Linear Dynamical Systems

Tutorial on Controllability: Part-II

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Controllable Canonical form

Problem 1

Given a system $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Find the controllable canonical form of the system.

[1] Antsaklis, Chapter 3, Example 4.11

Solution to Problem 1

Procedure to Compute Transformation Matrix

The representation $\{A_c, B_c, C_c, D_c\}$ in controller form is given by $A_c \triangleq \hat{A} = PAP^{-1}$ and $B_c \triangleq \hat{B} = PB$ with

$$A_c = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-1} \end{bmatrix}, \qquad B_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

where the coefficients α_i are the coefficients of the characteristic polynomial $\alpha(s)$ of A, that is,

$$\alpha(s) \triangleq |sI - A| = s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_1 s + \alpha_0$$

Note that $C_c \triangleq \hat{C} = CP^{-1}$ and $D_c = D$ do not have any particular structure. The structure of (A_c, B_c) is very useful (in control problems) and the representation $\{A_c, B_c, C_c, D_c\}$ shall be referred to as the *controller form* of the system.

Solution to Problem 1

Procedure to Compute Transformation Matrix

The similarity transformation matrix P is obtained as follows. The controllability matrix $\mathfrak{C} = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$ is in this case an $n \times n$ nonsingular matrix and $\mathfrak{C}^{-1} = \begin{bmatrix} \times \\ q \end{bmatrix}$, where q is the nth row of $\mathfrak{C}^{-1}$ and $\times$ indicates the remaining entries of $\mathfrak{C}^{-1}$. Then

$$P \triangleq \begin{bmatrix} q \\ qA \\ \vdots \\ qA^{n-1} \end{bmatrix}$$

To show that $PAP^{-1} = A_c$ and $PB = B_c$, note first that $qA^{i-1}B = 0,\ i = 1, \dots, n-1$, and $qA^{n-1}B = 1$. This can be verified from the definition of q, which implies that $q\mathfrak{C} = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}$.

Solution to Problem 1

Procedure to Compute Transformation Matrix

Now,

$$P\mathfrak{C} = P\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & \dots & 1 \\ 0 & 0 & \dots & 1 & \times \\ \vdots & 1 & & \vdots & \vdots \\ 1 & \times & \dots & \times & \times \end{bmatrix} = \mathfrak{C}_c$$

which implies that $|P\mathfrak{C}| = |P||\mathfrak{C}| \neq 0$ or that $|P| \neq 0$. Therefore, P qualifies as a similarity transformation matrix. In view of the above equation, $PB = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T = B_c$. Furthermore,

$$A_cP = \begin{bmatrix} qA \\ \vdots \\ qA^{n-1} \\ qA^n \end{bmatrix} = PA,$$

where in the last row of A_cP, the relation $-\sum_{i=0}^{n-1} \alpha_i A^i = A^n$ was used [which is the Cayley-Hamilton Theorem, namely, $\alpha(A) = 0$].

Solution to Problem 1

Calculating the Transformation matrix P,that reduces (A, B) to $(A_c = PAP^{-1}, B_c = PB)$, we have:

$$\mathfrak{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -2 & 4 \end{bmatrix}$$

and

$$\mathfrak{C}^{-1} = \begin{bmatrix} 1 & -1/3 & -1/3 \\ -1/2 & -1/2 & 0 \\ -1/2 & -1/6 & 1/3 \end{bmatrix}$$

The third (the nth)row of $\mathfrak{C}^{-1}$ is

$$q = \begin{bmatrix} -1/2 & -1/6 & 1/3 \end{bmatrix}$$

Solution to Problem 1

and therefore,

$$P = \begin{bmatrix} q \\ qA \\ qA^2 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/6 & 1/3 \\ 1/2 & -1/6 & -2/3 \\ -1/2 & -1/6 & 4/3 \end{bmatrix}$$

Calculating PAP^{-1} and PB gives:

$$A_c = PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}, B_c = PB = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

which is the controllable form.

Controllable Canonical Form

Problem 2

Consider the following third order SISO LTI system:

$$\dot{x} = Ax + Bu,\ x \in \mathbb{R}^3,\ u \in \mathbb{R}^1$$

Assume that the characteristic polynomial of A is given by

$$|sI - A| = s^3 + a_1 s^2 + a_2 s + a_3$$

and consider the matrix

$$T = \mathfrak{C} \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\mathfrak{C}$ is the system's controllability matrix.

[1] Hespanha, Problem 14.3

Problem 2

(a) Show that the following equality holds:

$$B = T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(b) Show that the following equality holds:

$$AT = T \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) Show that if the system is controllable then T is nonsingular.

Solution to Problem 2

(a)

$$
\begin{aligned}
T\begin{bmatrix}1\\0\\0\end{bmatrix} &= \mathfrak{C}\begin{bmatrix}1 & a_1 & a_2\\0 & 1 & a_1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\\
&= \begin{bmatrix}B\ AB\ A^2B\end{bmatrix}\begin{bmatrix}1 & a_1 & a_2\\0 & 1 & a_1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\\
&= \begin{bmatrix}B & AB & A^2B\end{bmatrix}]\begin{bmatrix}1\\0\\0\end{bmatrix} = B
\end{aligned}
$$

Solution to Problem 2

(b)

$$
\begin{aligned}
T &\begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= [B \; AB \; A^2B] \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & a_1 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}
$$

Solution to Problem 2

(b) Also,

$$AT = A\begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} AB & A^2B & A^3B \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Using Cayley Hamilton theorem:

$$AT = \begin{bmatrix} AB & A^2B & \left(-a_1A^2 - a_2A - a_3I\right)B \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Upon rearrangements and simplification we obtain the desired result.

Solution to Problem 2

- Note that, for the present case the matrix $\mathfrak{C}$ is a square matrix.If the system is controllable then the matrix $\mathfrak{C}$ is invertible. Also

$$T = \mathfrak{C}\begin{bmatrix}1 & a_1 & a_2\\0 & 1 & a_1\\0 & 0 & 1\end{bmatrix}$$

Since, the second matrix is also invertible (upper diagonal matrix with non zeros elements on the diagonal), the matrix T is also non-singular and invertible.

Controllable Canonical Form

Problem 2, continued

(d) Combining the above parts we have shown that if the system is controllable then the matrix T can be viewed as a similarity transformation that transforms the system into the controllable canonical form

$$T^{-1}AT = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Use this to find the similarity transformation that transforms the following pair into the controllable canonical form

$$A = \begin{bmatrix} 6 & 4 & 1 \\ -5 & -4 & 0 \\ -4 & -3 & -1 \end{bmatrix}; B = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Solution to Problem 2

(d) Use the MATLAB functions $poly(A)$ to compute the characteristic polynomial of A and $ctrb(A, B)$ to compute the controllability matrix of the pair (A, B):

$$|sI - A| = s^3 - s^2 - 2s - 3$$

and

$$\mathfrak{C} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix}$$

Finally the required matrix T is computed to be:

$$T = \mathfrak{C} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$$

PBH Test for Controllability

Problem 3

Show that the state equation

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

is controllable if and only if the pair (A_{22}, A_{21}) is controllable. Assume B_1 is a full-rank block matrix.

[1] Chen, Problem 6.4

Solution to Problem 3

Recall!

Recall from the lecture slide 50 that a LTI system is controllable if the $n \times (n+p)$ matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full row rank at every eigenvalue λ of A.

(A, B) is controllable if and only if

$\texttt{rank}\ M = \texttt{rank} \begin{bmatrix} A_{11} - \lambda I & A_{12} & B_1 \\ A_{21} & A_{22} - \lambda I & 0 \end{bmatrix} = n$ for every λ, where λ is an eigenvalue of A. Since B_1 is full rank, the statement holds for every λ if and only if $\begin{bmatrix} A_{22} - \lambda I & A_{21} \end{bmatrix}$ has full row rank, since then, the rank of M will definitely be n. This means that the pair (A, B) is controllable if and only if (A_{22}, A_{21}) is controllable.

PBH Test for Controllability

Problem 4

Given

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix},$$

Find the uncontrollable eigenvalues of the system using the PBH Test.

[1] Antsaklis, Chapter 3, Example 4.6

Solution to Problem 4

Recall!

Recall from the lecture slide 50 the PBH test of controllability:
The LTI systerm $\dot{x} = Ax + Bu$ is controllable iff

$$rank \; \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \; \forall \lambda \in \mathbb{C}$$

The eigenvalues of A are:$\lambda_1 = 0, \lambda_2 = -1$ and $\lambda_3 = -2$ rank

$$\begin{bmatrix} \lambda_i I - A & B \end{bmatrix}_{\lambda_3 = -2} = rank \begin{bmatrix} -2 & 2 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 1 \\ 0 & -1 & -1 & 1 & 2 \end{bmatrix} = 2 < 3 = n$$

Therefore, $\lambda_3 = -2$ is uncontrollable eigenvalue of A.

Stabilizable System

Problem 5

Consider the system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

a. Check the controllability of the system.
b. Comment on the stabilizability of the system using controllable decomposition procedure.

Solution to Problem 5

(a) The controllability matrix for the system is computed to be:

$$\mathfrak{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is a rank 2 matrix. Hence, the system is not controllable.

Solution to Problem 5

(b) We first construct the matrix T which will reduce the system to controllable decomposed form. Firstly, select two of the linearly independent columns of $\mathfrak{C}$ -

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Choosing a vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ which is independent of the above two columns, we get the desired transformation matrix as:

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is non-singular.

Solution of Problem 5

- (b) Using T as the transformation matrix, the new system is given as:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$

where

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and

$$\tilde{B} = T^{-1}B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Denote

$$A_1 \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \text{ and } b_1 \triangleq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution of Problem 5

(b) The new system can be decomposed as written as:

$$\dot{x} = \begin{bmatrix} A_1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} u$$

Is is evident that the eigen value -1 of this system is not controllable. However, being negative, it corresponds to a (asymptotically) stable system and hence the system is stabilizable but not controllable.

Recall!

Recall from the lecture slide 57 that the controllability property is invariant under similarity transformation.

Since the controllability property is invariant under similarity transformation, it can be safely concluded that the original system is stabilizable (but not controllable).

Controllable Decomposition

Problem 6

Consider the system

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \quad y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

Find the controllable part of the system using the decomposition method.

[1] Chen, Example 6.8

Solution to Question 6

B has a rank of 2, therefore we can directly use $\mathfrak{C}_2 = \begin{bmatrix} B & AB \end{bmatrix}$ to check controllability.

$$\mathtt{rank}(\mathfrak{C}_2) = \mathtt{rank}\begin{bmatrix} B & AB \end{bmatrix} = \mathtt{rank}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = 2 < \ 3$$

Therefore the system is not fully controllable. Let us choose arbitrarily,

$$P^{-1} = Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then,

$$\bar{A} = PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 0 & \vdots & 0 \\ 1 & 1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 1 \end{bmatrix}$$

$$\bar{B} = PB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \cdots & \cdots \\ 0 & 0 \end{bmatrix}$$

$$\bar{C} = CP^{-1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & \vdots & 1 \end{bmatrix}$$

Thus the system is reduced to:

$$\dot{\bar{x}}_c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \bar{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 2 \end{bmatrix} \bar{x}_c$$

Controllability of LTV system

Question 7

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} e^t \\ e^{2t} \end{bmatrix} u$$

with

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

Comment on the controllability of this LTV system.

[1] Terrell, Stability and Stabilization, Example 4.11

Solution to Question 7

Recall!

Recall from the lecture slide 39 the matrix test for the controllability of the linear time varying systems

For this problem :

$$M_0(t) = \begin{bmatrix} e^t \\ e^{2t} \end{bmatrix}$$

$$M_1(t) = -AM_0(t) + \frac{d}{dt}M_0(t)$$

$$\implies M_1(t) = \begin{bmatrix} -e^t \\ -2e^{2t} \end{bmatrix} + \begin{bmatrix} e^t \\ 2e^{2t} \end{bmatrix}$$

$$\implies M_1(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then clearly $rank(\begin{bmatrix} M_0(t) & M_1(t) \end{bmatrix}) = 1 < 2$ and it is not possible to comment on the controllability of the system using this test. The matrix test fails!

Solution to Question 7

The general solution is given by-

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^t(x_{10} + \int_0^t u(s)ds) \\ e^{2t}(x_{20} + \int_0^t u(s)ds) \end{bmatrix}$$

Solution to Question 7

$$\mathfrak{C}(t) = \begin{bmatrix} B(t) & AB(t) \end{bmatrix} = \begin{bmatrix} e^{t} & e^{t} \\ e^{2t} & 2e^{2t}, \end{bmatrix}$$

we obtain a nonsingular matrix. Therefore, the rank criterion of controllability is not applicable to linear time varying systems.