Linear Dynamical Systems

Tutorial on State-space solution and realization

Outline

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2. Time-Domain Solution for LTI Systems (Lecture1, Slides 9-16; Lecture 3, Slides 37-46)
3. Realization of an LTI system (Lecture 5, Slides 63-71)
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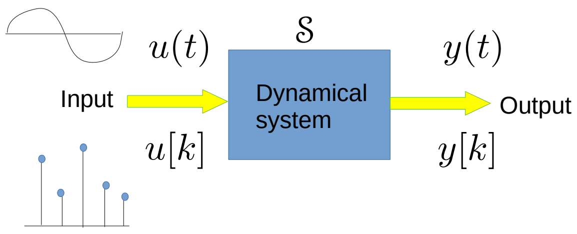
5. State Space representation of LTV systems (Lecture 1 Slides 4-7, 17)
6. State Space representation of LTV systems (Lecture 1 Slides 4-7, 17)
7. Equivalence of LTV systems (Lecture 4, Slides 56-61)
8. Time-Domain Solution for LTV Systems (Lecture 2, Slides 23-31)

State Space representation of LTI systems

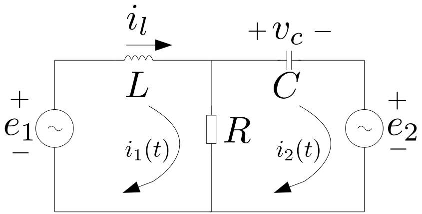
Problem 1

Consider the circuit shown below:



Find the state-space representation for the given circuit.

Solution to Problem 1



States: $i_l(t), v_c(t)$
Input: $e_1(t), e_2(t)$

$$e_1(t) = L\dot{i}_1(t)dt + R(i_1(t) - i_2(t)) \tag{1}$$

$$-e_2(t) = R(i_2(t) - i_1(t)) + v_c(t) \tag{2}$$

$$-e_2(t) = Ri_2(t) - Ri_1(t) + v_c(t)$$

Solution to Problem 1

$$
\begin{aligned}
e_1(t) &= L\dot{i}_1(t)dt + R(i_1(t) - i_2(t)) && (1)\\
-e_2(t) &= R(i_2(t) - i_1(t)) + v_c(t) && (2)
\end{aligned}
$$

$$i_1(t) = i_2(t) + \frac{v_c(t)}{R} + \frac{e_2(t)}{R} \tag{3}$$

Substituting (3) in (1),

$$e_1(t) = L\dot{i}_1(t) + R\left(i_2(t) + \frac{v_c(t)}{R} + \frac{e_2(t)}{R} - i_2(t)\right)$$

$$\dot{i}_1(t) = \frac{e_1(t)}{L} - \frac{e_2(t)}{L} - \frac{v_c(t)}{L} \tag{4}$$

From (2),

$$-e_2(t) = RC\dot{v}_c(t) - Ri_1(t) + v_c(t)$$

Solution to Problem 1

$$\dot{i}_1(t) = \frac{e_1(t)}{L} - \frac{e_2(t)}{L} - \frac{v_c(t)}{L} \tag{4}$$

$$\dot{v}_c(t) = \frac{-e_2(t)}{RC} + \frac{i_1(t)}{C} - \frac{v_c(t)}{RC} \tag{5}$$

From (4) and (5),

$$\dot{x} = Ax(t) + Bu(t)$$

$$\begin{bmatrix} \dot{i}_l(t) \\ \dot{v}_c(t) \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \begin{bmatrix} i_l(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} 1/L & -1/L \\ 0 & -1/RC \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

Taking $y_1(t) = i_l(t)$, $y_2(t) = i_c(t)$

$$y(t) = Cx(t) + Du(t)$$

$$\implies \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1/R \end{bmatrix} \begin{bmatrix} i_l(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1/R \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

Time-Domain Solution for LTI Systems

Problem 2

Solve the state-space equations obtained in Problem 1 to obtain the
(i) zero-state response and
(ii) zero-input response of the system
(iii) overall response.
Assume $R = \frac{2}{3}\Omega$, $L = 1H$, $C = \frac{1}{2}F$.
Take

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

as the initial state for (i)
and

$$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mathbf{u}(t)$$

for (ii), where $\mathbf{u}(t)$ is the standard unit step signal.

Solution to Problem 2

$$\begin{bmatrix} \dot{i}_l(t) \\ \dot{v}_c(t) \end{bmatrix} = \begin{bmatrix} 0 & -1/L \\ 1/C & -1/RC \end{bmatrix} \begin{bmatrix} i_l(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} 1/L & -1/L \\ 0 & -1/RC \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

$C = \frac{1}{2}F$, $L = 1H$, $R = \frac{2}{3}\Omega$

$$A = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{3}{2} \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} i_l(0) \\ v_c(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solution to Problem 2

(i) Zero Input Response:

Lecture Slides: 37

For a homogeneous LTI System,

$$\phi(t, t_0) = e^{A(t-t_0)}, t \geq t_0$$

$$y_{zi} = C[e^{At}x(0)]$$

$$\begin{aligned}
x(t) &= e^{At}x(0) \\
\implies x(t) &= \begin{bmatrix} -e^{-2t} + 2e^{-t} & e^{-2t} - e^{-t} \\ -2e^{-2t} + 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ 2e^{-2t} \end{bmatrix} \\
y_{zi} &= C[e^{At}x(0)] \\
&= \begin{bmatrix} 1 & 0 \\ 1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} e^{-2t} \\ 2e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \end{bmatrix}, \; t \geq 0
\end{aligned}$$

Solution to Problem 2

(ii) Zero State Response:

Lecture Slides: 39-41

For a non-homogeneous LTI System,

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

$$y_{zs}(t) = C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t), \quad t \geq t_0$$

$$y_{zs}(t) = \mathcal{L}^{-1}\left[C(sI - A)^{-1}BU(s) + DU(s)\right]$$

$$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\mathbf{u}(t)$$

$$y_{zs} = \mathcal{L}^{-1}\left[C(sI - A)^{-1}BU(s) + DU(s)\right]$$

Solution to Problem 2

$$
\begin{aligned}
=&\mathcal{L}^{-1}\left[\begin{bmatrix}1 & 0\\ 1 & -\frac{3}{2}\end{bmatrix}\begin{bmatrix}s & 1\\ -2 & s+3\end{bmatrix}^{-1}\begin{bmatrix}1 & -1\\ 0 & -3\end{bmatrix}\begin{bmatrix}\frac{1}{s}\\ \frac{2}{s}\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & -\frac{3}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{s}\\ \frac{2}{s}\end{bmatrix}\right]\\
=&\mathcal{L}^{-1}\Bigg(\begin{bmatrix}1 & 0\\ 1 & -\frac{3}{2}\end{bmatrix}\frac{1}{s^2+3s+2}\begin{bmatrix}s+3 & -1\\ 2 & s\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & -\frac{3}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{s}\\ \frac{2}{s}\end{bmatrix}\\
&+\begin{bmatrix}0 & 0\\ 0 & -\frac{3}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{s}\\ \frac{2}{s}\end{bmatrix}\Bigg)\\
=&\mathcal{L}^{-1}\left(\begin{bmatrix}1 & 0\\ 1 & -\frac{3}{2}\end{bmatrix}\frac{1}{s^2+3s+2}\begin{bmatrix}s+3 & -s-\frac{3}{2}\\ 2 & -2-\frac{3s}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{s}\\ \frac{2}{s}\end{bmatrix}+\begin{bmatrix}0\\ \frac{-3}{s}\end{bmatrix}\right)\\
=&\mathcal{L}^{-1}\left(\begin{bmatrix}1 & 0\\ 1 & -\frac{3}{2}\end{bmatrix}\frac{1}{s^2+3s+2}\begin{bmatrix}-1\\ -3-\frac{2}{s}\end{bmatrix}+\begin{bmatrix}0\\ \frac{-3}{s}\end{bmatrix}\right)
\end{aligned}
$$

Solution to Problem 2

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2+3s+2}\begin{bmatrix}-1\\ \frac{6+7s}{2s}\end{bmatrix}+\begin{bmatrix}0\\ \frac{-3}{s}\end{bmatrix}\right)$$

$$= \mathcal{L}^{-1}\left(\begin{bmatrix}\frac{-1}{(s+2)(s+1)}\\ \frac{6+7s}{2s(s+2)(s+1)}\end{bmatrix}\right)+\mathcal{L}^{-1}\left(\begin{bmatrix}0\\ \frac{-3}{s}\end{bmatrix}\right)$$

$$\implies y_{zs} = \begin{bmatrix}-e^{-2t}-e^{-t}\\ \frac{3}{2}-2e^{-2t}+\frac{1}{2}e^{-t}\end{bmatrix}+\begin{bmatrix}0\\ -3\end{bmatrix}$$

$$y_{zs} = \begin{bmatrix}-e^{-2t}-e^{-t}\\ \frac{3}{2}-2e^{-2t}+\frac{1}{2}e^{-t}-3\end{bmatrix},\ t\geq 0$$

Solution to Problem 2

Overall Response $y(t) = y_{zi} + y_{zs}$

$$
\begin{aligned}
&= \begin{bmatrix} e^{-2t} \\ -2e^{2t} \end{bmatrix} + \begin{bmatrix} -e^{-2t} - e^{-t} \\ \frac{3}{2} - 2e^{-2t} + \frac{1}{2}e^{-t} - 3 \end{bmatrix} \\
&= \begin{bmatrix} -e^{-t} \\ -2e^{2t} + \frac{3}{2} - 2e^{-2t} + \frac{1}{2}e^{-t} - 3 \end{bmatrix},\ t \geq 0
\end{aligned}
$$

Realization of an LTI system

Problem 3

Find the state-space realization of the Transfer Matrix given below:

$$\hat{G}(s) = \frac{1}{s^2+s+1}\begin{bmatrix} s+1 & -s \\ s & -s^2-s \end{bmatrix}$$

Solution to Problem 3

Given,

$$\hat{G}(s) = \frac{1}{s^2+s+1}\begin{bmatrix} s+1 & -s \\ s & -s^2-s \end{bmatrix}$$

Lecture Slides: 67,68

Transfer matrix $\hat{G}(s)$ can be decomposed as ,

$$\hat{G}(s) = \hat{G}(\infty) + \hat{G}_{sp}(s).$$

where,

$$\hat{G}_{sp}(s) = \frac{1}{d(s)}[N(s)] = \frac{1}{d(s)}\left[N_1 s^{r-1} + N_2 s^{r-2} + ... + N_{r-1}s + N_r\right]$$

and

$$d(s) = s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r$$

Solution to Problem 3

$$\hat{G}(s) = \frac{1}{s^2+s+1}\begin{bmatrix} s+1 & -s \\ s & -s^2-s \end{bmatrix}$$

$$\hat{G}(\infty) = \lim_{s\to\infty}\hat{G}(s) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{G}(s) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{s^2+s+1}\begin{bmatrix} s+1 & -s \\ s & 1 \end{bmatrix}$$

$$\Longrightarrow \hat{G}(s) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{s^2+s+1}\left(\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}s + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

and

$$d(s) = s^2+s+1$$

Solution to Problem 3

Lecture Slide: 68

$$\dot{x} = \begin{bmatrix} -\alpha_1 I_p & -\alpha_2 I_p & \dots & -\alpha_{r-1} I_p & -\alpha_r I_p \\ I_p & 0 & \dots & 0 & 0 \\ 0 & I_p & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_p & 0 \end{bmatrix} x + \begin{bmatrix} I_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} N_1 & N_2 & \dots & N_{r-1} & N_r \end{bmatrix} x + \hat{G}(\infty) u$$

Using the above expressions,

$$\implies \dot{x} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u \qquad (1)$$

Solution to Problem 3

$$\implies \dot{x} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u \tag{1}$$

$$y = \begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} u \tag{2}$$

The above set of equations are a realization of $\hat{G}(s)$.

Equivalence of LTI systems

Problem 4

Given a state-space representation with

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \ , \ B = \begin{bmatrix} 1 & -1 \\ 0 & -3/2 \end{bmatrix}$$

and another representation with

$$\bar{A} = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} \ , \ \bar{B} = \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix}$$

Prove the equivalence of these two systems.

Solution to Problem 4

Given
$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & -3/2 \end{bmatrix}$
and $\bar{A} = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix}$
$\textbf{Method 1}$:

$$B = \begin{bmatrix} 1 & -1 \\ 0 & -3/2 \end{bmatrix}$$

$$R_2 \rightarrow 2R_2$$

$$B = \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix} = \bar{B}$$

Therefore, Transformation matrix : $T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Solution to Problem 4

Verifying value of T matrix,

$$
\begin{aligned}
TAT^{-1} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} = \bar{A}
\end{aligned}
$$

$\mathbf{Method\ 2}$: For the two systems to be equivalent:

$$
\bar{A} = TAT^{-1}, \bar{B} = TB
$$

$$
\implies \bar{B}B^{-1} = TT = \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -3/2 \end{bmatrix}^{-1}
$$

Solution to Problem 4

$$T = \begin{bmatrix} 1 & -1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -2/3 \\ 0 & -2/3 \end{bmatrix}$$

$$\implies T = \begin{bmatrix} 2 & 0 \\ 1.5 & 0.5 \end{bmatrix}$$

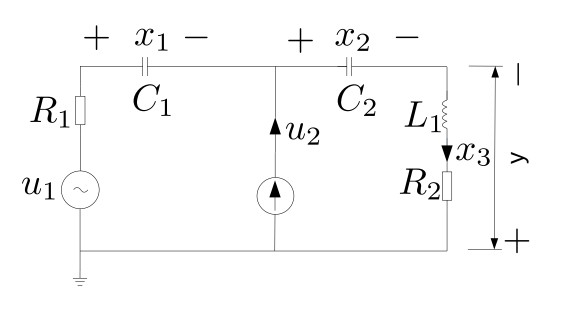
$$det T \neq 0$$

Since, T is a non-singular matrix, the corresponding two state equations are equivalent.

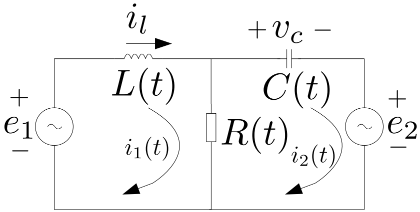
State Space representation of LTV systems

Problem 5

If the Resistance(R), Inductance(L), and Capacitance(C) of the circuit in Problem 1 are time-variant, find the state space representation of the corresponding LTV system.



Solution to Problem 5



Taking $R = R(t), L = L(t), C = C(t)$
States:$i_l(t), v_c(t)$
Input:$e_1(t), e_2(t)$

$$e_1(t) = \frac{d}{dt}(L(t)i_1(t)) + R(t)(i_1(t) - i_2(t)) \tag{1}$$

$$-e_2(t) = R(t)(i_2(t) - i_1(t)) + v_c(t) \tag{2}$$

Solution to Problem 5

$$e_1(t) = \frac{d}{dt}(L(t)i_1(t)) + R(t)(i_1(t) - i_2(t)) \tag{1}$$
$$-e_2(t) = R(t)(i_2(t) - i_1(t)) + v_c(t) \tag{2}$$

$$-e_2(t) = R(t)i_2(t) - R(t)i_1(t) + v_c(t)$$
$$i_1(t) = i_2(t) + \frac{v_c(t)}{R(t)} + \frac{e_2(t)}{R(t)} \tag{3}$$

Substituting (3) in (1),

$$e_1(t) = L(t)\dot{i}_1(t) + i_1(t)\dot{L}(t) + R(t)\left(i_2(t) + \frac{v_c(t)}{R(t)} + \frac{e_2(t)}{R(t)} - i_2(t)\right)$$

$$\dot{i}_1(t) = \frac{e_1(t)}{L(t)} - \frac{i_1(t)\dot{L}(t)}{L(t)} - \frac{v_c(t)}{L(t)} - \frac{e_2(t)}{L(t)} \tag{4}$$

Solution to Problem 5

$$-e_2(t) = R(t)(i_2(t) - i_1(t)) + v_c(t) \tag{2}$$

From equation (2),

$$-e_2(t) = R(t)(i_2(t) - i_1(t)) + v_c(t)$$

$$-e_2(t) = R(t)\frac{d}{dt}(C(t)v_c(t)) - R(t)i_1(t) + v_c(t)$$

$$-e_2(t) = R(t)C(t)\dot{v}_c(t) + R(t)v_c(t)\dot{C}(t) - R(t)i_1(t) + v_c(t)$$

$$\dot{v}_c(t) = \frac{-e_2(t)}{R(t)C(t)} - \frac{v_c(t)\dot{C}(t)}{C(t)} - \frac{v_c(t)}{R(t)C(t)} + \frac{i_1(t)}{C(t)} \tag{5}$$

Solution to Problem 5

$$\dot{i}_1(t) = \frac{e_1(t)}{L(t)} - \frac{i_1(t)\dot{L}(t)}{L(t)} - \frac{v_c(t)}{L(t)} - \frac{e_2(t)}{L(t)} \tag{4}$$

$$\dot{v}_c(t) = \frac{-e_2(t)}{R(t)C(t)} - \frac{v_c(t)\dot{C}(t)}{C(t)} - \frac{v_c(t)}{R(t)C(t)} + \frac{i_1(t)}{C(t)} \tag{5}$$

From equation (4) and (5),

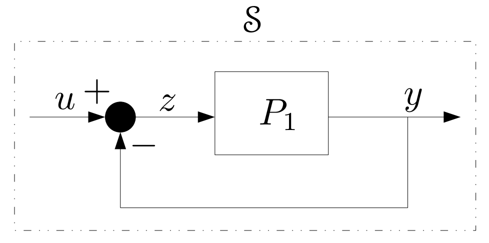
$$\begin{bmatrix} \dot{i}_l(t) \\ \dot{v}_c(t) \end{bmatrix} = \begin{bmatrix} -L^{-1}(t)\dot{L}(t) & -L^{-1}(t) \\ C^{-1}(t) & -[\dot{C}(t) + R^{-1}(t)]C^{-1}(t) \end{bmatrix} \begin{bmatrix} i_l(t) \\ v_c(t) \end{bmatrix}$$

$$+ \begin{bmatrix} L^{-1}(t) & -L^{-1}(t) \\ 0 & -R^{-1}(t)C^{-1}(t) \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

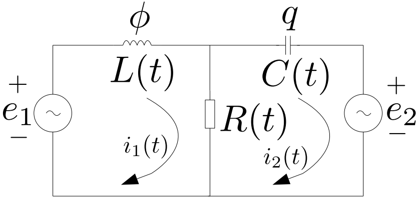
State Space representation of LTV systems

Problem 6

Taking $\phi(t)$(the magnetic flux through the inductor), $q(t)$(the charge on the capacitor) as the states, find the Time-Variant state space representation of the circuit given in Problem 1.



Solution to Problem 6



Taking $R = R(t), L = L(t), C = C(t)$
States:$\phi(t), q(t)$
Input:$e_1(t), e_2(t)$

$$v_l(t) = \frac{d\phi(t)}{dt}, i_2(t) = \frac{dq(t)}{dt}$$
$$\implies \phi(t) = \int v_l dt, q(t) = \int i_2 dt$$

Solution to Problem 6

$$e_1(t) = \frac{d}{dt}(L(t)i_1(t)) + R(t)(i_1(t) - i_2(t))$$

$$e_1(t) = \frac{d}{dt}\left(L(t)\frac{1}{L(t)}\int v_l dt\right) + \frac{R(t)\phi(t)}{L(t)} - R(t)\dot{q}(t)$$

$$e_1(t) = \dot{\phi}(t) + \frac{R(t)\phi(t)}{L(t)} - R(t)\dot{q}(t) \tag{1}$$

$$-e_2(t) = R(t)(i_2(t) - i_1(t)) + v_c(t)$$

$$-e_2(t) = R(t)\left(\dot{q}(t) - \frac{\phi(t)}{L(t)}\right) + \frac{q(t)}{C(t)}$$

$$-e_2(t) + \frac{R(t)\phi(t)}{L(t)} - \frac{q(t)}{C(t)} = R(t)\dot{q}(t)$$

$$\frac{-e_2(t)}{R(t)} + \frac{\phi(t)}{L(t)} - \frac{q(t)}{C(t)R(t)} = \dot{q}(t) \tag{2}$$

Solution to Problem 6

$$e_1(t) = \dot{\phi}(t) + \frac{R(t)\phi(t)}{L(t)} - R(t)\dot{q}(t) \tag{1}$$

$$\frac{-e_2(t)}{R(t)} + \frac{\phi(t)}{L(t)} - \frac{q(t)}{C(t)R(t)} = \dot{q}(t) \tag{2}$$

Replacing (2) in (1),

$$e_1(t) - e_2(t) - \frac{q(t)}{C(t)} = \dot{\phi}(t) \tag{3}$$

From (2) and (3),

$$\begin{bmatrix} \dot{\phi}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 0 & -C^{-1}(t) \\ L^{-1}(t) & R^{-1}(t)C^{-1}(t) \end{bmatrix} \begin{bmatrix} \phi(t) \\ q(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & -1 \\ 0 & -R^{-1}(t) \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

Equivalence of LTV systems

Problem 7

Prove the equivalence of State Space models obtained in Problem 5 and 6. Take: $L(t) = 0.5t$ H , $C(t) = 0.5t$ C, $R = 2\Omega$.

$$\begin{bmatrix} \dot{i}_l(t) \\ \dot{v}_c(t) \end{bmatrix} = \begin{bmatrix} -L^{-1}(t)\dot{L}(t) & -L^{-1}(t) \\ C^{-1}(t) & -[\dot{C}(t) + R^{-1}(t)]C^{-1}(t) \end{bmatrix} \begin{bmatrix} i_l(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} L^{-1}(t) & -L^{-1}(t) \\ 0 & -R^{-1}(t)C^{-1}(t) \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} \dot{\phi}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 0 & -C^{-1}(t) \\ L^{-1}(t) & R^{-1}(t)C^{-1}(t) \end{bmatrix} \begin{bmatrix} \phi(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & -R^{-1}(t) \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \tag{2}$$

Solution to Problem 7

Using $L(t) = 0.5t$ H , $C(t) = 0.5t$ C and $R = 2\Omega$

$$A = \begin{bmatrix} -1/t & -2/t \\ 2/t & -2/t \end{bmatrix}, B = \begin{bmatrix} 2/t & -2/t \\ 0 & -1/t \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 0 & -2/t \\ 2/t & -1/t \end{bmatrix}, \bar{B} = \begin{bmatrix} 1 & -1 \\ 0 & -0.5 \end{bmatrix}$$

Lecture Slide: 56

(A, B) and $(\bar{A}, \bar{B})$ are equivalent if there exists a non-singular matrix $P(t) \in \mathbb{R}^{2\times 2}$ such that:

$$\bar{A}(t) = [P(t)A(t) + \dot{P}(t)]P^{-1}(t) \quad (1)$$

$$\bar{B}(t) = P(t)B(t) \quad (2)$$

Solution to Problem 7

$$B = \begin{bmatrix} 2/t & -2/t \\ 0 & -1/t \end{bmatrix}$$

$$R_1 \to R_1 \frac{t}{2}, \quad R_2 \to R_2 \frac{t}{2}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & -0.5 \end{bmatrix} = \bar{B}$$

Elementary Row Matrices:

$$E_1(t) = \begin{bmatrix} t/2 & 0 \\ 0 & 1 \end{bmatrix}, E_2(t) = \begin{bmatrix} 1 & 0 \\ 0 & t/2 \end{bmatrix}$$

$$P(t) = E_1 E_2 = \begin{bmatrix} t/2 & 0 \\ 0 & t/2 \end{bmatrix}$$

Solution to Problem 7

To verify that $P(t)$ is the algebraic equivalent transformation matrix:
Substituting in (1),

$$\bar{A}(t) = [P(t)A(t) + \dot{P}(t)]P^{-1}(t)$$

RHS:

$$\left(\begin{bmatrix} t/2 & 0 \\ 0 & t/2 \end{bmatrix} \begin{bmatrix} -1/t & -2/t \\ 2/t & -2/t \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\right) \begin{bmatrix} 2/t & 0 \\ 0 & 2/t \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 2/t & 0 \\ 0 & 2/t \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2/t \\ 2/t & -1/t \end{bmatrix} = \bar{A}(t) \quad = \text{LHS. Hence Proved.}$$

Problem 8

Comment on the realizability of::

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

where,

$$A = \begin{bmatrix} 0 & -2/t \\ 2/t & -1/t \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 0 & -0.5 \end{bmatrix}$$

Use the concept of the fundamental matrix.

Solution to Problem 8

Given,

$$A = \begin{bmatrix} 0 & -2/t \\ 2/t & -1/t \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2$$

$$\implies \bar{A} = \begin{bmatrix} -4/t & 0 \\ 2/t & -1/t \end{bmatrix}$$

$$\dot{z}(t) = \bar{A}(t)z(t)$$

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} -4/t & 0 \\ 2/t & -1/t \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

$$\dot{z}_1(t) = -\frac{4}{t} z_1(t)$$

$$\dot{z}_2(t) = \frac{2}{t} z_1(t) - \frac{1}{t} z_2(t)$$

Solution to Problem 8

Taking $t_0 = 0$ and

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} z_1(t) &= z_1(t_0) + \frac{4}{t^2} z_1(t) \\ \implies \; z_1(t) &= \frac{1}{1 - \frac{4}{t^2}} \\ z_2(t) &= z_2(t_0) - \frac{2}{t^2} z_1(t) + \frac{1}{t^2} z_2(t) \end{aligned}$$

Substituting the value of $z_1(t)$

$$z_2(t) = -\frac{2}{t^2}\left(\frac{1}{1 - \frac{4}{t^2}}\right) + \frac{1}{t^2} z_2(t)$$

Solution to Problem 8

$$z_2(t)\left(1 - \frac{1}{t^2}\right) = -\frac{2}{t^2}\left(\frac{1}{1 - \frac{4}{t^2}}\right)$$

$$\implies z_2(t) = \frac{-2t^2}{(t^2 - 4)(t^2 - 1)}$$

Taking

$$\begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$z_1(t) = z_1(t_0) + \frac{4}{t^2} z_1(t)$$

$$z_1(t)\left(1 - \frac{4}{t^2}\right) = 0$$

$$\implies z_1(t) = 0$$

Solution to Problem 8

$$z_2(t) = z_2(t_0) - \frac{2}{t^2} z_1(t) + \frac{1}{t^2} z_2(t)$$

Substituting value of $z_1(t)$

$$z_2(t)\left(1 - \frac{1}{t^2}\right) = 1$$
$$\implies z_2(t) = \frac{1}{1 - \frac{1}{t^2}}$$

Fundamental Matrix $Z(t)$:

$$Z(t) = \begin{bmatrix} {z_1}^{(1)}(t) & {z_1}^{(2)}(t) \\ {z_2}^{(1)}(t) & {z_2}^{(2)}(t) \end{bmatrix} = \begin{bmatrix} \frac{t^2}{t^2-4} & 0 \\ \frac{-2t^2}{(t^2-4)(t^2-1)} & \frac{t^2}{t^2-1} \end{bmatrix}$$

Solution to Problem 8

Converting back to the original system using:
$X(t) = T^{-1}Z(t)$, where T is the elementary transformation matrix.

$$X(t) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{t^2}{t^2-4} & 1 \\ \frac{-2t^2}{(t^2-4)(t^2-1)} & \frac{t^2}{t^2-1} \end{bmatrix}$$

$$X(t) = \begin{bmatrix} \frac{t^2(t^2-5)}{(t^2-4)(t^2-1)} & \frac{2t^2}{t^2-1} \\ \frac{-2t^2}{(t^2-4)(t^2-1)} & \frac{t^2}{t^2-1} \end{bmatrix}$$

Recall

Recall from the lecture slide 74 that the impulse response $G(t,\tau)$ is realizable iff it can be decomposed as
$G(t,\tau) = M(t)N(\tau) + D(t)\delta(t-\tau)\ \forall t \geq \tau$

The impulse response for the system under consideration is :

$$G(t,\tau) = C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau)$$
$$= X(t)X^{-1}(\tau)B(\tau)$$

Clearly, by comparison, $M(t) = X(t)$, $N(\tau) = X^{-1}(\tau)B(\tau)$ and $D(t) = 0$ and hence the given impulse response is realizable.