Module 7:Data Representation

Lecture 34: DFT and DCT

Prev topic Next topic Prev page
Next page

The Lecture Contains:

- Fourier analysis
- Discrete Fourier Transform (DFT)
  - Properties
  - Coefficients
  - Dimensionality reduction
- Discrete Cosine Transform (DCT)

Prev topic Next topic Prev page
Next page

Fourier analysis

- Fourier analysis represents a periodic wave as a sum of (infinite) sine and cosine waves
- Way to analyze the frequency components in a signal
- Fourier transform is a transformation from time domain $f(x)$ to frequency domain $g(u)$

$$g(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ux\mathbf{i}}\, dx$$

- $f(x)$ can be obtained from g(u) by the inverse transformation

$$f(x) = \int_{-\infty}^{\infty} g(u) e^{2\pi ux\mathbf{i}}\, du$$

- $f(x)$ and $g(u)$ form a Fourier transform pair

Module 7:Data Representation

Lecture 34: DFT and DCT

Prev topic Next topic Prev page
Next page

Discrete Fourier Transform (DFT)

- For discrete case, assume vector x has N components
- DFT of x, denoted by X, has N components given by

$$X_k = 1.\sum_{n=0}^{N-1} x_n e^{-(2\pi/N)kn\mathbf{i}} \quad k = 0, \ldots, N-1$$

- Inverse transformation (IDFT) is given by

$$x_n = \frac{1}{N}.\sum_{k=0}^{N-1} X_k e^{(2\pi/N)kn\mathbf{i}} \quad n = 0, \ldots, N-1$$

- To avoid using separate scaling factors, both can be taken as $(1/\sqrt{N})$

Prev topic Next topic Prev page
Next page

Properties

- Parseval's theorem: $||x||_2 = ||X||_2$, i.e., length of vectors is preserved

$$\sum_{n=0}^{N-1} |x_n|^2 = \sum_{k=0}^{N-1} |X_k|^2$$

- When both scaling factors are $(1/\sqrt{N})$
- Contractive mapping
- Invertible, linear transformation
- Essentially, a rotation in N-dimensional space

Coefficients

- Expanding $e^{\theta \mathbf{i}} = \cos\theta + \mathbf{i}\sin\theta$,

$$X_k = \sum_{n=0}^{N-1} x_n e^{-(2\pi/N)kn\mathbf{i}}$$

$$= \sum_{n=0}^{N-1} x_n \left(\cos(\frac{2\pi}{N}kn) - \mathbf{i}\sin(\frac{2\pi}{N}kn)\right)$$

$$\therefore X_0 = \sum_{n=0}^{N-1} x_n \left(\cos 0 - \mathbf{i}\sin 0\right)$$

$$= \sum_{n=0}^{N-1} x_n$$

- First coefficient is (scaled) sum or average
- Other coefficients define the frequency components (cosine and sine) at frequencies $2\pi(k/N)$ for $k = 0, 1, \ldots, N-1$

Prev topic Next topic Prev page
Next page

Dimensionality reduction

- Retain k lower frequency components
- Discard higher frequency noise

Prev topic Next topic Prev page
Next page

Dimensionality reduction

- Retain k lower frequency components
- Discard higher frequency noise
- Alternatively, for a database of objects, retain those coefficients whose variances are highest

Module 7:Data Representation

Lecture 34: DFT and DCT

Prev topic Next topic Prev page
Next page

Discrete Cosine Transform (DCT)

- Way to represent a signal in terms of cosine waves of different frequencies (and amplitudes)
- Various definitions

$$X_k^{(1)} = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} u_n^{(1)} x_n \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right] \quad k = 0, \ldots, N-1$$

$$X_k^{(2)} = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} u_n^{(2)} x_n \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)\right] \quad k = 0, \ldots, N-1$$

$$u_n^{(1)} = 1/\sqrt{2} \quad n = 0$$

$$u_n^{(1)} = 1 \quad n = 1, \ldots, N-1$$

$$u_n^{(2)} = 1 \quad n = 0, \ldots, N-1$$

- Length preserving
- Inverses are the same functions