Module 11: Introduction to Optimal Control
Lecture Note 3

1 Linear Quadratic Regulator

Consider a linear system modeled by

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k), \quad \boldsymbol{x}(k_0) = \boldsymbol{x}_0$$

where $\boldsymbol{x}(k) \in R^n$ and $\boldsymbol{u}(k) \in R^m$. The pair $(A,\ B)$ is controllable.

The objective is to design a stabilizing linear state feedback controller $\boldsymbol{u}(k) = -K\boldsymbol{x}(k)$ which will minimize the quadratic performance index, given by,

$$J = \sum_{k=0}^{\infty} (\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k))$$

where, $Q = Q^T \geq 0$ and $R = R^T > 0$. Such a controller is denoted by $\boldsymbol{u}^*$.

We first assume that a linear state feedback optimal controller exists such that the closed loop system

$$\boldsymbol{x}(k+1) = (A - BK)\boldsymbol{x}(k)$$

is asymptotically stable.

This assumption implies that there exists a Lyapunov function $V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T P\boldsymbol{x}(k)$ for the closed loop system, for which the forward difference

$$\Delta V(\boldsymbol{x}(k)) = V(\boldsymbol{x}(k+1)) - V(\boldsymbol{x}(k))$$

is negative definite.

We will now use the theorem as discussed in the previous lecture which says if the controller $\boldsymbol{u}^*$ is optimal, then

$$\min_{\boldsymbol{u}} (\Delta V(\boldsymbol{x}(k)) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k)) = 0$$

Now, finding an optimal controller implies that we have to find an appropriate Lyapunov function which is then used to construct the optimal controller.

Let us first find the $\boldsymbol{u}^*$ that minimizes the function

$$f = f(\boldsymbol{u}(k)) = \Delta V(\boldsymbol{x}(k)) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k)$$

If we substitute ΔV in the above expression, we get

$$\begin{aligned} f(\boldsymbol{u}(k)) &= \boldsymbol{x}^T(k+1)P\boldsymbol{x}(k+1) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k) \\ &= (A\boldsymbol{x}(k) + B\boldsymbol{u}(k))^T P(A\boldsymbol{x}(k) + B\boldsymbol{u}(k)) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k) \end{aligned}$$

Taking derivative of the above function with respect to $\boldsymbol{u}(k)$,

$$\begin{aligned} \frac{\partial f(\boldsymbol{u}(k))}{\partial \boldsymbol{u}(k)} &= 2(A\boldsymbol{x}(k) + B\boldsymbol{u}(k))^T PB + 2\boldsymbol{u}^T R \\ &= 2\boldsymbol{x}^T(k)A^T PB + 2\boldsymbol{u}^T(k)(B^T PB + R) \\ &= \boldsymbol{0}^T \end{aligned}$$

The matrix $B^T PB + R$ is positive definite since R is positive definite, thus it is invertible. Hence,

$$\boldsymbol{u}^*(k) = -(B^T PB + R)^{-1} B^T PA\boldsymbol{x}(k) = -K\boldsymbol{x}(k)$$

where $K = (B^T PB + R)^{-1} B^T PA$. Let us denote $B^T PB + R$ by S. Thus

$$\boldsymbol{u}^*(k) = -S^{-1} B^T PA\boldsymbol{x}(k)$$

We will now check whether or not $\boldsymbol{u}^*$ satisfies the second order sufficient condition for minimization. Since

$$\begin{aligned} \frac{\partial^2 f(\boldsymbol{u}(k))}{\partial \boldsymbol{u}^2(k)} &= \frac{\partial}{\partial \boldsymbol{u}(k)}(2\boldsymbol{x}^T(k)A^T PB + 2\boldsymbol{u}^T(k)(B^T PB + R)) \\ &= 2(B^T PB + R) > 0 \end{aligned}$$

$\boldsymbol{u}^*$ satisfies the second order sufficient condition to minimize f.

The optimal controller can thus be constructed if an appropriate Lyapunov matrix P is found. For that let us first find the closed loop system after introduction of the optimal controller.

$$\boldsymbol{x}(k+1) = (A - BS^{-1}B^TPA)\boldsymbol{x}(k)$$

Since the controller satisfies the hypothesis of the theorem, discussed in the previous lecture,

$$\boldsymbol{x}^T(k+1)P\boldsymbol{x}(k+1) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^{*T}(k)R\boldsymbol{u}^*(k) = 0$$

Putting the expression of $\boldsymbol{u}^*$ in the above equation,

$$\begin{aligned}
&\boldsymbol{x}^T(k)(A - BS^{-1}B^TPA)^TP(A - BS^{-1}B^TPA)\boldsymbol{x}(k) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \\
&\quad \boldsymbol{x}^T(k)A^TPBS^{-1}RS^{-1}B^TPA\boldsymbol{x}(k) \\
&= \boldsymbol{x}^T(k)A^TPA\boldsymbol{x}(k) - \boldsymbol{x}^T(k)A^TPBS^{-1}B^TPA\boldsymbol{x}(k) - \boldsymbol{x}^T(k)A^TPBS^{-1}B^TPA\boldsymbol{x}(k) + \\
&\quad \boldsymbol{x}^T(k)A^TPBS^{-1}B^TPBS^{-1}B^TPA\boldsymbol{x}(k) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \\
&\quad \boldsymbol{x}^T(k)A^TPBS^{-1}RS^{-1}B^TPA\boldsymbol{x}(k) \\
&= \boldsymbol{x}^T(k)A^TPA\boldsymbol{x}(k) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) - 2\boldsymbol{x}^T(k)A^TPBS^{-1}B^TPA\boldsymbol{x}(k) + \\
&\quad \boldsymbol{x}^T(k)A^TPBS^{-1}(B^TPB + R)S^{-1}B^TPA\boldsymbol{x}(k) \\
&= \boldsymbol{x}^T(k)A^TPA\boldsymbol{x}(k) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) - \\
&\quad 2\boldsymbol{x}^T(k)A^TPBS^{-1}SS^{-1}B^TPA\boldsymbol{x}(k) + \boldsymbol{x}^T(k)A^TPBS^{-1}B^TPA\boldsymbol{x}(k) \\
&= \boldsymbol{x}^T(k)(A^TPA - P + Q - A^TPBS^{-1}B^TPA)\boldsymbol{x}(k) = 0
\end{aligned}$$

The above equation should hold for any value of $\boldsymbol{x}(k)$. Thus

$$A^TPA - P + Q - A^TPBS^{-1}B^TPA = 0$$

which is the well known discrete Algebraic Riccati Equation (ARE). By solving this equation we can get P to form the optimal regulator to minimize a given quadratic performance index.

Example 1: Consider the following linear system

$$\begin{aligned}
\boldsymbol{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ 0.5 & 0.8 \end{bmatrix}\boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k), \quad \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}(k)
\end{aligned}$$

Design an optimal controller to minimize the following performance index.

$$J = \sum_{k=0}^{\infty}(x_1^2 + x_1x_2 + x_2^2 + 0.1u^2)$$

Also, find the optimal cost.

Solution: The performance index J can be rewritten as

$$J = \sum_{k=0}^{\infty} (\boldsymbol{x}^T(k) \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \boldsymbol{x}(k) + 0.1u^2)$$

Thus, $Q = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ and $R = 0.1$.

Let us take P as

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

Then,

$$A^T PA - P = \begin{bmatrix} 0.25p_3 - p_1 & 0.5p_2 + 0.4p_3 - p_2 \\ 0.5p_2 + 0.4p_3 - p_2 & p_1 + 1.6p_2 + 0.64p_3 - p_3 \end{bmatrix}$$

$$A^T PA - P + Q = \begin{bmatrix} 0.25p_3 - p_1 + 1 & 0.4p_3 - 0.5p_2 + 0.5 \\ 0.4p_3 - 0.5p_2 + 0.5 & p_1 + 1.6p_2 - 0.36p_3 + 1 \end{bmatrix}$$

$$A^T PB = \begin{bmatrix} 0.5p_3 \\ p_2 + 0.8p_3 \end{bmatrix}, \quad B^T PA = \begin{bmatrix} 0.5p_3 & p_2 + 0.8p_3 \end{bmatrix}, \quad S = 0.1 + p_3$$

$$\begin{aligned} A^T PBS^{-1}B^T PA = &= \frac{1}{0.1 + p_3} \begin{bmatrix} 0.5p_3 \\ p_2 + 0.8p_3 \end{bmatrix} \begin{bmatrix} 0.5p_3 & p_2 + 0.8p_3 \end{bmatrix} \\ &= \frac{1}{0.1 + p_3} \begin{bmatrix} 0.25p_3^2 & 0.5p_2p_3 + 0.4p_3^2 \\ 0.5p_2p_3 + 0.4p_3^2 & p_2^2 + 1.6p_2p_3 + 0.64p_3^2 \end{bmatrix} \end{aligned}$$

The discrete ARE is

$$A^T PA - P + Q - A^T PBS^{-1}B^T PA \quad = \quad 0$$

Or,

$$\begin{bmatrix} 0.25p_3 - p_1 + 1 - \frac{0.25p_3^2}{0.1+p_3} & 0.4p_3 - 0.5p_2 + 0.5 - \dfrac{0.5p_2p_3 + 0.4p_3^2}{0.1 + p_3} \\ 0.4p_3 - 0.5p_2 + 0.5 - \frac{0.5p_2p_3+0.4p_3^2}{0.1+p_3} & p_1 + 1.6p_2 - 0.36p_3 + 1 - \frac{p_2^2+1.6p_2p_3+0.64p_3^2}{0.1+p_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We can get three equations from the discrete ARE. These are

$$0.25p_3 - p_1 + 1 - \frac{0.25p_3^2}{0.1 + p_3} = 0$$

$$0.4p_3 - 0.5p_2 + 0.5 - \frac{0.5p_2p_3 + 0.4p_3^2}{0.1 + p_3} = 0$$

$$p_1 + 1.6p_2 - 0.36p_3 + 1 - \frac{p_2^2 + 1.6p_2p_3 + 0.64p_3^2}{0.1 + p_3} = 0$$

Since the above three equations comprises three unknown parameters, these parameters can be solved uniquely, as

$$p_1 = 1.0238, \quad p_2 = 0.5513, \quad p_3 = 1.9811$$

The optimal control law can be found out as

$$\begin{aligned} u^*(k) &= -(R + B^T P B)^{-1} B^T P A \boldsymbol{x}(k) \\ &= -[0.4760 \quad 1.0265]\boldsymbol{x}(k) \\ &= -0.4760x_1(k) - 1.0265x_2(k) \end{aligned}$$

The optimal cost can be found as

$$\begin{aligned} J &= \boldsymbol{x}_0^T P \boldsymbol{x}_0 \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1.0238 & 0.5513 \\ 0.5513 & 1.9811 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 4.1075 \end{aligned}$$