Planar Graphs

Dr. Rajiv Misra
Associate Professor
Dept. of Computer Science & Engg.
Indian Institute of Technology Patna
rajivm@iitp.ac.in

Preface

Recap of Previous Lecture:

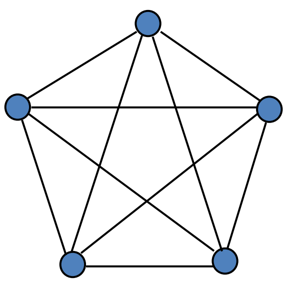
In previous lecture, we have discussed the properties of counting function, chromatic polynomial, chromatic recurrence, and further related topics.

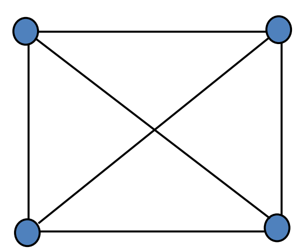
Content of this Lecture:

In this lecture, we will discuss planar graphs i.e. plane graph embeddings, Dual graphs, Euler's formula for plane graphs and Regular Polygedra.

Embeddings on Plane: Applications

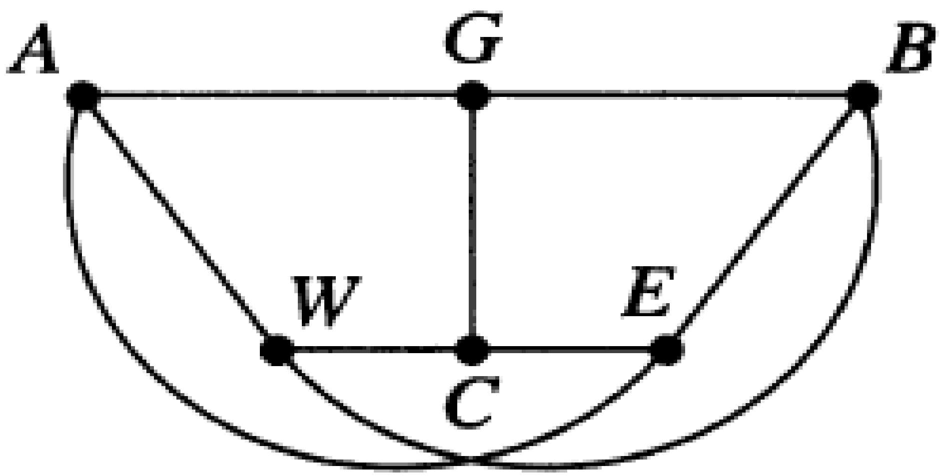
- Topological graph theory, broadly conceived, is the study of graph layouts.
- Initial motivation involved famous **Four Color Problem**: can the regions of every map on a globe be colored with four colors so that regions sharing a nontrivial boundary have different colors?
- Later motivation involves **circuit layouts** on silicon chips. Wire crossings cause problems in layouts, so we ask ; Which circuits have layouts without crossings?





Example: Gas-water-electricity 6.1.1

- Three sworn enemies A,B,C live in houses in the woods. We must cut paths so that each has a path to each of three utilities, which by tradition are gas, water, and electricity. In order to avoid confrontations, we don't want any of the paths to cross. Can this be done? This asks whether $K_{3,3}$ can be drawn in the plane without edge crossing.

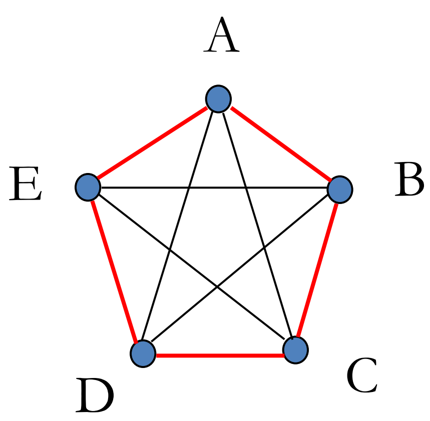


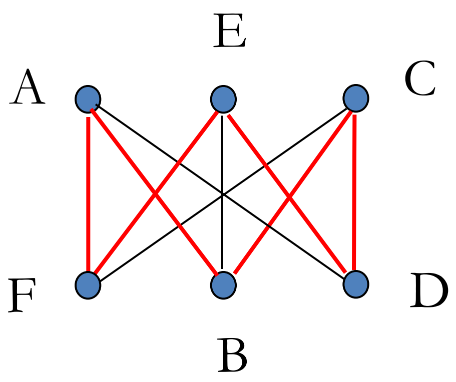
- Arguments about drawings of graphs in the plane are based on the fact that every closed curve in the plane separates the plane into **two regions (the inside and the outside).**
- Before discussing a way to make the arguments precise for graph theory, we will show informally how this result is used to prove impossibility for planar drawings.

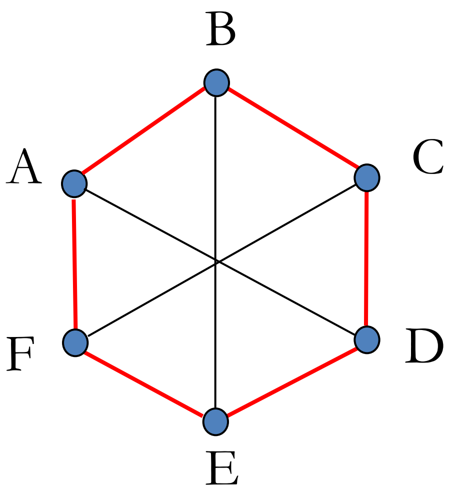
Proposition 6.1.2: K_5 and $K_{3,3}$ cannot be drawn without crossings

Proof:

- Considers a drawing of K_5 or $K_{3,3}$ in the plane.
- Let C be a **spanning cycle**. $\rightarrow$



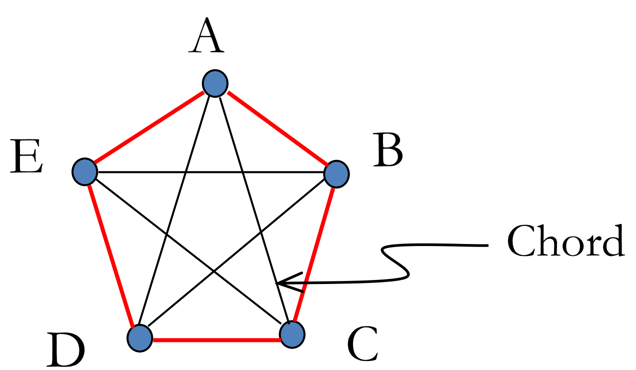




Proposition 6.1.2: K_5 and $K_{3,3}$ cannot be drawn without crossings

Proof: (continue)

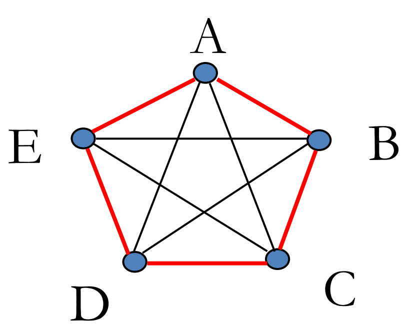
- If the drawing does not have crossing edges,
  - then C is drawn as a closed curve.
  - Chords of C must be drawn inside or outside this curve.
- Two chords conflict if their endpoints on C occur in alternating order.
- When two chords conflict, we can draw only one inside C and one outside C. $\rightarrow$

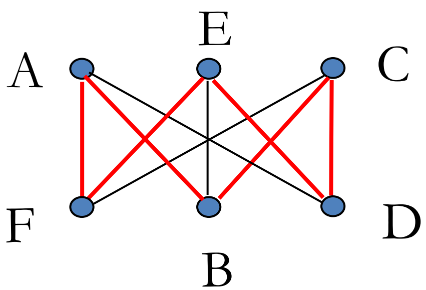


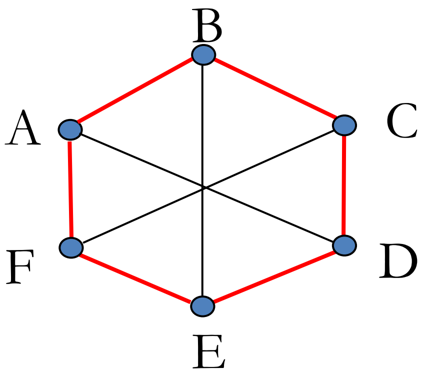
Proposition 6.1.2: K_5 and $K_{3,3}$ cannot be drawn without crossings Continued

Proof: continued

- A 6-cycle in $K_{3,3}$ has three pairwise conflicting chords.
- We can put at most one inside and one outside,
  - so it is not possible to complete the embedding.
- When C is a 5-cycle in K_5, at most two chords can go inside or outside.
  - Since there are five chords, again it is not possible to complete the embeddings.
- Hence neither of these graphs is planar.



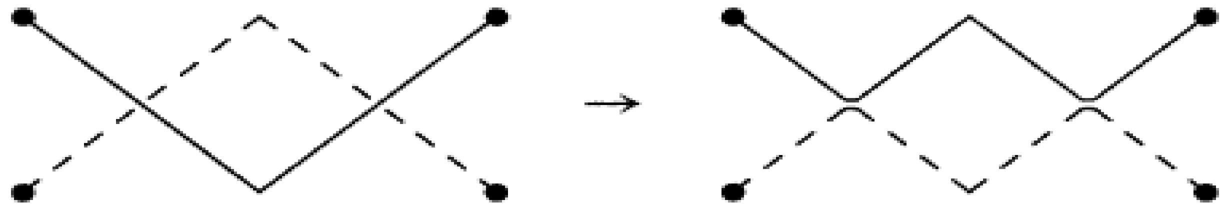




Definitions: Curve, Drawing 6.1.3

- A ***curve*** is the image of a continuous map from [0, 1] to R^2.
- A ***polygonal curve*** is a curve composed of finitely many line segments.
  - It is a **polygonal** u, v-**curve** when it starts at u and ends at v.
- A ***drawing*** of a graph G is a function f defined on $V(G) \cup E(G)$ that assigns each vertex v a point $f(v)$ in the plane and assigns each edge with endpoints u, v a polygonal $f(u)$, $f(v)$-curve.
  - The images of vertices are distinct.
  - A point in $f(e) \cap f(e')$ that is not a common endpoint is a **crossing.**

- It is common to use the same name for a graph G and a particular drawing of G, referring to the points and curves in the drawing as the vertices and edges of G. Since **the endpoint relation between the points and curves is the same as the incidence relation between the vertices and edges**, the drawing can be viewed as a member of the **isomorphism class** containing G.
- By moving edges slightly, we can ensure that no three edges have a common internal point, that **an edge contains no vertex except its endpoints, and that no two edges are tangent.**
- If two edges cross more than once, then modifying them as shown below reduces the number of crossings; thus we also require that edges cross at most once. We consider only drawings with these properties.

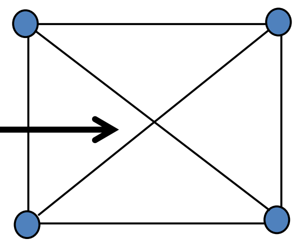


Definitions: Planar Graph, Plane Graph 6.1.4

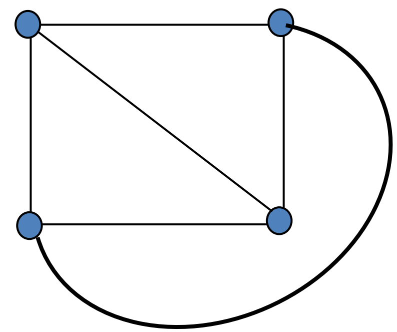
- A graph is ***planar*** if it has a drawing without crossings.
  - Such a drawing is a ***planar embedding*** of *G*.

- A ***plane graph*** is a particular planar embedding of a planar graph.

edge crossing



K4



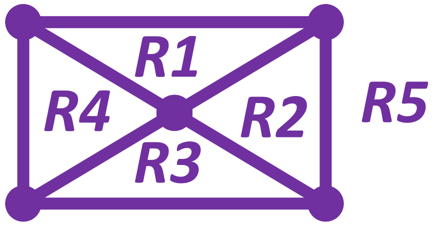
K4

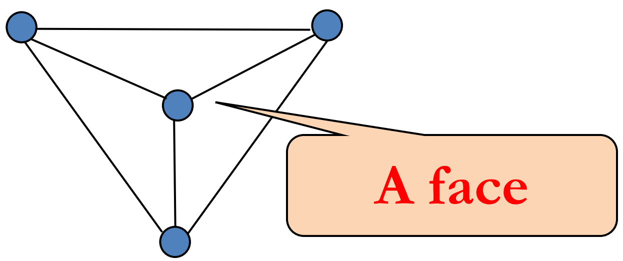
no edge crossing

- A *curve* is **closed** if its *first* and *last* points are the same.
  - It is **simple** if it has no repeated points except possibly *first* = *last*.
- A planar embedding of a graph ***cuts*** the plane into pieces.
  - These pieces are fundamental objects of study.

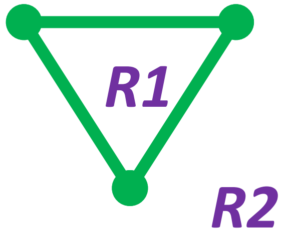
Definitions: Open set, Region, Faces 6.1.5

- An ***open set*** in the plane is a set $U \subseteq R^2$ such that for every $p \in U$, all points within some small distance from p belong to U.
- A ***region*** is an open set U that contains a polygonal u,v-curve for every pair $u,v \in U$.
- The ***faces*** of a plane graph are the maximal regions of the plane that contain no point used in the embedding.





Totally, 4 faces



- A finite plane graph G has one unbounded face (also called the **outer face**). The faces are pairwise disjoint. Points p,q $\in R^2$ lying in no edge of G are in the same face if and only if there is **a polygonal p,q-curve that crosses no edge**.

- In a plane graph, every cycle is embedded as a simple closed curve. Some faces lie inside it, some outside. This again relies on the fact that a simple closed curve cuts the plane into two regions. As we have suggested, this is not too difficult for polygonal curves. We will present some detail of this case in order to explain how to compute whether a point is in the inside or the outside. This proof appears in **Tveberg [1980]**

Dual Graphs

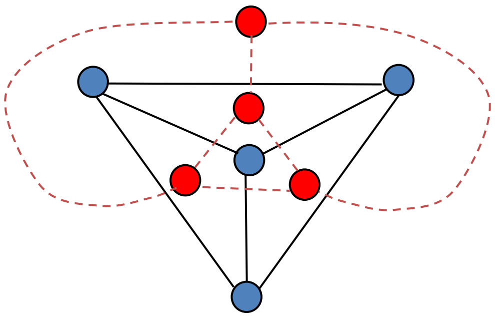
- A map on the plane or the sphere can be viewed as a plane graph in which the faces are the territories, the vertices are places where boundaries meet, and the edges are the portions of the boundaries that join two vertices.

- We allow the full generality of loops and multiple edges. From any plane graph G, we can form a related plane graph called its “**dual**”.

Definition: Dual Graphs 6.1.7

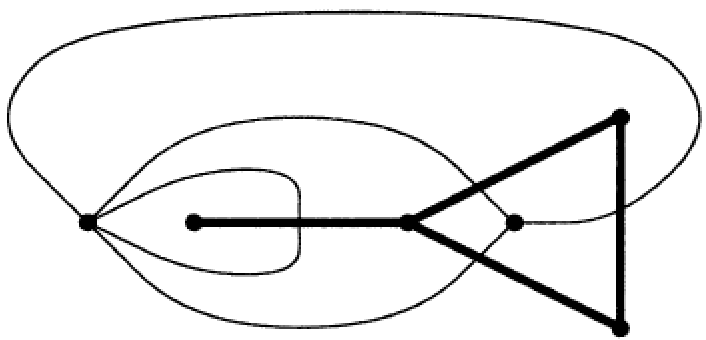
- The **dual graph** G^* of a plane graph G is a plane graph whose **vertices** correspond to the faces of G.

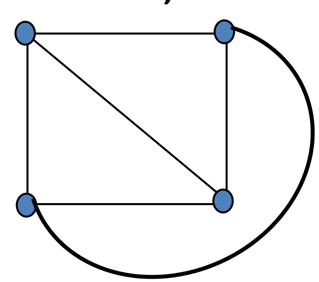
- The **edges** of G^* correspond to the edges of G as follows:
  - if e is an edge of G with face X on one side and face Y on the other side, then the endpoints of the dual edge $e^* \in E(G^*)$ are the vertices x, y of G^* that represent the faces X, Y of G.
  - The order in the plane of the edges incident to $x \in V(G^*)$ is the order of the edges bounding the face X of G in a walk around its boundary.



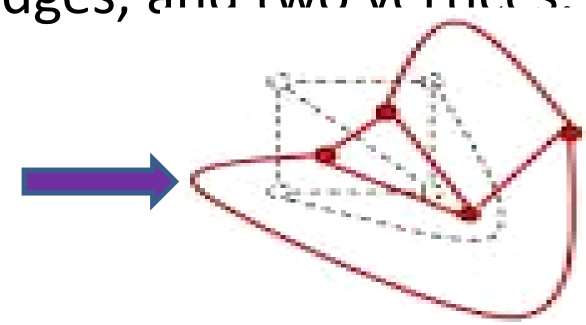
Example: Dual of Graph6.1.8

- Every planar embedding of K_4 has four faces, and these pairwise share boundary edges. Hence the dual is another copy of K_4.
- Every planar embedding of the cube Q_3 has eight vertices, 12 edges, and six faces. Opposite faces have no common boundary; the dual is a planar embedding of $K_{2,2,2}$, which has six vertices, 12 edges, and eight faces.
- Taking the dual can introduce loops and multiple edges. For example, let G be the paw, drawn below in bold edges as a plane graph. Its dual graph G* is drawn in solid edges. Since G has four vertices, four edges, and two faces, G* has four faces, four edges, and two vertices.





K4



Dual of K4

Remark 6.1.9

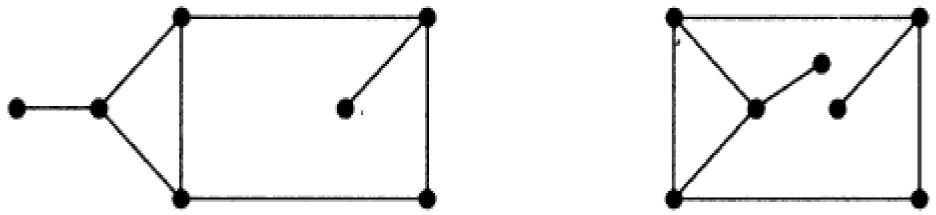
1) Example 6.1.8 shows that a simple plane graph may have loops and multiple edges in its dual. **A cut-edge** of G becomes a **loop** in G*, because the faces on both sides of it are the same. Multiple edges arise in the dual when distinct faces of G have more that one common boundary edge.

2) Some arguments require more careful geometric description of the dual. For each face X of G, we place the dual vertex x in the interior of X, so each face of G contains one vertex of G*. For each edge e in the boundary of X, we draw a curve from x to a point on e; these do not cross. Each such curve meets another from the other side of e at the same point on e to form the edge of G* that is dual to e. No other edges enter X. Hence G* is a plane graph, and each edge of G* in this layout crosses exactly one edge of G.

Such arguments lead to a proof that **(G*)* is isomorphic** to G if and only if G is connected. Mathematicians often use the word "**dual**" in a setting when performing an operation twice returns the original object.

Example 6.1.10 Non-isomorphic dual

- Two embeddings of a planar graph may have nonisomorphic duals. Each embedding shown below has three faces, so in each case the dual has three vertices. In the embedding on the right, the dual vertex corresponding to the outside face has degree 4. In the embedding on the left, no dual vertex has degree 4, so the duals are not isomorphic.
- This does not happen with 3-connected graphs. Every 3-connected planar graph has essentially one embedding.

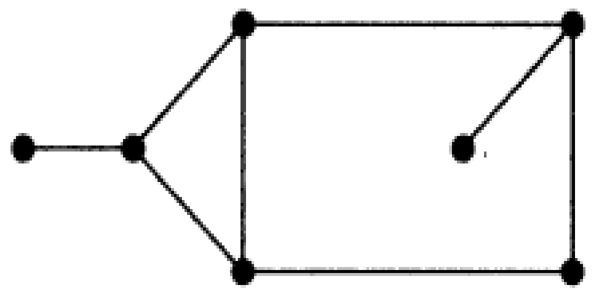


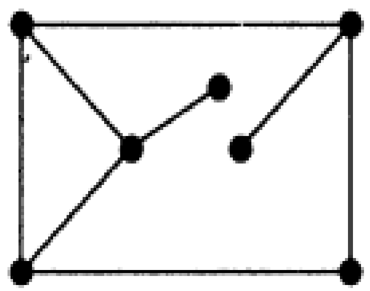
- When a plane graph is connected, the boundary of each face is a **closed walk**. When the graph is not connected, there are faces whose boundary consists of more than one closed walk.

Face and its length 6.1.11

- The ***length*** of a face in a plane graph *G* is the total length of the closed walk(s) in *G* bounding the face.

- **Example:** A cut-edge belongs to the boundary of only one face, and it contributes twice to its length. Each graph in the given example has three faces. In the embedding on the left the lengths are 3,6,7; on the right they are 3,4,9. The sum of the lengths is 16 in each case, which is twice the number of edges.

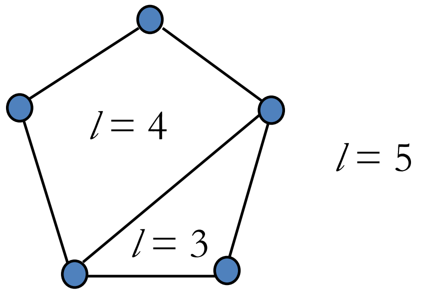




Proposition 6.1.13: If $l(F_i)$ denotes the length of face F_i in a plane graph G, then $2e(G) = \Sigma\, l(F_i)$.

Proof:

- The face lengths are the degrees of the dual vertices. Since $e(G) = e(G^*)$, the statement $2e(G) = \Sigma\, l(Fi)$ is thus the same as the degree-sum formula $2e(G^*) = \Sigma d_{G^*}(x)$ for G^*. (Both sums count each edge twice.)
- **Example:**



$e(G) = 6$

$\Sigma\, l(Fi) = 12$

Remark

- **Proposition 6.1.13** illustrates that statements about a connected plane graph becomes statements about the dual graph when we interchange the roles of vertices and faces. Edge incident to a vertex become edge bounding a face, and vice versa, so the roles of face lengths and vertex degrees interchange.

- We can also interpret coloring of G* in terms of G. The edges of G* represent shared boundaries between faces of G. Hence the chromatic number of G* equals the number of colors needed to properly color the faces of G. Since the dual of the dual of a connected plane graph is the original graph, this means that four colors suffice to properly color the regions in every planar map if and only if every planar graph has chromatic number at most four.

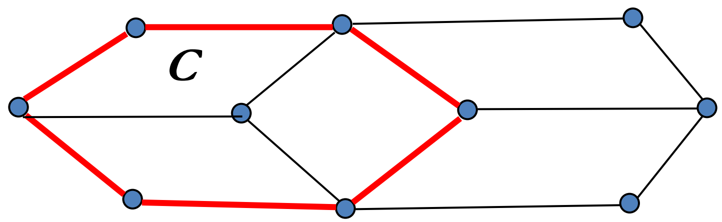
Theorem 6.1.16

The following are equivalent for a plane graph G.

A) G is bipartite.

B) Every face of G has even length.

C) The dual graph G^* is Eulerian.



Theorem 6.1.16 Continued

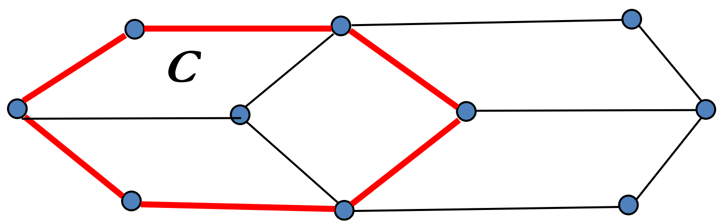
Proof: A⇒B

- A face boundary consists of closed walks.
- Every odd closed walk contains an odd cycle.
- Therefore, in a bipartite plane graph the contributions to the length of faces are all even.

Theorem 6.1.16 Continued

Proof: B$\Rightarrow$A

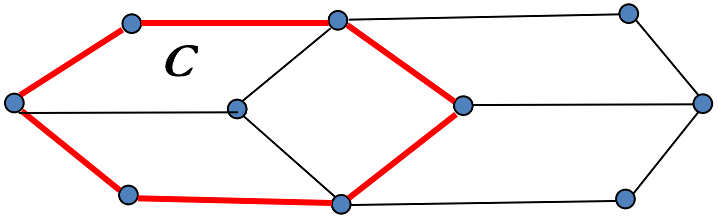
- Let *C* be a cycle in *G*. Since *G* has no crossings, *C* is laid out as a simple closed curve; let *F* be the region enclosed by *C*.
- Every region of *G* is wholly within *F* or wholly outside *F*.
- If we sum the face lengths for the regions inside *F*, we obtain an even number, since each face length is even.
- This sum counts each edge of *C* once. It also counts each edge inside *F* twice, since each such edge belongs twice to faces in *F*. Hence the parity of the length of *C* is the same as the parity of the full sum, which is even.



Theorem 6.1.16 Continued

Proof: B⇔C.

- The dual graph *G** is connected, and its vertex degrees are the face lengths of *G*.

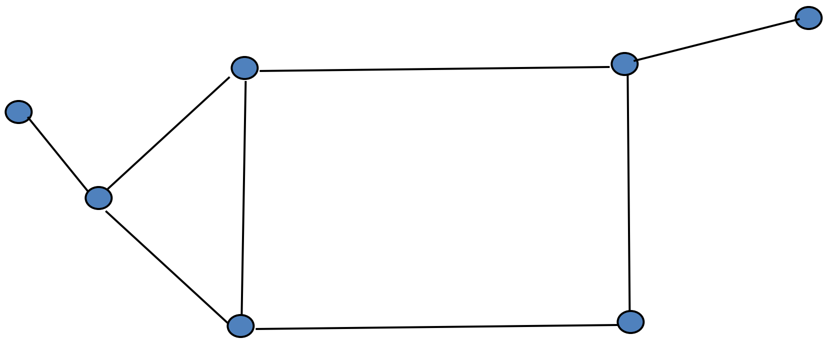


Definition: Outerplanar, Outerplane 6.1.17

- A graph is ***outerplanar*** if it has an embedding with every vertex on the boundary of the unbounded face.

- An ***outerplane*** *graph* is such an embedding of an outerplanar graph.

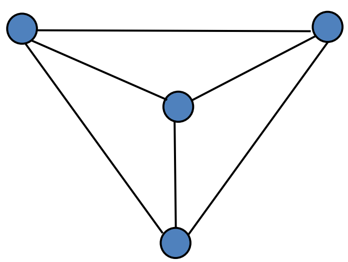
Example: Outerplanar

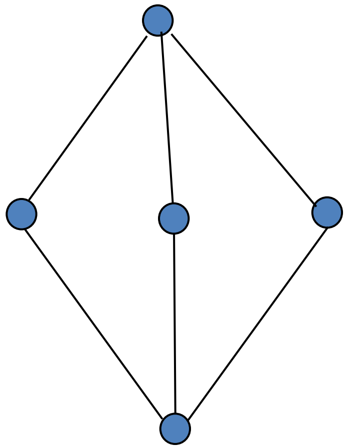


Proposition: K_4 and $K_{2,3}$ are planar but not outerplanar 6.1.19

Proof:

- The figure below shows that K_4 and $K_{2,3}$ are planar.

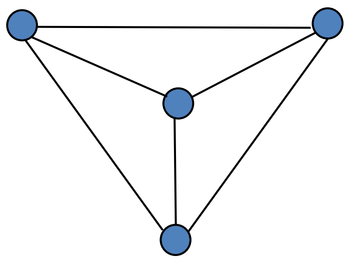


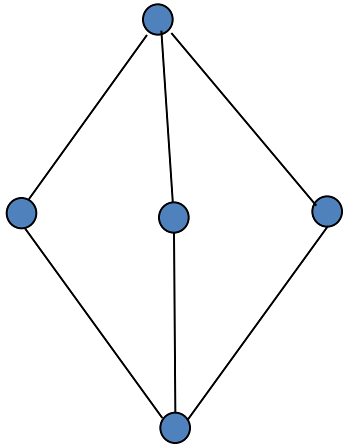


Proposition: K_4 and $K_{2,3}$ are planar but not outerplanar 6.1.19

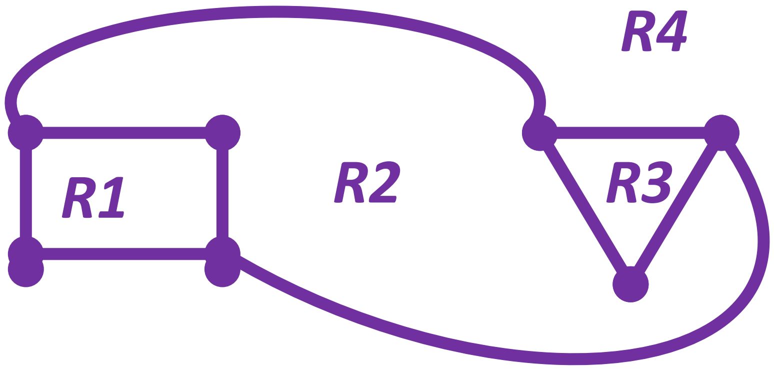
Proof: Continued

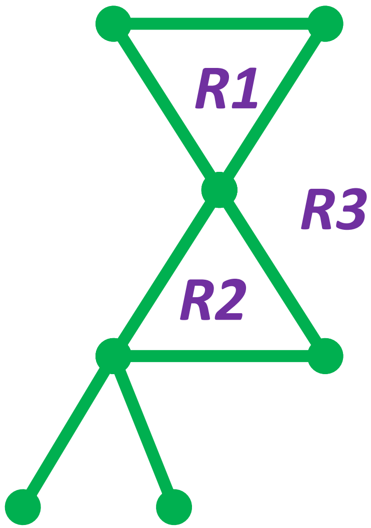
- To show that they are not outerplanar, observe that they are 2-connected. Thus an outerplane embedding requires a spanning cycle. There is no spanning cycle in $K_{2,3}$, since it would be a cycle of length 5 in a bipartite graph.
- There is a spanning cycle in K_4, but the endpoints of the remaining two edges alternate along it. Hence these chords conflict and cannot both be drawn inside. Drawing a chord outside separates a vertex from the outer face.





Euler's Formula: If a connected plane graph *with n* vertices, *m* edges, and *r* regions, then *n-m+r* = 2. 6.1.21





$n = 7$

$m = 9$

$r = 4$

$n-m+r = 2$

$n = 7$

$m = 8$

$r = 3$

$n-m+r = 2$

Euler's Formula: If a connected plane graph *with n* vertices, *m* edges, and *r* regions, then *n*-*m*+*r*=2. 6.1.21

Proof: (Induction on *m*)

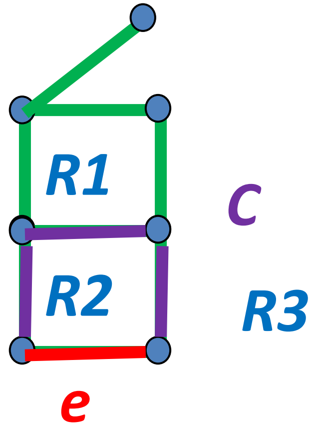
Basis: (m =0)

Then G = K_1 so

$$\left.\begin{array}{l} n=1 \\ m=0 \\ r=1 \end{array}\right\} n-m+r = 1-0+1=2$$

Euler's Formula: If a connected plane graph *with n* vertices, *m* edges, and *r* regions, then *n-m+r*=2. 6.1.21

- Induction Hypothesis: Suppose the theorem is true for all connected plane graphs with <m edges (where m ≥ 1)
- Now consider a connected plane graph G on m edges, n vertices, r regions.
- Case1: If G is a tree then m = n-1 and r =1
  so n-m+r = n-(n-1) +1 = 2
- Case2: If G is not a tree then G has a cycle C
  Let e be an edge of C
  Then e is not a cut=edge
  Hence G-e is connected and planar
  and has n vertices,
  m-1 edges,
  r-1 regions



Removing e means Regions R2 & R3 join into R2 region

By the Induction Hypothesis the theorem holds for G-e

Hence n – (m-1) + (r-1) = 2

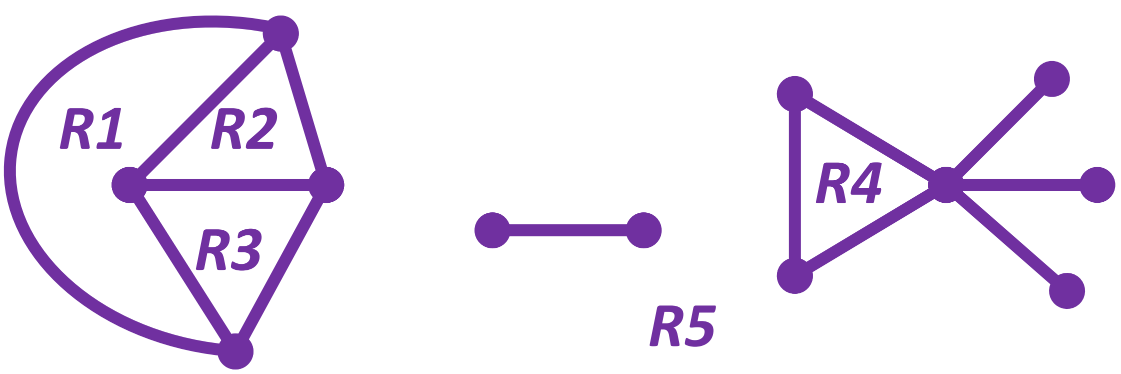
n – m + r = 2

Corollary

A connected plane graph satisfies n – m + r = 2

Corollary: If G is a plane graph with c(G) connected components then n-m+r = 1+ c(G)

Example:



n=12

m=13

r=5

C(G) = 3

n-m+r=12-13+5

Remark 6.1.22

Euler's Formula has many applications, particularly for simple plane graphs, where all faces have length at least 3.

Theorem 6.1.23: If G is a simple planar graph with at least three vertices, then $e(G) \leq 3n(G) - 6$. If also G is triangle-free, then $e(G) \leq 2n(G) - 4$.

Proof:

- It suffices to consider connected graphs; otherwise we could add edges. Euler's Formula will relate $n(G)$ and $e(G)$ if we can dispose of f.

- Proposition 6.1.13 provides an inequality between e and f. Every face boundary in a simple graph contains at least three edges (if $n(G) \geq 3$). Letting $\{ f_i \}$ be the list of face lengths, this yields $2e = \sum f_i \geq 3f$. Substituting into $n - e + f = 2$ yields $e \leq 3n - 6$.
- When G is triangle-free, the faces have length at least 4. In this case $2e = \sum f_i \geq 4f$, and we obtain $e \leq 2n-4$.

Example 6.1.24

- Nonplanarity of K_5 and $K_{3,3}$ follows immediately from Theorem 6.1.23. For K_5, we have

  $e = 10$ and $3n - 6 = 9$.
- Thus $e > 3n - 6$.

- Since $K_{3,3}$ is triangle-free, we have
- $e = 9$ and $2n - 4 = 8$.
- Thus $e > 2n - 4$
- These graphs have too many edges to be planar.

Definition: Maximal planar graph and Triangulation
6.1.25

- A ***maximal planar graph*** is simple planar graph that is not a spanning subgraph of another planar graph.

- A ***triangulation*** is a simple plane graph where every face boundary is a 3-cycle.

Proposition 6.1.26: For a simple *n*-vertex plane graph *G*, the following are equivalent.

A) *G* has $3n - 6$ edges.

B) *G* is a triangulation.

C) *G* is a maximal plane graph.

Proof:

A⇔B. For a simple n-vertex plane graph, the proof of Theorem 6.1.23 shows that having 3n-6 edges is equivalent to $2e = 3f$, which occurs if and only if each face is a 3-cycle.

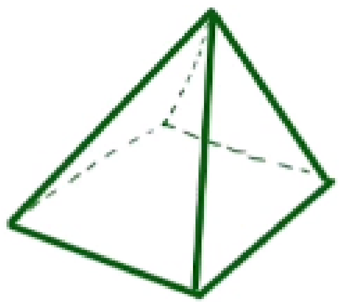
B⇔ C. there is a face that is longer than a 3-cycle if and only if there is a way to add an edge to the drawing and obtain a larger simple plane graph.

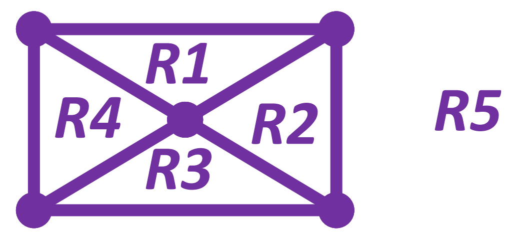
Planar Embeddings: method

- A graph embeds in the plane if and only of it embeds on a sphere. Given an embedding on a sphere, we can puncture the sphere inside a face and project the embedding onto a plane tangent to the opposite point.

- This yields a planar embedding in which the punctured face on the sphere becomes the unbounded face in the plane. The process is reversible.

Euler's Polyhedron Formula





Polyhedron with

5 vertices $V = 5$

8 edges $E = 8$

5 face $F = 5$

Associated graph

$n = 5$

$m = 8$

$r = 5$

If V, E, F are the number of vertices, edges and faces of a polyhedron then $\boxed{V - E + F = 2}$ (Euler's Polyhedron Formula)

Application: Regular Polyhedra 6.1.28

- Informally, we think of a regular polyhedron as a solid whose boundary consists of regular polygons of the same length, with the same number of faces meeting at each vertex.

- When we expand the polyhedron out to a sphere and then lay out the drawing in the plane as in Remark 6.1.27, we obtain a regular plane graph with faces of the same length. Hence the dual also is a regular graph.

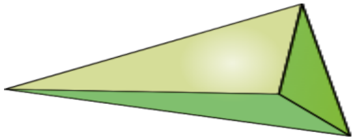
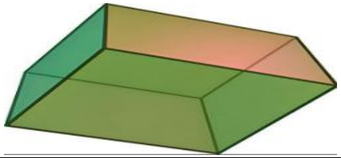
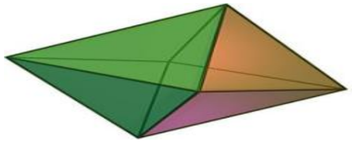
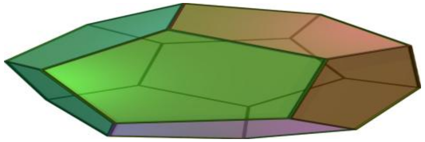
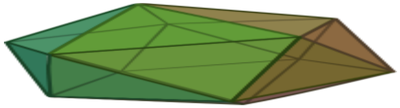
Regular Polyhedra continue

- Let G be a plane graph with n vertices, e edges, and f faces. Suppose that G is regular of degree k and that all faces have length l. The degree-sum formula for G and for G^* yields $kn=2e=lf$. By substituting for n and f in Euler's Formula, we obtain $e(2/k - 1 + 2/l)=2$. Since e and 2 are positive, the other factor must also be positive, which yields $(2/k) + (2/l) > 1$, and hence $2l + 2k > kl$. This inequality is equivalent to $(k-2)(l-2) < 4$.

- Because the dual of a 2-regular graph is not simple, we require that $k,l \geq 3$. Now $(k-2)(l-2) < 4$ also requires $k,l \leq 5$. The only integer pairs satisfying these requirements for (k,l) are (3,3), (3,4), (3,5), (4,3), and (5,3).

Application: Regular Polyhedra continue

- Once we specify k and l, there is only one way to lay out the plane graph when we start with any face. Hence there are only the **five Platonic solids** listed below, one for each pair (k,l) that satisfy the requirements.

k	l	$(k-2)(l-2)$	e	n	f	Name	Diagram
3	3	1	6	4	4	Tetrahedron	
3	4	2	12	8	6	Cube	
4	3	2	12	6	8	Octahedron	
3	5	3	30	20	12	Dodecahedron	
5	3	3	30	12	20	Icosahedron	

Conclusion

- In this lecture, we have discussed the **Planar Graphs** *i.e.* Drawings in the plane, Dual graphs and Euler's formula.

- In upcoming lecture, we will discuss the **Non Planar Graphs.**