Brooks’ Theorem and Color-Critical Graphs

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Preface

Recap of Previous Lecture:

In previous lecture, we have discussed k-coloring of a graph, optimal coloring, clique number, cartesian product, Upper bounds *i.e.* greedy coloring, register allocation and interval graphs.

Content of this Lecture:

In this lecture, we will discuss the Brooks' Theorem and elementary properties of k-critical graphs.

Brooks' Theorem

- **The bound $\chi(G) \leq 1 + \Delta(G)$** holds with equality for complete graphs and odd cycles.
- By choosing the vertex ordering more carefully, we can show that these are essentially the **only such graphs**.
- This implies, for example, that the Petersen graph is 3-colorable, without finding an explicit coloring. To avoid unimportant complications, we phrase the statement only for connected graphs.
- It extends to all graphs because the chromatic number of a graph is the maximum chromatic number of its components. Many proofs are known; we discuss a modification of the proof by Lovász [1975].

Theorem (Brooks [1941]) 5.1.22

- If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \le \Delta(G)$.

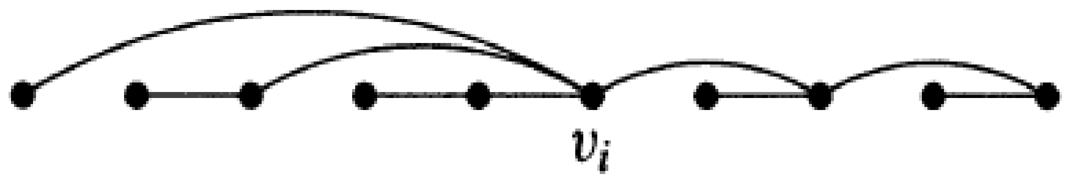
Proof:

- Let G be a connected graph, and let k = $\Delta(G)$. We may assume that k ≥ 3, since G is a complete graph when k ≤ 1, and G is an odd cycle or is bipartite when k = 2, in which case the bound holds.

- Our aim is to order the vertices so that each has at most k-1 lower-indexed neighbors; greedy coloring for such an ordering yields the bound.

Theorem (Brook [1941]) continue

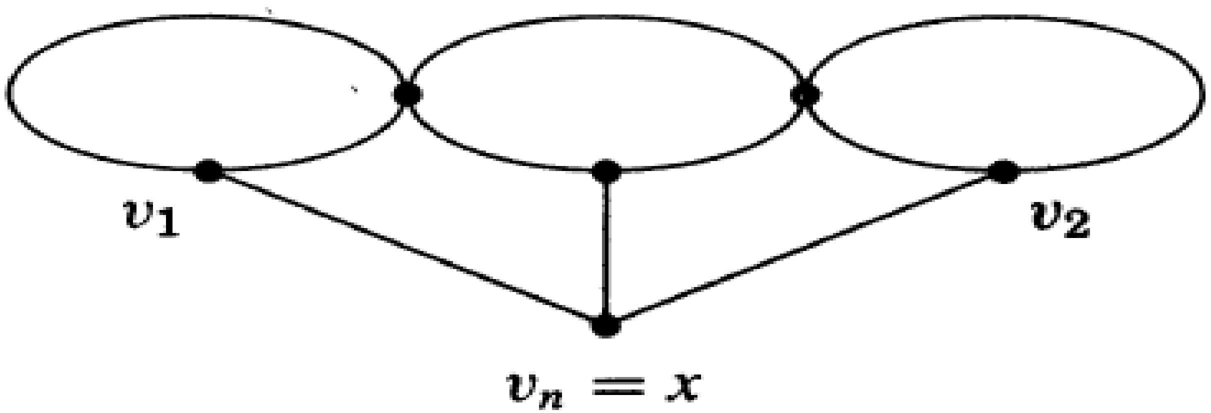
- **When G is not k-regular,** choose a vertex of degree less than k as v_n, Since G is connected, we can grow a spanning tree of G from v_n, assigning indices in decreasing order as we reach vertices. Each vertex other than v_n in the resulting ordering $v_1, \ldots, v_n$ has a higher-indexed neighbor along the path to v_n in the tree. Hence each vertex has at most k-1 lower-indexed neighbors, and the greedy coloring uses at most k colors.



- In the remaining case, **G is k-regular**, Suppose first that **G has a cut-vertex x,** and let G' be a subgraph consisting of a components of G-x together with its edges to x. The degree of x in G' is less than k, so the method above provides a proper k-coloring of G'. By permuting the names of colors in the subgraphs resulting in this way from components of G-x, we can make the colorings agree on x to complete a proper k-coloring of G.

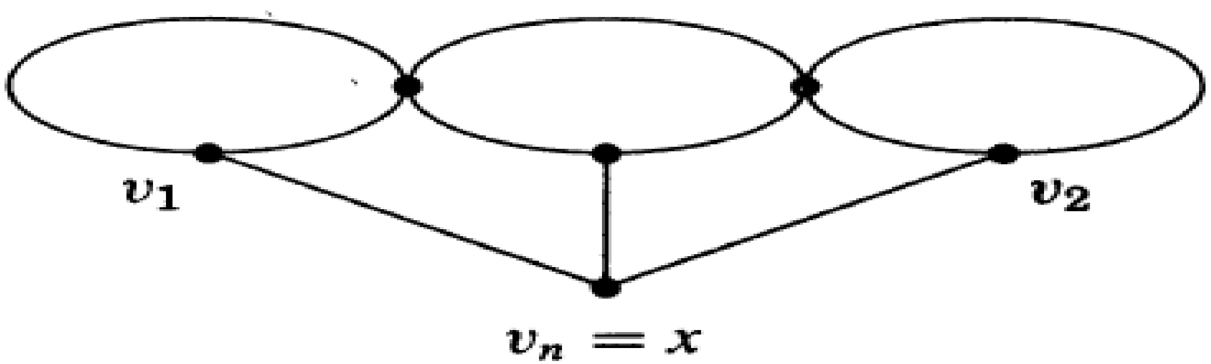
Theorem (Brook [1941]) continue

- We may thus **assume that G is 2-connected**. In every vertex ordering, the last vertex has k earlier neighbors. The greedy coloring idea may still work if we arrange that two neighbors of v_n get the same color.
- In particular, suppose that some vertex v_n has neighbors v_1, v_2 such that $v_1 \not\leftrightarrow v_2$ and G - $\{v_1,v_2\}$ using 3,…,n such that labels increase along paths to the root v_n. As before, each vertex before v_n has at most k-1 lower indexed neighbors. The greedy coloring also uses at most k-1 colors on neighbors of v_n, since v_1 and v_2 receive the same color.



Theorem (Brook [1941]) continue

- Hence it suffices to show that every 2-connected k-regular graph with $k \geq 3$ has such a triple v_1, v_2, v_n. Choose a vertex x. If $\kappa(G-x) \geq 2$, let v_1 be x and let v_2 be a vertex with distance 2 from x. Such a vertex v_2 exists because G is regular and is not a complete graph; let v_n be a common neighbor of v_1 and v_2.
- If $\kappa(G-x) = 1$, let $v_n = x$. Since G has no cut-vertex, x has a neighbor in every leaf block of G-x. Neighbors v_1, v_2 of x in two such blocks are nonadjacent. Also, $G - \{x, v_1, v_2]$ is connected, since blocks have no cut-vertices. Since $k \geq 3$, vertex x has another neighbor, and $G - \{v_1, v_2\}$ is connected.



Remark 5.1.23

- The bound $\chi(G) \leq \Delta(G)$ can be improved when G has no large clique. Brooks' Theorem implies that the complete graphs and odd cycles are the only k-1-regular k-critical graphs. Gallai [1963] strengthened this by proving that in the subgraph of a k-critical graph induced by the vertices of degree k-1, every block is a clique or an odd cycle.
- Brooks' Theorem states that $\chi(G) \leq \Delta(G)$ whenever $3 \leq \omega(G) \leq \Delta(G)$. Borodin and Kostochka [1977] conjectured that $\omega(G) < \Delta(G)$ implies $\chi(G) < \Delta(G)$ if $\Delta(G) \geq 9$ (example show that the condition $\Delta(G) \geq 9$ is needed). Reed [1999] proved that this is true when $\Delta(G) \geq 10^{14}$.
- Reed [1998] also conjectured that the chromatic number is bounded by the average of the trivial upper and lower bounds; that is, $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$

Remark 5.2.12

A graph G with no isolated vertices is **color-critical** if and only if $\chi(G - e) < \chi(G)$ for every e ϵ E(G).

Hence when we prove that a connected graph is color-critical, we need only compare it with subgraphs obtained by deleting a single edge.

Proposition 5.2.13

- Let G be a k-critical graph.

a) For v ϵ V(G), there is a proper k-coloring of G in which the color on v appears nowhere else, and the other k-1 colors appear on N(v).

b) For e ϵ E(G), every proper k -1- coloring of G-e gives the same color to the two endpoints of e.

Proof:

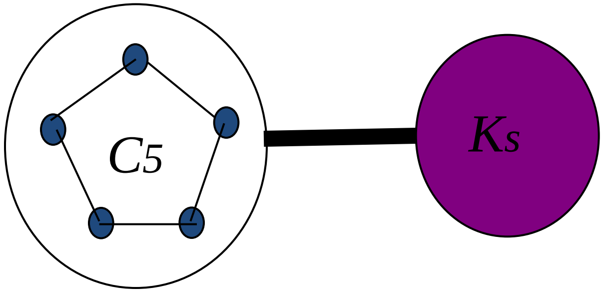
(a) Given a proper k-1-coloring *f* of G-v, adding color k on v alone completes a proper k-coloring of G. The other colors must all appear on N(v), since otherwise assigning a missing color to v would complete a proper k-1-coloring of G.

(b) If some proper k-1-coloring of G-e gave distinct colors to the endpoints of e, then adding e would yield a proper k-1-coloring of G.

For any graph G, Proposition 5.2.13a holds for every v ϵ V(G) such that $\chi(G - v) < \chi(G) = k$, and Proposition 5.2.13b holds for every e ϵE(G) such that $\chi(G - e) < \chi(G) = k$

Example 5.2.14

- The graph $C_5 \vee K_s$ of Example 5.1.8 is color critical. In general, the join of two color-critical graphs is always color-critical.
- This is easy to prove using Remark 5.2.12 and Proposition 5.2.13 by considering cases for the deletion of an edge; the deleted edge e may belong to G or H or have an endpoint in each.



Conclusion

- In this lecture, we have discussed the Brooks' Theorem and elementary properties of k-critical graph.

- In upcoming lecture, we will discuss the Properties of the counting function and further related topics.