Linear Dynamical Systems

Tutorial on Stability

Outline

1. Solvability of Lyapunov equations (Lecture Slide 22)
2. BIBO and internal stability (Lecture slides $38 - 43$)
3. Lyapunov's theory for linear systems (Lecture slide 30)
4. Margin of stability (Lecture slide 30)
5. Stability of linear time variant systems (Lecture slide 56)
6. Sampling and stability
7. Stability and Jordan canonical form (Lecture slides $53 - 54$)

Solvability of the Lyapunov matrix equation

Problem 1

Consider the system $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Comment on the solvability of the Lyapunov matrix equation $A^T P + PA = -Q$, $Q = Q^T \succeq 0$.

[1] Terrell, Page 76

Solvability of the Lyapunov matrix equation

Problem 1

Consider the system $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Comment on the solvability of the Lyapunov matrix equation $A^T P + PA = -Q$, $Q = Q^T \succeq 0$.

Recall! -Lecture Slide 22

Let $A \in \mathbb{R}^{n \times n}$ and let $\lambda_1, \ldots, \lambda_n$ denote the (not necessarily distinct) eigenvalues of A, then the equation

$$A^T P + PA = -Q, \qquad Q = Q^T \succ 0$$

has a unique solution for P corresponding to each Q if and only if $\cancel{\lambda_i \neq 0\ ,\ \lambda_i + \lambda_j \neq 0}$ ~~for all~~ $\cancel{i, j}$.

[1] Terrell, Page 76

Solution to Problem 1

The eigenvalues of A are λ_1, $\lambda_2 = \pm j$ and therefore the required condition is violated. Thus, the Lyapunov equation $A^T P + PA = -Q$ does not possess a *unique* solution for a given Q.

Solution to Problem 1

The eigenvalues of A are $\lambda_1,\ \lambda_2 = \pm j$ and therefore the required condition is violated. Thus, the Lyapunov equation $A^T P + PA = -Q$ does not possess a *unique* solution for a given Q.

We now verify this for two specific cases:

- When $Q = 0$, we obtain:

$$A^T P + PA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -2p_{12} & p_{11} - p_{22} \\ p_{11} - p_{22} & 2p_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

  or $p_{12} = 0$ and $p_{11} = p_{22}$. Therefore, for any $a \in \mathbb{R}$, the matrix $P = aI$ is a solution of the Lyapunov matrix equation.

Solution to Problem 1

- When $Q = 2I$, we obtain:

$$A^T P + PA = \begin{bmatrix} -2p_{12} & p_{11} - p_{22} \\ p_{11} - p_{22} & 2p_{12} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

  or $p_{11} = p_{22}$ and $p_{12} = 1$ and $p_{12} = -1$, which is impossible. Therefore, for $Q = -2I$ the Lyapunov equation has no solution at all.

BIBO stability and internal stability

Problem 2

Consider the continuous time linear time invariant (CT-LTI) system

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} x(t)$$

with $x(0) = \begin{bmatrix} x_{10} & x_{20} & -1.5x_{20} \end{bmatrix}^T$. Analyze the system for internal and BIBO stability.

Solution to Problem 2

Solution: BIBO stability

The dynamics can be written as:
$\dot{x}_1 = -x_1 + u(t), \dot{x}_2 = x_2 + 2x_3, \dot{x}_3 = 2x_2 + x_3$ thus,

$$
\begin{aligned}
x_1(t) &= e^{-t}x_{10} + e^{-t}\int_0^t e^{\tau}u(\tau)d\tau \\
x_2(t) &= 0.5e^{-t}\left(x_{20} + x_{30}\right) + e^{3t}\left(0.75x_{20} + 0.5x_{30}\right) \\
x_3(t) &= -e^{-t}\left(0.5x_{20} + 0.25x_{30}\right) + e^{3t}\left(0.5x_{20} + 0.25x_3\right) \\
&= -0.125e^{-t} + 0.125e^{3t}
\end{aligned}
$$

$$
y(t) = x_1 + x_2 - x_3 = e^{-t}x_{10} + e^{-t}\left(\int_0^t e^{\tau}u(\tau)d\tau\right) - 0.25e^{-t}x_{20}
$$

It is easy to see that the output $y(t)$ is bounded when $u(t)$ is bounded for all t. Thus, the system is clearly BIBO stable.

Solution to Problem 2

Solution: Internal stability

The matrix A is given as :

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

which has the eigen values $-1, -1$ and 3 and thus the system is not internally stable in the sense of Lyapunov (which requires the eigen values to be negative). Note that the transfer function has a zero at $s = 3$ and hence this pole-zero cancellation leads to the internal instability of the system , although the system is BIBO stable.

Recall!-Lecture slides 38-43

This example illustrates the fact that

External stability $\not\Longrightarrow$ Internal stability (in the sense of Lyapunov)

Lyapunov's theory of stability for linear systems

Problem 3

Assume that the origin of the system $\dot{x} = Ax$ is asymptotically stable. Then prove that the matrix A is similar to a matrix $\bar{A}$ which satisfies $\bar{A} + \bar{A}^T < 0$.

In other words, the system $\dot{x} = Ax$ is equivalent by a linear change of coordinates to a system $\dot{z} = \bar{A}z$ for which the Euclidean norm is strictly decreasing along non-zero solutions.

[1] Terrell, Theorem $3.7(d)$

Lyapunov's theory of stability for linear systems

Problem 3

Assume that the origin of the system $\dot{x} = Ax$ is asymptotically stable. Then prove that the matrix A is similar to a matrix $\bar{A}$ which satisfies $\bar{A} + \bar{A}^T < 0$.

In other words, the system $\dot{x} = Ax$ is equivalent by a linear change of coordinates to a system $\dot{z} = \bar{A}z$ for which the Euclidean norm is strictly decreasing along non-zero solutions.

Recall

This question is based on the lecture slide 30 which discusses the Lyapunov's theory of stability for linear systems

[1] Terrell, Theorem $3.7(d)$

Solution to Problem 3

Recall that since the matrix A is Hurwitz there exists positive definite solution of the equation

$$A^T P + PA + Q = 0 \tag{1}$$

where Q is positive definite. Setting $Q = I$, there exists a $P > 0 : A^T P + PA + I = 0$.

Also, there exists a positive definite matrix S such that $S^2 = P$; it is natural to write $S = P^{1/2}$ and call it the positive square root of P.

The matrix $P^{1/2}$ is invertible and we can write $P^{-1/2} \triangleq \left(P^{1/2}\right)^{-1}$.

Multiplying (1) on the right and on the left by $P^{-1/2}$ and rearranging it, we obtain:

$$P^{-1/2} A^T P^{1/2} + P^{1/2} A P^{-1/2} = -P^{-1}$$

Note that the right hand side is negative definite.

Now with $\bar{A} \triangleq P^{1/2} A P^{-1/2}$, we see that A is similar to $\bar{A}$ and $\bar{A} + \bar{A}^T < 0$ is negative definite. This completes the proof.

Margin of stability

Problem 4

Let $\sigma > 0$ be a positive number, Q be a positive definite matrix, and A a matrix of the same size as Q. Show that if there exists a positive definite matrix P such that

$$A^T P + PA + 2\sigma P = -Q$$

then every eigen values of A satisfies $\mathtt{Re}(\lambda) < -\sigma$

Recall

This question is based on the lecture slide 30 which discusses the Lyapunov's theory of stability for linear systems

[1] Terrell, Exercise 3.17

Solution to Problem 4

Let λ be a (possibly complex) eigenvlaue of A and v be the corresponding eigenvector, then

$$\begin{aligned}
& v^* \left(A^T P + PA + 2\sigma P\right) v = -v^* Q v \\
& \implies (Av)^* Pv + v^* P (Av) + 2\sigma v^* Pv = -v^* Qv \\
& \implies \bar{\lambda} v^* Pv + \lambda v^* Pv + 2\sigma v^* Pv = -v^* Qv \\
& \implies \left(\bar{\lambda} + \lambda + 2\sigma\right) v^* Pv = -v^* Qv
\end{aligned}$$

Since Q is positive definite matrix, the right hand side of the above equation is negative definite. Also, since P is positive definite it is necessary that

$$\left(\bar{\lambda} + \lambda + 2\sigma\right) < 0 \implies 2\mathtt{Re}(\lambda) < -2\sigma \implies \lambda < -\sigma$$

and since λ was an arbitrary eigenvalue pf A, every eigenvalue λ of A must satisfy $\lambda < -\sigma$.

Stability of linear time variant systems

Problem 5

Consider the system

$$\dot{x} = A(t)x = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} x, \qquad t \in (-\infty, \infty)$$

Analyze the system for stability.

[1] Terrell, Example 3.10

Stability of linear time variant systems

Problem 5

Consider the system

$$\dot{x} = A(t)x = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} x, \qquad t \in (-\infty, \infty)$$

Analyze the system for stability.

Recall

This question is based on the lecture slide 57- "*the fact that it is not possible to comment on the stability of a linear time varying system by merely computing the eigen values of the state matrix*".

[1] Terrell, Example 3.10

Solution to Problem 5

For each t, the matrix $A(t)$ has -1 as a repeated eigenvalue.

The solution for x_2 is $x_2(t) = e^{-t}x_{20}$. If we substitute this into the equation for x_1, then

$$
\begin{aligned}
x_1(t) &= e^{-t}x_{10} + e^{-t}\left(\int_0^t e^{3s}x_2(s)ds\right) \\
&= e^{-t}x_{10} + e^{-t}\left(e^{2s}x_{20}dss\right) \\
&= e^{-t}x_{10} + e^{-t}\frac{1}{2}\left(e^{2t}x_{20} - x_{20}\right) \\
&= e^{-t}x_{10} + \frac{1}{2}e^{t}x_{20} - \frac{1}{2}e^{-t}x_{20}
\end{aligned}
$$

Because of the exponential growth term, if $x_{20} \neq 0$ then $x_1(t) \to \infty$ as $t \to \infty$. Thus, negative real parts for all eigenvalues is not a sufficient condition for asymptotic convergence of all solutions to the origin in a linear time-varying system.

Sampling and stability

Problem 6

Compare the stability of the system

$$\dot{x} = Ax$$

with $A = \begin{bmatrix} 0 & 1 \\ -2 & -5 \end{bmatrix}$ with its discrete time counterpart (obtained using the Euler's method) with a sampling time $T = 0.5$ and $T = 0.1$.

Solution to Problem 6

Stable CT system

The eigen values of A are computed as -0.4384 and -4.5616 which clearly shows that the system is internally stable.

Solution to Problem 6

Discrete counterpart at $T = 0.5$

Using the Euler method the discrete system is given s:

$$\dot{x}_d(k+1) = (TA + I)\, x_d(k) = A_d x_d(k)$$

where T is the sampling time With $T = 0.5$ the state matrix is given as:

$$A_d = \begin{bmatrix} 1 & 0.5 \\ -1 & -1.5 \end{bmatrix}$$

with eigenvalues: -1.281 and 0.7808. Since one of the eigenvalue has magnitude greater than 1, the system is unstable.

Solution to Problem 6

Discrete counterpart at $T = 0.1$

With $T = 0.1$ the state matrix is given as:

$$A_d = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.5 \end{bmatrix}$$

with eigenvalues: 0.5438 and 0.9562. Since the eigenvalues have magnitude less than 1, the system is stable.

Observation

- It can be verified that the system obtained after discretizing using Euler method is stable as long as $T < 0.453$.
- Using another method of discretization or determining the stability of the discrete-time state matrix obtained using `c2d`-MATLAB command, the state matrix is always stable.

Jordan forms stability and minimal polynomial

Problem 7

Comment on the stability of the system $\dot{x} = Ax$ with

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Jordan forms stability and minimal polynomial

Problem 7

Comment on the stability of the system $\dot{x} = Ax$ with

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Recall

This question is based on the lecture slide $53 - 54$ which discuss relationship between stability, Jordan forms and minimal polynomial.

Solution to Problem 7

Jordan form computation

The Jordan form of a matrix can be computed using the concepts of eigenvalues, eigenvectors and the generalized eigenvectors. Or you can also use MATLAB command: `J = jordan(A)`.

The eigenvalues of A are computed to be $0, 0, 0, 0, -1$. For the given A the Jordan form is computed to be:

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly the Jordan blocks corresponding the zero eigenvalues are not 1×1 and hence the system under consideration is not marginally stable

Solution to Problem 7

Minimal polynomial

The characteristic polynomial is given as $s^4(s+1) = 0$.

Furthermore, it is easily verified that A satisfies $A^3(A+I) = 0$ and hence the minimal polynomial is $s^3(s+1)$ which has repeated roots at $s = 0$ and hence the system is unstable.