

Module 2: Modeling Discrete Time Systems by Pulse Transfer Function

Lecture Note 1

1 Motivation for using Z-transform

In general, control system design methods can be classified as:

conventional or classical control techniques

modern control techniques

Classical methods use transform techniques and are based on transfer function models, whereas modern techniques are based on modeling of system by state variable methods.

Laplace transform is the basic tool of the classical methods in continuous domain. In principle, it can also be used for modeling digital control systems. However typical Laplace transform expressions of systems with digital or sampled signals contain exponential terms in the form of e^{Ts} which makes the manipulation in the Laplace domain difficult. This can be regarded as a motivation of using Z-transform.

Let the output of an ideal sampler be denoted by $f^*(t)$.

$$\begin{aligned} L[f^*(t)] &= F^*(s) \\ &= \sum_{k=0}^{\infty} f(kT)e^{-kTs} \end{aligned}$$

Since $F^*(s)$ contains the term e^{-kTs} , it is not a rational function of s . When terms with e^{-Ts} appear in a transfer function other than a multiplying factor, difficulties arise while taking the inverse Laplace. It is desirable to transfer the irrational function $F^*(s)$ to a rational function for which one obvious choice is:

$$\begin{aligned} z &= e^{Ts} \\ \Rightarrow s &= \frac{1}{T} \ln z \end{aligned}$$

If, $s = \sigma + jw$,

$$\begin{aligned} Re\ z &= e^{T\sigma} \cos wT \\ Im\ z &= e^{T\sigma} \sin wT \end{aligned}$$

Z-transform:

$$\begin{aligned} F^* \left[s = \frac{1}{T} \ln z \right] &= F(z) \\ &= \sum_{k=0}^{\infty} f(kT) z^{-k} \end{aligned}$$

$F(z)$ is the Z-transform of $f(t)$ at the sampling instants k .

$$F(z) = \sum_{k=0}^{\infty} f(kT) z^{-k}$$

In general, we can say that if $f(t)$ is Laplace transformable then it also has a Z-transform.

$$\begin{aligned} L[f(t)] &= \int_0^{\infty} f(t) e^{-st} dt \\ Z[f(t)] &= \sum_{k=0}^{\infty} f(kT) z^{-k} \end{aligned}$$

2 Revisiting Z-Transforms

Z-transform is a powerful operation method to deal with discrete time systems. In considering Z-transform of a time function $x(t)$, we consider only the sampled values of $x(t)$, i.e., $x(0), x(T), x(2T), \dots$ where T is the sampling period.

$$\begin{aligned} X(z) = Z[x(t)] &= Z[x(kT)] \\ &= \sum_{k=0}^{\infty} x(kT) z^{-k} \end{aligned}$$

For a sequence of numbers $x(k)$

$$\begin{aligned} X(z) &= Z[x(k)] \\ &= \sum_{k=0}^{\infty} x(k) z^{-k} \end{aligned}$$

The above transforms are referred to as **one sided z-transform**. In one sided z-transform, we assume that $x(t) = 0$ for $t < 0$ or $x(k) = 0$ for $k < 0$. In **two sided z-transform**, we assume that $-\infty < t < \infty$ or $k = \pm 1, \pm 2, \pm 3, \dots$

$$\begin{aligned} X(z) &= Z[x(kT)] \\ &= \sum_{k=-\infty}^{\infty} x(kT) z^{-k} \end{aligned}$$

or for $x(k)$

$$\begin{aligned} X(z) &= Z[x(k)] \\ &= \sum_{k=-\infty}^{\infty} x(k)z^{-k} \end{aligned}$$

The one sided z-transform has a convenient closed form solution in its region of convergence (ROC) for most engineering applications. Whenever $X(z)$, an infinite series in z^{-1} , converges outside the circle $|z| = R$, where R is the radius of absolute convergence, it is not needed each time to specify the values of z over which $X(z)$ is convergent.

$$\begin{aligned} |z| > R &\Rightarrow \text{convergent} \\ |z| < R &\Rightarrow \text{divergent.} \end{aligned}$$

In one sided z-transform theory, while sampling a discontinuous function $x(t)$, we assume that the function is continuous from the right, i.e., if discontinuity occurs at 0 we assume that $x(0) = x(0+)$.

2.1 Z-Transforms of some elementary functions

Unit step function is defined as:

$$\begin{aligned} u_s(t) &= 1, \text{ for } t \geq 0 \\ &= 0, \text{ for } t < 0 \end{aligned}$$

Assuming that the function is continuous from right

$$\begin{aligned} X(z) &= \sum_{k=0}^{\infty} u_s(kT)z^{-k} \\ &= \sum_{k=0}^{\infty} z^{-k} \\ &= 1 + z^{-1} + z^{-2} + z^{-3} + \dots \\ &= \frac{1}{1 - z^{-1}} \\ &= \frac{z}{z - 1} \end{aligned}$$

The above series converges if $|z| > 1$.

One should note that the **Unit step sequence** is defined as

$$\begin{aligned} u_s(k) &= 1, \text{ for } k = 0, 1, 2 \dots \\ &= 0, \text{ for } k < 0 \end{aligned}$$

with a same Z-transform.

Unit ramp function is defined as:

$$\begin{aligned} u_r(t) &= t, \quad \text{for } t \geq 0 \\ &= 0, \quad \text{for } t < 0 \end{aligned}$$

The Z-transform is:

$$\begin{aligned} U_r(z) &= \frac{Tz}{(z-1)^2} \\ &= T \frac{z^{-1}}{(1-z^{-1})^2} \end{aligned}$$

with ROC $|z| > 1$.

For a **polynomial function** $x(k) = a^k$, the Z-transform is:

$$\begin{aligned} X(z) &= \frac{1}{1 - a.z^{-1}} \\ &= \frac{z}{z - a} \end{aligned}$$

where ROC is $|z| > a$.

Exponential function is defined as:

$$\begin{aligned} x(t) &= e^{-at}, \quad \text{for } t \geq 0 \\ &= 0, \quad \text{for } t < 0 \end{aligned}$$

We have $x(kT) = e^{-akT}$ for $k = 0, 1, 2 \dots$. Thus,

$$\begin{aligned} X(z) &= \frac{1}{1 - e^{-aT}z^{-1}} \\ &= \frac{z}{z - e^{-aT}} \end{aligned}$$

Similarly Z-transforms can be computed for sinusoidal and other compound functions. One should refer the Z-transform table provided in the appendix.

2.2 Properties of Z- transform

1. **Multiplication by a constant:** $Z[ax(t)] = aX(z)$, where $X(z) = Z[x(t)]$.

2. **Linearity:** If $x(k) = \alpha f(k) \pm \beta g(k)$, then $X(z) = \alpha F(z) \pm \beta G(z)$.

3. **Multiplication by a^k :** $Z[a^k x(k)] = X(a^{-1}z)$

4. **Real shifting:** $Z[x(t - nT)] = z^{-n}X(z)$ and $z[x(t + nT)] = z^n \left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k} \right]$

5. **Complex shifting:** $Z[e^{\pm at}x(t)] = X(ze^{\mp aT})$

6. **Initial value theorem:**

$$x(0) = \lim_{z \rightarrow \infty} X(z)$$

7. **Final value theorem:**

$$\lim_{k \rightarrow \infty} x(k) = \lim_{z \rightarrow 1} [(1 - z^{-1})X(z)]$$

2.3 Inverse Z-transforms

Single sided Laplace transform and its inverse make a unique pair, i.e. if $F(s)$ is the Laplace transform of $f(t)$, then $f(t)$ is the inverse Laplace transform of $F(s)$. But the same is not true for Z-transform. Say $f(t)$ is the continuous time function whose Z-transform is $F(z)$ then the inverse transform is not necessarily equal to $f(t)$, rather it is equal to $f(kT)$ which is equal to $f(t)$ only at the sampling instants. Once $f(t)$ is sampled by an ideal sampler, the information between the sampling instants is totally lost and we cannot recover actual $f(t)$ from $F(z)$,

$$\Rightarrow f(kT) = Z^{-1}[F(z)]$$

The transform can be obtained by using

→ Partial fraction expansion

→ Power series

→ Inverse formula.

Inverse Z-transform formula:

$$f(kT) = \frac{1}{2\pi j} \oint_{\Gamma} F(z) z^{k-1} dz$$

2.4 Other Z-transform properties

Partial differentiation theorem:

$$Z \left[\frac{\partial}{\partial a} [f(t, a)] \right] = \frac{\partial}{\partial a} F(z, a)$$

Real convolution theorem:

If $f_1(t)$ and $f_2(t)$ have z-transforms $F_1(z)$ and $F_2(z)$ and $f_1(t) = 0 = f_2(t)$ for $t < 0$, then

$$F_1(z)F_2(z) = Z \left[\sum_{n=0}^k f_1(nT)f_2(kT - nT) \right]$$

Complex convolution:

$$Z[f_1(t)f_2(t)] = \frac{1}{2\pi j} \oint_{\Gamma} \frac{F_1(\xi)F_2(z\xi^{-1})}{\xi} d\xi$$

Γ : circle / closed path in z-plane which lie in the region $\sigma < |\xi| < \frac{|z|}{\sigma_2}$

σ_1 : radius of convergence of $F_1(\xi)$

σ_2 : radius of convergence of $F_2(\xi)$

2.5 Limitation of Z-transform method

1. Ideal sampler assumption
 \Rightarrow z-transform represents the function only at sampling instants.
2. Non uniqueness of z-transform.
3. Accuracy depends on the magnitude of the sampling frequency w_s relative to the highest frequency component contained in the function $f(t)$.
4. A good approximation of $f(t)$ can only be interpolated from $f(kT)$, the inverse z-transform of $F(z)$ by connecting $f(kT)$ with a smooth curve.

2.6 Application of Z-transform in solving Difference Equation

One of the most important applications of Z-transform is in the solution of linear difference equations. Let us consider that a discrete time system is described by the following difference equation.

$$y(k+2) + 0.5y(k+1) + 0.06y(k) = -(0.5)^{k+1}$$

with the initial conditions $y(0) = 0$, $y(1) = 0$. We have to find the solution $y(k)$ for $k > 0$. Taking z-transform on both sides of the above equation:

$$\begin{aligned} z^2Y(z) + 0.5zY(z) + 0.06Y(z) &= -0.5 \frac{z}{z-0.5} \\ \text{or, } Y(z) &= -\frac{0.5z}{(z-0.5)(z^2+0.5z+0.06)} \\ &= -\frac{0.5z}{(z-0.5)(z+0.2)(z+0.3)} \end{aligned}$$

Using partial fraction expansion:

$$Y(z) = -\frac{0.8937z}{z-0.5} + \frac{7.143z}{z+0.2} - \frac{6.25z}{z+0.3}$$

$$\text{Taking Inverse Laplace: } y(k) = -0.893(0.5)^k + 7.143(-0.2)^k - 6.25(-0.3)^k$$

To emphasize the fact that $y(k) = 0$ for $k < 0$, it is a common practice to write the solution as:

$$y(k) = -0.893(0.5)^k u_s(k) + 7.143(-0.2)^k u_s(k) - 6.25(-0.3)^k u_s(k)$$

where $u_s(k)$ is the unit step sequence.