Module 5: Design of Sampled Data Control Systems
Lecture Note 6

1 Compensator Design Using Bode Plot

In this lecture we would revisit the continuous time design techniques using frequency domain since these can be directly applied to design for digital control system by transferring the loop transfer function in z-plane to w-plane.

1.1 Phase lead compensator

If we look at the frequency response of a simple PD controller, it is evident that the magnitude of the compensator continuously grows with the increase in frequency.

The above feature is undesirable because it amplifies high frequency noise that is typically present in any real system.

In lead compensator, a first order pole is added to the denominator of the PD controller at frequencies well higher than the corner frequency of the PD controller.

A typical lead compensator has the following transfer function.

$$C(s) = K\frac{\tau s + 1}{\alpha \tau s + 1}, \quad \text{where, } \alpha < 1$$

$\dfrac{1}{\alpha}$ is the ratio between the pole zero break point (corner) frequencies.

Magnitude of the lead compensator is $K\dfrac{\sqrt{1+\omega^2\tau^2}}{\sqrt{1+\alpha^2\omega^2\tau^2}}$. And the phase contributed by the lead compensator is given by

$$\phi = \tan^{-1}\omega\tau - \tan^{-1}\alpha\omega\tau$$

Thus a significant amount of phase is still provided with much less amplitude at high frequencies.

The frequency response of a typical lead compensator is shown in Figure 1 where the magnitude varies from $20\log_{10} K$ to $20\log_{10}\dfrac{K}{\alpha}$ and maximum phase is always less than 90^o (around 60^o in general).

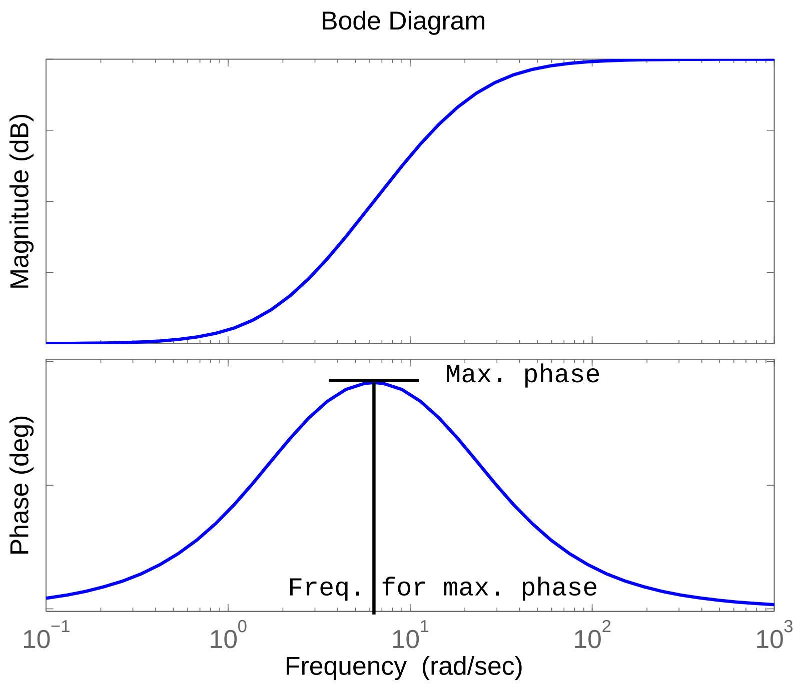

Figure 1: Frequency response of a lead compensator

It can be shown that the frequency where the phase is maximum is given by

$$\omega_{\max} = \frac{1}{\tau\sqrt{\alpha}}$$

The maximum phase corresponds to

$$\sin\phi_{\max} = \frac{1-\alpha}{1+\alpha}$$

$$\Rightarrow \alpha = \left(\frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})}\right)$$

The magnitude of $C(s)$ at $\omega_{\max}$ is $\dfrac{K}{\sqrt{\alpha}}$.

Example 1: Consider the following system

$$G(s) = \frac{1}{s(s+1)}, \quad H(s) = 1$$

Design a cascade lead compensator so that the phase margin (PM) is at least 45^o and steady state error for a unit ramp input is ≤ 0.1.

The lead compensator is

$$C(s) = K\frac{\tau s + 1}{\alpha \tau s + 1}, \quad \text{where, } \alpha < 1$$

When $s \to 0$, $C(s) \to K$.

Steady state error for unit ramp input is

$$\frac{1}{\lim_{s\to 0} sC(s)G(s)} = \frac{1}{C(0)} = \frac{1}{K}$$

Thus $\dfrac{1}{K} = 0.1$, or $K = 10$.

PM of the closed loop system should be 45^o. Let the gain crossover frequency of the uncompensated system with K be ω_g.

$$\begin{aligned}
G(j\omega) &= \frac{1}{j\omega(j\omega+1)} \\
Mag. &= \frac{1}{\omega\sqrt{1+\omega^2}} \\
Phase &= -90^o - \tan^{-1}\omega \\
\Rightarrow \frac{10}{\omega_g\sqrt{1+\omega_g^2}} &= 1 \\
\frac{100}{\omega_g^2(1+\omega_g^2)} &= 1 \\
\Rightarrow \omega_g &= 3.1
\end{aligned}$$

Phase angle at $\omega_g = 3.1$ is $-90 - \tan^{-1} 3.1 = -162^o$. Thus the PM of the uncompensated system with K is 18^o.

If it was possible to add a phase without altering the magnitude, the additional phase lead required to maintain PM=45^o is $45^o - 18^o = 27^o$ at $\omega_g = 3.1$ rad/sec.

However, maintaining same low frequency gain and adding a compensator would increase the crossover frequency. As a result of this, the actual phase margin will deviate from the designed one. Thus it is safe to add a safety margin of ϵ to the required phase lead so that if it devaites also, still the phase requirement is met. In general ϵ is chosen between 5^o to 15^o.

So the additional phase requirement is $27^o + 10^o = 37^o$. The lead part of the compensator will provide this additional phase at $\omega_{\max}$.

Thus

$$\begin{aligned} \phi_{\max} &= 37^o \\ \Rightarrow \alpha &= \left(\frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})}\right) = 0.25 \end{aligned}$$

The only parameter left to be designed is τ. To find τ, one should locate the frequency at which the uncompensated system has a logarithmic magnitude of $-20\log_{10}\dfrac{1}{\sqrt{\alpha}}$.

Select this frequency as the new gain crossover frequency since the compensator provides a gain of $20\log_{10}\dfrac{1}{\sqrt{\alpha}}$ at $\omega_{\max}$. Thus

$$\omega_{\max} = \omega_{g_{new}} = \frac{1}{\tau\sqrt{\alpha}}$$

In this case $\omega_{\max} = \omega_{g_{new}} = 4.41$. Thus

$$\tau = \frac{1}{4.41\sqrt{\alpha}} = 0.4535$$

The lead compensator is thus

$$C(s) = 10\,\frac{0.4535s+1}{0.1134s+1}$$

With this compensator actual phase margin of the system becomes 49.6^o which meets the design criteria. The corresponding Bode plot is shown in Figure 2

Example 2:

Now let us consider that the system as described in the previous example is subject to a sampled data control system with sampling time $T = 0.2$ sec. Thus

$$\begin{aligned} G_z(z) &= (1-z^{-1})\mathrm{Z}\left[\frac{1}{s^2(s+1)}\right] \\ &= \frac{0.0187z+0.0175}{z^2-1.8187z+0.8187} \end{aligned}$$

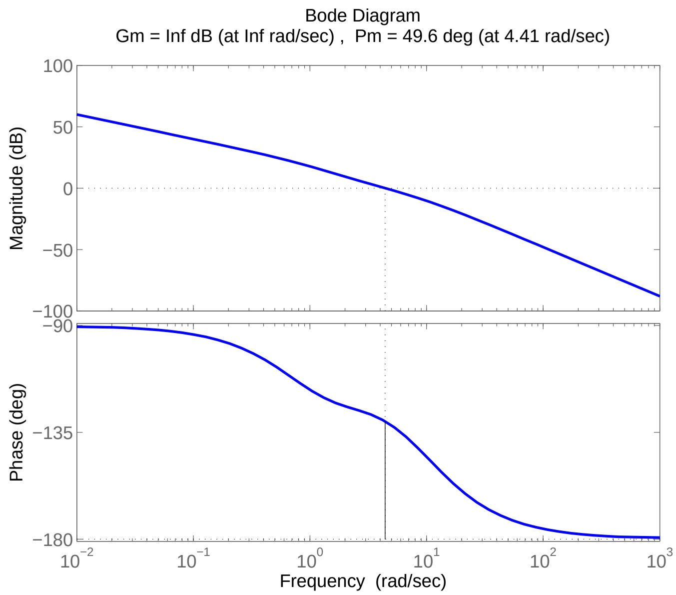

Figure 2: Bode plot of the compensated system for Example 1

The bi-linear transformation

$$z = \frac{1 + wT/2}{1 - wT/2} = \frac{(1 + 0.1w)}{(1 - 0.1w)}$$

will transfer $G_z(z)$ into w-plane, as

$$G_w(w) = \frac{\left(1 + \frac{w}{300}\right)\left(1 - \frac{w}{10}\right)}{w(w+1)} \quad \text{[please try the simplification]}$$

We need first design a phase lead compensator so that PM of the compensated system is at least 50^0 with $K_v = 2$. The compensator in w-plane is

$$C(w) = K\frac{1 + \tau w}{1 + \alpha\tau w} \qquad 0 < \alpha < 1$$

Design steps are as follows.

- K has to be found out from the K_v requirement.
- Compute the gain crossover frequency ω_g and phase margin of the uncompensated system after introducing K in the system.
- At ω_g check the additional/required phase lead, add safety margin, find out $\phi_{\max}$. Calculate α from the required $\phi_{\max}$.

- Since the lead part of the compensator provides a gain of $20\log_{10}\dfrac{1}{\sqrt{\alpha}}$, find out the frequency of the uncompensated system where the logarithmic magnitude is $-20\log_{10}\dfrac{1}{\sqrt{\alpha}}$. This will be the new gain crossover frequency where the maximum phase lead should occur.
- Make $\omega_{\max} = \omega_{g_{new}}$.
- Calculate τ from the relation

$$\omega_{g_{new}} = \omega_{\max} = \frac{1}{\tau\sqrt{\alpha}}$$

Now,

$$K_v = \lim_{w\to 0} wC(w)G_w(w) = 2$$

$$\Rightarrow K = 2$$

Using MATLAB command "margin", phase margin of the system with $K = 2$ is computed as 31.6^0 with $\omega_g = 1.26$ rad/sec, as shown in Figure 3.

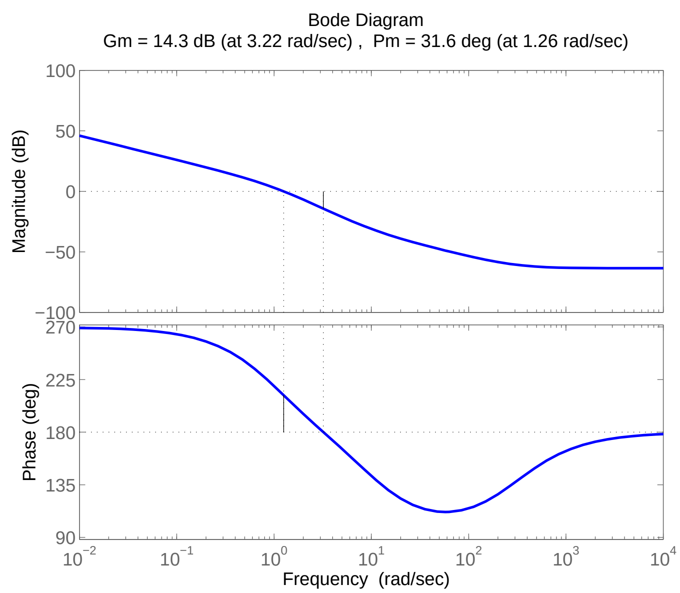

Figure 3: Bode plot of the uncompensated system for Example 2

Thus the required phase lead is $50^o - 31.6^o = 18.4^o$. After adding a safety margin of 11.6^o, $\phi_{\max}$ becomes 30^o. Hence

$$\alpha = \left(\frac{1-\sin(30^o)}{1+\sin(30^o)}\right) = 0.33$$

From the frequency response of the system it can be found out that at $\omega = 1.75$ rad/sec, the magnitude of the system is $-20\log_{10}\dfrac{1}{\sqrt{\alpha}}$. Thus $\omega_{g_{new}} = \omega_{\max} = 1.75$ rad/sec. This gives

$$1.75 = \frac{1}{\tau\sqrt{\alpha}}$$

Or,

$$\tau = \frac{1}{1.75\sqrt{0.33}} = 0.99$$

Thus the controller in w-plane is

$$C(w) = 2\frac{1+0.99w}{1+0.327w}$$

The Bode plot of the compensated system is shown in Figure 4.

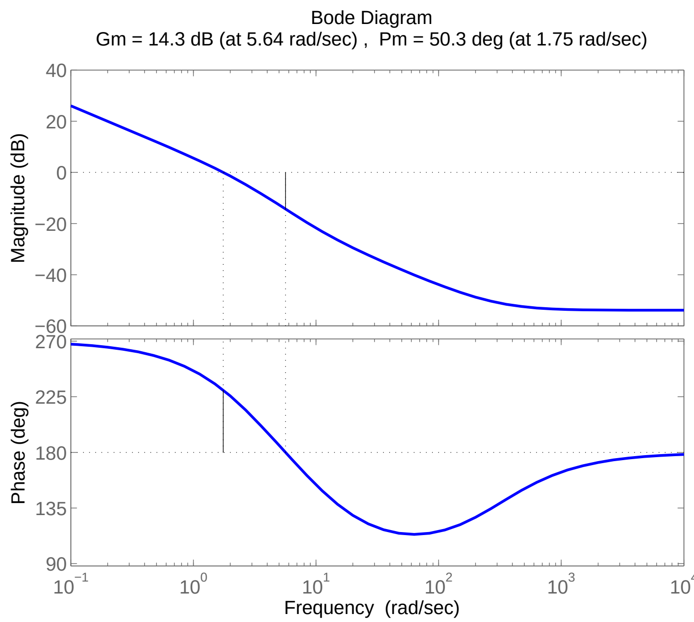

Figure 4: Bode plot of the compensated system for Example 2

Re-transforming the above controller into z-plane using the relation $w = 10\dfrac{z-1}{z+1}$, we get the controller in z-plane, as

$$C_z(z) \;\cong\; 2\,\frac{2.55z - 2.08}{z - 0.53}$$