Module 9: State Feedback Control Design
Lecture Note 4

In the last lecture we have already acquired some idea about observation and learned how a full order observer can be designed.

In this lecture we will discuss reduced order observers.

1 Reduced Order Observers

We know that an observer that estimates fewer than "n" states of the system is called reduced order observer.

Consider the following system

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k)\\ \mathbf{y}(k) &= C\mathbf{x}(k)\end{aligned}$$

where $\mathbf{x} \in R^{n\times 1}$, $\mathbf{u} \in R^{m\times 1}$ and $\mathbf{y} \in R^{p\times 1}$.

Since the output $\mathbf{y}$ is a vector with dimension p where $p < n$, we would like to use these p outputs to determine p states of the state vector and design an estimator of order $n-p$ to estimate the rest.

If $\text{rank}(C) = p$ then $\mathbf{y}(k) = C\mathbf{x}(k)$ can be used to solve for p of the x_i's in terms of y_i's and remaining $n-p$ state variables x_k's will be estimated.

Let us assume that the dynamics of the observer are given by

$$\bar{\mathbf{x}}(k+1) = D\bar{\mathbf{x}}(k) + E\mathbf{u}(k) + G\mathbf{y}(k) \tag{1}$$

Let us take a transformation P such that

$$\bar{\mathbf{x}} = P\mathbf{x}$$

Applying this transformation on the system dynamics,

$$P\mathbf{x}(k+1) = PA\mathbf{x}(k) + PB\mathbf{u}(k) \tag{2}$$

Subtracting (2) from (1),

$$(PA - DP - GC)\mathbf{x}(k) + (PB - E)\mathbf{u}(k) = 0$$

The above relation will be true for all k and any arbitrary input $\mathbf{u}(k)$, if

$$\begin{aligned} PA - DP &= GC \\ \text{and, } E &= PB \end{aligned}$$

If $\bar{\mathbf{x}} \neq P\mathbf{x}$ but the above equation holds true, then we can write

$$\begin{aligned} \bar{\mathbf{x}}(k+1) - P\mathbf{x}(k+1) &= D\bar{\mathbf{x}}(k) - PA\mathbf{x}(k) + GC\mathbf{x}(k) \\ &= D\bar{\mathbf{x}}(k) - DP\mathbf{x}(k) \\ &= D(\bar{\mathbf{x}}(k) - P\mathbf{x}(k)) \end{aligned}$$

If D has eigenvalues inside the unit circle, then we can write

$$\bar{\mathbf{x}}(k) \to P\mathbf{x}(k) \;\; \text{as, } k \to \infty$$

If we try $P = I_{n\times n}$ where $I_{n\times n}$ is the identity matrix with dimension $n \times n$, and $G = L$, we get

$$\begin{aligned} A - LC &= D \\ \text{and, } E &= B \end{aligned}$$

The resulting estimator is the Luenberger full order estimator. $A - LC = D$ can be solved for L such that D has eigenvalues at the prescribed locations.

The above is possible if and only if the pair $(A,\ C)$ is observable which was the only assumption in observer design.

The dimensions of P, D and G are as follows

$$P \in R^{(n-p)\times n}, \quad D \in R^{(n-p)\times(n-p)}, \quad G \in R^{(n-p)\times p}$$

Now we have

$$\begin{aligned} \mathbf{y}(k) &= C\mathbf{x}(k) \\ \text{and, } \bar{\mathbf{x}}(k) &= P\mathbf{x}(k) \quad (\text{as } \; k \to \infty) \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \mathbf{y} \\ \bar{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} C \\ P \end{bmatrix} \mathbf{x}$$

Thus the estimated state vector will be

$$\hat{\mathbf{x}} = \begin{bmatrix} C \\ P \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y} \\ \bar{\mathbf{x}} \end{bmatrix}$$

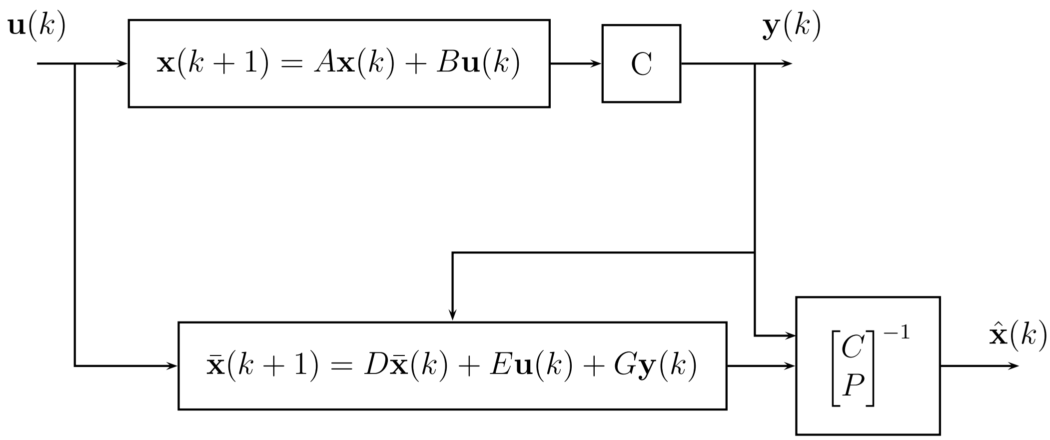

Figure 1: Reduced Order observer

Thus rank of $\begin{bmatrix} C \\ P \end{bmatrix}$ should be equal to n.

$PA - DP = GC$ can be uniquely solved if no eigenvalues of D is an eigenvalue of A.

Figure 1 shows the block diagram of a reduced order observer.

While choosing D and G, we have to make sure that $\begin{bmatrix} C \\ P \end{bmatrix}$ has rank n. The following example will illustrate the observer design.

Example 1: Let us take the following discrete time system

$$\begin{aligned}
\mathbf{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ 20 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(k)
\end{aligned}$$

Thus, $D \in R^{1\times 1}$. Let us take $D = 0.5$. We know,

$$PA - DP = GC$$

Let us assume $P = [p_1 \quad p_2] \in R^{1\times 2}$. Putting this in the above equation,

$$\begin{aligned}
[p_1 \quad p_2] \begin{bmatrix} 0 & 1 \\ 20 & 0 \end{bmatrix} - 0.5[p_1 \quad p_2] &= GC \\
\text{or, } [20p_2 - 0.5p_1 \quad p_1 - 0.5p_2] &= GC
\end{aligned}$$

If we take $G = 20$,
$$GC = [20 \quad 0]$$
Thus, we get

$$\begin{aligned} p_1 &= 0.5p_2 \\ \text{and, } 20p_2 - 0.5p_1 &= 20 \end{aligned}$$

Solving the above equations, $p_1 = 0.51$ and $p_2 = 1.01$ and $E = [0.51 \quad 1.01]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1.01$.

1.1 Controller with Reduced Order Observer

The observer dynamics:
$$\bar{\mathbf{x}}(k+1) = D\bar{\mathbf{x}}(k) + E\mathbf{u}(k) + GC\mathbf{x}(k)$$
The state feedback control
$$\mathbf{u}(k) = -K\hat{\mathbf{x}}(k)$$
where
$$\hat{\mathbf{x}}(k) = \begin{bmatrix} C \\ P \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix}$$
and $\mathbf{y}(k) = C\mathbf{x}(k)$. Let's assume $\begin{bmatrix} C \\ P \end{bmatrix}^{-1} = [Q_1 \;\; Q_2]$. Thus
$$\begin{aligned} \mathbf{u}(k) &= -K[Q_1 \;\; Q_2]\begin{bmatrix} \mathbf{y}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} \\ &= -KQ_1\mathbf{y}(k) - KQ_2\bar{\mathbf{x}}(k) \end{aligned}$$
Combining the observer with the system dynamics
$$\begin{aligned} \begin{bmatrix} \mathbf{x}(k+1) \\ \bar{\mathbf{x}}(k+1) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ GC & D \end{bmatrix}\begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} B \\ PB \end{bmatrix}(-KQ_1\mathbf{y}(k) - KQ_2\bar{\mathbf{x}}(k)) \\ &= \begin{bmatrix} A - BKQ_1C & -BKQ_2 \\ GC - PBKQ_1C & D - PBKQ_2 \end{bmatrix}\begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} \\ \mathbf{y}(k) &= [C \;\; 0]\begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} \end{aligned}$$
Let us define
$$\begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) - P\mathbf{x}(k) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -P & I_{n-p} \end{bmatrix}\begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix}$$
We can write
$$\begin{bmatrix} I_n & 0 \\ -P & I_{n-p} \end{bmatrix}^{-1} = \begin{bmatrix} I_n & 0 \\ P & I_{n-p} \end{bmatrix}$$

Again we can write

$$\begin{aligned}\mathbf{x}(k+1) &= (A - BKQ_1C)\mathbf{x}(k) - BKQ_2\bar{\mathbf{x}}(k)\\ &= (A - BKQ_1C)\mathbf{x}(k) - BKQ_2(\bar{\mathbf{x}}(k) - P\mathbf{x}(k)) - BKQ_2P\mathbf{x}(k)\\ &= (A - BKQ_1C - BKQ_2P)\mathbf{x}(k) - BKQ_2(\bar{\mathbf{x}}(k) - P\mathbf{x}(k))\end{aligned}$$

and

$$\begin{aligned}\bar{\mathbf{x}}(k+1) - P\mathbf{x}(k+1) &= D\bar{\mathbf{x}}(k) + E\mathbf{u}(k) + GC\mathbf{x}(k) - PA\mathbf{x}(k) - PB\mathbf{u}(k)\\ &= (DP - PA + GC)\mathbf{x}(k) + D(\bar{\mathbf{x}}(k) - P\mathbf{x}(k))\end{aligned}$$

Thus

$$\begin{bmatrix}\mathbf{x}(k+1)\\ \bar{\mathbf{x}}(k+1) - P\mathbf{x}(k+1)\end{bmatrix} = \begin{bmatrix}A - BKQ_1C - BKQ_2P & -BKQ_2\\ DP - PA + GC & D\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \bar{\mathbf{x}}(k) - P\mathbf{x}(k)\end{bmatrix}$$

Since $PA - DP - GC = 0$ and $Q_2P = I_n - Q_1C$, we can write

$$\begin{bmatrix}\mathbf{x}(k+1)\\ \bar{\mathbf{x}}(k+1) - P\mathbf{x}(k+1)\end{bmatrix} = \begin{bmatrix}A - BK & -BKQ_2\\ 0 & D\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \bar{\mathbf{x}}(k) - P\mathbf{x}(k)\end{bmatrix}$$

From the above we can say that if $(A - BK)$ and D have eigenvalues inside the unit circle then $\mathbf{x}(k) \to 0$ and $\bar{\mathbf{x}}(k) \to P\mathbf{x}(k)$

Again, the eigenvalues of $\begin{bmatrix}A - BK & -BKQ_2\\ 0 & D\end{bmatrix}$ are the eigenvalues of $A - BK$ together with eigenvalues of D.

Thus K and D can be separately designed to ensure that both $(A - BK)$ and D have eigenvalues inside the unit circle

Thus **separation principle** is valid for reduced order observer too.

Figure 2 shows the block diagram of controller with reduced order observer.

Points to remember

1. Separation principle assumes that the observer uses an exact dynamics of the plant. In reality, the precise dynamic model is hardly known.

2. The information known about the real process is often too complicated to be used in the observer.

3. The above points indicate that separation principle is not good enough for observer design, robustness of the observer must be checked as well.

4. K should be designed such that the resulting $\mathbf{u}$ is not much high because of hardware limitation. Also, large control increases the possibility of entering the system into nonlinear region.

5. Dynamics of the observer poles should be much faster than the controller poles.

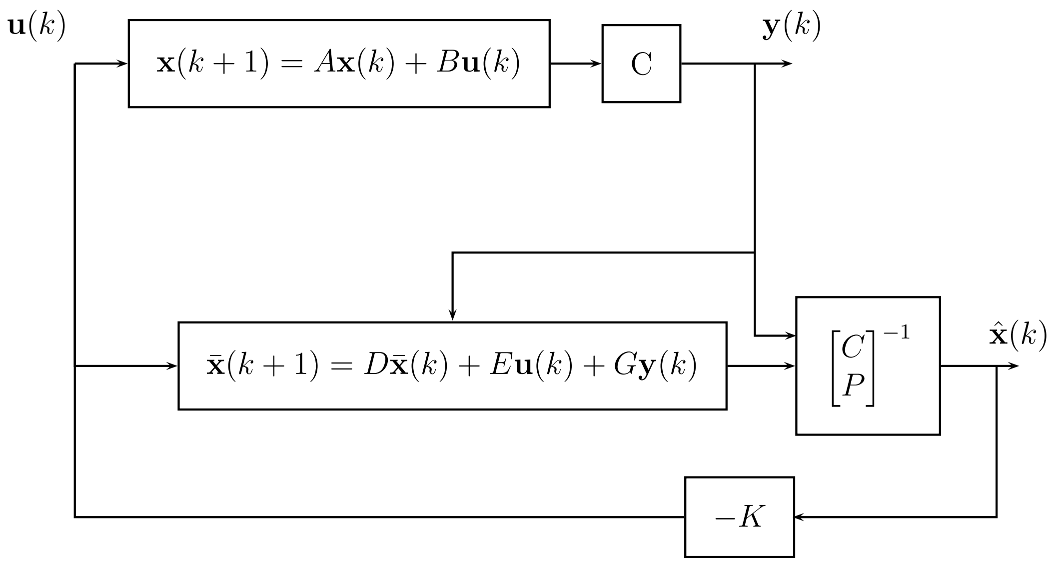

Figure 2: Controller with reduced order observer

2 Deadbeat Control by State Feedback and Deadbeat Observers

Consider the system

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k)\\ y(k) &= C\mathbf{x}(k)\end{aligned}$$

where $A \in R^{n\times n}$, $B \in R^{n\times 1}$ and $C \in R^{1\times n}$. With the state feedback control $u(k) = -K\mathbf{x}(k)$ the closed loop system becomes

$$\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k)$$

Desired characteristic equation:

$$z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1}z + \alpha_n = 0$$

We pick a K such that coefficients of $|zI - (A - BK)|$ match with those of the desired characteristic equation.

Let us consider a special case when $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. The desired characteristic equation in this case becomes

$$z^n = 0$$

By Cayley-Hamilton theorem:

$$(A - BK)^n = 0$$

Thus

$$\mathbf{x}(k) = (A - BK)^k \mathbf{x}(0) = 0, \quad \text{for } k \geq n$$

In other words, any initial state $\mathbf{x}(0)$ is driven to the equilibrium state $\mathbf{x} = 0$ in at most n steps.

Thus the control law that assigns all the poles to origin can be viewed as a deadbeat control.

Similarly when all observer poles are at zero, we refer to that a deadbeat observer.

In deadbeat response, settling time depends on the sampling period. For a very small T, settling time is also very small, but the control signal becomes very high. Designer has to make a trade off between the two.