

Module 3 : Frequency Control in a Power System

Lecture 12a : Solution of non-linear algebraic equations

Non-linear algebraic equations and their solution

In the next lecture, we will compute the steady state frequency of a power system, given the load characteristics. We shall see that in general, this will require us to carry out a "loadflow". A loadflow involves the solution of a set of non-linear algebraic equations. Therefore, in this lecture we revise the basic methods to solve non-linear algebraic equations.

We are aware that a transmission network in sinusoidal state can be modelled by linear algebraic equations in the node voltage phasors(**V**) and the nodal current phasor injections (**I**):

$$\mathbf{I} = \mathbf{Y}_{bus} \mathbf{V}$$

where \mathbf{Y}_{bus} is the bus admittance matrix.

However, in power system studies, nodal injections are not specified as current phasors but as real and reactive power injections (nonlinear functions of **V** and **I**) , and/or voltage magnitudes of some nodes. We have also seen that real and reactive power can be a function of frequency. In such a situation, obtaining the steady state solution (i.e. node voltage phasors and frequency) will require us to solve a set of *non-linear* equations.

Therefore we take a slight diversion from the main theme and review why and how we use numerical techniques for solving non-linear algebraic equations.

Let us consider the "why" question first. If we wish to solve an equation of the form:

$$e^{-x} - x = 0$$

Perhaps, "simplifying" it will help us solve it ?

$$e^{-x} = x$$

Perhaps, if we take the natural logarithm of both sides we may be able to do something ?

$$-x = \log x$$

But soon enough you will realize that we seem to be getting nowhere !

It is clear that some other way (guess work ?) may be required to get the solution.

Fixed Point Iteration Method

Since we have some idea of how the exponential function behaves we can try to guess the solution. We know that:

$$e^{-x} = \frac{1}{e^x}$$

and

$$e^1 \approx 2.7$$

and

$$e^{0.5} \approx \sqrt{2.7} \approx 1.6 \approx \frac{1}{0.6}$$

We can guess that the solution for $e^{-x} = x$ should lie between 0.5 and 1.

However, this is a rough estimate. Surprisingly if we take an initial guess value :

$$x_0 = x_{\text{init}} = 0.5$$

and iterate as follows starting with k=0:

$$x_{k+1} = e^{-x_k}$$

then $x_1=0.606$, $x_2=0.545$, $x_3=0.579$, $x_4=0.5600$, $x_5=0.571$, $x_6=0.565$, $x_7=0.568$, $x_8=0.566$, $x_9=0.567$

We seem to be "converging" to a solution which satisfies the equation $e^{-x} = x$!

Why does the Fixed Point Iterative Method Work ?

We can try to understand why we converge to the right solution by examining the behaviour of the iterative method

near the solution. Suppose the correct solution to the equation $e^{-x} = x$ is $x = x_s$, i.e.,

$$x_s = e^{-x_s}$$

Suppose the value of x at the kth iteration is near the solution x_s and differs from it by a small amount Δx_k , i.e.,

$$x_k = x_s + \Delta x_k$$

then:

$$x_s + \Delta x_{k+1} = e^{-(x_s + \Delta x_k)} = e^{-x_s} e^{-\Delta x_k} \approx e^{-x_s} (1 - \Delta x_k)$$

which yields :

$$\Delta x_{k+1} \approx -e^{-x_s} \Delta x_k$$

therefore if at k=0, $x = x_{\text{init}}$ then,

$$\Delta x_k = (-e^{-x_s})^k \Delta x_{\text{init}}$$

Since:

$$|-e^{-x_s}| < 1$$

Therefore as k tends to infinity, Dx_k tends to zero. This means that if we start close enough to the solution, we will converge to the correct solution after many iterations, i.e., $x_k = x_s$ if k is large. We say that the solution has converged if :

$$|e^{-x_k} - x_k| < \epsilon$$

where ϵ is the desired tolerance.

The convergence is affected by the properties of the function at the solution point.

You can check that if we wish to find out the solution of :

$$\frac{1}{x^2} - x = 0$$

by the iterative algorithm:

$$x_{k+1} = \frac{1}{x_k^2}$$

the solution will *diverge* for any value of the initial guess which is not the true solution ($x = 1$).

What happens if we formulate the iterative algorithm for the same equation as follows ?

$$x_{k+1} = \frac{1}{\sqrt{x_k}}$$

Fixed Point Iteration Method

We now formally write the algorithm down. If we wish to find the solution for a set of n algebraic equations in n variables:

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0 \\ f_2(x_1, x_2, \dots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \dots, x_n) &= 0 \end{aligned}$$

We first write the equations in the form (there is no unique way to do this!):

$$\begin{aligned} x_1 &= g_1(x_1, x_2, \dots, x_n) \\ x_2 &= g_2(x_1, x_2, \dots, x_n) \\ &\vdots \\ x_n &= g_n(x_1, x_2, \dots, x_n) \end{aligned}$$

We then guess the initial values of all n variables and iterate as follows:

$$\begin{aligned}
 x_{1_{k+1}} &= g_1(x_{1_k}, x_{2_k}, \dots, x_{n_k}) \\
 x_{2_{k+1}} &= g_2(x_{1_k}, x_{2_k}, \dots, x_{n_k}) \\
 &\vdots \\
 x_{n_{k+1}} &= g_n(x_{1_k}, x_{2_k}, \dots, x_{n_k})
 \end{aligned}$$

Unfortunately, this method does not converge always to the true solution for all nonlinear equations (as seen in the previous slide)

and even if it does it sometimes converges very slowly. Some variations are proposed to improve convergence.

For example, for the solution of $x = g(x)$, instead of using $x_{k+1} = g(x_k)$, one may first compute:

$$\bar{x}_{k+1} = g(x_k)$$

$$x_{k+1} = \alpha \bar{x}_{k+1} + (1 - \alpha)x_k$$

where α is a value which is between 0 and 1. It is found that this may speed up convergence for certain nonlinear equations.

Alternative methods like **Newton Raphson(N-R)** can also be used for solving nonlinear equations.

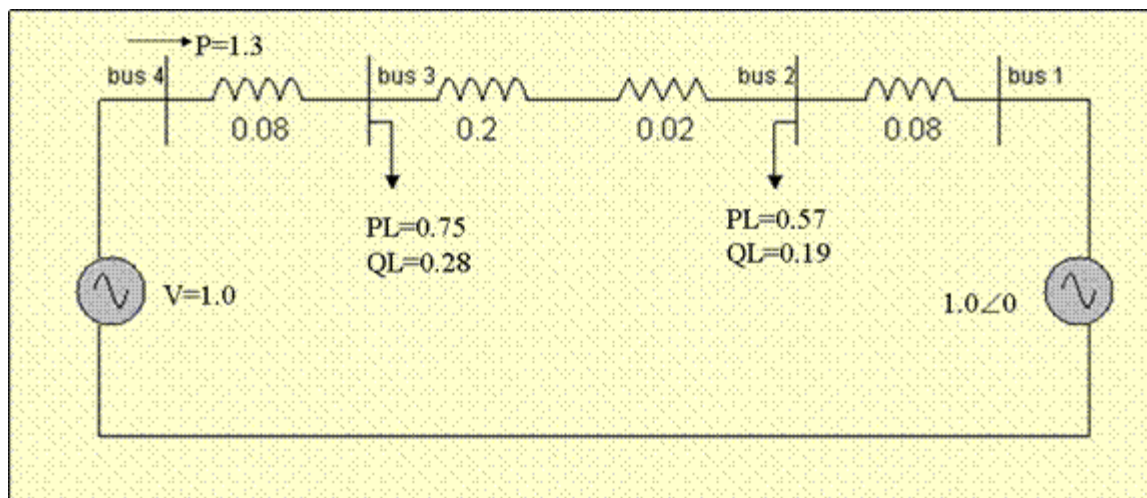
For an equation $f(x) = 0$, N-R algorithm is :

$$x_{k+1} = x_k + \left[\frac{\partial f}{\partial x} \right]^{-1} f(x_k)$$

where the partial derivative is **evaluated** at $x = x_k$.

Exercise:

Can you write down the nonlinear algebraic equations corresponding to the real and reactive balance equations at each node for the system shown below ? The numerical values of the "known quantities" are indicated on the figure.



Can you formulate the iterative algorithm to find out the unknown quantities, i.e., the bus voltage phasors at buses 2 and 3, and the bus voltage phasor angle at bus 4, using the **fixed point iterative method**? Note that the bus voltage phasor at bus 1, voltage magnitude and real power injection at bus 4 is indicated on the figure.

Recap

In this lecture you have learnt the following

- A recap on how to solve non-linear algebraic equations.

Congratulations, you have finished Lecture 12a. To view the next lecture select it from the left hand side menu of the page.