Module 1: Introduction to Digital Control
Lecture Note 4

1 Data Reconstruction

Most of the control systems have analog controlled processes which are inherently driven by analog inputs. Thus the outputs of a digital controller should first be converted into analog signals before being applied to the systems. Another way to look at the problem is that the high frequency components of $f(t)$ should be removed before applying to analog devices. A low pass filter or a data reconstruction device is necessary to perform this operation.

In control system, hold operation becomes the most popular way of reconstruction due to its simplicity and low cost. Problem of data reconstruction can be formulated as: **" Given a sequence of numbers,** $f(0), f(T), f(2T), \cdots, f(kt), \cdots$**, a continuous time signal** $f(t),\ t \geq 0$**, is to be reconstructed from the information contained in the sequence."**

Data reconstruction process may be regarded as an extrapolation process since the continuous data signal has to be formed based on the information available at past sampling instants. Suppose the original signal $f(t)$ between two consecutive sampling instants kT and $(k+1)T$ is to be estimated based on the values of $f(t)$ at previous instants of kT, i.e., $(k-1)T$, $(k-2)T$, $\cdots 0$.

Power series expansion is a well known method of generating the desired approximation which yields

$$\begin{aligned}
f_k(t) &= f(kT) + f^{(1)}(kT)(t-kT) + \frac{f^{(2)}(kT)}{2!}(t-kT)^2 + ...\\
\text{where, } f_k(t) &= f(t) \ \text{ for } \ kT \leq t \leq (k+1)T \ \text{ and}\\
f^{(n)}(kT) &= \frac{d^n f(t)}{dt^n}\Big|_{t=kT} \quad \text{for } \ n = 1, 2, ...
\end{aligned}$$

Since the only available information about $f(t)$ is its magnitude at the sampling instants, the derivatives of $f(t)$ must be estimated from the values of $f(kT)$, as

$$\begin{aligned}
f^{(1)}(kT) &\cong \frac{1}{T}[f(kT) - f((k-1)T)]\\
\text{Similarly,} \quad f^{(2)}(kT) &\cong \frac{1}{T}[f^{(1)}(kT) - f^{(1)}((k-1)T)]\\
\text{where,} \quad f^{(1)}((k-1)T) &\cong \frac{1}{T}[f((k-1)T) - f((k-2)T)]
\end{aligned}$$

1.1 Zero Order Hold

Higher the order of the derivatives to be estimated is, larger will be the number of delayed pulses required. Since time delay degrades the stability of a closed loop control system, using higher order derivatives of $f(t)$ for more accurate reconstruction often causes serious stability problem. Moreover a high order extrapolation requires complex circuitry and results in high cost.

For the above reasons, use of only the first term in the power series to approximate $f(t)$ during the time interval $kT \le t < (k+1)T$ is very popular and the device for this type of extrapolation is known as zero-order extrapolator or zero order hold. It holds the value of $f(kT)$ for $kT \le t < (k+1)T$ until the next sample $f((k+1)T)$ arrives. Figure 1 illustrates the operation of a ZOH where the green line represents the original continuous signal and brown line represents the reconstructed signal from ZOH.

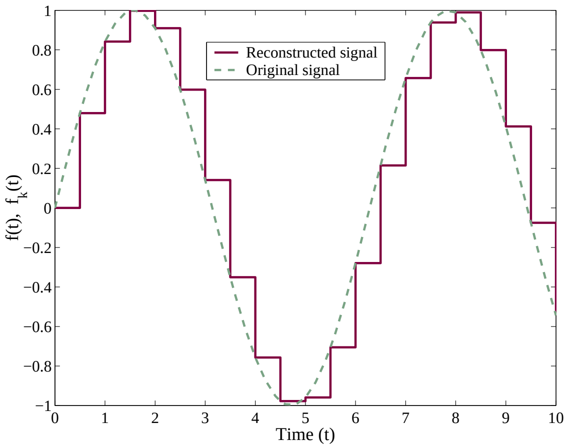

Figure 1: Zero order hold operation

The accuracy of zero order hold (ZOH) depends on the sampling frequency. When $T \to 0$, the output of ZOH approaches the continuous time signal. Zero order hold is again a linear device which satisfies the principle of superposition.

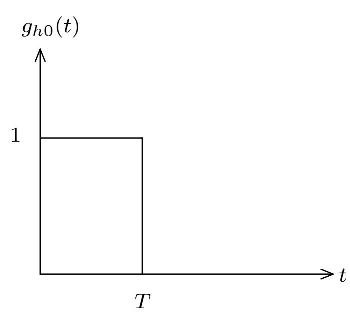

Figure 2: Impulse response of ZOH

The **impulse response** of a ZOH, as shown in Figure 2, can be written as

$$\begin{aligned} g_{ho}(t) &= u_s(t) - u_s(t-T) \\ \Rightarrow G_{ho}(s) &= \frac{1-e^{-TS}}{s} \\ G_{ho}(jw) &= \frac{1-e^{-jwT}}{jw} = T\,\frac{\sin(wT/2)}{wT/2}e^{-jwT/2} \end{aligned}$$

Since $T = \dfrac{2\pi}{w_s}$, we can write

$$G_{ho}(jw) = \frac{2\pi}{w_s}\frac{\sin(\pi w/w_s)}{\pi w/w_s}\, e^{-j\pi w/w_s}$$

Magnitude of $G_{ho}(jw)$:

$$|G_{ho}(jw)| = \frac{2\pi}{w_s}\left|\frac{\sin(\pi w/w_s)}{\pi w/w_s}\right|$$

Phase of $G_{ho}(jw)$:

$$\angle G_{h0}(jw) = \angle \sin(\pi w/w_s) - \frac{\pi w}{w_s} \quad \text{rad}$$

The sign of $\angle \sin(\pi w/w_s)$ changes at every integral value of $\dfrac{\pi w}{w_s}$. The change of sign from + to − can be regarded as a phase change of -180^0. Thus the phase characteristics of ZOH is linear with jump discontinuities of -180^0 at integral multiple of w_s. The magnitude and phase characteristics of ZOH are shown in Figure 3.

At the cut off frequency $w_c = \dfrac{w_s}{2}$, magnitude is 0.636. When compared with an ideal low pass filter, we see that instead of cutting of sharply at $w = \frac{w_s}{2}$, the amplitude characteristics of $G_{ho}(jw)$ is zero at $\dfrac{w_s}{2}$ and integral multiples of w_s.

1.2 First Order Hold

When the 1^{st} two terms of the power series are used to extrapolate $f(t)$, over the time interval $kT < t < (k+1)T$, the device is called a first order hold (FOH). Thus

$$\begin{aligned} f_k(t) &= f(kT) + f^1(kT)(t-kT) \\ \text{where, }\ f^1(kT) &= \frac{f(kT) - f((k-1)T)}{T} \\ \Rightarrow f_k(t) &= f(kT) + \frac{f(kT) - f((k-1)T)}{T}(t-kT) \end{aligned}$$

Impulse response of FOH is obtained by applying a unit impulse at $t = 0$, the corresponding output is obtained by setting $k = 0, 1, 2, ...$

$$\text{for } k = 0,\ \text{when } 0 \le t < T, \quad f_0(t) = f(0) + \frac{f(0) - f(-T)}{T}t$$

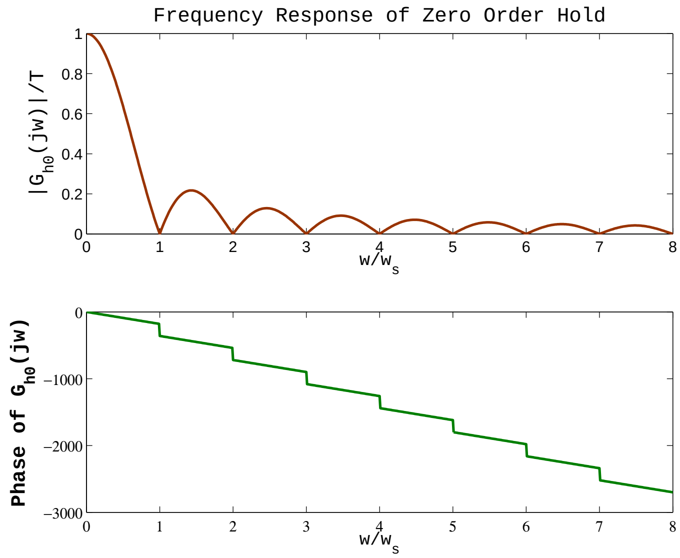

Figure 3: Frequency response of ZOH

$f(0) = 1$ [impulse unit] $f(-T) = 0$ $f_{h1}(t) = 1 + \dfrac{t}{T}$ in this region. When $T \le t < 2T$

$$f_1(t) = f(T) + \frac{f(T) - f(0)}{T}(t - T)$$

Since, $f(T) = 0$ and $f(0) = 1$, $f_{h1}(t) = 1 - \dfrac{t}{T}$ in this region. $f_{h1}(t)$ is 0 for $t \ge 2T$, since $f(t) = 0$ for $t \ge 2T$. Figure 4 shows the impulse response of first order hold.

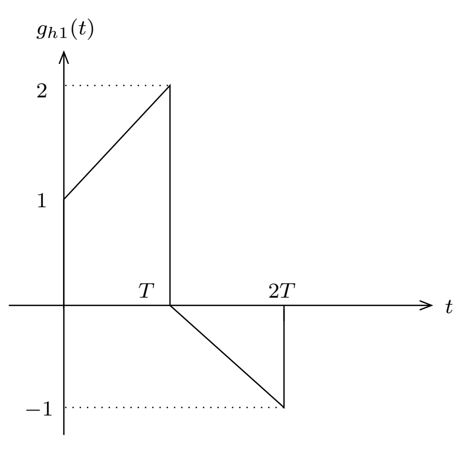

Figure 4: Impulse response of First Order Hold

If we combine all three regions, we can write the impulse response of a first order hold as,

$$\begin{aligned} g_{h1}(t) &= (1 + \frac{t}{T})u_s(t) + (1 - \frac{t}{T})u_s(t - T) - (1 + \frac{t}{T})u_s(t - T) - (1 - \frac{t}{T})u_s(t - 2T) \\ &= (1 + \frac{t}{T})u_s(t) - 2\frac{t}{T}u_s(t - T) - (1 - \frac{t}{T})u_s(t - 2T) \end{aligned}$$

One can verify that according to the above expression, when $0 \leq t < T$, only the first term produces a nonzero value which is nothing but $(1 + t/T)$. Similarly, when $T \leq t < 2T$, first two terms produce nonzero values and the resultant is $(1 - t/T)$. In case of $t \geq 2T$, all three terms produce nonzero values and the resultant is 0.

The transfer function of a first order hold is:

$$G_{h1}(s) = \frac{1+Ts}{T}\left[\frac{1-e^{-Ts}}{s}\right]^2$$

Frequency Response $\quad G_{h1}(jw) = \dfrac{1+jwT}{T}\left[\dfrac{1-e^{-jwT}}{s}\right]^2$

$$\begin{aligned}
\text{Magnitude: } |G_{h1}(jw)| &= \left|\frac{1+jwT}{T}\right| |G_{h0}(jw)|^2 \\
&= \frac{2\pi}{w_s}\sqrt{1+\frac{4\pi^2 w^2}{w_s^2}}\left|\frac{\sin(\pi w/w_s)}{\pi w/w_s}\right|^2 \\
\text{Phase: } \angle G_{h1}(jw) &= \tan^{-1}(2\pi w/w_s) - 2\pi w/w_s \quad \text{rad}
\end{aligned}$$

The frequency response is shown in Figure 5.

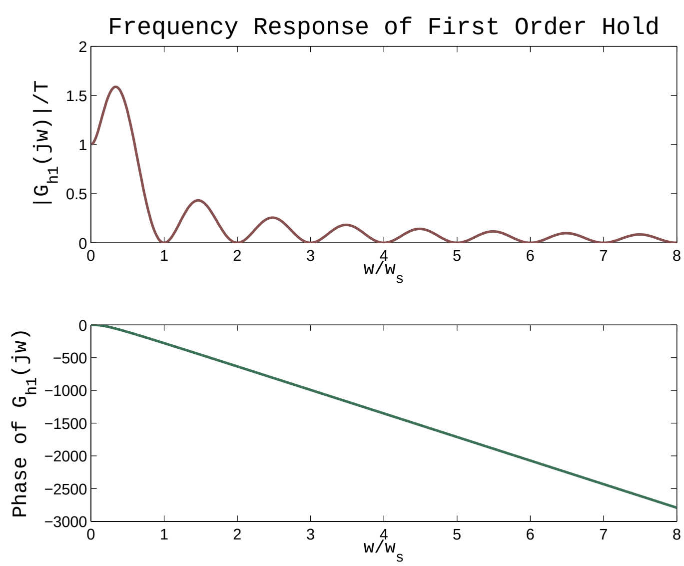

Figure 5: Frequency response of FOH

Figure 6 shows a comparison of the reconstructed outputs of ZOH and FOH.

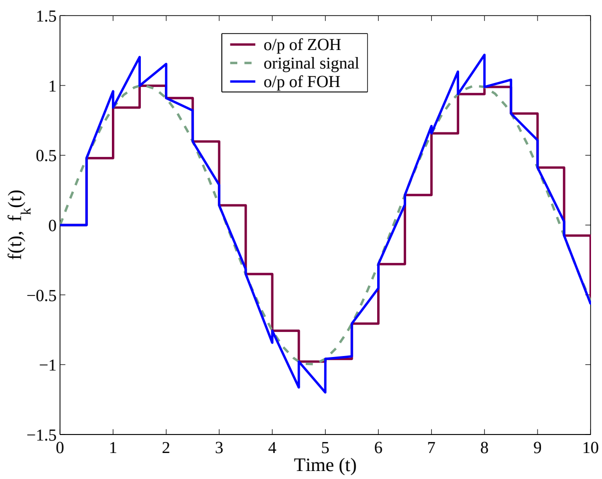

Figure 6: Operation of ZOH and FOH