lambda x_u, \qquad x_u \neq 0.$$

Then,

$$\bar{A}' \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \begin{bmatrix} A'_c & 0 \\ A'_{12} & A'_u \end{bmatrix} \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ x_u \end{bmatrix},$$
$$\bar{B}' \begin{bmatrix} 0 \\ x_u \end{bmatrix} = \begin{bmatrix} B'_c & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_u \end{bmatrix} = 0, \qquad \begin{bmatrix} 0 \\ x_u \end{bmatrix} \neq 0.$$

We have thus far found an "unstable" eigenvector of $\bar{A}'$ in the kernel of $\bar{B}'$, so $(\bar{A}, \bar{B})$ cannot be stabilizable.
To conclude that the original pair (A, B) is also not stabilizable we use the equivalence shown in part 1 to conclude that

$$x := (T')^{-1} \begin{bmatrix} 0 \\ x_u \end{bmatrix} \iff \begin{bmatrix} 0 \\ x_u \end{bmatrix} := T' x$$

is an "unstable" eigenvector of A' in the kernel of B'.

Popov-Belevitch-Hautus(PBH) Test for Stabilizability

For stabilizability, one can also reformulate the eigenvector test as a rank condition, similar to that for controllability.

Theorem (PBH test for stabilizability)

1. *The continuous-time LTI system* (AB-LTI) *is stabilizable if and only if*

$$\texttt{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n, \qquad \forall \lambda \in \mathbb{C} : \texttt{Re}[\lambda] \geq 0.$$

2. *The discrete-time LTI system* (AB-LTI) *is stabilizable if and only if*

$$\texttt{rank}\begin{bmatrix} A-\lambda I & B \end{bmatrix} = n, \qquad \forall \lambda \in \mathbb{C} : |\lambda| \geq 1.$$

The proof of this theorem is analogous to the earlier proof, except that now we need to *restrict* our attention to only the "unstable" portion of $\mathbb{C}$.

Lyapunov Test for Stabilizability

Theorem (Lyapunov test for stabilizability)

The LTI system (AB-LTI) *is stabilizable if and only if there is a positive-definite solution P to the following Lyapunov matrix inequality*

$$AP + PA' - BB' < 0 \quad / \quad APA' - P - BB' < 0 \qquad \text{(LMI)}$$

Proof: ((LMI) has positive-definite solution $P \implies$ (AB-LTI) is stabilizable)

Proof: ((LMI) has positive-definite solution $P \implies$ (AB-LTI) is stabilizable)

The simplest way to do this is by using the eigenvector test.

Proof: ((LMI) has positive-definite solution $P \implies$ (AB-LTI) is stabilizable)

The simplest way to do this is by using the eigenvector test.
Assume that

1. (LMI) holds, and
2. $x \neq 0$ be an eigenvector of A' associated with the "unstable" eigenvalue λ; i.e., $A'x = \lambda x$.

Proof: ((LMI) has positive-definite solution $P \implies$ (AB-LTI) is stabilizable)

The simplest way to do this is by using the eigenvector test.
Assume that

1. (LMI) holds, and
2. $x \neq 0$ be an eigenvector of A' associated with the "unstable" eigenvalue λ; i.e., $A'x = \lambda x$.

Then,

$$x^*(AP + PA')x < x^*BB'x = \|B'x\|^2 ,$$

where $(\cdot)^*$ denotes the complex conjugate transpose.

Proof: ((LMI) has positive-definite solution $P \implies$ (AB-LTI) is stabilizable)

The simplest way to do this is by using the eigenvector test.
Assume that

1. (LMI) holds, and
2. $x \neq 0$ be an eigenvector of A' associated with the "unstable" eigenvalue λ; i.e., $A'x = \lambda x$.

Then,

$$x^*(AP + PA')x < x^*BB'x = \|B'x\|^2 ,$$

where $(\cdot)^*$ denotes the complex conjugate transpose. But the left-hand side of this equation is equal to

$$(A'x^*)'Px + x^*PA'x = \lambda^* x^*Px + \lambda x^*Px = 2\mathtt{Re}[\lambda]x^*Px.$$

Proof: ((LMI) has positive-definite solution $P \implies$ (AB-LTI) is stabilizable)

The simplest way to do this is by using the eigenvector test.
Assume that

1. (LMI) holds, and
2. $x \neq 0$ be an eigenvector of A' associated with the "unstable" eigenvalue λ; i.e., $A'x = \lambda x$.

Then,

$$x^*(AP + PA')x < x^*BB'x = \|B'x\|^2 ,$$

where $(\cdot)^*$ denotes the complex conjugate transpose. But the left-hand side of this equation is equal to

$$(A'x^*)'Px + x^*PA'x = \lambda^* x^* Px + \lambda x^* Px = 2\texttt{Re}[\lambda] x^* Px.$$

Since P is positive-definite and $\texttt{Re}[\lambda] \geq 0$, we conclude that

$$0 \leq 2\texttt{Re}[\lambda] x^* Px < \|B'x\|^2 ,$$

and therefore x must not belong to the kernel of B'.

Proof: ((AB-LTI) is stabilizable $\implies$ (LMI) has positive-definite solution P)

Proof: ((AB-LTI) is stabilizable $\implies$ (LMI) has positive-definite solution P)

We saw earlier that controllability of the pair (A_c, B_c) guarantees the existence of a positive-definite P_c such that

$$A_c P_c + P_c A_c' - B_c B_c' = -Q_c < 0.$$

Proof: ((AB-LTI) is stabilizable $\Longrightarrow$ (LMI) has positive-definite solution P)

We saw earlier that controllability of the pair (A_c, B_c) guarantees the existence of a positive-definite P_c such that

$$A_c P_c + P_c A_c' - B_c B_c' = -Q_c < 0.$$

On the other hand, since A_u is a stability matrix, we conclude from the **Lyapunov stability theorem** that there exists a positive-definite matrix P_u such that

$$A_u P_u + P_u A_u' = -Q_u < 0.$$

Proof: ((AB-LTI) is stabilizable $\implies$ (LMI) has positive-definite solution P)

We saw earlier that controllability of the pair (A_c, B_c) guarantees the existence of a positive-definite P_c such that

$$A_c P_c + P_c A_c' - B_c B_c' = -Q_c < 0.$$

On the other hand, since A_u is a stability matrix, we conclude from the **Lyapunov stability theorem** that there exists a positive-definite matrix P_u such that

$$A_u P_u + P_u A_u' = -Q_u < 0.$$

Defining

$$\bar{P} = \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix}$$

for some scalar $\rho > 0$ to be determined shortly, we conclude that

$$\begin{aligned} \bar{A}\bar{P} + \bar{P}\bar{A}' - \bar{B}\bar{B}' &= \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix} + \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix} \begin{bmatrix} A_c' & 0 \\ A_{12}' & A_u' \end{bmatrix} - \begin{bmatrix} B_c \\ 0 \end{bmatrix} \begin{bmatrix} B_c' & 0 \end{bmatrix} \\ &= - \begin{bmatrix} Q_c & -\rho A_{12} P_u \\ -\rho P_u A_{12}' & \rho Q_u \end{bmatrix}. \end{aligned}$$

Proof: ((AB-LTI) is stabilizable $\Longrightarrow$ (LMI) has positive-definite solution P)

We saw earlier that controllability of the pair (A_c, B_c) guarantees the existence of a positive-definite P_c such that

$$A_c P_c + P_c A_c' - B_c B_c' = -Q_c < 0.$$

On the other hand, since A_u is a stability matrix, we conclude from the **Lyapunov stability theorem** that there exists a positive-definite matrix P_u such that

$$A_u P_u + P_u A_u' = -Q_u < 0.$$

Defining

$$\bar{P} = \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix}$$

for some scalar $\rho > 0$ to be determined shortly, we conclude that

$$\begin{aligned} \bar{A}\bar{P} + \bar{P}\bar{A}' - \bar{B}\bar{B}' &= \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix} + \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix} \begin{bmatrix} A_c' & 0 \\ A_{12}' & A_u' \end{bmatrix} - \begin{bmatrix} B_c \\ 0 \end{bmatrix} \begin{bmatrix} B_c' & 0 \end{bmatrix} \\ &= - \begin{bmatrix} Q_c & -\rho A_{12} P_u \\ -\rho P_u A_{12}' & \rho Q_u \end{bmatrix}. \end{aligned}$$

It turns out that by making ρ positive, but sufficiently small, the right-hand side can be made negative-definite.

Proof: ((AB-LTI) is stabilizable $\implies$ (LMI) has positive-definite solution P)

We saw earlier that controllability of the pair (A_c, B_c) guarantees the existence of a positive-definite P_c such that

$$A_c P_c + P_c A_c' - B_c B_c' = -Q_c < 0.$$

On the other hand, since A_u is a stability matrix, we conclude from the **Lyapunov stability theorem** that there exists a positive-definite matrix P_u such that

$$A_u P_u + P_u A_u' = -Q_u < 0.$$

Defining

$$\bar{P} = \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix}$$

for some scalar $\rho > 0$ to be determined shortly, we conclude that

$$\begin{aligned} \bar{A}\bar{P} + \bar{P}\bar{A}' - \bar{B}\bar{B}' &= \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix} + \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix} \begin{bmatrix} A_c' & 0 \\ A_{12}' & A_u' \end{bmatrix} - \begin{bmatrix} B_c \\ 0 \end{bmatrix} \begin{bmatrix} B_c' & 0 \end{bmatrix} \\ &= -\begin{bmatrix} Q_c & -\rho A_{12} P_u \\ -\rho P_u A_{12}' & \rho Q_u \end{bmatrix}. \end{aligned}$$

It turns out that by making ρ positive, but sufficiently small, the right-hand side can be made negative-definite. The proof is completed by verifying that the matrix.

$$P = T \begin{bmatrix} P_c & 0 \\ 0 & \rho P_u \end{bmatrix} T'$$

satisfies (LMI).

Controllability after sampling

Consider a continuous-time state equation

$$\dot{x} = Ax + Bu \qquad \text{(AB-LTI)}$$

If the input is piecewise constant or

$$u(k) := u(kT) = u(t) \qquad \text{for } kT \leq t < (k+1)T$$

then the equation can be described by

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) \qquad \text{(C2D)}$$

with $\bar{A} = e^{AT}, \bar{B} = \left(\int_0^T e^{At}dt\right) B \equiv MB$.

Controllability after sampling

Consider a continuous-time state equation

$$\dot{x} = Ax + Bu \qquad \text{(AB-LTI)}$$

If the input is piecewise constant or

$$u(k) := u(kT) = u(t) \qquad \text{for } kT \leq t < (k+1)T$$

then the equation can be described by

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) \qquad \text{(C2D)}$$

with $\bar{A} = e^{AT}, \bar{B} = \left(\int_0^T e^{At}dt\right) B \equiv MB$.

If (AB-LTI) is controllable, will its sampled equation (C2D) is controllable?

Controllability after sampling

Consider a continuous-time state equation

$$\dot{x} = Ax + Bu \qquad \text{(AB-LTI)}$$

If the input is piecewise constant or

$$u(k) := u(kT) = u(t) \qquad \text{for } kT \leq t < (k+1)T$$

then the equation can be described by

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) \qquad \text{(C2D)}$$

with $\bar{A} = e^{AT}, \bar{B} = \left(\int_0^T e^{At}dt\right) B \equiv MB$.

If (AB-LTI) is controllable, will its sampled equation (C2D) is controllable?

This problem is important in designing so-called dead-beat sampled-data systems and in computer control of continuous-time systems.

Controllability after sampling

Let λ_i and $\bar{\lambda}_i$ be, respectively, the eigenvalues of A and $\bar{A}$.

Controllability after sampling

Let λ_i and $\bar{\lambda}_i$ be, respectively, the eigenvalues of A and $\bar{A}$.

Theorem (Controllability after sampling)

Suppose (AB-LTI) *is controllable. A sufficient condition for* (C2D), *with sampling time* T, *to be controllable is that*

$$|\mathtt{Imag}(\lambda_i - \lambda_j)| \neq \frac{2\pi m}{T}, \quad m = 1, 2, \ldots$$

whenever $\mathtt{Re}(\lambda_i - \lambda_j) = 0$.
For the single input case, the condition is necessary as well.

Controllability after sampling

Let λ_i and $\bar{\lambda}_i$ be, respectively, the eigenvalues of A and $\bar{A}$.

Theorem (Controllability after sampling)

Suppose (AB-LTI) *is controllable. A sufficient condition for* (C2D), *with sampling time T, to be controllable is that*

$$|\texttt{Imag}(\lambda_i - \lambda_j)| \neq \frac{2\pi m}{T}, \quad m = 1, 2, \ldots$$

whenever $\texttt{Re}(\lambda_i - \lambda_j) = 0$.
For the single input case, the condition is necessary as well.

It is straightforward to verify that if A has only real eigenvalues, then the discretized equation with any sampling period $T > 0$ is always controllable.

Controllability after sampling

Further remarks:

Suppose A has complex conjugate eigenvalues $\alpha \pm j\beta$.

- If the sampling period T does not equal any integer multiple of π/β, then the discretized state equation is controllable.
- If $T = m\pi/\beta$ for some integer m, then the discretized equation $may\ not$ be controllable.

Controllability after sampling

Further remarks:

Suppose A has complex conjugate eigenvalues $\alpha \pm j\beta$.

- If the sampling period T does not equal any integer multiple of π/β, then the discretized state equation is controllable.
- If $T = m\pi/\beta$ for some integer m, then the discretized equation $may\ not$ be controllable.

Note (take it as an exercise)

Since $\bar{A} = e^{AT}$, if λ_i is an eigenvalue of A, then $\bar{\lambda}_i := e^{\lambda_i T}$ is an eigenvalue of $\bar{A}$.

Controllability after sampling

Further remarks:

Suppose A has complex conjugate eigenvalues $\alpha \pm j\beta$.

- If the sampling period T does not equal any integer multiple of π/β, then the discretized state equation is controllable.
- If $T = m\pi/\beta$ for some integer m, then the discretized equation $may\ not$ be controllable.

Note (take it as an exercise)

Since $\bar{A} = e^{AT}$, if λ_i is an eigenvalue of A, then $\bar{\lambda}_i := e^{\lambda_i T}$ is an eigenvalue of $\bar{A}$.

If $T = m\pi/\beta$ the two distinct eigenvalues $\lambda_1 = \alpha + j\beta$ and $\lambda_2 = \alpha - j\beta$ of A become a repeated eigenvalue $-e^{\alpha T}$ or $e^{\alpha T}$ of $\bar{A}$. This will cause the discretized equation to be uncontrollable.

Controllability after sampling

Further remarks:

Suppose A has complex conjugate eigenvalues $\alpha \pm j\beta$.

- If the sampling period T does not equal any integer multiple of π/β, then the discretized state equation is controllable.
- If $T = m\pi/\beta$ for some integer m, then the discretized equation $may\ not$ be controllable.

Note (take it as an exercise)

Since $\bar{A} = e^{AT}$, if λ_i is an eigenvalue of A, then $\bar{\lambda}_i := e^{\lambda_i T}$ is an eigenvalue of $\bar{A}$.

If $T = m\pi/\beta$ the two distinct eigenvalues $\lambda_1 = \alpha + j\beta$ and $\lambda_2 = \alpha - j\beta$ of A become a repeated eigenvalue $-e^{\alpha T}$ or $e^{\alpha T}$ of $\bar{A}$. This will cause the discretized equation to be uncontrollable.

Theorem

If the continuous time LTI state equation is not controllable, then its discretized state equation with any sampling period, is not controllable.