Characterization of Planar Graphs

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Preface

Recap of Previous Lecture:

In previous lecture, we have discussed planar graphs *i.e.* Plane graph embeddings, Dual graphs, Euler's formula for plane graphs and Regular Polyhedra.

Content of this Lecture:

In this lecture, we will discuss the characterization of planar graphs, Subdivision, Minor, Kuratowski's theorem and Wagner's Theorem.

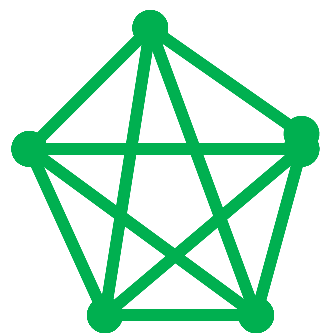
Characterization of Planar Graphs

Recall using Euler's Formula: K_5 , $K_{3,3}$ are not planar

If G contains a non-planar subgraph then G is non-planar

Example:

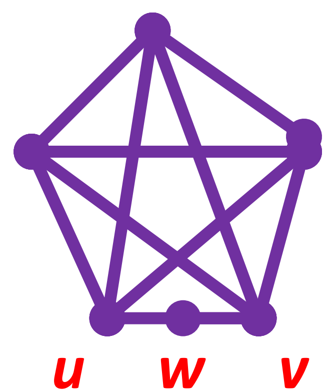
G



non-planar

Observe

G



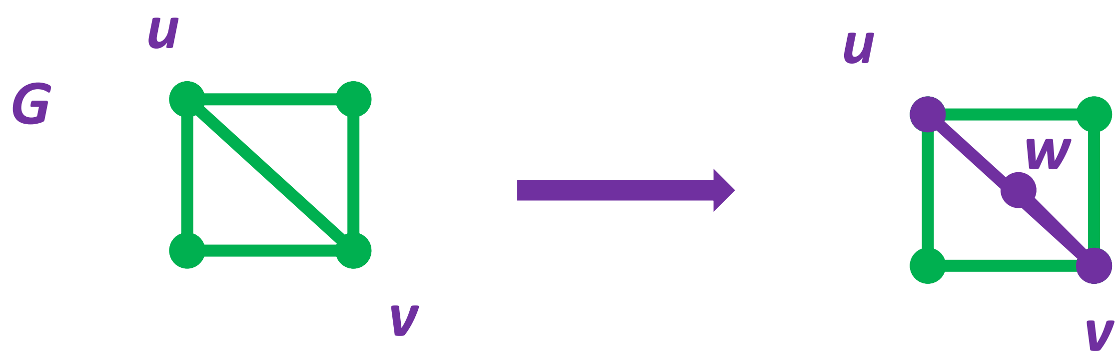
also non-planar

G is an elementary subdivision of K_5

Elementary Subdivision

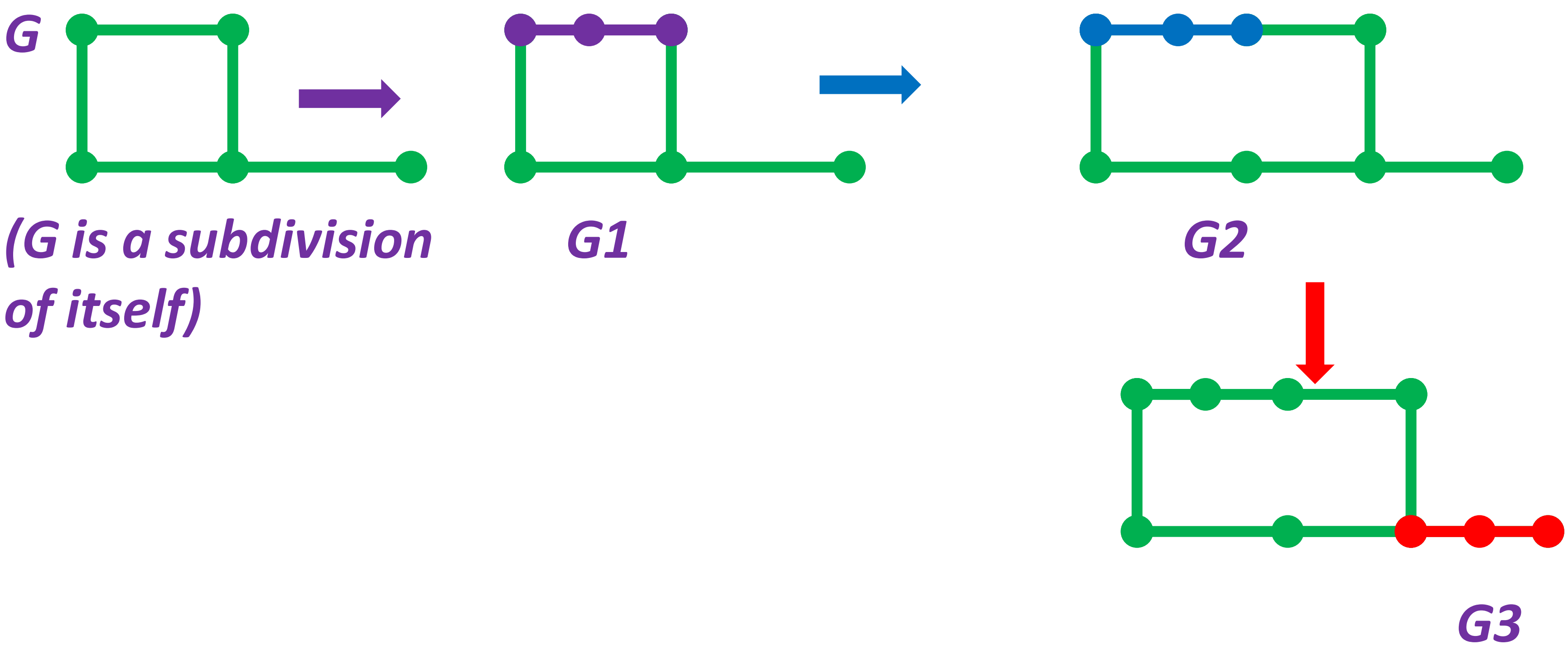
- An **elementary subdivision** of a nonempty graph G is a graph obtained from G by removing an edge e = uv and adding a new vertex w and new edges uw and vw.

Example:



- A **subdivision** of a graph G is a graph obtained from G by a sequence of zero or more elementary subdivisions.

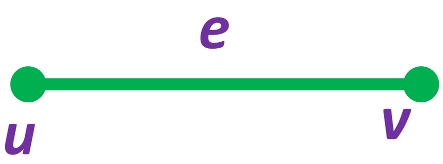
Example:



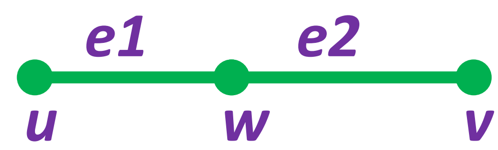
Example: Subdivision

- A **subdivision** of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths.

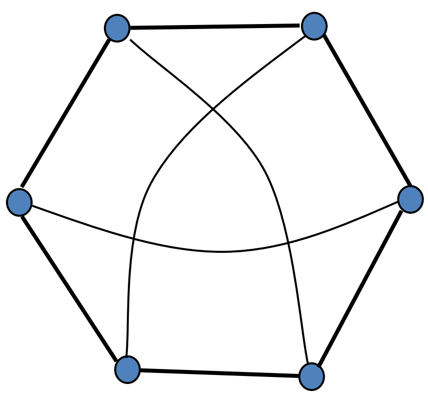
Example(1): Subdivision of an edge



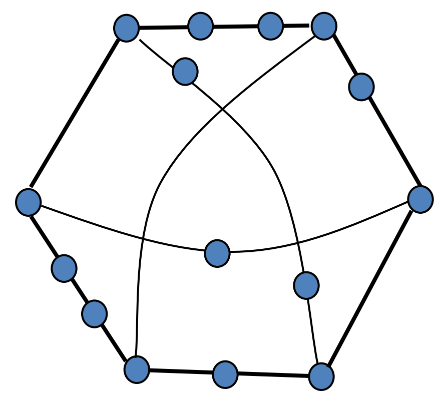
Subdivision



Example(2): Subdivision of $K_{3,3}$



Subdivision



Remarks

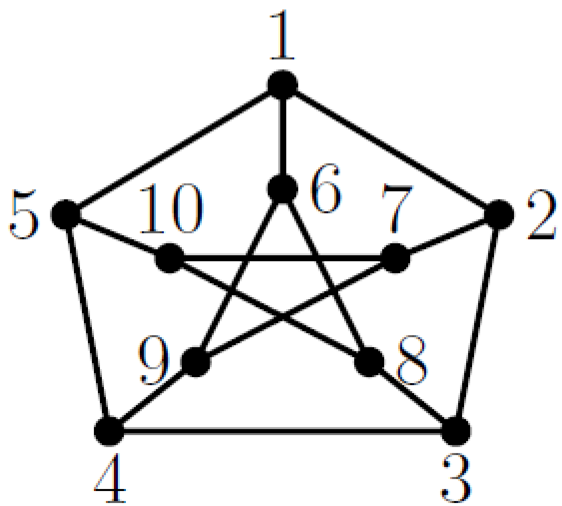
- **Remark 1:** Any subdivision H of a graph G is planar if and only if G is planar.
- **Remark 2:** If a graph G is a subdivision of K_5 or $K_{3,3}$ then G is non-planar.
- **Remark 3:** If a graph G contains a subgraph that is a subdivision of K_5 of $K_{3,3}$ then G is non-planar.

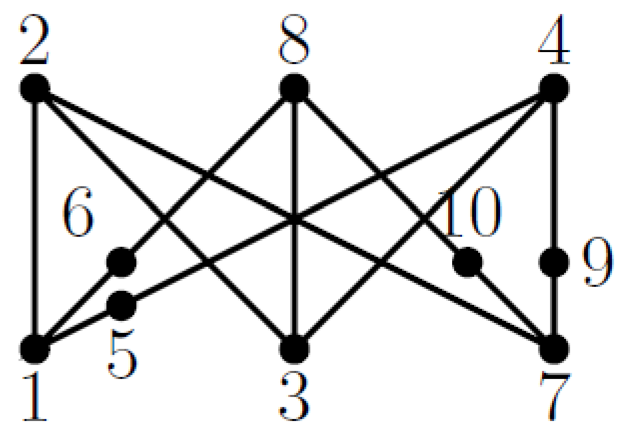
Kuratowski's Theorem

Theorem: (Kuratowski [1930])

A graph is planar **if and only if** it does not contain a subdivision of K_5 or $K_{3,3}$

Example:



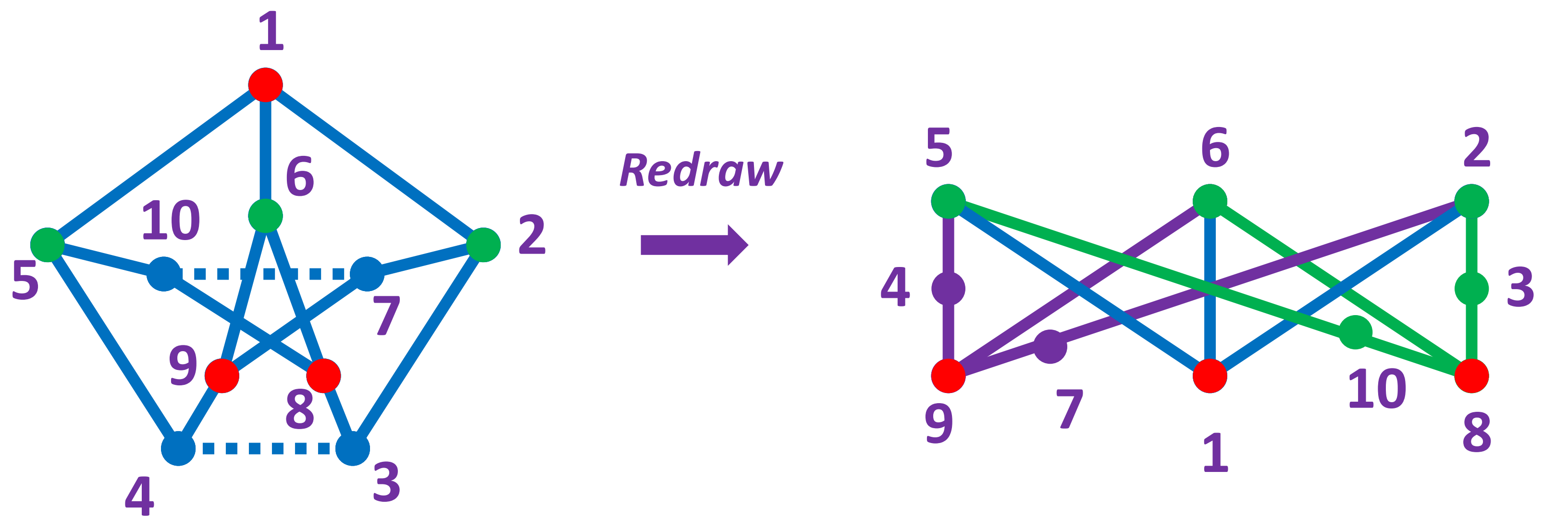


The Petersen graph contains a subdivision of $K_{3,3}$. Therefore, the Petersen graph is non-planar

Petersen graph is Non-planar By Kuratowski's Theorem

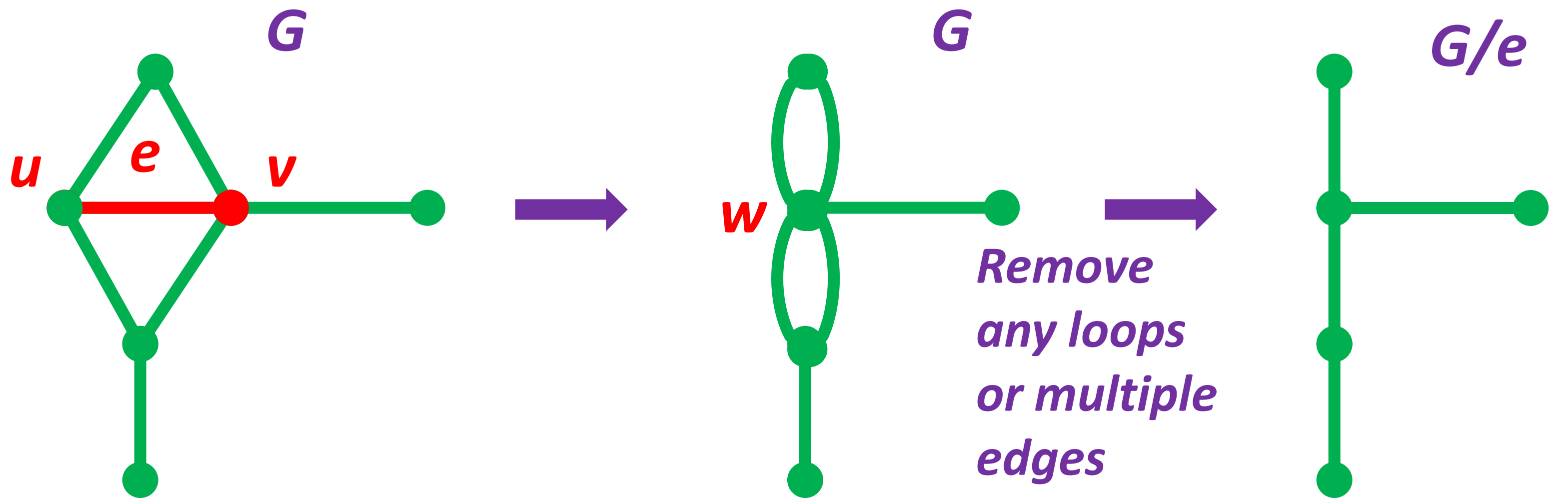
By Kuratowski's Theorem

- **Proof:** The Petersen graph contains a subgraph that is a subdivision of $K_{3,3}$.



- **By Kuratowski's Theorem, the Petersen graph is non-planar**

Edge Contraction



Original Graph G

Resulting multigraph after contracting edge e

Resulting Simple graph after contracting edge e

- Let G be a graph and e϶ E(G) with e=uv. Suppose w ∉V(G). Contracting edge e in G results in the graph G/e obtained from G by:
  - Removing edge e
  - Replacing vertices u and v with a new single vertex w
  - Vertex w is adjacent to the neighbors of u and to the neighbors of v

Definition: Minor

- A graph H is called a **minor** of G if it can be produced from G by successive application of these reductions:

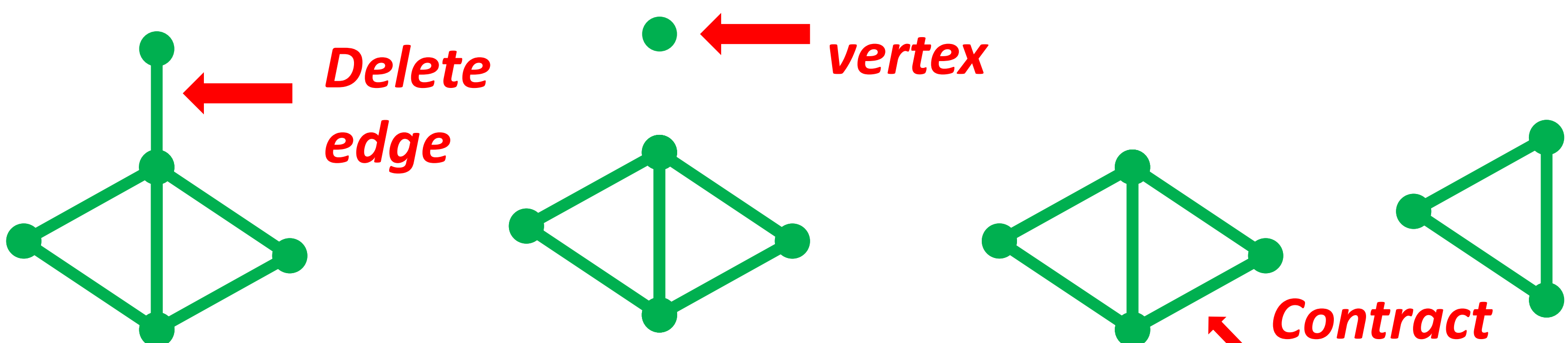
  (a) Deleting an edge

  (b) Contracting an edge

  (c) Deleting an isolated vertex

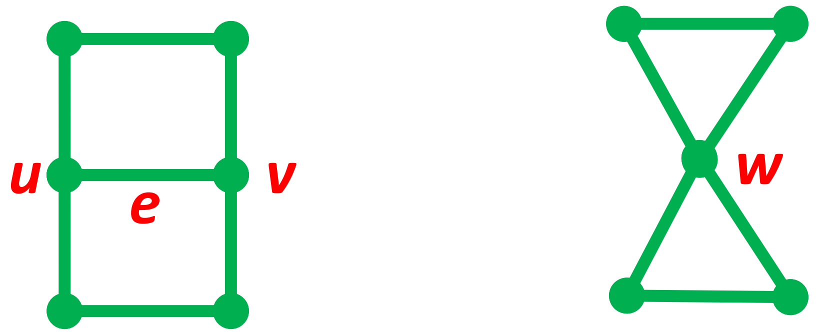
- **Note: G is a minor of itself**
- Every graph that is isomorphic to a minor of G is also called a **minor** of G.

Example:

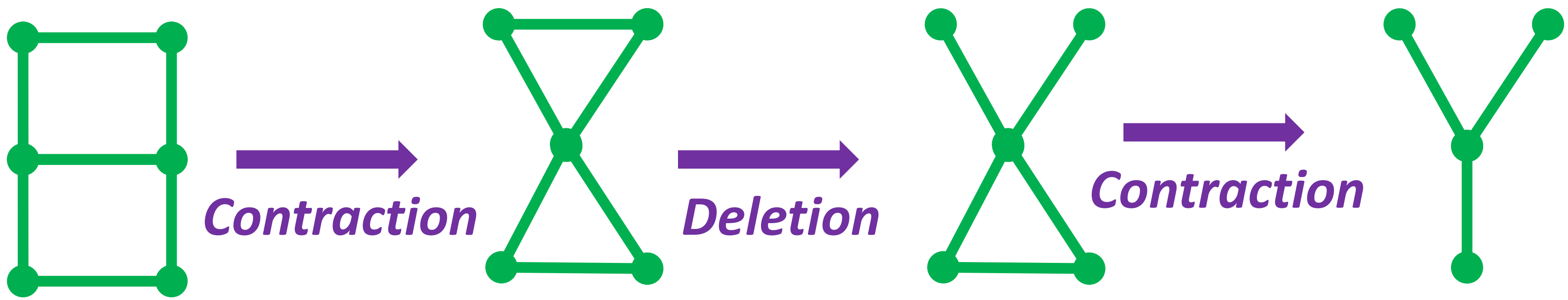


G has a K3-minor

Example: Edge contraction and Minors



(a) Contraction of e



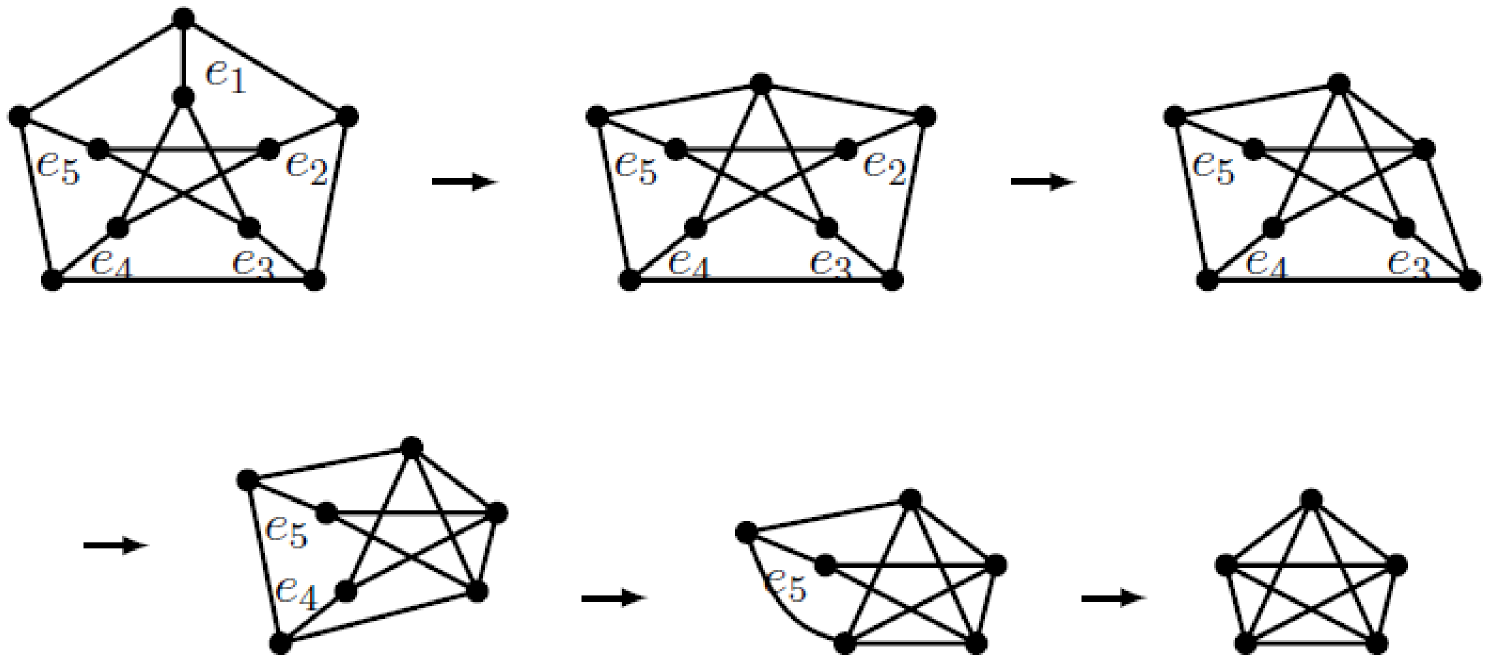
G

(b) $K_{1,3}$ is a minor of G

Wagner's Theorem (Wagner, 1937)

- A graph G is planar **if and only if** neither K_5 nor $K_{3,3}$ is a minor of G.

Example:

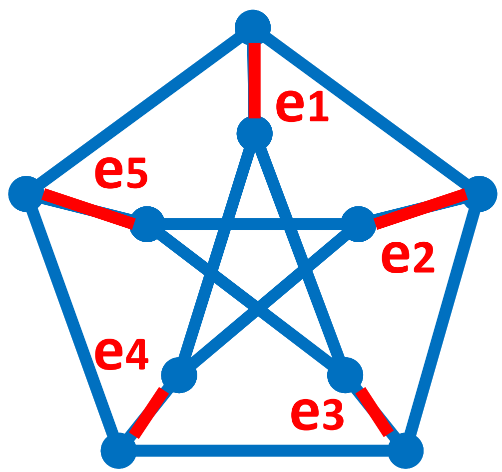


- The Petersen graph has a K_5-minor, Therefore, the Petersen graph is non-planar

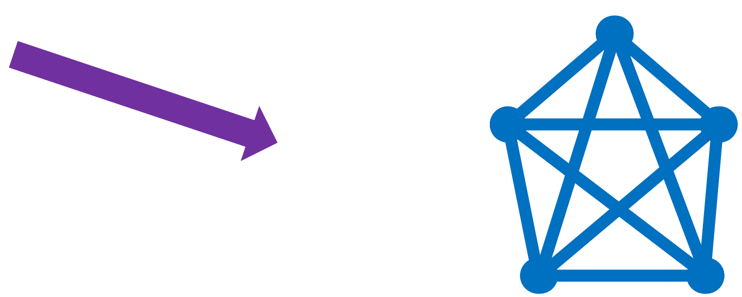
Petersen graph is Non-planar By Wagner’s Theorem

By Wagner’s Theorem

Proof: The Petersen graph has a K_5-minor



- **Perform edge contractions on edges e_1, e_2, e_3, e_4, e_5**
- **The resulting graph is isomorphic to K_5**



- **By Wagner’s Theorem, the Petersen graph is non-planar**

Wagner's Theorem vs. Kuratowski's Theorem

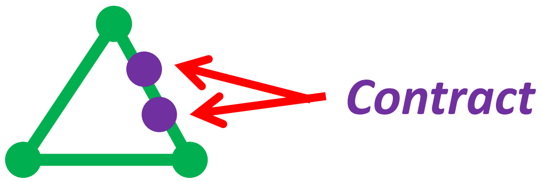
Wagner's Theorem: [1937]

- A graph is planar ⇔ It has no K_5 or $K_{3,3}$ minor

Kuratowski's Theorem: [1930]

- A graph is planar ⇔ It has no subgraph that is a subdivision of K_5 or $K_{3,3}$
- **Notes:** A subdivision of H can be converted into an H-minor by contracting all but one edge in each path formed by the subdivision process

Subdivision of K_3:



- BUT an H-minor cannot always be converted into a subdivision of H.
- For K_5 and $K_{3,3}$: If G has ≥ 1 of these as a minor, then it has ≥ 1of these as a subdivision.

Conclusion

- In this lecture, we have discussed the elementary properties of Subdivision, Minor, Kuratowski's theorem, Wagner's Theorem and also proved the Non-planarity of Peterson Graph.