Line Graphs and Edge-coloring

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Preface

Recap of Previous Lecture:

In previous lecture, we have discussed the elementary properties of Subdivision, Minor, Kuratowski's theorem and Wagner's Theorem and also proved the Non-planarity of Peterson Graph. .

Content of this Lecture:

In this lecture, we will discuss Line Graph, Edge-coloring and 1-factorization.

Line Graphs and Edge-coloring

- Many questions about vertices have natural analogues for edges. Independent sets have no adjacent vertices; matchings have no "**adjacent**" edges.

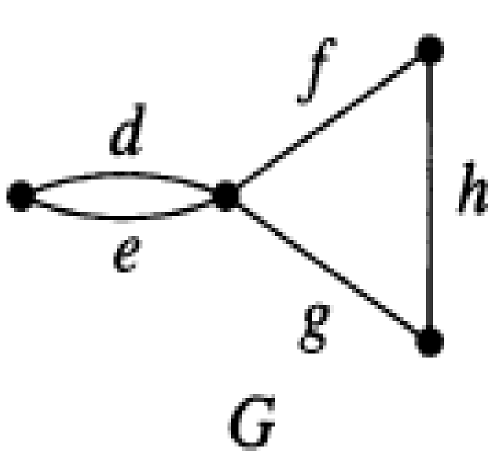
- Vertex colorings partition vertices into independent sets; we can instead partition edges into matchings. These pairs of problems are related via line graphs.

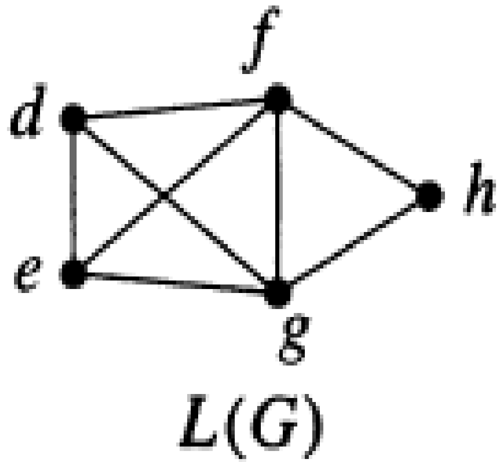
- Here we repeat the definition, emphasizing our return to the context in which a graph may have multiple edges. We use "**line graph**" and L(G) instead of "**edge graph**" because E(G) already denotes the edge set.

Definition: Line graph 7.1.11

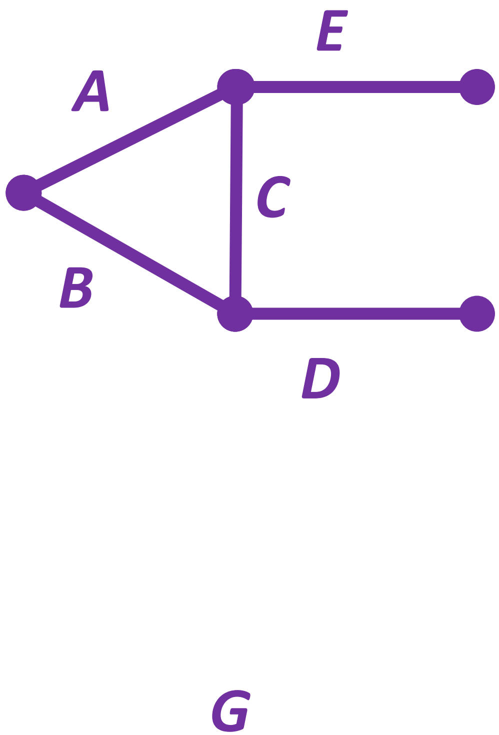
Definition: The **line graph** of G, written $L(G)$, is the simple graph whose vertices are the edges of G, with $ef \in E(L(G))$ when e and f have a common endpoint in G.

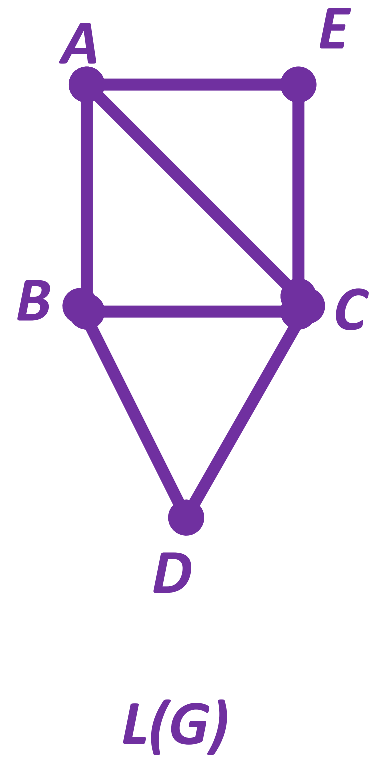
Example:





Example: Line Graph





Edge colorings

- Edge-coloring problems arise when the objects being scheduled are pairs of underlying elements.

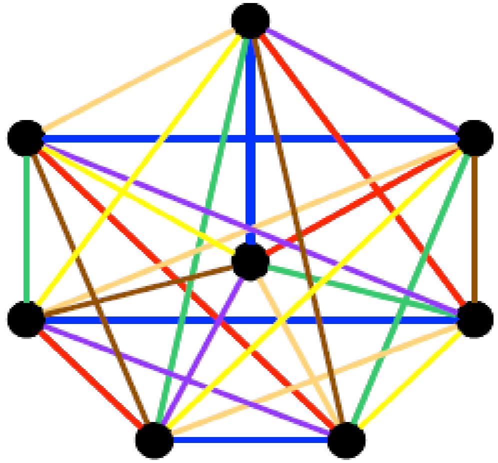
Example: Edge-coloring of K_{2n}.

- In a league with 2n teams, we want to schedule games so that each pair of teams plays a game, but each team plays at most once a week. Since each team must play 2n-1 others, the season lasts at least 2n-1 weeks. The games of each week must form a matching. We can schedule the season in 2n-1 weeks if and only if we can partition $E(K_{2n})$ into 2n-1 matchings. Since K_{2n} is 2n-1 regular, these must be perfect matchings.

Definition 7.1.3

- A k-**edge coloring** of G is labeling f: E(G) → S , where |S|=k (often we use S = [k].
- The labels are **colors**; the edges of one color form a **color class**.
- A k-edge-coloring is **proper** if incident edges have different labels; that is, if each color class is a matching.
- A graph is k-**edge colorable** if it has a proper k-edge coloring.
- The **edge chromatic number $\chi'(G)$** of a loopless graph G is the least k such that G is k-edge-colorable.

Example: Edge-coloring a complete graph



Chromatic index

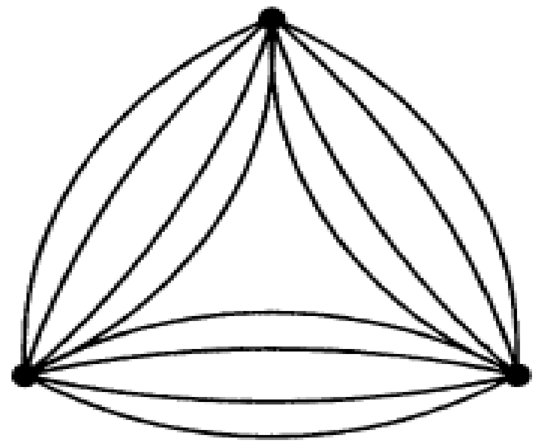
- **Chromatic index** is another name for $\chi'(G)$. Since edges sharing a vertex need different colors, $\chi'(G) \geq \Delta(G)$.
- Vizing [1964] and Gupta [1966] independently proved that $\Delta(G) + 1$ colors suffice when G is simple.
- A clique in **L(G)** is a set of pairwise-intersecting edges of G. When G is simple, such edges form a star or a triangle in G. For the hereditary class of line graphs of simple graphs, Vizing's Theorem thus states that $\chi(H) \leq \omega(H) + 1$; thus line graphs are "almost" perfect.
- In contrast to $\chi(G)$, multiple edges greatly affect $\chi'(G)$. A graph with a loop has no proper edge-coloring; the adjective "loopless" excludes loops but allows multiple edges.

Definition: Multiplicity 7.1.4

- In a graph G with multiple edges, we say that a vertex pair x, y is an edge of **multiplicity** m if there are m edges with endpoints x, y.

- We write $\mu(xy)$ for the multiplicity of the pair, and we write $\mu(G)$ for the maximum of the edge multiplicities in G.

Example: The "Fat Triangle"



Theorem: (König [1916]) If G is bipartite, then $\chi'(G) = \Delta(G)$ 7.1.7

Proof:

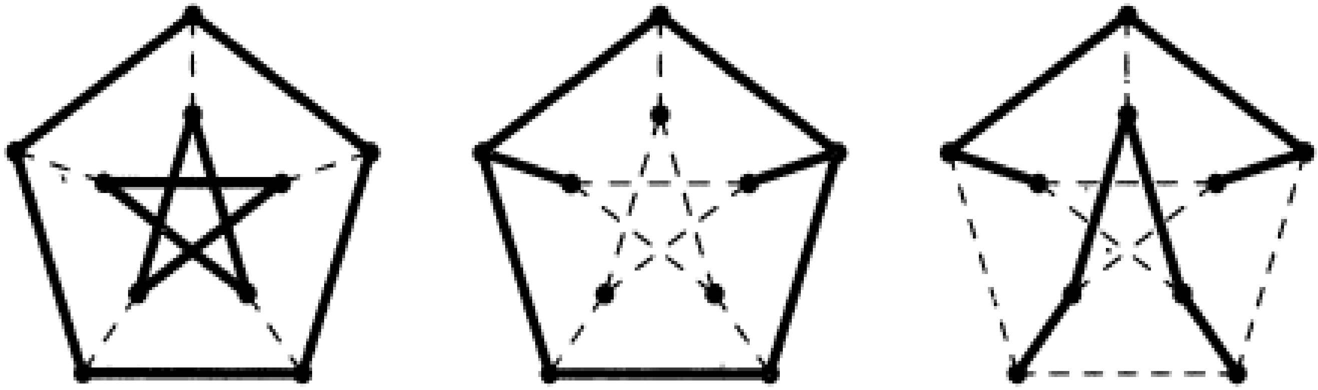
- Corollary 3.1.13 states that every regular bipartite graph H has a 1–factor. By induction on $\Delta(H)$, this yields a proper $\Delta(H)$-edge-coloring. It therefore suffices to show that for every bipartite graph G with maximum degree k, there is a k-regular bipartite graph H containing G.
- To construct such a graph, first add vertices to the smaller partite set of G, if necessary, to equalize the sizes. If the resulting graph G' is not regular, then each partite set has a vertex with degree less than k. Add an edge with these two vertices as endpoints. Continue adding such edges until the graph becomes k-regular; the resulting graph is H.
- For a regular graph G, proper edge-coloring with $\Delta(G)$ colors is equivalent to decomposition into 1-factors.

Definition: 1-factorization 7.1.8

- A decomposition of a regular graph G into 1-factors is a **1-factorization** of G.
- A graph with a 1-factorization is **1-factorable**.
- An odd cycle is not 1-factorable; $\chi'(C_{2m+1}) = 3 > \Delta(C_{2m+1})$. The Petersen graph also requires an extra color, but only one extra color.

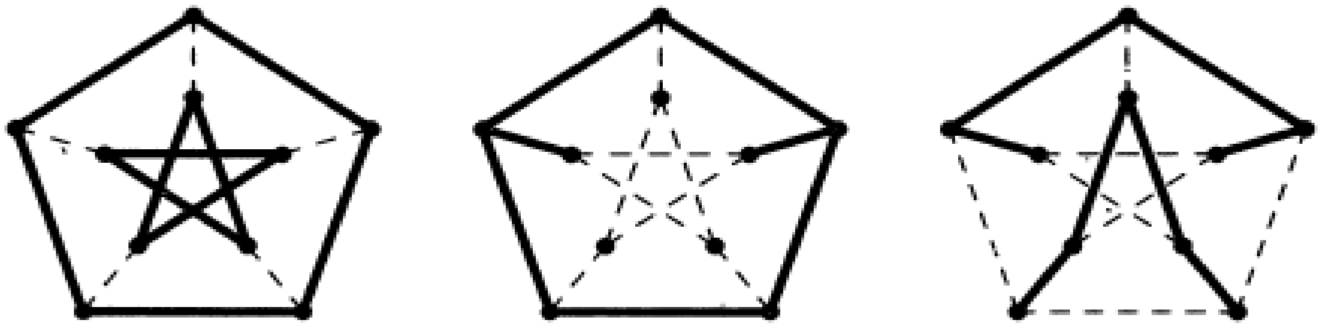
Example: The Petersen graph is 4-edge-chromatic.

- The Petersen graph is 3-regular; 3-edge-colorability requires a 1-factorization. Deleting a perfect matching leaves a 2-factor; all components are cycles. The 1-factorization can be completed only if these are all even cycles.
- Thus it suffices to show that every 2-factor is isomorphic to $2C_5$. Consider the drawing consisting of two 5-cycles and a matching (the **cross edges**) between them. We consider cases by the number of cross edges used.



Example: The Petersen graph is 4-edge-chromatic continue

- Every cycle uses an even number of cross edges, so a 2-factor H has an even number m of cross edges. **If m = 0 (left figure),** then H = $2C_5$.
- **If m = 2 (central figure),** then the two cross edges have nonadjacent endpoints on the inner cycle or the outer cycle. On the cycle where their endpoints are nonadjacent, the remaining three vertices force all five edges of that cycle into H, which violates the 2-factor requirement.
- **If m = 4 (right figure),** then the cycle edges forced into H by the unused cross edges form a $2P_5$ whose only completion to a 2-factor in H is $2C_5$.
- Note that since C_5 is 3-edge-colorable, the graph is 4-edge-colorable.



Vizing's Theorem

(Vizing [1964,1965], Gupta [1966])

Theorem: If G is a simple graph, then $\chi'(G) \leq \Delta(G) + 1$.

Definition: A simple graph G is **Class 1** if $\chi'(G) = \Delta(G)$. It is **Class 2** if $\chi'(G) = \Delta(G) + 1$

Determining whether a graph is Class 1 or Class2 is generally hard. Thus we seek conditions that forbid or guarantee $\Delta(G)$-edge-colorability.

Example: 1) All bipartite graphs are Class 1. (By König's line coloring theorem)

2) Class 2 graphs include the Petersen graph, complete graphs K_n for n= 3, 5, 7, ...,

Conclusion

- In this lecture, we have discussed the characterization of Line Graphs, Edge-coloring, Chromatic index, Multiplicity and 1-factorization.