Module 9: State Feedback Control Design
Lecture Note 3

1 State Estimators or Observers

- One should note that although state feedback control is very attractive because of precise computation of the gain matrix K, implementation of a state feedback controller is possible only when all state variables are directly measurable with help of some kind of sensors.
- Due to the excess number of required sensors or unavailability of states for measurement, in most of the practical situations this requirement is not met.
- Only a subset of state variables or their combinations may be available for measurements. Sometimes only output y is available for measurement.
- Hence the need for an estimator or observer is obvious which estimates all state variables while observing input and output.

Full Order Observer: If the state observer estimates all the state variables, regardless of whether some are available for direct measurements or not, it is called a full order observer.

Reduced Order Observer: An observer that estimates fewer than "n" states of the system is called reduced order observer.

Minimum Order Observer: If the order of the observer is minimum possible then it is called minimum order observer.

2 Full Order Observers

Consider the following system

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k)\\ \mathbf{y}(k) &= C\mathbf{x}(k)\end{aligned}$$

where $\mathbf{x} \in R^{n\times 1}$, $\mathbf{u} \in R^{m\times 1}$ and $\mathbf{y} \in R^{p\times 1}$.

Assumption: The pair $(A,\ C)$ is observable.

Goal: To construct a dynamic system that will estimate the state vector based on the information of the plant input $\mathbf{u}$ and output $\mathbf{y}$.

2.1 Open Loop Estimator

The schematic of an open loop estimator is shown in Figure 1.

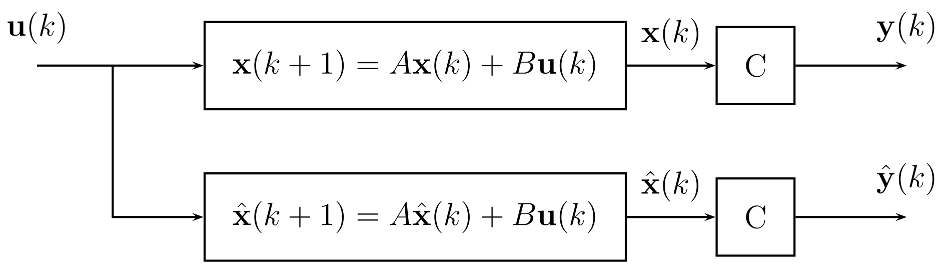

Figure 1: Open Loop Observer

The dynamics of this estimator are described by the following

$$\begin{aligned} \hat{\mathbf{x}}(k+1) &= A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) \\ \hat{\mathbf{y}}(k) &= C\hat{\mathbf{x}}(k) \end{aligned}$$

where $\hat{\mathbf{x}}$ is the estimate of $\mathbf{x}$ and $\hat{\mathbf{y}}$ is the estimate of $\mathbf{y}$.

Let $\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}$ be the estimation error. Then the error dynamics are defined by

$$\begin{aligned} \tilde{\mathbf{x}}(k+1) &= \hat{\mathbf{x}}(k+1) - \mathbf{x}(k+1) \\ &= A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) - A\mathbf{x}(k) - B\mathbf{u}(k) \\ &= A\tilde{\mathbf{x}}(k) \end{aligned}$$

with the initial estimation error as

$$\tilde{\mathbf{x}}(0) = \hat{\mathbf{x}}(0) - \mathbf{x}(0)$$

If the eigenvalues of A are inside the unit circle then $\tilde{\mathbf{x}}$ will converge to 0. But we have no control over the convergence rate.

Moreover, A may have eigenvalues outside the unit circle. In that case $\tilde{\mathbf{x}}$ will diverge from 0. Thus the open loop estimator is impractical.

2.2 Luenberger State Observer

Consider the system $\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k)$. Luenberger observer is shown in Figure 2. The observer dynamics can be expressed as:

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) + L(\mathbf{y}(k) - \hat{\mathbf{y}}(k)) \tag{1}$$

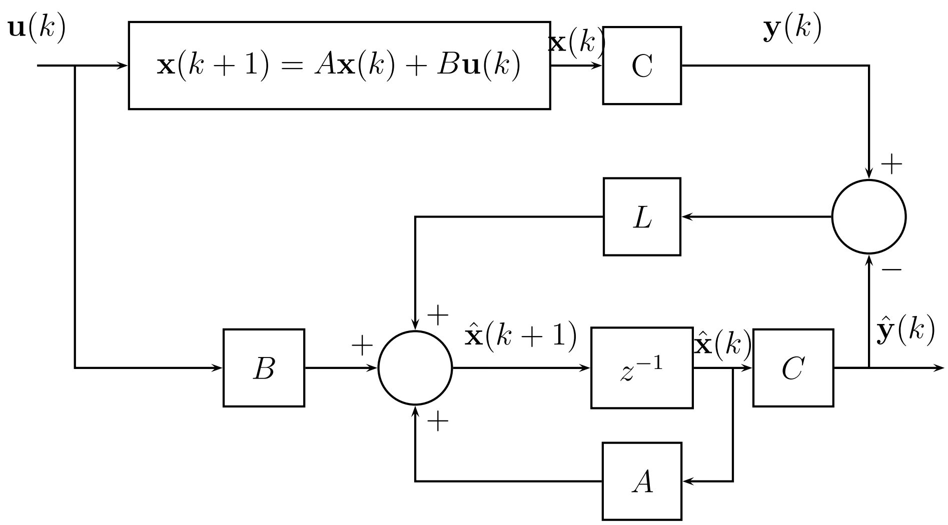

Figure 2: Luenberger observer

The closed loop error dynamics can be derived as:

$$\begin{aligned} \tilde{\mathbf{x}}(k) &= \hat{\mathbf{x}}(k) - \mathbf{x}(k) \\ \hat{\mathbf{x}}(k+1) - \mathbf{x}(k+1) &= A(\hat{\mathbf{x}}(k) - \mathbf{x}(k)) + LC(\mathbf{x}(k) - \hat{\mathbf{x}}(k)) \\ \tilde{\mathbf{x}}(k+1) &= (A - LC)\tilde{\mathbf{x}}(k) \end{aligned}$$

It can be seen that $\tilde{\mathbf{x}} \to 0$ if L can be designed such that $(A - LC)$ has eigenvalues inside the unit circle of z-plane.

The convergence rate can also be controlled by properly choosing the closed loop eigenvalues.

Computation of Observer gain matrix L

The task is to place the poles of $|A - LC|$. Necessary and sufficient condition for arbitrary pole placement is that the pair should be controllable.

Assumption: The pair $(A,\ C)$ is observable. Thus, from the theorem of duality, the pair $(A^T,\ C^T)$ is controllable.

You should note that the eigenvalues of $A^T - C^T L^T$ are same as that of $A - LC$. It is same as a hypothetical pole placement problem for the system $\bar{\mathbf{x}}(k+1) = A^T\bar{\mathbf{x}}(k) + C^T\bar{\mathbf{u}}(k)$, using a control law $\bar{u}(k) = -L^T\bar{\mathbf{x}}(k)$.

Example:

$$\begin{aligned}\mathbf{x}(k+1) &= \begin{bmatrix}0 & 1\\ 20 & 0\end{bmatrix}\mathbf{x}(k) + \begin{bmatrix}0\\1\end{bmatrix}u(k)\\ y(k) &= [1\ \ 0]\mathbf{x}(k)\end{aligned}$$

The observability matrix

$$U_O = \begin{bmatrix}C\\ CA\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

is non singular. Thus the pair $(A,\ C)$ is observable. The observer dynamics are

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + Bu(k) + LC(\mathbf{x}(k) - \hat{\mathbf{x}}(k))$$

L should be designed such that the observer poles are at 0.2 and 0.3.

We design L^T such that $A^T - C^T L^T$ has eigenvalues at 0.2 and 0.3.

$$A^T = \begin{bmatrix}0 & 20\\ 1 & 0\end{bmatrix}, \quad C^T = \begin{bmatrix}1\\0\end{bmatrix}$$

Using Ackermann's formula, $L^T = [-0.5\ \ \ 20.06]$. Thus

$$L = \begin{bmatrix}-0.5\\ 20.06\end{bmatrix}$$

2.3 Controller with Observer

The observer dynamics:

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) + LC(\mathbf{x}(k) - \hat{\mathbf{x}}(k))$$

Combining with the system dynamics

$$\begin{aligned}\begin{bmatrix}\mathbf{x}(k+1)\\ \hat{\mathbf{x}}(k+1)\end{bmatrix} &= \begin{bmatrix}A & 0\\ LC & A-LC\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \hat{\mathbf{x}}(k)\end{bmatrix} + \begin{bmatrix}B\\ B\end{bmatrix}\mathbf{u}(k)\\ \mathbf{y}(k) &= [C\ \ 0]\begin{bmatrix}\mathbf{x}(k)\\ \hat{\mathbf{x}}(k)\end{bmatrix}\end{aligned}$$

Since the states are unavailable for measurements, the control input is

$$\mathbf{u}(k) = -K\hat{\mathbf{x}}(k)$$

Putting the control law in the augmented equation

$$\begin{aligned}
\begin{bmatrix}\mathbf{x}(k+1)\\ \hat{\mathbf{x}}(k+1)\end{bmatrix} &= \begin{bmatrix}A & -BK\\ LC & A-LC-BK\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \hat{\mathbf{x}}(k)\end{bmatrix}\\
\mathbf{y}(k) &= [C\ \ 0]\begin{bmatrix}\mathbf{x}(k)\\ \hat{\mathbf{x}}(k)\end{bmatrix}
\end{aligned}$$

The error dynamics is

$$\tilde{\mathbf{x}}(k+1) = (A-LC)\tilde{\mathbf{x}}(k)$$

If we augment the above with the system dynamics, we get

$$\begin{aligned}
\begin{bmatrix}\mathbf{x}(k+1)\\ \tilde{\mathbf{x}}(k+1)\end{bmatrix} &= \begin{bmatrix}A-BK & -BK\\ 0 & A-LC\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \tilde{\mathbf{x}}(k)\end{bmatrix}\\
\mathbf{y}(k) &= [C\ \ 0]\begin{bmatrix}\mathbf{x}(k)\\ \tilde{\mathbf{x}}(k)\end{bmatrix}
\end{aligned}$$

where the dimension of the augmented system matrix is $R^{2n\times 2n}$. Looking at the matrix one can easily understand that $2n$ eigenvalues of the augmented matrix are equal to the individual eigenvalues of $A-BK$ and $A-LC$.

Conclusion: We can reach to a conclusion from the above fact is the design of control law, i.e., $A-BK$ is separated from the design of the observer, i.e., $A-LC$.

The above conclusion is commonly referred to as **separation principle**.

The block diagram of controller with observer is shown in Figure 3.

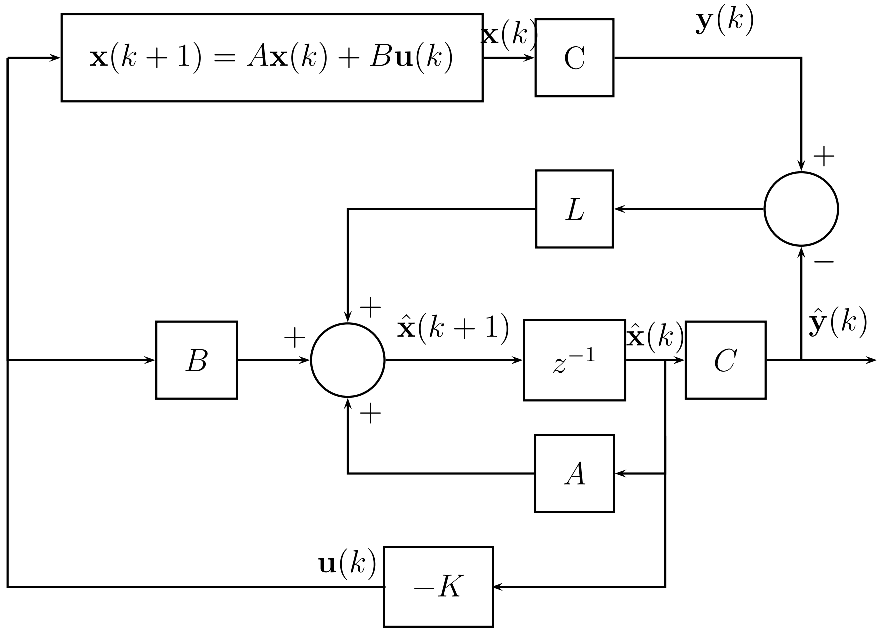

Figure 3: Controller with observer