COMBINATORICS- Lecture: 1

Some Books...

- Discrete and Combinatorial Mathematics: An Applied Introduction. (R. P. Grimaldi, B. V. Ramana)
- Extremal Combinatorics: with applications in computer science. (S. Jukna)
- A Walk Through Combinatorics, An Introduction to Enumeration and Graph Theory. (Miklós Bóna)
- Introduction to Enumerative Combinatorics (Miklós Bóna)
- A Course in Combinatorics (J. H. van Lint)
- Introductory Combinatorics (R. A. Brualdi)

Pigeonhole Principle

If a set consisting of more than n objects is partitioned to n classes then some class receives more than 1 object.

Simplest Examples

1. 2 of 13 students should have birth days during the same month.
2. 12 pairs of socks of different colors in a bag. At most howmany have to be taken out so that we are sure to get a pair of the same color ?
3. 50,000 words of 4 or fewer letters. Can they all be distinct ?
4. 13 persons. Their first names are Seeta, Geeta and Radha. Second names are Ramana, Raju, Rao, Naidu. Can they all have different (full) names ?

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Some questions involving numbers

There N numbers. If we devide these numbers by $N-1$ then at least 2 of the numbers should give same remainder.

101 integers selected from [200]. Then we have selected 2 numbers a and b such that a devides b.

Any subset of size 6 from the set $\{1, 2, \ldots, 9\}$ must contain 2 elements whose sum is 10.

Let S be a set of six positive integers whose maximum is at most 14. Then the sums of the elements in all the non-empty subsets of S cannot be all distinct.

Let m be an odd positive integer. Then there exists a positive integer n such that m devides $2^n - 1$.

There is an element in the sequence $7, 77, 777, 7777, \ldots,$ that is devisible by 2003.

Ramu goes on a 4-week vacation. He takes with him 40 chocolates in a box. He eats at least 1 each day, starting from day 1. Prove that there exists a span of consecutive days during which he eats exactly 15 chocolates.

(Erdös-Szekeres) In a sequence of $n^2 + 1$ distinct real numbers, there is either an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $n + 1$.

(Erdös-Szekeres) In a sequence of $rs + 1$ distinct real numbers, there is either an increasing subsequence of length $r + 1$ or a decreasing subsequence of length $s + 1$.

Some questions involving graphs

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Pigeonhole Principle- part 2

Let S be a set of six positive integers whose maximum is at most 14. Then the sums of the elements in all the non-empty subsets of S cannot be all distinct.

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Combinatorics- Lecture: 2

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Some questions involving graphs

In any graph there exists at least 2 vertices of the same degree.

The Generalized Pigeonhole Principle

If a set consisting of more than nk objects is partitioned to n classes then some class receives more than k object.

In any graph G with n vertices, $n \leq \alpha(G).\chi(G)$.

Let G be an n vertex graph. If every vertex has a degree of at least $\frac{n-1}{2}$, then G is connected.

(Mantel's Theorem:) If a graph G on $2n$ vertices contains n^2+1 edges, then G contains a triangle.

(Erdös-Szekeres) In a sequence of $n^2 + 1$ distinct real numbers, there is either an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $n + 1$.

(Erdös-Szekeres) In a sequence of $rs + 1$ distinct real numbers, there is either an increasing subsequence of length $r + 1$ or a decreasing subsequence of length $s + 1$.

(Dilworth 1950) In any partial order on a set P of $n \geq sr + 1$ elements, there exists a chain of length $s + 1$ or an antichain of length $r + 1$.

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Pigeonhole Principle- part 3

Some questions involving graphs

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Some Problems from Geometry

If 5 points are selected from the interior of an equilateral triangle, then 2 among them are such that the distance between them is less than $\frac{1}{2}$.

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10 points are given within a square of unit size.

1. Then there are two of them that are closer to each other than 0.48.
2. There are 3 among them that can be covered by a disk of radius 0.5

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Pigeonhole Principle- part 4

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Elementary Concepts

(Addition Principle) If A and B are two disjoint finite sets, then $|A \cup B| = |A| + |B|$.
(Generalized Addition Principle) Let $A_1, A_2, \ldots, A_n$ be finite sets that are pairwise disjoint. Then,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

.

Subtraction Principle: Let A be a finite set and $B \subseteq A$. Then $|A - B| = |A| - |B|$.

Find the number of positive integers ≤ 1000 that have at least 2 different digits.

Product Principle: Let X and Y be two finite sets. Then the number of pairs (x, y) satisfying $x \in X$ and $y \in Y$ is $|X| \times |Y|$.

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Generalized Product Principle: Let $X_1, X_2, \ldots, X_k$ be finite sets. Then the number of k-tuples $(x_1, x_2, \ldots, x_k)$ satisfying $x_i \in X_i$ is $|X_1| \times |X_2| \times \cdots \times |X_k|$.

For any positive integer k, the number of k-digit positive integers is 9.10^{k-1}.

Howmany 4 digit positive integers both start and end in even numbers?

Suppose that a password contains only digits from 0 to 9. Also number of digits should be at least 4, and atmost 7. Howmany passwords can be formed ?

Suppose that a password contains 5 digits, does not start with 0 and contains the digit 8. Then howmany possibilities are there ?

For any positive integer n, the number of ways to arrange all elements of the set $[n]$ in a line is $n!$.

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A permutation of a finite set S is a list of the elements of S containing each element of S exactly once.

Let n and k be positive integers, where $n \geq k$. Then the number of ways to make a k-element list from $[n]$ without repeating any elements is $(n)_k = n(n-1)\cdots(n-k+1)$.

Let S and T be finite sets, and let d be a fixed positive integer. We say that the function $f : T \to S$ is d-to-one if for each element $s \in S$, there exists exactly d elements $t \in T$ such that $f(t) = s$.

Division Principle: Let S and T be finite sets so that a d-to-one function $f : T \to S$ exists. Then $|S| = \frac{|T|}{d}$.

The number of different seating arrangements for n people around a circular table is $(n-1)!$.

Let n be a positive integer, and let $k \leq n$ be a non-negative integer. Then the number of all k-element subsets of $[n]$ is $\frac{n(n-1)\cdots(n-k+1)}{k!}$

(Binomial Theorem): If n is a positive integer then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{(n-k)}$$

Bijective Proofs

Howmany subsets are there for an n element set ?

For any positive integer n the number of divisors n that are larger than $\sqrt{n}$ is equal to the number of devisor of n that are smaller than $\sqrt{n}$.

Working with North-Eastern paths

1. Number of paths from $(0,0)$ to $(k, n-k)$, take $n = 10, k = 4$: Then to $(6,4)$.
2. Number of paths from $(0,0)$ to $(6,4)$ if we want to visit $(4,2)$ on the way.
3. Number of paths from $(0,0)$ to $(6,4)$, such that we either visit $(3,2)$ or $(2,3)$.

The number of north eastern Lattice paths from $(0,0)$ to (n,n) that never go above the diagonal $x = y$ (the main diagonal) is equal to the number of ways to fill a $2 \times n$ grid with the elements of $[2n]$ using each element once so that each row and column is increasing (to the right and down).
(Such a $2 \times n$ rectangle containing the elements of $[2n]$ so that each element is used once and each row and column is increasing (to the right and down) is called a Standard Young Tableau.

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Combinatorics- Lecture: 5

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Elementary Concepts: Part (2)

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Properties of Binomial Coefficients

Let n and k be non-negative integers so that $k \leq n$. Then $\binom{n}{k} = \binom{n}{n-k}$.

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

The Pascal Triangle

For all integers n,

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Let n be a positive integer. Then,

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$$

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Permutations with Repition

Assume we want to arrange n objects in a line, the n objects are of k different types, and objects of the same type are indistinguishable. Let a_i be the number of objects of type i. Then the number of different arrangements is:

$$\frac{n!}{a_1!a_2!\dots a_k!}$$

A quality controller has to visit one factory a day. In the next 8 days, she will visit each of 4 factories, A, B, C, and D, twice. The controller is free to choose the order in which she visits these factories, but the two visits to factory A cannot be on consecutive days. In howmay diffent orders can the controller proceed ?

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Example of a Bijective Proof

The number of north eastern Lattice paths from $(0,0)$ to (n,n) that never go above the diagonal $x = y$ (the main diagonal) is equal to the number of ways to fill a $2 \times n$ grid with the elements of $[2n]$ using each element once so that each row and column is increasing (to the right and down).
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Permutations with Repition
(Permutations of Multisets)

Let S be a multiset with objects of k different types with finite repetition numbers $a_1, \ldots, a_k$ respectively. Let $|S| = n = \sum_{i=1}^{k} a_i$. Then number of permutations of S equals

$$\frac{n!}{a_1!a_2!\ldots a_k!}$$

Assume we want to arrange n objects in a line, the n objects are of k different types, and objects of the same type are indistinguishable. Let a_i be the number of objects of type i. Then the number of different arrangements is:

$$\frac{n!}{a_1!a_2!\dots a_k!}$$

If S is a multiset with objects of k different types, where each has an infinite repition number. Then the number of r permutations of S is:
(Example: What is the number of ternary numerals with at most 4 digits? Set S her is $\{\infty.0, \infty.1, \infty.2\}$.)

A quality controller has to visit one factory a day. In the next 8 days, she will visit each of 4 factories, A, B, C, and D, twice. The controller is free to choose the order in which she visits these factories, but the two visits to factory A cannot be on consecutive days. In howmay diffent orders can the controller proceed ?

Combinatorics- Lecture: 8

The number of permutations of the letters in the word MISSISSIPPI:

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by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

For all integers n,

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Another view:
Let n be a positive integer and let $n_1, n_2, \ldots, n_k$ be positive integers with $n = n_1 + n_2 + \ldots + n_k$. The number of ways to partition a set of n objects into k labelled boxes $B_1, B_2, \ldots, B_k$ in which B_i contains n_i objects equals:

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

If the boxes are not labelled and $n_1 = n_2 = \cdots = n_k$, then the number of partitions equals

$$\frac{n!}{k!n_1!n_2!\dots n_k!}$$

Combinatorics- Lecture: 9

1. Howmany possibilities are there for 8 non-attacking rooks on an 8×8 chess board ?
2. If all the rooks are colored differently ?
3. If there are 1 red rook, 2 blue rooks and 4 yellow rooks ?

There are n rooks to be placed in a non-attacking configuration on an $n \times n$ chess board. $n = \sum_{i=1}^{k} n_i$ and there are n_i rooks of color C_i. The number of possible configurations are

$$\frac{(n!)^2}{n_1!n_2!\dots n_k!}$$

Binomial Coefficient vs Multinomial coeffient

For positive integers n, t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \dots + x_t)^n$ is

$$\frac{n!}{n_1! n_2! \dots n_t!}$$

What is the coefficient of $x^2y^2z^3$ in the expansion of $(x + y + z)^7$?
What is the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$?

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by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Combinatorics- Lecture: 10

For all integers n,

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Assume we want to arrange n objects in a line, the n objects are of k different types, and objects of the same type are indistinguishable. Let a_i be the number of objects of type i. Then the number of different arrangements is:

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Combinatorics- Lecture: 10

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What is the coefficient of $a^2 b^3 c^2 d^5$ in the expansion of $(a + 2b - 3c + 2d + 5)^{16}$?

Combinations with Repititions

Combinatorics- Lecture: 11

7 boys go to a shop to buy pens. In the shop there are 4 kinds of pens: red, blue, green and black. If each boy buys one pen, how many different purchases are possible from the shop owner's view point ?

Number of n objects taken r at a time, with repitition, is $\binom{n+r-1}{r}$.

A sweet shop offers 20 different kinds of sweets. Assuming that there are at least a dozen of each kind when we enter the shop, in howmany ways we can select a dozen sweets ?

10 RS should be distributed among 4 boys: A, B,C and D. In howmany ways we can do it ?
(a) Now, if each boy has to get at least one Rupee ?
(b) If A has to get at least 5 Rupees and each has to get at least 1 Rupee ?

In howmany ways can we distribute r identical balls among n distinct (or labelled) boxes ?

In howmany ways we can distribute 7 bananas and 6 oranges among 4 children so that each child receives at least one banana ?

A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 blank spaces between the symbols, with at least 3 symbols between each pair of consecutive symbols. In howmany ways can the transmitter send such a message ?

Determine all integer solutions to the equation

$$x_1 + x_2 + \ldots + x_4 = 7$$

where $x_i \geq 0$, for all $1 \leq i \leq 4$.

The following are equivalent:
(1) The number of integer solutions of the equation $x_1 + x_2 + \ldots + x_n = r$, where $x_i \geq 0, 1 \leq i \leq n$.
(2) The number of selections with repitition, of size r from a collection of size n.
(3) The number of ways r identical objects can be distributed among n distinct containers.

Howmany non-negative integer solutions are there to the inequality

$$x_1 + x_2 + \cdots + x_6 < 10$$

(The technique: Introduce a 7th variable, x_7).

Howmany terms are there in the expansion of $(w + x + y + z)^{10}$?

Different ways in which a positive integer n can be written as a sum of positive integers where the order of the summand is considered relevant. (There representations are called compositions).

The number of composition of $2n$ where each summand is even.

for $i = 1$ to 20 do,
for $j = 1$ to i do,
for $k = 1$ to j do,
print (something)
Howmany times will the print statement gets executed ?

A combinatorial proof to show $\sum_{i=1}^{n} = \frac{n(n+1)}{2}$.

The counter in a bar has to be 15 bar stools. In howmany ways can the stool be occupied if there has to 5 empty stools, 10 occupied stools and total 7 *runs*.
Example: OO E OOOO EEE OOO E O

COMBINATORICS- Lecture: 12

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Binomial Coefficient vs Multinomial coeffient

For positive integers n, t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + \ldots + x_t)^n$ is

$$\frac{n!}{n_1! n_2! \ldots n_t!}$$

What is the coefficient of $x^2y^2z^3$ in the expansion of $(x+y+z)^7$?
What is the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$?

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COMBINATORICS- Lecture: 13

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How big is $\binom{n}{r}$?

Combinatorics- Lecture: 13

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

Sterling's formula for factorial:

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

(From William Feller, Vol 1: An Introduction to Probability Theory and its applications)

$$\begin{aligned} 1! &\approx 0.9221(percentageerr : 8) \\ 2! &\approx 1.919(percentageerr : 4) \\ 5!(=120) &\approx 118.019(percentageerr : 2) \\ 10!(=3,628,800) &\approx 3,598,600(percenterr : 0.8) \end{aligned}$$

For 100! percentage error is 0.08

Sterling formular for factorial:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{\alpha_n}$$

where $\frac{1}{12n+1} < \alpha_n < \frac{1}{12n}$.

More general treatment of $\binom{n}{r}$

Falling factorial and rising factorial.

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots 1} \text{ for integer } k \geq 0$$

$$\binom{r}{k} = 0, \text{ for integer } k < 0$$

Some of the earlier identities we studied may not be valid in general:
Example: The symmetry identitity.

The absorption identity:

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}, \text{ for integer } k \neq 0$$

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Addition formula:

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$$

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Binomial Theorem:

$$(x+y)^r = \sum_k \binom{r}{k} x^k y^{r-k}$$

for integer $r \geq 0$ or when $|\frac{x}{y}| < 1$.

When r is not a non-negative integer, we often use the binomial theorem in the special case $y = 1$.

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(The general formula follows from this if we set $z = x/y$ and mutiply both sides by y^r)

Extending the Pascal's triangle for negative n.

(Negating the upper index)

$$\binom{r}{k} = (-1)^k \binom{k-r-1}{k}$$

More Techniques
Double Counting

Combinatorics- Lecture: 13

At a party, the number of guests who shake hands an odd number of times is even.

In every graph the sum of degrees of its vertices is two times the number of its edges, and hence is even.

Let $\mathcal{F}$ be a family of subsets of a set X. Then, $\sum_{x\in X} d(x) = \sum_{A\in\mathcal{F}} |A|$.

Turan's number $T(n,k,l), (l \leq k \leq n)$ is the smallest number of l element subsets of an n-element set X such that every k element subset of X contains at least one of these sets.
For all positive integers, $l \leq k \leq n$, $T(n,k,l) \geq \binom{n}{l} / \binom{k}{l}$

Combinatorics- Lecture: 13

Hall's Theorem for k-regular bipartite graphs.

Inclusion Exclusion Principle

COMBINATORICS- Lecture: 14

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

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Combinatorics- Lecture: 14

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More Techniques

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Inclusion Exclusion Principle

COMBINATORICS- Lecture: 15

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Combinatorics- Lecture: 15

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Hall's Theorem for k-regular bipartite graphs.

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COMBINATORICS- Lecture: 16

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The case of 2 sets: $|\overline{A} \cap \overline{B}|$?
(1) Proof using Venn diagram
(2) Another proof.

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General case: Let $A_1, A_2, \ldots, A_k \subseteq U$. Then $|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}| = |U| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| \cdots + (-1)^k |A_1 \cap A_2 \cap \cdots \cap A_k|$, where the first sum is over all 1-combinations of $[k]$, and the second sum is over all 2-combinations of $[k]$, and so on.

$|A_1 \cup A_2 \cup \cdots \cup A_k| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \cdots + (-1)^{k+1} |A_1 \cap A_2 \cap \cdots \cap A_k|$,

Find the number of integers between 1 and 1000, inclusive that are not divisible by $5, 6$ and 8.

Howmany permutations of the letters M,A,T,H,I,S,F,U,N are there such that none of the words MATH, IS and FUN occur as consecutive letters ?

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Combinatorics- Lecture: 16

How many integers between 0 and $99,999$ (inclusive) have among their digits each of $2, 5$ and 8.

Determine the number of 10-combinations of the multiset $T = \{3.a, 4.b, 5.c\}$.

Combinatorics- Lecture: 16

The number of r-combinations of the multi-set $\{n_1.a_1, n_2.a-2, \dots, n_k.a_k\}$ equals the number of integral solutions of the equation $x_1 + x_2 + \cdots + x_k = r$, that satisfy $0 \leq x_i \leq n_i$ for $i = 1, 2, \dots, k$.

What is the number of integral solutions of the equation $x_1 + x_2 + x_3 + x_4 = 18$ that satisfy $1 \leq x_1 \leq 5; -2 \leq x_2 \leq 4; 0 \leq x_3 \leq 5; 3 \leq x_4 \leq 9$.

COMBINATORICS- Lecture: 17

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

(Negating the upper index)

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More Techniques
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Hall's Theorem for k-regular bipartite graphs.

Combinatorics- Lecture: 17

Inclusion Exclusion Principle

The case of 2 sets: $|\overline{A} \cap \overline{B}|$?
(1) Proof using Venn diagram
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COMBINATORICS- Lecture: 18

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Double Counting technique

Hall's Theorem for k-regular bipartite graphs.

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The number of onto functions from an m-element set to an n element set $(m \geq n)$:
$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^n(n-n)^m = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^m = \sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^m$
Case when $n = m$ and $m < n$.

Euler's ϕ function: $\phi(n)$.
If $n = p_1^{e_1} p_2^{e_2} \dots p_t^{e_t}$, then
$\phi(n) = n \prod_{i=1}^{t} (1 - 1/p_i)$

6 married couples are to be seated at a circular table. In howmany ways can they arrange themselves to that no wife sits next to her husband ?

COMBINATORICS- Lecture: 19

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Combinatorics- Lecture: 19

Double Counting technique
Hall's Theorem for k-regular bipartite graphs.

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Combinatorics- Lecture: 19

Let α_i be the cardinality of the intersection of any collection of i sets from $A_1, A_2, \cdots, A_k$. Let $|U| = \alpha_0$. Then $|A_1 \cup A_2 \cup \cdots \cup A_k| = \alpha_0 - \binom{k}{1}\alpha_1 + \cdots + (-1)^i \binom{k}{i}\alpha_i + \cdots + (-1)^k \alpha_k$.

How many integers between 0 and $99,999$ (inclusive) have among their digits each of $2, 5$ and 8.

Determine the number of 10-combinations of the multiset $T = \{3.a, 4.b, 5.c\}$.

The number of r-combinations of the multi-set $\{n_1.a_1, n_2.a-2, \ldots, n_k.a_k\}$ equals the number of integral solutions of the equation $x_1 + x_2 + \cdots + x_k = r$, that satisfy $0 \leq x_i \leq n_i$ for $i = 1, 2, \ldots, k$.

What is the number of integral solutions of the equation $x_1 + x_2 + x_3 + x_4 = 18$ that satisfy
$1 \leq x_1 \leq 5; -2 \leq x_2 \leq 4; 0 \leq x_3 \leq 5; 3 \leq x_4 \leq 9.$

The number of onto functions from an m-element set to an n element set ($m \geq n$):
$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^n(n-n)^m =$
$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^m = \sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^m$
Case when $n = m$ and $m < n$.

Euler's ϕ function: $\phi(n)$.
If $n = p_1^{e_1}p_2^{e_2}\dots p_t^{e_t}$, then
$\phi(n) = n\prod_{i=1}^{t}(1 - 1/p_i)$

6 married couples are to be seated at a circular table. In howmany ways can they arrange themselves to that no wife sits next to her husband ?

COMBINATORICS- Lecture: 20

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

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Combinatorics- Lecture: 20

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In a certain area of the country side are 5 villages. An engineer is to devise a system of 2 way roads so that after the system is completed no village will be isolated. In how many ways can he do this ?

Derangements:
$D_n = n![1 - 1/1! + 1/2! - 1/3! + \cdots + (-1)^n 1/n!]$, for $n > 1$.
($e^x = 1 + x + x^2/2! + x^3/3! + \cdots$). So, $D_n \approx n!/e$,for large enough n.

COMBINATORICS- Lecture: 21

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Let α_i be the cardinality of the intersection of any collection of i sets from $A_1, A_2, \cdots, A_k$. Let $|U| = \alpha_0$. Then $|A_1 \cup A_2 \cup \cdots \cup A_k| = \alpha_0 - \binom{k}{1}\alpha_1 + \cdots + (-1)^i \binom{k}{i}\alpha_i + \cdots + (-1)^k \alpha_k$.

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Combinatorics- Lecture: 21

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COMBINATORICS- Lecture: 22

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

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Combinatorics- Lecture: 22

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Recurrence Relations

Fibonacci Numbers: $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, $f_1 = 1, f_0 = 0$.
The problem of Leonardo of Pisa.

The partial sum, $S_n = F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

F_n is even if and only if n is a multiple of 3.

The fibonacci numbers satisfy the formula:

$$F_n = \frac{1}{\sqrt{5}}\frac{1+\sqrt{5}}{2}^n - \frac{1}{\sqrt{5}}\frac{1-\sqrt{5}}{2}^n$$

for $n \geq 0$.

Changing the initial conditions to $f_0 = a$ and $f_1 = b$.

Determine the number of ways to perfectly cover a 2 by n board with dominoes.

Determine the number of ways to perfectly cover a 1 by n board with monominoes and dominoes.

$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \cdots + \binom{n-k}{k-1}$, where $k = \lfloor \frac{n+1}{2} \rfloor$.
In other words, the sequence $g_n = \sum_{k=0}^{n-1} \binom{n-1-k}{k}$ is the same as F_n.

COMBINATORICS- Lecture: 22

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

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Combinatorics- Lecture: 22

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COMBINATORICS- Lecture: 24

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Determine the number of ways to perfectly cover a 2 by n board with dominoes.

Combinatorics- Lecture: 24

Determine the number of ways to perfectly cover a 1 by n board with monominoes and dominoes.

For Let $S_0 = \emptyset$ and for $n > 0$, $S_n = [n]$. Let a_n denote the number of subsets that contain no consecutive integers. Find and solve a recurrence relation for a_n.

$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \cdots + \binom{n-k}{k-1}$, where $k = \left\lfloor \frac{n+1}{2} \right\rfloor$.
In other words, the sequence $g_n = \sum_{k=0}^{n-1} \binom{n-1-k}{k}$ is the same as F_n.

Linear Homogeneous Recurrence Relations

The sequence $h_0, h_1, \ldots, h_n, \ldots,$ is said to satisfy a linear recurrence relation of order k provided that there exists quantitites $a_1, a_2, \ldots, a_k$ with $a_k \neq 0$ and a quantity b_n such that $h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b_n$, for $n \geq k$.
(Here a_i and b_n may depend on n.)
When $b_n = 0$, it is called homogeneous.
When each a_i is constant, then it is said to have constant coefficients.

Let q be a non-zero number. Then $h_n = q^n$ is a solution of the linear homogeneous recurrence relation
$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \cdots - a_k h_{n-k} =,$
$a_k \neq 0, n \geq k$, with constant coefficients if and only if q is a root of the polynomial equation
$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k = 0$
If the polynomial equation has k distict roots $q_1, q_2, \cdots, q_k$ the $h_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n$ is the general solution in the following sense: No matter what initial values for $h_0, h_1, \cdots, h_{k-1}$ are given, there are constants $c_1, c_2, \ldots, c_k$ so that the above is the unique sequence that satisfies both the recurrence relation and the initial conditions.

COMBINATORICS- Lecture: 25

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

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Solve the recurrence relation:

$$h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}$$

for $n \geq 3$, and $h_0 = 1, h_1 = 2, h_2 = 0$.

Words of length n, using only the three letters a, b, c are to formed, subject to the condition that no word in which two a's appear consecutively is allowed. Howmany such words can be formed?

If the roots $q_1, q_2, \ldots, q_k$ of the characteristic equation are not distinct, then

$$h_n = c_1 q_1^n + \ldots + c_k q_k^n$$

is not a general solution of the equation.
Example: $h_n = 4h_{n-1} - 4h_{n-2}$ $(n \geq 2)$

If a (possibly complex) number q is a root of multiplicity s of the characteristic equation of a linear homogeneous recurrence relation with constant coefficients, then it can be shown that each of $h_n = q^n, h_n = nq^n, h_n = n^2q^n, \dots, h_n = n^{s-1}q^n$ is a solution and hence

$$h_n = c_1q^n + c_2nq^n + c_3n^2q^n + \cdots + c_sn^{s-1}q^n$$

for each choice of constants $c_1, c_2, \dots, c_s$.

Let $q_1, q_2, \dots, q_t$ be the distinct roots of the following characteristic equation of the linear homogeneous recurrence relation with constant coefficients:

$$h_n = a_1h_{n-1} + a_2h_{n-2} + \cdots + a_kh_{n-k}$$

where $a_k \neq 0, n \geq k$. Then if q_i is an s_i-fold root of the characteristic equation of the above recurrence relation, the part of the general solution of this recurrence relation corresponding to q_i is:

$$H_n^{(i)} = c_1q_i^n + c_2nq_i^n + c_3n^2q_i^n + \cdots + c_sn^{s-1}q_i^n$$

and the general solution of the recurrence relation is:

$$h_n = H_n^{(1)} + \cdots + H_n^{(t)}$$

Solve: $h_n = -h_{n-1} + 3h_{n-2} + 5h_{n-3} + 2h_{n-4}, n \geq 4$ subject to: $h_0 = 1, h_1 = 0, h_2 = 1, h_3 = 2.$

Non-Homogeneous Recurrence Relations

Towers of Hanoi Puzzle:

Combinatorics- Lecture: 25

Solve $h_n = 3h_{n-1} - 4n, (n \geq 1), h_0 = 2.$

Step (1) Find the general solution of the corresponding homogeneous relation.
Step (2) Find a particular solution of the non-homogeneous relation.
Step (3) Combine the general solution and the particular solution, and determine values of the constants arising in the general solution so that the combined solution satisfies the initial conditions.

If b_n is a polynomial of degree k in n, then look for a particular solution h_n that is also a polynomial of degree k in n. Try,

1. $h_n = r$ (a constant) if $b_n = d$ (a constant)
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3. $h_n = rn^2 + sn + t$ if $b_n = fn^2 + dn + e$

If b_n is an exponential, then look for a particular solution that is also an exponential: Try $h_n = pd^n$, if $b_n = d^n$.

Solve: $h_n = 2h_{n-1} + 3^n, (n \geq 1), h_0 = 2$

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COMBINATORICS- Lecture: 26

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Linear Homogeneous Recurrence Relations

Combinatorics- Lecture: 26

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Combinatorics- Lecture: 26

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$$h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}$$

for $n \geq 3$, and $h_0 = 1, h_1 = 2, h_2 = 0$.

Words of length n, using only the three letters a, b, c are to formed, subject to the condition that no word in which two a's appear consecutively is allowed. Howmany such words can be formed?

Combinatorics- Lecture: 26

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Combinatorics- Lecture: 26

Non-Homogeneous Recurrence Relations

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Combinatorics- Lecture: 26

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COMBINATORICS- Lecture: 27

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Non-Homogeneous Recurrence Relations

Combinatorics- Lecture: 27

Towers of Hanoi Puzzle: $h_n = 2h_{n-1} + 1$

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Generating Functions

Let $h_0, h_1, h_2, \ldots, h_n, \ldots,$ be an infinite sequence of numbers. Its generating function is defined to be the infinite series

$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots$$

Let m be a positive integer: The generating function for the binomial coefficients: $\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}$, is $(1+x)^m$

The generating function of the infinite sequence $1, 1, 1, 1, \ldots,$ is $\frac{1}{1-x}$

$$\frac{1-x^{n+1}}{1-x} = 1 + x + \cdots + x^n$$

$\frac{x}{(1-x)^2} = 0 + x + 2x^2 + 3x^3 + \cdots$

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$\frac{1}{1-2x} = 1 + 2x + 2^2x^2 + 2^3x^3 + \cdots$

Let α be a real number. The generating function for the infinite sequence of binomial coefficients

$$\binom{\alpha}{0}, \binom{\alpha}{1}, \ldots, \binom{\alpha}{n}, \ldots,$$

is $(1+x)^{\alpha}$

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Some basic facts to remember:
$(1-rx)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k}(-rx)^k$
(for $|x| < \frac{1}{|r|}$)
$(1-rx)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} r^k x^k$
(for $|x| < \frac{1}{|r|}$)

Let k be an integer, and let the sequence $h_0, h_1, \ldots, h_n, \ldots,$ be defined by letting h_n equal the number of non-negative integral solutions of $e_1 + e_2 + \cdots + e_k = n$.
The generating function for this sequence is $\frac{1}{(1-x)^k}$

In the above sequence let h_n be the number of integer solutions of $e_1 + e_2 + e_3 = n$, where $0 \leq e_1 \leq 5, 0 \leq e_2 \leq 2, 0 \leq e_3 \leq 4$. Then what is the generating function for this sequence ?

Determine the generating function for the number of n-combinations of apples, bananas, oranges and pears where in each n-combination, the number of apples is even, the number of bananas is odd, the number of oranges is between 0 and 4, and there is at least one pear.

Find the number h_n of bags of fruit that can be made out of apples, bananas, oranges and pears where in each bag the number of apples is even, the number of bananas is a multiple of 5, the number of oranges is at most 4, and the number of pears is 0 or 1.

Determine the generating function for the number h_n of solutions of the equation $e_1 + e_2 + \cdots + e_k = n$, in non-negative odd integers $e_1, e_2, \ldots, e_k$.

Combinatorics- Lecture: 27

Let h_n denote the number of non-negative integral solutions of the equation $3e_1 + 4e_2 + 2e_3 + 5e_4 = n$. Find the generating function for this sequence.

COMBINATORICS- Lecture: 28

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Combinatorics- Lecture: 28

$$\frac{1}{1-2x} = 1 + 2x + 2^2x^2 + 2^3x^3 + \cdots$$

Let α be a real number. The generating function for the infinite sequence of binomial coefficients

$$\binom{\alpha}{0}, \binom{\alpha}{1}, \ldots, \binom{\alpha}{n}, \ldots,$$

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Determine the sequence generated by $(1-4x)^{\frac{-1}{2}}$

Some basic facts to remember:
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Combinatorics- Lecture: 28

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Combinatorics- Lecture: 28

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Solving Recurrence Relations using Generating Functions

Solve the recurrence relation $h_n = 5h_{n-1} - 6h_{n-2}, (n \geq 2)$, subject to the initial values $h_0 = 1$ and $h_1 = -2$.

Combinatorics- Lecture: 28

Generalising the method to solve any linear homogenious recurrence relation of order k, with constant coefficients:
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Example: $h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0$, for $n \geq 3$, where $h_0 = 0, h_1 = 1, h_2 = -1$. Find a general form for h_n.

There is a relation between the characteristic equation $0 = r(x) = x^k + a_1 x^{k-1} + \cdots + a_k$ and $q(x)$.

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Combinatorics- Lecture: 28

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COMBINATORICS- Lecture: 29

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Combinatorics- Lecture: 29

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Combinatorics- Lecture: 29

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Combinatorics- Lecture: 29

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COMBINATORICS- Lecture: 30

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

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Let $n \in N$. For $r \geq 0$, let $a(n,r) =$ the number of ways we can select, with repititions allowed, r objects from a set of n distinct objects. Then
$a(n,r) = a(n-1,r) + a(n,r-1)$
$(a(n,0) = 1$ for $n \geq 0$ and $a(0,r) = 0$ for $r > 0)$
Here we define the generating function $f_n(x) = \sum_{r=0}^{\infty} a(n,r)x^r$
In particular $f_0(x) = 1$.

Finding the generating function for $\binom{n}{r}$ starting with the recurrence relation $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, for $r \geq 1$.
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Combinatorics- Lecture: 30

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Partition Numbers

Partition of a positive integer n is a representation of n as an unordered sum of one or more positive integers, called parts.

$$\begin{array}{lcl} 1 & \rightarrow & 1 \\ 2 & \rightarrow & 2; 1+1 \\ 3 & \rightarrow & 3; 2+1; 1+1+1 \\ 4 & \rightarrow & 4; 3+1; 2+2; 2+1+1; 1+1+1+1 \\ 5 & \rightarrow & 5; 4+1; 3+2; 3+1+1; 2+2+1; 2+1+1+1; 1+1+1+1 \end{array}$$

Combinatorics- Lecture: 30

Let p_n denote the number of different partitions of the positive integer n. For convinience let $p_0 = 1$.
p_n equals the number of solutions in non-negative integers $a_n, \ldots, a_2, a_1$, of the equation $na_n + \cdots + 2a_2 + a_1 = n$.

Ferrer's Diagram.

The number of partitions of an integer into m summands is equal to the number of partitions of n into summands where m is the largest summand.

The generating function for the sequence $p(0), p(1), p(2), \ldots$:

$$\pi_{i=1}^{\infty} \frac{1}{(1-x^i)}$$

Find the generating function for $p_d(n)$ the number of partitions of a positive integer n into distinct summands. (Take $p_d(0) = 1$).

Find the generating function for $p_o(n)$, the number of partitions of integer n into 'odd' summands, for $n \geq 1$. (Take $p_o(0) = 1$.)

$p_d(n) = p_o(n)$, for all $n \geq 0$

COMBINATORICS- Lecture: 31

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Combinatorics- Lecture: 31

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Combinatorics- Lecture: 31

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Combinatorics- Lecture: 31

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COMBINATORICS- Lecture: 32

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$p_k(n) = \sum_{s=1}^{k} p_s(n-k)$

$p_k(n) = p_{k-1}(n-1) + p_k(n-k)$
(Note: We have $p_k(n) = 0$ for $n < k$ and $p_k(k) = 1$. Also, $p_1(n) = 1$. What is $p_2(n)$?

$$\frac{1}{k!}\binom{n-1}{k-1} \leq p_k(n) \leq \frac{1}{k!}\binom{n+\frac{k(k-1)}{2}-1}{k-1}$$

Combinatorics- Lecture: 33

If k is fixed, then $p_k(n) \approx \frac{n^{k-1}}{k!(k-1)!}$, as $(n \to \infty)$.

Find the generating function for $p_D(n)$, the number of partitions of a positive integer n into distinct summands. (Take $p_D(0) = 1$).

Find the generating function for $p_o(n)$, the number of partitions of integer n into 'odd' summands, for $n \geq 1$. (Take $p_o(0) = 1$.)

$p_d(n) = p_o(n)$, for all $n \geq 0$

Ferrer's Diagram.

The number of partitions of an integer into m summands is equal to the number of partitions of n into summands where m is the largest summand.

The number of partitions of $n+k$ into k parts equals the number of partitions of n into at most k parts.

Number of partitions of n into an even number of unequal parts = Number of partitions of n into an odd number of unequal parts, unless $n \in \{\omega(m), \omega(-m) : \text{ integer } m \geq 1\}$, where $\omega(m) = \frac{3m^2 - m}{2}$ and $\omega(-m) = \frac{3m^2+m}{2}$
(Proof using Ferrer's diagram by Franklin (1881)).

Consequence: (Euler's Identity):

$$\prod_{k=1}^{\infty}(1 - x^k) = 1 + \sum_{m=1}^{\infty}(-1)^m(x^{\omega(m)} + x^{-\omega(m)})$$

Consequence:

$$p(n) = \sum_{m=1}^{\infty} (-1)^{m+1} [p(n - \omega(m)) + p(n - \omega(-m))]$$

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by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Number of partitions of n into an even number of unequal parts $=$ Number of partitions of n into an odd number of unequal parts, unless $n \in \{\omega(m), \omega(-m) : \text{ integer } m \geq 1\}$, where $\omega(m) = \frac{3m^2-m}{2}$ and $\omega(-m) = \frac{3m^2+m}{2}$
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Catalan Numbers

Counting the number of tree diagrams for rooted ordered binary trees with n vertices. Let this number be b_n

(Example: Three are 5 possible diagrams for $n = 3$ vertices, i.e. $b_3 = 5$). (Also, $b_0 = 1$, $b_1 = 1$, $b_2 = 2$).

The recurrence relation for the above problem:
$b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_n b_0$
The generating function for this sequence:
$f(x) = \frac{1-\sqrt{1-4x}}{2x}$
The nth Catalan number: $\frac{1}{n+1}\binom{2n}{n}$

COMBINATORICS- Lecture: 36

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Number of partitions of n into an even number of unequal parts = Number of partitions of n into an odd number of unequal parts, unless $n \in \{\omega(m), \omega(-m) : \text{ integer } m \geq 1\}$, where $\omega(m) = \frac{3m^2-m}{2}$ and $\omega(-m) = \frac{3m^2+m}{2}$
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Consequence: (Euler's Identity):

$$\prod_{k=1}^{\infty}(1-x^k) = 1 + \sum_{m=1}^{\infty}(-1)^m(x^{\omega(m)} + x^{-\omega(m)})$$

Consequence:

$$p(n) = \sum_{m=1}^{\infty} (-1)^{m+1} [p(n - \omega(m)) + p(n - \omega(-m))]$$

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The nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$

$C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14$ etc.

The number of sequences $a_1, a_2, \ldots, a_{2n}$ of $2n$ terms that can be formed using n, $+1$'s and n, -1's whose partial sums satisfy $a_1 + a_2 + \cdots + a_k \geq 0$, for $k = 1, 2, \ldots, 2n$ equals the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, $(n \geq 0)$.
(A combinatorial proof).

Combinatorics- Lecture: 36

There are $2n$ people in line to get into a theatre. The ticket charge is 50 RS. Of the $2n$ people n of them have a 50 RS note and n have a 100 RS note. If the ticket counter starts with no cash, then in how many ways can the people line up so that whenever a person with 100 RS buys a ticket, the ticket counter has a 50 RS note to make change ?

A big city lawer works n blocks north and n blocks east of her place of residence. Every day she walks $2n$ blocks to work. How many routes are possible if she never crosses, but may touch, the diagonal line from home to office.

n numbers are listed in the order $a_1, a_2, \ldots, a_n$. Howmany multiplication schemes are possible to get the product of these n numbers, if you have to keep this order in mind when multiplying.

The number of ways to divide a convex polygonal region with $n+1$ sides into triangular regions by inserting diagonals that do not intersect in the interior.

Difference Sequences...

Let the general term of a sequence be a polynomial of degree p in n, i.e.,
$h_n = a_p n^p + a_{p-1} n^{p-1} + \cdots + a_1 n + a_0, (n \geq 0)$
Then $\Delta^{p+1} h_n = 0$, for all $n \geq 0$.

Let g_n and f_n be general terms of two sequences. Let c, d be constants:
Then $\Delta^p (cg_n + df_n) = c\Delta^p g_n + d\Delta^p f_n$, $p \geq 0$.

The difference table is completely determined by its entries along the left edge, i.e. the 0th diagonal: $h_0 = \Delta^0 h_0, \Delta^1 h_0, \Delta^2 h_0, \ldots$.

Suppose the 0th diagonal of the difference table reads as follows: $0, 0, 0, \ldots, 0, 1, \ldots, 0, 0, \ldots$ where the single 1 appears at the pth position. Then the general term of the corresponding sequence is given by $h_n = \binom{n}{p}$.

If the 0the diagonal of the difference table is given by $c_0, c_1, c_2, \ldots, c_p, 0, 0, \ldots$, where $c_p \neq 0$. Then the general term of the corresponding sequence is a polynomial in n of degree p satisfying:
$h_n = c_0\binom{n}{0} + c_1\binom{n}{1} + \cdots + c_p\binom{n}{p}$

COMBINATORICS- Lecture: 37

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

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$$\prod_{k=1}^{\infty}(1-x^k) = 1 + \sum_{m=1}^{\infty}(-1)^m(x^{\omega(m)} + x^{-\omega(m)})$$

Consequence:

$$p(n) = \sum_{m=1}^{\infty} (-1)^{m+1} [p(n - \omega(m)) + p(n - \omega(-m))]$$

Combinatorics- Lecture: 37

Catalan Numbers

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The nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$

Combinatorics- Lecture: 37

$C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14$ etc.

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Combinatorics- Lecture: 37

n numbers are listed in the order $a_1, a_2, \ldots, a_n$. Howmany multiplication schemes are possible to get the product of these n numbers, if you have to keep this order in mind when multiplying.

The number of ways to divide a convex polygonal region with $n+1$ sides into triangular regions by inserting diagonals that do not intersect in the interior.

Difference Sequences...

Let the general term of a sequence be a polynomial of degree p in n, i.e.,
$h_n = a_p n^p + a_{p-1} n^{p-1} + \cdots + a_1 n + a_0, (n \geq 0)$
Then $\Delta^{p+1} h_n = 0$, for all $n \geq 0$.

Let g_n and f_n be general terms of two sequences. Let c, d be constants:
Then $\Delta^p(cg_n + df_n) = c\Delta^p g_n + d\Delta^p f_n$, $p \geq 0$.

The difference table is completely determined by its entries along the left edge, i.e. the 0th diagonal: $h_0 = \Delta^0 h_0, \Delta^1 h_0, \Delta^2 h_0, \ldots$.

Suppose the 0th diagonal of the difference table reads as follows: $0, 0, 0, \ldots, 0, 1, \ldots, 0, 0, \ldots$ where the single 1 appears at the pth position. Then the general term of the corresponding sequence is given by $h_n = \binom{n}{p}$.

If the 0the diagonal of the difference table is given by $c_0, c_1, c_2, \ldots, c_p, 0, 0, \ldots$, where $c_p \neq 0$. Then the general term of the corresponding sequence is a polynomial in n of degree p satisfying:
$h_n = c_0\binom{n}{0} + c_1\binom{n}{1} + \cdots + c_p\binom{n}{p}$

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by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

$n+1$ numbers are listed in the order $a_1, a_2, \ldots, a_n, a_{n+1}$. Howmany multiplication schemes are possible to get the product of these $n+1$ numbers, if you have to keep this order in mind when multiplying.

Combinatorics- Lecture: 38

The number of ways to divide a convex polygonal region with $n+2$ sides into triangular regions by inserting diagonals that do not intersect in the interior.

Sterling Numbers of the second kind
$S(p, k)$ counts the number of partitions of a set of p elements into k indistinguishable boxes in which no box is empty.

Combinatorics- Lecture: 38

Note the connection between $S(p, k)$ and the number of onto functions from a set of p elements to a set of k elements. For $0 \leq k \leq p$,

$$S(p, k) = \frac{1}{k!} \sum_{t=0}^{k} (-1)^t \binom{k}{t} (k-t)^p$$

$S(p, p) = 1 \quad (p \geq 0)$
$S(p, 0) = 0 \quad (p \geq 1)$
$S(p, 1) = 1, \quad (p \geq 1)$
$S(p, 2) = 2^{p-1} - 1, \quad (p \geq 2)$
$S(p, p-1) = \binom{p}{2}, \quad (p \geq 1)$

If $1 \leq k \leq p - 1$, then
$S(p, k) = kS(p-1, k) + S(p-1, k-1)$

Difference Sequences...

Let the general term of a sequence be a polynomial of degree p in n, i.e.,
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Suppose the 0th diagonal of the difference table reads as follows: $0, 0, 0, \ldots, 0, 1, \ldots, 0, 0, \ldots$ where the single 1 appears at the pth position. Then the general term of the corresponding sequence is given by $h_n = \binom{n}{p}$.

Combinatorics- Lecture: 38

If the 0the diagonal of the difference table is given by $c_0, c_1, c_2, \ldots, c_p, 0, 0, \ldots$, where $c_p \neq 0$. Then the general term of the corresponding sequence is a polynomial in n of degree p satisfying:

$h_n = c_0\binom{n}{0} + c_1\binom{n}{1} + \cdots + c_p\binom{n}{p}$

Let $h_n = n^3 + 3n^2 - 2n + 1,\ n \geq 0.$
Find c_0, c_1, c_2, c_3, so that $h_n = \sum_{i=0}^{3} c_i.\binom{n}{i}$

Combinatorics- Lecture: 38

Assume that the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ has a difference table whose 0th diagonal equals $c_0, c_1, c_2, \ldots, c_p, 0, 0, \ldots$. Then $\sum_{k=0}^{n} h_k = \sum_{i=0}^{p} c_i.\binom{n+1}{i+1}$

Example: Find an expression for $\sum_{i=1}^{n} i^4$

Sterling Number of the Second Kind:
$n^p = S(p,0)n^{\underline{0}} + S(p,1)n^{\underline{1}} + \cdots + S(p,p)n^{\underline{p}}$
The coefficients $S(p,i)$ are the Sterling Numbers of the second kind: They satisfy the same recurrence relation and the intial conditions.

The Bell numbers.

Sterling Numbers of the first kind.
The Sterling numbers of the first kind $s(p, k)$ counts the number of arrangements of p objects into k non-empty circular permutations.

If $1 \leq k \leq p-1$, then
$s(p,k) = (p-1)s(p-1,k) + s(p-1,k-1)$.

COMBINATORICS- Lecture: 39

by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

Let the general term of a sequence be a polynomial of degree p in n, i.e.,
$h_n = a_p n^p + a_{p-1} n^{p-1} + \cdots + a_1 n + a_0, (n \geq 0)$
Then $\Delta^{p+1} h_n = 0$, for all $n \geq 0$.

Combinatorics- Lecture: 39

Let g_n and f_n be general terms of two sequences. Let c, d be constants:
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$B(p) = S(p, 0) + S(p, 1) + \cdots + S(p, p)$

Sterling Numbers of the first kind.
The Sterling numbers of the first kind $s(p, k)$ counts the number of arrangements of p objects into k non-empty circular permutations.

Example: $s(4, 2) = 11$.

Combinatorics- Lecture: 39

$s(n, k) \geq S(n, k)$ for $n, k \geq 0$.

$s(n, 1) = (n-1)!$ for $n > 0$

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$\sum_{k=0}^{b} s(n, k) = n!$, for $n \geq 0$.

$s(n,n) = S(n,n) = 1,\ s(n,n-1) = S(n,n-1) = \binom{n}{2}$

Combinatorics- Lecture: 39

If $1 \leq k \leq p-1$, then
$s(p,k) = (p-1)s(p-1,k) + s(p-1,k-1)$.

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by Prof. L. Sunil Chandran, CSA, IISc, Bangalore.

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Combinatorics- Lecture: 40

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$\sum_{k=0}^{b} s(n, k) = n!$, for $n \geq 0$.

If $1 \leq k \leq p-1$, then
$s(p,k) = (p-1)s(p-1,k) + s(p-1,k-1)$.

$n^{\underline{p}} = s(p,p)n^p - s(p,p-1)n^{p-1} + s(p,p-2)n^{p-2} - \cdots + (-1)^{p-k}s(p,k)n^k + \cdots + (-1)^p s(p,0)n^0$