Module 7: Discrete State Space Models
Lecture Note 3

1 Characteristic Equation, eigenvalues and eigen vectors

For a discrete state space model, the characteristic equation is defined as

$$|zI - A| = 0$$

The roots of the characteristic equation are the eigenvalues of matrix A.

1. If $det(A) \neq 0$, i.e., A is nonsingular and λ_1, λ_2, $\cdots$, λ_n are the eigenvalues of A, then, $\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}$, $\cdots$, $\frac{1}{\lambda_n}$ will be the eigenvalues of A^{-1}.
2. Eigenvalues of A and A^T are same when A is a real matrix.
3. If A is a real symmetric matrix then all its eigenvalues are real.

The $n \times 1$ vector v_i which satisfies the matrix equation

$$Av_i = \lambda_i v_i \tag{1}$$

where $\lambda_i, \quad i = 1, 2, \cdots, n$ denotes the i^{th} eigenvalue, is called the eigen vector of A associated with the eigenvalue λ_i. If eigenvalues are distinct, they can be solved directly from equation (1).

Properties of eigen vectors

1. An eigen vector cannot be a null vector.
2. If v_i is an eigen vector of A then mv_i is also an eigen vector of A where m is a scalar.
3. If A has n distinct eigenvalues, then the n eigen vectors are linearly independent.

Eigen vectors of multiple order eigenvalues

When the matrix A an eigenvalue λ of multiplicity m, a full set of linearly independent may not exist. The number of linearly independent eigen vectors is equal to the degeneracy d of $\lambda I - A$. The degeneracy is defined as

$$d = n - r$$

where n is the dimension of A and r is the rank of $\lambda I - A$. Furthermore,

$$1 \leq d \leq m$$

2 Similarity Transformation and Diagonalization

Square matrices A and $\bar{A}$ are similar if

$$\begin{aligned} AP &= P\bar{A} \\ \text{or,}\quad \bar{A} &= P^{-1}AP \\ \text{and,}\quad A &= P\bar{A}P^{-1} \end{aligned}$$

The non-singular matrix P is called similarity transformation matrix. It should be noted that eigenvalues of a square matrix A are not altered by similarity transformation.

Diagonalization:

If the system matrix A of a state variable model is diagonal then the state dynamics are decoupled from each other and solving the state equations become much more simpler.

In general, if A has distinct eigenvalues, it can be diagonalized using similarity transformation. Consider a square matrix A which has distinct eigenvalues $\lambda_1,\ \lambda_2,\ \ldots\ \lambda_n$. It is required to find a transformation matrix P which will convert A into a diagonal form

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

through similarity transformation $AP = P\Lambda$. If $v_1,\ v_2,\ \ldots,\ v_n$ are the eigenvectors of matrix A corresponding to eigenvalues $\lambda_1,\ \lambda_2,\ \ldots\ \lambda_n$, then we know $Av_i = \lambda_i v_i$. This gives

$$A\left[v_1\ v_2\ \ldots\ v_n\right] = \left[v_1\ v_2\ \ldots\ v_n\right]\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

Thus $P = [v_1\ v_2\ \ldots\ v_n]$. Consider the following state model.

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + Bu(k)$$

If P transforms the state vector $\boldsymbol{x}(k)$ to $\boldsymbol{z}(k)$ through the relation

$$\boldsymbol{x}(k) = P\boldsymbol{z}(k), \text{or,}\quad \boldsymbol{z}(k) = P^{-1}\boldsymbol{x}(k)$$

then the modified state space model becomes

$$\boldsymbol{z}(k+1) = P^{-1}AP\boldsymbol{z}(k) + P^{-1}Bu(k)$$

where $P^{-1}AP = \Lambda$.

3 Computation of $\Phi(t)$

We have seen that to derive the state space model of a sampled data system, we need to know the continuous time state transition matrix $\Phi(t) = e^{At}$.

3.1 Using Inverse Laplace Transform

For the system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t)$, the state transition matrix e^{At} can be computed as,

$$e^{At} = \mathscr{L}^{-1}\big\{(sI - A)^{-1}\big\}$$

3.2 Using Similarity Transformation

If Λ is the diagonal representation of the matrix A, then $\Lambda = P^{-1}AP$. When a matrix is in diagonal form, computation of state transition matrix is straight forward:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

Given $e^{\Lambda t}$, we can show that

$$e^{At} = Pe^{\Lambda t}P^{-1}$$

$$\begin{aligned}
e^{At} &= I + At + \frac{1}{2!}A^2t^2 + \dots \\
\Rightarrow P^{-1}e^{At}P &= P^{-1}\left[I + At + \frac{1}{2!}A^2t^2 + \dots\right]P \\
&= I + P^{-1}APt + \frac{1}{2!}P^{-1}APP^{-1}APt^2 + \dots \\
&= I + \Lambda t + \frac{1}{2!}\Lambda^2t^2 + \dots \\
&= e^{\Lambda t} \\
\Rightarrow e^{At} &= Pe^{\Lambda t}P^{-1}
\end{aligned}$$

3.3 Using Caley Hamilton Theorem

Every square matrix A satisfies its own characteristic equation. If the characteristic equation is

$$\triangle(\lambda) = |\lambda I - A| = \lambda^n + \alpha_1\lambda^{n-1} + \dots + \alpha_n = 0$$

then,

$$\triangle(A) = A^n + \alpha_1A^{n-1} + \dots + \alpha_nI = 0$$

Application: Evaluation of any function $f(\lambda)$ and $f(A)$

$$f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_n\lambda^n + \dots \quad \text{order } \infty$$

$$\frac{f(\lambda)}{\triangle(\lambda)} = q(\lambda) + \frac{g(\lambda)}{\triangle(\lambda)}$$

$$\begin{aligned}
f(\lambda) &= q(\lambda)\triangle(\lambda) + g(\lambda) \\
&= g(\lambda) \\
&= \beta_0 + \beta_1\lambda + \dots + \beta_{n-1}\lambda^{n-1} \quad \text{order } n-1
\end{aligned}$$

If A has distinct eigenvalues $\lambda_1, \cdots, \lambda_n$, then,

$$f(\lambda_i) = g(\lambda_i), \quad i = 1, \cdots, n$$

The solution will give rise to $\beta_0, \beta_1, \cdots, \beta_{n-1}$, then

$$f(A) = \beta_0 I + \beta_1 A + \cdots + \beta_{n-1} A^{n-1}$$

If there are multiple roots (multiplicity = 2), then

$$f(\lambda_i) = g(\lambda_i) \tag{2}$$

$$\frac{\partial}{\partial \lambda_i} f(\lambda_i) = \frac{\partial}{\partial \lambda_i} g(\lambda_i) \tag{3}$$

Example 1:

$$\text{If } A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

then compute the state transition matrix using Caley Hamilton Theorem.

$$\triangle(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & 0 & 2 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda - 3 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2) = 0 \Rightarrow \lambda_1 = 1 \text{ (with multiplicity 2)}, \ \lambda_2 = 2$$

$$\text{Let } f(\lambda) = e^{\lambda t} \ \text{ and } \ g(\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2$$

Then using (2) and (3), we can write

$$\begin{aligned} f(\lambda_1) &= g(\lambda_1) \\ \frac{\partial}{\partial \lambda_1} f(\lambda_1) &= \frac{\partial}{\partial \lambda_1} g(\lambda_1) \\ f(\lambda_2) &= g(\lambda_2) \end{aligned}$$

This implies

$$\begin{aligned} e^t &= \beta_0 + \beta_1 + \beta_2 \quad (\lambda_1 = 1) \\ te^t &= \beta_1 + 2\beta_2 \quad (\lambda_1 = 1) \\ e^{2t} &= \beta_0 + 2\beta_1 + 4\beta_2 \quad (\lambda_2 = 2) \end{aligned}$$

Solving the above equations

$$\beta_0 = e^{2t} - 2te^t, \qquad \beta_1 = 3te^t + 2e^t - 2e^{2t}, \qquad \beta_2 = e^{2t} - e^t - te^t$$

Then

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A + \beta_2 A^2 \\ &= \begin{bmatrix} 2e^t - e^{2t} & 0 & 2e^t - 2e^{2t} \\ 0 & e^t & 0 \\ e^{2t} - e^t & 0 & 2e^{2t} - e^t \end{bmatrix} \end{aligned}$$

Example 2 For the system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t)$, where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. compute e^{At} using 3 different techniques.

Solution: Eigenvalues of matrix A are $1 \pm j1$.

Method 1

$$\begin{aligned}
e^{At} &= \mathscr{L}^{-1}(sI - A)^{-1} = \mathscr{L}^{-1} = \begin{bmatrix} s-1 & -1 \\ 1 & s-1 \end{bmatrix}^{-1} \\
&= \mathscr{L}^{-1} \frac{1}{s^2 - 2s + 2} \begin{bmatrix} s-1 & 1 \\ -1 & s-1 \end{bmatrix} \\
&= \mathscr{L}^{-1} \begin{bmatrix} \frac{s-1}{(s-1)^2+1} & \frac{1}{(s-1)^2+1} \\ \frac{-1}{(s-1)^2+1} & \frac{s-1}{(s-1)^2+1} \end{bmatrix} \\
&= \begin{bmatrix} e^t \cos\, t & e^t \sin\, t \\ -e^t \sin\, t & e^t \cos\, t \end{bmatrix}
\end{aligned}$$

Method 2

$e^{At} = Pe^{\Lambda t}P^{-1}$ where $e^{\Lambda t} = \begin{bmatrix} e^{(1+j)t} & 0 \\ 0 & e^{(1-j)t} \end{bmatrix}$. Eigen values are $1 \pm j$. The corresponding eigenvectors are found by using equation $Av_i = \lambda_i v_i$ as follows:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (1+j) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Taking $v_1 = 1$, we get $v_2 = j$. So, the eigenvector corresponding to $1 + j$ is $\begin{bmatrix} 1 \\ j \end{bmatrix}$ and the one corresponding to $1 - j$ is $\begin{bmatrix} 1 \\ -j \end{bmatrix}$. The transformation matrix is given by

$$P = [v_1 \;\; v_2] = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$$

Now,

$$\begin{aligned}
e^{At} &= Pe^{\Lambda t}P^{-1} \\
&= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{(1+j)t} & 0 \\ 0 & e^{(1-j)t} \end{bmatrix} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} e^{(1+j)t} & e^{(1-j)t} \\ je^{(1+j)t} & -je^{(1-j)t} \end{bmatrix} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} e^{(1+j)t} + e^{(1-j)t} & -j\left(e^{(1+j)t} - e^{(1-j)t}\right) \\ j\left(e^{(1+j)t} - e^{(1-j)t}\right) & e^{(1+j)t} + e^{(1-j)t} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 2e^t \cos\, t & -j(j)e^t 2 \sin\, t \\ e^t (j)(j) 2 \sin\, t & 2e^t \cos\, t \end{bmatrix} \\
&= \begin{bmatrix} e^t \cos\, t & e^t \sin\, t \\ -e^t \sin\, t & e^t \cos\, t \end{bmatrix}
\end{aligned}$$

Method 3: Caley Hamilton Theorem

The eigenvalues are $\lambda_{1,2} = 1 \pm j$.

$$\begin{aligned} e^{\lambda_1 t} &= \beta_0 + \beta_1 \lambda_1 \\ e^{\lambda_2 t} &= \beta_0 + \beta_1 \lambda_2 \end{aligned}$$

Solving,

$$\begin{aligned} \beta_0 &= \frac{1}{2}(1+j)e^{(1+j)t} + \frac{1}{2}(1-j)e^{(1-j)t} \\ \beta_1 &= \frac{1}{2j}\left(e^{(1+j)^t} - e^{(1-j)^t}\right) \end{aligned}$$

Hence,

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A \\ &= \begin{bmatrix} e^t \cos\ t & e^t \sin t \\ -e^t \sin t & e^t \cos\ t \end{bmatrix} \end{aligned}$$

We will now show through an example how to derive discrete state equation from a continuous one.

Example: Consider the following state model of a continuous time system.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= x_1(t) \end{aligned}$$

If the system is under a sampling process with period T, derive the discrete state model of the system.

To derive the discrete state space model, let us first compute the state transition matrix of the continuous time system using Caley Hamilton Theorem.

$$\triangle(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2$$

$$\text{Let } f(\lambda) = e^{\lambda t}$$

This implies

$$\begin{aligned} e^t &= \beta_0 + \beta_1 \quad (\lambda_1 = 1) \\ e^{2t} &= \beta_0 + 2\beta_1 \quad (\lambda_2 = 2) \end{aligned}$$

Solving the above equations

$$\beta_1 = e^{2t} - e^t \qquad \beta_0 = 2e^t - e^{2t}$$

Then

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A \\ &= \begin{bmatrix} e^t & e^{2t} - e^t \\ 0 & e^{2t} \end{bmatrix} \end{aligned}$$

Thus the discrete state matrix A is given as

$$A = \Phi(T) = \begin{bmatrix} e^T & e^{2T} - e^T \\ 0 & e^{2T} \end{bmatrix}$$

The discrete input matrix B can be computed as

$$\begin{aligned} B &= \Theta(T) = \int_0^T \Phi(T - t') \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt' \\ &= \int_0^T \begin{bmatrix} e^T.e^{-t'} & e^{2T}.e^{-2t'} - e^T.e^{-t'} \\ 0 & e^{2T}.e^{-2t'} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt' \\ &= \begin{bmatrix} e^T - 1 & 0.5e^{2T} - e^T + 0.5 \\ 0 & 0.5e^{2T} - 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.5e^{2T} - e^T + 0.5 \\ 0.5e^{2T} - 0.5 \end{bmatrix} \end{aligned}$$

The discrete state equation is thus described by

$$\begin{aligned} \mathbf{x}((k+1)T) &= \begin{bmatrix} e^T & e^{2T} - e^T \\ 0 & e^{2T} \end{bmatrix} \mathbf{x}(kT) + \begin{bmatrix} 0.5e^{2T} - e^T + 0.5 \\ 0.5e^{2T} - 0.5 \end{bmatrix} u(kT) \\ y(kT) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(kT) \end{aligned}$$

When $T = 1$, the state equations become

$$\begin{aligned} \mathbf{x}(k+1) &= \begin{bmatrix} 2.72 & 4.67 \\ 0 & 7.39 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1.48 \\ 3.19 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k) \end{aligned}$$