

# Module 8: Controllability, Observability and Stability of Discrete Time Systems

## Lecture Note 2

The utmost important requirement in control system design is the stability. We would revisit some of the definitions related to stability of a system.

### 1 Revisiting the basics

Let us consider the following system.

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k), & \mathbf{x}(0) &= \mathbf{x}_0 \\ y(k) &= C\mathbf{x}(k)\end{aligned}\tag{1}$$

where  $A \in R^{n \times n}$ ,  $B \in R^{n \times 1}$ ,  $C \in R^{1 \times n}$ .

**Zero State Response:** The output response of system (1) that is due to the input only (initial states are zero) is called zero state response.

**Zero Input Response:** The output response of system (1) that is driven by the initial states only (in absence of any external input) is called zero input response.

**BIBO Stability:** If for any bounded input  $u(k)$ , the output  $y(k)$  is also bounded, then the system is said to be BIBO stable.

**Bounded Input Bounded State Stability:** If for any bounded input  $u(k)$ , the states are also bounded, then the system is said to be Bounded Input Bounded State stable.

**$L^2$  Norm:**  $L^2$  norm of a state vector  $\mathbf{x}(k)$  is defined as

$$\|\mathbf{x}(k)\|_2 = \left[ \sum_{i=1}^n x_i^2(k) \right]^{\frac{1}{2}}$$

$\mathbf{x}(k)$  is said to be bounded if  $\|\mathbf{x}(k)\| < M$  for all  $k$  where  $M$  is finite.

**Zero Input or Internal Stability:** If the zero input response of a system subject

to a finite initial condition is bounded and reaches zero as  $k \rightarrow \infty$ , then the system is said to be internally stable.

The above condition can be formulated as

1.  $|y(k)| \leq M < \infty$
2.  $\lim_{k \rightarrow \infty} |y(k)| = 0$

The above conditions are also requirements for **asymptotic stability**.

To ensure all possible stability for an LTI system, the only requirement is that the roots of the characteristic equations are inside the unit circle.

## 2 Definitions Related to Stability for A Generic System

We know that a general time invariant system (linear or nonlinear) with no external input can be modeled by the following equation

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k)) \quad (2)$$

**Equilibrium Point:** The equilibrium point or equilibrium state of a system is that point in the state space where the dynamics of the system is zero which implies that the states will remain there forever once brought.

Thus the equilibrium points are the solutions of the following equation.

$$\mathbf{f}(\mathbf{x}(k)) = \mathbf{0} \quad (3)$$

One should note that since an LTI system with no external input can be modeled by  $\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k)$ ,  $\mathbf{x}(k) = \mathbf{0}$  is the only equilibrium point for such a system.

Nonlinear systems can have multiple equilibrium points. Thus when we talk about the stability of a nonlinear system, we do so with respect to the equilibrium points.

For convenience, we state all definitions for the case when the equilibrium point is at the origin. There is no loss of generality if we do so because any equilibrium point can be shifted to origin via a change of variables.

### Example:

Find out the equilibrium points of the following nonlinear system.

$$\begin{aligned} x_1(k+1) &= x_1(k) - x_1^3(k) \\ x_2(k+1) &= -x_2(k) \end{aligned}$$

Equating  $x_1(k) - x_1^3(k)$  and  $-x_2(k)$  to 0, we get  $x_2 = 0$  always.  $x_1$  can take 3 values, which are 0, 1 and  $-1$  respectively. Thus the system has three equilibrium points, located at  $(0, 0)$ ,  $(1, 0)$  and  $(-1, 0)$  respectively.

**Stability in the sense of Lyapunov:** The equilibrium point  $\mathbf{x} = \mathbf{0}$  of (2) is stable if, for each  $\epsilon > 0$ , there exists a  $\delta = \delta(\epsilon, k_0) > 0$  such that

$$\|\mathbf{x}(k_0)\| < \delta \Rightarrow \|\mathbf{x}(k)\| < \epsilon, \forall k \geq k_0 \geq 0$$

The above condition is illustrated in Figure 1.

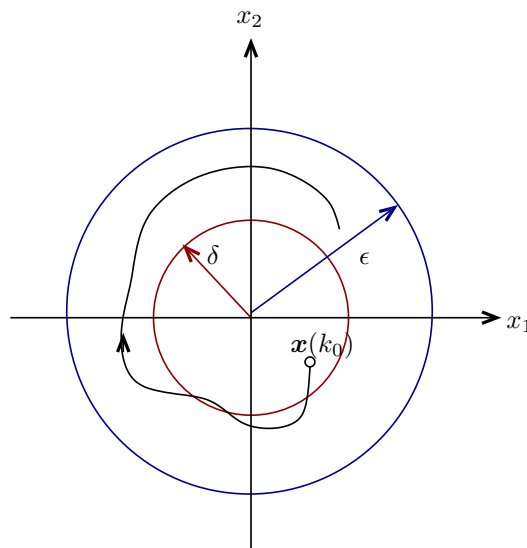


Figure 1: Illustration of stable equilibrium in the sense of Lyapunov in two dimension

**Asymptotic Stability:** The equilibrium point  $\mathbf{x} = \mathbf{0}$  of (2) is asymptotically stable if it is stable and there is a positive constant  $c = c(k_0)$  such that

$$\mathbf{x}(k) \rightarrow \mathbf{0} \text{ as } k \rightarrow \infty, \quad \forall \|\mathbf{x}(k_0)\| < c$$

The above condition is illustrated in Figure 2.

**Instability:** The equilibrium point  $\mathbf{x} = \mathbf{0}$  of (2) is unstable if it is not stable.

The above condition is illustrated in Figure 3.

**Uniform Stability:** The equilibrium point  $\mathbf{x} = \mathbf{0}$  of (2) is uniformly stable if, for each  $\epsilon > 0$ , there exists a  $\delta = \delta(\epsilon) > 0$ , independent of  $k_0$ , such that

$$\|\mathbf{x}(k_0)\| < \delta \Rightarrow \|\mathbf{x}(k)\| < \epsilon, \forall k \geq k_0 \geq 0$$

**Uniform Asymptotic Stability:** The equilibrium point  $\mathbf{x} = \mathbf{0}$  of (2) is uniformly

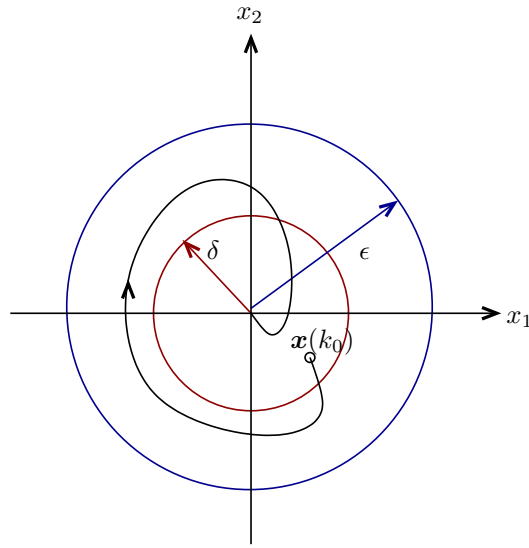


Figure 2: Illustration of asymptotically stable equilibrium in two dimension

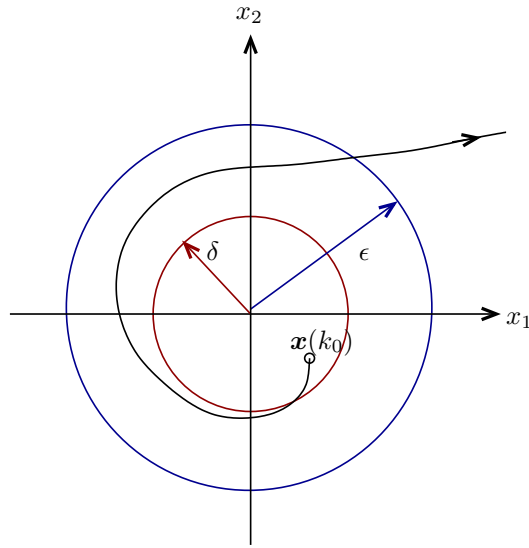


Figure 3: Illustration of unstable equilibrium in two dimension

asymptotically stable if it is uniformly stable and there is a positive constant  $c$ , independent of  $k_0$ , such that for all  $\|\mathbf{x}(k_0)\| < c$ ,  $\mathbf{x}(k) \rightarrow \mathbf{0}$  as  $k \rightarrow \infty$  uniformly in  $k_0$ .

**Global Uniform Asymptotic Stability:** The equilibrium point  $\mathbf{x} = \mathbf{0}$  of (2) is globally uniformly asymptotically stable if it is uniformly asymptotically stable for such a  $\delta$  when  $\delta(\epsilon)$  can be chosen to satisfy the following condition

$$\lim_{\epsilon \rightarrow \infty} \delta(\epsilon) = \infty$$

**Exponential Stability:** The equilibrium point  $\mathbf{x} = \mathbf{0}$  of (2) is exponentially stable if

there exist positive constants  $c$ ,  $\gamma$  and  $\lambda$  such that

$$\|\mathbf{x}(k)\| \leq \gamma \|\mathbf{x}(k_0)\| e^{-\lambda(k-k_0)}, \quad \forall \|\mathbf{x}(k_0)\| < c$$

**Global Exponential Stability:** The equilibrium point  $\mathbf{x} = \mathbf{0}$  of (2) is globally exponentially stable if it is exponentially stable for any initial state  $\mathbf{x}(k_0)$ .