Counting Proper Colorings

Dr. Rajiv Misra
Associate Professor
Dept. of Computer Science & Engg.
Indian Institute of Technology Patna
rajivm@iitp.ac.in

Preface

Recap of Previous Lecture:

In previous lecture, we have discussed the Brooks' Theorem and elementary properties of k-critical graphs.

Content of this Lecture:

In this lecture, we will discuss the properties of counting function, chromatic polynomial, chromatic recurrence, and further related topics.

Enumerative Aspects of Coloring

- The chromatic number $\chi(G)$ is the minimum *k* such that the count is positive; knowing the count for all *k* would tell us the chromatic number.

- Birkhoff [1912] introduced this counting problem as a possible way to attack the Four Color Problem.

- In this lecture, we will discuss properties of the counting function, classes where it is easy to compute, and further related topics.

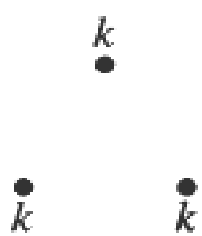
Counting Proper Colorings $\chi(G ; k)$

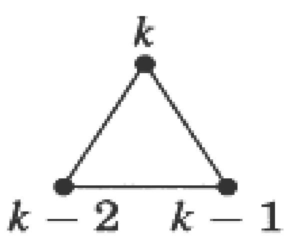
Definition:

- Given $k \in N$ and a graph G, the value $\chi(G ; k)$ is the number of proper colorings $f : V(G) \rightarrow [k]$.
- The set of available colors is $[k] = \{1,\ldots.,k\}$; the k colors need not all be used in a coloring f. Changing the names of the colors that are used produces a different coloring.

Example 5.3.2

- $\chi(\overline{K_n}; k) = k^n$ and $\chi(K_n; k) = k(k-1) \ldots (k-n+1)$.
- When coloring the vertices of $\overline{K_n}$,we can use any of the k colors at each vertex no matter what colors we have used at other vertices. Each of the k^n functions from the vertex set to [k] is a proper coloring, and hence $\chi(\overline{K_n}; k) = k^n$.
- When we color the vertices of K_n , the colors chosen earlier cannot be used on the ith vertex. There remain k-i+1 choices for the color of the ith vertex no matter how the earlier colors were chosen. Hence $\chi(K_n; k) = k(k-1)\ldots(k-n+1)$.





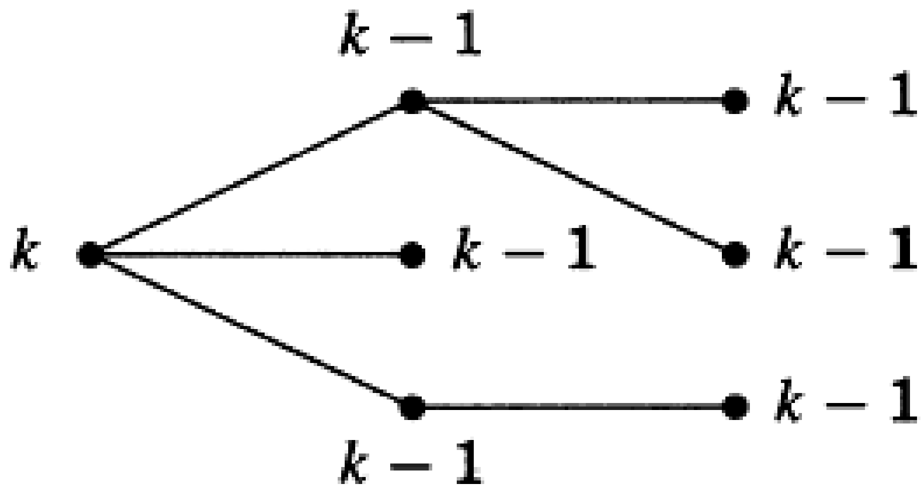
Example continue

- We can also count this as $\binom{k}{n}n!$ by first choosing n distinct colors and then multiplying by n! to count the ways to assign the chosen colors to the vertices. For example, $\chi(K_3;3)=6$ and $\chi(K_3;4)=24$.
- The value of the product is 0 when $k < n$. This makes sense, since K_n has no proper k-colorings when $k < n$.

Proposition 5.3.3: If T is a tree with n vertices, then $\chi(T\,;k) = k(k-1)^{n-1}$.

Proof:

- Choose some vertex v of T as a root. We can color v in k ways. If we extend a proper coloring to new vertices as we grow the tree from v, at each step only the color of the parent is forbidden, and we have k-1 choices for the color of the new vertex. Furthermore, deleting a leaf shows inductively that every proper k-coloring arises in this way. Hence $\chi(T; k) = k(k-1)^{n-1}$.



Chromatic Polynomial $\chi(G; k)$

- Another way to count the colorings is to observe that the color classes of each proper coloring of G partition V(G) into independent sets. Grouping the colorings according to this partition leads to a formula for $\chi(G; k)$ that is a polynomial in k of degree n(G).

- Note that this holds for the answers in Example 5.3.2 and Proposition 5.3.3. Since every graph has this property, $\chi(G; k)$ as a function of k is called the **chromatic polynomial** of G.

Proposition 5.3.4

- Let $x_{(r)}=x(x-1)\dots(x-r+1)$. If $p_r(G)$ denotes the number of partitions of $V(G)$ into r nonempty independent sets,
- then $\chi(G;k) = \sum_{r=1}^{n(G)} p_r(G)\cdot k_{(r)}$, which is a polynomial in k of degree $n(G)$.

Proof:

- When r colors are actually used in a proper coloring, the color classes partition $V(G)$ into exactly r independent sets, which can happen in $p_r(G)$ ways. When k colors are available, there are exactly $k_{(r)}$ ways to choose colors and assign them to the classes. All the proper colorings arise in this way, so the formula for $\chi(G; k)$ is correct.
- Since $k_{(r)}$ is a polynomial in k and $p_r(G)$ is a constant for each r, this formula implies that $\chi(G; k)$ is a polynomial function of k. When G has n vertices, there is exactly one partition of G into n independent sets and no partition using more sets, so the leading term is k^n.

Example 5.3.5

- Always $p_n(G) = 1$, using independent sets of size 1. Also $p_1(G) = 0$ unless G has no edges, since only for $\overline{K_n}$ is the entire vertex set an independent set.
- Consider $G = C_4$. There is exactly one partition into two independent sets: opposite vertices must be in the same independent set, When $r = 3$, we put two opposite vertices together and leave the other two in sets by themselves; we can do this in two ways. Thus $p_2 = 1$, $p_3 = 2$, $p_4 = 1$.

$$\chi(C_4; k) = 1\cdot k(k-1) + 2\cdot k(k-1)(k-2) + 1\cdot k(k-1)(k-2)(k-3)$$
$$= k(k-1)(k^2-3k+3)$$

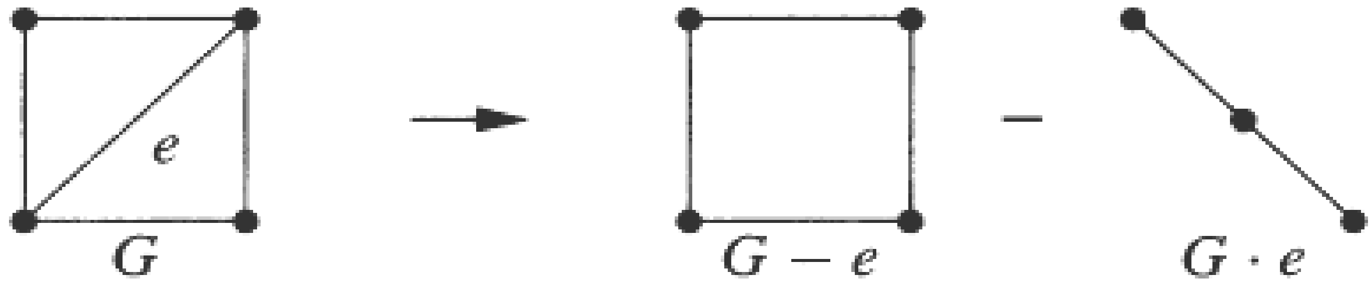
- Computing the chromatic polynomial in this way is not generally feasible, since there are too many partitions to consider. There is a recursive computation much like that used in Proposition 2.2.8 to count spanning trees,

- Again G·e denotes the graph obtained by contracting the edge e in G. Since the number of proper k-colorings is unaffected by multiple edges, **we discard multiple copies of edges that arise from the contraction**, keeping only one copy of each to form a simple graph

Theorem: (Chromatic recurrence) If G is a simple graph and $e \in E(G)$, then $\chi(G; k) = \chi(G\text{-}e; k) - \chi(G \cdot e; k)$. 5.3.6

Proof:

- Every proper k-coloring of G is a proper k-coloring of G-e. A proper k-coloring of G-e is a proper k-coloring of G if and only if it gives distinct colors to the endpoints u,v of e. Hence we can count the proper k-colorings of G by subtracting from $\chi(G\text{-}e; k)$ the number of proper k-colorings of G-e that give u and v the same color.
- Colorings of G-e in which u and v have the same color correspond directly to proper k-colorings of G·e, in which the color of the contracted vertex is the common color of u and v, Such a coloring properly colors all the edges of G·e if and only if it properly colors all the edges of G other than e.



Example: Proper k-colorings of C_4 5.3.7

- Deleting an edge of C_4 produces P_4, while contracting an edge produces K_3. Since P_4 is a tree and K_3 is a complete graph, we have $\chi(P_4;\ k) = k(k-1)^3$ and $\chi(K_3;\ k) = k(k-1)(k-2)$. Using the chromatic recurrence, we obtain

$$\chi(C_4\,;k) = \chi(P_3;k) - \chi(K_3;k) = k(k-1)(k^2-3k+3)$$

- Because both G-e and G·e have fewer edges than G, we can use the chromatic recurrence inductively to compute $\chi(G\,;\ k)$. We need initial conditions for graphs with no edges, which we have already computed: $\chi(\overline{K_n}\,;\ k) = k^n$

Theorem: (Whitney [1933]) 5.3.8

Theorem: The chromatic polynomial $\chi(G; k)$ of a simple graph G has degree n(G), with integer coefficients alternating in sign and beginning 1, -e(G) ,….

Proof:

We use induction on e(G). The claims hold trivially when e(G) = 0, where $\chi(\overline{K_n}; k) = k^n$. For the induction step, let G be an n-vertex graph with e(G) ≥ 1. Each of G-e and G·e has fewer edges than G, and G·e has n-1 vertices. By the induction hypothesis, there are nonnegative integers $\{a_i\}$ and $\{b_i\}$ such that

$\chi(G - e; k) = \sum_{i=0}^{n}(-1)^i a_i k^{n-i}$ and $\chi(G \cdot e; k) = \sum_{i=0}^{n-1}(-1)^i b_i k^{n-1-i}$

Theorem: (Whitney [1933]) continue

By the chromatic recurrence,

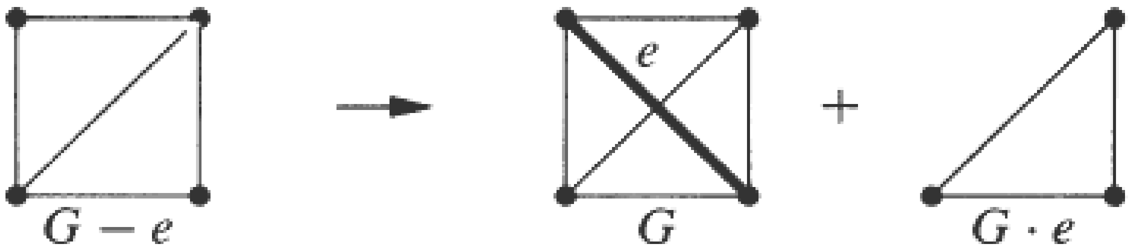
$$\begin{array}{lllll}
\chi(G - e; k): & k^n - [e(G)-1]k^{n-1} + a_2k^{n-2} - \ldots + (-1)^i & a_ik^{n-i}\ldots \\
-\ \chi(G\cdot e; k): & -\ (\qquad\qquad k^{n-1} - b_1k^{n-2} + \ldots + (-1)^{i-1} & b_{i-1}k^{n-i}\ldots) \\
\hline
= \chi(G; k): & k^n - e(G)k^{n-1} + (a_2+b_1)k^{n-2} - \ldots + (-1)^i (a_i+b_{i-1})k^{n-i}\ldots
\end{array}$$

Hence $\chi(G; k)$ is a polynomial with leading coefficient $a_0 = 1$ and next coefficient $-(a_1 + b_0) = -e(G)$, and its coefficients alternate in sign.

Example 5.3.9

- When adding an edge yields a graph whose chromatic polynomial is easy to compute, we can use the chromatic recurrence in a different way. Instead of $\chi(G;\ k) = \chi(G\text{-}e;\ k) - \chi(G\cdot e;\ k)$. We can write $\chi(G\text{-}e;\ k) = \chi(G;\ k) + \chi(G\cdot e;\ k)$. Thus we may be able to compute $\chi(G\text{-}e;\ k)$ using $\chi(G;\ k)$.
- To compute $\chi(K_n - e;\ k)$, for example, we let G be K_n in this alternative formula and obtain

$$\chi(K_n - e; k) = \chi(K_n; k) + \chi(K_{n-1}; k) = (k - n + 2)^2 \prod_{i=0}^{n-3} (k - i).$$



- We close our general discussion of $\chi(G; k)$ with an explicit formula. It has exponentially many terms, so its uses are primarily theoretical.

- The formula summarizes what happens if we iterate the chromatic recurrence until we dispose of all the edges.

Theorem (Whitney [1932]) 5.3.10

- Let c(G) denote the number of components of a graph G. Given a set $S \subseteq E(G)$ of edges in G, let G(S) denote the spanning subgraph of G with edge set S. Then the number $\chi(G; k)$ of proper k-colorings of G is given by:

$$\chi(G;k) = \sum_{S \subseteq E(G)} (-1)^{|S|} k^{c(G(S))}$$

Proof:

- In applying the chromatic recurrence, contraction may produce multiple edges. We have observed that dropping these does not affect $\chi(G; k)$. We claim that deleting extra copies of edges also does not change the claimed formula.

Theorem (Whitney [1932]) continue

- Let e and e' be edges in G with the same endpoints. When $e' \in S$ and $e \notin S$, we gave $c(G(S \cup \{e\})) = c(G(S))$, since both endpoints of e are in the same component of G(S). However, $|S \cup \{e\}| = |S| + 1$. Thus the terms for S and $S \cup \{e\}$ in the sum cancel. Therefore, omitting all terms for sets of edges containing e' does not change the sum. This implies that we can keep or drop e' from the graph without changing the formula.

- When computing the chromatic recurrence, we therefore obtain the same result if we do not discard multiple edges or loops and instead retain all edges for contraction or deletion. Iterating the recurrence now yields $2^{e(G)}$ terms as we dispose of all edges; each in turn is deleted or contracted.

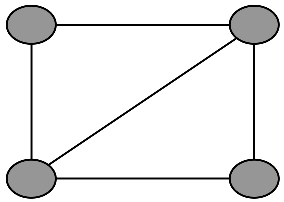
Theorem (Whitney [1932]) continue

- When all edges have been deleted or contracted, the graph that remains consists of isolated vertices. Let S be the set of edges that were contracted, The remaining vertices correspond to the components of G(S); each such component becomes one vertex when the edges of S are contracted and the other edges are deleted. The c(G(S)) isolated vertices at the end yield a term with $k^{c(G(S))}$ colorings. Furthermore, the sign of the contribution changes for each contracted edge, so the contribution is positive if and only if $|S|$ is even.

- Thus the contribution when S is the set of contracted edges is $(-1)^{|S|} k^{c(G(S))}$, and this accounts for all terms in the sum.

Example: A chromatic polynomial 5.3.11

- When G is a simple graph with n vertices, every spanning subgraph with 0, 1, or 2 edges has n, n-1, or n-2 components, respectively. When $|S| = 3$, the number of components is n-2 if and only if the three edges form a triangle; otherwise it is n-3.
- For example, when G is a kite (four vertices, five edges) there are ten sets of three edges. For two of these, G(S) consists of a triangle plus one isolated vertex. The other eight sets of three edges yield spanning subgraphs with one component. Both types of triple are counting negatively, since $|S| = 3$. All spanning subgraphs with four or five edges have only one component, Hence Theorem 5.3.10 yields



$$\chi(G; k) = k^4 - 5k^3 + 10k^2 - (2k^2 + 8k^1) + 5k - k$$
$$= k^4 - 5k^3 + 8k^2 - 4k$$

- This agrees with $\chi(G; k) = k(k-1)(k-2)(k-2)$, computed by counting colorings directly or by using $\chi(G; k) = \chi(C_4; k) - \chi(P_3; k)$

- Whitney proved Theorem 5.3.10 using the **inclusion-exclusion principle of elementary counting**, Among the universe of k-colorings, the proper colorings are those not assigning the same color to the endpoints of any edge.

- Letting A_i be the set of k-colorings assigning the same color to the endpoints of edge e_i, we want to count the colorings that lie in none of $A_1, \ldots\ldots, A_m$.

Conclusion

- In this lecture, we have discussed the properties of counting function, chromatic polynomial, chromatic recurrence, and theorems based on these.