

Module 5: Design of Sampled Data Control Systems

Lecture Note 4

1 Frequency Domain Analysis

When a sinusoidal input is given to a stable LTI system it produces a sinusoidal output of same frequency but with different magnitude and phase.

The variation of output magnitude and phase with input frequency is known as frequency response of the system.

Frequency domain analysis provides a good design in presence of uncertainty in plant model. Experimental results can be used to construct frequency response even if the plant model is unknown.

Analysis of digital control systems in frequency domain depends on the extension of the existing techniques in continuous time case.

Two most popular graphical representations in frequency domain are **Nyquist plot** and **Bode diagram**.

1.1 Nyquist plot

The Nyquist plot of a transfer function, usually the loop transfer function $GH(z)$, is a mapping of Nyquist contour in z -plane onto $GH(z)$ plane which is in polar coordinates. Thus it is sometimes known as polar plot.

Absolute and relative stabilities can be determined from the Nyquist plot using Nyquist stability criterion.

Given the loop transfer function $GH(z)$ of a digital control system, the polar plot of $GH(z)$ is obtained by setting $z = e^{j\omega T}$ and varying ω from 0 to ∞ .

Nyquist stability criterion: The closed loop transfer function of single input single output

digital control system is described by

$$M(z) = \frac{G(z)}{1 + GH(z)}$$

Characteristic equation $1 + GH(z) = 0$

The stability of the system depends on the roots of the characteristic equation or poles of the system. All the roots of the characteristic equation must lie inside the unit circle for the system to be stable.

Before discussing Nyquist stability criterion for the digital system, following steps are necessary.

1. Defining the Nyquist path in the z -plane that encloses the exterior of the unit circle. Here the region to the left of a closed path is considered to be enclosed by that path when the direction of the path is taken anticlockwise.
2. Mapping the Nyquist path in z -plane onto the $GH(z)$ plane which results in Nyquist plot of $GH(z)$.
3. Stability of the closed loop system is investigated by studying the behavior of Nyquist plot with respect to the critical point $(-1, j0)$ in the $GH(z)$ plane.

Two Nyquist paths are defined. The Nyquist path **z1** as shown in Figure 1 does not enclose poles on the unit circle whereas the Nyquist path **z2** as shown in Figure 2 encloses poles on the unit circle.

These figures are the mapping of the Nyquist contours in s -plane where the entire right half of the s -plane, without or with the imaginary axis poles, is enclosed by the contours.

Let us now define the following parameters.

Z_{-1} = number of zeros of $1 + GH(z)$ outside the unit circle in the z -plane.

P_{-1} = number of poles of $1 + GH(z)$ outside the unit circle in the z -plane.

P_0 = number of poles of $GH(z)$ (same as number of poles of $1 + GH(z)$) that are on the unit circle.

N_1 = number of times the $(-1, j0)$ point is encircled by the Nyquist plot of $GH(z)$ corresponding to **z1**.

N_2 = number of times $(-1, j0)$ point is encircled by Nyquist plot of $GH(z)$ corresponding to **z2**.

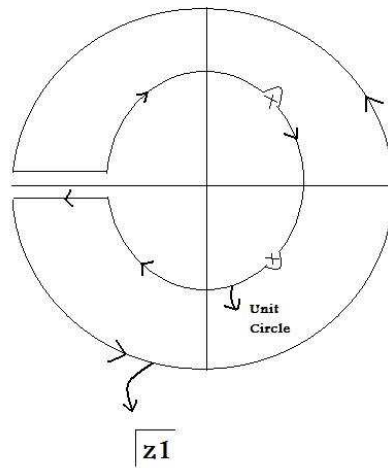


Figure 1: Nyquist path that does not enclose poles on the unit circle

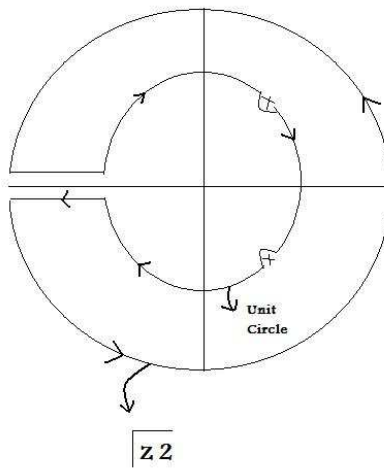


Figure 2: Nyquist path that encloses poles on the unit circle

According to the principle of argument in complex variable theory

$$N_1 = Z_{-1} - P_{-1}$$

$$N_2 = Z_{-1} - P_{-1} - P_0$$

Now let us denote the angle traversed by the phasor drawn from $(-1, j0)$ point to the Nyquist plot of $GH(z)$ as ω varies from $\frac{\omega_s}{2}$ to 0, on the unit circle of **z1** excluding the small indentations, by ϕ .

It can be shown that

$$\phi = (Z_{-1} - P_{-1} - 0.5P_0)180^\circ \quad (1)$$

For the closed loop digital control system to be stable, Z_{-1} should be equal to zero. Thus the Nyquist criterion for stability of the closed loop digital control systems is

$$\phi = -(P_{-1} + 0.5P_0)180^\circ \quad (2)$$

Hence, we can conclude that ***for the closed loop digital control system to be stable, the angle, traversed by the phasor drawn to the $GH(z)$ plot from $(-1, j0)$ point as ω varies from $\frac{\omega_s}{2}$ to 0, must satisfy equation (2).***

Example 1: Consider a digital control system for which the loop transfer function is given as

$$GH(z) = \frac{0.095Kz}{(z-1)(z-0.9)}$$

where K is a gain parameter. The sampling time $T = 0.1$ sec.

Since $GH(z)$ has one pole on the unit circle and does not have any pole outside the unit circle,

$$P_{-1} = 0 \quad \text{and} \quad P_0 = 1$$

Nyquist path has a small indentation at $z = 1$ on the unit circle.

Nyquist plot of $GH(z)$, as shown in Figure 3, intersects the negative real axis at $-0.025K$ when $\omega = \frac{\omega_s}{2} = 31.4$ rad/sec.

ϕ can be computed as

$$\phi = -(0 + 0.5 \times 1)180^\circ = -90^\circ$$

It can be seen from Figure 3 that for ϕ to be -90° , $(-1, j0)$ point should be located at the left of $-0.025K$ point. Thus for stability

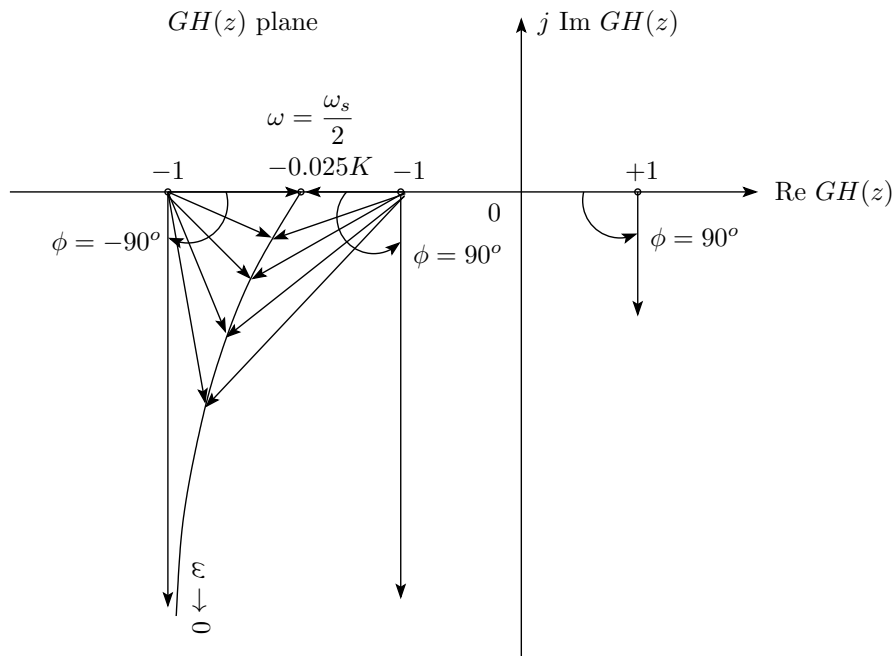


Figure 3: Nyquist plot for Example 1

$$-1 < -0.025K$$

$$\Rightarrow K < 40$$

If $K > 40$, $(-1, j0)$ will be at the right of $-0.025K$ point, hence making $\phi = 90^\circ$.

If $\phi = 90^\circ$, we get from (1)

$$Z_{-1} = \frac{\phi}{180^\circ} + 0 + 0.5 = 1$$

Thus for $K > 40$, one of the closed loop poles will be outside the unit circle.

If K is negative we can still use the same Nyquist plot but refer $(+1, j0)$ point as the critical point. ϕ in this case still equals $+90^\circ$ and the system is unstable. Hence the stable range of K is

$$0 \leq K < 40$$

More details can be found in **Digital Control Systems** by B. C. Kuo.