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The Lecture Contains:

Wavelets

- Fourier transform analyzes frequency resolutions, but not time
- Wavelets analyze a function in both time and frequency domains
    - Good time resolution and poor frequency resolution at high frequencies
    - Good frequency resolution and poor time resolution at low frequencies
- Wavelets are useful for short duration signals of high frequency and long duration signals of short frequency
- Wavelets are generated from a mother wavelet function ψ
- Zero mean (oscillatory, i.e., wave nature): $\int \psi(x)\, dx = 0$
- Unit length: $\int \psi^2(x)\, dx = 1$
- Basis functions are generated by scaling (s) and shifting (*l*) the mother wavelet

$$\psi_{s,l}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-l}{s}\right)$$

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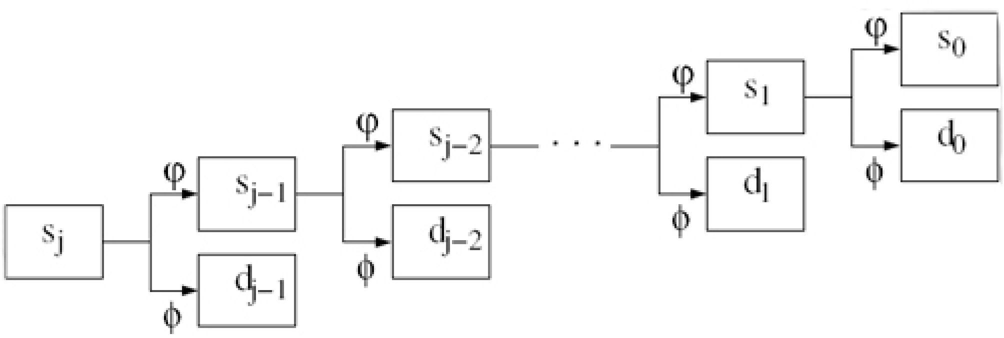
Discrete Wavelet Transform (DWT)

- DWT generates a set of basis function or vectors
- Two functions:
    - Wavelet function
    - Scaling function
- Space spanned by $n = 2^j$ basis vectors at level j can be spanned by two sets of basis vectors ψ and ϕ at level $j-1$
- ψ and ϕ are wavelet and scaling functions respectively
- DWT generates basis vectors for wavelet and scaling functions at different levels



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Haar wavelets: Example

- Compute sum and difference between each consecutive pairs of entries (and scale)
- Repeat the steps for the sum coefficients

$$f_3 = \{2, 5, 8, 9, 7, 4, -1, 1\}$$

$$f_2 = \frac{1}{\sqrt{2}}\{2+5, 8+9, 7+4, -1+1, 2-5, 8-9, 7-4, -1-1\}$$

$$= \left\{\frac{7}{\sqrt{2}}, \frac{17}{\sqrt{2}}, \frac{11}{\sqrt{2}}, \frac{0}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{-2}{\sqrt{2}}\right\}$$

$$f_1 = \left\{\frac{24}{2}, \frac{11}{2}, \frac{-10}{2}, \frac{11}{2}, \frac{-3}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{-2}{\sqrt{2}}\right\}$$

$$f_0 = \left\{\frac{35}{2\sqrt{2}}, \frac{13}{2\sqrt{2}}, \frac{-10}{2}, \frac{11}{2}, \frac{-3}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{-2}{\sqrt{2}}\right\}$$

- Length is preserved: $||f_0||_2 = 15.524 = ||f_3||_2$
- Invertible: f_3 is losslessly obtained from f_0

Haar wavelets: Theory

- Wavelet (ψ) and scaling (ϕ) functions:

$$\psi(x) = \begin{cases} +1 & \text{when} \quad 0 \le x < 1/2 \\ -1 & \text{when} \quad 1/2 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi(x) = \begin{cases} 1 & \text{when} \quad 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

- Shifting (i) and scaling (j):

$$\psi_{j,i}(x) = 2^{j/2}\psi(2^j x - i) \quad j = 0, \ldots \quad i = 0, \ldots, 2^j - 1$$

$$\phi_{j,i}(x) = 2^{j/2}\phi(2^j x - i) \quad j = 0, \ldots \quad i = 0, \ldots, 2^j - 1$$

- Binary dilation (scaling) and dyadic translation (shifting)
- Coefficients corresponding to $\psi_{j,i}$ are averages or sum coefficients
- Coefficients corresponding to $\phi_{j,i}$ are differences or detail coefficients

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Matrix form

- $\psi_{j,i}, \phi_{j,i}$ etc. form the basis vectors
- Only $\psi_{0,0}$ is needed since others are expressed in terms of $\phi_{\cdot,\cdot}$
- When size of vector is $n = 2^j$, j levels of basis vectors are needed
- There are $2^{j-1} + 2^{j-2} + \cdots + 2^0 = 2^j - 1$ basis vectors corresponding to detail coefficients and 1 basis vector corresponding to sum coefficient at 0^{th} level
- Transformation matrix H has these basis vectors as columns
- Transformed vector $v' = v.H$ for data (row) vector v

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- Transformation matrix H has these basis vectors as columns
- Transformed vector $v' = v.H$ for data (row) vector v
- Each step can be defined as multiplication by a matrix
- Composition of these matrices gives the final transformation matrix
- H is orthonormal
- $H^{-1} = H^T$
- Hence, inverse operation is easy

Haar wavelet matrices

$$H_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$f_2 = f_3.H_2$$

Haar wavelet matrices

$$H_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$f_1 = f_2.H_1$$

Haar wavelet matrices

$$H_0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$f_0 = f_1 . H_0$$

Haar wavelet matrices

$$H_3 = H_2.H_1.H_0 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{-1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{-1}{2} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 & \frac{1}{2} & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 & \frac{-1}{2} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 & \frac{-1}{2} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$f_0 = f_3.H_3$$

Inversely, $= f_0.H_3^{-1} = f_0.H_3^T$

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Dimensionality reduction using Haar wavelets

- Retain sum and lower level detail coefficients
- For example, retain 1 sum (at level 0) and 3 detail (1 at level 0 and 2 at level 1) coefficients
- Contractive

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- Another option is to zero out coefficients whose absolute values are below a threshold
- What happens when dimensionality is not a power of 2?

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  - Pad zeros at end: latter half becomes less important
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- What happens when dimensionality is not a power of 2?
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  - Pad equal amount of zeros in each half: should be done recursively
- Interestingly, shuffing the dimensions produce different wavelet coefficients
  - How to shuffle to satisfy some critera?