

Module 5: Design of Sampled Data Control Systems

Lecture Note 7

1 Lag Compensator Design

In the previous lecture we discussed lead compensator design. In this lecture we would see how to design a phase lag compensator

1.1 Phase lag compensator

The essential feature of a lag compensator is to provide an increased low frequency gain, thus decreasing the steady state error, without changing the transient response significantly.

For frequency response design it is convenient to use the following transfer function of a lag compensator.

$$C_{lag}(s) = \alpha \frac{\tau s + 1}{\alpha \tau s + 1}, \quad \text{where, } \alpha > 1$$

The above expression is only the lag part of the compensator. The overall compensator is

$$C(s) = K C_{lag}(s)$$

$$\text{when, } s \rightarrow 0, \quad C_{lag}(s) \rightarrow \alpha$$

$$\text{when, } s \rightarrow \infty, \quad C_{lag}(s) \rightarrow 1$$

Typical objective of lag compensator design is to provide an additional gain of α in the low frequency region and to leave the system with sufficient phase margin.

The frequency response of a lag compensator, with $\alpha = 4$ and $\tau = 3$, is shown in Figure 1 where the magnitude varies from $20 \log_{10} \alpha$ dB to 0 dB.

Since the lag compensator provides the maximum lag near the two corner frequencies, to maintain the PM of the system, zero of the compensator should be chosen such that $\omega = 1/\tau$ is much lower than the gain crossover frequency of the uncompensated system.

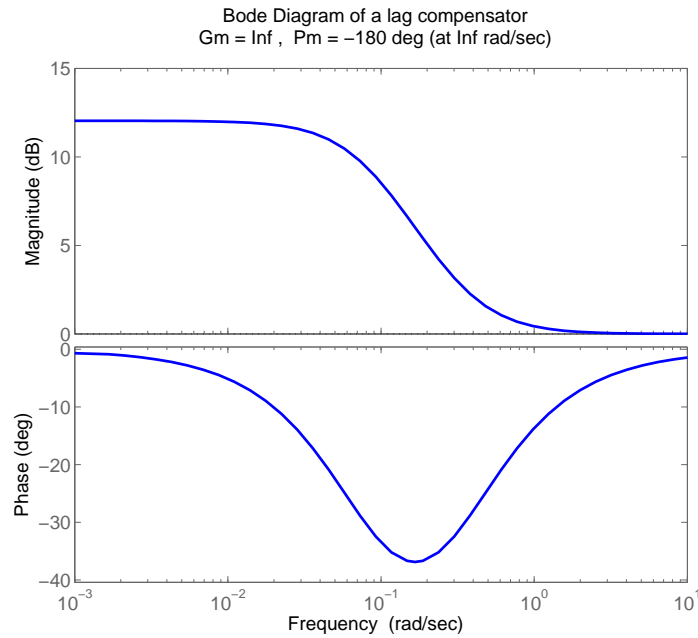


Figure 1: Frequency response of a lag compensator

In general, τ is designed such that $1/\tau$ is at least one decade below the gain crossover frequency of the uncompensated system. Following example will be comprehensive to understand the design procedure.

Example 1: Consider the following system

$$G(s) = \frac{1}{(s+1)(0.5s+1)}, \quad H(s) = 1$$

Design a lag compensator so that the phase margin (PM) is at least 50° and steady state error to a unit step input is ≤ 0.1 .

The overall compensator is

$$C(s) = KC_{lag}(s) = K\alpha \frac{\tau s + 1}{\alpha \tau s + 1}, \quad \text{where, } \alpha > 1$$

When $s \rightarrow 0$, $C(s) \rightarrow K\alpha$.

Steady state error for unit step input is

$$\frac{1}{1 + \lim_{s \rightarrow 0} C(s)G(s)} = \frac{1}{1 + C(0)} = \frac{1}{1 + K\alpha}$$

Thus, $\frac{1}{1 + K\alpha} = 0.1$, or, $K\alpha = 9$.

Now let us modify the system transfer function by introducing K with the original system. Thus the modified system becomes

$$G_m(s) = \frac{K}{(s + 1)(0.5s + 1)}$$

PM of the closed loop system should be 50° . Let the gain crossover frequency of the uncompensated system with K be ω_g .

$$\begin{aligned} G_m(j\omega) &= \frac{K}{(j\omega + 1)(0.5j\omega + 1)} \\ \text{Mag.} &= \frac{K}{\sqrt{1 + \omega^2}\sqrt{1 + 0.25\omega^2}} \\ \text{Phase} &= -\tan^{-1}\omega - \tan^{-1}0.5\omega \end{aligned}$$

Required PM is 50° . Since the PM is achieved only by selecting K , it might be deviated from this value when the other parameters are also designed. Thus we put a safety margin of 5° to the PM which makes the required PM to be 55° .

$$\Rightarrow 180^\circ - \tan^{-1}\omega_g - \tan^{-1}0.5\omega_g = 55^\circ$$

$$\text{or, } \tan^{-1} \frac{\omega_g + 0.5\omega_g}{1 - 0.5\omega_g^2} = 125^\circ$$

$$\text{or, } \tan^{-1} \frac{1.5\omega_g}{1 - 0.5\omega_g^2} = \tan 125^\circ = -1.43$$

$$\text{or, } 0.715\omega_g^2 - 1.5\omega_g - 1.43 = 0$$

$$\Rightarrow \omega_g = 2.8 \text{ rad/sec}$$

To make $\omega_g = 2.8 \text{ rad/sec}$, the gain crossover frequency of the modified system, magnitude at ω_g should be 1. Thus

$$\frac{K}{\sqrt{1 + \omega_g^2}\sqrt{1 + 0.25\omega_g^2}} = 1$$

Putting the value of ω_g in the last equation, we get $K = 5.1$.

Thus,

$$\alpha = \frac{9}{K} = 1.76$$

The only parameter left to be designed is τ .

Since the desired PM is already achieved with gain K , we should place $\omega = 1/\tau$ such that it does not much effect the PM of the modified system with K . If we place $1/\tau$ one decade below the gain crossover frequency, then

$$\frac{1}{\tau} = \frac{2.8}{10}, \quad \text{or, } \tau = 3.57$$

The overall compensator is

$$C(s) = 9 \frac{3.57s + 1}{6.3s + 1}$$

With this compensator actual phase margin of the system becomes 52.7° , as shown in Figure 2, which meets the design criteria.

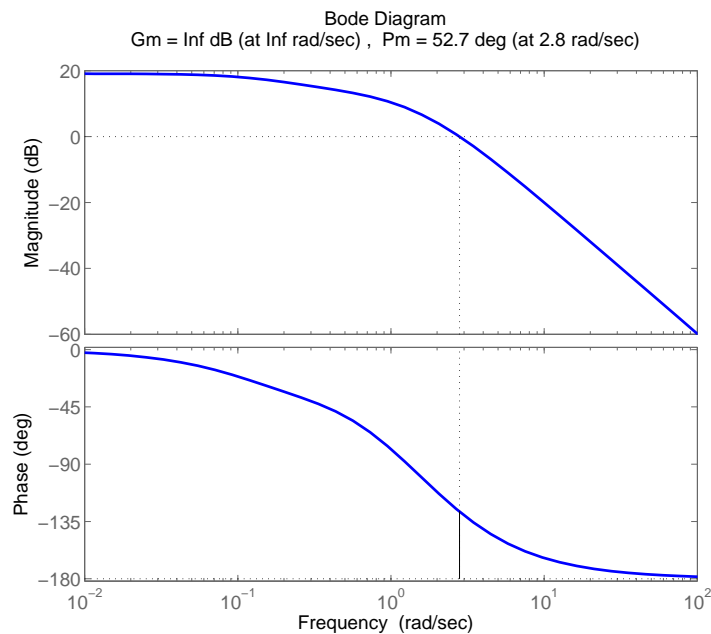


Figure 2: Bode plot of the compensated system for Example 1

Example 2:

Now let us consider that the system as described in the previous example is subject to a sampled data control system with sampling time $T = 0.1$ sec. We would use MATLAB to derive the plant transfer function w -plane.

Use the below commands.

```
>> s=tf('s');
>> gc=1/((s+1)*(0.5*s+1));
>> gz=c2d(Gp,0.1,'zoh');
```

You would get

$$G_z(z) = \frac{0.009z + 0.0008}{z^2 - 1.724z + 0.741}$$

The bi-linear transformation

$$z = \frac{1 + wT/2}{1 - wT/2} = \frac{(1 + 0.05w)}{(1 - 0.05w)}$$

will transfer $G_z(z)$ into w -plane. Use the below commands

```
>> aug=[0.1,1];
>> gwss = bilin(ss(gz),-1,'S_Tust',aug)
>> gw=tf(gwss)
```

to find out the transfer function in w -plane, as

$$\begin{aligned} G_w(w) &= \frac{1.992 - 0.09461w - 0.00023w^2}{w^2 + 2.993w + 1.992} \\ &\cong \frac{-0.00025(w - 20)(w + 400)}{(w + 1)(w + 2)} \end{aligned}$$

The Bode plot of the uncompensated system is shown in Figure 3.

We need to design a phase lag compensator so that PM of the compensated system is at least 50° and steady state error to a unit step input is ≤ 0.1 . The compensator in w -plane is

$$C(w) = K\alpha \frac{1 + \tau w}{1 + \alpha \tau w} \quad \alpha > 1$$

where,

$$C(0) = K\alpha$$

Since $G_w(0) = 1$, $K\alpha = 9$ for 0.1 steady state error.

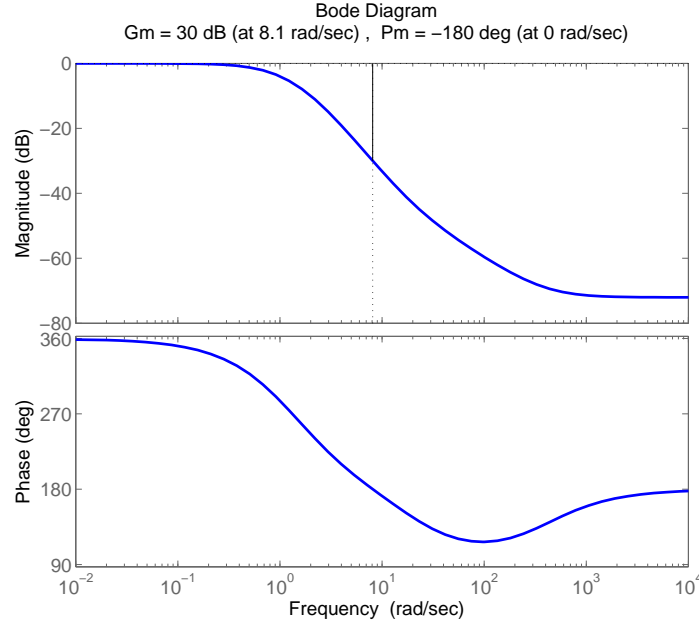


Figure 3: Bode plot of the uncompensated system for Example 2

Now let us modify the system transfer function by introducing K to the original system. Thus the modified system becomes

$$G_m(w) = \frac{-0.00025K(w - 20)(w + 400)}{(w + 1)(w + 2)}$$

PM of the closed loop system should be 50° . Let the gain crossover frequency of the uncompensated system with K be ω_g . Then,

$$\begin{aligned} \text{Mag.}(G_m) &= \frac{0.00025K \sqrt{400 + \omega^2} \sqrt{160000 + \omega^2}}{\sqrt{1 + \omega^2} \sqrt{4 + \omega^2}} \\ \text{Phase}(G_m) &= -\tan^{-1} \omega - \tan^{-1} 0.5\omega - \tan^{-1} 0.05\omega + \tan^{-1} 0.0025\omega \end{aligned}$$

Required PM is 50° . Let us put a safety margin of 5° . Thus the PM of the system modified with K should be 55° .

$$\Rightarrow 180^\circ - \tan^{-1} \omega_g - \tan^{-1} 0.5\omega_g - \tan^{-1} 0.05\omega_g + \tan^{-1} 0.0025\omega_g = 55^\circ$$

$$\text{or, } \tan^{-1} \frac{\omega_g + 0.5\omega_g}{1 - 0.5\omega_g^2} + \tan^{-1} \frac{0.05\omega_g - 0.0025\omega_g}{1 + 0.000125\omega_g^2} = 125^\circ$$

By solving the above, $\omega_g = 2.44$ rad/sec. Thus the magnitude at ω_g should be 1.

$$\Rightarrow \frac{0.00025K \sqrt{400 + \omega_g^2} \sqrt{160000 + \omega_g^2}}{\sqrt{1 + \omega_g^2} \sqrt{4 + \omega_g^2}} = 1$$

Putting the value of ω_g in the last equation, we get $K = 4.13$. Thus,

$$\alpha = \frac{9}{K} = 2.18$$

If we place $1/\tau$ one decade below the gain crossover frequency, then

$$\frac{1}{\tau} = \frac{2.44}{10}, \quad \text{or, } \tau = 4.1$$

Thus the controller in w -plane is

$$C(w) = 9 \frac{1 + 4.1w}{1 + 8.9w}$$

Re-transforming the above controller into z -plane using the relation $w = 20 \frac{z-1}{z+1}$, we get

$$\begin{aligned} C_z(z) &= 9 \frac{1 + 20 \times 4.1 \times \frac{z-1}{z+1}}{1 + 20 \times 8.9 \times \frac{z-1}{z+1}} \\ &= 9 \frac{83z - 81}{179z - 177} \end{aligned}$$

The Bode plot of the uncompensated system is shown in Figure 3.

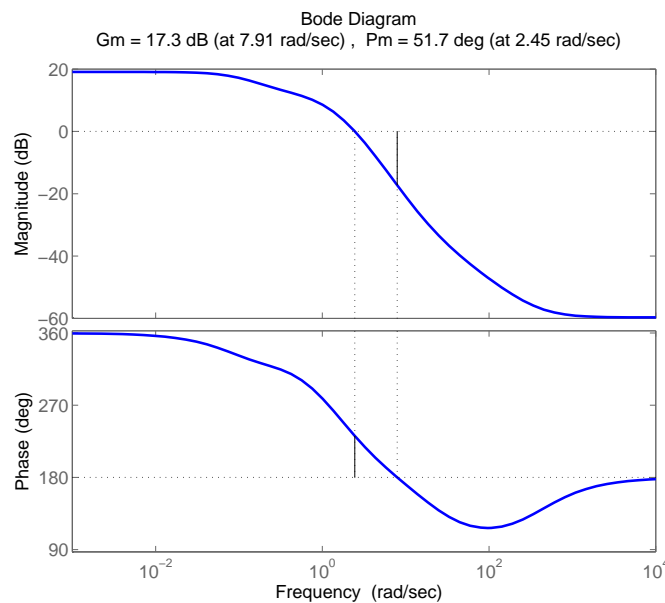


Figure 4: Bode plot of the compensated system for Example 2

In the next lecture, we would discuss lag-lead and PID controllers and conclude the topic of compensator design.