Module 9: State Feedback Control Design
Lecture Note 2

1 Set Point Tracking

We discussed in the last lecture that a state feedback design can be done to place poles such that the system is stable. However, the tracking is not guaranteed.

1.1 Feed Forward Gain Design

Consider the state space model

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k) \\ y(k) &= C\mathbf{x}(k) \end{aligned}$$

A control law is selected

$$u(k) = -K\mathbf{x}(k) + Nr(k)$$

as shown in figure 1 so that output can track any step reference command signal r. The closed loop dynamics of this configuration becomes

$$\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k) + BNr(k)$$

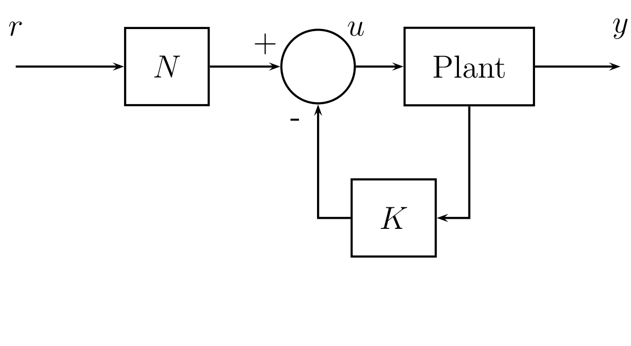

Figure 1: State feedback controller with feed forward gain for set point tracking

At steady state, say $\mathbf{x} = \mathbf{x}_{ss}$, $y = C\mathbf{x}_{ss} = r$ and $u = u_{ss}$. Since the states or the output do not change with time in steady state, we can write

$$\mathbf{x}_{ss} = A\mathbf{x}_{ss} + Bu_{ss}$$

Let us assume

$$\tilde{u}(k) = u(k) - u_{ss}, \quad \tilde{\mathbf{x}}(k) = \mathbf{x}(k) - \mathbf{x}_{ss}, \quad \tilde{y}(k) = y(k) - r$$

Thus in shifted domain,

$$\begin{aligned}
\tilde{\mathbf{x}}(k+1) &= A\mathbf{x}(k) + Bu(k) - A\mathbf{x}_{ss} - Bu_{ss} \\
&= A\tilde{\mathbf{x}}(k) + B\tilde{u}(k) \\
\tilde{y}(k) &= C\tilde{\mathbf{x}}(k)
\end{aligned}$$

If we design a stable control $\tilde{u}(k) = -K\tilde{\mathbf{x}}(k)$ in the shifted domain, it will drive the state variables in shifted domain to 0.

$$\tilde{\mathbf{x}}(k) \to 0 \Rightarrow \mathbf{x}(k) \to \mathbf{x}_{ss} \Rightarrow y(k) \to r$$

The problem of tracking is thus converted into a simple regulator problem.

$$\begin{aligned}
u(k) - u_{ss} &= -K\mathbf{x}(k) + K\mathbf{x}_{ss} \\
\Rightarrow u(k) &= -K\mathbf{x}(k) + u_{ss} + K\mathbf{x}_{ss}
\end{aligned}$$

We know that $\mathbf{x}_{ss} = A\mathbf{x}_{ss} + Bu_{ss}$. Thus

$$\begin{aligned}
0 &= (A - I)\mathbf{x}_{ss} + Bu_{ss} \\
&= (A - I)\mathbf{x}_{ss} + BK\mathbf{x}_{ss} - BK\mathbf{x}_{ss} + Bu_{ss} \\
& \quad (A - BK - I)\mathbf{x}_{ss} + B(u_{ss} + K\mathbf{x}_{ss}) \\
\Rightarrow \mathbf{x}_{ss} &= -(A - BK - I)^{-1}B(u_{ss} + K\mathbf{x}_{ss})
\end{aligned}$$

Now,

$$C\mathbf{x}_{ss} = -C(A - BK - I)^{-1}B(u_{ss} + K\mathbf{x}_{ss}) = r$$

Thus the possible solution for N is $u_{ss} + K\mathbf{x}_{ss} = Nr$ where

$$N^{-1} = -C(A - BK - I)^{-1}B$$

and the control input is

$$u(k) = -K\mathbf{x}(k) + Nr$$

Example 1: Consider the following system

$$\begin{aligned}
\mathbf{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ -0.47 & 1.47 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\
y(k) &= [0.007 \quad 0.009]\, \mathbf{x}(k)
\end{aligned}$$

Design a state feedback controller such that the output follows a step input with the desired closed loop poles at 0.5 and 0.6.

Solution: Desired characteristic equation:

$$z^2 - 1.1z + 0.3 = 0$$

Open loop characteristic equation:

$$z^2 - 1.47z + 0.47 = 0$$

The controller can be designed using the following expression

$$u(k) = -K\mathbf{x}(k) + Nr$$

Since the system is in controllable canonical form, where the controllability matrix is $U_C = \begin{bmatrix} 0 & 1 \\ 1 & 1.47 \end{bmatrix}$ (non singular), the state feedback gain can be straight away designed as

$$K = [0.3 - 0.47 \quad -1.1 + 1.47] = [-0.17 \quad 0.37]$$

N can be designed as

$$N^{-1} = -C(A - BK - I)^{-1}B = 0.08$$

Thus

$$u(k) = -[-0.17 \quad 0.37]\,\mathbf{x}(k) + 12.5\,r$$

1.2 State Feedback with Integral Control

Calculation of feed forward gain requires exact knowledge of the system parameters. Change in parameters will effect the steady state error.

In this scheme we feedback the states as well as the integral of the output error which will eventually make the actual output follow the desired one.

One way to introduce the integrator is to augment the integral state v with the plant state vector $\mathbf{x}$.

v integrates the difference between the output $y(k)$ and reference r. By using backward rectangular integration, it can be defined as

$$v(k) = v(k-1) + y(k) - r$$

Thus

$$\begin{aligned} v(k+1) &= v(k) + y(k+1) - r \\ &= v(k) + C[A\mathbf{x}(k) + Bu(k)] - r \end{aligned}$$

If we augment the above with the state equation,

$$\begin{bmatrix} \mathbf{x}(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} \\ CA & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} B \\ CB \end{bmatrix} u(k) + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r$$

Since r is constant, if the system is stable, then $\mathbf{x}(k+1) = \mathbf{x}(k)$ and $v(k+1) = v(k)$ in the steady state. So, in steady state,

$$\begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r = \begin{bmatrix} \mathbf{x}_{ss} \\ v_{ss} \end{bmatrix} - \begin{bmatrix} A & \mathbf{0} \\ CA & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{ss} \\ v_{ss} \end{bmatrix} - \begin{bmatrix} B \\ CB \end{bmatrix} u_{ss}$$

Let us define $\tilde{\mathbf{x}}_a = [\mathbf{x} - \mathbf{x}_{ss} \quad v - v_{ss}]^T$ and $\tilde{u} = u - u_{ss}$. This implies

$$\tilde{\mathbf{x}}_a(k+1) = \bar{A}\tilde{\mathbf{x}}_a(k) + \bar{B}\tilde{u}(k)$$

where $\bar{A} = \begin{bmatrix} A & \mathbf{0} \\ CA & 1 \end{bmatrix}$ and $\bar{B} = \begin{bmatrix} B \\ CB \end{bmatrix}$

The design problem is now converted to a standard regulation problem. We need to design

$$\tilde{u}(k) = -K\tilde{\mathbf{x}}_a(k)$$

where $K = [K_p \quad K_i]$, K_i is the integral gain. Now,

$$\begin{aligned} u - u_{ss} &= -[K_p \quad K_i] \begin{bmatrix} \mathbf{x} - \mathbf{x}_{ss} \\ v - v_{ss} \end{bmatrix} \\ &= -K_p(\mathbf{x} - \mathbf{x}_{ss}) - K_i(v - v_{ss}) \end{aligned}$$

The steady state terms in the above expression must balance, which implies

$$u(k) = -K_p\mathbf{x}(k) - K_i v(k)$$

At steady state,

$$\begin{aligned} \tilde{\mathbf{x}}_a(k+1) - \tilde{\mathbf{x}}_a(k) &= 0 \\ v(k+1) - v(k) &= = 0 \end{aligned}$$

We can write from the above expression $y(k) - r = 0$ at steady state. In other words, $y(k)$ follows r at steady state. The block diagram of state feedback with integral control is shown in Figure 2.

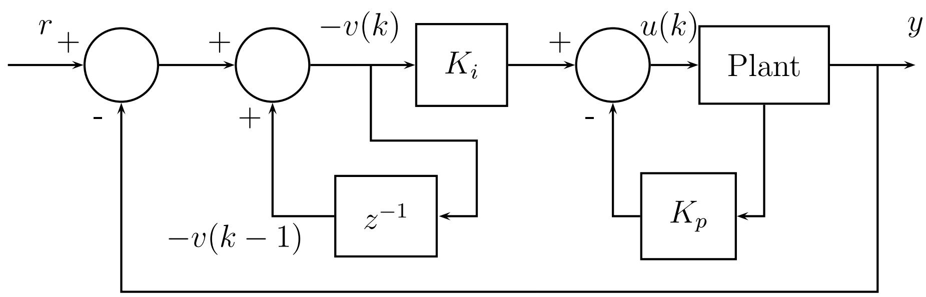

Figure 2: State feedback controller with integral control for set point tracking

Example: Consider the problem of digital control of the following plant

$$G(s) = \frac{1}{s(s+2)}$$

for a sampling period $T = 0.1$ sec using a state feedback with integral control such that the plant output follows a step input. The closed loop continuous poles of the system must be located at $-1 \pm j$ and -5 respectively.

Solution:

Discretization of the plant model gives

$$\begin{aligned} G(z) &= Z\left[\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{1}{s(s+2)}\right)\right] \\ &= \frac{0.005z + 0.004}{z^2 - 1.82z + 0.82} \end{aligned}$$

The discrete state space model of the plant is:

$$\begin{aligned} \mathbf{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ -0.82 & 1.82 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= [0.004 \quad 0.005]\, \mathbf{x}(k) \end{aligned}$$

Augmenting the plant state vector with the integral state $v(k)$, defined by

$$v(k) = v(k-1) + y(k) - r$$

we obtain

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -0.82 & 1.82 & 0 \\ -0.004 & 0.013 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0.005 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} r$$

In terms of the state variables representing deviations from the steady state,

$$\tilde{\mathbf{x}}_a(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ -0.82 & 1.82 & 0 \\ -0.004 & 0.013 & 1 \end{bmatrix} \tilde{\mathbf{x}}_a(k) + \begin{bmatrix} 0 \\ 1 \\ 0.005 \end{bmatrix} \tilde{u}(k)$$

where $\tilde{\mathbf{x}}_a = [\mathbf{x} - \mathbf{x}_{ss} \quad v - v_{ss}]^T$ and $\tilde{u} = u - u_{ss}$. $\tilde{u}(k)$ can be designed for the desired pole locations, $0.9 \pm 0.1j$ and 0.6 (in discrete domain) using Ackermann's formula, as,

$$\tilde{u}(k) = -K\tilde{\mathbf{x}}_a(k) = -[-0.328 \quad 0.416 \quad 0.889]\, \tilde{\mathbf{x}}_a(k)$$

Thus

$$u(k) = -[-0.328 \quad 0.416]\, \mathbf{x}(k) - 0.889\, v(k)$$