Linear Dynamical Systems

Tutorial on State Feedback, Part-II

Outline

1. Cyclic Design (Lecture slides $43 - 52$)
2. Cyclic Design (Lecture slides $43 - 52$)
3. State Feedback Design for Multi-Input system
4. State Feedback and Disturbance Rejection (Lecture Slides $27 - 36$)
5. Feedback Invariant of Nonlinear system
6. Linear Quadratic Regulator(LQR) (Lecture slides $53 - 62$)

Cyclic Design

Problem 1

Given a plant defined by (A, B) pair,

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

Is matrix A cyclic? Can a state feedback controller be designed using single input variable controller design method?

Solution to Problem 1

Recall (Lecture Slide 43)

A matrix A is called *cyclic* whenever the Jordan form of A has one and only Jordan block associated with each distinct eigenvalue.

For

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The eigenvalues of A are $\{-1, -1, -1\}$, forming two Jordan blocks of size (2×2) and (1×1). Therefore, A is not cyclic.

Solution to Problem 1

Recall (Lecture Slide 43)

A matrix A is called *cyclic* whenever the Jordan form of A has one and only Jordan block associated with each distinct eigenvalue.

For

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The eigenvalues of A are $\{-1, -1, -1\}$, forming two Jordan blocks of size (2×2) and (1×1). Therefore, A is not cyclic.

Recall(Lecture slide 48)

If (A, B) is controllable, then for almost any $p \times n$ real constant matrix K, the matrix $(A - BK)$ has only distinct eigenvalues and is, consequently cyclic.

Further, we can also calculate the controllability matrix of the pair (A, B) as

$$\mathfrak{C} = \begin{bmatrix} 2 & 1 & -2 & 1 & 2 & -3 \\ 0 & 2 & 0 & -2 & 0 & 2 \\ 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}$$

which has a rank equal to 3. Therefore, pair (A, B) is controllable.

Solution to Problem 1

Recall(Lecture slide 48)

If (A, B) is controllable, then for almost any $p \times n$ real constant matrix K, the matrix $(A - BK)$ has only distinct eigenvalues and is, consequently cyclic.

Suppose K is arbitrarily selected as

$$K = \begin{bmatrix} 1 & 0.5 & -1 \\ -1 & 0.8 & 1 \end{bmatrix}$$

Then,

$$\begin{aligned}(A - BK) &= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & -1 \\ -1 & 0.8 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & -0.8 & 1 \\ 2 & -2.6 & -2 \\ -1 & -0.5 & 0 \end{bmatrix}\end{aligned}$$

which can be written in the Jordan canonical form as $\begin{bmatrix} -1.6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Therefore, $(A - BK)$ is cyclic.

Cyclic Design

Problem 2

Given a plant defined by (A, B) pair,

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 3 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

Is A cyclic? Comment on the controllability of (A, Bv) pair, where $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution to Problem 2

Recall (Lecture Slide 43)

A matrix A is called $cyclic$ whenever the Jordan form of A has one and only Jordan block associated with each distinct eigenvalue.

Given,

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 3 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

The eigenvalues of A are $\{4, 2, -1\}$, forming 3 jordan blocks of size (1×1). Therefore, A is cyclic as the sufficient condition is satisfied.

Solution to Problem 2

Recall(Lecture slide 44)

If the n-dimensional p-input pair (A, B) is controllable and if A is cyclic, then for almost any $p \times l$ vector v, the single-input pair (A, Bv) is controllable.

The controllability matrix of pair (A, B) is given as

$$\mathfrak{C} = \begin{bmatrix} 2 & 1 & -2 & 1 & 8 & 6 \\ 0 & 2 & 6 & 7 & 6 & 17 \\ 1 & 0 & 8 & 2 & 28 & 10 \end{bmatrix}$$

which has a rank $= 3$. Since pair (A, B) is controllable and A is cyclic, it implies that (A, Bv) is also controllable.

Solution to Problem 2

Numerically verifying the claim, we have the controllability matrix of pair (A, Bv), where $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, equal to

$$\begin{bmatrix} 2 & -2 & 8 \\ 0 & 6 & 6 \\ 1 & 8 & 28 \end{bmatrix}$$

which also has a rank $= 3$.
Similarly, this theorem can also be verified for other values of v.

State Feedback Design for Multi-Input system

Problem 3

Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix}$$

Find two different constant matrices K such that $(A - BK)$ has eigenvalues $-4 \pm 3j$ and $-5 \pm 4j$.

Solution to Problem 3

Given,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix}$$

Calculating the Controllability matrix of pair (A, B)

$$\mathfrak{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 2 & 10 \\ 0 & 0 & 1 & 2 & 2 & 10 & 5 & 22 \\ 1 & 2 & 2 & 10 & 5 & 22 & 12 & 54 \\ 0 & 2 & 0 & 0 & 1 & 2 & 4 & 14 \end{bmatrix}$$

which has a rank$= 4$. Therefore the pair (A, B) is controllable.

Solution to Problem 3

The Jordan form of A matrix is then given as

$$\begin{bmatrix} 0.3215 - 1.2581i & 0 & 0 & 0 \\ 0 & 0.3215 + 1.2581i & 0 & 0 \\ 0 & 0 & -1.3262 & 0 \\ 0 & 0 & 0 & 2.6833 \end{bmatrix}$$

Since all the Jordan blocks are of size 1×1, A is cyclic. Arbitrarily selecting v as $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, we can calculate the controllability matrix of pair (A, Bv) to find that it is also a controllable pair (having rank $= 4$).

We can now proceed forward to design the state feedback controller for the reduced single-input system (A, Bv).

Solution to Problem 3

The state feedback gain vector (k) for system defined by (A, Bv) for eigenvalues $-4 \pm 3j$ and $-5 \pm 4j$, calculated by eigenvalue placement method (equivalent to using `place` command in MATLAB) is

$$\begin{bmatrix} 90.7152 & 10.5868 & 6.0939 & -2.6174 \end{bmatrix}$$

Solution to Problem 3

The state feedback gain vector (k) for system defined by (A, Bv) for eigenvalues $-4 \pm 3j$ and $-5 \pm 4j$, calculated by eigenvalue placement method (equivalent to using `place` command in MATLAB) is

$$\begin{bmatrix} 90.7152 & 10.5868 & 6.0939 & -2.6174 \end{bmatrix}$$

Then, the overall state feedback for the multi-input system is $u(t) = vu'(t)$, where $u'(t) = -kx(t)$.
The gain matrix becomes

$$K_1 = vk = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 90.7152 & 10.5868 & 6.0939 & -2.6174 \end{bmatrix}$$

$$\implies K_1 = \begin{bmatrix} 90.7152 & 10.5868 & 6.0939 & -2.6174 \\ 181.4304 & 21.1735 & 12.1878 & -5.2348 \end{bmatrix}$$

Solution to Problem 3

Selecting another arbitrary value of v as $\begin{bmatrix} 0.8 \\ -1 \end{bmatrix}$, the state feedback gain vector (k) for the same eigenvalues become:

$$\begin{bmatrix} -168.9968 - 62.8041 - 13.2899 - 2.0261 \end{bmatrix}$$

and the multi-input system's gain matrix becomes

$$K_2 = vk = \begin{bmatrix} 0.8 \\ -1 \end{bmatrix} \begin{bmatrix} -168.9968 - 62.8041 - 13.2899 - 2.0261 \end{bmatrix}$$

$$\implies K_2 = \begin{bmatrix} -135.1975 & -50.2433 & -10.6319 & -1.6209 \\ 168.9968 & 62.8041 & 13.2899 & 2.0261 \end{bmatrix}$$

State Feedback and Disturbance Rejection

Problem 4

Consider the system

$$A = \begin{bmatrix} -10 & 1 \\ -0.02 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

a. Design a controller such that the desired eigenvalues are located at $s = -5 \pm j$ and the output tracks a unit step input, i.e. $r = 1$.
b. Plot the step response of the system under the effect of an external step disturbance (w) as in

$$\dot{x} = Ax + bu + bw$$

Solution to Problem 4

(a) For the system

$$A = \begin{bmatrix} -10 & 1 \\ -0.02 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Firstly, we calculate the desired feedback gains for eigenvalues at $s = -5 \pm j$.

Solution to Problem 4

(a) For the system

$$A = \begin{bmatrix} -10 & 1 \\ -0.02 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Firstly, we calculate the desired feedback gains for eigenvalues at $s = -5 \pm j$.

The characteristic equation of the closed-loop state feedback system becomes

$$\begin{aligned} |sI - A + bk| &= 0 \\ \left| \begin{bmatrix} s+10 & -1 \\ 0.02 & s+2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right| &= 0 \\ \begin{vmatrix} s+10 & -1 \\ 0.02 + 2k_1 & s + 2 + 2k_2 \end{vmatrix} &= 0 \\ \implies s^2 + s(12 + 2k_2) + 19.98 + 20k_2 - 2k_1 &= 0 \end{aligned}$$

Solution to Problem 4

(a) For the system

$$A = \begin{bmatrix} -10 & 1 \\ -0.02 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Firstly, we calculate the desired feedback gains for eigenvalues at $s = -5 \pm j$.

The characteristic equation of the closed-loop state feedback system becomes

$$\begin{aligned} |sI - A + bk| &= 0 \\ \left| \begin{bmatrix} s+10 & -1 \\ 0.02 & s+2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right| &= 0 \\ \begin{vmatrix} s+10 & -1 \\ 0.02 + 2k_1 & s + 2 + 2k_2 \end{vmatrix} &= 0 \\ \implies s^2 + s(12 + 2k_2) + 19.98 + 20k_2 - 2k_1 &= 0 \end{aligned}$$

The desired characteristic equation is

$$(s + 5 + j)(s + 5 - j) = 0$$

$$s^2 + 10s + 26 = 0$$

Solution to Problem 4

(a) Comparing the desired and actual characteristic equations, we get

$$k = \begin{bmatrix} 12.99 & -1 \end{bmatrix}$$

Solution to Problem 4

(a) Comparing the desired and actual characteristic equations, we get

$$k = \begin{bmatrix} 12.99 & -1 \end{bmatrix}$$

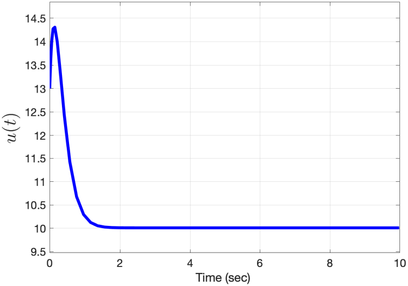
Now, the transfer function of the closed-loop system becomes

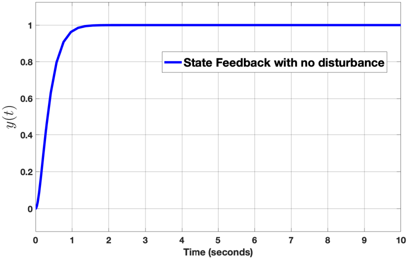
$$\begin{aligned} \hat{g}(s) &= c(sI - A - bk)^{-1} b \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s+10 & -1 \\ 0.02 & s+2 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} 12.99 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ \implies \hat{g}(s) &= \frac{2}{s(s-10)+26} \end{aligned}$$

The steady state value of output $y(t) = \hat{g}(0) = \frac{1}{13}$.
Since $\hat{g}(0) \neq (r(t) = 1)$, we will use p (feedforward gain) equal to $\frac{1}{\hat{g}(0)} = 13$ to make the output track the unit step input $(r = 1)$.

Solution to Problem 4

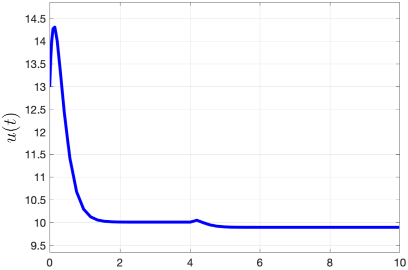
(a) Consequently, the unit step response becomes:

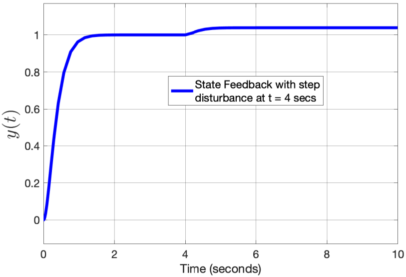




Solution to Problem 4

- (b) Adding a step disturbance at $t = 4$ secs to the system, the step response becomes





It is evident from the step response that the state feedback control is unable to track the reference$(r = 1)$ due to the effect of this external disturbance.

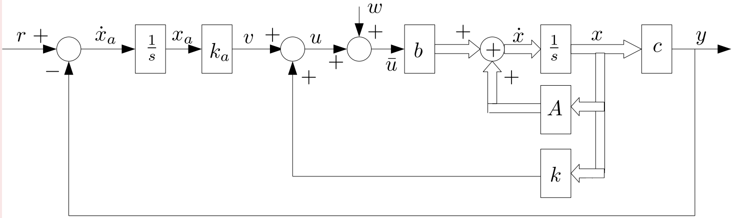
State Feedback and Disturbance Rejection

Problem 4, continued

(c) Design a robust controller to reject the effect of disturbances.

Solution to Problem 4

Recall (Lecture Slides 29)



In the above closed-loop control design, the output y will track asymptotically and robustly any step reference input $r(t) = a$ and reject any step disturbance with unknown magnitude.

Solution to Problem 4

(c) Adding an integral control action to the closed-loop system, the (A, b) pair becomes,

$$\bar{A} = \begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

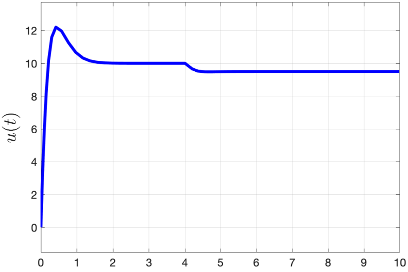
Again, calculating the feedback gain vector $k_{\mathsf{fb}} = \begin{bmatrix} k & k_a \end{bmatrix}$, we get

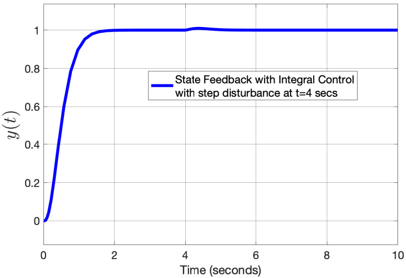
$$\begin{bmatrix} 12.99 & 2 & -78 \end{bmatrix}$$

where the last element represents the integral gain k_a of the controller.

Solution to Problem 4

- The step response now becomes,





which clearly shows that the state feedback control with integral action is capable of disturbance rejection.

Feedback Invariant of Nonlinear system

Problem 5

Consider the non-linear system

$$\dot{x} = f(x, u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k \tag{NLS}$$

and continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ with $V(0) = 0$. Verify that the functional

$$H(x(\cdot), u(\cdot)) \triangleq -\int_0^\infty \frac{\partial V(x(t))}{\partial x} f\left(x(t), u(t)\right) dt$$

is a feedback invariant as long as $\lim_{t\to\infty} x(t) = 0$.

[1] Hespanha Exercise 20.1

Solution to Problem 5

Recall!

Recall from the lecture slide 56 , that a functional $H\left(x(\cdot), u(\cdot)\right)$ that involves system's input and state is a feedback invariant for a given dynamical system if when computed along a solution to the system, its value depends only on the initial condition and not on the specific input signal.

Solution to Problem 5

Recall!

Recall from the lecture slide 56 , that a functional $H\left(x(\cdot), u(\cdot)\right)$ that involves system's input and state is a feedback invariant for a given dynamical system if when computed along a solution to the system, its value depends only on the initial condition and not on the specific input signal.

Given

$$H\left(x(\cdot), u(\cdot)\right) = -\int_0^\infty \frac{\partial V(x(t)}{\partial x} f\left(x(t), u(t)\right) dt$$

Solution to Problem 5

Recall!

Recall from the lecture slide 56 , that a functional $H\left(x(\cdot), u(\cdot)\right)$ that involves system's input and state is a feedback invariant for a given dynamical system if when computed along a solution to the system, its value depends only on the initial condition and not on the specific input signal.

Given

$$H\left(x(\cdot), u(\cdot)\right) = -\int_0^\infty \frac{\partial V(x(t)}{\partial x} f\left(x(t), u(t)\right) dt$$

The derivative of V along the trajectories of (NLS) denoted by $\dot{V}(x)$, is given by

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x)$$

$$= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \frac{\partial V}{\partial x} f(x)$$

The derivative of V along the trajectories of a system is dependent on the system's equation.

Solution to Problem 5

Consider

$$H\left(x(\cdot), u(\cdot)\right) = -\int_0^{\infty} \frac{\partial V(x(t))}{\partial x} f\left(x(t), u(t)\right) dt$$

Solution to Problem 5

Consider

$$
\begin{aligned}
H\left(x(\cdot), u(\cdot)\right) &= -\int_0^\infty \frac{\partial V(x(t))}{\partial x} f\left(x(t), u(t)\right) dt \\
&= -\int_0^\infty \dot{V}(x(t)) dt \\
&= -\big(V(x(\infty)) - V(x(0))\big) \\
&= V(x(0)) \qquad \text{as long as } \lim_{t\to\infty} x(t) = 0
\end{aligned}
$$

Since $H\left(x(\cdot), u(\cdot)\right)$ depends only on the initial state of the system, it is a feedback invariant.

Linear Quadratic Regulator(LQR)

Problem 6

Given a system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & x_1(0) &= 5 \\ \dot{x}_2(t) &= -2x_1(t) + 5x_2(t) + u(t), & x_2(0) &= 10 \end{aligned}$$

and the performance index (PI)

$$J = \frac{1}{2} \int_0^\infty \left[2x_1^2(t) + 6x_1(t)x_2(t) + 5x_2^2(t) + 0.25u^2(t) \right] dt$$

obtain the feedback control law. Compare the performance for different input and state weighting matrices.

[1] Naidu, Optimal Control Systems, Example 3.1

Solution to Problem 6

From the given system and performance index, the various quantities are

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix};\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; Q = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix};\quad r = \frac{1}{4}; t_0 = 0.$$

It is easy to check that the system is unstable.

Solution to Problem 6

From the given system and performance index, the various quantities are

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix};\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; Q = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix};\quad r = \frac{1}{4}; t_0 = 0.$$

It is easy to check that the system is unstable. Let P be the 2×2 symmetric matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Then, the optimal control is given by

$$u^* = -r^{-1} b' P x^*$$

where P is the solution of the algebraic Riccati equation

$$A'P + PA + Q - Pbr^{-1}b'P = 0$$

Solution to Problem 6

From the given system and performance index, the various quantities are

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix};\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; Q = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix};\quad r = \frac{1}{4}; t_0 = 0.$$

It is easy to check that the system is unstable. Let P be the 2×2 symmetric matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

Then, the optimal control is given by

$$u^* = -r^{-1} b' P x^*$$

where P is the solution of the algebraic Riccati equation

$$A'P + PA + Q - Pbr^{-1}b'P = 0$$

This equation can be solved using the `care` command in MATLAB.

Solution to Problem 6

The next simulation results show the variation of the trajectories for different weighting matrices

1. $Q_1 = Q = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, $r_1 = r = 0.25$ and
2. $Q_2 = 4Q$ and $r_2 = r = 0.25$.

Solution to Problem 6

Simulations

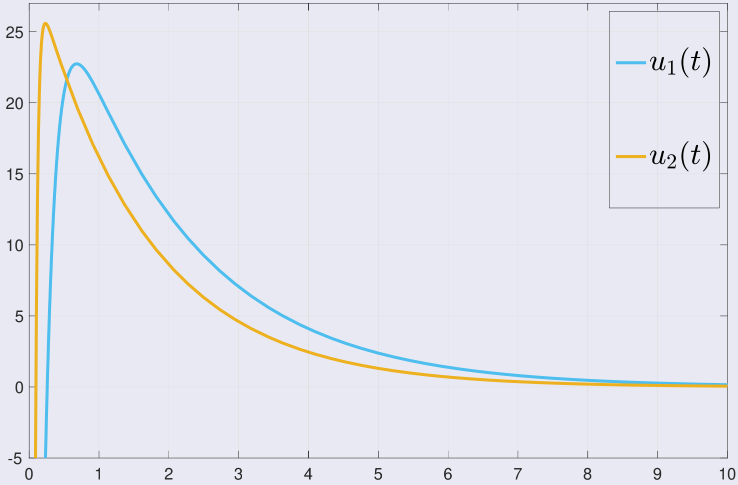


Figure: Input trajectories (u_1 for Q_1 and r_1 and u_2 for Q_2 and r_2)

Solution to Problem 6

Simulations

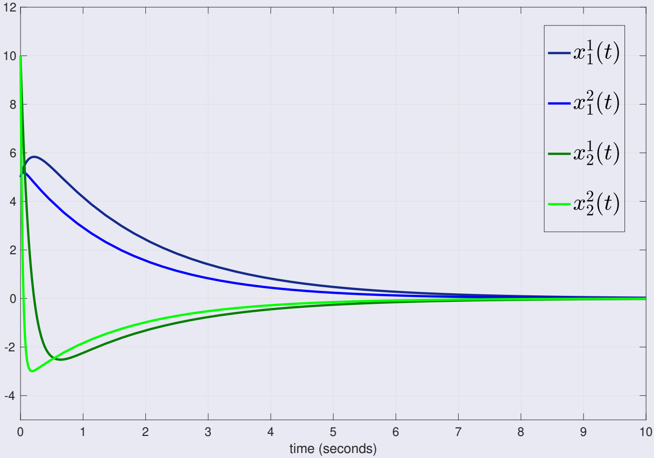


Figure: State and trajectories (x_1^1 and x_2^1 for Q_1, r_1 and x_1^2 and x_2^2 for Q_2 and r_2)