Module 1: Introduction to Digital Control
Lecture Note 3

1 Mathematical Modeling of Sampling Process

Sampling operation in sampled data and digital control system is used to model either the sample and hold operation or the fact that the signal is digitally coded. If the sampler is used to represent S/H (Sample and Hold) and A/D (Analog to Digital) operations, it may involve delays, finite sampling duration and quantization errors. On the other hand if the sampler is used to represent digitally coded data the model will be much simpler. Following are two popular sampling operations:

1. Single rate or periodic sampling
2. Multi-rate sampling

We would limit our discussions to periodic sampling only.

1.1 Finite pulse width sampler

In general, a sampler is the one which converts a continuous time signal into a pulse modulated or discrete signal. The most common type of modulation in the sampling and hold operation is the pulse amplitude modulation.

The symbolic representation, block digram and operation of a sampler are shown in Figure 1. The pulse duration is p second and sampling period is T second. Uniform rate sampler is a linear device which satisfies the principle of superposition. As in Figure 1, $p(t)$ is a unit pulse train with period T.

$$p(t) = \sum_{k=-\infty}^{\infty} [u_s(t-kT) - u_s(t-kT-p)]$$

where $u_s(t)$ represents unit step function. Assume that leading edge of the pulse at $t = 0$ coincides with $t = 0$. Thus $f_p^*(t)$ can be written as

$$f_p^*(t) = f(t) \sum_{k=-\infty}^{\infty} [u_s(t-kT) - u_s(t-kT-p)]$$

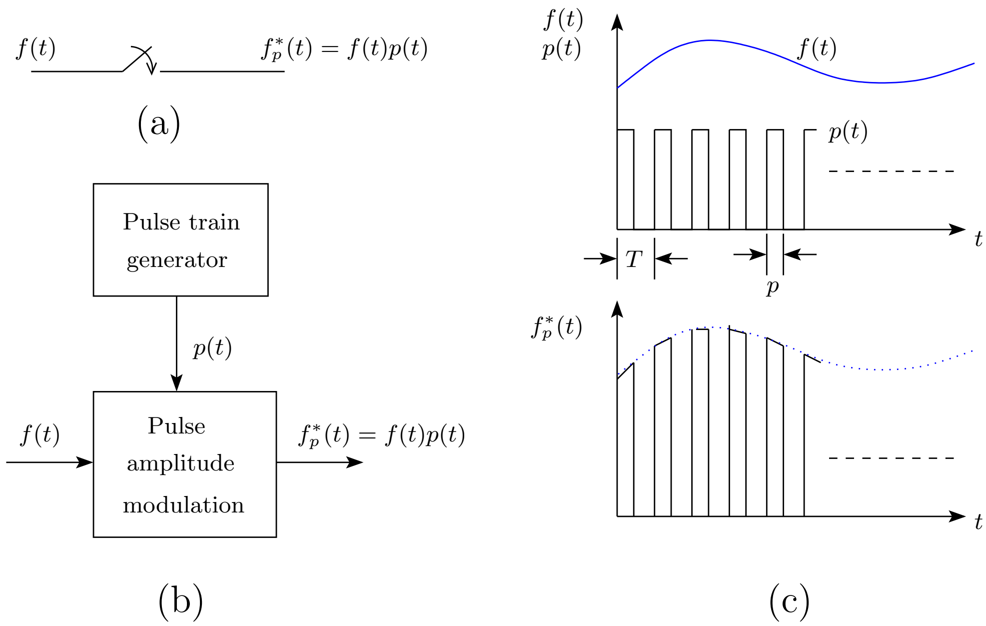

Figure 1: Finite pulse width sampler:(a)Symbolic representation (b)Block diagram (c)Operation

Frequency domain characteristics:

Since $p(t)$ is a periodic function, it can be represented by a Fourier series, as

$$p(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_s t}$$

where

$w_s = \dfrac{2\pi}{T}$ is the sampling frequency and C_n's are the complex Fourier series coefficients.

$$C_n = \frac{1}{T}\int_0^T p(t)e^{-jnw_s t}dt$$

Since $p(t) = 1$ for $0 \le t \le p$ and 0 for rest of the period,

$$\begin{aligned} C_n &= \frac{1}{T}\int_0^p e^{-jnw_s t}dt \\ &= \left[\frac{1}{-jnw_s T}.e^{-jnw_s t}\right]_0^p \\ &= \frac{1 - e^{-jnw_s p}}{jnw_s T} \end{aligned}$$

C_n can be rearranged as,

$$\begin{aligned} C_n = &= \frac{e^{-jnw_s p/2}(e^{jnw_s p/2} - e^{-jnw_s p/2})}{jnw_s T} \\ &= \frac{2je^{-jnw_s p/2}\sin(nw_s p/2)}{jnw_s T} \\ &= \frac{p}{T}\frac{\sin(nw_s p/2)}{nw_s p/2}e^{-jnw_s p/2} \end{aligned}$$

Since $f_p^*(t)$ is also periodic, it can be written as

$$f_p^*(t) = \sum_{n=-\infty}^{\infty} C_n f(t) e^{jnw_s t}$$

$$\begin{aligned} \Rightarrow F_p^*(jw) &= \mathcal{F}[f_p^*(t)], \quad \text{where } \mathcal{F} \text{ represents Fourier transform} \\ &= \int_{-\infty}^{\infty} f_p^*(t)e^{-jwt}dt \end{aligned}$$

Using complex shifting theorem of Fourier transform

$$\mathcal{F}\left[e^{jnw_s t}f(t)\right] = F(jw - jnw_s)$$

$$\Rightarrow F_p^*(jw) = \sum_{n=-\infty}^{\infty} C_n F(jw - jnw_s)$$

Since n is from $-\infty$ to ∞, the above equation can also be written as

$$F_p^*(jw) = \sum_{n=-\infty}^{\infty} C_n F(jw + jnw_s)$$

where,

$$\begin{aligned} C_o &= \lim_{n\to 0} C_n \\ &= \frac{p}{T} \\ F_p^*(jw)|_{n=0} &= C_0 F(jw) = \frac{p}{T}F(jw) \end{aligned}$$

The above equation implies that the frequency contents of the original signal $f(t)$ are still present in the sampler output except that the amplitude is multiplied by the factor $\frac{p}{T}$.
For $n \neq 0$, C_n is a complex quantity, the magnitude of which is,

$$|C_n| = \frac{p}{T}\left|\frac{\sin(nw_s p/2)}{nw_s p/2}\right|$$

Magnitude of $F_p^*(jw)$

$$\begin{aligned} \left|F_p^*(jw)\right| &= \left|\sum_{n=-\infty}^{\infty} C_n F(jw + jnw_s)\right| \\ &\leq \sum_{n=-\infty}^{\infty} |C_n||F(jw + jnw_s)| \end{aligned}$$

Sampling operation retains the fundamental frequency but in addition, sampler output also contains the harmonic components.

$$F(jw + jnw_s) \quad \text{for} \;\; n = \pm 1, \pm 2, ...$$

According to Shannon's **sampling theorem**, "if a signal contains no frequency higher than w_c rad/sec, it is completely characterized by the values of the signal measured at instants of time separated by $T = \pi/w_c$ sec."

Sampling frequency rate should be greater than the **Nyquist rate** which is twice the highest frequency component of the original signal to avoid aliasing.

If the sampling rate is less than twice the input frequency, the output frequency will be different from the input which is known as **aliasing**. The output frequency in that case is called **alias frequency** and the period is referred to as **alias period**.

The overlapping of the high frequency components with the fundamental component in the frequency spectrum is sometimes referred to as **folding** and the frequency $\dfrac{w_s}{2}$ is often known as **folding frequency**. The frequency w_c is called **Nyquist frequency**.

A low sampling rate normally has an adverse effect on the closed loop stability. Thus, often we might have to select a sampling rate much higher than the theoretical minimum.

1.2 Flat-top approximation of finite-pulsewidth sampling

The Laplace transform of $f_p^*(t)$ can be written as

$$F_p^*(s) = \sum_{n=-\infty}^{\infty} \frac{1 - e^{-jnw_s p}}{jnw_s T} F(s + jnw_s)$$

If the sampling duration p is much smaller than the sampling period T and the smallest time constant of the signal $f(t)$, the sampler output can be approximated by a sequence of rectangular pulses since the variation of $f(t)$ in the sampling duration will be less significant. Thus for $k = 0, 1, 2, ...$, $f_p^*(t)$ can be expressed as an infinite series

$$f_p^*(t) = \sum_{k=0}^{\infty} f(kT)\left[u_s(t - kT) - u_s(t - kT - p)\right]$$

Taking Laplace transform,

$$F_p^*(s) = \sum_{k=0}^{\infty} f(kT)\left[\frac{1-e^{-ps}}{s}\right]e^{-kTs}$$

Since p is very small, e^{-ps} can be approximated by taking only the first 2 terms, as

$$\begin{aligned}1-e^{-ps} &= 1-[1-ps+\frac{(ps)^2}{2!}.......]\\ &\cong ps\\ \text{Thus, }\ F_p^*(s) &\cong p\sum_{k=0}^{\infty} f(kT)e^{-kTs}\end{aligned}$$

In time domain,

$$f_p^*(t) = p\sum_{k=0}^{\infty} f(kT)\delta(t-kT)$$

where, $\delta(t)$ represents the unit impulse function. Thus the finite pulse width sampler can be viewed as an impulse modulator or an ideal sampler connected in series with an attenuator with attenuation p.

1.3 The ideal sampler

In case of an ideal sampler, the carrier signal is replaced by a train of unit impulse as shown in Figure 2. The sampling duration p approaches 0, i.e., its operation is instantaneous.

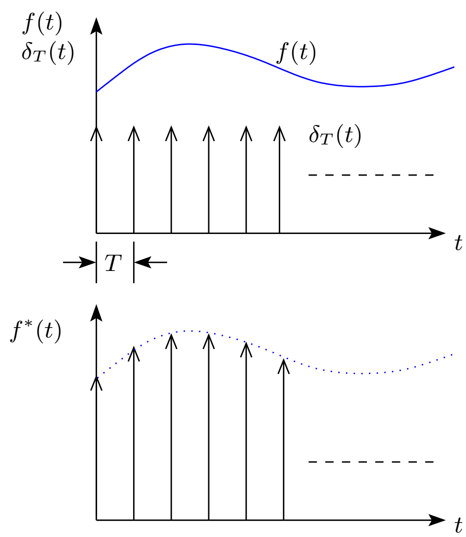

Figure 2: Ideal sampler operation

The output of an ideal sampler can be expressed as

$$
\begin{aligned}
f^*(t) &= \sum_{k=0}^{\infty} f(kT)\delta(t-kT) \\
\Rightarrow F^*(s) &= \sum_{k=0}^{\infty} f(kT)e^{-kTs}
\end{aligned}
$$

One should remember that practically the output of a sampler is always followed by a hold device which is the reason behind the name sample and hold device. Now, the output of a hold device will be the same regardless the nature of the sampler and the attenuation factor p can be dropped in that case. Thus the sampling process can be be always approximated by an ideal sampler or impulse modulator.