

Module 11: Introduction to Optimal Control

Lecture Note 2

1 Performance Indices

Whenever we use the term optimal to describe the effectiveness of a given control strategy, we do so with respect to some performance measure or index.

We generally assume that the value of the performance index decreases with the quality of the control law.

Constructing a performance index can be considered as a part of the system modeling. We would now discuss some typical performance indices which are popularly used.

Let us first consider the following system

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k), \quad \mathbf{x}(k_0) = \mathbf{x}_0 \\ \mathbf{y}(k) &= C\mathbf{x}(k)\end{aligned}$$

Suppose that the objective is to control the system such that over a fixed interval $[N_0, N_f]$, the components of the state vector are as small as possible. A suitable performance to be minimized is

$$J_1 = \sum_{k=N_0}^{N_f} \mathbf{x}^T(k)\mathbf{x}(k)$$

When J_1 is very small, $\|\mathbf{x}(k)\|$ is also very small.

If we want to minimize the output over a fixed interval $[N_0, N_f]$, a suitable performance would be

$$\begin{aligned}J_2 &= \sum_{k=N_0}^{N_f} \mathbf{y}^T(k)\mathbf{y}(k) \\ &= \sum_{k=N_0}^{N_f} \mathbf{x}^T(k)C^TC\mathbf{x}(k)\end{aligned}$$

If $C^T C = Q$, which is a symmetric matrix,

$$J_2 = \sum_{k=N_0}^{N_f} \mathbf{x}^T(k) Q \mathbf{x}(k)$$

When the objective is to control the system in such a way that the control input is not too large, the corresponding performance index is

$$J_3 = \sum_{k=N_0}^{N_f} \mathbf{u}^T(k) \mathbf{u}(k)$$

Or,

$$J_4 = \sum_{k=N_0}^{N_f} \mathbf{u}^T(k) R \mathbf{u}(k)$$

where the weight matrix R is symmetric positive definite.

We cannot simultaneously minimize the performance indices J_1 and J_3 because minimization of J_1 requires large control input whereas minimization of J_3 demands a small control. A compromise between the two conflicting objects is

$$\begin{aligned} J_5 &= \lambda J_1 + (1 - \lambda) J_3 \\ &= \sum_{k=N_0}^{N_f} [\lambda \mathbf{x}^T(k) \mathbf{x}(k) + (1 - \lambda) \mathbf{u}^T(k) \mathbf{u}(k)] \end{aligned}$$

A generalization of the above performance index is

$$J_6 = \sum_{k=N_0}^{N_f} [\mathbf{x}^T(k) Q \mathbf{x}(k) + \mathbf{u}^T(k) R \mathbf{u}(k)]$$

which is the most commonly used quadratic performance index.

In certain applications, we may wish the final state to be close to 0. Then a suitable performance index is

$$J_7 = \mathbf{x}^T(N_f) F \mathbf{x}(N_f)$$

When the control objective is to keep the state small, the control input not too large and the final state as close to 0 as possible, we can combine J_6 and J_7 , to get the most general performance index

$$J_8 = \frac{1}{2} \mathbf{x}^T(N_f) F \mathbf{x}(N_f) + \frac{1}{2} \sum_{k=N_0}^{N_f} [\mathbf{x}^T(k) Q \mathbf{x}(k) + \mathbf{u}^T(k) R \mathbf{u}(k)]$$

$1/2$ is introduced to simplify the manipulation.

Sometimes we want the system state to track a desired trajectory throughout the interval $[N_0, N_f]$. In that case the performance index J_8 can be modified as

$$J_9 = \frac{1}{2} \left[(\mathbf{x}_d(N_f) - \mathbf{x}(N_f))^T F (\mathbf{x}_d(N_f) - \mathbf{x}(N_f)) \right] \\ + \frac{1}{2} \sum_{k=N_0}^{N_f} [(\mathbf{x}_d(k) - \mathbf{x}(k))^T Q (\mathbf{x}_d(k) - \mathbf{x}(k)) + \mathbf{u}^T(k) R \mathbf{u}(k)]$$

For infinite time problem, the performance index is

$$J = \sum_{k=N_0}^{\infty} [\mathbf{x}^T(k) Q \mathbf{x}(k) + \mathbf{u}^T(k) R \mathbf{u}(k)]$$

In most cases, N_0 is considered to be 0.

Example: Consider the dynamical system

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \quad \mathbf{x}(0) = \mathbf{x}_0 \\ y(k) = [2 \quad 0] \mathbf{x}(k)$$

Suppose that we want to minimize the output as well as the input with equal weightage along the convergence trajectory. Construct the associated performance index.

Since the initial condition of the system is $\mathbf{x}(0) = \mathbf{x}_0$ and we have to minimize the performance index over the whole convergence trajectory, we need to take summation from 0 to ∞ .

Again, since the output and input are to be minimized with equal weightage, we can write the cost function or performance index as

$$J = \sum_{k=0}^{\infty} (y^2(k) + u^2(k)) \\ = \sum_{k=0}^{\infty} (\mathbf{x}^T(k) [2 \quad 0]^T [2 \quad 0] \mathbf{x}(k) + u^2(k)) \\ = \sum_{k=0}^{\infty} (\mathbf{x}^T(k) \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x}(k) + u^2(k))$$

Comparing with the standard cost function, we can say that here $Q = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = 1$.

In the next lecture we will discuss design of Linear Quadratic Regulator (LQR) by solving Algebraic Riccati Equation (ARE). To derive ARE, we need the following theorem.

Consider the system

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k)$$

where $\mathbf{x}(k) \in R^n$, $\mathbf{u}(k) \in R^m$ and $\mathbf{x}(0) = \mathbf{x}_0$.

Theorem 1: If the state feedback controller $\mathbf{u}^*(k) = -K\mathbf{x}(k)$ is such that

$$\min_{\mathbf{u}} (\Delta V(\mathbf{x}(k)) + \mathbf{x}^T(k)Q\mathbf{x}(k) + \mathbf{u}^T(k)R\mathbf{u}(k)) = 0 \quad (1)$$

for some Lyapunov function $V(k) = \mathbf{x}^T(k)P\mathbf{x}(k)$, then $\mathbf{u}^*(k)$ is optimal. Here the cost function is

$$J(u) = \sum_{k=0}^{\infty} (\mathbf{x}^T(k)Q\mathbf{x}(k) + \mathbf{u}^T(k)R\mathbf{u}(k))$$

and we assume that the closed loop system is asymptotically stable.

Proof: Equation (1) can also be represented as

$$\Delta V(\mathbf{x}(k))|_{\mathbf{u}=\mathbf{u}^*} + \mathbf{x}^T(k)Q\mathbf{x}(k) + \mathbf{u}^{*T}(k)R\mathbf{u}^*(k)$$

Hence, we can write

$$\Delta V(\mathbf{x}(k))|_{\mathbf{u}=\mathbf{u}^*} = (V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)))|_{\mathbf{u}=\mathbf{u}^*} = -\mathbf{x}^T(k)Q\mathbf{x}(k) - \mathbf{u}^{*T}(k)R\mathbf{u}^*(k)$$

We can sum both sides of the above equation from 0 to ∞ and get

$$V(\mathbf{x}(\infty)) - V(\mathbf{x}(0)) = - \sum_{k=0}^{\infty} (\mathbf{x}^T(k)Q\mathbf{x}(k) + \mathbf{u}^{*T}(k)R\mathbf{u}^*(k))$$

Since the closed loop system is stable by assumption, $\mathbf{x}(\infty) = 0$ and hence $V(\mathbf{x}(\infty)) = 0$. Thus

$$V(\mathbf{x}(0)) = \sum_{k=0}^{\infty} (\mathbf{x}^T(k)Q\mathbf{x}(k) + \mathbf{u}^{*T}(k)R\mathbf{u}^*(k))$$

Now, $V(\mathbf{x}(0)) = \mathbf{x}_0^T P \mathbf{x}_0$.

Thus if a linear state feedback controller satisfies the hypothesis of the theorem the value of the resulting cost function is

$$J(\mathbf{u}^*) = \mathbf{x}_0^T P \mathbf{x}_0$$

To show that such a controller is indeed optimal, we will use a proof by contradiction.

Assume that the hypothesis of the theorem holds true but the controller is not optimal. Thus there exists a controller $\bar{\mathbf{u}}$ such that

$$J(\bar{\mathbf{u}}) < J(\mathbf{u}^*)$$

Using the theorem, we can write

$$\Delta V(\mathbf{x}(k))|_{\mathbf{u}=\bar{\mathbf{u}}} + \mathbf{x}^T(k)Q\mathbf{x}(k) + \bar{\mathbf{u}}^T(k)R\bar{\mathbf{u}}(k) \geq 0$$

The above can be rewritten as

$$\Delta V(\mathbf{x}(k))|_{\mathbf{u}=\bar{\mathbf{u}}} \geq -\mathbf{x}^T(k)Q\mathbf{x}(k) - \bar{\mathbf{u}}^T(k)R\bar{\mathbf{u}}(k)$$

Summing the above from 0 to ∞ ,

$$V(\mathbf{x}(0)) \leq \sum_{k=0}^{\infty} (\mathbf{x}^T(k)Q\mathbf{x}(k) + \bar{\mathbf{u}}^T(k)R\bar{\mathbf{u}}(k))$$

The above inequality implies that

$$J(\mathbf{u}^*) \leq J(\bar{\mathbf{u}})$$

which is a contradiction of our earlier assumption. Thus \mathbf{u}^* is optimal.

For more details one may consult **Systems and Control** by Stanislaw H. Żak