Linear Dynamical Systems

Week 8 - Observer Design and Output Feedback

Outline of Week 8

1. State Estimation
    - Full-order design
    - Reduced-order design
2. Feedback from estimated states
3. State Estimation - Multivariable case
4. Unknown Input Observers (UIOs)

Introduction and Motivation

Problem Statement

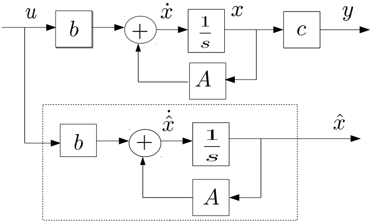
State estimation problem

Consider the n-dimensional state equation

$$\dot{x} = Ax(t) + bu(t), \qquad y = cx(t) \qquad \text{(CLTI)}$$

where A, b, c are given and the input $u(t)$ and output $y(t)$ are available to us. The state x, however is not available to us. The problem is to estimate x from u and y with the knowledge of A, b, c.

Introduction and Motivation



Introduction and Motivation

Gramians provide only the value of the state at a particular instant of time, instead of the continuous estimate.

Theorem (Gramian-based reconstruction)

Suppose we are given two times $t_1 > t_0 \geq 0$ *and an input/output pair* $u(t), y(t), \forall t \in [t_0, t_1]$*. When the system (CLTV) is observable*

$$x(t_0) = W_O(t_0, t_1)^{-1} \int_{t_0}^{t_1} \Phi(t, t_0)^T C(t)^T \tilde{y}(t) dt,$$

where

$$\tilde{y}(t) := y(t) - \int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau - D(t)u(t), \quad \forall t \in [t_0, t_1].$$

Introduction and Motivation

Two disadvantages in using the open-loop estimator

- the initial state must be computed and set each time we use the estimator.

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- the initial state must be computed and set each time we use the estimator.
- if the matrix A has eigenvalue with positive real parts, then even for a very small difference between $x(t_0)$ and $\hat{x}(t_0)$ for some t_0 which may be caused by a disturbance between $x(t)$ and $\hat{x}(t)$ will grow with time.

State Estimator

The open-loop estimator is now modified as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + bu(t) + l(y(t) - c\hat{x}(t))$$

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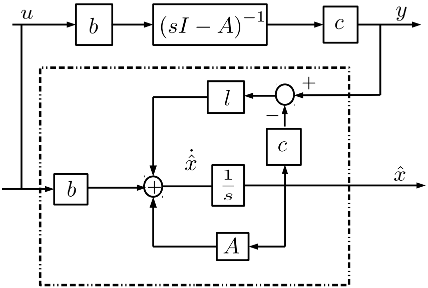

State Estimator

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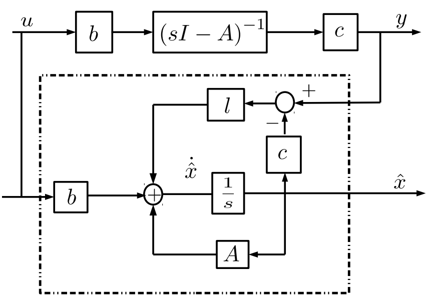
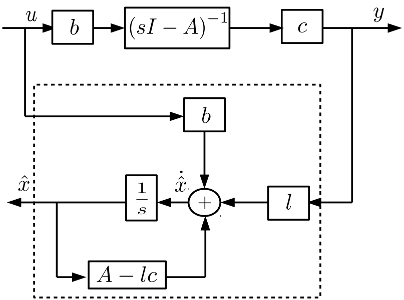
$$\dot{\hat{x}}(t) = A\hat{x}(t) + bu(t) + l(y(t) - c\hat{x}(t))$$

which can be written as

$$\dot{\hat{x}}(t) = (A - lc)\hat{x}(t) + bu(t) + ly(t). \tag{SE}$$





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&= (A - lc)x(t) - (A - lc)\hat{x}(t)) = (A - lc)(x(t) - \hat{x}(t))
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or,

$$\dot{e}(t) = (A - lc)e(t)$$

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Observation

If all eigenvalues of $(A - lc)$ can be assigned arbitrarily, then we can control the rate for $e(t)$ to approach zero or equivalently, for the estimated state to approach the actual state.

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Even if there is a large error between $\hat{x}(t_0)$ and $x(t_0)$ at the initial time t_0 the estimated state will approach the actual state rapidly. Thus, there is *no need to compute the initial state* of the original state equation.

State estimation

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + bu(t) + l(y(t) - c\hat{x}(t)) \\ &= (A - lc)\hat{x}(t) + bu(t) + ly(t). \end{aligned} \tag{SE}$$

Theorem

Consider the closed-loop state estimator (SE). If the output injection matrix gain $l \in \mathbb{R}^{n\times 1}$ makes $A - lc$ a stability matrix, then the state estimation error $e(t)$ converges to zero exponentially fast, for $every$ input signal $u(t)$.

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Note: The "correcting term" $l(\hat{y} - y)$ is used to correct any deviations of $\hat{x}$ from the true value x. When $\hat{x} = x$, we have $\hat{y} = y$ and this term disappears.

State estimation

Further questions

1. Does there exists a vector l?
2. How to compute l?
3. Under what conditions $A - lc$ is a stability matrix?
4. Can the eigenvalues of $A - lc$ be placed arbitrarily?
5. Can the eigenvalues of $A - lc$ be placed at least on the LHS of the complex plane?
6. ...

State Estimator

Theorem

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- ((A, c) is observable) $\iff$ ((A', c') is controllable)

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Observation

The procedure for computing state feedback gains can be used to compute the gain l in the state estimators.

Eigenvalue assignment by output injection

The following results can also be obtained by duality from the eigenvalue assignment results that we proved for controllable and stabilizable systems.

Theorem

When the system pair (A, c) is detectable, it is always possible to find a matrix gain $l \in \mathbb{R}^{n\times 1}$ such that $A - lc$ is a stability matrix.

Theorem

Assume that the pair (A, c) is observable. Given any set of n complex numbers $\lambda_1, \lambda_2, \ldots \lambda_n$ there exists a state feedback matrix $l \in \mathbb{R}^{n\times 1}$ such that $A - lc$ has the eigenvalues equal to the λ_i.

Lyapunov Equation Method

Consider n-dimensional state equation

$$\dot{x}(t) = Ax(t) + bu(t), \qquad y(t) = cx(t) \tag{CLTI}$$

1. Select an arbitrary $n \times n$ matrix F that has no eigenvalues in common with those of A.
2. Select an arbitrary $n \times 1$ vector l such that (F, l) is controllable.
3. Solve the unique T in the Lyapunov equation $TA - FT = lc$.
4. Then the state-space equation

$$\begin{aligned}\dot{z}(t) &= Fz(t) + Tbu(t) + ly(t)\\ \hat{x}(t) &= T^{-1}z(t)\end{aligned}$$

generates an estimate of x.

Lyapunov Equation Method

Justification of the procedure:

Let us define

$$e(t) := z(t) - Tx(t)$$

Lyapunov Equation Method

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Then we have, replacing TA by $FT + lc$,

$$\begin{aligned}\dot{e}(t) := \dot{z}(t) - T\dot{x}(t) &= Fz(t) + Tbu(t) + lcx(t) - TAx(t) - Tbu(t) \\ &= Fz(t) + lcx(t) - (FT + lc)x(t) = F(z(t) - Tx(t)) = Fe(t)\end{aligned}$$

If F is stable, for any $e(0)$, the error vector $e(t)$ approaches zero as $t \to \infty$. Thus z approaches $Tx(t)$ or, equivalently, $T^{-1}z(t)$ is an estimate of $x(t)$.

Full-order state estimator

Reduced-Dimensional State Estimator

Consider n-dimensional state equation

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If it is observable, then it can be transformed into the observable canonical form as

$$\dot{x} = \begin{bmatrix} -\alpha_1 & 1 & 0 & 0 \\ -\alpha_2 & 0 & 1 & 0 \\ -\alpha_3 & 0 & 0 & 1 \\ -\alpha_4 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

We see that y equals x_1.

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Therefore, it is sufficient to construct an $(n-1)$ dimensional state estimator to estimate x_i for $i = 2, 3, \ldots, n$. This estimator with output equation can then be used to estimate all n state variables . This estimator has a lesser dimension than (CLTI) and is called a *reduced-dimensional estimator*.

Reduced-Dimensional State Estimator

Reduced dimensional estimators can be designed by transformations or by solving Lyapunov equations.

- Select an arbitrary $(n-1) \times (n-1)$ stable matrix F that has no eigenvalues in common with those of A.
- Select an arbitrary $(n-1) \times 1$ vector l such that (F, l) is controllable.
- Solve the unique T in the Lyapunov equation $TA - FT = lc$. Note that T is an $(n-1) \times n$ matrix .
- Then the $(n-1)$-dimensional state equation

$$\dot{z}(t) = Fz(t) + Tbu(t) + ly(t)$$
$$\hat{x} = \begin{bmatrix} c \\ T \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$$

  is an estimate of $x(t)$.

Reduced-Dimensional State Estimator

Justification of the procedure:

We write $\hat{x} = \begin{bmatrix} c \\ T \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$ as

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} c \\ T \end{bmatrix} \hat{x}(t) =: P\hat{x}(t)$$

which implies $y = c\hat{x}(t)$ and $z = T\hat{x}(t)$. Clearly $y(t)$ is an estimate of $cx(t)$.

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$$e(t) = z(t) - Tx(t)$$

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Then we have

$$\dot{e}(t) = \dot{z}(t) - T\dot{x}(t) = Fz(t) + Tbu(t) + lcx(t) - TAx(t) - Tbu(t) = Fe(t)$$

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Clearly if F is stable, then $e(t) \to 0$ as $t \to \infty$. Thus, z is an estimate of Tx.

Reduced-Dimensional State Estimator

Theorem

If A and F have no common eigenvalues then the square matrix

$$P = \begin{bmatrix} c \\ T \end{bmatrix}$$

where T is the unique solution of $TA - FT = lc$, is nonsingular if and only if (A, c) is observable and (F, L) is controllable.

Feedback from estimated states

Consider a plant described by the n-dimensional state equation

$$\dot{x} = Ax + bu, \qquad y = cx \tag{CLTI}$$

If (A, b) is controllable state feedback $u = r - kx$ can place the eigenvalues of $(A - bk)$ in any desired positions.

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If the state variables are not available for feedback, we can design a state estimator.

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If the state variables are not available for feedback, we can design a state estimator.

If (A, c) is observable, a full or reduced dimensional estimator with arbitrary eigenvalue can be constructed.

Feedback from estimated states

Consider the n-dimensional state estimator

$$\dot{\hat{x}} = (A - lc)\hat{x} + bu + ly \qquad \text{(Estimator)}$$

The estimated state can approach the actual state with any rate by selecting the vector l.

Feedback from estimated states

Consider the n-dimensional state estimator

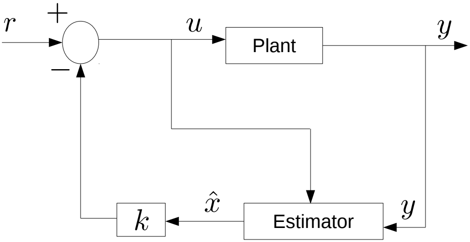
$$\dot{\hat{x}} = (A - lc)\hat{x} + bu + ly \qquad \text{(Estimator)}$$

The estimated state can approach the actual state with any rate by selecting the vector l.

If x is not available it is natural to apply the feedback gain to the estimated state as

$$u = r - k\hat{x} \qquad \text{(Controller)}$$

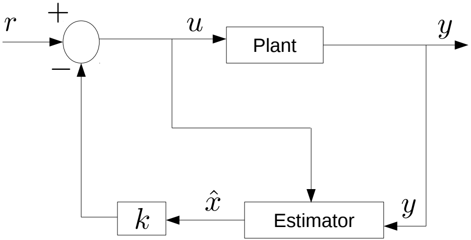
as shown in the figure below. The connection is called the *controller-estimator* configuration.



Feedback from estimated states

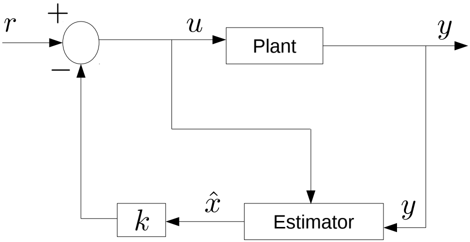
Questions raised in this connection

1. The eigenvalues of $(A - bk)$ are obtained from $u = r - kx$. Do we still have the same set of eigenvalues in using $u = r - k\hat{x}$?
2. Will the eigenvalues of the estimator be affected by the connection?
3. What is the effect of the estimator on the transfer function from r to y?.
4. ...



Feedback from estimated states

Let us develop a state equation to describe the overall system.



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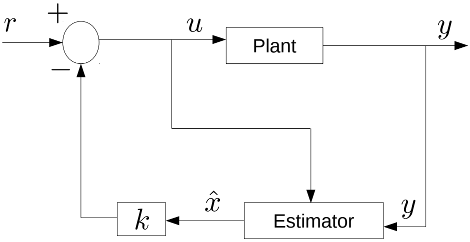

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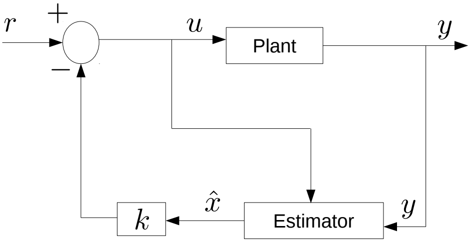

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Feedback from estimated states

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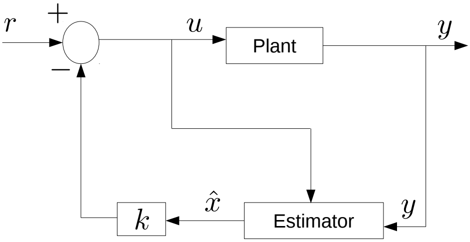

$$\dot{x} = Ax - bk\hat{x} + br$$
$$\dot{\hat{x}} = (A - lc)\hat{x} + b(r - k\hat{x}) + lcx$$

They can be combined as

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -bk \\ lc & A - lc - bk \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} r$$
$$y = \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

This $2n$ dimensional state equation describe the feedback system.

Feedback from estimated states

Let us introduce the following equivalence transformation

$$\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} =: P \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

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Computing P^{-1} which happens to equal P, we can obtain the following equivalent state equation

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} A - bk & bk \\ 0 & A - lc \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} r \\ y &= \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \end{aligned} \qquad \text{(Estimate-Control)}$$

Feedback from estimated states

Let us introduce the following equivalence transformation

$$\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} =: P \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

Computing P^{-1} which happens to equal P, we can obtain the following equivalent state equation

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} A - bk & bk \\ 0 & A - lc \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} r \\ y &= \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \end{aligned} \qquad \text{(Estimate-Control)}$$

Theorem (Seperation)

The closed-loop of the process (Estimate-Control) *with the output feedback controller results in a system whose eigenvalues are the union of the eigenvalues of the state feedback closed-loop matrix* $(A - bk)$ *with the eigenvalues of the state estimator matrix* $(A - lc)$*.*

Feedback from estimated states

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - bk & bk \\ 0 & A - lc \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} r$$

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Some observations

Feedback from estimated states

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Some observations

- Inserting the state estimator does not affect the eigenvalues of the original state feedback; nor are the eigenvalues of the state estimator affected by the connection.

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- Inserting the state estimator does not affect the eigenvalues of the original state feedback; nor are the eigenvalues of the state estimator affected by the connection.
- The design of the state feedback and the design of the estimator can be carried out *independently*.

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- Inserting the state estimator does not affect the eigenvalues of the original state feedback; nor are the eigenvalues of the state estimator affected by the connection.
- The design of the state feedback and the design of the estimator can be carried out *independently*.
- The state equation in (Estimate-Control) is not controllable.
- The transfer function of (Estimate-Control) equals the transfer function of the reduced equation

  $$\dot{x} = (A - bk)x + br, \qquad y = cx$$

  or,

  $$\hat{g}_f(s) = c(sI - A + bk)^{-1}b.$$
- The estimator is completely canceled in the transfer function from r to y.

State Estimators - Multivariable Case

Consider the n-dimensional p-input q-output state equation

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

The problem is to use available input u and output y to drive a system whose output gives an estimate of the state x. We extend the previous study to the multi-variable case as

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This is a full-dimensional state estimator.
Let us define the error vector as

$$e(t) := x(t) - \hat{x}(t)$$

Then we have

$$\dot{e} = (A - LC)e$$

If (A, C) is observable, then all eigenvalues of $(A - LC)$ can be assigned arbitrarily by choosing an L. Thus the convergence rate for the estimated state $\hat{x}$ to approach the actual state x can be as fast as desired.

Procedure for computing L - Reduced state estimator

Procedure for computing L - Reduced state estimator

Consider the n-dimensional q-output observable pair (A, C). It is assumed that C has rank q.

1. Select an arbitrary $(n-q) \times (n-q)$ stable matrix F that has no eigenvalues in common with those of A.
2. Select an arbitrary $(n-q) \times q$ matrix L such that (F, L) is controllable.
3. Solve the unique $(n-q) \times n$ matrix T in the Lyapunov equation $TA - FT = LC$
4. If the square matrix of order n

$$P = \begin{bmatrix} C \\ T \end{bmatrix}$$

   is singular, go back to step 2 and repeat the process.
5. If P is nonsingular, then the $(n-q)$-dimensional state equation

$$\dot{z} = Fz + TBu + Ly$$
$$\hat{x} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} y \\ z \end{bmatrix}$$

   generates an estimate of x.

Justification of the procedure

Let us write

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix} \hat{x}$$

which implies $y = C\hat{x}$ and $z = T\hat{x}$. Clearly y is an estimate of Cx. We now show that z is an estimate of Tx. Let us define

$$e := z - Tx$$

Then we have

$$\begin{aligned} \dot{e} = z - T\dot{x} &= Fz + TBu + LCx - TAx - TBu \\ &= Fz + (LC - TA)x = F(z - Tx) = Fe \end{aligned}$$

If F is stable, then $e(t) \to 0$ as $t \to \infty$. Thus z is an estimate of Tx.

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If F is stable, then $e(t) \to 0$ as $t \to \infty$. Thus z is an estimate of Tx.

Theorem (Necessary condition)

If A and F have no common eigenvalues, then the square matrix

$$P := \begin{bmatrix} C \\ T \end{bmatrix}$$

where T is the unique solution of $TA - FT = LC$, is non-singular only if (A, C) is observable and (F, L) is controllable.

Introduction - UIO

Problem statement

Consider a system in which the system uncertainty can be summarized as an additive unknown disturbance term as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \\ y(t) = Cx(t) \end{cases} \qquad \text{(disturbed-CLTI)}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^m$ is the output vector, $u(t) \in \mathbb{R}^r$ is the known input vector and $d(t) \in \mathbb{R}^q$ is the unknown input (or disturbance) vector. A, B, C and E are known matrices with appropriate dimensions.
The problem is to estimate the state of the system such that the disturbances have no effect on the state-estimation error.

Extended formulations

1. There is no loss of generality in assuming that the unknown input distribution matrix E should be full column rank. When this is not the case, the following rank decomposition can be applied to the matrix E

$$Ed(t) = E_1 E_2 d(t)$$

where E_1 is a full column rank matrix and $E_2 d(t)$ can now be considered as a new unknown input.

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2. The term $Ed(t)$ can be used to describe an additive disturbance as well as a number of other different kinds of modeling uncertainties. Examples are: noise, interconnecting terms in large scale systems, non-linear terms in system dynamics, terms arise from time-varying system dynamics, linearization and model reduction errors, parameter variations.

Extended formulations

3. The disturbance term may also appear in the output equation, i.e.,

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This case is not considered here because the disturbance term $E_y d(t)$ in the output equation can be nulled by simply using a transformation of the output signal $y(t)$, i.e.

$$y_E(t) = T_y y(t) = T_y Cx(t) + T_y E_y d(t) = T_y Cx(t)$$

where $T_y E_y = 0$, if one replaces $y(t)$ and C with $y_E(t)$ and $T_y C$, the problem will be equivalent to one without output disturbances.

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As the control input $u(t)$ is known, a new output can be constructed as:

$$\bar{y}(t) = y(t) - Du(t) = Cx(t)$$

If the output $y(t)$ is replaced by $\bar{y}(t)$, the problem will be equivalent to the one without the term $Du(t)$.

Unknown Input Observers

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \\ y(t) = Cx(t) \end{cases} \qquad \text{(disturbed-CLTI)}$$

Definition (Unknown Input Observer (UIO))

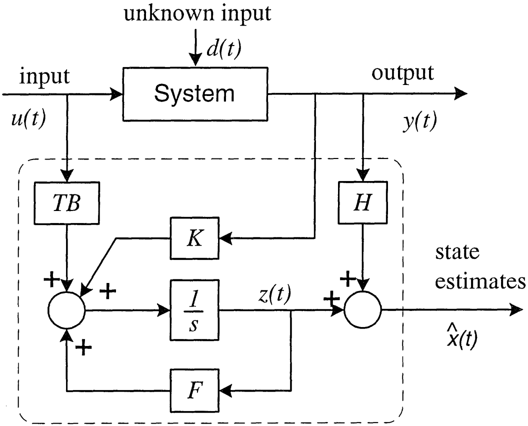
An observer is defined as an *unknown input observer* for the system described by (disturbed-CLTI), whenever its state estimation error vector $e(t)$ approaches zero asymptotically, regardless of the presence of the unknown input (disturbance) in the system.

Unknown Input Observers

The structure for a full-order observer is described as

$$\begin{cases} \dot{z}(t) = Fz(t) + TBu(t) + Ky(t) \\ \hat{x}(t) = z(t) + Hy(t) \end{cases} \qquad \text{(UIO)}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimated state vector and $z \in \mathbb{R}^n$ is the state of this full-order observer, and F, T, K, H are matrices *to be designed* for achieving unknown input de-coupling and other design requirements.



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When the observer (UIO) is applied to the system (disturbed-CLTI), the estimation error ($e(t) = x(t) - \hat{x}(t)$) is governed by the equation

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$$\begin{aligned} \dot{e}(t) = (A - HCA - K_1C)e(t) + [F - (A - HCA - K_1C)]z(t) \\ + [K_2 - (A - HCA - K_1C)H]y(t) \\ + [T - (I - HC)]Bu(t) + (HC - I)Ed(t) \end{aligned}$$

where

$$K = K_1 + K_2$$

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If one can make the following relations hold true:

$$(HC - I)E = 0 \tag{1}$$

$$T = I - HC \tag{2}$$

$$F = A - HCA - K_1C \tag{3}$$

$$K_2 = FH \tag{4}$$

Unknown Input Observers

The state estimation error will then be:

$$\dot{e}(t) = Fe(t)$$

If all eigenvalues of F are stable, $e(t)$ will approach zero asymptotically, i.e. $\hat{x} \to x$. This means that the observer (UIO) is an unknown input observer for the system.

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Questions to address

- Does a solution to eqs. (1-4) exists?
- How to compute it?
- How to ensure that F is Hurwitz?
- ...

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Theorem

Equation (1) is solvable if and only if

$$\mathtt{rank}(CE) = \mathtt{rank}(E)$$

and a special solution is

$$H^* = E[(CE)^T CE]^{-1}(CE)^T$$

Proof

Necessity:
When equation (1) has a solution H, one has $HCE = E$

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Sufficiency:

When $\texttt{rank}(CE) = \texttt{rank}(E)$ holds true, CE is a full column rank matrix (because E is assumed to be full column rank), and a left inverse of CE exists

$$(CE)^+ = [(CE)^T CE]^{-1}(CE)^T$$

Clearly $H = E(CE)^+$ is a solution to equation (1)

Unknown Input Observers

$$(HC - I)E = 0 \tag{1}$$

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Substituting H into (3), we get

$$\begin{aligned} F &= A - HCA - K_1C \\ &= (I_n - E(CE)^+C)A - K_1C \\ &= A_1 - K_1C \end{aligned}$$

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$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \\ y(t) = Cx(t) \end{cases} \qquad \text{(disturbed-CLTI)}$$

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Theorem

The necessary and sufficient conditions for (UIO) *to be a UIO for the system* (disturbed-CLTI) *are*

- $\mathtt{rank}(CE) = \mathtt{rank}(E)$
- (C, A_1) *is detectable pair, where*

$$A_1 = A - E[(CE)^T CE]^{-1}(CE)^T CA$$

Unknown Input Observers

$$(HC - I)E = 0 \quad (1)$$

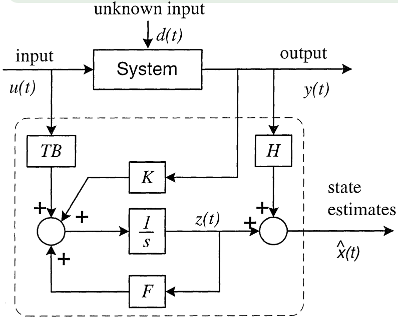
$$T = I - HC \quad (2)$$

$$F = A - HCA - K_1C \quad (3)$$

$$K_2 = FH \quad (4)$$

Observations

- K_1 is a free parameter in the design of a UIO. The only restriction on K_1 is that it must stabilize the system dynamics matrix F.
- The matrix K_1 is not unique.
- (UIO) will be a simple full-order Luenberger observer by setting $T = I$ and $H = 0$, when $E = 0$.



Design procedure for UIOs

One of the most important steps in designing a UIO is to stabilise $F = A_1 - K_1 C$ by choosing the matrix K_1, when the pair (C, A_1) is detectable.

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One of the most important steps in designing a UIO is to stabilise $F = A_1 - K_1 C$ by choosing the matrix K_1, when the pair (C, A_1) is detectable.

- If (C, A_1) is observable, this can be achieved easily by using a pole placement routine.
- If (C, A_1) is not observable, an observable canonical decomposition procedure should be applied to (C, A_1) which is

$$PA_1P^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix} \; ; A_{11} \in \mathbb{R}^{n_1 \times n_1}$$

$$CP^{-1} = \begin{bmatrix} C^* & 0 \end{bmatrix} \; ; C^* \in \mathbb{R}^{m \times n_1}$$

where n_1 is the rank of the observability matrix for (C, A_1), and (C^*, A_{11}) is observable.

If all eigenvalues of A_{22} are stable, (C, A_1) is detectable and the matrix F can be stabilized.

UIO design procedure

1. Check the rank condition for E and CE: If $\texttt{rank}(CE) \neq \texttt{rank}(E)$, a UIO does not exist, go to 10
2. Compute H, T and A_1:

$$H = E[(CE)^T CE]^{-1}(CE)^T; \qquad T = I - HC; \qquad A_1 = TA$$

3. Check the observability: If (C, A_1) observable, a UIO exists and K_1 can be computed using pole placement, go to 9.
4. Construct a transformation matrix P for the observable canonical decomposition: To select independent $n_1 = \texttt{rank}(W_0)$ (W_0 is the observability matrix of (C, A_1) row vector $p_1^T, \dots, p_{n_1}^T$ from W_0, together other $n - n_1$ row vector $p_{n_1+1}^T, \dots, p_n^T$ to construct a non-singular matrix as:

$$P = [p_1, \cdots, p_{n0}; p_{n_0+1}, \cdots, p_n]^T$$

5. Perform an observable canonical decomposition on (C, A_1):

$$PA_1P^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{12} & A_{22} \end{bmatrix} \qquad CP^{-1} = [C^* \quad 0]$$

6. Check the detectability of (C, A_1): If any one of the eigenvalues of A_{22} is unstable a UIO does not exist and go to 10.
7. Select n_1 desirable eigenvalues and assign them to $A_{11} - K_p^1 C^*$ using pole placement.
8. Compute $K_1 = P^{-1}K_p = P^{-1}\left[(K_p^1)^T \quad (K_p^2)^T\right]^T$ where K_p^2 can be any $(n - n_1) \times m$ matrix.
9. Compute F and K: $F = A_1 - K_1C$, $K = K_1 + K_2 = K_1 + FH$
10. STOP