Linear Dynamical Systems

Week 2 - Stability

Outline of Week 2

1. Introduction and Origin of Stability Analysis
2. Brief review of norms and definite matrices
3. Definitions of Stability
4. Lyapunov stability
5. Eigenvalue condition for Lyapunov stability
6. Bounded Input Bounded Output Stability
7. BIBO vs Lyapunov stability

Origin of Stability Analysis

One of the first significant feedback control systems in modern Europe

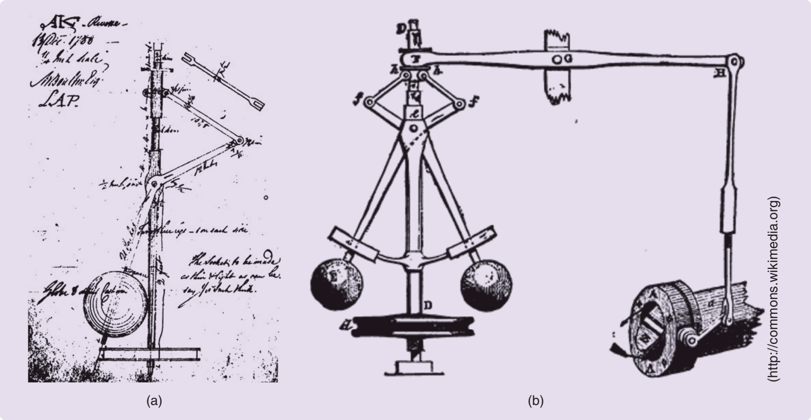

Figure: The flyball governor invented by James Watt in 1788. (a) The original design, and (b) the improved design.[1]

[1] C. G. Kang, "Origin of Stability Analysis: "On Governors" by J.C. Maxwell [Historical Perspectives]," in IEEE Control Systems, 36(5), pp. 77-88, 2016.

Origin of Stability Analysis

- At the height of industrial revolution around 1868, many governors were installed.
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- The governor system was soon discovered to be plagued by problems of *instability* and *inaccuracy* that could apparently not be overcome by either theoretical or practical approaches
- Maxwell's 1868 paper[2]:
  - Stability concept
  - Simple mathematical models
  - Importance of integral actions
  - Linearization
  - Stability is an algebraic problem
  - Criteria for 1st, 2nd, and 3rd order systems
  - Posed stability problem in competition

[2] J. C. Maxwell, "On governors," Proc. R. Soc. London, 16, 270–283, 1868.

The equation of motion of the machine itself is

$$M\frac{d^2x}{dt^2}=P-R-F\left(\frac{dx}{dt}-V_1\right)-Gy. \quad \ldots \ldots \quad (10)$$

This must be combined with equation (7) to determine the motion of the whole apparatus. The solution is of the form

$$x=A_1e^{n_1t}+A_2e^{n_2t}+A_3e^{n_3t}+Vt, \quad \ldots \ldots \quad (11)$$

where n_1, n_2, n_3 are the roots of the cubic equation

$$MBn^3+(MY+FB)n^2+FYn+FG=0. \quad \ldots \ldots \quad (12)$$

If n be a pair of roots of this equation of the form $a\pm\sqrt{-1}b$, then the part of x corresponding to these roots will be of the form

$$e^{at}\cos(bt+\beta).$$

If a is a negative quantity, this will indicate an oscillation the amplitude of which continually decreases. If a is zero, the amplitude will remain constant, and if a is positive, the amplitude will continually increase.

One root of the equation (12) is evidently a real negative quantity. The condition that the real part of the other roots should be negative is

$$\left(\frac{F}{M}+\frac{Y}{B}\right)\frac{Y}{B}-\frac{G}{B}=\text{a positive quantity.}$$

This is the condition of stability of the motion. If it is not fulfilled there will be a dancing motion of the governor, which will increase till it is as great as the limits of motion of the governor. To ensure this stability, the value of Y must be made sufficiently great, as compared with G, by placing the weight W in a viscous liquid if the viscosity of the lubri-

Taxonomy - Stability Concept[3]

"the motion of a machine with its governor consist in general of a uniform motion, combined with a disturbance that may be expressed as the sum of several component motions. These components may be of four different kinds:

1. continually increase,
2. continually diminish,
3. be an oscillation of continually increasing amplitude, and
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The second and fourth kinds are admissible in a good governor, and are mathematically equivalent to the condition that all the possible roots, and all the possible parts of the impossible roots of a characteristic equation shall be negative.

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Matrix Norms (Review)

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Thus, the matrix norm is a function $\|\cdot\| : K^{m\times n} \to \mathbb{R}$ that must satisfy the following properties:

For all scalars $\alpha \in K$ and for all matrices $A, B \in K^{m\times n}$,

$$\begin{aligned}
\|\alpha A\| &= |\alpha|\|A\| && \text{(homogeneity)}\\
\|A + B\| &\leq \|A\| + \|B\| && \text{(triangle inequality)}\\
\|A\| &\geq 0 && \text{(positive-valued)}\\
\|A\| &= 0 \text{ iff } A = 0_{m,n} && \text{(definite)}
\end{aligned}$$

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$$\|A\|_1 \triangleq \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|.$$

For a (column) vector $v = [v_i] \in \mathbb{R}^\ell$, $\|v\|_1 \triangleq \sum_{i=1}^{\ell} |v_i|$.

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Matrix Norms (Review)

3. The *two-norm*,

$$\|A\|_2 \triangleq \sigma_{\max}[A], \\ \triangleq \sqrt{\lambda_{\max}[A'A]}$$

where $\sigma_{\max}[A]$ denotes the largest singular value, and $\lambda_{\max}[A]$ denotes the largest eigenvalue of A

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4. The *Frobenius norm*,

$$\|A\|_F \triangleq \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{\sum_{i=1}^{n} \sigma_i[A]^2},$$

where the $\sigma_i[A]$ are the singular values of A.

For (column) vectors, the Frobenius norm coincides with the two-norm, but in general this is not true for matrices.

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$$\frac{\|A\|_\infty}{\sqrt{n}} \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty$$

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For any submultiplicative norm $\| \bullet \|_p$, we have

$$\|Ax\|_p \leq \|A\|_p \|x\|_p, \qquad \forall x$$

and therefore

$$\|A\|_p \geq \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

Positive-Definite Matrices (Review)

A symmetric $n \times n$ matrix Q is *positive-definite* if

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When (1) holds only for $\leq$ or $\geq$, the matrix is said to be *negative-semidefinite* or *positive-semidefinite*, respectively.

Positive-Definite Matrices (Review)

The following statements are equivalent for a symmetric $n \times n$ matrix Q.

1. Q is positive-definite.
2. All eigenvalues of Q are strictly positive.
3. The determinants of all upper left submatrices of Q are positive.
4. There exists an $n \times n$ nonsingular real matrix H such that

$$Q = H'H$$

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$$Q = H'H$$

For a positive-definite matrix Q we have

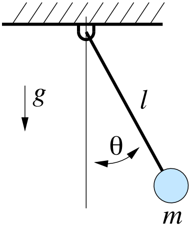
$$0 < \lambda_{\min}[Q]\|x\|^2 \leq x'Qx \leq \lambda_{\max}[Q]\|x\|^2, \qquad \forall x \neq 0,$$

where $\lambda_{\min}[Q]$ and $\lambda_{\max}[Q]$ denote the smallest and largest eigenvalues of Q, respectively.

- Equations of motion

$$I\ddot{\theta}(t) + mgl\sin\theta(t) = Q,$$

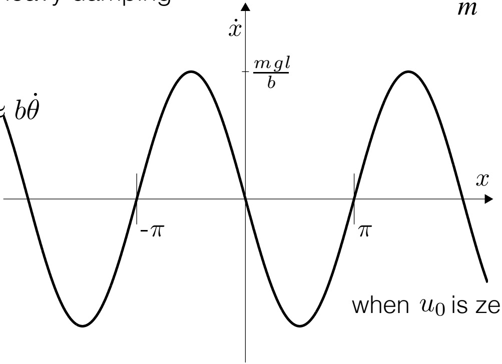
$$Q = -b\dot{\theta}(t) + u(t)$$



Consider a special case of heavy damping
i.e. $\frac{b}{I} \gg 1$

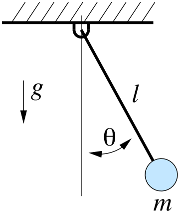
$$u_0 - mgl\sin\theta = I\ddot{\theta} + b\dot{\theta} \approx b\dot{\theta}$$

$$b\dot{x} = u_0 - mgl\sin x$$



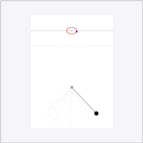
when u_0 is zero

Simple Pendulum

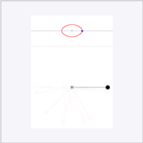


- Two equilibrium position (or steady states):
  - at $\theta = 0$
  - at $\theta = \pi$

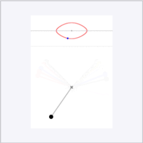
Initial angle of 0°, a



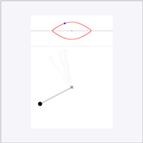
Initial angle of 45°



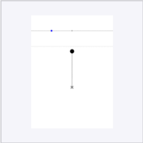
Initial angle of 90°



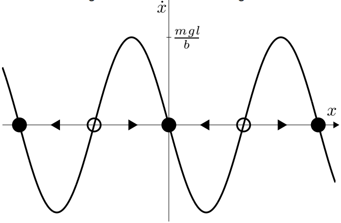
Initial angle of 135°



Initial angle of 170°



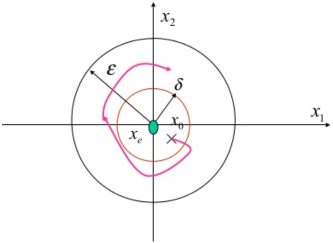
Initial angle of 180°,

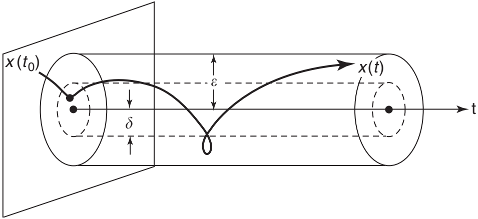


Definitions of stability

In the sense of Lyapunov

A system steady state $\mathbf{x}_s$ is said to be *stable* if for each possible region of radius $\epsilon > 0$ around the steady state, there is an initial state $\mathbf{x}_0$ at $t = t_0$ falling within a radius $\delta > 0$ around the steady state that causes the dynamic trajectory to stay within the region $|(\mathbf{x} - \mathbf{x}_s)| < \epsilon$ for all times $t > t_0$.

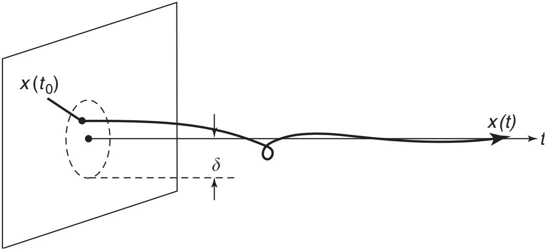




Definitions of stability

In the sense of Lyapunov

A system steady state $\mathbf{x}_s$ is said to be *asymptotically stable* if it is both stable and in addition, there exists a region of initial conditions of radius $\delta_0 > 0$ around $\mathbf{x}_s$ for which the system approaches $\mathbf{x}_s$ as $t \to \infty$.



Definitions of stability

Definition 3

A system steady state is said to be unstable if it is not stable.

Lyapunov Stability

Consider the following continuous-time LTV system

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k, y \in \mathbb{R}^m.$$

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Definition (Lyapunov Stability)

The system (CLTV) is said to be

1. *(marginally) stable in the sense of Lyapunov or internally stable* whenever, for every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$, the homogeneous state response

$$x(t) = \phi(t, t_0)x_0, \quad \forall t \geq 0$$

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3. *exponentially stable* whenever, in addition, there exist constants $c, \lambda > 0$ such that, for every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$, we have

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4. *unstable* whenever it is not marginally stable in the Lyapunov sense.

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Attention!

1. For marginally stable systems, the effect of initial conditions does not grow unbounded with time (but it may grow temporarily during a transient phase).
2. For asymptotically stable systems, the effect of initial conditions eventually disappears with time.
3. For unstable systems, the effect of initial conditions (may) grow over time (depending on the specific initial conditions and the value of the matrix C).

Eigenvalue Conditions for Lyapunov Stability

The overall objective is to determine simple conditions to classify the continuous-time homogeneous LTI system

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$

in terms of its Lyapunov stability, *without explicitly computing the solution to the system*.

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2. *asymptotically stable if and only if all the eigenvalues of* A *have strictly negative real parts,*
3. *unstable if and only if at least one eigenvalue of* A *has a positive real part or zero real part, but the corresponding Jordan block is larger than* 1×1*.*

Eigenvalue Conditions for Lyapunov Stability

Asymptotic and Exponential stability of LTI systems

When all the eigenvalues of A have strictly negative real parts, all entries of e^{At} converge to zero exponentially fast, and therefore $\|e^{At}\|$ converges to zero exponentially fast (for every matrix norm); i.e., there exist constants $c, \lambda > 0$ such that

$$\|e^{At}\| \leq ce^{-\lambda t} \qquad \forall t \in \mathbb{R}$$

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In this case, for a submultiplicative norm, we have

$$\|x(t)\| = \|e^{A(t-t_0)}x_0\| \leq \|e^{A(t-t_0)}\|\|x_0\| \leq ce^{-\lambda(t-t_0)}\|x_0\|, \quad \forall t \in \mathbb{R}$$

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Attention!

These conditions do not generalize to time-varying systems. One can find matrix-valued signals $A(t)$ that are stability matrices for every fixed $t \geq 0$, but the time-varying system $\dot{x} = A(t)x$ is not even stable.

Lyapunov Stability Theorem

The Lyapunov stability theorem provides an alternative condition to check whether or not the CT homogeneous LTI system is asymptotically stable.

Lyapunov Stability Theorem

Theorem (Lyapunov stability)

The following conditions are equivalent:

1. *The system (H-CLTI) is asymptotically stable.*
2. *The system (H-CLTI) is exponentially stable.*
3. *All the eigenvalues of A have strictly negative real parts.*
4. *For every symmetric positive-definite matrix Q, there exists a unique solution P to the following Lyapunov equation*

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1. *The (improper) integral in* (2) *is well defined (i.e., it is finite)*
2. *The matrix P in* (2) *solves the equation* (Lyapunov Eq.)
3. *The matrix P in* (2) *is symmetric and positive-definite*
4. *No other matrix solves this equation*

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therefore

$$\begin{aligned} A'P + PA &= \int_0^\infty \frac{d}{dt}\left(e^{A'\,t} Q e^{At} \right) dt = \left[e^{A'\,t} Q e^{At} \right]_0^\infty \\ &= \left(\lim_{t\to\infty} e^{A'\,t} Q e^{At} \right) - e^{A'0} Q e^{A0}. \end{aligned}$$

Equation (Lyapunov Eq.) follows from this and the facts that $\lim_{t\to\infty} e^{At} = 0$ because of asymptotic stability and that $e^{A0} = I$.

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where $w(t) = e^{At}z, \forall t \geq 0$. Since Q is positive-definite, we conclude that $z'Pz \geq 0$. Moreover,

$$z'Pz = 0 \implies \int_0^\infty w(t)'Qw(t)dt = 0,$$

which can only happen if $w(t) = e^{At}z = 0, \forall t \geq 0$, from which one concludes that $z = 0$, because e^{At} is nonsingular. Therefore P is positive-definite.

Proof: $2 \implies 4$

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On the other hand,

$$\frac{d}{dt}\left(e^{A't}(P - \bar{P})e^{At}\right) = e^{A't}A'(P - \bar{P})e^{At} + e^{A't}(P - \bar{P})Ae^{At} = 0$$

and therefore $e^{A't}(P - \bar{P})e^{At}$ must remain constant for all times. But, because of stability, this quantity must converge to zero as $t \to \infty$, so it must be always zero. Since e^{At} is nonsingular, this is only if $P = \bar{P}$.

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$$v(t) = x'(t)Px(t) \geq 0, \qquad \forall t \geq 0.$$

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Therefore, $v(t)$ is a nonincreasing signal

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$$\|x\|^2 \leq \frac{x'(t)Px(t)}{\lambda_{\min}[P]} = \frac{v(t)}{\lambda_{\min}[P]} \leq \frac{v(0)}{\lambda_{\min}[P]}, \qquad \forall t \geq 0, \tag{4}$$

which means that the H-CLTI system is stable.

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To verify that it is actually exponentially stable, we go back to (3) and, using the facts that $x'Qx \geq \lambda_{\min}[Q]\|x\|^2$ and that $v = x'Px \leq \lambda_{\max}[P]\|x\|^2$

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To proceed, we need the Comparison lemma.

Theorem (Comparison Lemma)

Let $v(t)$ be a differentiable scalar signal for which

$$\dot{v}(t) \leq \mu v(t), \qquad \forall t \geq t_0$$

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Applying the Comparison lemma to (5), we conclude that

$$v(t) \leq e^{-\lambda(t-t_0)}v(t_0), \quad \forall t \geq 0, \qquad \lambda \triangleq -\frac{\lambda_{\min}[Q]}{\lambda_{\max}[P]},$$

which shows that $v(t)$ converges to zero exponentially fast and so does $\|x(t)\|$ [see (4)].

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which is precisely equivalent to (6). □

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Discrete-time case

Consider now the following discrete-time LTV system

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t). \tag{DLTV}$$

Definition (Lyapunov stability)

The system (DLTV) is said to be

1. *(marginally) stable in the sense of Lyapunov or internally stable* whenever, for every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$, the homogeneous state response

$$x(t) = \phi(t, t_0)x_0, \quad \forall t \geq t_0$$

   is uniformly bounded,
2. *asymptotically stable (in the Lyapunov sense)* whenever, in addition, for every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$, we have $x(t) \to 0$ as $t \to \infty$,
3. *exponentially stable* whenever, in addition, there exist constants $c > 0, 0 < \lambda < 1$ such that, for every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$,

$$\|x(t)\| \leq c\lambda^{t-t_0}\|x(t_0)\|, \forall t \geq t_0,$$

4. *unstable* whenever it is not marginally stable in the Lyapunov sense.

Discrete-time case

The matrices $B(\bullet), C(\bullet)$, and $D(\bullet)$ play no role in this definition; therefore, one often simply talks about the Lyapunov stability of the homogeneous system

$$x(t+1) = A(t)x, \qquad x \in \mathbb{R}^n. \tag{H-DLTV}$$

Theorem (Eigenvalue conditions)

The discrete-time homogeneous LTI system

$$x^+ = Ax, \qquad x \in \mathbb{R}^n \tag{H-DLTI}$$

is

1. *marginally stable if and only if all the eigenvalues of A have magnitude smaller than or equal to 1 and all the Jordan blocks corresponding to eigenvalues with magnitude equal to 1 are 1×1,*
2. *asymptotically and exponentially stable if and only if all the eigenvalues of A have magnitude strictly smaller than 1, or*
3. *unstable if and only if at least one eigenvalue of A has magnitude larger than 1 or magnitude equal to 1, but the corresponding Jordan block is larger than 1×1.*

Discrete-time case

Theorem (Lyapunov stability in discrete time)

The following five conditions are equivalent:

1. *The system* (H-DLTI) *is asymptotically stable.*
2. *The system* (H-DLTI) *is exponentially stable.*
3. *All the eigenvalues of A have magnitude strictly smaller than 1.*
4. *For every symmetric positive-definite matrix Q, there exists a unique solution P to the following Stein equation (more commonly known as the discrete-time Lyapunov equation)*

$$A'PA - P = -Q. \tag{DT Lyapunov Eq.}$$

   Moreover, P is symmetric and positive-definite.

5. *There exists a symmetric positive-definite matrix P for which the following Lyapunov matrix inequality holds:*

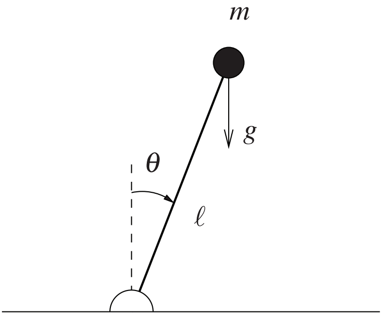
$$A'PA - P < 0. \tag{DT LMI}$$

Lyapunov Stability Tests for LTI systems

	Definition	Continuous time		Discrete time	
		Eigenvalue test	Lyapunov test	Eigenvalue test	Lyapunov test
Unstable	For some $t_0, x(t_0)$, $x(t)$ can be unbounded.	For some $\lambda_i[A]$, $\Re\lambda_i[A] > 0$ or $\Re\lambda_i[A] = 0$ with Jordan block larger than 1×1.		For some $\lambda_i[A]$, $\lvert\lambda_i[A]\rvert > 1$ or $\lvert\lambda_i[A]\rvert = 1$ with Jordan block larger than 1×1.	
Marginally stable	For every $t_0, x(t_0)$, $x(t)$ is uniformly bounded.	For every $\lambda_i[A]$, $\Re\lambda_i[A] < 0$ or $\Re\lambda_i[A] = 0$ with 1×1 Jordan block.		For every $\lambda_i[A]$, $\lvert\lambda_i[A]\rvert < 1$ or $\lvert\lambda_i[A]\rvert = 1$ with 1×1 Jordan block.	
Asymptotically stable	For every $t_0, x(t_0)$, $\lim_{t\to\infty} x(t) = 0$.	For every $\lambda_i[A]$, $\Re\lambda_i[A] < 0$.	For every $Q > 0$, $\exists P = P' > 0$: $A'P + PA = -Q$ or	For every $\lambda_i[A]$, $\lvert\lambda_i[A]\rvert < 1$.	For every $Q > 0$, $\exists P = P' > 0$: $A'PA - P = -Q$ or
Exponentially stable	$\exists c, \lambda > 0$: for every $t_0, x(t_0)$, $\lVert x(t)\rVert \le ce^{-\lambda t}\lVert x(t_0)\rVert$, $\forall t \ge t_0$.		$\exists P = P' > 0$: $A'P + PA < 0$.		$\exists P = P' > 0$: $A'PA - P < 0$.

Example: Inverted Pendulum

Consider the inverted pendulum and assume that $u = T$ and $y = \theta$ are its input and output, respectively.



From Newton's law,

$$m\ell^2\ddot{\theta} = mg\ell\sin\theta - b\dot{\theta} + T,$$

where T denotes a torque applied at the base and g is the gravitational acceleration.

Example: Inverted Pendulum

At equilibrium point $\theta = \pi$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The eigenvalues of A are given by

$$\det(\lambda I - A) = \lambda \left(\lambda + \frac{b}{m\ell^2} \right) + \frac{g}{\ell} = 0 \Leftrightarrow \lambda = -\frac{b}{2m\ell^2} \pm \sqrt{\left(\frac{b}{2m\ell^2} \right) - \frac{g}{\ell}}$$

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At equilibrium point $\theta = \pi$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{b}{m\ell^2} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

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$$P = \begin{bmatrix} \frac{b^2 + g^2\ell^2 m^2 + g\ell^3 m^2}{2bg\ell m} & \frac{\ell}{2g} \\ \frac{\ell}{2g} & \frac{\ell^2 m(g+\ell)}{2bg} \end{bmatrix}$$

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and therefore the linearized system is exponentially unstable, because

$$-\frac{b}{2m\ell^2} + \sqrt{\left(\frac{b}{2m\ell^2}\right) + \frac{g}{\ell}} > 0.$$

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Bounded-Input, Bounded Output Stability

Internal or Lyapunov stability is concerned only with the effect of the initial conditions on the response of the system.

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We shall see that for LTI systems these two notions of stability are closely related.

Consider the continuous-time LTV system

$$\dot{x} = A(t)x + B(t)u, \quad y = C(t)x + D(t)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^k, y \in \mathbb{R}^m$$

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Definition (BIBO stability)

The system (CLTV) is said to be *(uniformly) BIBO stable* if there exists a finite constant g[1] such that, for every input u, its forced response y_f satisfies

$$\sup_{t\in[0,\infty)} \|y_f(t)\| \leq g \sup_{t\in[0,\infty)} \|u(t)\|$$

[1] The factor g can be viewed as a system "gain".

Time-domain condition for BIBO stability

Theorem (Time domain BIBO stability condition)

The following two statements are equivalent.

1. *The system (CLTV) is uniformly BIBO stable.*
2. *Every entry of $D(t)$ is uniformly bounded* [1] *and*

$$\sup_{t \geq 0} \int_0^t |g_{ij}(t,\tau)| d\tau < \infty$$

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[1] A signal $x(t)$ is uniformly bounded if there exists a finite constant c such that $\|x(t)\| \leq c, \forall t \geq 0$.

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Finally
$$g = \sup_{t\geq 0}\int_0^t \|C(t)\phi(t,\tau)B(\tau)\|d\tau + \delta \leq \sup_{t\geq 0}\sum_{i,j}\int_0^t |g_{i,j}(t,\tau)|d\tau + \delta < \infty$$

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To do this, pick an arbitrary time T and consider the following step input:

$$u_T(\tau) \triangleq \begin{cases} 0 & 0 \leq \tau < T \\ e_j & \tau \geq T \end{cases} \quad \forall \tau \geq 0,$$

where $e_j \in \mathbb{R}^k$ is the jth vector in the canonical basis of $\mathbb{R}^k$.

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We thus have found an input for which

$$\sup_{t\in[0,\infty)} \|u_T(t)\| = 1$$

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To do this, pick an arbitrary time T and consider the following "switching" input:

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Since (7) is unbounded, also now we conclude that we can make $\sup_{t\in[0,\infty)} \|y_f(t)\|$ arbitrarily large by using inputs $u_T(\bullet)$ for which $\sup_{t\in[0,\infty)} u_T(t) = 1$, which is not compatible with the existence of a finite gain g. This means that condition (2) must hold for a system to be BIBO stable.

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The following two statements are equivalent.

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The ijth entry of this matrix will be a strictly proper rational function of the general form

$$\hat{g}_{ij}(s) = \frac{\alpha_0 s^q + \alpha_1 s^{q-1} + \cdots + \alpha_{q-1}s + \alpha_q}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_k)^{m_k}},$$

where the λ_ℓ are the (distinct) pole of $\hat{g}_{ij}(s)$ and the m_ℓ are the corresponding multiplicities.

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To determine whether a time-invariant system (CLTI) is BIBO stable, we need to compute the entries $g_{ij}(t)$ of $Ce^{At}B$. To do this, we compute its Laplace transform, $\mathcal{L}[Ce^{At}B] = C(sI - A)^{-1}B$.

The ijth entry of this matrix will be a strictly proper rational function of the general form

$$\hat{g}_{ij}(s) = \frac{\alpha_0 s^q + \alpha_1 s^{q-1} + \cdots + \alpha_{q-1}s + \alpha_q}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_k)^{m_k}},$$

where the λ_ℓ are the (distinct) pole of $\hat{g}_{ij}(s)$ and the m_ℓ are the corresponding multiplicities. Perform the partial fraction as

$$\begin{aligned}\hat{g}_{ij}(s) = \frac{a_{11}}{(s-\lambda_1)} + \frac{a_{12}}{(s-\lambda_1)^2} + \cdots + \frac{a_{1m_1}}{(s-\lambda_1)^{m_1}} + \cdots \\ + \frac{a_{k1}}{(s-\lambda_k)} + \frac{a_{k2}}{(s-\lambda_k)^2} + \cdots + \frac{a_{km_k}}{(s-\lambda_K)^{m_k}}.\end{aligned}$$

The inverse Laplace transform is then given by

$$\begin{aligned}g_{ij}(t) &= \mathcal{L}^{-1}\left[\hat{g}_{ij}(s)\right] \\ &= \begin{array}{c} a_{11}e^{\lambda_1 t} + a_{12}te^{\lambda_1 t} + \cdots + a_{1m_1}t^{m_1-1}e^{\lambda_1 t} + \cdots \\ a_{k1}e^{\lambda_k t} + a_{k2}te^{\lambda_k t} + \cdots + a_{km_k}t^{m_k-1}e^{\lambda_k t}.\end{array}\end{aligned}$$

Frequency Domain Conditions for BIBO Stability

We therefore conclude the following.

1. If for all $\hat{g}_{ij}(s)$, all the poles λ_ℓ have strictly negative real parts, then $g_{ij}(t)$ converges to zero exponentially fast and the system (CLTI) is BIBO stable.

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2. If at least one of the $\hat{g}_{ij}(s)$ has a pole λ_ℓ with a zero or positive real part, then $|g_{ij}(t)|$ does not converge to zero and the system (CLTI) is not BIBO stable.

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Note that adding a constant D term will not change its poles.

Theorem (Frequency domain BIBO condition)

The following two statements are equivalent:

1. *The system (CLTI) is uniformly BIBO stable.*
2. *Every pole of every entry of the transfer function of the system (CLTI) has a strictly negative real part.*

BIBO vs Lyapunov stability

We now know that the LTI system

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

is uniformly BIBO stable if and only if every entry $\bar{g}_{ij}(t)$ of $Ce^{At}B$ satisfies

$$\int_0^\infty |\bar{g}_{ij}(t)|dt < \infty \tag{8}$$

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When the system (CLTI) is exponentially stable, then it must also be BIBO stable.

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Attention!

In general, the converse of the above theorem is not true, because there are systems that are BIBO stable but not exponentially stable.

BIBO vs Lyapunov stability

Example[1]

Consider the system[2]

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

for which

$$e^{At} = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

is unbounded and therefore Lyaunov unstable, but

$$Ce^{At}B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{-2t}$$

and therefore the system is BIBO stable

---

[1] We shall see in later lectures that this discrepancy between Lyapunov and BIBO stability is always associated with *lack of controllability or observability*, two concepts that will be introduced later.

[2] The system is not controllable.

Discrete-time case

Consider now the following discrete-time LTV system

$$x(t+1) = A(t)x(t) + B(t)u(t), \qquad y(t) = C(t)x(t) + D(t)u(t)$$

The forced response of this system is given by

$$y_f(t) = \sum_{\tau=0}^{t-1} C(t)\phi(t,\tau+1)B(\tau)u(\tau)d\tau + D(t)u(t), \quad \forall t \geq 0,$$

Definition (BIBO stability)

The system (DLTV) is said to be (uniformly) BIBO stable whenever there exists a finite constant g[1] such that, for every input $u(\bullet)$, its forced response $y_f(\bullet)$ satisfies

$$\sup_{t\in\mathbb{N}} \|y_f(t)\| \leq g \sup_{t\in\mathbb{N}} \|u(t)\|.$$

[1] The factor g can be viewed as the "gain" of the system.

Discrete-time case

Theorem (Time domain BIBO condition)

The following two statements are equivalent.

1. *The system (DLTV) is uniformly BIBO stable.*
2. *Every entry of $D(\bullet)$ is uniformly bounded and*

$$\sup_{t \geq 0} \sum_{\tau=0}^{t-1} |g_{ij}(t,\tau)| < \infty$$

for every entry $g_{ij}(t,\tau)$ of $C(t)\phi(t,\tau)B(\tau)$.

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Theorem (BIBO LTI conditions)

The following three statements are equivalent.

1. *The system (DLTI) is uniformly BIBO stable.*
2. *For every entry $\bar{g}_{ij}(\rho)$ of $CA^{\rho}B$. we have*

$$\sum_{\rho=1}^{\infty} |\bar{g}_{ij}(\rho)| < \infty$$

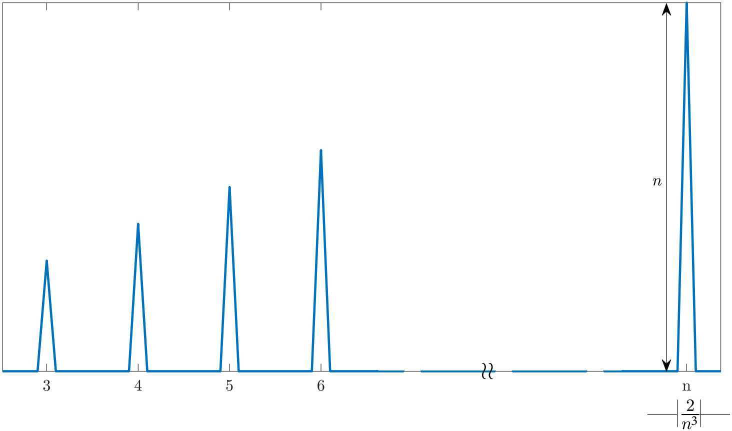
3. *Every pole of every entry of the transfer function of the system (DLTI) has magnitude strictly smaller than 1.*

Interesting facts

Consider the function defined by

$$f(t-n) = \begin{cases} n + (t-n)n^4, & \text{for } n - \frac{1}{n^3} \le t \le n \\ n - (t-n)n^4, & \text{for } n < t \le n + \frac{1}{n^3} \end{cases}$$

for $n = 2, 3, \ldots$. The area under each triangle is $1/n^2$. Thus the absolute integration of the function equals $\sum_{n=2}^{\infty}(1/n^2) < \infty$. This function is absolutely integrable but is ***not*** bounded and does not approach zero as $t \to \infty$.



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In the discrete-time case, if $g(t)$ is absolutely summable, then it ***must*** be bounded and approach zero as $t \to \infty$. However, the converse is not true.

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Example

Consider $g(t) = 1/t$, for $t = 1, 2, \ldots$ and $g(0) = 0$. We compute

$$\begin{aligned} S = \sum_{t=1}^{\infty} |g(t)| &= \sum_{t=1}^{\infty} \frac{1}{t} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots \end{aligned}$$

We notice that

$$S > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$$

This impulse response sequence is bounded and approaches 0 as $t \to \infty$ but is not absolutely summable.

Additional results

Theorem

The equation $\dot{x} = Ax$ is marginally stable if and only if all the eigenvalus of A have zero or negative real parts and those with zero real parts are simple roots of the minimal polynomial of A.

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Theorem (Eigenvalue conditions - Slide 20)

The system (H-CLTI) is

1. *$marginally\ stable$ if and only if all the eigenvalues of A have negative or zero real parts and all the Jordan blocks corresponding to eigenvalues with zero real parts are 1×1,*
2. *$asymptotically\ stable$ if and only if all the eigenvalues of A have strictly negative real parts,*
3. *$unstable$ if and only if at least one eigenvalue of A has a positive real part or zero real part, but the corresponding Jordan block is larger than 1×1.*

Additional results

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1. The characteristic polynomial of A is defined as $f(s) = det(s.I - A)$. $f(s)$ is a monic polynomial of degree n.
2. The minimal polynomial of A, which we will denote by $\psi(s)$, is defined by the following properties:
    - $\psi(s)$ is monic (i.e., its leading coefficent is 1),
    - $\psi(A) = 0$,
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They also satisfy the following properties:

- If $g(s)$ is another polynomial, then $g(A) = 0$ if and only if $\psi(s)$ divides $g(s)$.
- $f(s)$ is a multiple of $\psi(s)$.

Additional results (Example)

Consider

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x$$

Its characteristic polynomial is $\Delta(\lambda) = \lambda^2(\lambda + 1)$

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The equation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is not marginally stable, however, because its minimal polynomial is $\psi(\lambda) = \lambda^2(\lambda + 1)$ and $\lambda = 0$ is not a simple root of the minimal polynomial.

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A time-invariant system is asymptotically stable if all eigenvalues of A have negative real parts. Is this also true for the time-varying case?

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Consider

$$\dot{x} = A(t)x = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix}$$

The characteristic polynomial of $A(t)$ and the eigenvalues are

$$\det(\lambda.I - A(t)) = (\lambda + 1)^2 \implies \lambda = -1, -1$$

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It can be verified directly that

$$\phi(t, 0) = \begin{bmatrix} e^{-t} & 0.5(e^{t} - e^{-t}) \\ 0 & e^{-t} \end{bmatrix}$$

Note that determination of the stability using the eigenvalues of matrix $A(t)$ is not applicable in the time varying case.