Linear Dynamical Systems

Tutorial on Controllability: Part I

Outline

1. Motion of an orbiting satellite (Controllability test discussed in lecture slides 21, 43)
2. Controllability and control input design (lecture slides 20)
3. Reachability and control input design (lecture slides $30 - 31$)
4. Reachability for a discrete time system (lecture slide 27)
5. Reachability of a time varying system (lecture slides $15 - 16$)
6. Hot air balloon (lecture slides $20 - 23$)
7. Rank equivalence (lecture slides $12 - 16$)
8. Controllability Check (lecture slides $34 - 49$)
9. Controllability Check (lecture slides $34 - 49$)

Motion of an orbiting satellite

Problem 1

- Consider the state equation $\dot{x} = Ax + Bu$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which was obtained by linearizing the nonlinear equations of motion of an orbiting satellite about a steady-state solution. In the state vector $x = [x_1, x_2, x_3, x_4]'$, x_1 is the differential radius, while x_3 is the differential angle. In the input vector $u = [u_1, u_2]'$, u_1 is the radial thrust and u_2 is the tangential thrust.

  a. Is the system controllable?
  b. Can the system be controlled if the radial thruster fails? What if the tangential thruster fails?

[1] Antsaklis, Problem 3.2

Solution to Problem 1

Recall!

Recall from the lecture slide 43 that an LTI system is controllable if the controllability matrix has full rank.

(a) since $\mathtt{rank}\begin{bmatrix}B & AB & A^2B & A^3B\end{bmatrix} = 4$ the system is controllable from u, (can also use MATLAB command $\mathtt{rank(ctrb(A,B))}$)

(b) If the radial thruster fails, that is $u_1 = 0$, consider the controllability matrix

$$\mathfrak{C}_2 = \begin{bmatrix}b_2 & Ab_2 & A^2b_2 & A^3b_2\end{bmatrix}$$

, where $b_2 = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^T$. Since $\mathtt{rank}(\mathfrak{C}_2) = 4$, the system is controllable from u_2. Similarly, if the tangential thruster fails, consider :

$$\mathfrak{C}_1 = \begin{bmatrix}b_1 & Ab_1 & A^2b_1 & A^3b_1\end{bmatrix}$$

where $b_1 = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}^T$. since $\mathtt{rank}(\mathfrak{C}_1) < 4$, the system is not controllable from u_1.

Controllability and control input Design

Problem 2

- Consider the state equation $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} u$
  1. If $x_0 = x(0) = \begin{bmatrix} a \\ b \end{bmatrix}$, derive an input that will drive the state to $x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in T seconds.
  2. For $x_0 = x(0) = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$ compare the plots of the input and state trajectories for $T = 1, 2$ and 5. How do the trajectories appear after $t = T$?

[1] Antsaklis, Problem 3.3

Solution to Problem 2

Recall!

Recall from the lecture slides $17-21$ the formula of the controllability matrix and the subsequent computation of the control input.

(a) The required control input is given by: $u(t) = B'e^{A'(T-t)}W_r^{-1}(0,T)(x_1 - e^{AT}x_0)$ where W_r is the $n \times n$ reachability gramian given as:

$$W_r(0,T) = \int_0^T e^{(T-\tau)A}BB'e^{(T-\tau)A'}d\tau$$

and x_0 is the initial state and x_1 is the final state.

Solution to Problem 2

For $x_0 = \begin{bmatrix} a & b \end{bmatrix}'$, $x_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}'$, we obtain

$$\begin{aligned} u(t) =& \frac{1}{\Delta}\Bigg\{\frac{1}{2}e^{t/2}\left[\frac{b}{3}e^{-3T/2}(1-e^{-3T/2}) - \frac{a}{2}e^{-T}(1-e^{-2T})\right] \\ &+ e^{t}\left[\frac{a}{3}e^{-3T/2}(1-e^{-3T/2}) - \frac{b}{4}e^{-2T}(1-e^{-T})\right]\Bigg\} \end{aligned}$$

where $\Delta = \frac{1}{72} - \frac{1}{8}e^{-T} + \frac{2}{9}e^{-3T/2} - \frac{1}{8}e^{-2T} + \frac{1}{72}e^{-3eT}$

Solution to Problem 2

(b) The next figures summarize the answer:

Simulations

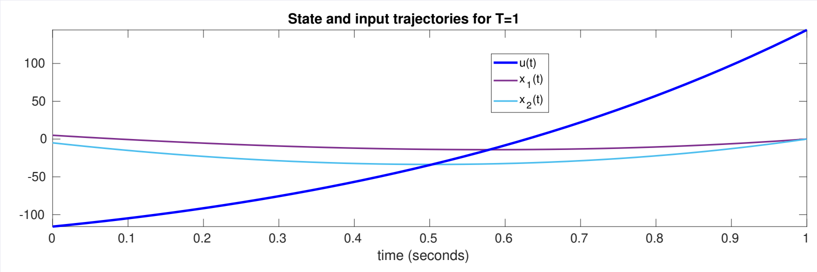


Figure: State and input trajectories for $T = 1$

Solution to Problem 2

Simulations

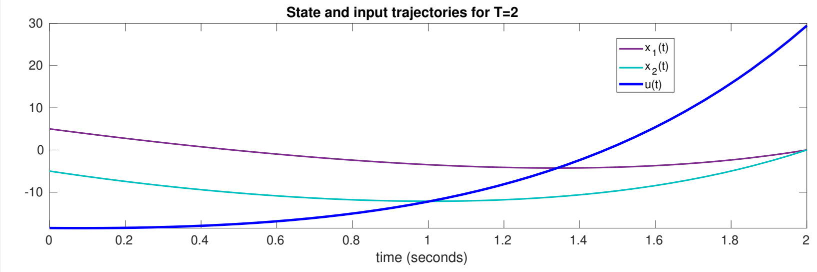


Figure: State ad input trajectories for $T = 2$

Solution to Problem 2

Simulations

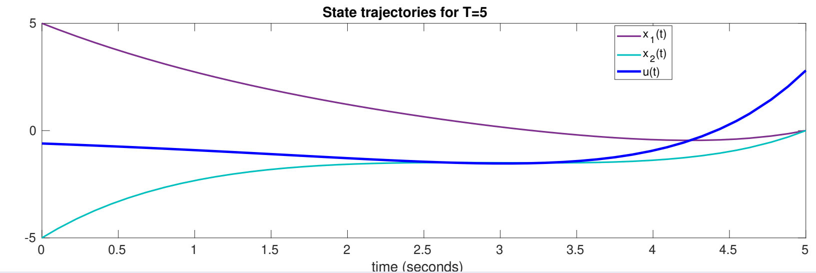


Figure: State trajectory for $T = 5$

Solution to Problem 2

Simulations

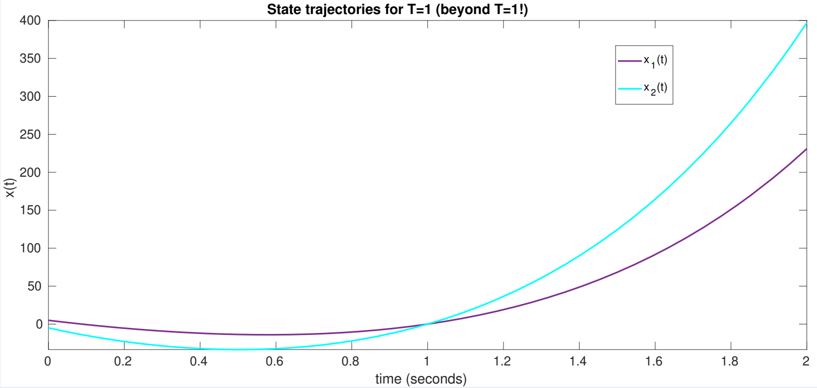


Figure: State trajectory beyond $T = 1$

Solution to Problem 2

Simulations

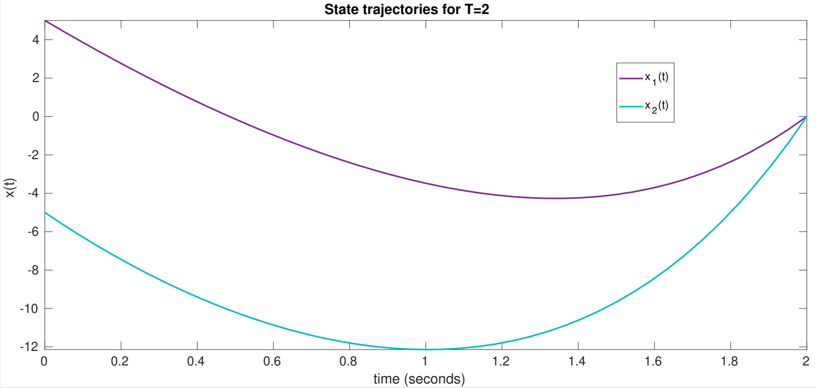


Figure: State trajectory for $T = 2$

Solution to Problem 2

Simulations

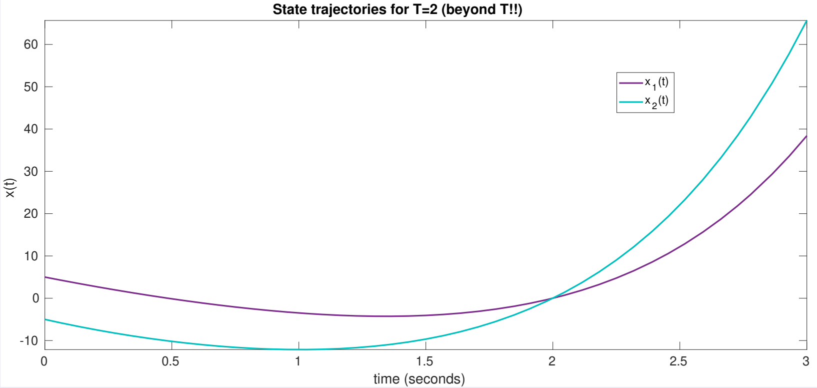


Figure: State trajectory beyond $T = 2$

Solution to Problem 2

Simulations

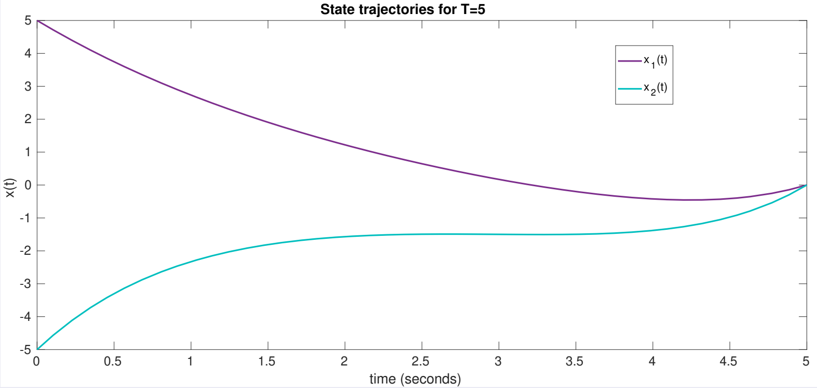


Figure: State trajectory for $T = 5$

Solution to Problem 2

Simulations

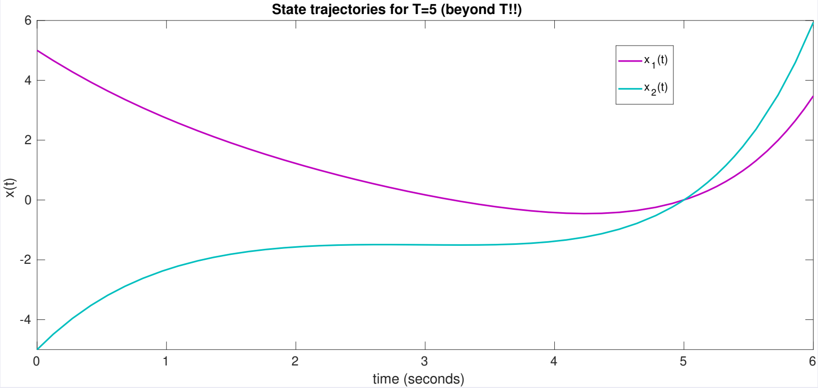


Figure: State trajectory beyond $T = 5$

Solution to Problem 2

Comparison of energies

Figure: Comparison of the energies of the input $u(t)$ for $T = 1, 2$ and 5

T	Energy (computed using the formula $\int_0^T u^2(t)dt$)
1	5858.90
2	440.80
5	7.12

Reachability and Control Input Design

Problem 3

Consider the equation $x(k+1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(k)$,

$y(k) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k)$.

a) Is $x_1 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$ reachable? If yes, what is the minimum number of steps required to transfer the state from the zero state to x_1? What inputs do you need?

b) Determine all states that are reachable.

c) For the discrete-time system $x(k+1) = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u(k)$, comment on reachability and controllability subspaces. Are they equal?

[1] Antsaklis, Problem 3.4

Solution to Problem 3

Recall!

Recall from the lecture slides $28 - 29$ the formula of the controllability and the subsequent computation of the control input for the discrete time systems.

(a) The controllability matrix for this case is given as:

$$\mathfrak{C} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

which is a rank 2 matrix. But

$$x = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = \mathfrak{C} \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

Thus, $x_1 \in \mathtt{Im}(\mathfrak{C})$, it is reachable. Further, $x_1 \in \begin{bmatrix} B & AB \end{bmatrix}$, it can be reached in two steps.

Solution to Problem 3

(b) Note that,

$$x(1) = Ax(0) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(0) \implies x(1) = \begin{bmatrix} 0 \\ u(0) \\ u(0) \end{bmatrix}$$

and

$$x(2) = A \begin{bmatrix} 0 \\ u(0) \\ u(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(1) \implies x(2) = \begin{bmatrix} u(0) \\ u(0) + u(1) \\ u(0) + u(1) \end{bmatrix}$$

Thus, the state $x_1 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$ can be reached with $u(0) = 3$ and $u(1) = -1$.

Solution to Problem 3

- (c) A basis for the reachability subspace range($\begin{bmatrix} B & AB & A^2B \end{bmatrix}$) is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$. Thus, any $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a \\ a \end{bmatrix}$ will be reachable.

Solution to Problem 3

(d) The controllability matrix is computed to be:

$$\mathfrak{C} = \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix}$$

which has rank 1 and hence the system is not reachable. However, for any $x_1[0] = a$ and $x_2[0] = b$, the input $u[0] = 2a + b$ transfers the state $x[0]$ to $x[1] = 0$ and hence the system is controllable to the origin.

Recall!

Recall from the lecture slide 31 that for a singular matrix A, the reachable space of the system $x(k+1) = Ax(k) + Bu(k)$ is subset of the controllability space.

Reachability for a discrete system

Problem 4

Given $x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$, $y(k) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(k)$, and assume zero initial conditions.

(a) Is there a sequence of inputs $u(0), u(1), \ldots$ that transfers the output from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in finite time? If the answer is yes, determine such a sequence.

(b) Characterize all outputs that can be reached from the zero output $\left(y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$ in one step.

[1] Antsaklis, Problem 3.10

Solution to Problem 4

Recall!

Recall from the lecture slides $28 - 29$ the formula of the controllability and the subsequent computation of the control input for the discrete time systems.

(a) This is essentially a problem of output reachability. Consider: $x(k+1) = Ax(k) + Bu(k)$ with

$$\begin{aligned} & y(k) = Cx(k) \\ \Longrightarrow\ & y(k+1) = Cx(k+1) \\ \Longrightarrow\ & y(k+1) = CAx(k) + CBu(k) \\ \Longrightarrow\ & y(k+1) = CAC^{-1}y(k) + CBu(k) \\ & \text{(assuming that } C \text{ is invertible)} \end{aligned}$$

This system shall be reachable if the matrix $\begin{bmatrix} CB & CAC^{-1}CB \end{bmatrix} = \begin{bmatrix} CB & CAB \end{bmatrix}$ is full rank.

Solution to Problem 4

Computation gives:

$$[CB \quad CAB] = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Since this matrix is of full rank, the system is reachable and the required input exists.
The required input is easily computed as $u(0) = -1$, $u(1) = 2$.

- (b) $y(0) = [0\ 0]'$ implies $x(0) = [0\ 0]'$ as well. Therefore

$$y(1) = Cx(1) = CBu(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(0).$$

where $u(0)$ is arbitrary.

Reachability of a time varying system

Problem 5

Consider the state equation

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix} u(t)$$

a. Show that it is controllable at any $t_0 \in (-\infty, \infty)$.
b. Suppose we are interested only in $x_2(t)$. Consider, therefore, $\dot{x}_2(t) = x_2(t) + e^{-t}u(t)$. Is it possible to determine $u(t)$ so that the state $x_2(t)$ is transferred from x_{20} at $t = t_0$ $[x_2(t_0) = x_{20}]$ to the zero state at some $t = t_1$ $[x_2(t_1) = 0]$ and then stay there? If the answer is yes, find such a $u(t)$.
c. In (b), let $t_0 = 0$ and study the effects of the sizes of t_1 and x_0 on the magnitude of $u(t)$.
d. For the system in (b), determine, if possible, a $u(t)$ so that the state is transferred from x_{20} at $t = t_0$ to x_{21} at $t = t_1$ and then stay there.

[1] Antsaklis, Problem 3.11

Solution to Problem 5

Recall!

See the lecture slide 17 for the computation of the controllability gramian. The controllability gramian is given as

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \tau) B(\tau) B(\tau)^T \phi(t_0, \tau)^T d\tau$$

(a) The controllability gramian is computed to be :

$$W_c\,(t_0, t_1) = \begin{bmatrix} t_1 - t_0 & \frac{e^{-t_0}}{2} - \frac{e^{t_0 - 2t_1}}{2} \\ \frac{e^{-t_0}}{2} - \frac{e^{t_0 - 2t_1}}{2} & \frac{e^{-2t_0}}{4} - \frac{e^{2t_0 - 4t_1}}{4} \end{bmatrix}$$

For any $t_0 \in (-\infty, \infty)$ we can find $t_1 > t_0$ such that rank$(W_c(t_0, t_1)) = 2$. Therefore the system is controllable at any $t_0 \in (-\infty, \infty)$.

(b) We need to find an input that will satisfy $x_2(t) = 0$, $t \geq t_1$. Since:

Solution to Problem 5

$$x_2(t) = e^{t-t_0}x_{20} + \int_{t_0}^{t} e^{t-\tau}e^{-\tau}u(\tau)d\tau$$
$$\implies x_2(t_1) = e^{t_1-t_0}x_{20} + e^{t_1}\int_{t_0}^{t_1} e^{-2\tau}u(\tau)d\tau$$

In order to have $x_2(t_1) = 0$, we need $e^{t_1-t_0}x_{20} + \int_{t_0}^{t_1} e^{-2\tau}u(\tau)d\tau = 0$. We also need $u(t) = 0, t \geq t_1$, because $e^{-t}u(t) = 0$ for $t \geq t_1$. Let

$$u(t) = \begin{cases} e^{2t}(at+b) & t_0 \leq t \leq t_1 \\ 0 & t \geq t_1 \end{cases}$$

Then,

$$e^{-t_0}x_{20} + \frac{a}{2}(t_1^2 - t_0^2) + b(t_1 - t_0) = 0 \text{ and } at_1 + b = 0$$
$$\implies a = \frac{2x_{20}e^{-t}}{(t_1-t_0)^2}, b = -\frac{2t_1x_{20}e^{-t_0}}{(t_1-t_0)^2}$$

Hence,

$$u(t) = \begin{cases} \frac{2x_{20}e^{-t}}{(t_1-t_0)^2}\left(t-t_1\right)e^{2t} & t_0 \leq t \leq t_1 \\ 0 & t \geq t_1 \end{cases}$$

Solution to Problem 5

Simulation result for part (b) with $t_1 = 2$, $t_0 = 1$ and $x_0 = 5$

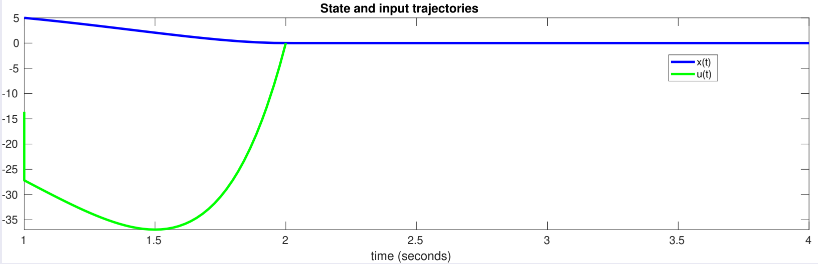


Figure: State and input trajectories for (b)

Solution to Problem 5

(c) For

$$t_0 = 0, u(t) = 2\frac{x_{20}(t-t_1)}{t_1^2}e^{2t}; t_0 \leq t \leq t_1$$

The energy of $u(t)$ is computed to be:

$$E_u = \int_0^{t_1} (u(t))^2\,dt = \frac{x_{20}^2}{8t_1^4}\left[e^{4t_1} - \left(1 + 2t_1 + 8t_1^2\right)\right]$$

Obviously, the energy of $u(t)$ increases as x_{20} increases or t_1 decreases.

Solution to Problem 5

- (d) We need $x_2(t_1) = x_{21}$ that is

$$x_2(t_1) = x_{21} = e^{t_1 - t_0} x_{20} + \int_{t_0}^{t_1} e^{t_1-\tau} e^{-\tau} u(\tau) d\tau$$

and $\dot{x}_2(t) = x_2 + e^{-t} u(t) = 0, \quad t \geq t_1$

Let

$$u(t) = \begin{cases} e^{2t}(at+b);\ t_0 \leq t \leq t_1 \\ -e^{-t} x_{21};\ t \geq t_1 \end{cases}$$

Then

$$e^{-t} x_{21} = e^{-t_0} x_{20} + \frac{a}{2}\left(t_1^2 - t_0^2\right) + b\left(t_1 - t_0\right)$$

$$e^{2t_1}\left(at+b\right) = -e^{-t} x_{21}$$

$\implies a = \frac{2}{(t_1-t_0)^2}\left[-x_{21}e^{t_1}(t_1 - t_0) + x_{20}e^{-t_0} - x_{21}e^{-t_1}\right]$ and

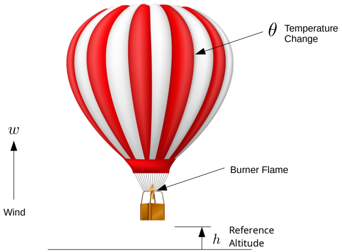
$b = \frac{(t_1+t_0)x_{21}e^{t_1}}{t_1-t_0} - \frac{2t_1\left(x_{20}e^{-t_0} - x_{21}e^{-t_1}\right)}{(t_1-t_0)^2}$

Hot air Balloon

Problem 6

Approximate equations of motion for a hot air balloon are

$$\dot{\theta} = -\frac{1}{\tau_1}\theta + u$$
$$\dot{v} = -\frac{1}{\tau_2}v + \sigma\theta + \frac{1}{\tau_2}w$$
$$\dot{h} = v$$



Here θ = temperature change of air in balloon away from equilibrium temperature, u is proportional to change in heat added to air in balloon (control), v = vertical velocity, h = change in altitude from equilibrium altitude, and w = vertical wind velocity (disturbance). Determine the transfer function from u to h and from w to h. Is the system completely controllable by u? Is it completely controllable by w?

[1] Example 2.3 (Linear Systems- T. Kailath)

Solution to Problem 6

We consider w as a second input, the state equation are

$$\begin{bmatrix}\dot{\theta}\\ \dot{v}\\ \dot{h}\end{bmatrix} = \begin{bmatrix}-1/\tau_1 & 0 & 0\\ \sigma & -1/\tau_2 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\theta\\ v\\ h\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1/\tau_2\\ 0 & 0\end{bmatrix}\begin{bmatrix}u\\ w\end{bmatrix}; \qquad y = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}\theta\\ v\\ h\end{bmatrix}$$

Note that the eigenvalues are $-1/\tau_1, -1/\tau_2$ and 0. The transfer function to the output from one input of a multi-input system is defined with all other input zero. Then when all inputs are present we merely use superposition.

Recall that the transfer function from $\frac{y(s)}{u(s)}$ for the system $\dot{x} = Ax + Bu, y = Cx$ is given by

$$\frac{y(s)}{u(s)} = C\left(sI - A\right)^{-1} B$$

Using this formula for the present system with

$$A = \begin{bmatrix}-1/\tau_1 & 0 & 0\\ \sigma & -1/\tau_2 & 0\\ 0 & 1 & 0\end{bmatrix}$$

Solution to Problem 6

The quantity $(sI - A)^{-1}$ is computed to be

$$(sI - A)^{-1} = \begin{bmatrix} \frac{\tau_1}{(s\tau_1+1)} & 0 & 0 \\ \frac{\sigma\tau_1\tau_2}{(s\tau_1+1)(s\tau_2+1)} & \frac{\tau_2}{(s\tau_2+1)} & 0 \\ \frac{\sigma\tau_1\tau_2}{s(s\tau_1+1)(s\tau_2+1)} & \frac{\tau_2}{s(s\tau_2+1)} & \frac{1}{s} \end{bmatrix}$$

Taking transforms with $w = 0$, we obtain, after some algebraic manipulations

$$\frac{h(s)}{u(s)}\Big|_{w=0} = \frac{h(s)v(s)\theta(s)}{v(s)\theta(s)u(s)} = \frac{\sigma}{s(s + 1/\tau_2)(s + 1/\tau_1)}$$

Similarly, with $u = 0$, we obtain

$$\frac{h(s)}{w(s)}\Big|_{u=0} = \frac{h(s)v(s)}{v(s)w(s)} = \frac{1/\tau_2}{s(s + 1/\tau_2)} = \frac{1}{s(\tau_2 s + 1)}$$

The eigenvalues at $-1/\tau_1$ has evidently been canceled by the numerator, as the eigenvalue $s = \frac{-1}{\tau_1}$ is the present in the expression of $(sI - A)^{-1}$.

Solution to Problem 6

Rank equivalence

Problem 7

Is it true that the rank of $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ equals the rank of $\begin{bmatrix} AB & A^2B & \cdots & A^nB \end{bmatrix}$? If not, under what condition will it be true?

[1] Chen, Problem 6.3

Solution to Problem 7

Recall!

Recall the lecture slide $12 - 16$ for revisiting the linear algebra concepts required for this problem.

It is not always true that the rank of $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ equals the rank of $\begin{bmatrix} AB & A^2B & \cdots & A^nB \end{bmatrix}$. Only if A is nonsingular

$$\begin{aligned}\rho(\begin{bmatrix} AB & A^2B & \cdots & A^nB \end{bmatrix}) =& \rho(A \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}) \\ =& \rho(\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix})\end{aligned}$$

will be true.
Let A be $m \times n$ matrix and C and D be any $n \times n$ and $m \times m$ nonsingular matrices. Then we have

$$\rho(AC) = \rho(A) = \rho(DA)$$

In other words, the rank of a matrix will not change after pre or post-multiplication by a nonsingular matrix.

Controllability Check

Problem 8

Consider the state equation

$$\dot{x} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_{11} & c_{12} & c_{21} & c_{22} \end{bmatrix}$$

It is the modal form. It has one real eigenvalue and two pairs of complex conjugate eigenvalues. It is assumed that they are distinct. Show that the state equation is controllable if and only if $b_1 \neq 0$; $b_{i1} \neq 0$ or $b_{i2} \neq 0$ for $i = 1, 2$.

[1] Chen, Problem 6.16

Solution to Problem 8

Controllability is invariant under any equivalence transformation. So we introduce a nonsingular matrix transformed into Jordan form

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.5j & 0 & 0 \\ 0 & 0.5 & 0.5j & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -0.5j \\ 0 & 0 & 0 & 0.5 & 0.5j \end{bmatrix},$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & j & -j & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & j & -j \end{bmatrix},$$

Solution to Problem 8

$$\bar{A} = PAP^{-1} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 + j\beta_1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 - j\beta_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 + j\beta_2 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 - j\beta_2 \end{bmatrix},$$

$$\bar{B} = PB = \begin{bmatrix} b_1 \\ 0.5(b_{11} - jb_{12}) \\ 0.5(b_{11} + jb_{12}) \\ 0.5(b_{21} - jb_{22}) \\ 0.5(b_{21} - jb_{22}) \end{bmatrix}.$$

Recall!

Recall from the lecture slide 43 that a LTI system is controllable if the controllability matrix has full rank.

Solution to Problem 8

The state equation is controllable if and only if $b_1 \neq 0$, $0.5(b_{11} \pm jb_{12}) \neq 0$ and $0.5(b_{21} \pm jb_{22}) \neq 0$; equivalently if and only if $b_1 \neq 0$, $b_{i1} \neq 0$ or $b_{i2} \neq 0$ for $i = 1, 2$

Controllability Check

Problem 9

For time-invariant systems, show that (A, B) is controllable if and only if (-A,B) is controllable. Is this true for time-varying systems?

[1] Chen, Problem 6.23

Solution to Problem 9

Recall!

Recall from the lecture slide 43 that a LTI system is controllable if the controllability matrix has full rank.

For time-invariant system (A, B) is controllable if and only if $\rho(\mathfrak{C}_1) = \rho \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$ Assuming A is $n \times n$, $(-A, B)$ is controllable if and only if

$$\begin{aligned} \rho(\mathfrak{C}_2) =& \rho \begin{bmatrix} B & -AB & A^2B & -A^3B \cdots & A^{n-1}B \end{bmatrix} \\ =& \rho \begin{bmatrix} B & AB & A^2B & A^3B \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & -I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ & & & \ddots & \end{bmatrix} \end{aligned}$$

Solution to Problem 9

We know that any column of a matrix multiplied by nonzero constant does not change the rank of the matrix, so the conditions $\rho(\mathfrak{C}_1) = \rho(\mathfrak{C}_2)$ is identically the same, thus we conclude that (A, B) is controllable if and only if $(-A, B)$ is controllable and for the time-varying system this is not true.

Solution to Problem 9

Recall!

Recall from the lecture slide 35 that a linear system is controllable if the controllability gramian W_c is non-singular $\forall t$.

For example consider $(A(t), B(t)) = \left(\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix} \right)$,

$\phi(t,\tau) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(t-\tau)} \end{bmatrix}$,

$\phi(t,\tau)B(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}$

$W_c(t_0, t_1) = \int_{t_0}^{t_1} \begin{bmatrix} 1 \\ e^{-\tau} \end{bmatrix} \begin{bmatrix} 1 & e^{-\tau} \end{bmatrix} d\tau =$

$\begin{bmatrix} t_1 - t_0 & e^{-t_1}(t_1 - t_0) \\ e^{-t_1}(t_1 - t_0) & e^{-2t_1}(t_1 - t_0) \end{bmatrix}$

$\det W_c(t_0, t_1) = 0$ for all t_0 and $t_1 \geq t_0$. Thus the equation is not controllable at any t.

Solution to Problem 9

$(-A(t), B(t))=\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1\\e^{-t}\end{bmatrix}\right)$, we have $\phi(t,\tau) = \begin{bmatrix}1 & 0\\0 & e^{t-\tau}\end{bmatrix}$

$\phi(t,\tau)B(\tau) = \begin{bmatrix}1\\e^{t-\tau}e^{-\tau}\end{bmatrix} = \begin{bmatrix}1\\e^{t-2\tau}\end{bmatrix}$

$$
\begin{aligned}
W_c(t_0,t_1) &= \int_{t_0}^{t_1} \begin{bmatrix}1\\e^{t_1-2\tau}\end{bmatrix}\begin{bmatrix}1 & e^{t_1-2\tau}\end{bmatrix} d\tau \\
&= \int_{t_0}^{t_1} \begin{bmatrix}1 & e^{t_1-2\tau}\\ e^{t_1-2\tau} & e^{2(t_1-2\tau)}\end{bmatrix} d\tau \\
&= \begin{bmatrix} t_1 - t_0 & \frac{1}{3}e^{t_1}(e^{-3t_0} - e^{-3t_1}) \\ \frac{1}{3}e^{t_1}(e^{-3t_0} - e^{-3t_1}) & \frac{1}{5}e^{2t_1}(e^{-5t_0} - e^{-5t_1}) \end{bmatrix}
\end{aligned}
$$

For any t_0, we can find a t_1 so that $W_c(t_0,t_1)$ is nonsingular and $(-A(t), B(t))$ is controllable at any t although $(A(t), B(t))$ is not.