Module 2: Modeling Discrete Time Systems by Pulse Transfer Function
Lecture Note 3

1 Pulse Transfer Function

Transfer function of an LTI (Linear Time Invariant) continuous time system is defined as

$$G(s) = \frac{C(s)}{R(s)}$$

where $R(s)$ and $C(s)$ are Laplace transforms of input $r(t)$ and output $c(t)$. We assume that initial condition are zero.

Pulse transfer function relates z-transform of the output at the sampling instants to the Z-transform of the sampled input. When the same system is subject to a sampled data or digital signal $r^*(t)$, the corresponding block diagram is given in Figure 1.

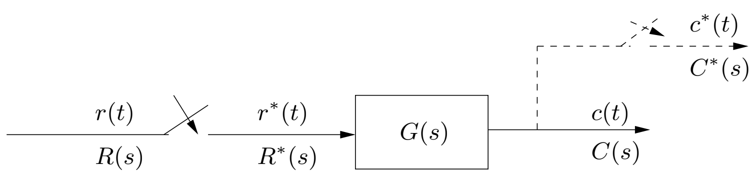

Figure 1: Block diagram of a system subject to a sampled input

The output of the system is $C(s) = G(s)R^*(s)$. The transfer function of the above system is difficult to manipulate because it contains a mixture of analog and digital components. Thus, it is desirable to express the system characteristics by a transfer function that relates $r^*(t)$ to $c^*(t)$, a fictitious sampler output as shown in Figure 1. One can then write:

$$\begin{aligned} C^*(s) &= \sum_{k=0}^{\infty} c(kT)e^{-kTs} \\ \text{Since } c(kT) \text{ is periodic,} \quad C^*(s) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s + jnw_s) \;\; \text{with } c(0) = 0 \end{aligned}$$

The detailed derivation of the above expression is omitted. Similarly,

$$\begin{aligned}
R^*(s) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} R(s+jnw_s) \\
\text{Again, } C^*(s) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s+jnw_s) \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} R^*(s+jnw_s)G(s+jnw_s)
\end{aligned}$$

Since $R^*(s)$ is periodic $R^*(s+jnW_s) = R^*(s)$. Thus

$$\begin{aligned}
C^*(s) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} R^*(s)G(s+jnw_s) \\
&= R^*(s)\frac{1}{T}\sum_{n=-\infty}^{\infty} G(s+jnw_s)
\end{aligned}$$

If we define $G^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s+jnw_s)$, then $C^*(s) = R^*(s)G^*(s)$.

$$G^*(s) = \frac{C^*(s)}{R^*(s)}$$

is known as **pulse transfer function**. Sometimes it is also referred to as the **starred transfer function**. If we now substitute $z = e^{Ts}$ in the previous expression we will directly get the **z-transfer function** $G(z)$ as

$$G(z) = \frac{C(z)}{R(z)}$$

$G(z)$ can also be defined as

$$G(z) = \sum_{k=0}^{\infty} g(kT)z^{-k}$$

where $g(kT)$ denotes the sequence of the impulse response $g(t)$ of the system of transfer function $G(s)$. The sequence $g(kT), k = 0, 1, 2, ..$ is also known as impulse sequence.

Overall Conclusion

1. Pulse transfer function or z-transfer function characterizes the discrete data system responses only at sampling instants. The output information between the sampling instants is lost.

2. Since the input of discrete data system is described by output of the sampler, for all practical purposes the samplers can be simply ignored and the input can be regarded as $r^*(t)$.

Alternate way to arrive at $G(z) = \dfrac{C(z)}{R(z)}$:

$$\begin{aligned} c^*(t) &= g^*(t)\big|_{\text{when } r^*(t) \text{ is an impulse function}} \\ &= \sum_{k=0}^{\infty} g(kT)\delta(t-kT) \end{aligned}$$

When the input is $r^*(t)$,

$$\begin{aligned} c(t) &= r(0)g(t) + r(T)g(t-T) + ... \\ \Rightarrow c(kT) &= r(0)g(kT) + r(T)g((k-1)T) + ... \\ \Rightarrow c(kT) &= \sum_{n=0}^{k} r(nT)g(kT-nT) \\ \Rightarrow C(z) &= \sum_{k=-\infty}^{\infty}\sum_{n=0}^{k} r(nT)g(kT-nT)z^{-k} \end{aligned}$$

Using real convolution theorem

$$C(z) = R(z)G(z)$$
$$\Rightarrow G(z) = \frac{C(z)}{R(z)}$$

1.1 Pulse transfer of discrete data systems with cascaded elements

Care must be taken when the discrete data system has cascaded elements. Following two cases will be considered here.

- Cascaded elements are separated by a sampler
- Cascaded elements are not separated by a sampler

The block diagram for the first case is shown in Figure 2.

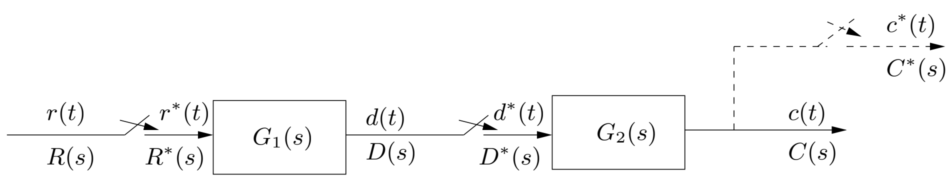

Figure 2: Discrete data system with cascaded elements, separated by a sampler

The input-output relations of the two systems G_1 and G_2 are described by

$$D(z) = G_1(z)R(z)$$

and

$$C(z) = G_2(z)D(z)$$

Thus the input-output relation of the overall system is

$$C(z) = G_1(z)G_2(z)R(z)$$

We can therefore conclude that the z-transfer function of two linear system separated by a sampler are the products of the individual z-transfer functions.

Figure 3 shows the block diagram for the second case. The continuous output $C(s)$ can be

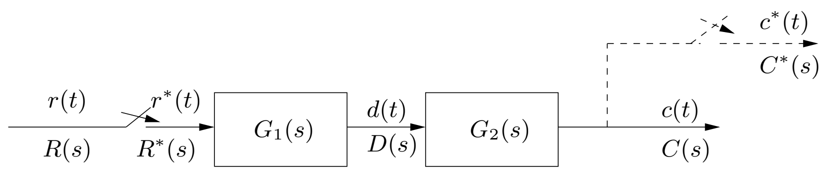

Figure 3: Discrete data system with cascaded elements, not separated by a sampler

written as

$$C(s) = G_1(s)G_2(s)R^*(s)$$

The output of the fictitious sampler is

$$C(z) = Z\left[G_1(s)G_2(s)\right]R(z)$$

z-transform of the product $G_1(s)G_2(s)$ is denoted as

$$Z\left[G_1(s)G_2(s)\right] = G_1G_2(z) = G_2G_1(z)$$

One should note that in general $G_1G_2(z) \neq G_1(z)G_2(z)$, except for some special cases. The overall output is thus,

$$C(z) = G_1G_2(z)R(z)$$

1.2 Pulse transfer function of ZOH

As derived in lecture 4 of module 1, transfer function of zero order hold is

$$G_{ho}(s) = \frac{1-e^{-Ts}}{s}$$

$$\begin{aligned}
\Rightarrow \text{Pulse transfer function } \; G_{ho}(z) &= Z\left[\frac{1-e^{-Ts}}{s}\right] \\
&= (1-z^{-1})Z\left[\frac{1}{s}\right] \\
&= (1-z^{-1})\frac{z}{z-1} \\
&= 1
\end{aligned}$$

This result is expected because zero order hold simply holds the discrete signal for one sampling period, thus taking z-transform of ZOH would revert back its original sampled signal.

A common situation in discrete data system is that a sample and hold (S/H) device precedes a linear system with transfer function $G(s)$ as shown in Figure 4. We are interested in finding the transform relation between $r^*(t)$ and $c^*(t)$.

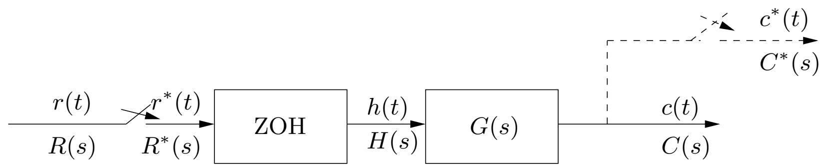

Figure 4: Block diagram of a system subject to a sample and hold process

z-transform of output $c(t)$ is

$$\begin{aligned} C(z) &= Z\left[G_{ho}(s)G(s)\right]R(z) \\ &= Z\left[\frac{1-e^{-Ts}}{s}G(s)\right]R(z) \\ &= (1-z^{-1})Z\left[\frac{G(s)}{s}\right]R(z) \end{aligned}$$

where $(1-z^{-1})Z\left[\frac{G(s)}{s}\right]$ is the z-transfer function of an S/H device and a linear system.

It was mentioned earlier that when sampling frequency reaches infinity a discrete data system may be regarded as a continuous data system. However, this does not mean that if the signal $r(t)$ is sampled by an ideal sampler then $r^*(t)$ can be reverted to $r(t)$ by setting the sampling time T to zero. This simply bunches all the samples together. Rather, if the output of the sampled signal is passed through a hold device then setting the sampling time T to zero the original signal $r(t)$ can be recovered. In relation with Figure 4,

$$\lim_{T\to 0} H(s) = R(s)$$

Example

Consider that the input is $r(t) = e^{-at}u_s(t)$, where $u_s(t)$ is the unit step function.

$$\Rightarrow R(s) = \frac{1}{s+a}$$

Laplace transform of sampled signal $r^*(t)$ is

$$R^*(s) = \frac{e^{Ts}}{e^{Ts} - e^{-aT}}$$

Laplace transform of the output after the ZOH is

$$\begin{aligned} H(s) &= G_{ho}(s)R^*(s) \\ &= \frac{1-e^{-Ts}}{s} \cdot \frac{e^{Ts}}{e^{Ts}-e^{-aT}} \end{aligned}$$

When $T \to 0$,

$$\lim_{T\to 0} H(s) = \lim_{T\to 0} \frac{1-e^{-Ts}}{s} \frac{e^{Ts}}{e^{Ts}-e^{-aT}}$$

The limit can be calculated using **L' hospital's rule**. It says that:

If $\lim_{x\to a} f(x) = 0/\infty$ *and if* $\lim_{x\to a} g(x) = 0/\infty$, *then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

For the given example, $x = T$, $f(T) = \dfrac{1-e^{-Ts}}{s}$ and $g(T) = \dfrac{e^{Ts}-e^{-aT}}{e^{Ts}}$. Both the expressions approach zero as $T \to 0$. So,

$$\begin{aligned} H(s) &= \lim_{T\to 0} \frac{f(T)}{g(T)} \\ &= \lim_{T\to 0} \frac{f'(T)}{g'(T)} \\ &= \lim_{T\to 0} \frac{e^{Ts}}{(s+a)e^{-T(s+a)}} \\ &= \frac{1}{s+a} \\ &= R(s) \end{aligned}$$

which implies that the original signal can be recovered from the output of the **sample and hold** device if the sampling period approaches zero.