Matchings and Covers

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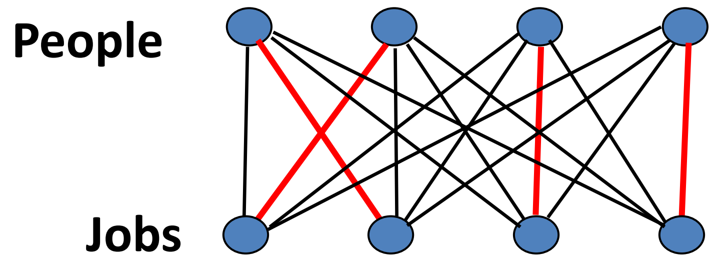
Preface

Content of this Lecture:

- In this lecture, we will discuss the Concept of Matching, Perfect matchings, Maximal matchings, Maximum Matchings, M-alternating path, M-augmenting path, Symmetric difference, Hall’s Matching condition and Vertex covers.

Matchings and Covers

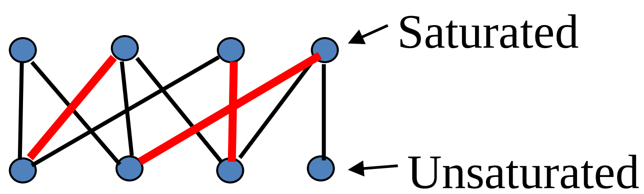
- Within a set of people, some pairs are compatible as roommates; under what conditions can be pair them all up? Many applications of graphs involve such pairings.
- **Example:** Problem of filling jobs with qualified candidates



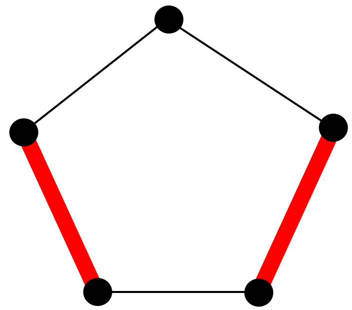
- Bipartite graphs have a natural vertex partition into two sets, and we want to know whether the two sets can be paired using edges. In the roommate question, the graph need not be bipartite.

Matching 3.1.1

- A **matching** in a graph G is a set of non-loop edges with no shared endpoints.
- The vertices incident to the edges of a matching M are **saturated** by M; the others are **unsaturated** (we say *M- saturated* and *M- unsaturated*)
- **Example:**



A matching in bipartite graph

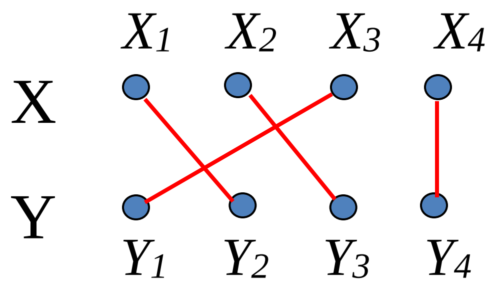


A matching in general graph

Perfect Matching 3.1.2

- A **perfect matching** in a graph is a matching that saturates every vertex.

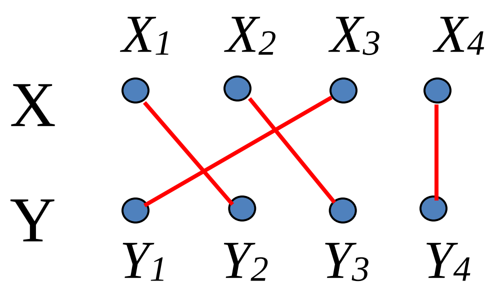
- **Example:**



Example: Perfect matchings in $K_{n,n}$ 3.1.2

- Consider $K_{n,n}$ with partite sets $X=\{x_1,\ldots, x_n\}$ and $Y=\{y_1,\ldots, y_n\}$. Perfect matching bijection from X to Y.
- Successively finding mates for $x_1, x_2,\ldots$ yields $n!$ perfect matchings.

We can express the matchings as matrices



$$\begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{array}
\begin{array}{c} \begin{array}{cccc} Y_1 & Y_2 & Y_3 & Y_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

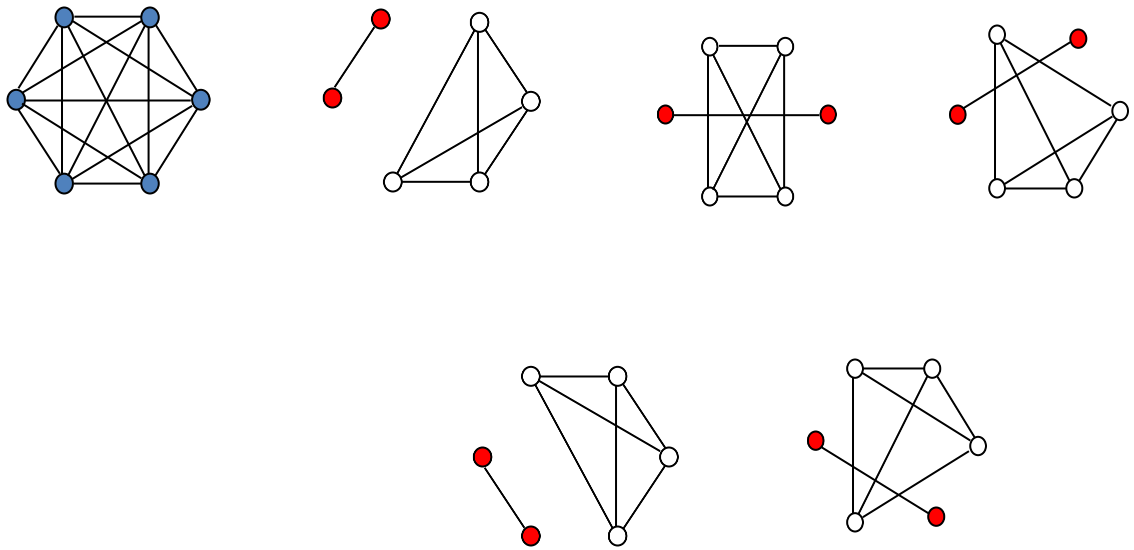
Example: Perfect Matchings in Complete graphs 2.1.3

- K_{2n+1} has no perfect matching. Since it has odd order
- K_{2n} ways to pair up $2n$ distinct people is f_n
  - There are $2n$-1 choices for partner of v_{2n}, and for each such choices there are f_{n-1} ways to complete the matching.
  - Hence $f_n=(2n-1)f_{n-1}$ for $n\geq 1$. With $f_0=1$, it follows by induction that $f_n=(2n-1)\cdot(2n-3)\cdot\cdot\cdot(1)$.

There is also a counting argument for f_n . From an ordering of $2n$ people, we form a matching by pairing the first two, the next two, and so on. Each ordering thus yields one matching. Each matching is generated by $2^n n!$ orderings, since changing the order of the pairs or the order within a pair does not change the resulting matching.
Thus there are $f_n = (2n)!/(2^n n!)$ perfect matchings.

Example: Perfect Matchings in K_6

- For K_6 , *number of perfect matchings* is f_3,
  - f_3 is the number of perfect matchings
  - f_3= (2n-1) * f_2 = 5* f_2=5*3*f_1

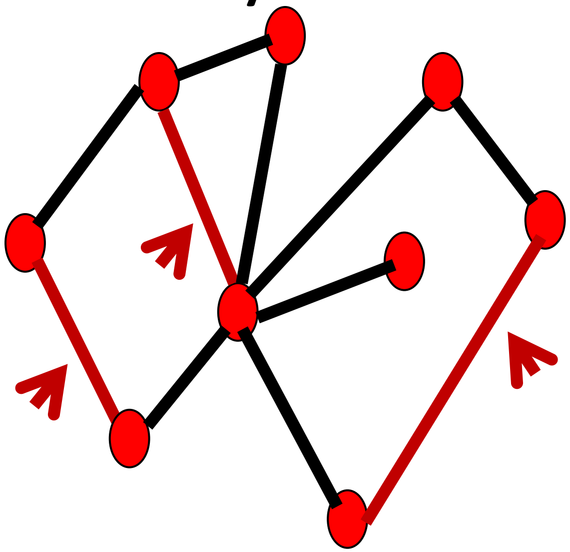


Maximal Matching and Maximum Matching 3.1.4

- A ***maximal matching*** in a graph is matching that can not be enlarged by adding an edge.
- A ***maximum matching*** is a matching of maximum size among all matchings in the graph.
- A matching M is *maximal* if every edge not in M is incident to an edge already in M.
- Every maximum matching is a maximal matching, but the converse need not hold.

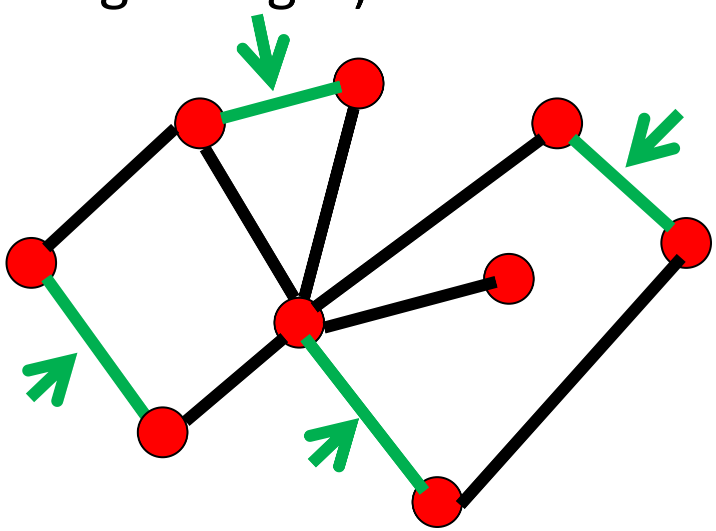
Example: Maximal Matching

- A ***matching*** is ***maximal*** if no more edges may be added.



Maximal

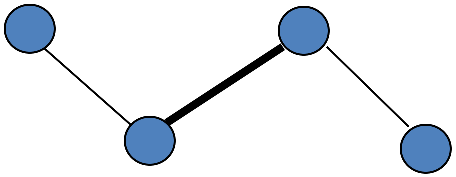
- A graph's ***maximum*** matching is its largest (more edges or total edge weight)



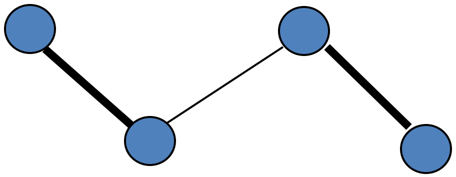
Maximum

Maximal ≠ Maximum 3.1.5

- The smallest graph having a maximal matching that is not a maximum matching is P_4.
  - If we take the middle edge, then we can add no other, but the two end edges form a larger matching.
  - Below we show this phenomenon in P_4.



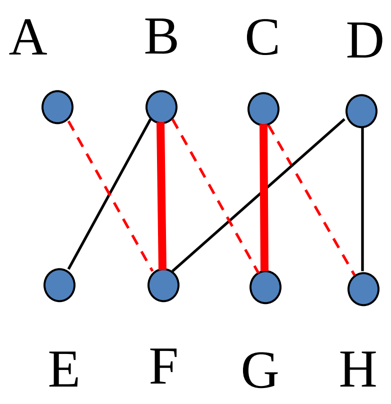
Maximal



Maximum

Alternating path & Augmenting path 3.1.6

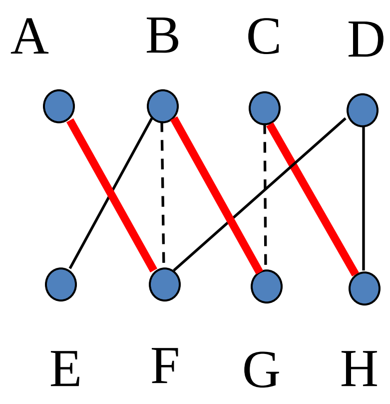
- Given a matching *M*, an ***M-alternating path*** is a path that alternates between edges in *M* and edges not in *M.*
- An *M*-alternating path whose endpoints are unsaturated by *M* is a ***M*-augmenting path.**



Augmenting path:

1) E-B-F-D

2) A-F-B-G-C-H



Alternating path & Augmenting path 3.1.6

- Given an **M-augmenting path P**, we can replace the edges of M in P with the other edges of P to obtain a new matching M' with one more edge. Thus when M is a maximum matching, there is no M-augmenting path.

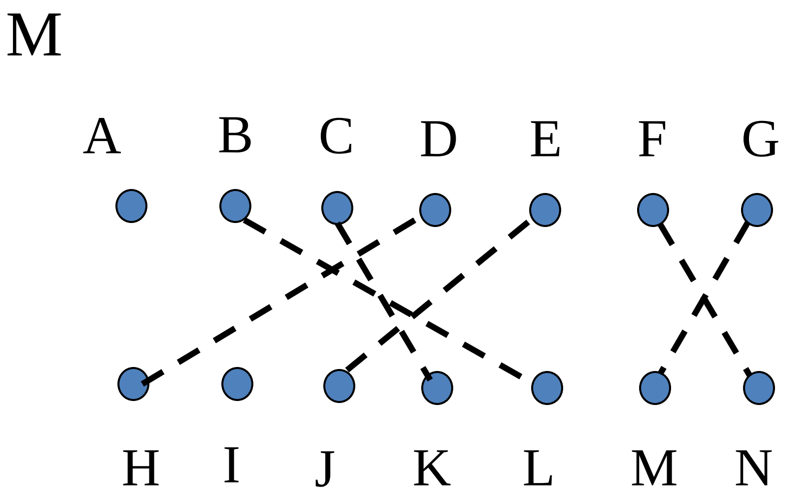
- Maximum matchings are characterized by the absence of augmenting paths. It can be proved by considering two matchings and examining the set of edges belonging to exactly one of them. This operation can be defined for any two graphs with the same vertex set.

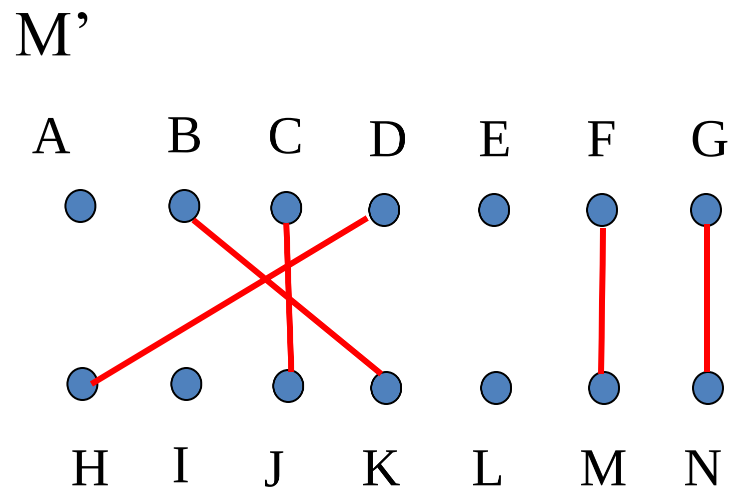
Symmetric Difference 3.1.7

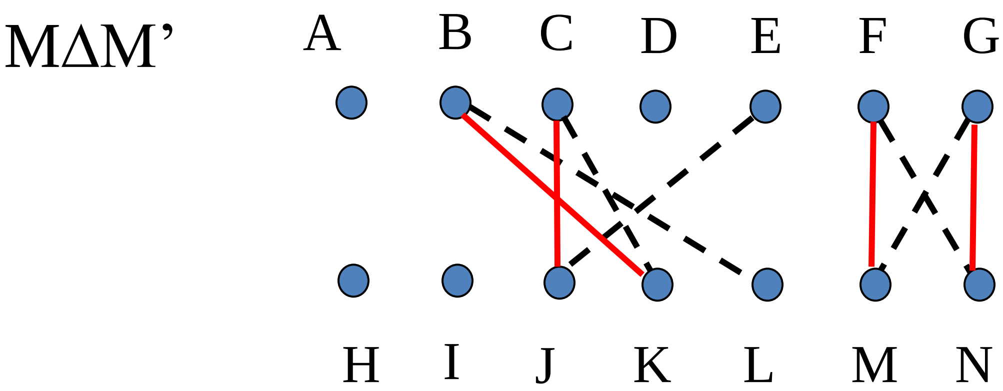
- If G and H are graphs with vertex set V, then the ***symmetric difference*** $G \Delta H$ is the graph with vertex set V whose edges are those edges appearing in exactly one of G and H.

- We also use this notation for sets of edges ; in particular, if M and M' are matchings, then $M \Delta M' = (M - M') \cup (M' - M)$.

Example of Symmetric Difference (1)

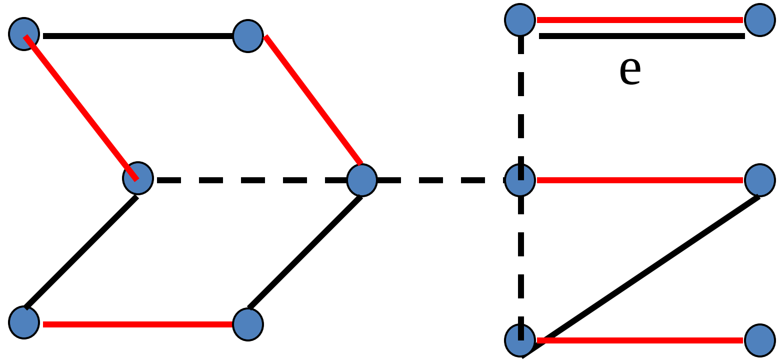






Example of Symmetric Difference (2)

- In the graph below, M is the matching with five solid edges M' is the one with six bold edges, and the dashed edges belong to neither M not M'. The two matchings have one common edge e; it is not in their symmetric difference. The edges of M∆M' form a cycle of length 6 and a path of length 3.



Lemma: Every component of the symmetric difference of two matchings is a path or an even cycle 3.1.9

Proof: Let M and M' be matchings, and $F = M \Delta M'$.

- Since M and M' are matchings, every vertex has at most one incident edge from each of them.
- Thus F has at most two edges at each vertex.
- Since $\Delta(F) \leq 2$, every component of F is a path or a cycle.
- Furthermore, every path or cycle in F alternates between edges $M - M'$ and edges of $M' - M$.
- Thus each cycle has even length, with an equal number of edges from M and from M'.

Theorem: (Berge [1957)] A matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path. 3.1.10

Proof: We prove **the contrapositive** of each direction;

***G* has a matching larger than *M* if and only if *G* has an *M*-augmenting path.**

- **(sufficiency)** an M-augmenting path can be used to produce a matching larger than M.
- **(necessity)** Let M' be a matching in G larger than M; we construct an M-augmenting path
  - Let $F=M\Delta M'$. By Lemma 3.1.9, F consists of paths and even cycles; the cycles have the same number of edges from M and M'.
  - Since $|M'|>|M|$, F must have a component with more edges of M' than of M. Such a component can only be a path that starts and ends with an edge of M'; thus it is a M-augmenting path in G.

Hall's Matching Condition

- When we are filling jobs with applicants, there may be many more applicants than jobs; successfully filling the jobs will not use all applicants.

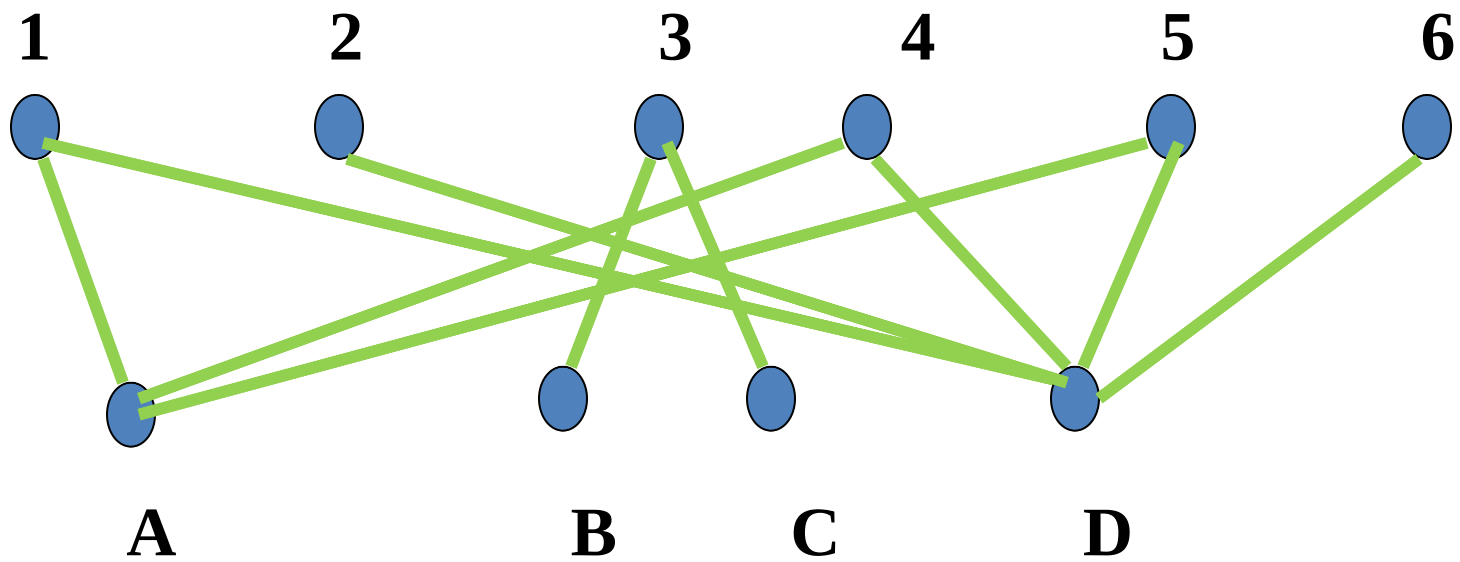
- To model this problem, we consider an X, Y-bigraph (bipartite graph with bipartition X, Y), and we seek a matching that saturates X.

Hall's Matching Condition

- If a matching M saturates X, then for every $S \subseteq X$ there must be at least $|S|$ vertices that have neighbors in S, because the vertices matched to S must be chosen from that set.
- We use $N_G(S)$ or simply $N(S)$ to denote the set of vertices having a neighbor is S. Thus $|N(S)| \geq |S|$ is a necessary condition.
- The condition "For all $S \subseteq X$, $|N(S)| \geq |S|$" is **Hall's Condition**, Hall proved that this obvious necessary condition is also sufficient **(TONCAS)**

Example

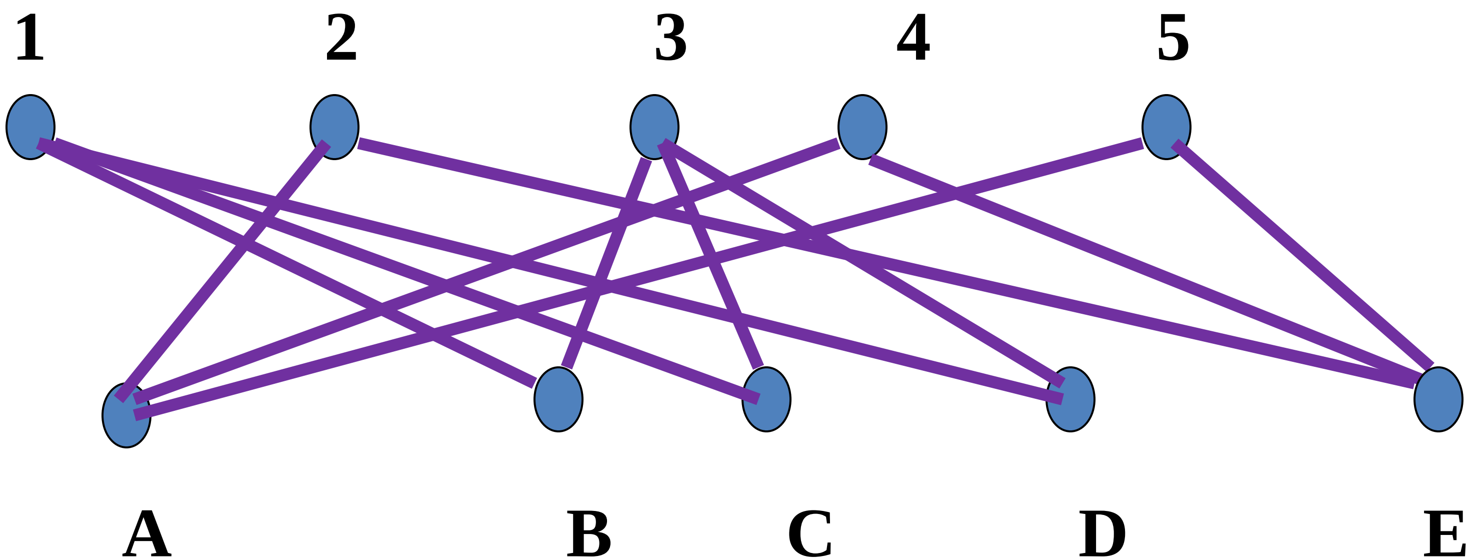
- Does G have a matching of size 4?



- Let X={B,C}   N(X) = {3}. |N(X)|=1 |X|=2
- Since |N(X)|<|X|, Hence violates Hall's Condition i.e no matching of size 4 exists.

Example

- Does G have a matching of size 5?



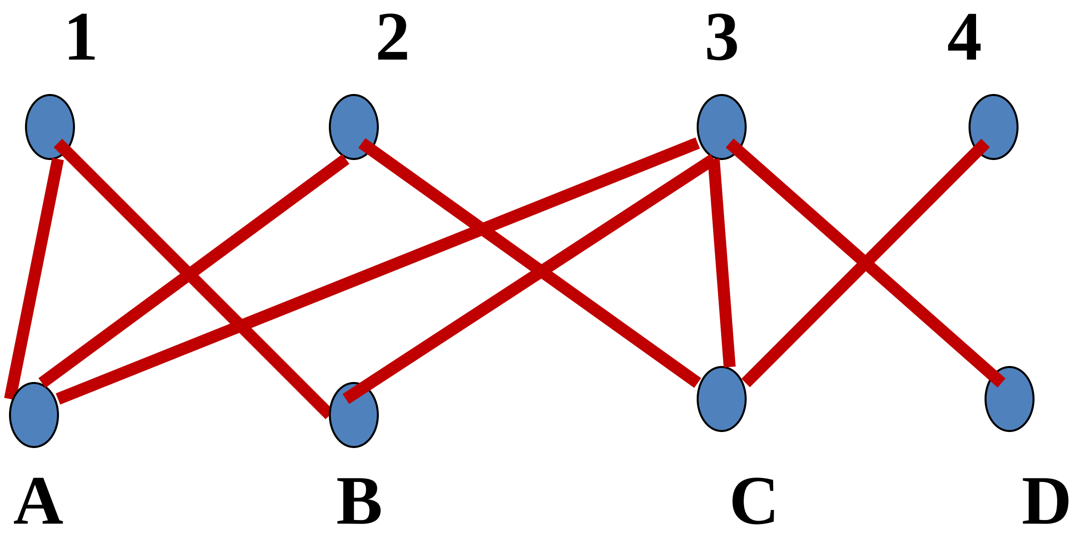
- Let X={B,C, D} N(X) = {1, 3}. $|N(X)| \not\geq |X|$

  $2 \not\geq 3$

- Violates Hall's Condition: No matching of size 5.

Example

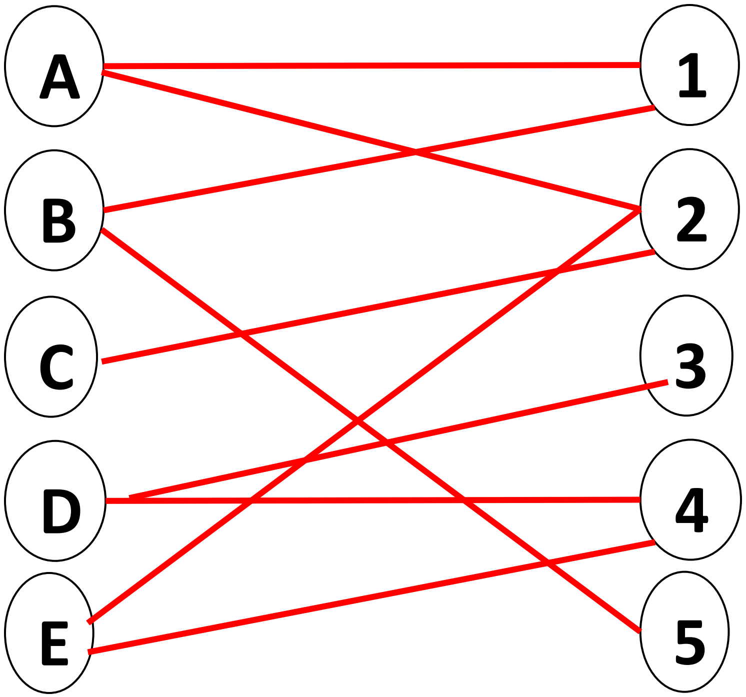
- Does G have a matching of size 4?

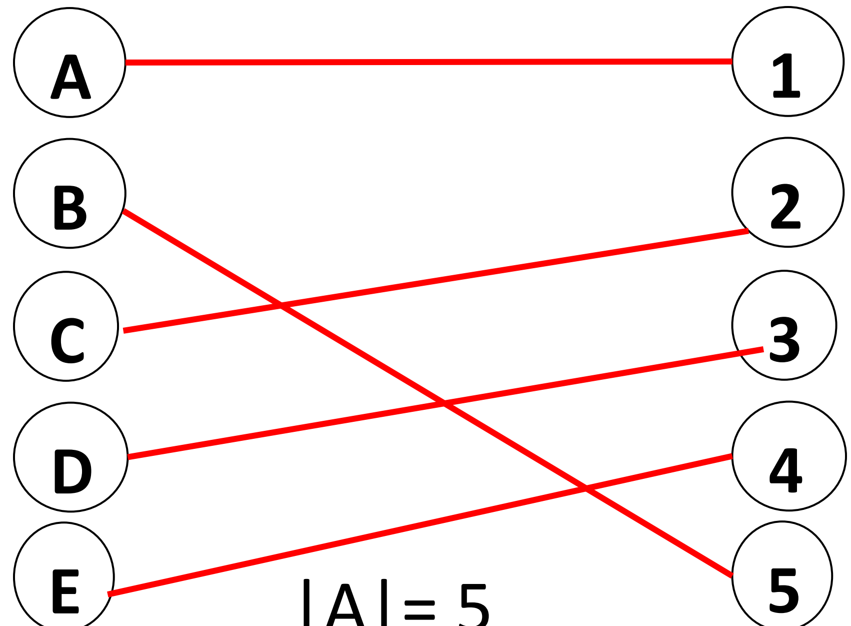


- Yes. {C4, D3, A2, B1}

Example

- You have a group of 5 friends and you only want to offer your friends chocolate if you know for sure that each of them can get a piece





|A|= 5

|N(A)|=9

|N(A)|≥ |A|

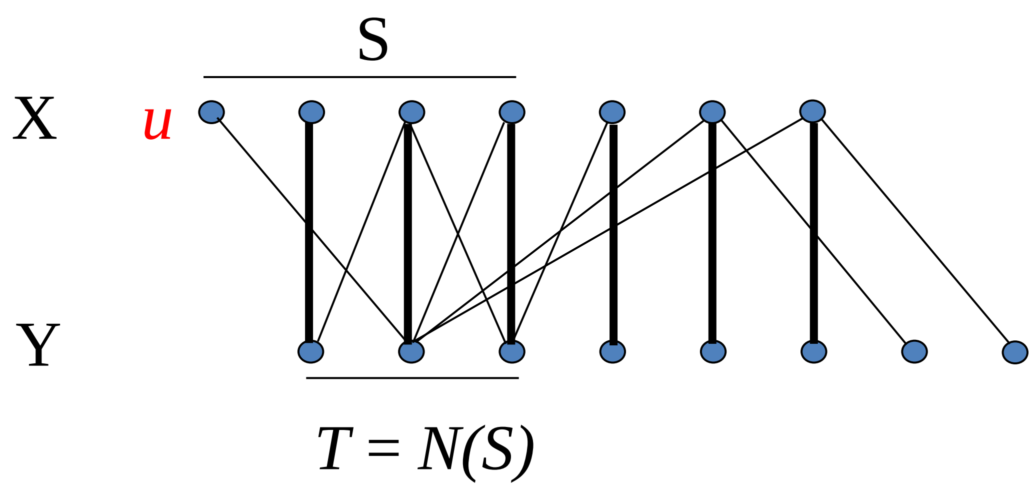
9 > 5

Theorem: (Hall's Theorem- P. Hall [1935] An *X,Y*-bigraph *G* has a matching that saturates *X* if and only if $|N(S)| \geq |S|$ for *all* $S \subseteq X$. 3.1.11

- ***Necessity.*** The $|S|$ vertices matched to S must lie in $N(S)$. i.e **$|N(S)| \geq |S|$ for all $S \subseteq X$**
- ***Sufficiency.*** To prove that Hall's Condition is sufficient, we prove the contrapositive:

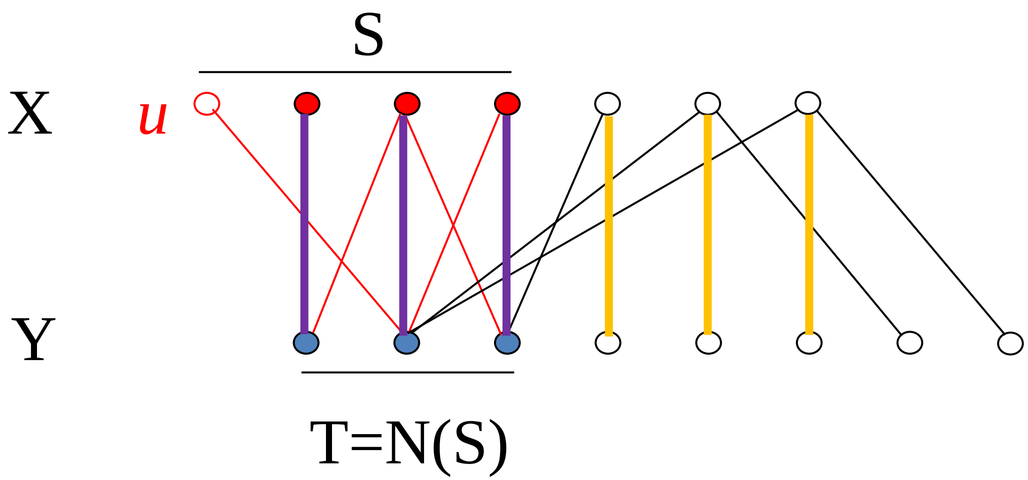
  If *M* is a maximum matching in *G* and *M* does not saturate *X*, then there is a set $S \subseteq X$ such that $|N(S)|<|S|$.

  - Let $u \in X$ be a vertex unsaturated by M. Among all the vertices reachable from u by M-alternating paths in G, let S consist of those in X, and let T consist of those in Y. Note that $u \in S$.



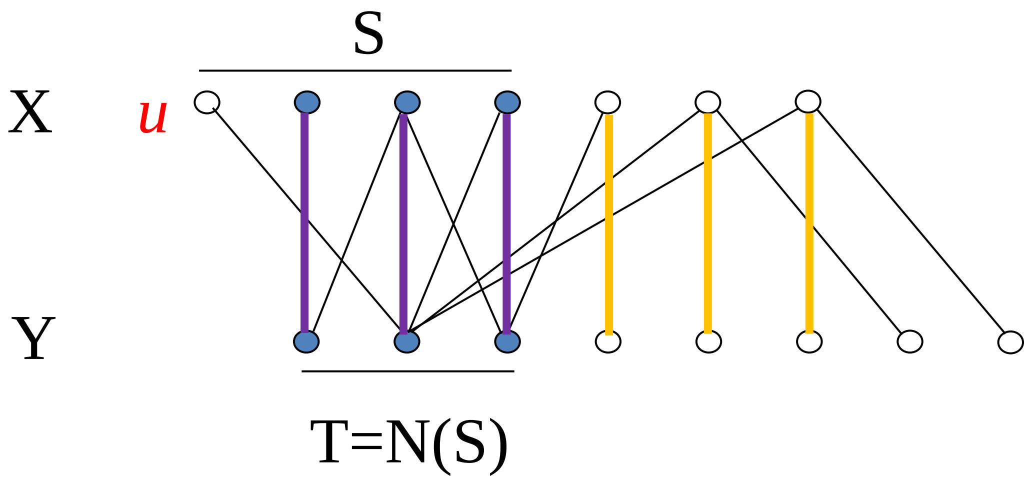
Theorem 3.1.11 Continue

- We claim that M matches T with S - $\{u\}$. The M-alternating paths from u reach Y along edges not in M and return to X along edges in M.
- Hence every vertex of S - $\{u\}$ is reached by an edge in M from a vertex in T.



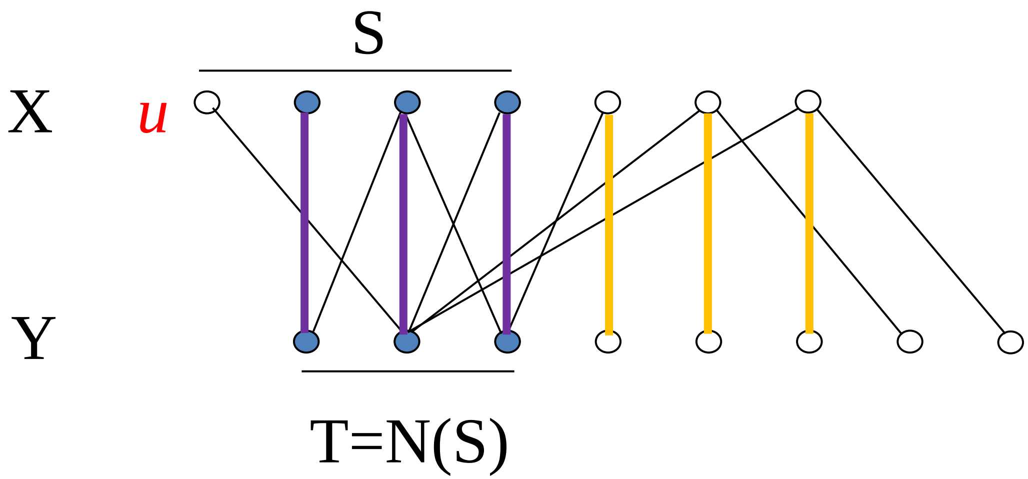
Theorem 3.1.11 Continue

- Since there is no M-augmenting path, every vertex of T is saturated; thus an M-alternating path reaching $y \in T$ extends via M to a vertex of S.
- Hence these edges of M yield a bijection from T to S - $\{u\}$, and we have $|T|=|S - \{u\}|$.



Theorem 3.1.11 Continue

- The matching between T and S - $\{u\}$ yields $T \subseteq N(S)$. In fact, $T = N(S)$. Suppose that $y \in Y - T$ has a neighbor $v \in S$. The edge vy can not be in M, since u is unsaturated and the rest of S is matched to T by M. Thus adding vy to an M-alternating path reaching v yields an M-alternating path to y. This contradicts $y \notin T$, and hence vy cannot exist
- With $T = N(S)$, we have proved that $|N(S)|=|T|=|S|-1 < |S|$ for this choice of S. This completes the proof of the contrapositive.



Remark 3.1.12

- Theorem 3.1.11 implies that whenever an X,Y-bigraph has no matching saturating *X*, we can verify this by exhibiting a subset of *X* with too few neighbors.
- When the sets of the bipartition have the same size, Hall's Theorem is the **Marriage Theorem**(proved originally by Frobenius [1917].) The name arises from the setting of the compatibility relation between a set of *n* men and a set of *n* women, If every man is compatible with *k* women and every woman is compatible by *k* men, then a perfect matching must exist.

Corollary: for $k>0$, every k-regular bipartite graph has a perfect matching. 3.1.13

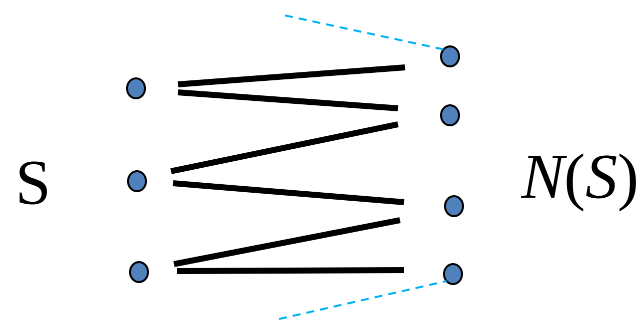
Proof: Let G be a k-regular X,Y- bigraph.

- Counting the edges by endpoints in X and by endpoints in Y shows that $k|X|=k|Y|$, so $|X|=|Y|$. Hence it suffices to verify Hall's Condition; a matching that saturates X will also saturate Y and be perfect matching.

Corollary: for $k>0$, every k-regular bipartite graph has a perfect matching. 3.1.13

Proof: Continued

- Consider $S \subseteq X$. Let m be the number of edges from S to $N(S)$. Since G is k-regular, $m=k|S|$. These m edges are incident to $N(S)$, so $m \le k|N(S)|$. Hence $k|S| \le k|N(S)|$, which yields $|N(S)|\ge|S|$ when $k>0$. Having chosen $S \subseteq X$ arbitrarily, we have established Hall's condition.



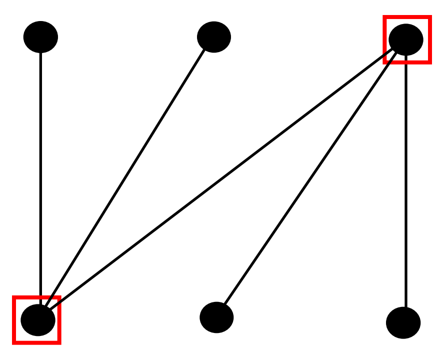
$$m = k|S|, \quad m \le k|N(S)|$$

Min-Max Theorems

- When a graph G does not have a perfect matching, Theorem 3.1.10 allows us to prove that M is a maximum matching by proving that G has no M-augmenting path.
- Exploring all M-alternating paths to eliminate the possibility of augmentation could take a long time.
- Instead of exploring all M-alternating paths, we would prefer to exhibit an explicit structure in G that forbids a matching larger than M.

Vertex Cover 3.1.14

- A ***vertex cover*** of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. The vertices in Q *cover* $E(G)$.
- **Example:**



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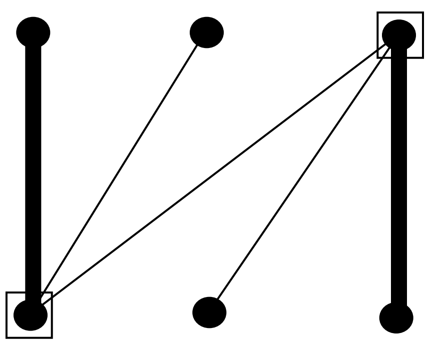
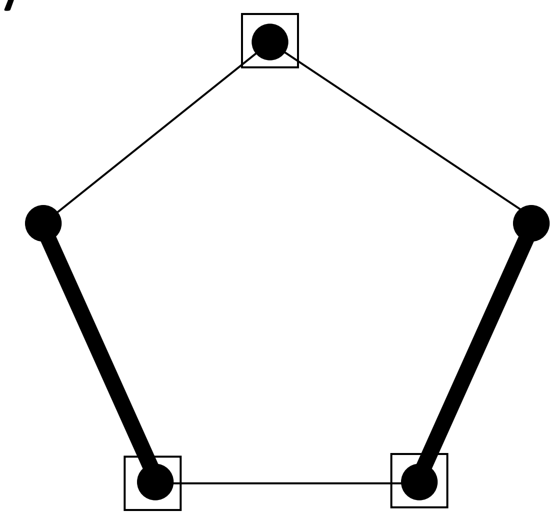
- Since no vertex can cover two edges of a matching, the **size of every vertex cover is at least the size of every matching**.

- Therefore, obtaining a matching and a vertex cover of the same size PROVES that each is optimal Such proofs exist for bipartite graphs, but not for all graphs.

Example: Matchings and Vertex covers

- In the graph on the left below,
  - We mark a vertex cover of size 2 and show a matching of size 2 in bold. The vertex cover of size 2 prohibits matchings with more than 2 edges, and the matching of size 2 prohibits vertex covers with fewer than 2 vertices.
  - |vertex cover| ≥ | matching|
- As illustrated on the right in the next page, the optimal values differ by 1 for an odd cycle. The difference can be arbitrarily large.

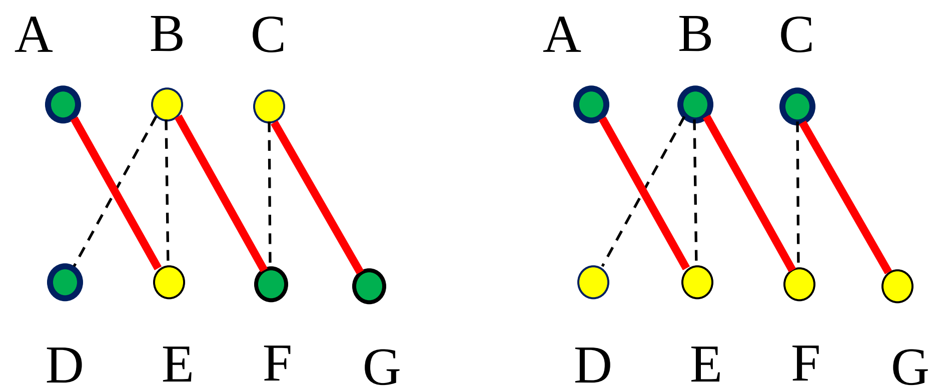




Theorem: (König [1931], Egerváry [1931]) 3.1.16

- If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.

Green: Vertex cover
Red: Matching
$|Q| \geq |M|$



- Since distinct vertices must be used to cover the edges of a matching, $|Q| \geq |M|$ whenever Q is a vertex cover and M is a matching in G.

Conclusion

- In this lecture, we have discussed the Concept of Matching, Perfect matchings, Maximal matchings, Maximum Matchings, M-alternating path, M-augmenting path, Symmetric difference, Hall's Matching condition and Vertex covers.

- In upcoming lectures, we will discuss Min-Max Theorems, Independent sets and covers.