k-Connected Graphs

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Preface

Recap of Previous Lecture:

In previous lecture, we have discussed Connectivity *i.e.* vertex connectivity, edge connectivity, bond, blocks and also discuss the theorems based on the cuts and connectivity.

Content of this Lecture:

In this lecture, we will discuss the k-Connected Graphs.

k-Connected Graphs

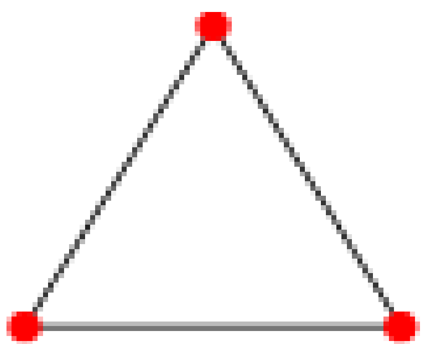
- A communication network is fault-tolerant if it has alternative paths between vertices: the more disjoint paths, the better.

- In this lecture, we will prove that this alternative measure of connection is essentially the same as k-connectedness. When k=1, the definition already states that a graph G is 1-connected iff each pair of vertices is connected by a path. For larger k the equivalence is more subtle.

2-Connected Graphs

- **Definition:** Two paths from *u* to *v* **are internally disjoint** if they have no common internal vertex.

- **Example:** 2-Connected Graph



Theorem 4.2.2

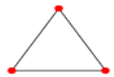
- **(Whitney [1932])** A graph G having at least three vertices is 2-connected if and only if for each pair $u, v \in V(G)$ there exist internally disjoint u, v-paths in G.

Proof: Sufficiency: When G has internally disjoint u, v-paths, deletion of one vertex cannot separate u from v. Since this condition is given for every pair u, v, deletion of one vertex cannot make any vertex unreachable from any other. We conclude that G is 2-connected.

Proof continue

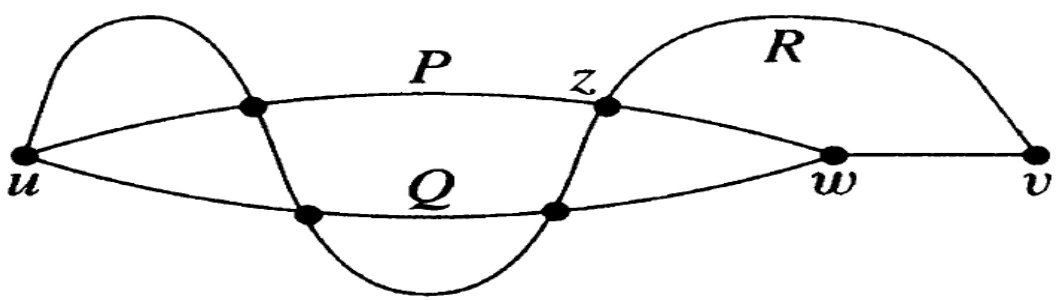
Necessity: Suppose that G is 2-connected. We prove by induction on d(u,v) that G has internally disjoint u, v -paths.

Basis step (d(u, v) = 1). When d(u,v)=1, the graph G-uv is connected, since $\kappa'(G) \geq \kappa(G) \geq 2$. A u,v-path in G-uv is internally disjoint in G from the u, v-path formed by the edge uv itself.



Induction step (d(u,v) > 1). Let k=d(u,v). Let w be the vertex before v on a shortest u,v-path; we have d(u,w)=k-1. By the induction hypothesis, G has internally disjoint u,w-paths P and Q. If v ϵ V(P) U V(Q), then we find the desired paths in the cycle P U Q. Suppose not.

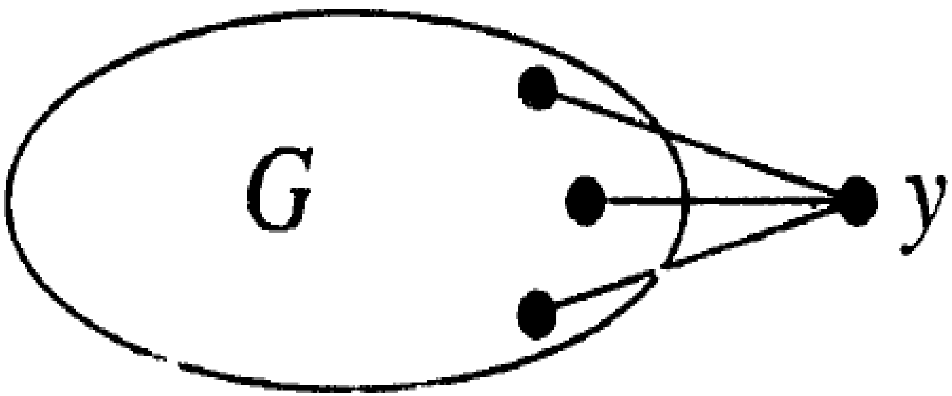
Since G is 2-connected, G-w is connected and contains a u,v-path R. If R avoids P or Q, we are done, but R may share internal vertices with both P and Q. Let z be the last vertex of R (before v) belonging to P U Q. By symmetry, we may assume that z ϵ P. We combine the u, z-subpath of P with the z, v-subpath of R to obtain a u,v-path internally disjoint from Q U w v.



Lemma (Expansion Lemma) 4.2.3

- If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected.

Proof: We prove that a separating set S of G' must have size at least k. If y $\in$ S, then S-{y} separates G, so $|S| \geq k+1$, If y $\notin$ S and N(y) $\subseteq$ S, then $|S| \geq k$. Otherwise, y and N(y)-S lie in a single component of G'-S. Thus again S must separate G and $|S| \geq k$.



Theorem 4.2.4

For a graph G with at least three vertices, the following conditions are equivalent (and characterize 2-connected graphs).

A) G is connected and has no cut-vertex.

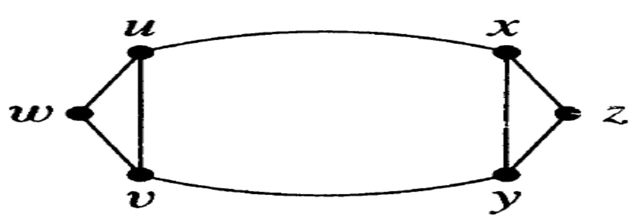
B) For all x, y∈ V(G), there are internally disjoint x, y-paths.

C) For all x, y∈ V(G), there is a cycle through x and y.

D) $\delta(G) \geq 1$, and every pair of edges in G lies on a common cycle.

Proof:

- **Theorem 4.2.2** proves A⇔B
- **For B⇔ C**, note that cycles containing x and y correspond to pairs of internally disjoint x, y-paths.
- **For D⇔ C,** the condition $\delta(G) \geq 1$ implies that vertices x and y are not isolated; we then apply the last part of D to edges incident to x and y. If there is only one such edge, then we use it and any edge incident to a third vertex.
- To complete the proof, we assume that G satisfies the equivalent properties A and C and then derive D. Since G is connected, $\delta(G) \geq 1$. Now consider two edges uv and xy. Add to G the vertices w with neighborhood {u, v} and z with neighborhood {x, y}. Since G is 2-connected, the **Expansion Lemma (Lemma 4.2.3)** implies that the resulting graph G' is 2-connected.
- Hence condition C holds in G', so w and z lie on a cycle C in G'. Since w, z each have degree 2, C must contain the paths u, w, v and x, z, y but not the edges uv or xy. Replacing the paths u, w, v and x, z, y in C with the edges uv and xy yields the desired cycle through uv and xy in G.

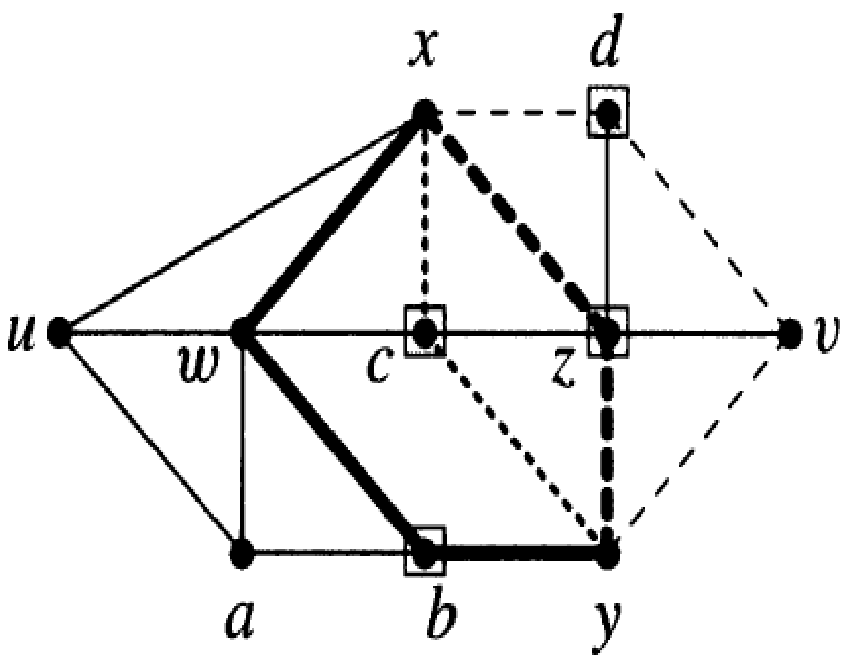


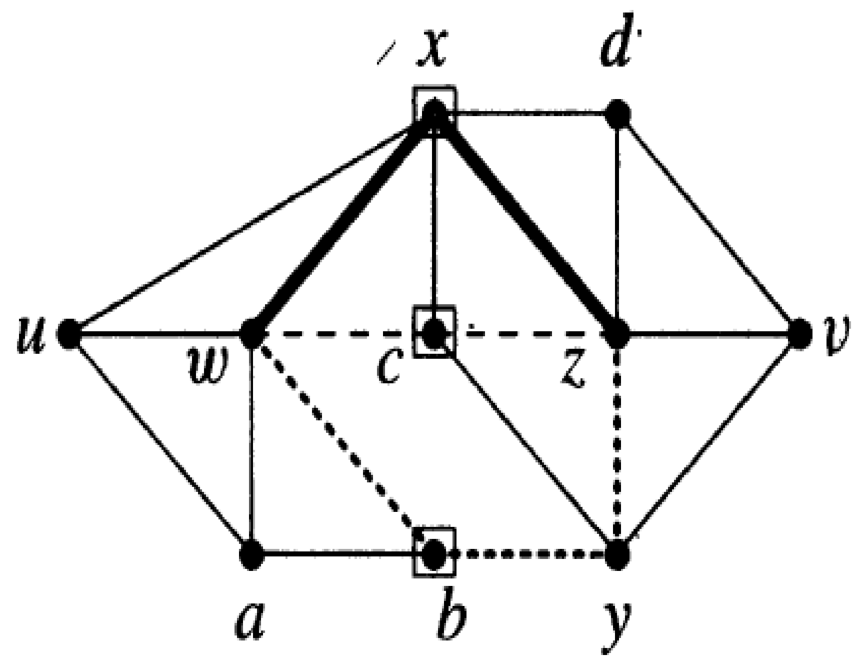
k-Connected and k-Edge-Connected Graphs

- **Def:** Given $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an ***x,y*-separator** or ***x, y*-cut** if G-S has no x, y-path.
- Let $\kappa(x, y)$ be the minimum size of an x, y-cut.
- Let $\lambda(x, y)$ be the maximum size of a set of pairwise internally disjoint x, y-paths.
- For $X, Y \subseteq V(G)$, an ***X, Y*-path** is a path having first vertex in X, last vertex in Y, and no other vertex in $X \cup Y$.
- An x,y-cut must contain an internal vertex of every x,y-path, and no vertex can cut two internally disjoint x, y-paths. Therefore, always $\kappa(x, y) \geq \lambda(x, y)$.
- Thus the problem of finding the smallest cut and the largest set of paths are dual problems.

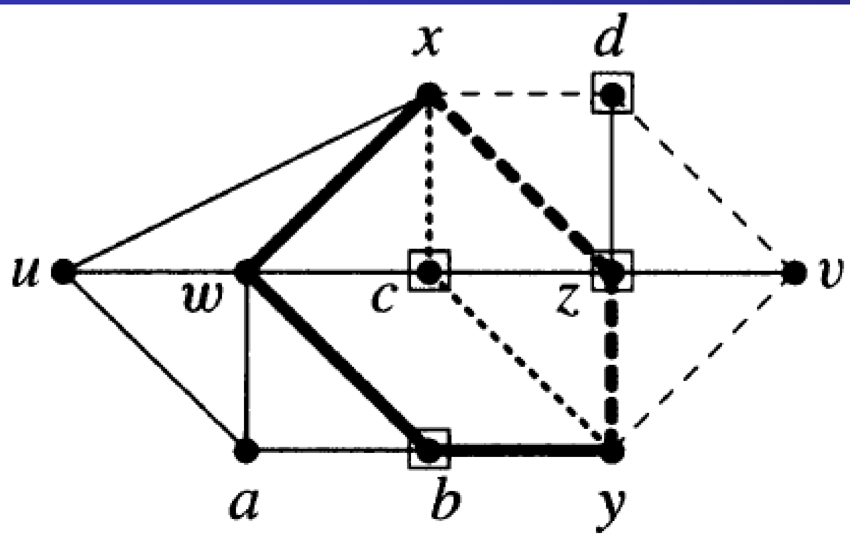
Example 4.2.16

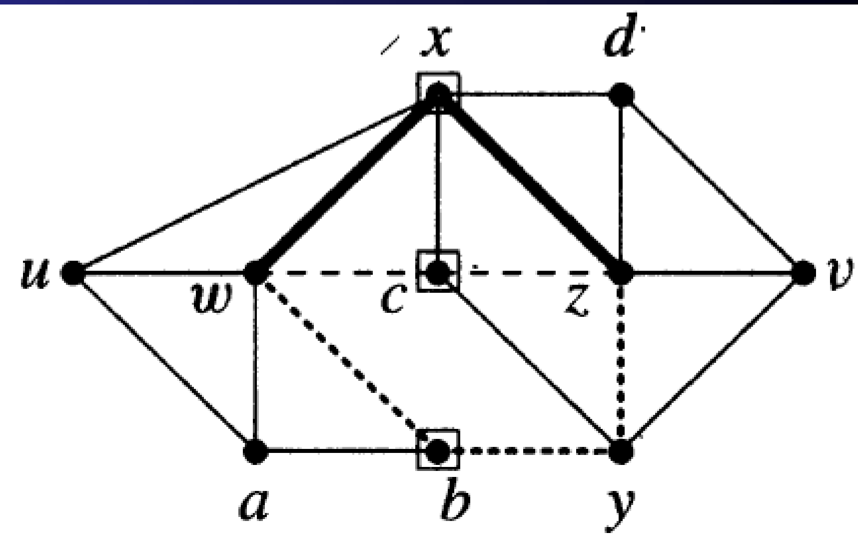
- In the graph G below, the set $S = \{b, c, z, d\}$ is an x, y-cut of size 4; thus $\kappa(x, y) \le 4$. As shown on the left, G has four pairwise internally disjoint x, y-paths; thus $\lambda(x, y) \ge 4$. Since $\kappa(x, y) \ge \lambda(x, y)$ always, we have $\kappa(x, y) = \lambda(x, y) = 4$.





Example continue





- Consider also the pair *w*, *z*. As shown on the right, κ(*w*, *z*) = λ(*w*, *z*) =3, with {*b*, *c*, *x*} being a minimum *w*, *z*-cut. The graph *G* is 3-connected; for every pair *u*, *v* ϵ *V*(*G*), we can find three pairwise internally disjoint *u*, *v*-paths.
- From the equality for internally disjoint paths, we will obtain an analogous equality for edge-disjoint paths. Although κ(*w*, *z*)=3 above, it takes four edges to break all *w*, *z*-paths, and there are four pairwise edge-disjoint *w*, *z*-paths.

Theorem (Menger [1927]) 4.2.17

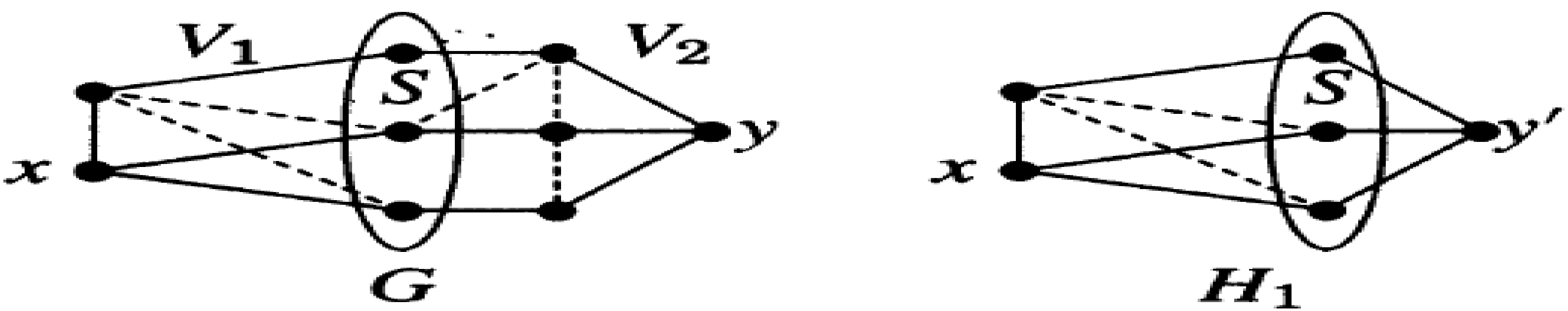
If x, y are vertices of a graph G and $xy \notin E(G)$, then the minimum size of an x, y-cut equals the maximum number of pairwise internally disjoint x, y-paths.

Proof: An x, y-cut must contain an internal vertex from each path in a set of pairwise internally disjoint x, y-paths. These vertices must be distinct, so $\kappa(x, y) \geq \lambda(x, y)$.

- To prove equality, we use induction on $n(G)$. **Basis step:** $n(G) = 2$. Here $xy \notin E(G)$ yields $\kappa(x, y) = \lambda(x, y)=0$.
- **Induction step:** $n(G) > 2$. Let $k= \kappa_G(x, y)$. We construct k pairwise internally disjoint x, y-paths. Note that since $N(x)$ and $N(y)$ are x, y-cuts, no minimum cut properly contains $N(x)$ or $N(y)$.

Case 1: G has a minimum x, y-cut S other than N(x) or N(y)

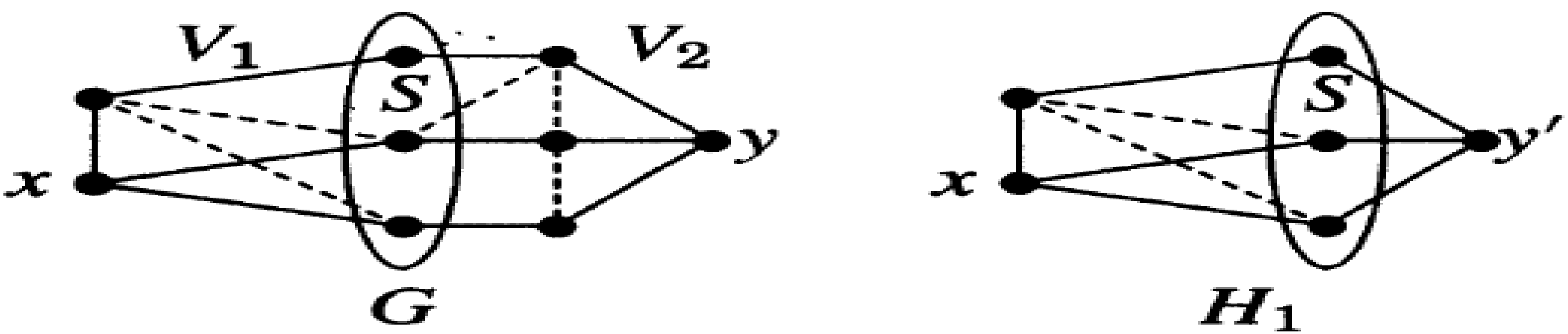
- To obtain the k desired paths, we combine x, S-paths and S, y-paths obtained from the induction hypothesis (as formed by solid edges shown below). Let V_1 be the set of vertices on x, S-paths, and let V_2 be the set of vertices on S, y-paths. We claim that $S = V_1 \cap V_2$. Since S is a minimal x, y-cut, every vertex of S lies on an x, y-path, and hence $S \subseteq V_1 \cap V_2$. If $v \in (V_1 \cap V_2) - S$, then following the x, v-portion of some x, S-path and then the v, y-portion of some S, y-path yields an x, y-path that avoids the x, y-cut S. This is impossible, so $S = V_1 \cap V_2$. By the same argument, V_1 omits N(y)-S and V_2 omits N(x)-S.



Case 1

Case1 continue

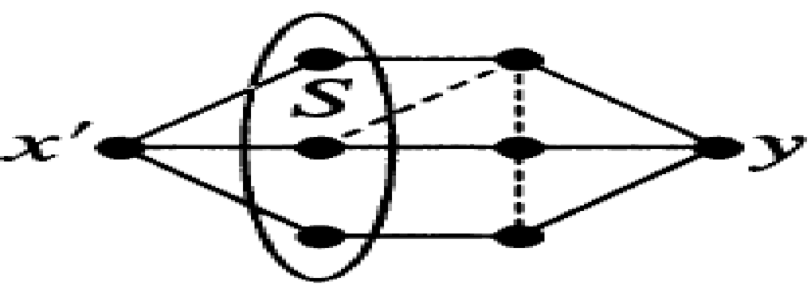
- Form H_1, by adding to $G[V_1]$ a vertex y' with edges from S. From H_2 by adding to $G[V_2]$ a vertex x' with edges to S. Every x, y-path in G starts with an x, S-path (contained in H_1), so every x, y'-cut in H_1 is an x, y-cut in G. Therefore, $\kappa_{H1}(x, y')$=k, and similarly $\kappa_{H2}(x', y)$=k.
- Since V_1 omits N(y)-S and V_2 omits N(x)-S, both H_1 and H_2 are smaller than G. Hence the induction hypothesis yields $\lambda_{H1}(x, y')=k=\lambda_{H2}(x', y)$. Since $V_1 \cap V_2=S$, deleting y' from the k paths in H_1 and x' from the k paths in H_2 yields the desired x, S-paths and S, y-paths in G that combine to form k pairwise internally disjoint x, y-paths in G.



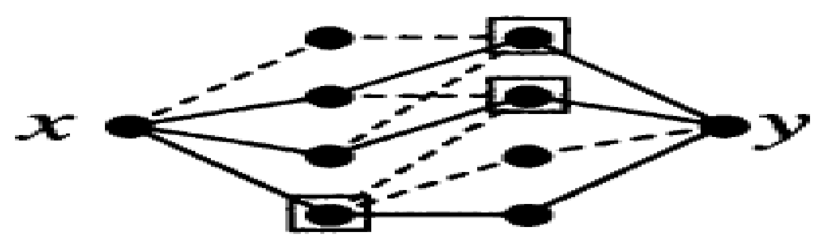
Case 1

Case 2: Every minimum x, y-cut is N(x) or N(y)

- Again we construct the k desired paths. In this case, every vertex outside {x} U N(x) U N(y) U {y} is in no minimum x, y-cut. If G has such a vertex v, then $\kappa_{G-v}(x, y)$=k, and applying the induction hypothesis to G-v yields the desired x, y-paths in G. Also, if there exists u ϵ N(x) ∩ N(y), then u appears in every x, y-cut, and $\kappa_{G-u}(x, y)$=k-1. Now applying the induction hypothesis to G-u yields k-1 paths to combine with the path x, u, y.
- We may thus assume that N(x) and N(y) partitions V(G) - {x, y}. Let G' be the bipartite graph with bipartition N(x), N(y) and edge set [N(x), N(y)]. Every x, y-path in G uses some edge from N(x) to N(y), so the x, y-cuts in G are precisely the vertex covers of G'. Hence β(G')=k. By the König-Egerváry Theorem, G' has a matching of size k. These k edges yield k pairwise internally disjoint x, y-paths of length 3.



H_2

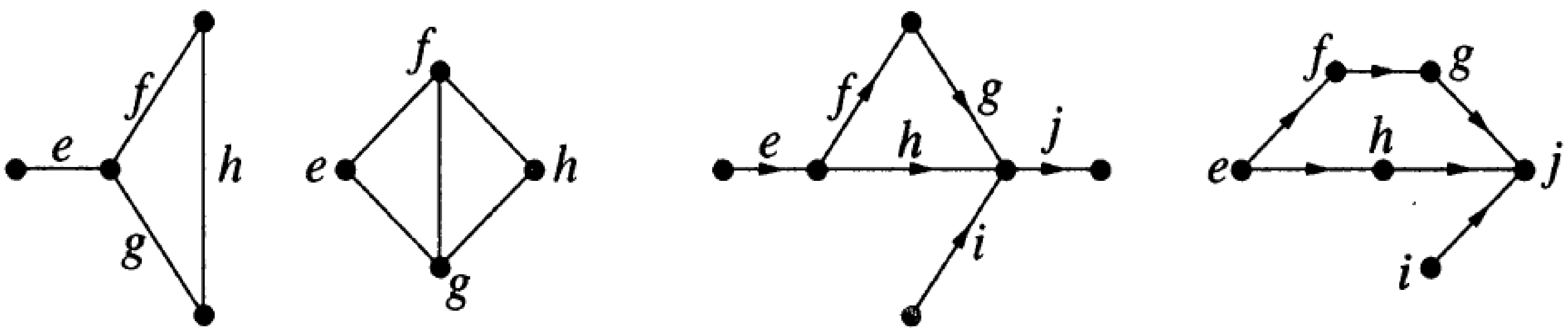


G'

Case 2

Definition: Line Graph 4.2.18

- The **line graph** of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e=uv$ and $f=vw$ in G. Substituting "digraph" for "graph" in this sentence yields the definition of **line digraph**. For graphs, e and f share a vertex; for digraphs, the head of e must be the tail of f.
- **Example:**

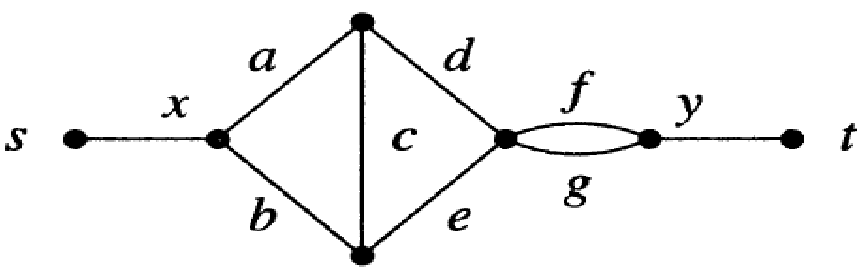


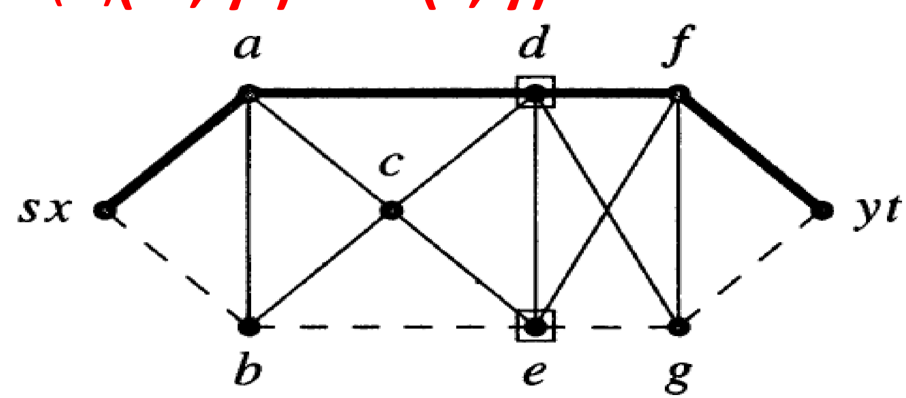
G $\quad$ $L(G)$ $\quad$ H $\quad$ $L(H)$

Theorem 4.2.19

- If x and y are distinct vertices of a graph or digraph G, then the minimum size of an x, y-disconnecting set of edges equals the maximum number of pairwise edge-disjoint x, y-paths.

Proof: Modify G to obtain G' by adding two new vertices s,t and two new edges sx and yt. This does not change $\kappa'(x, y)$ or $\lambda'(x,y)$, and we can think of each path as starting from the edge sx and ending with the edge yt. A set of edges disconnects y from x in G if and only if the corresponding vertices of L(G') form an sx, yt-cut, Similarly, edge-disjoint x,y-paths in G become internally disjoint sx,yt-paths in L(G'), and vice versa, Since $x \neq y$, we have no edge from sx to yt in L(G'). Applying theorem 4.2.17 to L(G') yields $\kappa'_G(x, y) = \kappa_{L(G')}(sx, yt) = \lambda_{L(G')}(sx, yt) = \lambda'_G(x, y)$





Lemma: Deletion of an edge reduces connectivity by at most 1. 4.2.20

- **Proof:** Since every separating set of G is a separating set of G-xy, we have $\kappa(G\text{-}xy) \leq \kappa(G)$. Equality holds unless G – xy has a separating set S that has size less than $\kappa(G)$ and hence is not a separating set of G. Since G-S is connected, G-xy-S has two components G[X] and G[Y], with $x \in X$ and $y \in Y$. In G-S, the only edge joining X and Y is xy.
- If $|X| \geq 2$, then $S \cup \{x\}$ is a separating set of G, and $\kappa(G) \leq \kappa(G\text{-}xy)+1$. If $|Y| \geq 2$, then again the inequality holds. In the remaining case, $|S| = n(G)-2$. Since we have assumed that $|S| < \kappa(G)$, $|S| = n(G)-2$ implies that $\kappa(G) \geq n(G)-1$, which holds only for a complete graph, Thus $\kappa(G\text{-}xy)= n(G)-2= \kappa(G)-1$, as desired.

Theorem 4.2.21

- The connectivity of G equals the maximum k such that $\lambda(x,y) \geq k$ for all $x, y \in V(G)$. The edge-connectivity of G equals the maximum k such that $\lambda'(x,y) \geq k$ for all $x, y \in V(G)$.
- **Proof:** Since $\kappa'(G) = \min_{x,y \in V(G)} \kappa'(x,y)$, Theorem 4.2.19 immediately yields the claim for edge-connectivity.
- For connectivity, we have $\kappa(x,y) = \lambda(x,y)$ for $xy \notin E(G)$, and $\kappa(G)$ is the minimum of these values. We need only show that $\lambda(x,y)$ cannot be less than $\kappa(G)$ when $xy \in E(G)$. Certainly deletion of xy reduces $\lambda(x,y)$ by 1, since xy itself is an x, y-path and cannot lie in any other x, y-path. With this, Theorem 4.2.17, and Lemma 4.2.20, we have

$$\lambda_G(x,y) = 1 + \lambda_{G-xy}(x,y) = 1 + \kappa_{G-xy}(x,y) \geq 1 + \kappa(G-xy) \geq \kappa(G)$$

Conclusion

- In this lecture we have discussed the k-connected graphs, k-edge-connected graphs, Menger's theorem and Line graph.