Linear Dynamical Systems

Week 5 - State Feedback Controller Design

Outline of Week 5

1. Open-loop control
2. State-feedback controller design
3. Regulation and Tracking
4. Extension to Multivariable case
5. Preview of Optimal control

Control Problem

Open-loop minimum-energy control

Suppose that a particular state x_1 belongs to the reachable subspace $\mathcal{R}[t_0, t_1]$ of the system (AB-CLTV).

Theorem (Reachable subspace)

Given two times $t_1 > t_0 \geq 0$,

$$\mathcal{R}[t_0, t_1] = \mathtt{Im} W_R(t_0, t_1).$$

Moreover, if $x_1 = W_R(t_0, t_1)\eta_1 \in \mathtt{Im} W_R(t_0, t_1)$, *the control*

$$u(t) = B(t)^T \phi(t_1, t)^T \eta_1, \qquad t \in [t_0, t_1] \qquad \text{(Min-energy control)}$$

can be used to transfer the state from $x(t_0) = 0$ *to* $x(t_1) = x_1$.

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can be used to transfer the state from $x(t_0) = 0$ to $x(t_1) = x_1$.

In general, there may be other control that achieve the first goal, but controls of the form (Min-energy control) are desirable because they $minimize\ control\ energy$.

Open-loop minimum-energy control

Suppose that $\bar{u}(\cdot)$ is another control that transfers the state to x_1 and therefore

$$x_1 = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) u(\tau) d\tau = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) \bar{u}(\tau) d\tau.$$

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For this to hold, we must have

$$\int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) v(\tau) d\tau = 0$$

where $v = \bar{u} - u$.

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where $v = \bar{u} - u$.The "energy" of $\bar{u}(\cdot)$ can be related to the energy of $u(\cdot)$ as follows

$$\int_{t_0}^{t_1} \|\bar{u}(\tau)\|^2 d\tau = \int_{t_0}^{t_1} \| \overbrace{B'(t)\phi'(t_1, \tau)\eta_1}^{u(\tau)} + v(\tau)\|^2 d\tau$$

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$$= \eta_1' W_R(t_0,t_1)\eta_1 + \int_{t_0}^{t_1} \|v(\tau)\|^2 d\tau + 2\eta_1' \int_{t_0}^{t_1} \phi(t_1,\tau)B(\tau)v(\tau)d\tau$$

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$$\begin{aligned}\int_{t_0}^{t_1} \|\bar{u}(\tau)\|^2 d\tau &= \int_{t_0}^{t_1} \| \overbrace{B'(t)\phi'(t_1,\tau)\eta_1}^{u(\tau)} + v(\tau)\|^2 d\tau \\ &= \eta_1' W_R(t_0,t_1)\eta_1 + \int_{t_0}^{t_1} \|v(\tau)\|^2 d\tau + 2\eta_1' \int_{t_0}^{t_1} \phi(t_1,\tau)B(\tau)v(\tau)d\tau\end{aligned}$$

Note the last term is equal to zero, and we conclude that the energy of $\bar{u}$ is minimized for $v(\cdot) = 0$, i.e., for $\bar{u} = u$.

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$$= \eta_1' W_R(t_0, t_1)\eta_1 + \int_{t_0}^{t_1} \|v(\tau)\|^2 d\tau + 2\eta_1' \int_{t_0}^{t_1} \phi(t_1, \tau)B(\tau)v(\tau)d\tau$$

Note the last term is equal to zero, and we conclude that the energy of $\bar{u}$ is minimized for $v(\cdot) = 0$, i.e., for $\bar{u} = u$.Moreover, for $v(\cdot) = 0$, we conclude that the energy required for the optimal control $u(\cdot)$ in (Min-energy control) is given by

$$\int_{t_0}^{t_1} \|u(\tau)\|^2 d\tau = \eta_1' W_R(t_0, t_1)\eta_1.$$

Open-loop minimum-energy control

Theorem (Reachable and Controllable subspaces)

1. *if* $x_1 = W_R(t_0, t_1)\eta_1 \in \mathtt{Im} W_R(t_0, t_1)$, *the control*

$$u(t) = B(t)^T \phi(t_1, t)^T \eta_1, \qquad t \in [t_0, t_1] \tag{1}$$

can be used to transfer the state from $x(t_0) = 0$ *to* $x(t_1) = x_1$.

2. *if* $x_0 = W_C(t_0, t_1)\eta_0 \in \mathtt{Im} W_C(t_0, t_1)$, *the control*

$$u(t) = -B(t)^T \phi(t_0, t)^T \eta_0, \qquad t \in [t_0, t_1] \tag{2}$$

can be used to transfer the state $x(t_0) = x_0$ *to* $x(t_1) = 0$.

Open-loop minimum-energy control

Theorem (Reachable and Controllable subspaces)

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can be used to transfer the state $x(t_0) = x_0$ *to* $x(t_1) = 0$.

Theorem (Minimum-energy control)

Given two times $t_1 > t_0 \geq 0$,

1. *when* $x_1 \in \mathcal{R}[t_0, t_1]$, *the control* (1) *transfers the state from* $x(t_0) = 0$ *to* $x(t_1) = x_1$ *with the smallest amount of control energy, which is given by*

$$\int_{t_0}^{t_1} \|u(\tau)\|^2 d\tau = \eta_1' W_R(t_0, t_1)\eta_1,$$

2. *when* $x_1 \in \mathcal{C}[t_0, t_1]$, *the control* (2) *transfers the state from* $x(t_0) = x_0$ *to* $x(t_1) = 0$ *with the smallest amount of control energy, which is given by*

$$\int_{t_0}^{t_1} \|u(\tau)\|^2 d\tau = \eta_0' W_C(t_0, t_1)\eta_0.$$

Solution by Inversion

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State Feedback

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$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= Cx\end{aligned} \tag{LTI}$$

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$$u = r - Kx = r - \begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix} x = r - \sum_{i=1}^{n} k_i x_i. \tag{3}$$

Each feedback gain k_i is a real constant. This is called the *constant gain negative state feedback* or, simply, *state feedback*.

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$$\begin{aligned} \dot{x} &= (A - bk)x + br \\ y &= cx \end{aligned} \tag{CL-LTI}$$

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$$\begin{aligned} \dot{x} &= (A - bk)x + br \\ y &= cx \end{aligned} \tag{CL-LTI}$$

Theorem

The pair $(A - bk, b)$, for any $1 \times n$ real constant vector k, is controllable if and only if (A, b) is controllable.

State Feedback

Proof.

We show the theorem for $n = 4$. Define

$$\mathfrak{C} = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix}$$

and

$$\mathfrak{C}_f = \begin{bmatrix} b & (A - bk)b & (A - bk)^2b & (A - bk)^3b \end{bmatrix}$$

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It is straightforward to verify

$$\mathfrak{C}_f = \mathfrak{C} \begin{bmatrix} 1 & -kb & -k(A-bk)b & -k(A-bk)^2b \\ 0 & 1 & -kb & -k(A-bk)b \\ 0 & 0 & 1 & -kb \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Note that k is $1 \times n$ and b is $n \times 1$. Thus kb is scalar; so is every entry in the rightmost matrix. Because the right most matrix is nonsingular for any k, the rank of $\mathfrak{C}_f$ equals the rank of $\mathfrak{C}$. Thus (CL-LTI) is controllable if and only if (LTI) is controllable. $\square$

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Note

The input r does not control the state x directly; it generates u to control x. Therefore, if u cannot control x, neither can r.

State Feedback

Theorem

Consider the (LTI) *system with* $n = 4$ *and the characteristic polynomial*

$$\Delta(s) = \det(sI - A) = s^4 + \alpha_1 s^3 + \alpha_1 s^2 + \alpha_3 s + \alpha_4$$

If the system is controllable, then it can be transformed by the transformation $\bar{x} = Px$ *with*

$$Q := P^{-1} = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

into the controllable canonical form

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \bar{c}\bar{x} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \bar{x}$$

Furthermore, the transfer function of the system with $n = 4$ *equals*

$$\hat{g}(s) = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}$$

State Feedback

Theorem (Eigenvalue Assignment)

If the $n-$dimensional (LTI) *system is controllable, then by state feedback $u = r - kx$, where k is a $1 \times n$ real constant vector, the eigenvalues of $A - bk$ can arbitrarily be assigned provided that complex conjugate eigenvalues are assigned in pairs.*

Proof

Let $n = 4$, if (LTI) is controllable then it can be transformed into the CCF $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}\bar{u}$ where $\bar{A} = PAP^{-1}$, $\bar{b} = Pb$, and $\bar{\mathfrak{C}} = P\mathfrak{C}$.

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Substituting $\bar{x} = Px$ in u yields

$$u = r - kx = r - kP^{-1}\bar{x} = r - \bar{k}\bar{x}.$$

Since $\bar{A} - \bar{b}\bar{k} = P(A - bk)P^{-1}$, it implies $\lambda[A - bk] = \lambda[\bar{A} - \bar{b}\bar{k}]$.

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From any set of desired eigenvalues, we can form

$$\Delta_f(s) = s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4.$$

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If $\bar{k}$ is chosen as

$$\bar{k} = \begin{bmatrix} \bar{\alpha}_1 - \alpha_1 & \bar{\alpha}_2 - \alpha_2 & \bar{\alpha}_3 - \alpha_3 & \bar{\alpha}_4 - \alpha_4 \end{bmatrix}$$

the state feedback equation becomes

$$\dot{\bar{x}} = \left(\bar{A} - \bar{b}\bar{k}\right)\bar{x} + \bar{b}r = \begin{bmatrix} -\bar{\alpha}_1 & -\bar{\alpha}_2 & -\bar{\alpha}_3 & -\bar{\alpha}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \bar{x}$$

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Let $n = 4$, if (LTI) is controllable then it can be transformed into the CCF $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}\bar{u}$ where $\bar{A} = PAP^{-1}$, $\bar{b} = Pb$, and $\bar{\mathfrak{C}} = P\mathfrak{C}$.
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$$y = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{bmatrix} \bar{x}$$

Because of the companion form, the characteristic polynomial of $(\bar{A} - \bar{b}\bar{k})$ and of $(A - bk)$ equals $\Delta_f(s)$. Thus the state feedback equation has the set of desired eigenvalues. The feedback gain k can be computed from

$$k = \bar{k}P = \bar{k}\bar{\mathfrak{C}}\mathfrak{C}^{-1}.$$

Feedback transfer function

$$(A, b, c) \implies \hat{g}(s) = c(sI - A)^{-1}b = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}$$

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After state feedback

$$(A - bk, b, c) \implies \hat{g}_f(s) = c(sI - A + bk)^{-1}b = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4}$$

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Note

- the numerators are the same, state feedback can shift the poles of a plant but has *no effect on the zeros*,
- state feedback *may alter the observability* property because one or more poles are shifted to coincide with the zeros of $\hat{g}(s)$.

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Attention

The command `K=place(A,B,v)` computes a matrix K such that the eigenvalues of $A - BK$ are those specified in the vector v. The pair (A, B) should be controllable, and the vector v should have no repeated eigenvalues. This command should be used with great caution (and generally avoided), because it is numerically badly conditioned.

Selection of desired eigenvalues[1]

1. depends on the performance criteria
   - rise time
   - overshoot
   - settling time
2. response depends upon the poles and zeros both
3. factors affecting the selection of poles
   - zeros of the plant
   - magnitude of u: saturation or burn out
   - rise time, settling time, overshoot
   - bandwidth of the closed-loop
4. involve compromises among many conflicting objectives

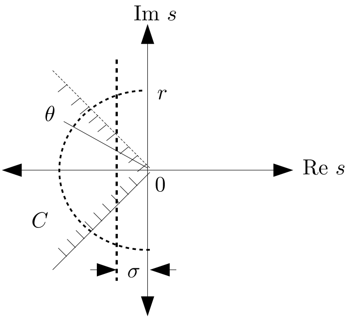
[1] **Boyd** *et al*. Linear controller design: limits of performance. Englewood Cliffs, NJ: Prentice Hall, 1991.

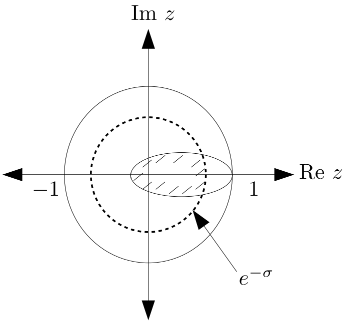
Åström. Limitations on control system performance. European Journal of Control. 6(1) 2000.

Some guidelines

As a guide, place all the poles inside the region denoted by C

- larger the σ, faster the response
- large the θ, larger the overshoot
- larger the r, faster the response , u will also be larger, BW will also be larger and the resulting system will be more susceptible to noise





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[a] Once F and $\bar{k}$ are selected, we may use the MATLAB function `lyap` to solve the Lyapunov equation

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What remains to be proved is the nonsingularity of T!

Nonsingularity of T

Theorem

If A and F have no eigenvalues in common, then the unique solution T of $AT - TF = b\bar{k}$ is nonsingular if and only if (A, b) and $(F', \bar{k}')$ are controllable pairs.

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We shall prove the theorem for $n = 4$.
Recall, the characteristic polynomial of A is given by

$$\Delta(s) = s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4$$

then from Cayley-Hamilton theorem we have

$$\Delta(A) = A^4 + \alpha_1 A^3 + \alpha_2 A^2 + \alpha_3 A + \alpha_4 I = 0$$

Proof

Let us consider

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$$\det \Delta(F) = \prod_i \Delta(\bar{\lambda}_i) \neq 0$$

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$$\begin{aligned} A^2T - TF^2 &= A(TF + b\bar{k}) - TF^2 = Ab\bar{k} + (AT - TF)F \\ &= Ab\bar{k} + b\bar{k}F \end{aligned}$$

Proof (Cont...)

Proceeding forward, we can obtain the following set of equations:

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If (A, b) and $(F', \bar{k}')$ are controllable, then all three matrices are nonsingular, which implies that T is nonsingular.

If (A, b) and/or $(F, \bar{k})$ are uncontrollable, then the product of the three matrices is singular. Therefore T is singular. This establishes the theorem.

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- If the desired eigenvalues are all distinct, we can also use the modal form. For example, if $n = 5$, and if the five distinct desired eigenvalues are selected as $\lambda_1, \alpha_1 \pm j\beta_1$ and $\alpha_2 \pm j\beta_2$, then we can select F as

$$F = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}$$

It is a block-diagonal matrix.

Regulation

Regulation problem

Suppose the reference signal r is zero, and the response of the system is caused by some nonzero initial conditions. The problem is to find a state feedback gain so that the response will die out at a desired rate.

Examples:

- Aircraft cruise control
- Liquid level control in tanks

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Consider a plant described by (A, b, c). If A is unstable, then the response excited by any nonzero initial conditions will grow unbounded.

Let $u = r - kx$. Then the state feedback equation becomes $(A - bk, b, c)$ and the response caused by $x(0)$ is

$$y(t) = ce^{(A-bk)t}x(0)$$

Tracking

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Suppose the reference signal r is a constant or $r(t) = a$, for $t \geq 0$. The problem is to design an overall system so that $y(t)$ approaches $r(t) = a$ as t approaches infinity. This is called *asymptotic tracking* of a step reference input.

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Tracking a non-constant reference signal is called a *servomechanism* problem and is a much more difficult problem.

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After the state feedback and feedforward, it will now become

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If (A, b) is controllable, all eigenvalues of $(A - bk)$ or, equivalently, all poles of $\hat{g}_f(s)$ can be assigned arbitrarily.

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$$\hat{g}_f(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = p\frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \bar{\alpha}_1 s^3 + \bar{\alpha}_2 s^2 + \bar{\alpha}_3 s + \bar{\alpha}_4}$$

If (A, b) is controllable, all eigenvalues of $(A - bk)$ or, equivalently, all poles of $\hat{g}_f(s)$ can be assigned arbitrarily.Under this assumption, if the reference input is a step function with magnitude a, then the output $y(t)$ will approach the constant $\hat{g}_f(0).a$ as $t \to \infty$.

Tracking

To address the tracking problem, in addition to the state feedback, we also need a *feedforward* gain p as

$$u(t) = pr(t) - kx.$$

Consider again the transfer function

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$$1 = \hat{g}_f(0) = p\frac{\beta_4}{\bar{\alpha}_4} \quad \text{or} \quad p = \frac{\bar{\alpha}_4}{\beta_4}$$

which requires $\beta_4 \neq 0$, which is possible if and only if the plant transfer function $\hat{g}(s)$ has no zero at $s = 0$.

Robust Tracking and Disturbance Rejection

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a different controller design that can achieve robust tracking and disturbance rejection.

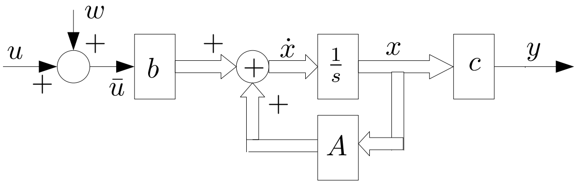
Robust Tracking and Disturbance Rejection

Control problem

Consider a plant described by (LTI) affected by a constant disturbance w with "unknown magnitude" enters at the plant input. Then the state equation is given as

$$\dot{x} = Ax + bu + bw, \qquad y = cx \qquad \text{(Disturbed LTI)}$$

The problem is to design an overall system so that the output $y(t)$ will track asymptotically any step reference input even with the presence of disturbance $w(t)$ and with "plant parameter variations".



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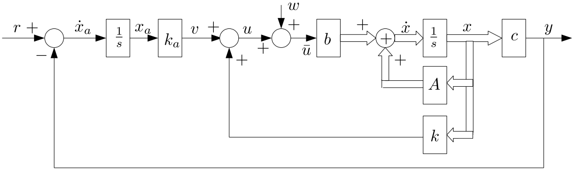

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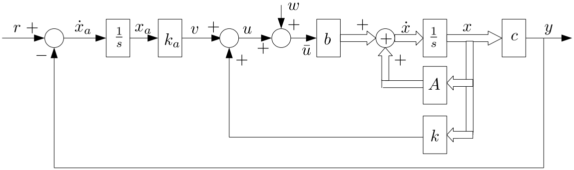

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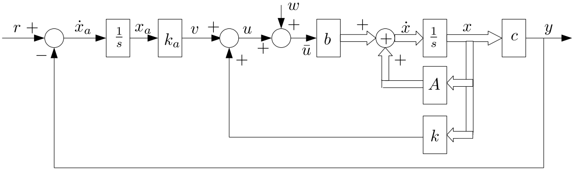

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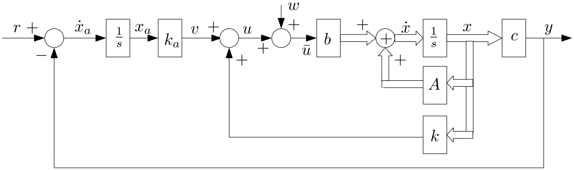

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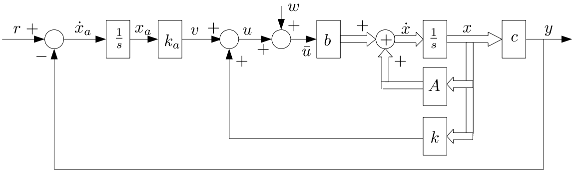

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Substituting these into (Disturbed LTI) yields

$$\begin{bmatrix} \dot{x} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} A + bk & bk_a \\ -c & 0 \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r + \begin{bmatrix} b \\ 0 \end{bmatrix} w$$

$$y = \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} \qquad \text{(Aug. Dist. CL-LTI)}$$

Robust Tracking and Disturbance Rejection

Theorem (Lemma)

If (A, b) *is controllable and if* $\hat{g}(s) = c(sI - A)^{-1}b$ *has no zero at* $s = 0$*, then all eigenvalues of the "new"* $A-$*matrix can be assigned arbitrarily by selecting a feedback gain* $\begin{bmatrix} k & k_a \end{bmatrix}$*.*

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We show the theorem for $n = 4$.

Assumption

1. A, b and c have been transformed into controllable canonical form.
2. The transfer function has no zeros at $s = 0$ if and only if $\beta_4 \neq 0$.

$$(A, b, c) \implies \hat{g}(s) = c(sI - A)^{-1}b = \frac{\beta_1 s^3 + \beta_2 s^2 + \beta_3 s + \beta_4}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}$$

Proof

We now show that the pair

$$\left(\begin{bmatrix} A & 0 \\ -c & 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix} \right) \qquad \text{(New pair)}$$

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Its determinant is -β_4. Thus the matrix is nonsingular if and only if $\beta_4 \neq 0$.

Robust Controller Design (1/5)

Consider again (Aug. Dist. CL-LTI)

$$\begin{bmatrix} \dot{x} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} A + bk & bk_a \\ -c & 0 \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r + \begin{bmatrix} b \\ 0 \end{bmatrix} w$$

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Assume that a set of $n+1$ desired stable eigenvalues has been selected and the feedback gain $\begin{bmatrix} k & k_a \end{bmatrix}$ has been found such that

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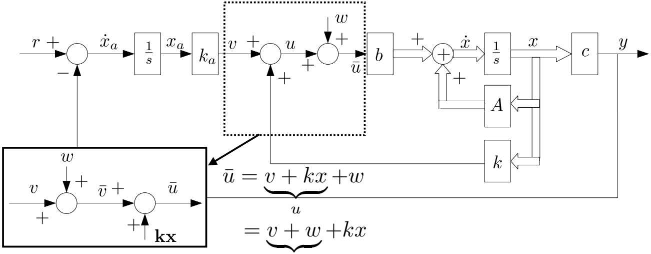


$$\bar{u} = \underbrace{v + kx}_{u} + w$$

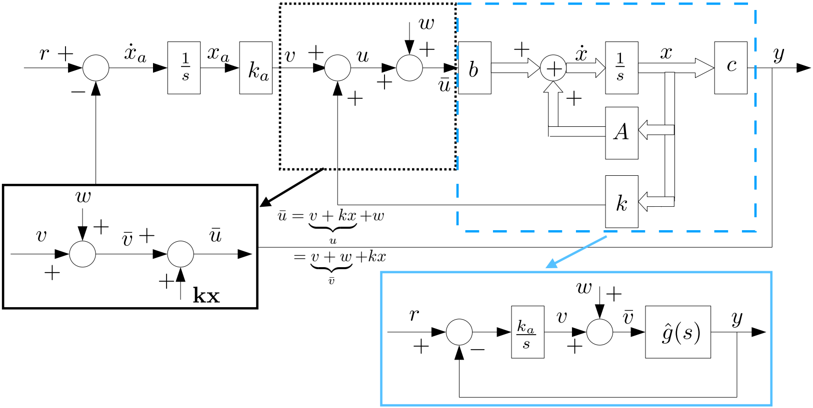
$$= \underbrace{v + w}_{\bar{v}} + kx$$

Robust Controller Design (2/5)

The transfer function from $\bar{v}$ to y is

$$\hat{\bar{g}}(s) := \frac{\bar{N}(s)}{\bar{D}(s)} := c(sI - A - bk)^{-1}b$$

with $\bar{D}(s) = \det(sI - A - bk)$.



Robust Controller Design (3/5)

It is straight forward to verify the following equality:

$$\underbrace{\begin{bmatrix} I & 0 \\ c(sI - A - bk)^{-1} & 1 \end{bmatrix}}_{\text{unimodular}} \overbrace{\begin{bmatrix} sI - A - bk & -bk_a \\ c & s \end{bmatrix}}^{(sI - A_{\text{ADCL-LTI}})}$$

$$= \begin{bmatrix} sI - A - bk & -bk_a \\ 0 & s + c(sI - A - bk)^{-1}bk_a \end{bmatrix}$$

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Taking its determinant, we obtain

$$1 \cdot \Delta_f(s) = \bar{D}(s)\left(s + \frac{\bar{N}(s)}{\bar{D}(s)}k_a\right) = s\bar{D}(s) + k_a\bar{N}(s).$$

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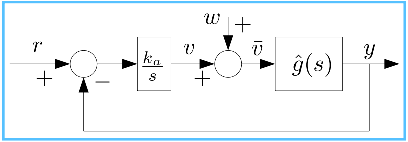
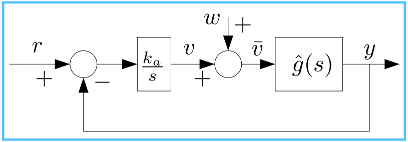
$$\boxed{\triangle_f(s) = s\bar{D}(s) + k_a\bar{N}(s)}$$

This is a key equation.

Robust Controller Design (4/5)

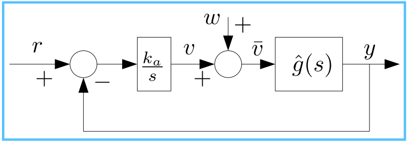


Robust Controller Design (4/5)



$$\hat{g}_{yw} = \frac{\frac{\bar{N}(s)}{\bar{D}(s)}}{1 + \frac{k_a}{s}\frac{\bar{N}(s)}{\bar{D}(s)}} = \frac{s\bar{N}(s)}{s\bar{D}(s) + k_a\bar{N}(s)} = \frac{s\bar{N}}{\Delta_f(s)}$$

Robust Controller Design (4/5)



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If the disturbance is $w(t) = \bar{w}$ for all $t \geq 0$, where $\bar{w}$ is unknown constant, then $\hat{w}(s) = \bar{w}/s$ and the corresponding output is given by

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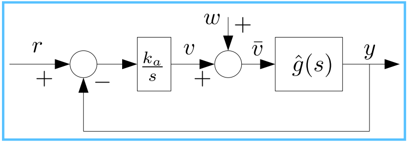

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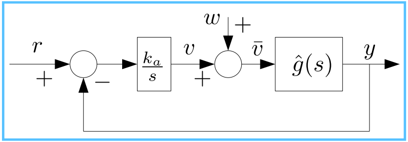
Because the pole s is cancelled, all remaining poles of $\hat{y}_w(s)$ are stable poles. Therefore the corresponding time response, for any $\bar{w}$, will die out as $t \to \infty$.

Robust Controller Design (4/5)



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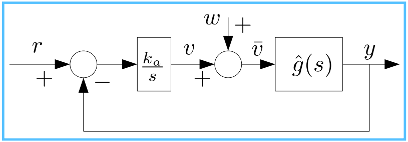
Because the pole s is cancelled, all remaining poles of $\hat{y}_w(s)$ are stable poles. Therefore the corresponding time response, for any $\bar{w}$, will die out as $t \to \infty$.

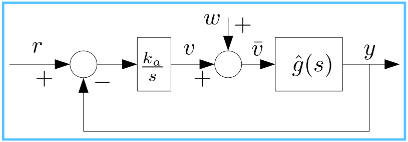
If there are plant parameter variations and variations in the feedforward gain k_a and feedback gain k, the rejection still holds as long as overall system remains stable.

Robust Controller Design (5/5)



$\hat{g}_{yr}(s)$

Robust Controller Design (5/5)



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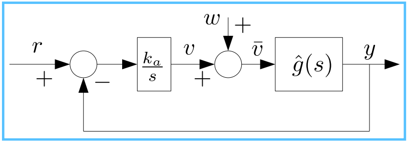

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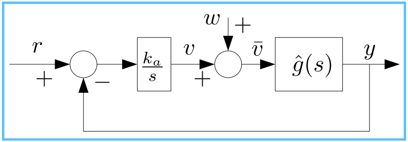

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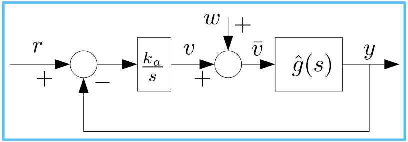

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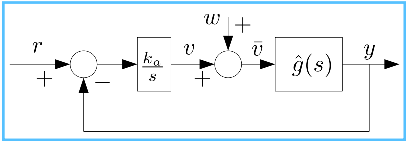

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The above equation holds even when there are parameter perturbations in the plant transfer function and the gains. Thus asymptotic tracking of any step reference input is robust.

Note that this robust tracking holds even for very large parameter perturbations as long as overall system remains stable.

Key Observation

The integrator is in fact a model of the step reference input and constant disturbance. Thus it is called the *internal model principle*.

Stabilization

Stabilization problem

If a state equation is controllable, all eigenvalues can be assigned arbitrarily by introducing the state feedback. The problem is to design a *stabilizing controller* whenever the state equation is not state-controllable .

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Every uncontrollable state equation can be transformed into

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_u \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_u \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_u \end{bmatrix} + \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u \qquad \text{(Uncontrollable Decomposition)}$$

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Observation

Because the "new $A-$matrix" is block triangular, the eigenvalues of the original $A-$matrix are the union of the eigenvalues of $\bar{A}_c$ and $\bar{A}_u$.

Stabilization

Let introduce the state feedback controller

$$u = r - kx = r - \bar{k}\bar{x} = r - \begin{bmatrix} \bar{k}_1 & \bar{k}_2 \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_u \end{bmatrix}$$

where we have partitioned $\bar{k}$ as in $\bar{x}$

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We see that $\bar{A}_u$ and consequently, its eigenvalues are not affected by the state feedback. Thus we conclude that controllability condition of (A, b) is not only sufficient (as stated earlier, see slide $\#13$) but also necessary to assign all eigenvalues of $(A - bk)$ to any desired positions.

Stabilization

Recall stabilizability

Consider again the (Uncontrollable Decomposition) state equation. If $\bar{A}_u$ is stable, and if $(\bar{A}_c, \bar{b}_c)$ is controllable then the state equation is said to be stabilizable.

Comment on Tracking and Rejection problems

1. The controllability condition for tracking and disturbance rejection can be replaced by the $weaker$ condition of stabilizability.
2. But, we do not have complete control of the $rate\ of\ tracking\ and\ rejection$.
3. If the uncontrollable stable eigenvalues have large imaginary parts or are close to imaginary axis, then the tracking and rejection $may\ not$ be satisfactory.

State feedback

Consider a plant described by the n-dimensional p-input state equation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \text{(Plant)}$$

In state feedback, the input u is given by

$$u = r - Kx \qquad \text{(Controller)}$$

where K is a $p \times n$ real constant matrix and r is a reference signal. Substituting (Controller) in (Plant) yields

$$\begin{aligned} \dot{x} &= (A - BK)x + Br \\ y &= Cx \end{aligned} \qquad \text{(Closed-loop)}$$

State feedback

Theorem

The pair $(A - BK, B)$*, for any* $p \times n$ *real constant matrix* K*, is controllable if and only if* (A, B) *is controllable.*

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The pair $(A - BK, B)$, for any $p \times n$ real constant matrix K, is controllable if and only if (A, B) is controllable.

Logical idea.

1. The proof of this theorem follows closely the proof of the earlier result. The only difference is that we must modify the key equation as

$$\mathfrak{C}_f = \mathfrak{C} \begin{bmatrix} I_p & -KB & -K(A-BK)B & -K(A-BK)^2B \\ 0 & I_p & -KB & -K(A-BK)B \\ 0 & 0 & I_p & -KB \\ 0 & 0 & 0 & I_p \end{bmatrix}$$

   where $\mathfrak{C}_f$ and $\mathfrak{C}$ are $n \times np$ controllability matrices with $n = 4$ and I_p is the unit matrix of order p.

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2. Because the rightmost $4p \times 4p$ matrix is nonsingular, $\mathfrak{C}_f$ has rank n if and only if $\mathfrak{C}$ has rank n. Thus the controllability property is preserved in any state feedback.

State feedback

Theorem

All eigenvalues of $(A - BK)$ *can be assigned arbitrarily (provided complex conjugate eigenvalues assigned in pairs) by selecting a real constant* K *if and only if* (A, B) *is controllable.*

State feedback

Theorem

All eigenvalues of $(A - BK)$ can be assigned arbitrarily (provided complex conjugate eigenvalues assigned in pairs) by selecting a real constant K if and only if (A, B) is controllable.

If (A, B) is not controllable, then (A, B) can be transformed into the form shown in (Uncontrollable Decomposition) and the eigenvalues of $\bar{A}_u$ will not be affected by any state feedback.

Lyapunov-Equation Method

Problem

Consider an n-dimensional p-input pair (A, B). Find a $p \times n$ real constant matrix K so that $(A - BK)$ has any set of desired eigenvalues as long as the set does not contain any eigenvalue of A.

Lyapunov-Equation Method

1. Select an $n \times n$ matrix F with a set of desired eigenvalues that contains no eigenvalues of A.
2. Select an arbitrary $p \times n$ matrix $\bar{K}$ such that $(F', \bar{K}')$ is controllable.
3. Solve the unique T in the Lyapunov equation $AT - TF = B\bar{K}$.

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     $$(A - BK)T = TF \text{ or } A - BK = TFT^{-1}$$

     Thus $A - BK$ and F are similar and have the same set of eigenvalues.

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   - Unlike the SISO case where T is *always* nonsingular, the T here may not be nonsingular even if (A, B) is controllable and $(F', \bar{K}')$ is controllable.

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   - Unlike the SISO case where T is *always* nonsingular, the T here may not be nonsingular even if (A, B) is controllable and $(F', \bar{K}')$ is controllable.

$$\det T \neq 0 \quad \underset{\not\Leftarrow\ \text{suf}}{\overset{\text{nec}}{\Longrightarrow}} \quad \mathtt{rank}\mathfrak{C}_{(A,B)} = \mathtt{rank}\mathfrak{C}_{(F',\bar{K}')} = n$$

Lyapunov-Equation Method

Theorem

If A and F have no eigenvalues in common, then the unique solution T of $AT - TF = B\bar{K}$ is nonsingular $only\ if$ the pairs (A, B) and $(F', \bar{K}')$ are controllable.

Lyapunov-Equation Method

Theorem

If A and F have no eigenvalues in common, then the unique solution T of $AT - TF = B\bar{K}$ is nonsingular only if the pairs (A, B) and $(F', \bar{K}')$ are controllable.

Proof.

The proof is similar to that of the previous except that

$$\Delta(A)T - T\Delta(F) = -T\Delta(F) = \begin{bmatrix} b & Ab & A^2b & A^3b \end{bmatrix} \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 & 1 \\ \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{k} \\ \bar{k}F \\ \bar{k}F^2 \\ \bar{k}F^3 \end{bmatrix}$$

now modifies to

$$= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \begin{bmatrix} \alpha_3 I & \alpha_2 I & \alpha_1 I & I \\ \alpha_2 I & \alpha_1 I & I & 0 \\ \alpha_1 I & I & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{K} \\ \bar{K}F \\ \bar{K}F^2 \\ \bar{K}F^3 \end{bmatrix}$$

$$= \mathfrak{C}_{(A,B)} \Sigma \mathfrak{C}_{(F',\bar{K}')}$$

where $\Delta(F)$ is nonsingular and $\mathfrak{C}_{(A,B)}$, Σ, and $\mathfrak{C}_{(F',\bar{K}')}$ are, respectively, $n \times np$, $np \times np$ and $np \times n$. If $\mathfrak{C}_{(A,B)}$ or $\mathfrak{C}_{(F',\bar{K}')}$ has rank less than n, then T is singular. However the conditions that $\mathfrak{C}_{(A,B)}$ and $\mathfrak{C}_{(F',\bar{K}')}$ have rank n do not imply the nonsingularity of T. Thus the controllability of (A, B) and $(F', \bar{K}')$ are only necessary conditions for T to be nonsingular. □

Lyapunov-Equation Method

Cyclic Design

Idea

We change the multi-input problem into a single input problem and then apply earlier results.

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Definition (Cyclic matrix)

A matrix A is called $cyclic$ whenever the Jordan form of A has one and only Jordan block associated with each distinct eigenvalue.

Cyclic Design

Theorem

If the n-dimensional p-input pair (A, B) is controllable and if A is cyclic, then for almost any $p \times l$ vector v, the single-input pair (A, Bv) is controllable.

Cyclic Design

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If the n-dimensional p-input pair (A, B) is controllable and if A is cyclic, then for almost any $p \times l$ vector v, the single-input pair (A, Bv) is controllable.

Controllability Invariance

Controllability is invariant under any equivalence transformation; thus we may assume A to be in Jordan form.

Cyclic Design: Logical idea behind the proof

To see the basic idea, we use the following example:

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 2 \\ 4 & 3 \\ 1 & 0 \end{bmatrix} \quad Bv = B \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ \alpha \\ \times \\ \beta \end{bmatrix}$$

There is only one Jordan block associated with each distinct eigenvalue; thus A is cyclic.

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Exercise

The necessary and sufficient conditions for (A, Bv) to be controllable are $\alpha \neq 0$ and $\beta \neq 0$.

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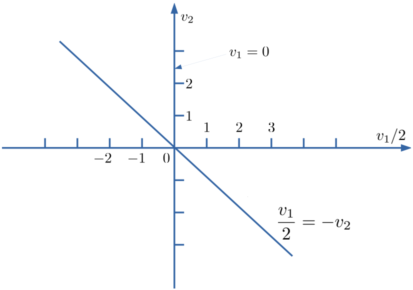
The necessary and sufficient conditions for (A, Bv) to be controllable are $\alpha \neq 0$ and $\beta \neq 0$.

Because $\alpha = v_1 + 2v_2$ and $\beta = v_1$, either α or β is zero if and only if $v_1 = 0$ or $v_1/v_2 = -2/1$. Thus any v other than $v_1 = 0$ and $v_1 = -2v_2$ will make (A, Bv) controllable.

Cyclic Design: Logical idea behind the proof

The vector v can assume any value in the two-dimensional real space.



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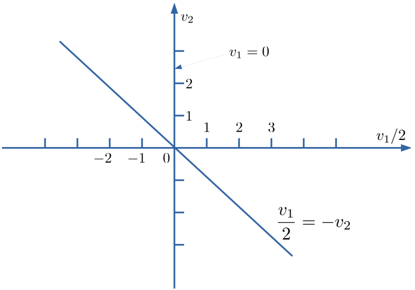

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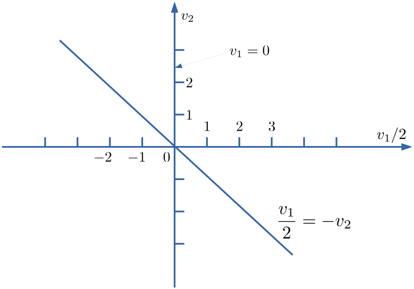

The cyclicity assumption in this theorem is essential. For example, the pair

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

is controllable. However, there is no v such that (A, Bv) is controllable.

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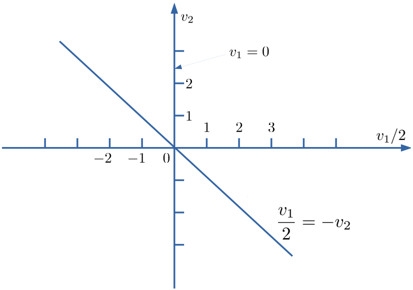

The cyclicity assumption in this theorem is essential. For example, the pair

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$$

is controllable. However, there is no v such that (A, Bv) is controllable.

If all eigenvalues of A are distinct, then there is only one Jordan block associated with each eigenvalue. Thus a sufficient condition for A to be cyclic is that all eigenvalues of A are distinct.

Cyclic Design

Theorem

If (A, B) is controllable, then for almost any $p \times n$ real constant matrix K, the matrix $(A - BK)$ has only distinct eigenvalues and is, consequently cyclic.

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If the n-dimensional p-input pair (A, B) is controllable and if A is cyclic, then for almost any $p \times l$ vector v, the single-input pair (A, Bv) is controllable.

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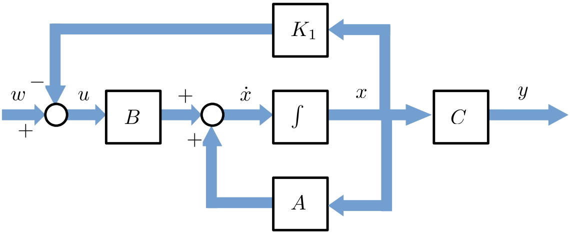
With these two theorems, we can now find a K to place all eigenvalues of $(A - BK)$ in any desired positions.

If A is not cyclic, we introduce $u = w - K_1 x$, such that $\bar{A} := A - BK_1$ in

$$\dot{x} = (A - BK_1)x + Bw =: \bar{A}x + Bw$$

is cyclic.



Cyclic Design

Because (A, B) is controllable, so is $(\bar{A}, B)$. Thus there exists a $p \times 1$ real vector v such that $(\bar{A}, Bv)$ is controllable[1].

[1] The choice of K_1 and v are not unique. They can be chosen arbitrarily.

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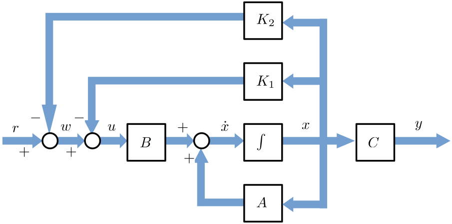
Next we introduce "another" state feedback $w = r - K_2 x$ with $K_2 = vk$, where k is a $1 \times n$ real vector. Then

$$\dot{x} = (\bar{A} - BK_2)x + Br = (\bar{A} - Bvk)x + Br$$



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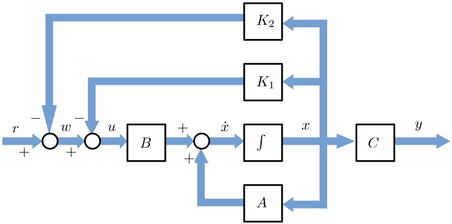

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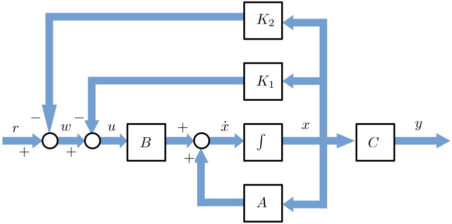
Combining the two state feedback $u = w - K_1 x$ and $w = r - K_2 x$ as

$$u = r - (K_1 + K_2)x =: r - Kx$$

we obtain a $K := K_1 + K_2$ that achieves arbitrary eigenvalue assignment.



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The Linear Quadratic Regulator Problem

Problem

Given a continuous-time LTI system:

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

the *linear quadratic regulation (LQR)* problem consists of finding the control signal $u(t)$ that makes the following criterion as small as possible:

$$J_{LQR} \triangleq \int_0^\infty y^T(t)Qy(t) + u^T(t)Ru(t)dt, \qquad \text{(Cost function)}$$

where Q and R are the positive-definite weighting matrices.

The following terms provides a *measure*

$$\int_0^\infty y^T(t)Qy(t) \qquad \text{(Output energy)}$$

$$\int_0^\infty u^T(t)Ru(t) \qquad \text{(Control energy)}$$

The Linear Quadratic Regulator Problem

$$\boxed{J_{LQR} \triangleq \int_0^\infty y^T(t)Qy(t) + u^T(t)Ru(t)dt}$$

In LQR one seeks a controller that minimizes both energies. However, decreasing the energy of the output requires a large control signal, and a small control signal leads to large outputs.

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The role of the weighting matrices Q and R is to establish a trade-ff between these two conflicting goals.

1. When R is much larger than Q, the most effective way to decrease J_{LQR} is to employ a small control input at the expense of a large output.
2. When R is much smaller than Q, the most effective way to decrease J_{LQR} is to obtain a very small output, even if this is acheived at the expense of employing a large control input.

Feedback Invariants

Definition (Feedback Invariant)

Given a continuous-time LTI system:

$$\dot{x} = Ax + Bu, \qquad y = Cx, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k, y \in \mathbb{R}^m \quad \text{(CLTI)}$$

we say that a functional

$$H(x(\cdot), u(\cdot))$$

that involves the system's input and state is a *feedback invariant* for the system (CLTI) whenever its value depends only on the initial condition $x(0)$ and not on the specific input $u(\cdot)$.

Feedback Invariants

Theorem (Feedback Invariant)

For a symmetric matrix P, the functional

$$H(x(\cdot), u(\cdot)) \triangleq -\int_0^\infty (Ax(t) + Bu(t))^T Px(t) + x^T(t)P(Ax(t) + Bu(t))\,dt$$

is a feedback invariant for CLTI as long as $\lim_{t\to\infty} x(t) = 0$.

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Proof.

We can write H as

$$\begin{aligned} H(x(\cdot), u(\cdot)) &= -\int_0^\infty \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t)dt \\ &= -\int_0^\infty \frac{d(x^T Px)}{dt} \\ &= x(0)^T Px(0) - \lim_{t\to\infty} x^T Px = x^T(0)Px(0), \end{aligned}$$

as long as $\lim_{t\to\infty} x(t) = 0$. $\square$

Feedback Invariants in Optimal Control

Suppose that we are able to express a criterion J to be minimized by an appropriate choice of the input $u(\cdot)$ in the following form:

$$J = H(x(.), u(.)) + \int_0^\infty \Lambda(x(t), u(t))dt, \qquad \text{(criterion)}$$

where H is a feedback invariant and the function $\Lambda(x, u)$ has the property that for every $x \in \mathbb{R}^n$

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Note that it is not possible to get a lower value for J since

1. the feedback invariant H is never affected by u and
2. a smaller value for J would require the integral in the right hand side of (criterion) to be negative, which is not possible since $\Lambda(x(t), u(t))$ can at best be as low as zero.

Optimal State Feedback

$$J = H(x(.), u(.)) + \int_0^\infty \Lambda(x(t), u(t))dt, \qquad \text{(criterion)}$$

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To check that this is so, we add and subtract this feedback invariant to the LQR criterion and conclude that

$$\begin{aligned} J_{LQR} &= \int_0^\infty \left(x^T C^T Q C^T x + u^T R u\right) dt \\ &= H + \int_0^\infty \left(x^T C^T Q C^T x + u^T R u + (Ax + Bu)^T P x + x^T P (Ax + Bu)\right) dt \end{aligned}$$

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By completing the squares as follows, we group the quadratic term in u with the cross-term in u times x

$$\left(u^T + x^T K^T\right) R \left(u + Kx\right) = u^T R u + x^T P B R^{-1} B^T P x + 2u^T B^T P x,$$

$$K = R^{-1} B^T P.$$

Optimal State Feedback

We conclude that

$$J_{LQR} = H(x(.), u(.)) + \int_0^\infty x^T \left(A^T P + PA + C^T Q C^T - PBR^{-1} B^T P \right) x \\ + \left(u^T + x^T K^T \right) R \left(u + Kx \right) dt.$$

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If we are able to select the matrix P so that

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leading to the closed-loop system

$$\dot{x} = Ax + BKx = \left(A - BR^{-1}B^T P \right) x.$$

Optimal state feedback

Theorem (Optimal State Feedback)

Assume that there exists a symmetric matrix P to the following algebraic Riccati equation (ARE)

$$A^T P + PA + C^T QC - PBR^{-1}B^T P = 0 \qquad \text{(ARE)}$$

for which $A - BR^{-1}B^T P$ is a stability matrix. Then the feedback control law

$$u = -Kx, \qquad K = R^{-1}B^T P,$$

stabilizes the closed-loop system while minimizing the LQR criterion

$$J_{LQR} \triangleq \int_0^\infty y^T(t)Qy(t) + u^T(t)Ru(t)dt.$$

Note: Asymptotic stability of the closed loop system is needed because we assumed that $\lim_{t\to\infty} x(t)Px(t) = 0$.

---

[1] Kumar and Jain. Some Insights on Synthesizing Optimal Linear Quadratic Controllers using Krotov Sufficient Conditions, IEEE Control Systems Letters, 2020.

Optimal state feedback

Attention

The ARE itself already provides the clues about whether or not the closed-loop system is stable. Indeed if we write the Lyapunov equation for the closed loop, we get

$$\begin{aligned}(A - BR^{-1}B^TP)^TP + P(A - BR^{-1}B^TP) \\ = A^TP + PA - 2PBR^{-1}B^TP \\ = -\bar{Q} \leq 0\end{aligned}$$

for $\bar{Q} = C^TQC + PBR^{-1}B^TP \geq 0$. In case $P > 0$ and $\bar{Q} > 0$, we could immediately conclude that the closed loop system was stable by Lyapunov stability theorem.

LQR with MATLAB

The command

```
[K,P,E] = lqr(A,B,QQ,RR,NN)
```

computes the optimal state feedback LQR controller for the process

$$\dot{x} = \mathtt{A}x + \mathtt{B}u$$

with the criterion

$$J = \int_0^\infty x(t)'\mathtt{QQ}x(t) + u(t)'\mathtt{RR}u(t) + 2x(t)'\mathtt{NN}u(t)dt.$$

For the criterion in (Cost function), one should select

$$\mathtt{QQ} = C'QC, \quad \mathtt{RR} = R, \quad \mathtt{NN} = 0.$$

This command returns the optimal state feedback matrix `K`, the solution `P` to the corresponding algebraic Riccati equation, and the poles `E` of the closed-loop system.