Module 10: Output Feedback Design
Lecture Note 2

1 Output feedback design examples

In the last lecture, we have discussed about the incomplete state feedback design and output feedback design. In this lecture we would solve some examples to make the procedure properly understood.

Example 1: Let us consider the following system

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k)\\ \mathbf{y}(k) &= C\mathbf{x}(k)\end{aligned}$$

for which the A, B, C matrices are as follows

$$A = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1\\ 1 & 0\\ 0 & 0\end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}$$

We know, for output feedback,

$$\mathbf{u}(k) = -G\mathbf{y}(k)$$

where the matrix G has to be designed. Since C has rank 2 and the rank of B is also 2, minimum 2 eigenvalues can be placed at desired locations. Let these two be 0.1 and 0.2. The characteristic equation of A is

$$|zI - A| = z^3 + 1$$

B^* is written as

$$B^* = BW = \begin{bmatrix} 0 & 1\\ 1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} w_1\\ w_2\end{bmatrix} = \begin{bmatrix} w_2\\ w_1\\ 0\end{bmatrix}$$

which has two independent parameters in terms of w_1 and w_2. Controllability matrix for the pair $(A,\ B^*)$ is

$$U_c^* = \begin{bmatrix} B^* & AB^* & A^2B^* \end{bmatrix} = \begin{bmatrix} w_2 & w_1 & 0 \\ w_1 & 0 & -w_2 \\ 0 & -w_2 & -w_1 \end{bmatrix}$$

It will be non singular if ${w_1}^3 - {w_2}^3 \neq 0$. Let

$$G^* = \begin{bmatrix} {g_1}^* & {g_2}^* \end{bmatrix} C$$

Then

$$G^*C = \begin{bmatrix} {g_1}^* + {g_2}^* & {g_2}^* & 0 \end{bmatrix}$$

Since C has a rank of 2, G^*C has two independent parameters in terms of g_1^* and g_2^*. The closed loop characteristic equation is

$$\phi(z) = z^3 + \alpha_3 z^2 + \alpha_2 z + \alpha_1 = 0$$

Thus

$$G^*C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} {U_c^*}^{-1} \phi(A)$$

or,

$$\begin{bmatrix} {g_1}^* + {g_2}^* \\ {g_2}^* \\ 0 \end{bmatrix} = \frac{1}{w_1^3 - w_2^3} \begin{bmatrix} -\alpha_3 w_2^2 + \alpha_2 w_1^2 - (\alpha_1 - 1) w_1 w_2 \\ \alpha_3 w_1^2 - \alpha_2 w_1 w_2 + (\alpha_1 - 1) w_2^2 \\ -\alpha_3 w_1 w_2 + \alpha_2 w_2^2 - (\alpha_1 - 1) w_1^2 \end{bmatrix}$$

The last row in the above equation corresponds to the following constraint equation.

$$-\alpha_3 w_1 w_2 + \alpha_2 w_2^2 - (\alpha_1 - 1) w_1^2 = 0 \tag{1}$$

Since 2 of the three eigenvalues can be arbitrarily placed, w_1 and w_2 can be arbitrary provided the condition $w_1^3 \neq w_2^3$ is satisfied. But they should be selected such that the third eigenvalue is stable. This puts an additional constraint on w_1 and w_2.

For example, the necessary condition for the closed loop system to be stable is $|\alpha| < 1$. To satisfy this condition, w_2 cannot be equal to zero.

For $z = 0.1$ and 0.2 to be the roots of the characteristic equation

$$z^3 + \alpha_3 z^2 + \alpha_2 z + \alpha_1 = 0$$

the following equations must be satisfied

$$\alpha_1 + 0.001 + \alpha_3 0.01 + 0.1\alpha_2 = 0$$
$$\alpha_1 + 0.008 + \alpha_3 0.04 + 0.2\alpha_2 = 0$$

Simplifying the above equations,

$$\alpha_2 + 0.3\alpha_3 + 0.07 = 0 \tag{2}$$

$$\alpha_1 - 0.02\alpha_3 - 0.006 = 0 \tag{3}$$

Solving equations (1), (2) and (3) together

$$\begin{aligned}
\alpha_1 &= \frac{0.02w_1^2 + 0.0004w_2^2 + 0.006w_1w_2}{0.3w_2^2 + w_1w_2 + 0.02w_1^2} \\
\alpha_2 &= \frac{-0.2996w_1^2 - 0.07w_1w_2}{0.3w_2^2 + w_1w_2 + 0.02w_1^2} \\
\alpha_3 &= \frac{0.994w_1^2 - 0.07w_2^2}{0.3w_2^2 + w_1w_2 + 0.02w_1^2}
\end{aligned}$$

If we set $w_1 = 0$ and $w_2 = 1$, we get $\alpha_1 = 0.00133$, $\alpha_2 = 0$ and $\alpha_3 = -0.23333$.

With the above coefficients we find the roots to be $z_1 = 0.1$, $z_2 = 0.2$ and $z_3 = -0.0667$. Thus the third pole is placed within the unit circle and the closed loop system is stable.

There also exist some other combinations of w_1 and w_2 for which $z_1 = 0.1$, $z_2 = 0.2$ and the closed loop system is stable.

Putting the values of w_1 and w_2 and corresponding α_1, α_2 and α_3 in the expression of G^*C, we get

$$\begin{bmatrix} {g_1}^* + {g_2}^* & {g_2}^* & 0 \end{bmatrix} = \begin{bmatrix} \alpha_3 \\ -\alpha_1 + 1 \\ \alpha_2 \end{bmatrix}^T = \begin{bmatrix} -0.23333 \\ 0.99867 \\ 0 \end{bmatrix}^T$$

Thus the feedback matrix can be calculated as

$$G^* = \begin{bmatrix} -1.232 & 0.99867 \end{bmatrix}$$

Hence,

$$G = WG^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1.232 & 0.99867 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ -1.232 & 0.99867 \end{bmatrix}$$

Example 2: Consider the same system as in the previous example except for the fact that now

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which has rank 1. This implies that

$$G^*C = \begin{bmatrix} {g_1}^* + {g_2}^* & 0 & 0 \end{bmatrix}$$

which has only one independent parameter in terms of g_1^* and g_2^*. Thus

$$\begin{bmatrix} {g_1}^* + {g_2}^* \\ 0 \\ 0 \end{bmatrix} = \frac{1}{w_1^3 - w_2^3} \begin{bmatrix} -\alpha_3 w_2^2 + \alpha_2 w_1^2 - (\alpha_1 - 1) w_1 w_2 \\ \alpha_3 w_1^2 - \alpha_2 w_1 w_2 + (\alpha_1 - 1) w_2^2 \\ -\alpha_3 w_1 w_2 + \alpha_2 w_2^2 - (\alpha_1 - 1) w_1^2 \end{bmatrix}$$

Last two rows of the above equation are constrained to be zero. Thus we can only assign w_1 or w_2 arbitrarily, not both. The constraint equations are as follows.

$$\alpha_3 w_1^2 - \alpha_2 w_1 w_2 + (\alpha_1 - 1) w_2^2 = 0$$

$$-\alpha_3 w_1 w_2 + \alpha_2 w_2^2 - (\alpha_1 - 1) w_1^2 = 0$$

If we want two closed loop eigenvalues to be placed at $z = 0.1$ and $z = 0.2$, we will altogether have four equations with five unknowns. Only one of these five unknowns can be assigned arbitrarily.

But these four equations would be nonlinear in w_1 and w_2, hence difficult to solve. The simpler way would be to use the following equation

$$|zI - A + BGC| = z^3 + (g_{21} + g_{22})z^2 + (g_{11} + g_{12})z + 1 = 0$$

Only two coefficients can be arbitrarily assigned. Since the constant term is equal to 1, the system cannot be stabilized with output feedback.