Linear Dynamical Systems

Week 7 - Observability and Minimal Realization

Outline of Week 7

1. Observability and its tests
2. Kalman Decomposition
3. Detectability and its tests
4. Minimal Realization

Motivation: Output Feedback

Consider the continuous-time LTI system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \qquad x \in R^n, u \in R^k, y \in R^m \qquad \text{(CLTI)}$$

We know that if the pair (A, B) is stabilizable, then there exists a state feedback law

$$u = -Kx \qquad \text{(Control law)}$$

that asymptotically stabilize the system (CLTI), i.e., for which $(A - BK)$ is a stability matrix.

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However, when only the output y can be measured (as opposed to the whole state x), the (Control law) cannot be implemented.

Possible solution

In principle, this difficulty can be overcome if it is possible to reconstruct the state of the system based on its measured output and perhaps also on the control input that is applied.

Motivation: Output Feedback

When the matrix C is invertible, instantaneous reconstruction of x from y and u is possible by solving the output equation for x

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When the number of outputs is strictly less than number of states, instantaneous reconstruction of x is not possible, but it may still be possible to reconstruct the state from the output $y(t)$ and input $u(t)$ over the time interval $[t_0, t_1]$.

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Two formulations are usually considered.

1. *Observability* refers to determining $x(t_0)$ from the *future* inputs and outputs, $u(t)$ and $y(t), t \in [t_0, t_1]$.
2. *Constructibility* refers to determining $x(t_1)$ from the *past* inputs and outputs, $u(t)$ and $y(t), t \in [t_0, t_1]$.

Unobservable Subspace

Consider the continuous-time LTV system

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u, \qquad x \in R^n, u \in R^k, y \in R^m \tag{CLTV}$$

We know that the system's state $x_0 := x(t_0)$ at time t_0 is related to its input and output on the interval $[t_0, t_1]$ by the variation of constants formula:

$$y(t) = C(t)\phi(t,t_0)x_0 + \int_{t_0}^{t_1} C(t)\phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t), \qquad \forall t \in [t_0, t_1],$$

where $\phi(.)$ denotes the system's state transition matrix.

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where $\phi(.)$ denotes the system's state transition matrix.

To study the system's observability, we need to determine under which conditions we can solve

$$\tilde{y}(t) = C(t)\phi(t,t_0)x_0, \qquad \forall t \in [t_0, t_1]$$

for the unknown $x_0 \in R^n$, where

$$\tilde{y}(t) = y(t) - \int_{t_0}^{t} C(t)\phi(t,\tau)B(\tau)u(\tau)d\tau - D(t)u(t), \qquad \forall t \in [t_0, t_1].$$

Unobservable Subspace

Definition (Unobservable subspace)

Given two times $t_1 > t_0 \geq 0$, the *unobservable subspace on* $[t_0, t_1]$, *i.e.*, $\mathcal{UO}[t_0, t_1]$ consists of all states $x_0 \in \mathbb{R}^n$ for which

$$\boxed{C(t)\phi(t, t_0)x_0 = 0}, \qquad \forall \in [t_0, t_1].$$

Unobservable Subspace

Properties (Unobservable subspace)

Suppose we are given two times $t_1 > t_0 \geq 0$ and an input/output pair $u(t), y(t), [t_0, t_1]$.

1. When a particular initial state $x_0 = x(t_0)$ is compatible with the input/output pair, then every initial state of the form

$$x_0 + x_u, \quad x_u \in \mathcal{UO}[t_0, t_1]$$

is also compatible with the same input/output pair. This is because

$$\begin{cases} \tilde{y} = C(t)\phi(t, t_0)x_0, & \forall t \in [t_0, t_1] \\ 0 = C(t)\phi(t, t_0)x_u, & \forall t \in [t_0, t_1] \end{cases}$$

$$\Rightarrow \tilde{y} = C(t)\phi(t, t_0)(x_0 + x_u) \quad \forall t \in [t_0, t_1].$$

Unobservable Subspace

Properties (Unobservable subspace)

Suppose we are given two times $t_1 > t_0 \geq 0$ and an input/output pair $u(t), y(t), [t_0, t_1]$.

2. When the unobservable subspace contains only the zero vector, then there exists at most one initial state that is compatible with the input/output pair [1].

[1] Because of this property, it is possible to uniquely reconstruct the state of an observable system from (future) inputs/outputs.

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2. When the unobservable subspace contains only the zero vector, then there exists at most one initial state that is compatible with the input/output pair [1].

   This is because if two different states $x_0, \bar{x}_0 \in \mathbb{R}^n$ were compatible with the same input/output pair, i.e.,

$$\begin{cases} \tilde{y} = C(t)\phi(t,t_0)x_0, & \forall t \in [t_0, t_1] \\ \tilde{y} = C(t)\phi(t,t_0)\bar{x}_0, & \forall t \in [t_0, t_1] \end{cases}$$

$$\Rightarrow 0 = C(t)\phi(t,t_0)(x_0 - \bar{x}_0) \quad \forall t \in [t_0, t_1],$$

   and therefore $x_0 - \bar{x}_0 \neq 0$ would have to belong to the unobservable subspace.

[1] Because of this property, it is possible to uniquely reconstruct the state of an observable system from (future) inputs/outputs.

Unobservable Subspace

The above properties motivate the following definition.

Definition (Observable system)

Given two times $t_1 > t_0 \geq 0$, the system (CLTV) is *observable* whenever its unobservable subspace contains only the zero vector; i.e., $\mathcal{UO}[t_0, t_1] = 0$.

The matrices $B(.)$ and $D(.)$ play no role in the definition of the unobservable subspace; therefore one often simply talks about the unobservable subspace or the observability of the system

$$\dot{x} = A(t)x, \quad y = C(t)x \qquad x \in \mathbb{R}^n, y \in \mathbb{R}^m \qquad \text{(AC-CLTV)}$$

Unconstructible Subspace

The "future" system's state $x_1 := x(t_1)$ at time t_1 can also be related to the system's input and output on the interval $[t_0, t_1]$ by the variation of constants formula:

$$y(t) = C(t)\phi(t, t_1)x_1 + \int_{t_1}^{t} C(t)\phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t), \quad \forall \in [t_0, t_1]$$

Definition (Unconstructible subspace)

Given two times $t_1 > t_0 \geq 0$, the *unconstructible subspace on* $[t_0, t_1]$, *i.e.*, $\mathcal{UC}[t_0, t_1]$ consists of all states x_1 for which

$$C(t)\phi(t, t_1)x_1 = 0, \qquad \forall t \in [t_0, t_1].$$

Unconstructible Subspace

Properties(Unconstructible subspace)

Suppose we are given two times $t_1 > t_0 \geq 0$ and an input/output pair $u(t), y(t), t \in [t_0, t_1]$.

1. When a particular final state $x_1 = x(t_1)$ is compatible with the input/output pair, then every final state of the form

$$x_1 + x_u, \qquad x_u \in \mathcal{UC}[t_0, t_1]$$

   is also compatible with the same input/output pair.

2. When the unconstructible subspace contains only the zero vector, then there exists at most one final state that is compatible with the input/output pair.

Definition (Constructible system)

Given two times $t_1 > t_0 \geq 0$, the system (CLTV) is *constructible* whenever its unconstructible subspace contains only the zero vector, i.e., $\mathcal{UC}[t_0, t_1] = 0$.

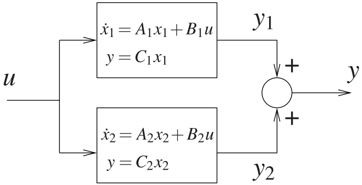
Physical example

Parallel interconnection:

Consider the below interconnection of two systems with states $x_1, x_2 \in \mathbb{R}^n$.
The overall system corresponds to following state space model

$$\dot{x} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} x$$

where we chose for state $x := \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T \in \mathbb{R}^{2n}$.



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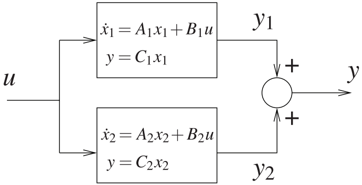
$$y(t) = C_1 e^{A_1 t} x_1(0) + C_1 e^{A_2 t} x_2(0) + \int_0^t (C_1 e^{A_1(t-\tau)} B_1 + C_2 e^{A_2(t-\tau)} B_2) u(\tau) d\tau.$$



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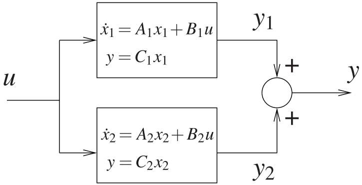
When $A_1 = A_2 = A$ and $C_1 = C_2 = C$, we have

$$y(t) = Ce^{At}\left(x_1(0) + x_2(0)\right) + \int_0^t Ce^{A(t-\tau)}(B_1 + B_2) u(\tau) d\tau.$$



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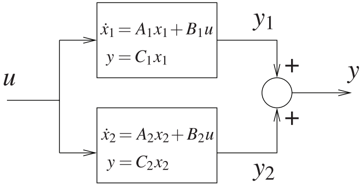
$$y(t) = Ce^{At}\left(x_1(0) + x_2(0)\right) + \int_0^t Ce^{A(t-\tau)}(B_1 + B_2)u(\tau)d\tau.$$

This shows that, solely by knowing the input and output of the system, we cannot distinguish between initial states for which $x_1(0) + x_2(0)$ is the same.



Subspace characterization using Gramians

Definition (Observability and Constructibility Gramians)

Given two times $t_1 > t_0 \geq 0$, the *observability* and *constructibility* Gramians[1] of the system (CLTV) are defined by

$$W_O(t_0, t_1) := \int_{t_0}^{t_1} \Phi(\tau, t_0)^T C(\tau)^T C(\tau) \Phi(\tau, t_0) d\tau,$$

$$W_{Cn}(t_0, t_1) := \int_{t_0}^{t_1} \Phi(\tau, t_1)^T C(\tau)^T C(\tau) \Phi(\tau, t_1) d\tau.$$

[1] Both Gramians are symmetric positive-semidefinite $n \times n$ matrices.

Subspace characterization using Gramians

Theorem (Unobservable and Unconstructible subspaces)

Given two times $t_1 > t_0 \geq 0$*,*

$$\mathcal{UO}[t_0, t_1] = \texttt{ker} W_O(t_0, t_1), \qquad \mathcal{UC}[t_0, t_1] = \texttt{ker} W_{Cn}(t_0, t_1).$$

Proof.

From the definition of the observability Gramian, for every $x_0 \in \mathbb{R}^n$, we have

$$\begin{aligned} x_0^T W_O(t_0, t_1) x_0 &= \int_{t_0}^{t_1} x_0^T \Phi(\tau, t_0)^T C(\tau)^T C(\tau) \Phi(\tau, t_0) x_0 d\tau \\ &= \int_{t_0}^{t_1} \|C(\tau)\Phi(\tau, t_0) x_0\|^2 d\tau. \end{aligned}$$

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Therefore

$$\begin{aligned} x_0 \in \mathtt{ker} W_O(t_0, t_1) &\implies C(\tau)\Phi(\tau, t_0) x_0 = 0, \quad \forall \tau \in [t_0, t_1] \\ &\implies x_0 \in \mathcal{UO}[t_0, t_1] \quad \text{from definition.} \end{aligned}$$

Conversely,

$$\begin{aligned} x_0 \in \mathcal{UO}[t_0, t_1] &\implies C(\tau)\Phi(\tau, t_0) x_0 = 0, \quad \forall \tau \in [t_0, t_1] \\ &\implies x_0 \in \mathtt{ker} W_O(t_0, t_1). \end{aligned}$$

For the second implication, we are using the fact, for any given positive-semidefinite matrix W, $x^T W x = 0$ implies that $Wx = 0$. This implication is *not true for nonsemidefinite matrices.* A similar argument can be made for the unconstructible subspace. □

Subspace characterization using Gramians

This result provides a first method to determine whether a system is observable or constructible, because the kernel of a square matrix contains only the zero vector when the matrix is nonsingular.

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Theorem (Observable and Constructible systems)

Suppose we are given two times $t_1 > t_0 \geq 0$.

1. *The system (CLTV) is observable if and only if* $\boxed{\texttt{rank} W_O(t_0, t_1) = n}$.
2. *The system (CLTV) is constructible if and only if* $\boxed{\texttt{rank} W_{Cn}(t_0, t_1) = n}$.

Gramian-based reconstruction

Consider the continuous-time LTV system

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k,\ y \in \mathbb{R}^m. \tag{CLTV}$$

We have seen that the system's state $x_0 := x(t_0)$ at time t_0 is related to its input and output on the interval $[t_0, t_1]$ by

$$\tilde{y}(t) = C(t)\Phi(t, t_0)x_0, \quad \forall t \in [t_0, t_1],$$

where

$$\tilde{y}(t) = y(t) - \int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau - D(t)u(t), \quad \forall t \in [t_0, t_1].$$

Gramian-based reconstruction

Premultiplying by $\Phi(t, t_0)^T C(t)^T$ and integrating between t_0 and t_1 yields

$$\int_{t_0}^{t_1} \Phi(t, t_0)^T C(t)^T \tilde{y}(t) dt = \int_{t_0}^{t_1} \Phi(t, t_0)^T C(t)^T C(t) \Phi(t, t_0) x_0 dt,$$

which can be written as

$$\int_{t_0}^{t_1} \Phi(t, t_0)^T C(t)^T \tilde{y}(t) dt = W_O(t_0, t_1) x_0,$$

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which can be written as

$$\int_{t_0}^{t_1} \Phi(t,t_0)^T C(t)^T \tilde{y}(t)dt = W_O(t_0,t_1)x_0,$$

If the system is observable, $W_O(t_0,t_1)$ is invertible, and we conclude that

$$x_0 = W_0(t_0,t_1)^{-1} \int_{t_0}^{t_1} \Phi(t,t_0)^T C(t)^T \tilde{y}(t)dt,$$

which allows us to reconstruct $x(t_0)$ from the future inputs and outputs on $[t_0,t_1]$. A similar construction can be carried out to reconstruct $x(t_1)$ from past inputs and outputs for reconstructible systems.

Gramian-based reconstruction

Theorem (Gramian-based reconstruction)

Suppose we are given two times $t_1 > t_0 \geq 0$ and an input/output pair $u(t), y(t), \forall t \in [t_0, t_1]$.

1. *When the system (CLTV) is observable*

$$\boxed{x(t_0) = W_O(t_0, t_1)^{-1} \int_{t_0}^{t_1} \Phi(t, t_0)^T C(t)^T \tilde{y}(t) dt},$$

where

$$\tilde{y}(t) := y(t) - \int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau - D(t)u(t), \quad \forall t \in [t_0, t_1].$$

2. *When the system (CLTV) is constructible*

$$\boxed{x(t_1) = W_{Cn}(t_0, t_1)^{-1} \int_{t_0}^{t_1} \Phi(t, t_1)^T C(t)^T \bar{y}(t) dt},$$

where

$$\bar{y}(t) := y(t) - \int_{t_1}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau - D(t)u(t), \quad \forall t \in [t_0, t_1].$$

Discrete-time Case

Consider the discrete time LTV system

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t), \qquad \text{(DLTV)}$$

for which the system's state $x_0 := x(t_0)$ at time t_0 is related to its input and output on the interval $t_0 \leq t \leq t_1$ by the variations of constant formula,

$$y(t) = C(t)\Phi(t, t_0)x_0 + \sum_{\tau=t_0}^{t-1} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t) \ \ \forall t_0 \leq t \leq t_1$$

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Definition (Unobservable and unconstructible subspaces)

Given two times $t_1 > t_0 \geq 0$, the *unobservable subspace* on $[t_0, t_1)$, $\mathcal{UO}[t_0, t_1)$ consists of all states x_0 for which

$$C(t)\Phi(t,t_0)x_0 = 0, \quad \forall t_0 \leq t < t_1.$$

The *unconstructible subspace on* $[t_0, t_1)$, $\mathcal{UC}[t_0, t_1)$ consists of all states x_1 for which

$$C(t)\Phi(t,t_1)x_1 = 0, \quad \forall t_0 \leq t < t_1.$$

Discrete-time Case

Attention!

The definition of the discrete-time unconstructible subspace requires a backward-in-time state transition matrix $\Phi(t, t_1)$ from time t_1 to time $t \leq t_1 - 1 < t_1$. This matrix is well defined only when

$$x(t_1) = A(t_1 - 1)A(t_1 - 2) \cdots A(\tau)x(t), \quad t_0 \leq \tau \leq t_1 - 1$$

can be solved for $x(t)$, i.e., when all the matrices $A(t_0), A(t_0 + 1), \ldots, A(t_1 - 1)$ are nonsingular. When this does not happen, *the unconstructibility subspace cannot be defined.*

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can be solved for $x(t)$, i.e., when all the matrices $A(t_0), A(t_0 + 1), \ldots, A(t_1 - 1)$ are nonsingular. When this does not happen, *the unconstructibility subspace cannot be defined.*

Definition (Observable and Constructible systems)

Given two times $t_1 > t_0 \geq 0$, the system (DLTV) is *observable* whenever its unobservable subspace contains only the zero vector, and it is *constructible* whenever its unconstructible subspace contains only the zero vector.

Discrete-time Case

Definition (Observability and constructibility Gramians)

Given two times $t_1 > t_0 \geq 0$, the *observability* and *constructibility Gramians* of the system (AC-DLTV) are defined by

$$W_O(t_0, t_1) := \sum_{\tau=t_0}^{t_1-1} \Phi(\tau, t_0)^T C(\tau)^T C(\tau) \Phi(\tau, t_0),$$

$$W_{Cn}(t_0, t_1) := \sum_{\tau=t_0}^{t_1-1} \Phi(\tau, t_1)^T C(\tau)^T C(\tau) \Phi(\tau, t_1)$$

Theorem (Unobservable and Unconstructible subspaces)

Given two times $t_1 > t_0 \geq 0$,

$$\mathcal{UO}[t_0, t_1) = \texttt{ker} W_O(t_0, t_1), \qquad \mathcal{UC}[t_0, t_1) = \texttt{ker} W_{Cn}(t_0, t_1)$$

Dicrete-time Case

Theorem (Gramian-based reconstruction)

Suppose we are given two times $t_1 > t_0 \geq 0$ *and an input/output pair* $u(t), y(t), t_0 \leq t < t_1$.

1. *When the system (DLTV) is observable*

$$x(t_0) = W_O(t_0, t_1)^{-1} \sum_{t=t_0}^{t_1 - 1} \Phi(t, t_0)^T C(t)^T \tilde{y}(t).$$

2. *When the system (DLTV) is constructible*

$$x(t_1) = W_{Cn}(t_0, t_1)^{-1} \sum_{t=t_0}^{t_1 - 1} \Phi(t, t_1)^T C(t)^T \bar{y}(t).$$

Duality (LTI)

Consider the continuous-time LTI system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^k, y \in \mathbb{R}^m. \tag{CLTI}$$

So far we have shown the following

The system (CLTI) is controllable $\iff \texttt{rank} W_C(t_0, t_1) = n,$

where $W_C(t_0, t_1) := \int_{t_0}^{t_1} e^{A(\tau - t_0)} BB^T e^{A^T(\tau - t_0)} d\tau$.

The system (CLTI) is observable on $[t_0, t_1] \iff \texttt{rank} W_O(t_0, t_1) = n,$

where $W_O(t_0, t_1) := \int_{t_0}^{t_1} e^{A^T(\tau - t_0)} C^T C e^{A(\tau - t_0)} d\tau$.

Duality (LTI)

Suppose we construct the following dual system.

$$\dot{\bar{x}} = A^T\bar{x} + C^T\bar{u}, \quad \bar{y} = B^T\bar{x} + D^T\bar{u}, \quad \bar{x} \in \mathbb{R}^n, \bar{u} \in \mathbb{R}^m, \bar{y} \in \mathbb{R}^k. \tag{DUAL-CLTI}$$

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For this system we have the following.

The system (DUAL-CLTI) is controllable $\iff \texttt{rank}\bar{W}_C(t_0, t_1) = n,$

where $\bar{W}_C(t_0, t_1) := \int_{t_0}^{t_1} e^{A^T(\tau - t_0)} C^T C e^{A(\tau - t_0)} d\tau.$

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The system (DUAL-CLTI) is observable on $[t_0, t_1] \iff \mathtt{rank}\bar{W}_O(t_0, t_1) = n,$

where $\bar{W}_O(t_0, t_1) := \int_{t_0}^{t_1} e^{A(\tau - t_0)}BB^Te^{A^T(\tau-t_0)}d\tau.$

Recall!

$$W_C(t_0, t_1) := \int_{t_0}^{t_1} e^{A(\tau-t_0)}BB^Te^{A^T(\tau-t_0)}d\tau$$
$$W_O(t_0, t_1) := \int_{t_0}^{t_1} e^{A^T(\tau-t_0)}C^TCe^{A(\tau-t_0)}d\tau.$$

Duality (LTI)

Theorem (Duality controllability/observability)

Suppose we are given two times $t_1 > t_0 \geq 0$

1. *The system (CLTI) is controllable if and only if the system (DUAL-CLTI) is observable on* $[t_0, t_1]$.
2. *The system (CLTI) is observable on* $[t_0, t_1]$ *if and only if the system (DUAL-CLTI) is controllable.*

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2. *The system (CLTI) is observable on* $[t_0, t_1]$ *if and only if the system (DUAL-CLTI) is controllable.*

Theorem (Duality reachability/constructability)

Suppose we are given two times $t_1 > t_0 \geq 0$

1. *The system (CLTI) is reachable if and only if the system (DUAL-CLTI) is constructible on* $[t_0, t_1]$*.*
2. *The system (CLTI) is constructible on* $[t_0, t_1]$ *if and only if the system (DUAL-CLTI) is reachable.*

Duality (LTI)

Theorem (Duality controllability/observability)

Suppose we are given two times $t_1 > t_0 \geq 0$

1. *The system (CLTI) is controllable if and only if the system (DUAL-CLTI) is observable on $[t_0, t_1]$.*
2. *The system (CLTI) is observable on $[t_0, t_1]$ if and only if the system (DUAL-CLTI) is controllable.*

Theorem (Duality reachability/constructability)

Suppose we are given two times $t_1 > t_0 \geq 0$

1. *The system (CLTI) is reachable if and only if the system (DUAL-CLTI) is constructible on $[t_0, t_1]$.*
2. *The system (CLTI) is constructible on $[t_0, t_1]$ if and only if the system (DUAL-CLTI) is reachable.*

Theorem (Duality)

The pair (A, B) is controllable if and only if the pair (A', B') is observable.

Observability Tests

Consider the LTI systems

$$\dot{x}/x^{+} = Ax, \quad y = Cx, \qquad x \in \mathbb{R}^{n}, y \in \mathbb{R}^{m} \tag{AC-LTI}$$

From the duality theorems , we can conclude that a pair (A, C) is observable if and only if the pair (A^{T}, C^{T}) is controllable.

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This allows us to use *all* previously discussed tests for controllability to determine whether or not a system is observable.

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This allows us to use *all* previously discussed tests for controllability to determine whether or not a system is observable.

To apply the controllability matrix test to the pair (A^T, C^T), we construct the corresponding controllability matrix

$$\mathfrak{C} = \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T & \cdots & (A^T)^{n-1} C^T \end{bmatrix}_{(kn)\times n} = \mathfrak{O}^T$$

where $\mathfrak{O}$ denotes the *observability matrix* of the system (AC-LTI), which is defined by

$$\mathfrak{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Observability Tests

Since $\texttt{rank}\mathfrak{C} = \texttt{rank}\mathfrak{O}^T = \texttt{rank}\mathfrak{O}$, we obtain the following tests.

Theorem (Observability tests)

The following statements are equivalent.

1. *The system (AC-LTI) is observable.*
2. $\texttt{rank}\mathfrak{O} = n.$
3. *No eigenvector of A is in the kernel of C.*
4. $\texttt{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n, \qquad \forall \lambda \in \mathbb{C}.$

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Theorem (Lyapunov test for observability)

Assume that A is a stability matrix/Schur stable. The system (AC-LTI) is observable if and only if there is a unique positive-definite solution W to the Lyapunov equation

$$A^T W + WA = -C^T C \quad / \quad A^T W A - W = -C^T C$$

Moreover, the unique solution to this equation is

$$W = \int_0^\infty e^{A^T \tau} C^T C e^{A\tau} d\tau = \lim_{t_1 - t_0 \to \infty} W_O(t_0, t_1)$$

$$/ \quad W = \sum_{\tau=0}^{\infty} (A^T)^\tau C^T C A^\tau d\tau = \lim_{t_1 - t_0 \to \infty} W_O(t_0, t_1).$$

Observability Tests: LTV

Theorem (Necessary and Sufficient condition)

The pair $(A(t), C(t))$ is observable at time t_0 if and only there exists a finite $t_1 > t_0$ such that the $n \times n$ matrix

$$W_O(t_0, t_1) := \int_{t_0}^{t_1} \Phi(\tau, t_0)^T C(\tau)^T C(\tau) \Phi(\tau, t_0) d\tau,$$

is nonsingular.

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is nonsingular.

Theorem (Sufficient condition)

Let $A(t)$ and $C(t)$ be $n-1$ times continuously differentiable. Then the n-dimensional pair $(A(t), C(t))$ is observable at t_0 if there exists a finite $t_1 > t_0$ such that

$$\texttt{rank} \begin{bmatrix} N_0(t_1) \\ N_1(t_1) \\ \vdots \\ N_{n-1}(t_1) \end{bmatrix} = n$$

where

$$N_{m+1}(t) = N_m(t)A(t) + \frac{d}{dt} N_m(t) \quad m = 0, 1, \dots, n-1$$

with

$$N_0 = C(t).$$

Observability Tests: LTV

Theorem (Necessary and Sufficient condition)

The pair $(A(t), C(t))$ is observable at time t_0 if and only there exists a finite $t_1 > t_0$ such that the $n \times n$ matrix

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Attention!

For $time\text{-}varying\ systems$, duality is more "complicated", because the state transition matrix of the dual system must be the transpose of the state transition matrix of the original system, but this is not obtained by simply transposing $A(t)$.

Observable Decomposition

Consider the LTI system

$$\dot{x}/x^+ = Ax + Bu, \quad y = Cx + Du, \qquad x \in \mathbb{R}^n, y \in \mathbb{R}^m \qquad \text{(AC-LTI)}$$

and a similarity transformation $\bar{x} := T^{-1}x$, leading to :

$$\begin{aligned} \dot{\bar{x}}/x^+ &= \bar{A}\bar{x} + \bar{B}u, \quad y = \bar{C}\bar{x} + Du, \\ \bar{A} &= T^{-1}AT, \ \bar{B} = T^{-1}B, \ \bar{C} = CT \end{aligned}$$

The observability matrices of the system $\bar{\mathfrak{O}}$ and $\mathfrak{O}$ of the above two systems are related by

$$\bar{\mathfrak{O}} = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T = \mathfrak{O}T$$

Theorem (Invariance with respect to similarity transformations)

The pair (A, C) is observable if and only if the pair $(\bar{A}, \bar{C}) = (T^{-1}AT, CT)$ is observable.

Theorem (Invariance with respect to similarity transformations)

The pair (A, C) is observable if and only if the pair $(\bar{A}, \bar{C}) = (T^{-1}AT, CT)$ is observable.

Theorem (Observable decomposition)

For every LTI system (AC-LTI) there is a similarity transformation that takes the system to the form

$$\begin{bmatrix} A_o & 0 \\ A_{21} & A_u \end{bmatrix} = T^{-1}AT, \quad \begin{bmatrix} B_o & B_u \end{bmatrix} = T^{-1}B, \quad \begin{bmatrix} C_o & 0 \end{bmatrix} = CT$$

for which

1. *the unobservable subspace of the transformed system is given by*

$$\bar{\mathcal{UO}} = \mathtt{Im} \begin{bmatrix} 0 \\ I_{\bar{n} \times \bar{n}} \end{bmatrix}$$

   where $\bar{n}$ denotes the dimension of the unobservable subspace $\mathcal{UO}$ of the original system, and

2. *the pair (A_o, C_o) is observable*

Observable Decomposition

By partitioning the state of the transformed system as

$$\bar{x} = T^{-1}x = \begin{bmatrix} x_0 \\ x_u \end{bmatrix}$$

its state space model can be written as follows:

$$\begin{bmatrix} \dot{x}_0 \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_u \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + \begin{bmatrix} B_o \\ B_u \end{bmatrix} u, \quad y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + Du$$

The figure below shows a block representation of this system, which highlights the fact that the x_u component of the state $x(t)$ cannot be reconstructed from the output.

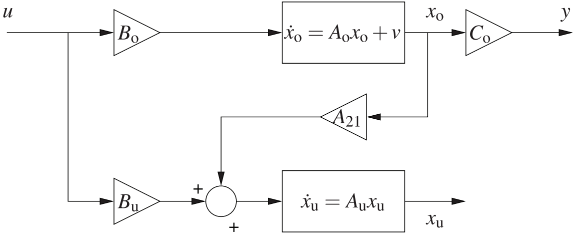

Figure: Observable Decomposition. The direct feed-through term D was omitted to simplify the diagram.

Kalman Decomposition

Consider the LTI system

$$\dot{x}/x^+ = Ax + Bu; \quad y = Cx + Du, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k, y \in \mathbb{R}^m \qquad \text{(LTI)}$$

We know that every LTI system can be transformed through a similarity transformation into the following standard form for uncontrollable systems:

$$\begin{bmatrix} \dot{x}_c/x_c^+ \\ \dot{x}_{\bar{c}}/x_{\bar{c}}^+ \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + Du$$

in which the pair (A_c, B_c) is controllable .This was obtained by choosing a similarity transformation

$$\begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} := T^{-1}x$$

$$T := \begin{bmatrix} V_c & V_{\bar{c}} \end{bmatrix},$$

where leftmost columns V_c form a basis for the (A-invariant) controllable subspace $\mathcal{C}$ of the pair (A, B).

Definition (A-invariant)

Given an $n \times n$ matrix A, a linear subspace $\mathcal{V}$ of $\mathbb{R}^n$ is said to be A-*invariant* whenever for every vector $v \in \mathcal{V}$ we have $Av \in \mathcal{V}$.

Kalman Decomposition

Using duality, we further concluded that every LTI system can also be transformed into the following standard form for unobservable systems:

$$\begin{bmatrix} \dot{x}_o/x_o^+ \\ \dot{x}_{\bar{o}}/x_{\bar{o}}^+ \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix} u$$

$$y = \begin{bmatrix} C_o & 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} + Du$$

in which the pair (A_o, C_o) is observable.
This is obtained by choosing the similarity transformation:

$$\begin{bmatrix} x_o \\ x_{\bar{o}} \end{bmatrix} := T^{-1}x, \qquad T := \begin{bmatrix} V_o & V_{\bar{o}} \end{bmatrix}$$

whose rightmost columns $V_{\bar{o}}$ form a basis for the (A-invariant) unobservable subspace $\mathcal{UO}$ of the pair (A, C).

Kalman Decomposition

Suppose now that we choose a similarity transformation

$$\bar{x} := T^{-1}x, \quad T := \begin{bmatrix} V_{co} & V_{c\bar{o}} & V_{\bar{c}o} & V_{\bar{c}\bar{o}} \end{bmatrix}$$

Kalman Decomposition

Suppose now that we choose a similarity transformation

$$\bar{x} := T^{-1}x, \quad T := \begin{bmatrix} V_{co} & V_{c\bar{o}} & V_{\bar{c}o} & V_{\bar{c}\bar{o}} \end{bmatrix}$$

such that

1. the columns of $V_{c\bar{o}}$ form a basis for the (A-invariant) subspace $\mathcal{C} \cap \mathcal{UO}$,
2. the columns of $[V_{co}\ V_{c\bar{o}}]$ form a basis for the (A-invariant) controllable subspace $\mathcal{C}$ of the pair (A, B), and
3. the columns of $[V_{c\bar{o}}\ V_{\bar{c}\bar{o}}]$ form a basis for the (A-invariant) unobservable subspace $\mathcal{UO}$ of the pair (A, C).

Kalman Decomposition

Suppose now that we choose a similarity transformation

$$\bar{x} := T^{-1}x, \quad T := \begin{bmatrix} V_{co} & V_{c\bar{o}} & V_{\bar{c}o} & V_{\bar{c}\bar{o}} \end{bmatrix}$$

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3. the columns of $[V_{c\bar{o}}\ V_{\bar{c}\bar{o}}$ form a basis for the (A-invariant) unobservable subspace $\mathcal{UO}$ of the pair (A, C).

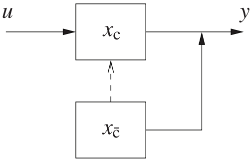
This similarity transformation leads to the system in the form:

$$\begin{bmatrix} \dot{x}_{co}/x^+_{co} \\ \dot{x}_{c\bar{o}}/x^+_{c\bar{o}} \\ \dot{x}_{\bar{c}o}/x^+_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}}/x^+_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{\times o} & 0 \\ A_{c\times} & A_{c\bar{o}} & A_{\times\times} & A_{\times\bar{o}} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{\bar{c}\times} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

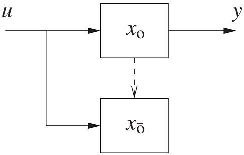
$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du$$

Kalman Decomposition

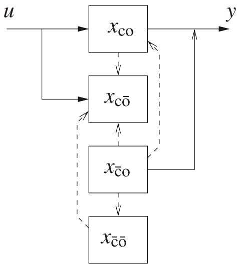
This similarity transformation is called a *canonical Kalman decomposition*, and it is represented schematically in figure below.



(a) Controllable decomposition



(b) Observable decomposition



(c) Kalman decomposition

Figure: Schematic representation of the structural decompositions

Kalman Decomposition

Theorem (Kalman Decomposition)

For every LTI system (AB-LTI), there is a similarity transformation that takes it to the form

$$\begin{bmatrix} \dot{x}_{co}/x^+_{co} \\ \dot{x}_{c\bar{o}}/x^+_{c\bar{o}} \\ \dot{x}_{\bar{c}o}/x^+_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}}/x^+_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{\times o} & 0 \\ A_{c\times} & A_{c\bar{o}} & A_{\times\times} & A_{\times\bar{o}} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{\bar{c}\times} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{co} & 0 & C_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + Du$$

for which

1. *the pair* $\left(\begin{bmatrix} A_{co} & 0 \\ A_{c\times} & A_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} B_{co} \\ B_{c\bar{o}} \end{bmatrix} \right)$ *is controllable*
2. *the pair* $\left(\begin{bmatrix} A_{co} & A_{\times o} \\ 0 & A_{\bar{c}o} \end{bmatrix}, \begin{bmatrix} C_{co} & C_{c\bar{o}} \end{bmatrix} \right)$ *is observable*
3. *the triple* (A_{co}, B_{co}, C_{co}) *is both controllable and observable, and*
4. *the tranfer function* $C(sI - A)^{-1}B + D$ *of the original system is the* $same$ *as the transfer function* $C_{co}(sI - A_{co})^{-1}B_{co} + D$ *of the controllable and observable system*

Detectability

We just saw that any LTI system is algebraically equivalent to a system in the following standard form for unobservable systems

$$\begin{bmatrix} \dot{x}_o / x_0^+ \\ \dot{u} / x_u^+ \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_u \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + \begin{bmatrix} B_o \\ B_u \end{bmatrix} u, \qquad x_0 \in \mathbb{R}^{\bar{n}}, x_0 \in \mathbb{R}^{n-\bar{n}}$$

$$y = \begin{bmatrix} C_o \\ 0 \end{bmatrix} \begin{bmatrix} x_o \\ x_u \end{bmatrix} + Du, u \in \mathbb{R}^k, m \in \mathbb{R}^m$$

Definition (Detectable system)

The pair (A, C) is *detectable* whenever it is algebraically equivalent to a system in the standard form for unobservable systems with $n = \bar{n}$ i.e A_u non-existent or with A_u a stability matrix.

Detectability

For a continuous-time system, the evolution of the unobservable component of the state is determined by

$$\dot{x}_u = A_u x_u + A_{21} x_o + B_u u$$

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Regarding $A_{21} x_o + B_u u$ as the input, we can use the variation of constants formula to conclude that

$$x_u(t) = e^{A_u (t-t_0)} x_u(t_0) + \int_{t_0}^{t} e^{A_u (t-\tau)} \left(A_{21} x_o(\tau) + B_u u(\tau)\right) d\tau$$

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For detectable systems, the term $e^{A_u(t-t_0)} x_u(t_0)$ eventually converges to zero, and therefore one can guess that $x_u(t)$ up to an error converges to zero exponentially fast.

Detectability tests

Investigating the detectability of an LTI system

$$\dot{x}/x^{+} = Ax, \quad y = Cx, \qquad x \in \mathbb{R}^{n}, y \in \mathbb{R}^{m} \tag{AC-LTI}$$

from the definition requires the computation of the observable decomposition.

Detectability tests

Investigating the detectability of an LTI system

$$\dot{x}/x^+ = Ax, \quad y = Cx, \qquad x \in \mathbb{R}^n, y \in \mathbb{R}^m \tag{AC-LTI}$$

from the definition requires the computation of the observable decomposition. However, there are alternative tests that avoid this intermediate step. These tests can be deduced by duality from the stabilizability tests.

Theorem (Eigenvector test for detectability)

1. *The* $\underline{continuous-time}$ *LTI system* (AC-LTI) *is detectable if and only if every eigenvector of* A *corresponding to an eigenvalue with a positive or zero real part is not in the kernel of* C*.*
2. *The* $\underline{discrete-time}$ *LTI system* (AC-LTI) *is detectable if and only if every eigenvector of* A *corresponding to an eigenvalue with magnitude larger than or equal to 1 is not in the kernel of* C*.*

Detectability tests

Theorem (Popov-Belevitch-Hautus(PBH) test for detectability)

1. *The continuous-time system* (AC-LTI) *is detectable if and only if*

$$\texttt{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n, \qquad \forall \lambda \in \mathbb{C} : \texttt{Re}[\lambda] \geq 0.$$

2. *The discrete-time system* (AC-LTI) *is detectable if and only if*

$$\texttt{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n, \qquad \forall \lambda \in \mathbb{C} : |\lambda| \geq 1$$

Detectability tests

Theorem (Lyapunov test for detectability)

1. *The continuous-time system* (AC-LTI) *is detectable if and only if there is a positive-definite solution* P *to the Lyapunov matrix inequality*

$$A'P + PA - C'C \prec 0$$

2. *The discrete-time system* (AC-LTI) *is detectable if and only if there is a positive-definite solution* P *to the Lyapunov matrix inequality*

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Minimal Realizations

Recall that, given a transfer function $\hat{G}(s)$, we say that

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^k, y \in \mathbb{R}^m \tag{1}$$

is a realization of $\hat{G}(s)$ if

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

The size n of the state-space vector x is called the order of the realization.

Attention!

A transfer function can have realizations of different orders!

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Attention!

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Definition (Minimal Realization)

A realization of $\hat{G}(s)$ is called *minimal* or *irreducible* whenever there is no realization of $\hat{G}(s)$ of smaller order.

Minimal Realizations

Theorem

Every minimal realization must be both controllable and observable.

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Every minimal realization must be both controllable and observable.

Proof.

This theorem can be easily proved by $contradiction$. Assuming that a realisation is either not controllable or not observable, by Kalman decomposition theorem one could find another realisation of smaller order that realises the same transfer function, which would contradict minimality. □

Minimal Realizations

Theorem

A realization is minimal if and only if it is both controllable and observable.

Proof.

$1 \implies 2$ We have already shown previously that if a realization is minimal, then it must be controllable and observable. $\square$

$1 \Longleftarrow 2$ OR $\neg 1 \Longrightarrow \neg 2$

Assume that

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \qquad x \in \mathbb{R}^n \tag{LTI}$$

is a controllable and observable realization of $\hat{G}(s)$, but this realization is not minimal; i.e., there exists another realization

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad y = \bar{C}\bar{x} + \bar{D}u, \qquad x \in \mathbb{R}^{\bar{n}} \tag{$\overline{\text{LTI}}$}$$

for $\hat{G}(s)$ with $\bar{n} < n$.

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for $\hat{G}(s)$ with $\bar{n} < n$.For (LTI), compute
$$\mathfrak{O}\mathfrak{C} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = \underbrace{\begin{bmatrix} CB & CAB & \dots & CA^{n-1}B \\ CAB & CA^2B & \dots & CA^nB \\ \vdots & \vdots & & \vdots \\ CA^{n-1}B & CA^nB & \dots & CA^{2n-2}B \end{bmatrix}}_{\text{Markov parameters}}$$

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Since (LTI) is controllable and observable, both $\mathfrak{C}$ and $\mathfrak{O}$ have rank n, and therefore the above matrix also has rank n.

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Since (LTI) and ($\overline{\text{LTI}}$) realize the same transfer function, they must have the same Markov parameters, and therefore $\mathfrak{O}\mathfrak{C} = \bar{\mathfrak{O}}\bar{\mathfrak{C}}$.

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Since (LTI) and ($\overline{\text{LTI}}$) realize the same transfer function, they must have the same Markov parameters, and therefore $\mathfrak{O}\mathfrak{C} = \bar{\mathfrak{O}}\bar{\mathfrak{C}}$. But since $\bar{\mathfrak{C}}$ has only $\bar{n} < n$ columns, its rank must be lower than n and therefore

$$\texttt{rank}\bar{\mathfrak{O}}\bar{\mathfrak{C}} \leq \texttt{rank}\bar{\mathfrak{C}} \leq \bar{n} \leq n.$$

which contradicts the fact that $\texttt{rank}\bar{\mathfrak{O}}\bar{\mathfrak{C}} = \texttt{rank}\mathfrak{O}\mathfrak{C} = n$.

Similarity of Minimal Realizations

The definition of minimal realization automatically guarantees that all minimal realizations have the same order, but minimal realizations are even more closely related.

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Theorem

All minimal realizations of a transfer function are algebraically equivalent.

Order of a Minimal SISO realization

Any proper SISO rational function $\hat{g}(s)$ can be written as

$$\hat{g}(s) = \frac{n(s)}{d(s)},$$

where $d(s)$ is a monic[1] polynomial, and $n(s)$ and $d(s)$ are coprime[2]. In this case, the right-hand side of the above is called a *coprime fraction*, $d(s)$ is called the *pole* (or *characteristic polynomial*) of $\hat{g}(s)$, and the degree of $d(s)$ is called the *degree of the transfer function* $\hat{g}(s)$. The roots of $d(s)$ are called the *poles of the transfer function* and the roots of $n(s)$ are called the *zeros of the transfer function*.

[1] A polynomial is *monic* if its highest order coefficient is equal to 1.

[2] Two polynomial are *coprime* if they have no common roots.

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Theorem

A SISO realizaion

$$\dot{x}/x^+ = Ax + bu, \quad y = cx + du, \qquad x \in \mathbb{R}^n, u, y \in \mathbb{R},$$

of $\hat{g}(s)$ is minimal if and only if its order n is equal to the degree of $\hat{g}(s)$. In this case, the pole polynomial $d(s)$ of $\hat{g}(s)$ is equal to the characteristic polynomial of A; i.e., $d(s) = \det(sI - A)$.

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[2] Two polynomial are *coprime* if they have no common roots.

Comment on BIBO and Asymptotic stability

[1] This result also holds for MIMO systems.

Comment on BIBO and Asymptotic stability

Theorem

If the SISO realization (LTI) of $\hat{g}(s)$ is minimal or the pair (A, b, c, d) is controllable and observable, then we have

$$\textit{Asymptotic stability} \iff \textit{BIBO stability}^{1}$$

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`MATLAB` commands

The command `msys=minreal(sys)` computes a minimal realization of the system `sys`, which can either be in state-space or transfer function form.

When `sys` is in state-space form, `msys` is a state-space system from which all uncontrollable and unobservable modes were removed.

When `sys` is in transfer function form, `msys` is a transfer function from which all common poles and zeros have been canceled.

[1] This result also holds for MIMO systems.