Factors & Perfect Matching in General Graphs

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Preface

Recap of Previous Lecture:

In previous lecture, we have discussed Stable Matchings, Gale-Shapley Algorithm and Faster Bipartite Matching *i.e.* Hopcroft-Karp algorithm.

Content of this Lecture:

In this lecture, we will discuss Factors & Perfect Matchings in General Graphs, Tutte's 1-Factor Theorem and f-Factor of Graphs.

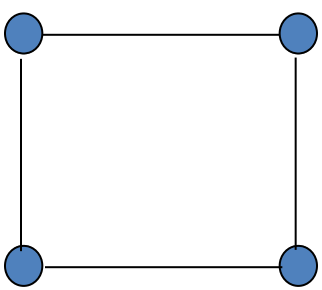
Definitions

Factor: A *factor* of graph G is a spanning subgraph of G.

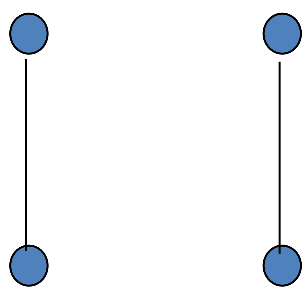
k-factor: A *k-factor* is a spanning k-regular subgraph.

Odd component: An odd component of a graph is a component of odd order; the number of odd components of H is $o(H)$.

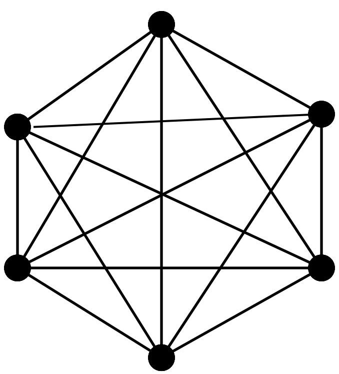
Example:

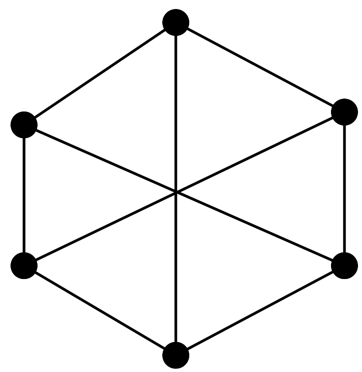


G



1-factor

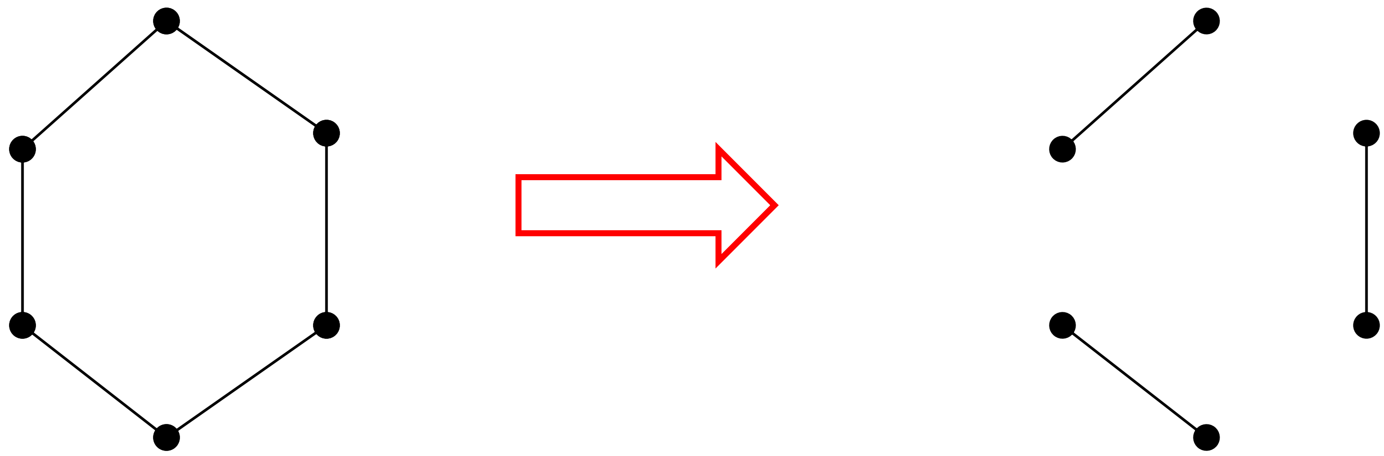




3 factor

Perfect Matching (1-Factor)

- A collection of edges such that every vertex is incident with exactly one edge.
- **Example:**



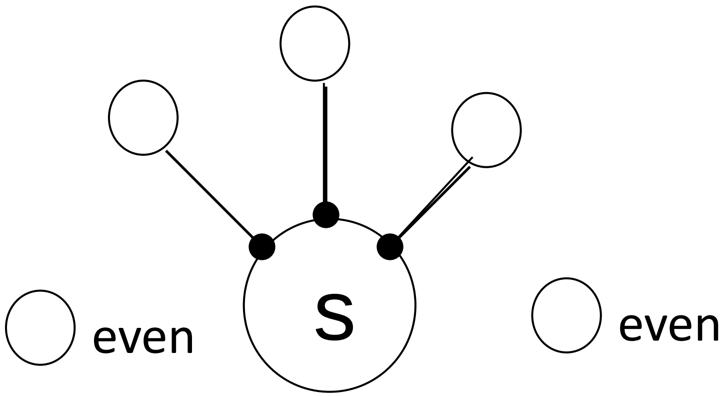
Remark 3.3.2

- A 1-factor and a perfect matching are almost the same thing. The precise distinction is that "1-factor" is a spanning 1-regular subgraph of G, while "perfect matching" is the set of edges in such a subgraph.

- A 3-regular graph that has a perfect matching decomposes into a 1-factor and a 2-factor.

Tutte's 1-factor Theorem

- Tutte found a necessary and sufficient condition for which graphs have 1-factors.
  - If G has a 1-factor and we consider a set $S \subseteq V(G)$, then every odd component of G-S has a vertex matched to something outside it, which can only belong to S.
  - Since these vertices of S must be distinct, $o(G\text{-}S) \leq |S|$.



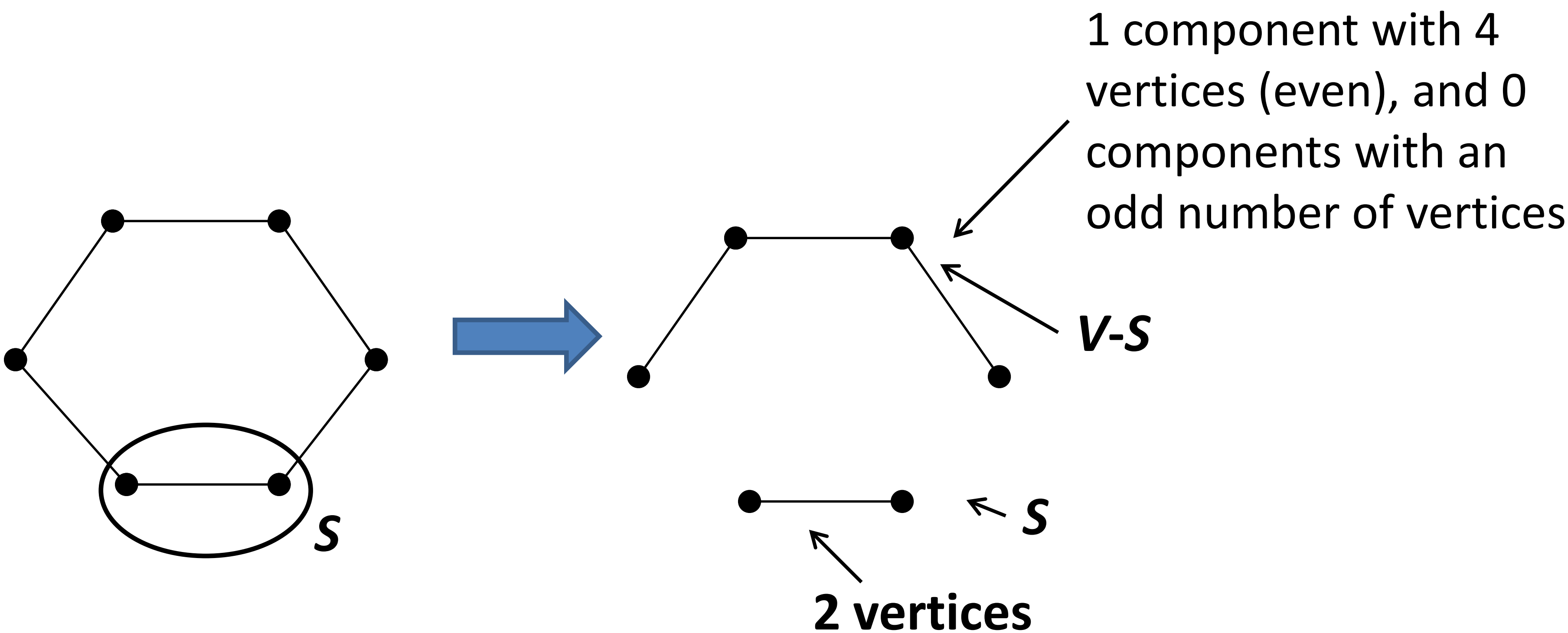
Tutte's Condition

The condition "For all S $\subseteq$ $V(G)$, $o(G\text{-}S)\leq|S|$" is **Tutte's Condition**.

Tutte proved that this obvious necessary condition is also sufficient (TONCAS).

Tutte's Theorem: Example

The condition "For all $S \subseteq V(G)$, $o(G\text{-}S) \leq |S|$" is **Tutte's Condition**.



O < 2 Works for any subset S

Theorem: (Tutte [1947]) A graph G has a 1-factor if and only if $o(G-S) \leq |S|$ for every $S \subseteq V(G)$ 3.3.3

- **Proof:** (Lovász [1975]). ***Necessity***. The odd components of G-S must have vertices matched to distinct vertices of S. **(shown previously)**
- **Sufficiency:** When we add an edge joining two components of G-S, the number of odd components does not increase **(odd and even together become one odd component, two components of the same parity become one even component)**. Hence Tutte's Condition is preserved by addition of edges:
- if $G'=G+e$ and $S \subseteq V(G)$, then $o(G'-S) \leq o(G-S) \leq |S|$.

Also, if $G'=G+e$ has no 1-factor, then G has no 1-factor.

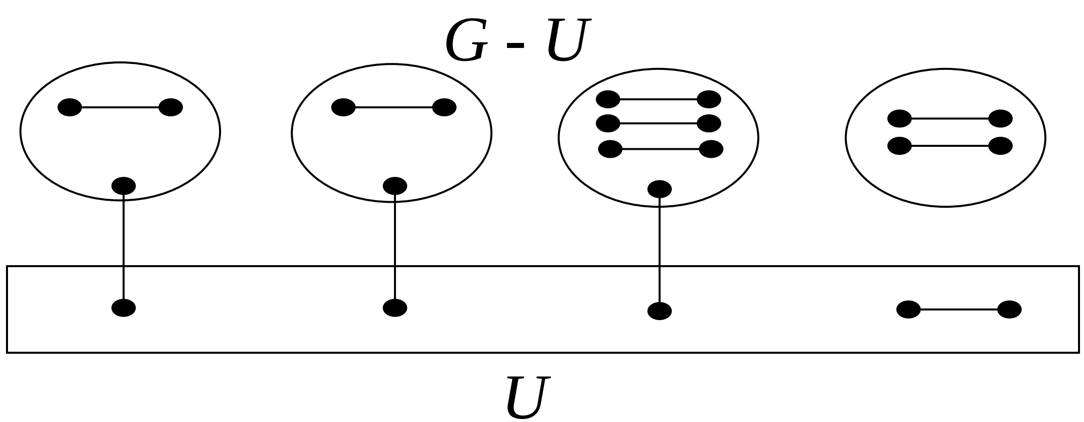
- Therefore, the theorem holds unless there exists a simple graph G such that G satisfies Tutte's Condition, **G has no 1-factor, and adding any missing edge to G yields a graph with a 1-factor**.

Let G be such a graph. *We obtain a* ***contradiction*** *by showing that* ***G actually does contain a 1-factor.***

Case 1: G-U consists of disjoint complete graphs.

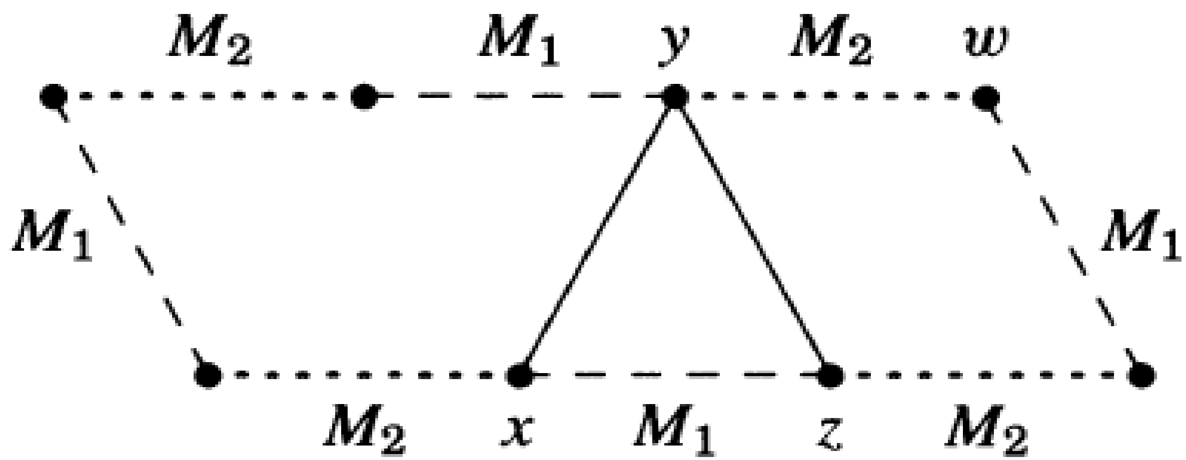
Let U be the set of vertices in G that have degree $n(G)$-1.

- **Case-1:** The vertices in each component of G-U can be paired in any way, with one extra in the odd components, Since $o(G\text{-}U) \leq |U|$ and each vertex of U is adjacent to all of G-U, we can match the leftover vertices to vertices of U.
- The remaining vertices are in U, which is a clique. To complete the 1-factor, we need only show that an even number of vertices remain in U. We have matched an even number, so it suffices to show that $n(G)$ is even. This follows by invoking Tutte's Condition for $S = \varnothing$, since a graph of odd order would have a component of odd order.



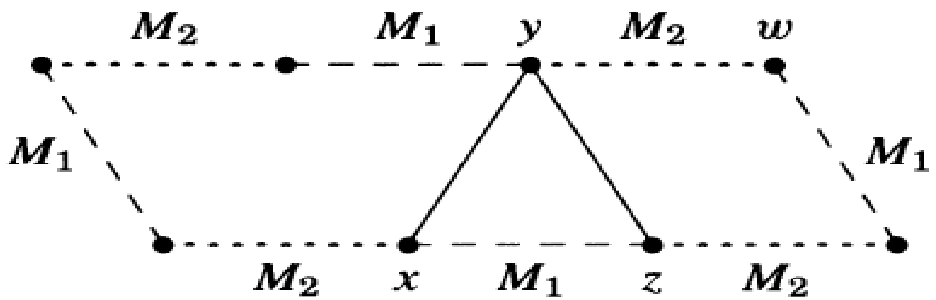
Case 2: G-U is not a disjoint union of cliques.

- **Case-2:** In this case, G-U has two vertices at distance 2; these are nonadjacent vertices x, z with a common neighbor $y \notin U$. Furthermore, G-U has another vertex w not adjacent to y, since $y \notin U$.
- By the choice of G, adding an edge to G creates a 1-factor; let M_1 and M_2 be 1-factors in $G + xz$ and $G + yw$, respectively. It suffices to show that $M_1 \Delta M_2 \cup \{xy, yz\}$ contains a 1-factor avoiding xz and yw, because this will be a 1-factor in G.



Contd…

- Let $F = M_1 \Delta M_2$. Since $xz \in M_1 - M_2$ and $yw \in M_2 - M_1$, both xz and yw are in F. Since every vertex of G has degree 1 in each of M_1 and M_2, every vertex of G has degree 0 or 2 in F. Hence the components of F are even cycles and isolated vertices (Lemma 3.1.9). Let C be the cycle of F containing xz. If C does not also contain yw , then the desired 1-factor consists of the edges of M_2 from C and all of M_1 not in C
- If C contains both yw and xz, as shown below, then to avoid them we use yx or yz. In the portion of C starting from y along yw, we use edges of M_1 to avoid using yw. When we reach $\{x,z\}$, we use zy if we arrive at z (as shown); otherwise, we use xy. In the remainder of C we use the edges of M_2. We have produced a 1-factor of $C + \{xy, yz\}$ that does not use xz or yw. Combined with M_1 or M_2 outside $V(C)$, we have a 1-factor of G.

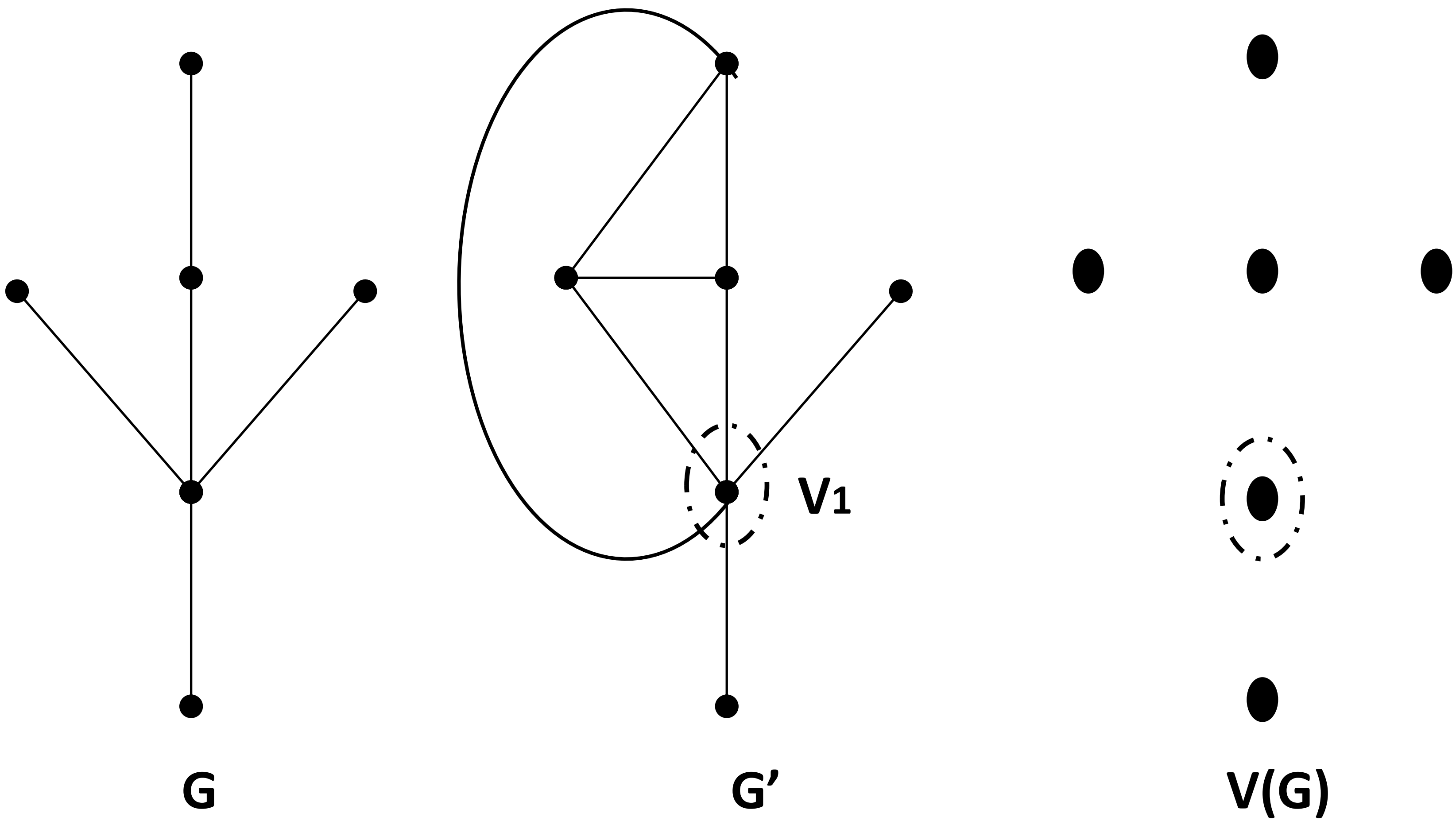


Remarks

- **Remark 3.3.4:** Like other characterization theorems, Theorem 3.3.3 yields short verifications both when the property holds *and* when it doesn't. We prove that G has a 1-factor by exhibiting one. When it doesn't exist, Theorem 3.3.3 gurantees that we can exhibit a set whose deletion leaves too many odd components.

- **Remark 3.3.5:** For a graph G and any $S \subseteq V(G)$, counting the vertices modulo 2 shows that $|S|+o(G\text{-}S)$ has the same parity as $n(G)$. Thus also the difference $o(G\text{-}S) - |S|$ has the same priority as $n(G)$. We conclude that if $n(G)$ is even, and G has no 1-factor, then $o(G\text{-}S)$ exceeds $|S|$ by at least 2 for some S.

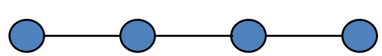
Examples: Tutte's Theorem



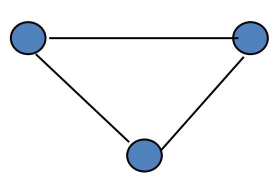
Definition: Join 3.3.6

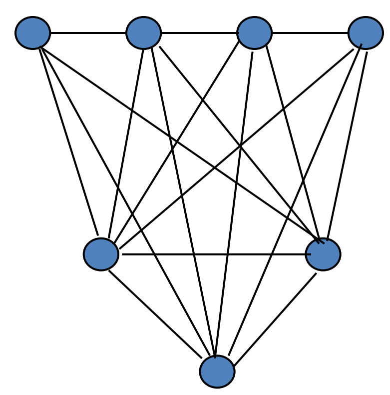
- The join of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G+H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$.

P4



K3

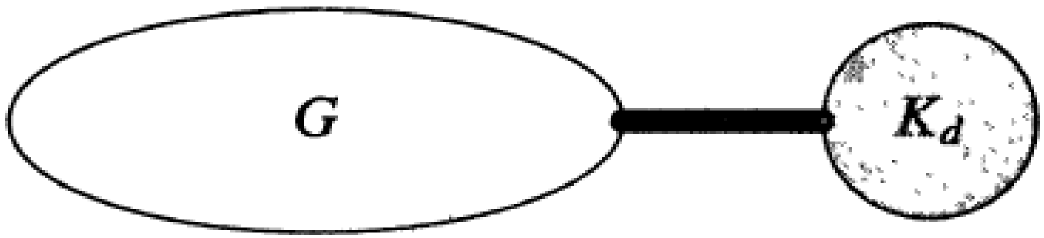




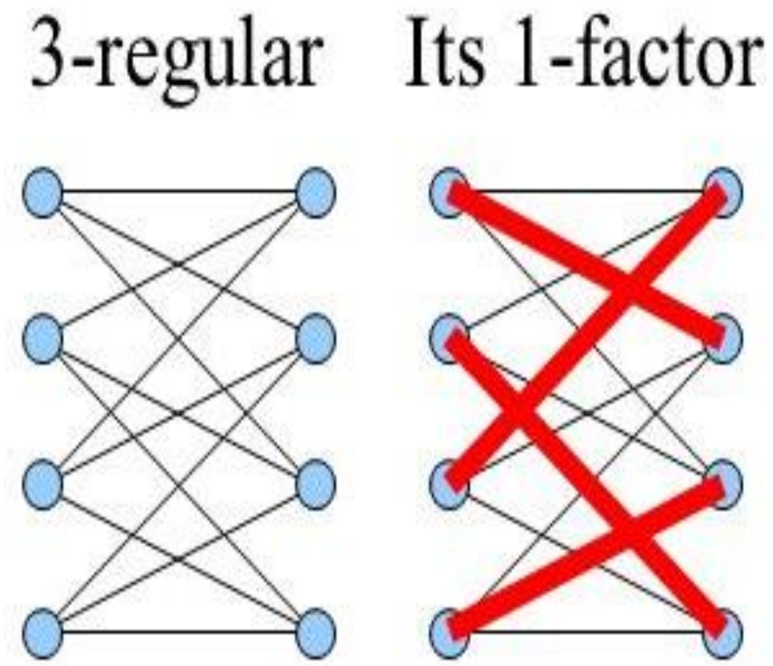
P4 ∨ K3

Corollary: (Burge-Tutte Formula-Berge [1958]) 3.3.7

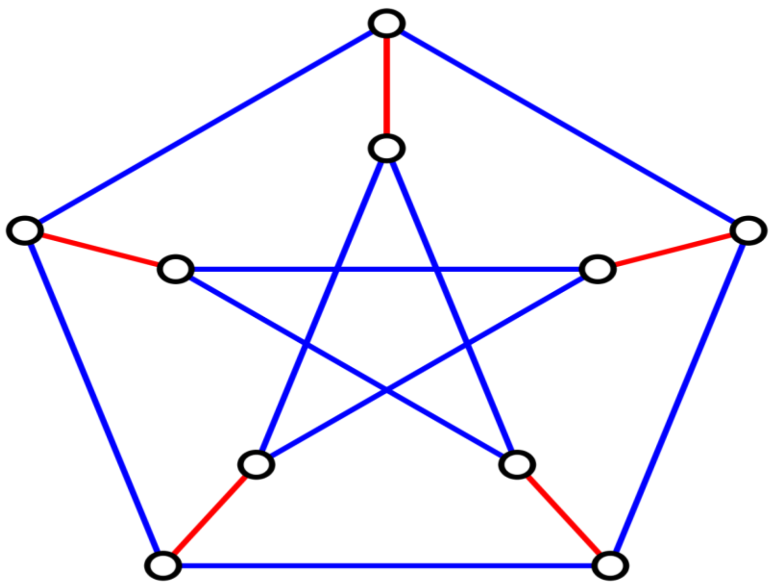
The largest number of vertices saturated by a matching in G is $\min_{S \subseteq V(G)}\{n(G)-d(S)\}$, where $d(S) = o(G-S) - |S|$.



Corollary: (Peterson [1891]) Every 3-regular graph with no cut-edge has a 1-factor 3.3.8



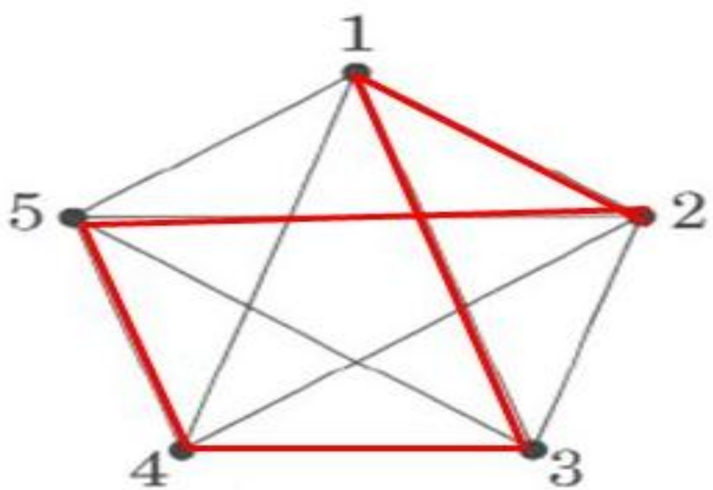
Peterson Graph: 1 factor

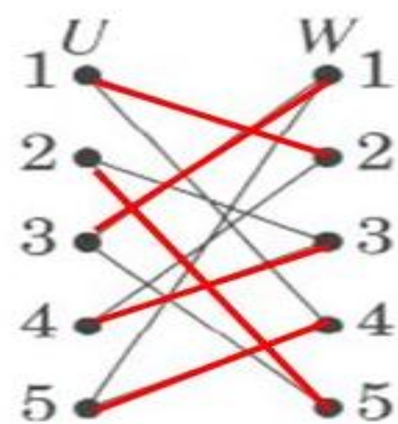


Theorem: (Petersen [1891]) Every regular graph with positive even degree has a 2-factor. 3.3.9

Example: Construction of a 2-factor

- Consider the Eulerian circuit in $G=K_5$ that successively visits 1231425435. The corresponding bipartite graph H is on the right. For the 1-factor whose u, w-pairs are 12,43,25,31,54, the resulting 2-factor is the cycle (1,2,5,4,3). The remaining edges form another 1-factor, which corresponds to the 2-factor (1,4,2,3,5) that remains in G.





Definition: *f*-Factors of Graphs

- Given a function f: $V(G) \rightarrow$ N U {0}, an ***f*-factor** of a graph G is a spanning subgraph H such that $d_H(v) = f(v)$ for all $v \in V(G)$.

- **Tutte [1952]** proved a necessary and sufficient condition for a graph G to have an f-factor. He later reduced the problem to checking for a 1-factor in a related simple graph.

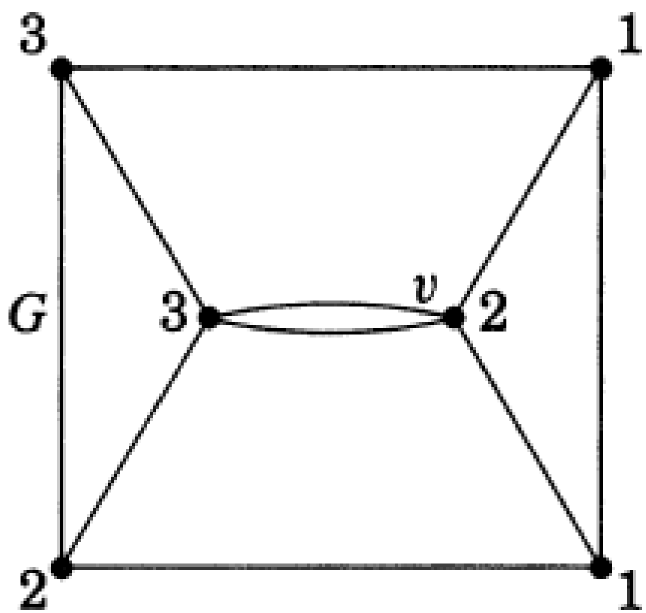
Example 3.3.12

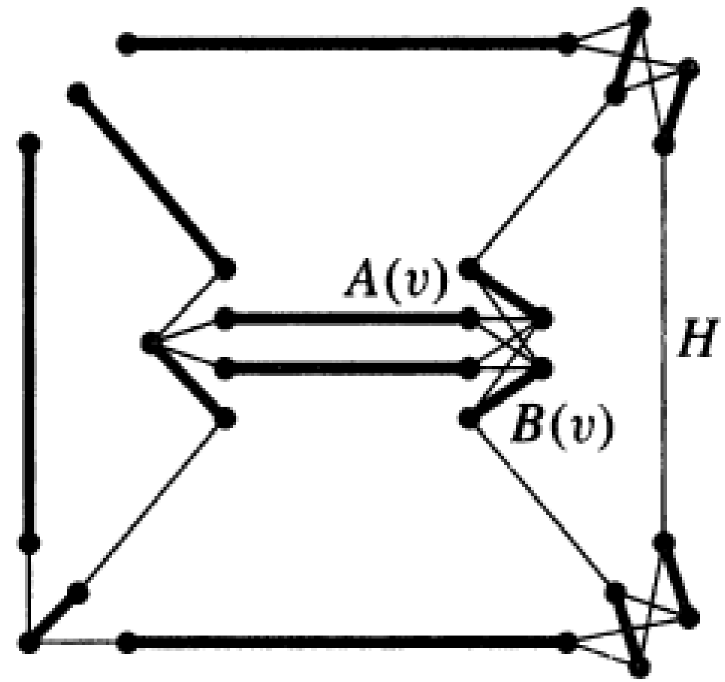
- **A graph transformation (Tutte [1954a]).** We assume that $f(w) \le d(w)$ for all w; otherwise G has too few edges at w to have an f-factor. We then construct a graph H that has a 1-factor if and only if G has an f-factor. Let $e(w)=d(w)-f(w)$; this is the excess degree at w and is nonnegative.

- To construct H, replace each vertex v with a biclique $K_{d(v),\, e(v)}$ having partite sets A(v) of size $d(v)$ and $B(v)$ of size $e(v)$. For each $vw \in E(G)$, add an edge joining one vertex of $A(v)$ to one vertex of $A(w)$. Each vertex of $A(v)$ participate in one such edge.

Example continue

- The figure below show a graph *G*, vertex labels given by *f*, and the resulting simple graph *H*. The bold edges in *H* form a 1-factor that corresponds to an *f*-factor of *G*. In this example, the *f*-factor is not unique .





Theorem: A graph G has an *f*-factor if and only if the graph *H* constructed from *G* and *f* as in Example 3.3.12 has a 1-factor. 3.3.13

- **Proof: Necessity:**

  If G has an f-factor, then the corresponding edges in H leave $e(v)$ vertices of $A(v)$ unmatched; match them arbitrarily to the vertices of $B(v)$ to obtain a 1-factor of H.

- **Sufficiency:**

  From a 1-factor of H, deleting $B(v)$ and the vertices of $A(v)$ matched into $B(v)$ leaves $f(v)$ vertices of degree 1 corresponding to v. Doing this for each v and merging the remaining $f(v)$ vertices of each $A(v)$ yields a subgraph of G with degree $f(v)$ at v. It is an f-factor of G.

Conclusion

- In this lecture, we have discussed Matchings in General Graphs, Tutte's 1-Factor Theorem and *f*-Factor of Graphs

- In upcoming lecture, we will discuss the Matchings in General Graphs *i.e.* Edmonds' Blossom Algorithm.