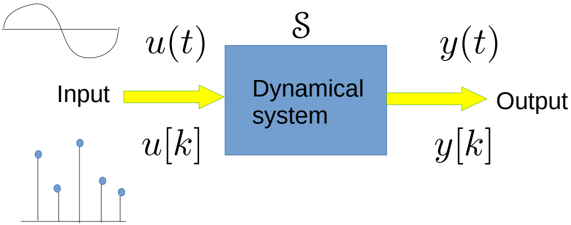
# Linear Dynamical Systems

## Week 1- State-space solutions and realizations

# Outline of Week 1

1. Introduction
2. State-space solution of linear systems
   - Linear Time Varying (LTV) systems
   - Linear Time Invariant (LTI) systems
3. Equivalent representation of linear state-space systems
4. Realization problem and its solution

# Dynamical System




Continuous-Time (CT): accepts CT signals and generates CT signals

Discrete-Time (DT): accepts DT signals and generates DT signals

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the concept of state deals with this problem!

# Dynamical System

**State:**

1. The state $x(t_0)$ of a system at time $t_0$ is the information at $t_0$ that, together with the input $u(t)$, for $t \geqslant t_0$ determines uniquely the output $y(t)\,\forall t \geqslant t_0$

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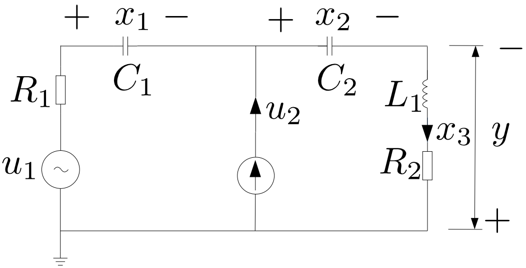
2. no need to know the input $u(t)$ applied before $t_0$ in determining the output $y(t)$ after $t_0$.

3. the state summarizes the effect of past input on future output

# Example

Consider the electrical circuit




If we know the voltages $x_1(t_0)$ and $x_2(t_0)$ across the two capacitors and the current $x_3(t_0)$ passing through the inductor...

# Example: Continued...

...then for any input applied on and after $t_0$ you can determine uniquely the output for $t \geq t_0$

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1. State Variables $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$

2. $\mathcal{S}$ : using the state at $t_0$

$$\left.\begin{matrix} x(t_0) \\ u(t), t_0 \leq t \end{matrix}\right\} \xrightarrow{\mathcal{S}} y(t), t \geq t_0$$

# Dynamical Systems: **Linearity**

- $\mathcal{S}$ is linear if (Superposition Property)

$$\left.\begin{array}{r}\alpha_1 x_1(t_0) + \alpha_2 x_2(t_0) \\ \alpha_1 u_1(t) + \alpha_2 u_2(t), t \geq t_0\end{array}\right\} \xrightarrow{\mathcal{S}} \alpha_1 y_1(t) + \alpha_2 y_2(t),\ t \geq t_0$$

  for any real constants $\alpha_1, \alpha_2$

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Based on the input-state-output variables, two types of responses can now be defined

1. Zero Input Response:

$$\left.\begin{array}{l}x(t_0)\\ u(t)=0, t\geq t_0\end{array}\right\}\xrightarrow{\mathcal{S}} y_{z_i}, t\geq t_0$$

2. Zero State Response:

$$\left.\begin{array}{l}x(t_0)=0\\ u(t), t\geq t_0\end{array}\right\}\xrightarrow{\mathcal{S}} y_{z_s}(t), t\geq t_0$$

# Response of linear systems

- The additivity property implies that:

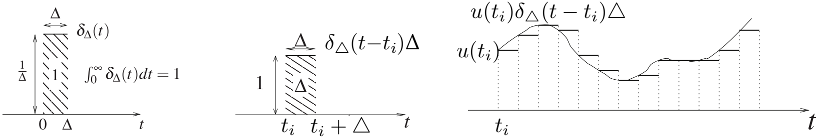
$$\text{output due to} \begin{Bmatrix} x(t_0) \\ u(t), t \geq 0 \end{Bmatrix} = \text{output due to} \begin{Bmatrix} x(t_0) \\ u(t) = 0, t \geq 0 \end{Bmatrix}$$
$$+ \text{output due to} \begin{Bmatrix} x(t_0) = 0 \\ u(t), t \geq 0 \end{Bmatrix}$$

i.e.,

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

# Zero-state response of linear systems

Consider the SISO system.
Let $\delta_\Delta(t - t_1)$ be the pulse as shown in the figure, then every input can be approximated by a sequence of the pulses




(a) Pulse (b) Time-shifted pulse (c) Step approximation

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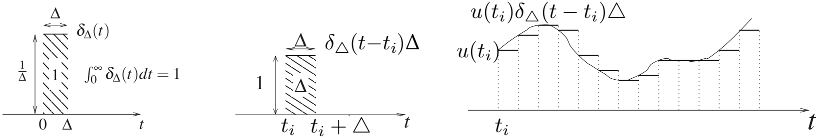


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The input can be expressed symbolically as :

$$u(t) \approx \sum_i u(t_i)\delta_\Delta(t-t_i)\Delta$$

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$$\sum_i u(t_i)\delta_\Delta(t - t_i)\Delta \xrightarrow{\mathcal{S}} \sum_i g_\Delta(t, t_i)u(t_i)\Delta \qquad \text{(additivity)}$$

Thus,

$$\boxed{y(t) \approx \sum_i g_\Delta(t, t_i)u(t_i)\Delta}$$

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$$\boxed{y(t) = \lim_{\Delta \to 0} \sum_i g_\Delta(t, t_i) u(t_i) \Delta = \int_{-\infty}^{\infty} g(t, \tau) u(\tau) d\tau}$$

where $t$ is the time at which the output is observed; $\tau$ is the time at which the impulse input is applied; and $g(t, \tau)$ is the impulse response

[1] The last equation is a consequence of the definition of the Riemann integral, i.e. $\int_{-\infty}^{\infty} f(\tau) d\tau = \lim_{\Delta \to 0} \sum_i f(k\Delta)\Delta$. It implicitly assumes that the limit and the integral both exist.

# Zero-state response of linear systems

If a system is causal, the output will not appear before the input is applied. Thus

$$\boxed{\text{Causal} \iff g(t,\tau) = 0 \text{ for } t < \tau} \implies y(t) = \int_{t_0}^{t} g(t,\tau)u(\tau)d\tau$$

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**Theorem (Impulse Response)**

Consider a continuous-time linear system with $m$ inputs and $p$ outputs. There exists a matrix-valued signal $G(t,\tau) \in \mathbb{R}^{p\times m}$ such that for every input $u$ a corresponding output is given by

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If the system is time-invariant as well, then

$$G(t,\tau) = G(t+T,\tau+T) = G(t-\tau,0) = G(t-\tau) \text{ for any } T$$

and assuming $t_0 = 0$

$$y(t) = \int_{0}^{t} G(t-\tau)u(\tau)d\tau \triangleq (G \star u)(t) \;\forall t \geq 0$$

where $\star$ denotes the convolution operator.

# Zero-state response of linear systems

For Discrete-time systems:

**Theorem (Impulse Response)**

Consider a discrete-time linear system with $m$ inputs and $p$ outputs. There exists a matrix valued function $G(t,\tau) \in \mathbb{R}^{p \times m}$ such that for every input $u$ a corresponding output is given by

$$y(t) = \sum_{t_0}^{t} G(t,\tau)u(\tau); \ \forall t \geq t_0, t, \tau \in \mathbb{N}$$

If the system is time-invariant as well, then the time-shifting property holds and assuming $t_0 = 0$

$$y(t) = \sum_{0}^{t} G(t-\tau)u(\tau)d\tau \triangleq (G \star u)(t) \ \forall t \in \mathbb{N} \geq 0$$

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Taking its Laplace transform, one obtains

$$\hat{y}(s) = \int_0^\infty \int_0^\infty e^{-st} G(t-\tau)u(\tau)d\tau dt$$

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But because of causality,

$$\int_0^\infty e^{-s(t-\tau)}G(t-\tau)dt = \int_{-\tau}^\infty e^{-s\bar{t}}G(\bar{t})d\bar{t} = \int_0^\infty e^{-s\bar{t}}G(\bar{t})d\bar{t} = \hat{G}(s) \tag{2}$$

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Substituting (2) into (1) and removing $\hat{G}(s)$ from the integral, we conclude that

$$\hat{y}(s) = \int_0^\infty \hat{G}(s)e^{-s\tau} u(\tau)d\tau = \hat{G}(s) \int_0^\infty e^{-s\tau} u(\tau)d\tau = \hat{G}(s)\hat{u}(s)$$

# Zero-state response of linear systems: Transfer Function

**Definition (Transfer function)**

The transfer function of a CT causal LTI system is the Laplace transform

$$\hat{G}(s) = \mathcal{L}[G(t)] = \int_0^\infty e^{-st} G(t) dt, \quad s \in \mathbb{C}$$

of an impulse response $G(t_2, t_1) = G(t_2 - t_1), \forall t_2 \geq t_1 \geq 0$.

**Definition (Transfer function)**

The transfer function of a DT causal LTI system is the $\mathcal{Z}$-transform

$$\hat{G}(z) = \mathcal{Z}[G(t)] = \sum_{t=0}^{\infty} z^{-t} G(t), \quad z \in \mathbb{C}$$

of an impulse response $G(t_2, t_1) = G(t_2 - t_1), \forall t_2 \geq t_1 \geq 0$.

# State-space systems

State-space representation of linear systems

Using the state variable, as introduced earlier, a continuous-time state-space linear system is represented by the following two equations:

$$\left.\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}\right\} \triangleq \text{LTV} \qquad (3)$$

where

$$u : [0, \infty) \to \mathbb{R}^m, \quad x : [0, \infty) \to \mathbb{R}^n, \quad y : [0, \infty) \to \mathbb{R}^p$$

are called the input, state, and output signals of the system and the time-varying matrices $(A, B, C, D)(t)$ are of appropriate dimensions.

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**Note:**

- The first equation of (3) is called the *state equation* and the second equation of (3) is called *the output equation*.
- when all the matrices $(A, B, C, D)(t)$ are constant $\forall t \geq 0$, the system is LTI

# Interconnections

Interconnections of block diagrams are especially useful to highlight special structures in state-space equations.

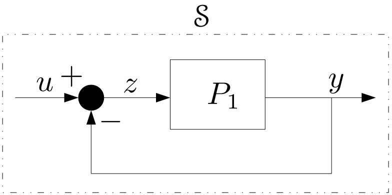


Figure: Negative feedback interconnection

Given $P_1 : z \mapsto y \quad \dot{x} = A_1 x + B_1 z, \quad y_1 = C_1 x + D_1 z$

Compute $\mathcal{S} : u \mapsto y$

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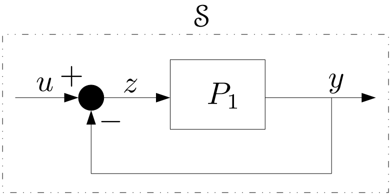


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Compute $\mathcal{S} : u \mapsto y$

$$\begin{aligned} \dot{x} &= (A_1 - B_1(I + D_1)^{-1}C_1)x + B_1(I - (I + D_1)^{-1}D_1)u \\ y &= (I + D_1)^{-1}C_1 x + (I + D_1)^{-1}D_1 u \end{aligned}$$

——— Show By Yourself! ———

# Impulse Response and Transfer function for LTI system

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Coming back to the time domain by applying inverse Laplace transforms, we obtain

$$y(t) = \Psi(t)x(0) + (G \star u)(t) \quad \text{where} \quad \begin{aligned} G(t) &= \mathcal{L}^{-1}[\hat{G}(s)], \\ \Psi(t) &= \mathcal{L}^{-1}[\hat{\Psi}(s)]. \end{aligned}$$

# Impulse Response and Transfer function for LTI system

**Theorem (In continuous-time domain)**

*The impulse response and transfer function of the CLTI system are given by:*

$$G(t) = \mathcal{L}^{-1}\left[C(sI - A)^{-1}B + D\right] \quad \textit{and} \quad \hat{G}(s) = C\left(sI - A\right)^{-1}B + D$$

*respectively Moreover, the response* $y(t) = (G \star u)(t)$ *corresponds to the zero initial condition* $x(0) = 0$*.*

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### Theorem (In discrete-time domain)

*The impulse response and transfer function of the DLTI system are given by:*

$$G(t) = \mathcal{Z}^{-1}\left[C(zI - A)^{-1}B + D\right] \quad \textit{and} \quad \hat{G}(z) = C\,(zI - A)^{-1}\,B + D$$

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Laplace transforms can be used for solving the LTI state-space systems, however for time-varying linear systems, this tool cannot be used

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1. The Laplace transform of $G(t, \tau)$ is a function of two variables
2. $\mathcal{L}\left[A(t)x(t)\right] \neq \mathcal{L}[A(t)]\mathcal{L}[x(t)]$

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First we will see the solution of LTV systems and then tailor it for LTI systems

In the last lecture, we discussed

- Key properties of dynamical systems and the physical significance of the state
- Zero-state response of linear (TI and TV) systems in (CT and DT ) -domain
- Zero-state response of LTI system in frequency domain and its relation with the state-space representation

# Solution to homogeneous LTV systems

We start by considering the solution to a CTLTV system with a given initial condition but zero input

$$\dot{x}(t) = A(t)x(t); \qquad x(t_0) = x_0 \in \mathbb{R}^n; \qquad t \geq 0 \tag{4}$$

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A key property of homogeneous systems is that the map from the initial condition $x(t_0) = x_0 \in \mathbb{R}^n$ to the solution $x(t) \in \mathbb{R}^n$ at a given time $t \geq 0$ is always $linear$.

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## Theorem (Peano-Baker Series)

*The unique solution to* (4) *is given by* $x(t) = \phi(t, t_0)x_0, x_0 \in \mathbb{R}^n, t \geq 0$ *where*

$$\phi(t, t_0) = I + \int_{t_0}^{t} A(\tau_1)d\tau_1 + \int_{t_0}^{t} \left( A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2)d\tau_2 \right) d\tau_1 + \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2) \int_{t_0}^{\tau_2} A(\tau_3)d\tau_3 d\tau_2 d\tau_1 + \dots$$

*The* $n \times n$ *matrix* $\phi(t, t_0)$ *is called the state transition matrix.*

# Solution to homogeneous LTV systems

## Properties of the state-transition matrix

1. For every $t_0 \geq 0$, $\phi(t, t_0)$ is the unique solution to

$$\frac{d}{dt}\phi(t, t_0) = A(t)\phi(t, t_0) \qquad \phi(t_0, t_0) = I, \qquad t \geq 0.$$

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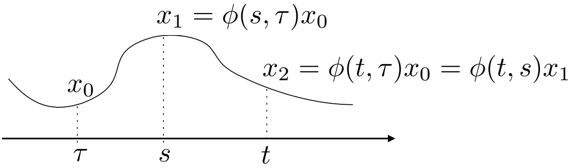
2. For every $t, s, \tau \geq 0$,

$$\phi(t, s)\phi(s, \tau) = \phi(t, \tau)$$



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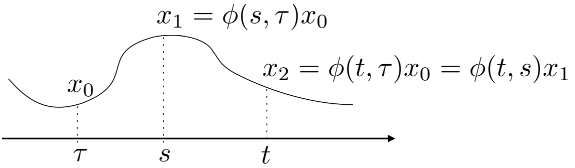


3. For every $t, \tau \geq 0$, $\phi(t, \tau)$ is non-singular and $\phi(t, \tau)^{-1} = \phi(\tau, t)$

# Computation of $\phi(t, t_0)$

Consider

$$\dot{x} = A(t)x \tag{5}$$

where $A \in \mathbb{R}^{n \times n}$ is a continuous function, then for every initial state $x^i(t_0) \in \mathbb{R}^n$, there exists a unique solution $x^i(t) \in \mathbb{R}^n$ for $i = 1, 2, 3, \ldots, n$

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1. Arrange these $n$ solutions as $X = \begin{bmatrix} x^1 & x^2 & \ldots & x^n \end{bmatrix}$ a square matrix of order $n$. Because every $x^i$ satisfies (5), we have

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**Question**

1. Is $X(t)$ unique?
2. Is $X(t)$ non-singular for all $t$ ?

# Computation of $\phi(t, t_0)$

**Example:** Consider the homogeneous equation

$$\dot{x}_1 = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} x(t)$$

or

$$\dot{x}_1(t) = 0, \qquad \dot{x}_2 = t x_1(t)$$

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$$x_2(t) = \int_0^t \tau x_1(0) d\tau + x_2(0) = 0.5 t^2 x_1(0) + x_2(0)$$

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Choose

$$x^1(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$x^2(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The two initial states are linearly independent.

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Thus

$$x^1(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 \\ 0.5t^2 \end{bmatrix} = x^1(t)$$

and

$$x^2(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \implies \begin{bmatrix} 1 \\ 0.5t^2 + 2 \end{bmatrix} = x^2(t)$$

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The two initial states are linearly independent. Thus

$$X(t) = \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix}$$

# Computation of $\phi(t, t_0)$

**Theorem**

*Let $X(t)$ be a fundamental matrix of $\dot{x} = A(t)x$. Then*

$$\phi(t, t_0) = X(t)X^{-1}(t_0).$$

Because $X(t)$ is non-singular for all $t$, its inverse is well defined

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Revisit the last example:

$$X(t) = \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix}$$

The state-transition matrix is given by

$$\phi(t, t_0) = \begin{bmatrix} 1 & 0 \\ 0.5(t^2 - t_0^2) & 1 \end{bmatrix}$$

Verify the three earlier listed properties of $\phi(t, t_0)$.

# Solution of non-homogeneous LTV systems

We now go back to the original non-homogeneous LTV system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}, \qquad x(t_0) = x_0 \in \mathbb{R}^n,\ t \geq 0 \tag{6}$$

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**Theorem (Variation of constants)**

*The unique solution to (6) is given by*

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau \tag{7}$$

$$y(t) = C(t)\phi(t, t_0)x_0 + C(t)\int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t) \tag{8}$$

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*where $\phi(t, t_0)$ is the state-transition matrix.*

- Equation (7) is known as the *variation of constants formula.*
- Homogeneous response: $y_{zi}(t) \equiv y_h(t) = C(t)\phi(t, t_0)x_0$
- Forced response: $y_{zs}(t) \equiv y_f(t) = C(t)\int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)$

# Solution of non-homogeneous LTV systems

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## Proof

To verify (7) is a solution to (6), note that at $t = t_0$, the integral in (7) disappears and we get $x(t_0) = x_0$.

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Taking the derivative of (7), we obtain

$$\begin{aligned}\dot{x} &= \frac{d\phi(t, t_0)}{dt}x_0 + \phi(t, t)B(t)u + \int_{t_0}^{t} \frac{d\phi(t, \tau)}{dt}B(\tau)u(\tau)d\tau \\ &= A(t)\phi(t, t_0)x_0 + B(t)u(t) + A(t)\int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau \\ &= A(t)x(t) + B(t)u(t)\end{aligned}$$

which shows that (7) is indeed a solution to (6).

# Solution of non-homogeneous LTV systems

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}, \qquad x(t_0) = x_0 \in \mathbb{R}^n,\ t \geq 0 \tag{6}$$

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(8) is obtained by direct substitution of $x(t)$ in $y(t) = C(t)x + D(t)u$.

# Solution of non-homogeneous LTV systems: Facts

Relation between input-output and state-space descriptions:

The zero-state response is given as

$$y_{zs}(t) = C(t)\int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

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and is equivalent to

$$y_{zs}(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

implying

$$G(t,\tau) \triangleq C(t)\phi(t,\tau)B(\tau) + D(t)\delta(t-\tau)$$

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Computing $\phi(t, t_0)$ is generally difficult

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If $A(t)$ is triangular such as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & 0 \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

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Because $x_1(t)$ has been solved, the preceding scalar equation can be solved for $x_2(t)$. This is what we did in the example on slide# 26.

# Solution of homogeneous **DT**LTV systems

$$x(t+1) = A(t)x(t), \qquad x(t_0) = x_0 \in \mathbb{R}^n, \qquad t \in \mathbb{N}$$

# Solution of homogeneous **DT**LTV systems

The (unique) solution to the homogeneous discrete-time linear time-varying system

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where

$$\phi(t, t_0) = \left\{ \begin{array}{ll} I, & \text{for } t = t_0 \\ A(t-1)A(t-2)\ldots A(t_0+1)A(t_0), & \text{for } t > t_0 \end{array} \right\}$$

is called the *(discrete-time) state transition matrix*

# Solution of homogeneous **DT**LTV systems

**Note**

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Properties of $\phi(t, t_0)$

1. For every $t_0 \geq 0, \phi(t, t_0)$ is the unique solution to
$$\phi(t+1, t_0) = A(t)\phi(t, t_0), \quad \phi(t_0, t_0) = I, \quad t \geq t_0$$
2. For every $t \geq s \geq \tau \geq 0$,
$$\phi(t, s)\phi(s, \tau) = \phi(t, \tau)$$
3. $\phi(t, t_0)$ may be singular

# Solution of non-homogeneous **DT**LTV systems

## Theorem (Variation of constants)

*The unique solution to*

$$x(t+1) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

*with* $x(t_0) = x_0 \in \mathbb{R}^n, t \in \mathbb{N}$ *is given by*

$$x(t) = \phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} \phi(t, \tau+1)B(\tau)u(\tau), \qquad \forall t \geq t_0$$

$$y(t) = C(t)\phi(t, t_0)x_0 + C(t)\sum_{\tau=t_0}^{t-1} \phi(t, \tau+1)B(\tau)u(\tau) + D(t)u(t), \forall t \geq t_0$$

*where* $\phi(t, t_0)$ *is the discrete-time state transition matrix.*

———–Show by yourself———–

# Solution of non-homogeneous LTV systems: Facts

Relation between input-output and state-space descriptions:

The zero-state response is given as

$$y_{zs}(t) = C(t) \sum_{\tau=t_0}^{t-1} \phi(t, \tau + 1) B(\tau) u(\tau) + D(t) u(t), \forall t \geq t_0$$

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$$y_{zs}(t) = \sum_{\tau=t_0}^{t} \left[ C(t)\phi(t, \tau+1) B(\tau) + D(\tau)\delta(t-\tau) \right] u(\tau)$$

where the impulse sequence $\delta(t-\tau)$ equals $1$ if $t = \tau$ and $0$ if $t \neq \tau$.

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where the impulse sequence $\delta(t-\tau)$ equals $1$ if $t = \tau$ and $0$ if $t \neq \tau$. Therefore,

$$G(t, \tau) \triangleq C(t) \phi(t, \tau+1) B(\tau) + D(\tau) \delta(t-\tau)$$

for $t \geq \tau$.

# Week 1 - Lecture 3

In the last lecture, we discussed

- solution of LTV state-space system in CT
- solution of LTV state-space system in DT
- properties and implications of these solution

# Solution to LTI systems: Homogeneous case

$$\dot{x} = Ax, \qquad x(t_0) = x_0 \in \mathbb{R}^n, \qquad t \geq 0 \tag{9}$$

# Solution to LTI systems: Homogeneous case

By applying the earlier results to the homogeneous *time-invariant* systems

$$\dot{x} = Ax, \qquad x(t_0) = x_0 \in \mathbb{R}^n, \qquad t \geq 0 \tag{9}$$

we have the following result.

## Theorem (Peano-Baker Series)

*The unique solution to* (9) *is given by* $x(t) = \phi(t, t_0)x_0,\ x_0 \in \mathbb{R}^n,\ t \geq 0$ *where*

$$\phi(t, t_0) = I + \int_{t_0}^{t} A d\tau_1 + \int_{t_0}^{t} \left( A \int_{t_0}^{\tau_1} A d\tau_2 \right) d\tau_1 + \int_{t_0}^{t} A \int_{t_0}^{\tau_1} A \int_{t_0}^{\tau_2} A d\tau_3 d\tau_2 d\tau_1 + \dots$$

*The* $n \times n$ *matrix* $\phi(t, t_0)$ *is called the state transition matrix.*

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*The* $n \times n$ *matrix* $\phi(t, t_0)$ *is called the state transition matrix.*

Since

$$\int_{t_0}^{t} \int_{t_0}^{\tau_1} \cdots \int_{t_0}^{\tau_{k-2}} \int_{t_0}^{\tau_{k-1}} A^k d\tau_k d\tau_{k-1} \cdots d\tau_2 d\tau_1 = \frac{(t-t_0)^k}{k!} A^k$$

we conclude that

$$\phi(t, t_0) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} A^k.$$

# Solution to LTI systems: Homogeneous case

$$\boxed{\phi(t, t_0) = \sum_{k=0}^{\infty} \frac{(t - t_0)^k}{k!} A^k} \tag{10}$$

# Solution to LTI systems: Homogeneous case

$$\boxed{\phi(t,t_0)=\sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}A^k} \tag{10}$$

Define the matrix exponential of a given $n \times n$ matrix $M$ by

$$e^M=\sum_{k=0}^{\infty}\frac{1}{k!}M^k$$

# Solution to LTI systems: Homogeneous case

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Define the matrix exponential of a given $n \times n$ matrix $M$ by

$$e^M = \sum_{k=0}^{\infty} \frac{1}{k!} M^k$$

which allows us to rewrite (10) as

$$\phi(t, t_0) = e^{A(t-t_0)}$$

# Solution to LTI systems: Non-homogeneous case

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}, \qquad x(t_0) = x_0 \in \mathbb{R}^n, \qquad t \geq t_0$$

# Solution to LTI systems: Non-homogeneous case

From the variation of constants formula, the solution to

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}, \qquad x(t_0) = x_0 \in \mathbb{R}^n, \qquad t \geq t_0$$

is given by

$$\boxed{\begin{aligned}x(t) &= e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}B(\tau)u(\tau)d\tau, \\ y(t) &= Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}B(\tau)u(\tau)d\tau\end{aligned}}$$

# Properties of the matrix exponential

1. The function $e^{At}$ is the unique solution to

$$\frac{d}{dt}e^{At} = Ae^{At}, \qquad e^{A.0} = I, \qquad t \geq 0$$

2. For every $t, \tau \in \mathbb{R}$,

$$e^{At}e^{A\tau} = e^{A(t+\tau)}$$

   In general, $e^{At}e^{Bt} \neq e^{(A+B)t}$

3. For every $t \in \mathbb{R}, e^{At}$ is nonsingular and

$$\left(e^{At}\right)^{-1} = e^{-At}$$

4. For every $n \times n$ matrix $A$,

$$Ae^{At} = e^{At}A, \qquad \forall t \in \mathbb{R}$$

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Taking the Laplace Transform, we conclude that

$$\begin{aligned}
\mathcal{L}\left[\frac{d}{dt}e^{At}\right] &= \mathcal{L}\left[Ae^{At}\right] \\
s\widehat{e^{At}} - e^{At}\Big|_{t=0} &= A\widehat{e^{At}} \\
(sI - A)\widehat{e^{At}} &= I \\
\widehat{e^{At}} &= (sI - A)^{-1}
\end{aligned}$$

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(sI - A)\widehat{e^{At}} &= I \\
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\end{aligned}$$

Therefore,

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

# Importance of the Characteristic polynomial

$$(sI - A)^{-1} = \frac{1}{\mathtt{det}(sI - A)} \left[\mathtt{adj}(sI - A)\right]'$$

where

$$\mathtt{det}(sI - A) = (s - \lambda_1)^{m_1}(s - \lambda_2)^{m_2} \cdots (s - \lambda_k)^{m_k}$$

1. is the characteristic polynomial of $A$, whose roots $\lambda_i$ are the eigenvalues of $A$ and,
2. $\mathtt{adj}(sI - A)$ is the adjoint matrix of $sI - A$ whose entries are polynomials in $s$ of degree $(n - 1)$ or lower

To compute $\mathcal{L}^{-1}[(sI - A)^{-1}]$, we need to perform the partial fraction expansion.

# Importance of the Characteristic polynomial

These are of the forms

$$
\begin{aligned}
&\frac{\alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_k)^{m_k}} \\
&\qquad = \frac{a_{11}}{s-\lambda_1} + \frac{a_{12}}{(s-\lambda_1)^2} + \cdots + \frac{a_{1m_1}}{(s-\lambda_1)^{m_1}} + \cdots \\
&\qquad\qquad + \frac{a_{k1}}{(s-\lambda_k)} + \cdots + \frac{a_{km_k}}{(s-\lambda_k)^{m_k}}
\end{aligned}
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The inverse Laplace transform is then given by

$$\mathcal{L}^{-1}\left[\frac{\alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n}{(s-\lambda_1)^{m_1}\cdots(s-\lambda_k)^{m_k}}\right]$$
$$= a_{11}e^{\lambda_1 t} + \cdots + a_{1m_1}t^{m_1-1}e^{\lambda_1 t} + \cdots + a_{k1}e^{\lambda_k t} + \cdots + a_{km_k}t^{m_k-1}e^{\lambda_k t}$$

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\end{aligned}
$$

Thus when all the eigenvalues $\lambda_i$ of $A$ have strictly negative real parts, all entries of $e^{At}$ converge to zero as $t \to \infty$, i.e., $y(t)$ converges to the forced response

$$
y_f(t) = \int_{t_0}^{t} C e^{A(t-\tau)} B u(\tau) d\tau + D u(t)
$$

# Discretization

Consider the continuous-time state equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{11}$$

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For discretization, since

$$\dot{x}(t) = \lim_{T \to 0} \frac{x(t+T) - x(t)}{T}$$

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If we compute $x(t)$ and $y(t)$ only at $t = kT$ for $k = 1, 2, \ldots$, then

$$\begin{aligned} x((k+1)T) &= (I + TA)x(kT) + TBu(kT) \\ y(kT) &= Cx(kT) + Du(kT) \end{aligned}$$

This discretization is easy to carry out but yields the least accurate results for the same $T$.

# Discretization: Another method

Let

$$u(t) = u(kT) \triangleq u[k], \qquad \text{for } kT \leq t \leq (k+1)T \tag{12}$$

for $k = 0, 1, 2, \ldots$. This input changes values only at discrete-time instants.

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$$x[k] \triangleq x(kT) = e^{AkT}x(0) + \int_0^{kT} e^{A(kT-\tau)}Bu(\tau)d\tau \tag{13}$$

and

$$x[k+1] \triangleq x((k+1)T) = e^{A(k+1)T}x(0) + \int_0^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)d\tau$$

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Substituting (12) and (13) and introducing the new variable $\alpha = kT + T - \tau$, we get

$$x[k+1] = e^{AT}x[k] + \left(\int_0^T e^{A\alpha}d\alpha\right)Bu[k]$$

# Discretization: Another method

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$$\boxed{x[k+1] = e^{AT}x[k] + \left(\int_0^T e^{A\alpha}d\alpha\right)Bu[k]}$$

Thus,

$$x[k+1] = A_d x[k] + B_d u[k], \qquad y[k] = C_d x[k] + D_d u[k]$$

with

$$A_d = e^{At} \qquad B_d = \left(\int_0^T e^{A\alpha}d\alpha\right)B \qquad C_d = C \qquad D_d = D$$

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<u>Computation of $B_d$:</u>
Note that

$$\int_0^T \left(I + A\tau + A^2\frac{\tau^2}{2!} + \cdots\right)d\tau = TI + \frac{T^2}{2!}A + \frac{T^3}{2!}A^2 + \cdots$$

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If $A$ is nonsingular, then the series can be written as

$$A^{-1}\left(TA + \frac{T^2}{2!}A^2 + \frac{T^3}{2!}A^3 + \cdots + I - I\right) = A^{-1}\left(e^{AT} - I\right)$$

Thus, we have

$$B_d = A^{-1}(A_d - I)B$$

MATLAB code: $[\mathtt{ad}, \mathtt{bd}] = \mathtt{c2d}(\mathtt{a}, \mathtt{b}, \mathtt{T})$

# Solution of Discrete-time Equations

Consider the discrete-time state-space equations

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}, \qquad x(t_0) = x_0 \in \mathbb{R}^n, \quad k \geq 0$$

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Once $x[0]$ and $u[k], k = 0, 1, \ldots$, are given, the response can be computed recursively from the equations.

$$\begin{aligned} x[1] &= Ax[0] + Bu[0] \\ x[2] &= Ax[1] + Bu[1] = A^2x[0] + ABu[0] + Bu[1] \end{aligned}$$

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Proceeding forward, we can readily obtain, for $k > 0$,

$$x[k] = A^k x[0] + \sum_{m=0}^{k-1} A^{k-1-m} Bu[m]$$

$$y[k] = CA^k x[0] + \sum_{m=0}^{k-1} CA^{k-1-m} Bu[m] + Du[k]$$

State transition matrix, $\phi[k, k_0] = A^{k-k_0}, \forall k \geq k_0$

# Computation of $\phi[k, k_0]$

The matrix power can be computed using $\mathcal{Z}$-transforms.

$$\mathcal{Z}\left[A^{k+1}\right] \triangleq \sum_{k=0}^{\infty} z^{-k} A^{k+1} = z \sum_{k=0}^{\infty} z^{-(k+1)} A^{k+1} = z \left( \sum_{k=0}^{\infty} z^{-k} A^{k} - I \right)$$

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$$\mathcal{Z}\left[A^{k+1}\right] \triangleq \sum_{k=0}^{\infty} z^{-k} A^{k+1} = z \sum_{k=0}^{\infty} z^{-(k+1)} A^{k+1} = z\left(\sum_{k=0}^{\infty} z^{-k} A^{k} - I\right)$$
$$= z\left(\mathcal{Z}\left[A^{k}\right] - I\right)$$

Also, $\mathcal{Z}\left[A^{k+1}\right] = A\mathcal{Z}\left[A^{k}\right]$.

# Computation of $\phi[k, k_0]$

The matrix power can be computed using $\mathcal{Z}$-transforms.

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Taking inverse $\mathcal{Z}$-transform, we obtain

$$\boxed{A^k = \mathcal{Z}^{-1}\left[z(zI - A)^{-1}\right]}$$

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Taking inverse $\mathcal{Z}$-transform, we obtain

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Now, when all eigenvalues of $A$ have magnitude smaller than $1$, all entries of $A^k$ will converge to zero as $t \to \infty$, which means that the output will converge to the forced response.

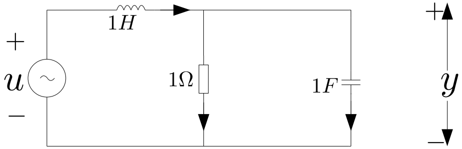
# Week 1 - Lecture 4

In the last lecture, we discussed

- Solution of LTI system both in CT and DT domain
- Two methods of discretization
- Importance of characteristic polynomial

# Equivalent state equations

Consider the network shown below:



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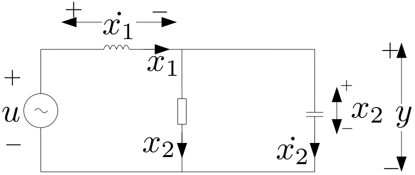


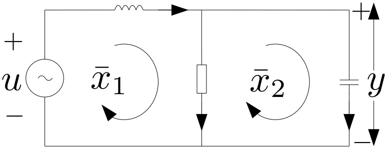


State variable, $x(t)$

$x_1$: Inductor current; $x_2$: Capacitor voltage

State variable, $\bar{x}(t)$

$\bar{x}_1, \bar{x}_2$: Loop currents

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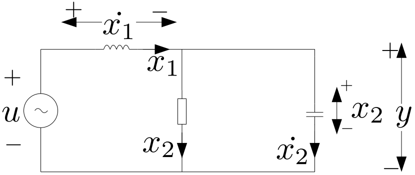


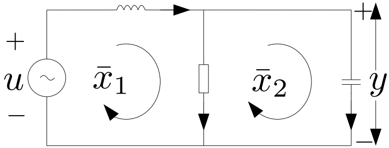


State variable, $x(t)$

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$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

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$\bar{x}_1, \bar{x}_2$: Loop currents

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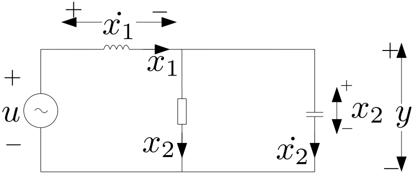


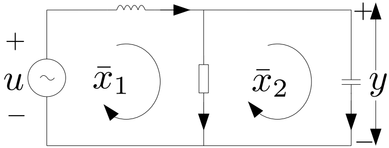


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**Problem**

Given two or more state-space equations, when can we say that these equations are equivalent or describe the same system?

# Equivalent LTI state equations

Consider the $n$-dimensional continuous-time LTI system

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

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The same system can be defined using $\bar{x}$ as the state,

$$\dot{\bar{x}} = T\dot{x} = TAx + TBu = TAT^{-1}\bar{x} + TBu$$
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which can be written as

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \qquad\qquad y = \bar{C}\bar{x} + \bar{D}u$$

for

$$\bar{A} \triangleq TAT^{-1}, \quad \bar{B} \triangleq TB, \quad \bar{C} \triangleq CT^{-1}, \quad \bar{D} \triangleq D$$

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Let $T$ be an $n \times n$ real nonsingular matrix and let $\bar{x} = Tx$, then the state equation

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**Property**

The equivalent transformations have the *same*

- set of eigenvalues
- transfer functions

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$$\begin{aligned}\bar{\Delta}(\lambda) = \det(\lambda I - \bar{A}) &= \det(\lambda T T^{-1} - TAT^{-1}) \\ &= \det\left[T\left(\lambda I - A\right)T^{-1}\right] = \det(\lambda I - A) = \Delta(\lambda)\end{aligned}$$

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$$\hat{\bar{G}}(s) = \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B} + \bar{D} = CT^{-1}\left[T(sI - A)T^{-1}\right]^{-1}TB + D$$
$$= \bar{C}T^{-1}T\left(sI - A\right)^{-1}T^{-1}TB + D = C\left(sI - A\right)^{-1}B + D = \hat{G}(s)$$

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**Definition (Zero-state equivalent)**

Two state equations are said to be zero state equivalent whenever they have the same transfer function matrix

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|---|---|---|

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|---|---|---|

Under what conditions we can ensure the zero-state equivalence?

# Markov parameters

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We know that

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Therefore,

$$\hat{G}(s) = C\,(sI - A)^{-1}\,B + D = D + \sum_{i=0}^{\infty} s^{-(i+1)} C A^i B$$

The matrices $D$, $CA^iB$, $i \geq 0$ are called the *Markov parameters*

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The matrices $D$, $CA^iB$, $i \geq 0$ are called the $Markov\ parameters$ which are also related to the system's impulse response i.e.

$$G(t) = \mathcal{L}^{-1}[\hat{G}(s)] = \mathcal{L}^{-1}[C(sI - A)^{-1}B + D] = Ce^{At}B + D\delta t$$

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Taking derivative of the RHS, we get

$$\frac{d^i G(t)}{dt^i} = CA^i e^{At} B, \qquad \forall i \geq 1, t > 0$$

from which we obtain the relationship: $\lim_{t\to 0} \frac{d^i G(t)}{dt^i} = CA^iB, \qquad \forall i \geq 1$

# Equivalent LTI state equations

## Theorem

*Two state-space representations*

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

*and*

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \qquad y = \bar{C}\bar{x} + \bar{D}u$$

*are* $zero\text{-}state\ equivalent$ *or have the same transfer function matrix if and only if they have the same Markov parameters i.e.,*

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Prove it by yourself!!

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**Definition (Algebraically Equivalent)**

Let $P(t) \in \mathbb{R}^{n\times n}$ be a non-singular matrix and both $P(t)$ and $\dot{P}(t)$ are continuous for all $t$. Let $\bar{x} \triangleq P(t)x$, then the state equation

$$\dot{\bar{x}} = \bar{A}(t)\bar{x} + \bar{B}(t)u, \qquad y = \bar{C}(t)\bar{x} + \bar{D}(t)u \tag{16}$$

where

$$\begin{aligned} \bar{A}(t) &= \left[P(t)A(t) + \dot{P}(t)\right] P^{-1}(t), & \bar{C}(t) &= C(t)P^{-1}(t) \\ \bar{B}(t) &= P(t)B(t), & \bar{D}(t) &= D(t) \end{aligned}$$

is said to be *algebraically equivalent* to (15) and $P(t)$ is called an *algebraic equivalent transformation*.

# Equivalent LTV state equations

**Theorem (Equivalence of fundamental matrix)**

*Let $X(t)$ be a fundamental matrix of* (15)*, then $\bar{X}(t) = P(t)X(t)$ is a fundamental matrix of* (16).

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\bar{A}(t) &= \left[P(t)A(t) + \dot{P}(t)\right] P^{-1}(t) \\
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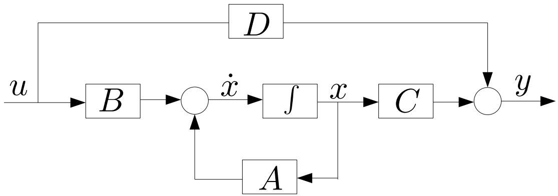
&= \left[e^{A_0 t}X^{-1}(t)A(t) + A_0 e^{A_0 t}X^{-1}(t) + e^{A_0 t}\dot{X}^{-1}(t)\right] X(t)e^{-A_0 t}
\end{aligned}$$

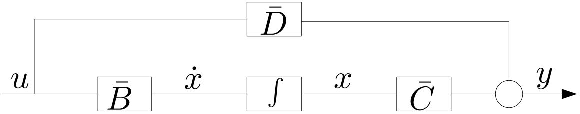
which becomes after substituting (17)

$$\bar{A}(t) = A_0 e^{A_0 t}X^{-1}(t)X(t)e^{-A_0 t} = A_0.$$

# Equivalent LTV state equations: Additional points

If $A_0$ is chosen as zero matrix, then $P(t) = X^{-1}(t)$, thus
$\bar{A}(t) = 0$, $\bar{B}(t) = X^{-1}(t)B(t)$, $\bar{C}(t) = C(t)X(t)$, $\bar{D}(t) = D(t)$



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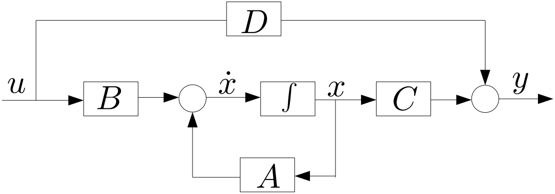


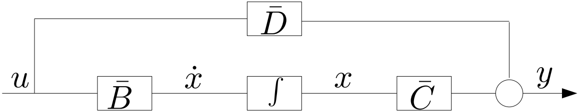


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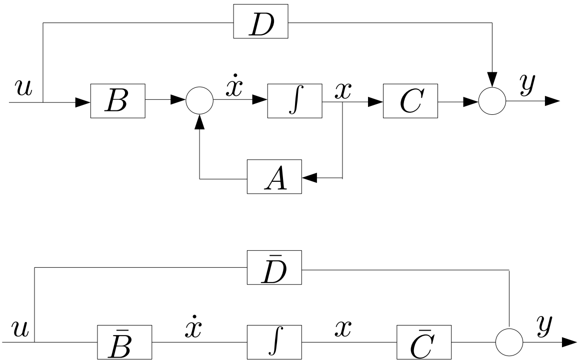


1. Every time-varying state equation can be transformed into such a block diagram
2. However, the challenge is to determine its fundamental matrix.

# Equivalent LTV state equations: Additional points

Invariance of Impulse Response matrix

$$\begin{aligned} G(t,\tau) &= C(t)\phi(t,\tau)B(\tau) + D(t)\delta(t-\tau) \\ &= C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau) \end{aligned}$$

Using the above substitutions, we get

$$\begin{aligned} \bar{G}(t,\tau) &= \bar{C}(t)\bar{X}(t)\bar{X}^{-1}(\tau)\bar{B}(\tau) + \bar{D}(t)\delta(t-\tau) \\ &= CP^{-1}PXX^{-1}(\tau)P^{-1}(\tau)P(\tau)B(\tau) + D(t)\delta(t-\tau) \\ &= CXX^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau) = G(t,\tau) \end{aligned}$$

Thus, the impulse response is invariant under any equivalence transformation.

# Concluding remarks on equivalence

## Definition (Lyapunov transformation)

A matrix $P(t)$ is called a Lyapunov transformation whenever

1. $P(t)$ is non singular
2. $P(t)$ and $\dot{P}(t)$ are continuous
3. $P(t)$ and $P^{-1}(t)$ are bounded for all $t$.

# Concluding remarks on equivalence

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1. $P(t)$ is non singular
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Recall

$$\dot{x} = A(t)x + B(t)u, \qquad y = C(t)x + D(t)u \tag{18}$$

$$\dot{\bar{x}} = \bar{A}(t)\bar{x} + \bar{B}(t)u, \qquad y = \bar{C}(t)\bar{x} + \bar{D}(t)u \tag{19}$$

**Definition (Lyapunov equivalent)**

Equations (18) and (19) are said to be Lyapunov equivalent whenever $P(t)$ is a Lyapunov transformation

In the last lecture, we discussed

- What is the equivalent representation problem?
- Algebraic equivalence of LTI and LTV systems
- Zero-state equivalence of LTI and LTV systems
- Relationship between algebraic and zero-state equivalence.

# Realization: LTI systems

Every LTI system can be described by the input-output description

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s)$$

and if the system is *lumped* as well, by input-system-output description

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

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**Realization problem**

Find a state-space equation from a given transfer matrix.

# Realization: LTI systems

## Definition (Realization)

A transfer function matrix $\hat{G}(\xi), \xi \in \{s, z\}$ is said to be *realizable* whenever there exists a finite-dimensional state equation or simply $\{A, B, C, D\}$ such that

$$\hat{G}(\xi) = C(\xi I - A)^{-1}B + D, \quad \xi \in \{s, z\}$$

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## Note

1. If $\hat{G}(\xi)$ is realizable then it has "infinitely" many realizations, not necessarily of the same dimension

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## Note

1. If $\hat{G}(\xi)$ is realizable then it has "infinitely" many realizations, not necessarily of the same dimension

   the realization problem is fairly complex

2. Here we shall study the "realizability condition" and compute one realization

# Realization: LTI systems

**Theorem**

*A transfer function matrix $\hat{G}(s)$ is realizable if and only if $\hat{G}(s)$ is a proper rational matrix.*

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The proof shall be done in two parts.

**Theorem (Necessary part)**

*If $\hat{G}(s)$ is realizable then $\hat{G}(s)$ is a proper rational matrix.*

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If $\hat{G}$ is realizable, then we can write

$$\hat{G}_{sp}(s) = C(sI - A)^{-1}B = \frac{1}{det(sI - A)} C[Adj(sI - A)]'B$$

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- Every entry of $Adj(sI - A)$ has at most degree $(n - 1)$

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- Every entry of $Adj(sI - A)$ has at most degree $(n - 1)$

Thus $C(sI - A)^{-1}B$ is a strictly proper rational matrix.

If D is non-zero, then $C(sI - A)^{-1}B + D \triangleq \hat{G}(s)$ is proper. $\square$

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Let

$$d(s) = s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1} s + \alpha_r$$

be the LCD of all entries of $\hat{G}_{sp}(s)$.

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be the LCD of all entries of $\hat{G}_{sp}(s)$.Then $\hat{G}_{sp}(s)$ can be expressed as

$$\hat{G}_{sp}(s) = \frac{1}{d(s)}[N(s)] = \frac{1}{d(s)}\left[N_1 s^{r-1} + N_2 s^{r-2} + \ldots + N_{r-1}s + N_r\right] \quad (20)$$

where $N_i$ are $q \times p$ constant matrix.

# Realization: LTI systems

## Theorem (Sufficient part)

*If $\hat{G}(s)$ is a proper rational matrix then $\hat{G}(s)$ is realizable.*

## Proof (Cont.)

We claim that the set of equations

$$\dot{x} = \begin{bmatrix} -\alpha_1 I_p & -\alpha_2 I_p & \dots & -\alpha_{r-1} I_p & -\alpha_r I_p \\ I_p & 0 & \dots & 0 & 0 \\ 0 & I_p & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_p & 0 \end{bmatrix} x + \begin{bmatrix} I_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \tag{21}$$

$$y = \begin{bmatrix} N_1 & N_2 & \dots & N_{r-1} & N_r \end{bmatrix} x + \hat{G}(\infty) u$$

where $I_p \in \mathbb{R}^{p\times p}, 0 \in \mathbb{R}^{p\times p}, A \in \mathbb{R}^{rp\times rp}, B \in \mathbb{R}^{rp\times p}, C \in \mathbb{R}^{q\times rp}$ is a realization of $\hat{G}(s)$ with dimension $rp$.

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# Realization: LTI systems

## Theorem (Sufficient part)

*If $\hat{G}(s)$ is a proper rational matrix then $\hat{G}(s)$ is realizable.*

## Proof (Cont.)

Let us define

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_r \end{bmatrix} \triangleq (sI - A)^{-1} B \tag{22}$$

where $Z_i$ is $p \times p$ and $Z$ is $rp \times p$.

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where $Z_i$ is $p \times p$ and $Z$ is $rp \times p$. Then the transfer matrix of (21) equals

$$C(sI - A)^{-1} B + \hat{G}(\infty) = N_1 Z_1 + N_2 Z_2 + \cdots + N_r Z_r + \hat{G}(\infty) \tag{23}$$

Write (22) as $(sI - A)Z = B$ or

$$sZ = AZ + B$$

## Theorem (Sufficient part)

*If $\hat{G}(s)$ is a proper rational matrix then $\hat{G}(s)$ is realizable.*

## Proof (Cont.)

Using the shifting property of the matrix $A$, from the second to the last block, we can readily obtain,

$$sZ = AZ + B \equiv s \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_r \end{bmatrix} = \begin{bmatrix} -\alpha_1 I_p & -\alpha_2 I_p & \dots & -\alpha_{r-1} I_p & -\alpha_r I_p \\ I_p & 0 & \dots & 0 & 0 \\ 0 & I_p & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_p & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_r \end{bmatrix} + \begin{bmatrix} I_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$sZ_2 = Z_1, \quad sZ_3 = Z_2, \quad \dots, \quad sZ_r = Z_{r-1}$$

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$$sZ_2 = Z_1, \quad sZ_3 = Z_2, \quad \dots, \quad sZ_r = Z_{r-1}$$

which implies

$$Z_2 = \frac{Z_1}{s}, \quad Z_3 = \frac{Z_1}{s^2}, \quad \dots, \quad Z_r = \frac{Z_1}{s^{r-1}}$$

Substituting these into the first block of $A$ yields

$$\begin{aligned} sZ_1 &= -\alpha_1 Z_1 - \alpha_2 Z_2 - \cdots - \alpha_r Z_r + I_p \\ &= -\left(\alpha_1 + \frac{\alpha_2}{s} + \cdots + \frac{\alpha_r}{s^{r-1}}\right) Z_1 + I_p \end{aligned}$$

# Realization: LTI systems

## Theorem (Sufficient part)

*If $\hat{G}(s)$ is a proper rational matrix then $\hat{G}(s)$ is realizable.*

## Proof (Cont.)

Using $d(s)$

$$\left(s + \alpha_1 + \frac{\alpha_2}{s} + \cdots + \frac{\alpha_r}{s^{r-1}}\right) Z_1 = \frac{d(s)}{s^{r-1}} Z_1 = I_p$$

Thus,

$$Z_1 = \frac{s^{r-1}}{d(s)} I_p, \quad Z_2 = \frac{s^{r-2}}{d(s)} I_p, \quad \cdots, \quad Z_r = \frac{1}{d(s)} I_p$$

Substituting these into

$$C(sI - A)^{-1} B + \hat{G}(\infty) = N_1 Z_1 + N_2 Z_2 + \cdots + N_r Z_r + \hat{G}(\infty)$$

yields

$$C\,(sI - A)^{-1}\,B + \hat{G}(\infty) = \frac{1}{d(s)}\left[N_1 s^{r-1} + N_2 s^{r-2} + \cdots + N_r\right] + \hat{G}(\infty)$$

□

# Realization: LTV systems

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$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau$$

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- input-state-output description

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}$$

If the state equation is available, the impulse response can be computed from

$$G(t,\tau) = C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau), \qquad \forall t \geq \tau$$

where $X(t)$ is the fundamental matrix.

# Realization: LTV systems

**Theorem**

*A $q \times p$ impulse response matrix $G(t,\tau)$ is* $realizable$ *if and only if $G(t,\tau)$ can be decomposed as*

$$G(t,\tau) = M(t)N(\tau) + D(t)\delta(t-\tau), \forall t \geq \tau$$

*where $M, N$ and $D$ are respectively $q \times n, n \times p$ and $q \times p$ matrices for some integer n.*

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Proof shall be done in two parts.

**Theorem (Necessary part)**

*If $G(t, \tau)$ is realizable then there exists a realization that satisfies*

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**Theorem (Sufficient part)**

*If $G(t, \tau)$ can be decomposed as mentioned above then $G(t, \tau)$ is realizable.*

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**Theorem (Sufficient part)**

*If $G(t,\tau)$ can be decomposed as mentioned above then $G(t,\tau)$ is realizable.*

**Proof.**

If $G(t,\tau)$ can be decomposed as above, then the $n-$dimensional state equation

$$\dot{x} = N(t)u, \qquad y = M(t)x + D(t)u$$

is a realization. Indeed, a fundamental matrix of $\dot{x} = 0.x$ is $X(t) = I$. Thus the impulse response is

$$M(t)I.I^{-1}N(\tau) + D(s)\delta(t-\tau) = G(t,\tau)$$