Module 5 : Real and Reactive Power Scheduling

Lecture 23 : Optimization

Objectives

In this lecture you will learn the following

- How does one solve a constrained optimization problem ?

- An illustrative example

Unconstrained and Constrained Optimization

In an *unconstrained* system, the usual approach to minimize the cost function is to set the function derivatives to zero and then solve for the control and auxiliary variables from the set of resulting equations, i.e., if we wish to

Minimize J(u)

then the solution is obtained by setting the partial derivative of J with respect to every control variable u to zero. The number of equations will be equal to the number of control variables.

Suppose we wish to minimize a cost function subject to some equality and inequality constraints. If constraints are specified, then the the above-mentioned procedure will not work because the solution obtained from the procedure may not satisfy the constraints. Therefore an alternative method is required in order to take constraints into account.

Let us first consider the case wherein **only equality constraints** are present:

Minimize

$J(x,u)$

subject to

$g(x,u) = 0$

In such a case the procedure is to form a composite cost function

$C = J(x,u) - l' * g(x,u)$

where l' is a row vector of the variables called the Lagrange Multipliers. The number of multipliers is equal to the number of equality constraints $g(x,u)=0$.

The composite cost function is minimized by treating the Lagrange mulitpliers as additional variables. *This solution is a minimum solution which satisfies the equality constraints.*

We shall now illustrate this using a simple example.

Equality Constrained Optimization

We will use the same example as in the previous lecture, but solve it by using the formal Lagrange Multipliers formulation.

Consider two generators with the following cost functions:

$C1 = 50 * P1 + 3 * P1^2$

$C2 = 50 * P2 + 2.5 * P2^2$

Let us assume that the generators (G1 and G2) do not have any maximum limits. Find the least cost schedule if a load demand of 550 MW is to be met.

Total cost is given by : C1+C2 which is to be minimized.

subject to $P1+P2 - 550 = 0$

The composite cost function is given by : $C = C1+C2 - l * (P1+P2-550)$

Since there is only one constraint equation, *l* is a scalar. In this example, P1 and P2 are the control variables and there are no auxiliary variables (to keep things simple!).

Now by differentiating with the reference to the control, auxiliary (there are none in this example) and *l*, and equating all derivatives to zero, we obtain :

$dC/dP1 = 50 + 6 * P1 - l = 0$, $dC/dP2 = 50 + 5 * P2 - l = 0$, and $dC/dl = -(P1+P2-550) = 0$, or equivalently

$$\begin{bmatrix} 6 & 0 & -1 \\ 0 & 5 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P1 \\ P2 \\ \lambda \end{bmatrix} = \begin{bmatrix} -50 \\ -50 \\ 550 \end{bmatrix}$$

These are three independent equations in three variables and can be solved by pre-multiplying the column vector on the RHS by the inverse of the matrix on the LHS. Verify that that P1=250 MW, P2=300 MW, and *l* = Rs 1550 / MW-hr is the solution.

Also note that since at the optimum solution, $dC/dP1 = dC1/dP1 - l = 0 = dC/dP2 = dC2/dP2 - l$. In other words, at the optimum solution, $dC1/dP1 = dC2/dP2 = l$

The equality of $dC1/dP1 = dC2/dP2$ was an observation that we made in the previous lecture too.

It is not difficult to see why this is true at the optimum (least cost) solution. If $dC1/dP1 > dC2/dP2$ at a certain value of P1 and P2, then it would be possible to reduce total cost by slightly increasing P2, and reducing P1 by the same amount. Therefore these values of P1 and P2 CANNOT be the least cost solution. The same can be argued for values of P1 and P2 such that $dC1/dP1 < dC2/dP2$.

Only if $dC1/dP1 = dC2/dP2$ can we state that no improvement in cost will result for (small) changes in P1 and P2.

General Constrained Optimization with Inequality Constraints

What happens variables are constrained by an *inequality* constraint ?

An inequality constraint usually arises when one wants to place some limits on the value of a variable. For example, power output of a generator is limited by its capacity. A simple and *approximate* way of ensuring that limits are not hit (i.e. inequality constraints are not violated) is to augment the composite cost function discussed earlier, so that costs become extremely high if any limit is violated.

Consider the following problem:

Minimize

$$J(x,u)$$

subject to

$$g(x,u) = 0$$

and

$$h_{\min} \le h(x,u) \le h_{\max}$$

The last equation can be rewritten (for convenience) as follows:

$$h(x,u) \le h_{\max}$$
$$-h(x,u) \le -h_{\min}$$

Therefore all the inequality equations can be re-written as: $q(x,u) \le q_{\max}$

Now, the procedure for miminizing J subject to the constraints is to form a composite (augmented) cost function as follows:

$$C = J(x,u) - \lambda' * g(x,u) + p(q(x,u) - q_{\max})$$

Let us define $t = q(x,u) - qmax$.

The function $p(t)$ is chosen such that its value is very small if t is less than zero, otherwise it is very large.

A "penalty" function like

$p(t) = e^{\beta t}$ where b >0 is suitable for this purpose. (How does one decide the value of b ?)

General Constrained Optimization with Inequality Constraints

Let us again consider the previous example, but now we will also incorporate inequality constraints.

Consider two generators with the following cost functions:

C1 = 50 *P1 + 3 * P1 *P1

C2 = 50 *P2 + 2.5 * P2 *P2

Let us assume that the generators (G1 and G2) have maximum limits: P1 < 290 MW and P2 <290 MW

Find the least cost schedule if a load demand of 550 MW is to be met.

Clearly, the solution without the inequality constraints is not feasible since P2=300 MW violates the constraint P2 <290 MW. The correct solution can be obtained by following the procedure outlined below (can you guess the answer right away ?)

Total cost is given by : C1+C2 which is to be minimized, subject to P1+P2 - 550 =0 and P1 < 290 MW and P2 <290 MW

The augmented cost function considering equality and inequality constraints is

C = C1+C2 - *l* * (P1+P2-550) + *p*(P1 -290) + *p*(P2- 290)

$p(t) = e^{\beta t}$, where b is chosen to be 3.

The minimum value of C is obtained by calculating the derivative with respect to the variables, P1, P2 and *l.*

.As a result we obtain the following (nonlinear) equations:

$$\begin{bmatrix} 6 & 0 & -1 \\ 0 & 5 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P1 \\ P2 \\ \lambda \end{bmatrix} + \begin{bmatrix} 3 * e^{3*(P1-290)} \\ 3 * e^{3*(P2-290)} \\ 0 \end{bmatrix} - \begin{bmatrix} -50 \\ -50 \\ 550 \end{bmatrix} = 0$$

These equations are non-linear and cannot be solved by direct matrix inversion. In such circumstances, one can try to employ numerical techniques like Gauss-Siedel or Newton-Raphson method. Alternatively one can use a "gradient descent method', in which the LHS of the above equation is evaluated at a guess value, say P1 = 250 MW , P2 = 250 MW and *l = 1550. Note that the LHS is the "gradient" of the cost function, i.e.,*

$$\nabla C_k = \begin{bmatrix} 6 & 0 & -1 \\ 0 & 5 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P1_k \\ P2_k \\ \lambda_k \end{bmatrix} + \begin{bmatrix} 3 * e^{3*(P1_k-290)} \\ 3 * e^{3*(P2_k-290)} \\ 0 \end{bmatrix} - \begin{bmatrix} -50 \\ -50 \\ 550 \end{bmatrix}$$

Recall from vector calculus that the gradient of a function points towards the direction of maximum change of the function for small changes in the value of the variables. *The trick is to change P1, P2 and l in a direction which is negative to the gradient direction, i.e.,*

$$\begin{bmatrix} P1_{k+1} \\ P2_{k+1} \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} P1_k \\ P2_k \\ \lambda_k \end{bmatrix} - \alpha \nabla C_k$$

a is chosen to be a small value. One keeps updating the values of P1, P2 and l in this fashion ("descending") till there is very little difference between the values of variables for 2 consecutive iterations. Consider a MATLAB program which implements this (you may copy this program and run it):

```
A=[6 0 -1;0 5 -1;1 1 0];
B=[-50;-50;550];
x=[250;250;1550];
alpha=0.01; beta=3;
for k=1:2000
C=[beta*exp(beta*(x(1)-290));        beta*exp(beta*(x(2)-
290));0];
grad=A*x+C-B;
xold=x;
x=x-grad*alpha;
if norm(x-xold,2)<0.01
break
end
end
```

What values of x does it converge to? Do the generator powers remain within the limits specified ? If not why ?

What is the effect of varying a and the value of b ? What if beta is chosen to be a) too low or b) too high ?

sRecap

In this lecture you have learnt the following

- How to solve a constrained optimzation problem using Lagrange Multipliers ?

- Handling inequality constraints using penalty function approach.

Congratulations, you have finished Lecture 23. To view the next lecture select it from the left hand side menu of the page.