Module 7: Discrete State Space Models
Lecture Note 4

In this lecture we would discuss about the solution of discrete state equation, computation of discrete state transition matrix and state diagram.

1 Solution to Discrete State Equation

Consider the following state model of a discrete time system:

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + Bu(k)$$

where the initial conditions are $\boldsymbol{x}(0)$ and $u(0)$. Putting $k = 0$ in the above equation, we get

$$\boldsymbol{x}(1) = A\boldsymbol{x}(0) + Bu(0)$$

Similarly if we put $k = 1$, we would get

$$\begin{aligned} \boldsymbol{x}(2) &= A\boldsymbol{x}(1) + Bu(1) \\ \text{Putting the expression of } \boldsymbol{x}(1) \Rightarrow \boldsymbol{x}(2) &= A^2\boldsymbol{x}(0) + ABu(0) + Bu(1) \end{aligned}$$

For $k = 2$,

$$\begin{aligned} \boldsymbol{x}(3) &= A\boldsymbol{x}(2) + Bu(2) \\ &= A^3\boldsymbol{x}(0) + A^2Bu(0) + ABu(1) + Bu(2) \end{aligned}$$

and so on. If we combine all these equations, we would get the following expression as a general solution:

$$\boldsymbol{x}(k) = A^k\boldsymbol{x}(0) + \sum_{i=0}^{k-1} A^{k-1-i}Bu(i)$$

As seen in the above expression, $\boldsymbol{x}(k)$ has two parts. One is the contribution due to the initial state $\boldsymbol{x}(0)$ and the other one is the contribution of the external input $u(i)$ for $i = 0, 1, 2, \cdots, k-1$.

When the input is zero, solution of the homogeneous state equation $\boldsymbol{x}(k+1) = A\boldsymbol{x}(k)$ can be written as

$$\boldsymbol{x}(k) = A^k \boldsymbol{x}(0)$$

where $A^k = \phi(k)$ is the state transition matrix.

2 Evaluation of $\phi(k)$

Similar to the continuous time systems, the state transition matrix of a discrete state model can be evaluated using the following different techniques.

1. **Using Inverse Z-transform**:

$$\phi(k) = \mathscr{Z}^{-1}\left\{(zI - A)^{-1}\right\}$$

2. **Using Similarity Transformation** If Λ is the diagonal representation of the matrix A, then $\Lambda = P^{-1}AP$. When a matrix is in diagonal form, computation of state transition matrix is straight forward:

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

Given Λ^k, we can compute $A^k = P\Lambda^k P^{-1}$

3. **Using Caley Hamilton Theorem**

Example Compute A^k for the following system using three different techniques and hence find $y(k)$ for $k \geq 0$.

$$\begin{aligned} \boldsymbol{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ -0.21 & -1 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-1)^k; \quad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ y(k) &= x_2(k) \end{aligned}$$

Solution: $A = \begin{bmatrix} 0 & 1 \\ -0.21 & -1 \end{bmatrix}$ and eigenvalues of A are -0.3 and -0.7.

Method 1

$$A^k = \mathscr{Z}^{-1}(zI - A)^{-1} = \mathscr{Z}^{-1}\left\{ \begin{bmatrix} z-1 & -1 \\ 1 & z-1 \end{bmatrix}^{-1} \right\}$$

$$
\begin{aligned}
A^k &= \mathscr{Z}^{-1}\begin{bmatrix} \frac{z+1}{z^2+z+0.21} & \frac{1}{z^2+z+0.21} \\ \\ \frac{-0.21}{z^2+z+0.21} & \frac{z}{z^2+z+0.21} \end{bmatrix} \\
&= \mathscr{Z}^{-1}\begin{bmatrix} \frac{1.75}{z+0.3} - \frac{0.75}{z+0.7} & \frac{2.5}{z+0.3} - \frac{2.5}{z+0.7} \\ \\ \frac{-0.525}{z+0.3} + \frac{0.525}{z+0.7} & \frac{-0.75}{z+0.3} + \frac{1.75}{z+0.7} \end{bmatrix} \\
&= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k & 2.5(-0.3)^k - 2.5(-0.7)^k \\ \\ -0.525(-0.3)^k + 0.525(-0.7)^k & -0.75(-0.3)^k + 1.75(-0.7)^k \end{bmatrix}
\end{aligned}
$$

Method 2

$A^k = P\Lambda^k P^{-1}$ where $\Lambda^k = \begin{bmatrix} (-0.3)^k & 0 \\ 0 & (-0.7)^k \end{bmatrix}$. Eigen values are -0.3 and -0.7. The corresponding eigenvectors are found, by using equation $Av_i = \lambda_i v_i$, as $\begin{bmatrix} 1 \\ -0.3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -0.7 \end{bmatrix}$ respectively. The transformation matrix is given by

$$
P = \begin{bmatrix} 1 & 1 \\ -0.3 & -0.7 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1.75 & 2.5 \\ -0.75 & -2.5 \end{bmatrix}
$$

Thus,

$$
\begin{aligned}
A^k &= P\Lambda^k P^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ -0.3 & -0.7 \end{bmatrix} \begin{bmatrix} (-0.3)^k & 0 \\ 0 & (-0.7)^k \end{bmatrix} \begin{bmatrix} 1.75 & 2.5 \\ -0.75 & -2.5 \end{bmatrix} \\
&= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k & 2.5(-0.3)^k - 2.5(-0.7)^k \\ \\ -0.525(-0.3)^k + 0.525(-0.7)^k & -0.75(-0.3)^k + 1.75(-0.7)^k \end{bmatrix}
\end{aligned}
$$

Method 3: Caley Hamilton Theorem

The eigenvalues are -0.3 and -0.7.

$$
\begin{aligned}
(-0.3)^k &= \beta_0 - 0.3\beta_1 \\
(-0.7)^k &= \beta_0 - 0.7\beta_1
\end{aligned}
$$

Solving,

$$
\begin{aligned}
\beta_0 &= 1.75(-0.3)^k - 0.75(-0.7)^k \\
\beta_1 &= 2.5(-0.3)^k - 2.5(-0.7)^k
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\phi(k) = A^k &= \beta_0 I + \beta_1 A \\
&= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k & 2.5(-0.3)^k - 2.5(-0.7)^k \\ \\ -0.525(-0.3)^k + 0.525(-0.7)^k & -0.75(-0.3)^k + 1.75(-0.7)^k \end{bmatrix}
\end{aligned}
$$

The solution $\boldsymbol{x}(k)$ is

$$\begin{aligned}\boldsymbol{x}(k) &= A^k\boldsymbol{x}(0) + \sum_{i=0}^{k-1} A^{k-1-i}Bu(i)\\ &= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k \\ -0.525(-0.3)^k + 0.525(-0.7)^k \end{bmatrix} + \sum_{i=0}^{k-1}\begin{bmatrix} 2.5(-0.3)^{k-1-i} - 2.5(-0.7)^{k-1-i} \\ -0.75(-0.3)^{k-1-i} + 1.75(-0.7)^{k-1-i}\end{bmatrix}(-1)^i\end{aligned}$$

Since $y(k) = x_2(k)$, we can write

$$\begin{aligned}y(k) &= -0.525(-0.3)^k + 0.525(-0.7)^k + \sum_{i=0}^{k-1}[-0.75(-0.3)^{k-1-i} + 1.75(-0.7)^{k-1-i}](-1)^i\\ &= -0.525(-0.3)^k + 0.525(-0.7)^k - 0.75(-0.3)^{k-1}\sum_{i=0}^{k-1}(1/0.3)^i + 1.75(-0.7)^{k-1}\sum_{i=0}^{k-1}(1/0.7)^i\end{aligned}$$

Now,

$$\begin{aligned}\sum_{i=0}^{k-1}(1/0.3)^i = \sum_{i=0}^{k-1}(3.33)^i = \frac{1-(3.33)^k}{1-3.33} = -0.43[1-(3.33)^k]\\ \sum_{i=0}^{k-1}(1/0.7)^i = \sum_{i=0}^{k-1}(1.43)^i = \frac{1-(1.43)^k}{1-1.43} = -2.33[1-(1.43)^k]\end{aligned}$$

Putting the above expression in $y(k)$

$$y(k) = 0.475(-0.3)^k - 5.3(-0.7)^k + (-0.3)^k(3.33)^k + 5.825(-0.7)^{k-1}(1.43)^k$$

3 State Diagram

Conventional signal flow graph method was meant for only algebraic equation, thus these are generally used for the derivation of input output relation in a transformed domain.

State diagram or state transition signal flow graph is an extension of conventional signal flow graph which can be applied to represent differential and difference equations as well.

Example 1: Draw the state diagram for the following differential equation.

$$\ddot{y}(t) + 2\dot{y}(t) + y(t) = u(t)$$

Considering the state variables as $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, we can write

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -x_1(t) - 2x_2(t) + u(t)\end{aligned}$$

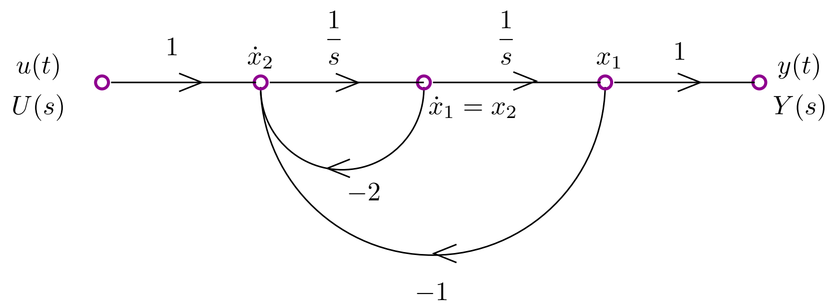

Figure 1: State Diagram of Example 1

The state diagram is shown in Figure 1.

Example 2: Consider a discrete time system described by the following state difference equations.

$$\begin{aligned} x_1(k+1) &= -x_1(k) + x_2(k) \\ x_2(k+1) &= -x_1(k) + u(k) \\ y(k) &= x_1(k) + x_2(k) \end{aligned}$$

Draw the state diagram.

The state diagram is shown in Figure 2.

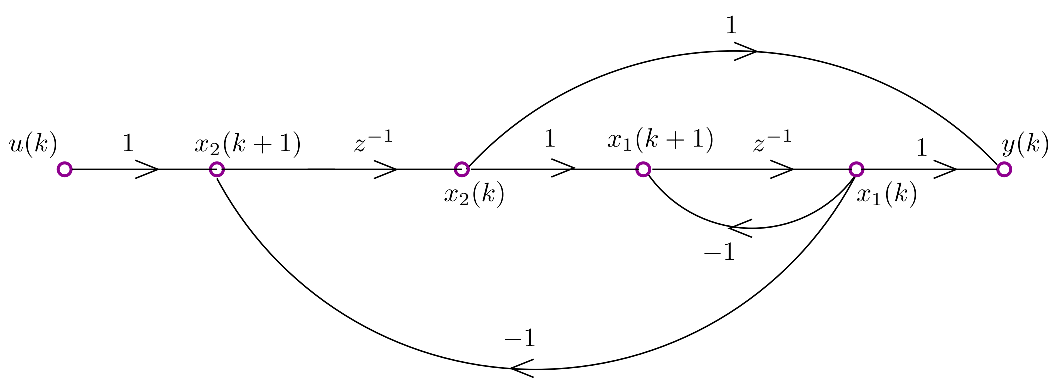

Figure 2: State Diagram of Example 2

3.1 State Diagram of Zero Order Hold

State diagram of zero order hold is important for sampled date control systems. Let the input to and output of a ZOH is $e^*(t)$ and $h(t)$ respectively. Then, for the inetrval $kT \leq t \leq (k+1)T$,

$$h(t)e(kT)$$

Or,

$$H(s) = \frac{e(kT)}{s}$$

Therefore, the state diagram, as shown in Figure 3, consists of a single branch with gain s^{-1}.

$$e(kT) \circ\!\xrightarrow{\;s^{-1}\;}\!\circ\, H(s)$$

Figure 3: State Diagram of Zero Order Hold

4 System Response between Sampling Instants

State variable method is a convenient way to evaluate the system response between the sampling instants of a sampled data system. State transition equation is given as:

$$\boldsymbol{x}(t) = \phi(t-t_0)\boldsymbol{x}(t_0) + u(t_0)\int_{t_0}^{t}\phi(t-\tau)Bd\tau$$

where $\boldsymbol{x}(t_0)$ is the initial state of the system and $u(t)$ is the external input.

$$\text{when } t_0 = kT, \qquad \boldsymbol{x}(t) = \phi(t-kT)\boldsymbol{x}(kT) + u(kT)\int_{kT}^{t}\phi(t-\tau)Bd\tau$$

Since we are interested in response between the sampling instants, let us consider $t = (k+\Delta)T$ where $k = 0, 1, 2, \cdots$ and $0 \leq \Delta \leq 1$. This implies

$$\boldsymbol{x}((k+\Delta)T) = \phi(\Delta T)\boldsymbol{x}(kT) + \theta(\Delta T)u(kT)$$

where $\theta(\Delta T) = \displaystyle\int_{kT}^{(k+\Delta)T}\phi((k+\Delta)T-\tau)Bd\tau$. By varying the value of Δ between 0 and 1 all information on $\boldsymbol{x}(t)$ for all t can be obtained.

Example 3: Consider the following state model of a continuous time system.

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \\
y(t) &= x_1(t)
\end{aligned}$$

which undergoes through a sampling process with period T. To derive the discrete state space model, let us first compute the state transition matrix of the continuous time system using Caley Hamilton Theorem.

$$\triangle(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda+1 & 1 \\ 0 & \lambda+2 \end{vmatrix} = (\lambda+1)(\lambda+2) = 0 \Rightarrow \lambda_1 = -1,\ \lambda_2 = -2$$

$$\text{Let } f(\lambda) = e^{\lambda t}$$

This implies

$$\begin{aligned}
e^{-t} &= \beta_0 - \beta_1 \quad (\lambda_1 = -1) \\
e^{-2t} &= \beta_0 - 2\beta_1 \quad (\lambda_2 = -2)
\end{aligned}$$

Solving the above equations

$$\beta_1 = e^{-t} - e^{-2t} \qquad \beta_0 = 2e^{-t} - e^{-2t}$$

Then

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A \\ &= \begin{bmatrix} e^{-t} & e^{-2t} - e^{-t} \\ 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

Thus the discrete state matrix A is given as

$$A = \phi(T) = \begin{bmatrix} e^{-T} & e^{-2T} - e^{-T} \\ 0 & e^{-2T} \end{bmatrix}$$

The discrete input matrix B can be computed as

$$\begin{aligned} B &= \theta(T) = \int_0^T \Phi(T - t') \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt' \\ &= \begin{bmatrix} e^{-T} - 0.5e^{-2T} - 0.5 \\ 0.5 - 0.5e^{-2T} \end{bmatrix} \end{aligned}$$

When $t = (k+1)T$, the discrete state equation is described by

$$\mathbf{x}((k+1)T) = \begin{bmatrix} e^{-T} & e^{-2T} - e^{-T} \\ 0 & e^{-2T} \end{bmatrix} \mathbf{x}(kT) + \begin{bmatrix} e^{-T} - 0.5e^{-2T} - 0.5 \\ 0.5 - 0.5e^{-2T} \end{bmatrix} u(kT)$$

When $t = (k + \Delta)T$,

$$\mathbf{x}(kT + \Delta T) = \begin{bmatrix} e^{-\Delta T} & e^{-2\Delta T} - e^{-\Delta T} \\ 0 & e^{-2\Delta T} \end{bmatrix} \mathbf{x}(kT) + \begin{bmatrix} e^{-\Delta T} - 0.5e^{-2\Delta T} - 0.5 \\ 0.5 - 0.5e^{-2\Delta T} \end{bmatrix} u(kT)$$

If the sampling period $T = 1$,

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.37 & -0.23 \\ 0 & 0.14 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -0.2 \\ 0.43 \end{bmatrix} u(k)$$

At the sampling instants we can find $\mathbf{x}(k)$ by putting $k = 0, 1, 2 \cdots$. If $\Delta = 0.5$, then between the sampling instants,

$$\mathbf{x}(k+0.5) = \begin{bmatrix} 0.61 & -0.24 \\ 0 & 0.37 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -0.08 \\ 0.32 \end{bmatrix} u(k)$$

The responses in between the sampling instants, i.e., $\mathbf{x}(0.5)$, $\mathbf{x}(1.5)$, $\mathbf{x}(2.5)$ etc., can be found by putting $k = 0, 1, 2 \cdots$.