# Advanced Solid Mechanics

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# Contents

# Chapter 1

# Introduction

This course builds upon the concepts learned in the course "Mechanics of Materials" also known as "Strength of Materials". In the "Mechanics of Materials" course one would have learnt two new concepts "stress" and "strain" in addition to revisiting the concept of a "force" and "displacement" that one would have mastered in a first course in mechanics, namely "Engineering Mechanics". Also one might have been exposed to four equations connecting these four concepts, namely strain-displacement equation, constitutive equation, equilibrium equation and compatibility equation. Figure 1.1 pictorially depicts the concepts that these equations relate. Thus, the strain displacement relation allows one to compute the strain given a displacement; constitutive relation gives the value of stress for a known value of the strain or vice versa; equilibrium equation, crudely, relates the stresses developed in the body to the forces and moment applied on it; and finally compatibility equation places restrictions on how the strains can vary over the body so that a continuous displacement field could be found for the assumed strain field.

In this course too we shall be studying the same four concepts and four equations. While in the "mechanics of materials" course, one was introduced to the various components of the stress and strain, namely the normal and shear, in the problems that was solved not more than one component of the stress or strain occurred simultaneously. Here we shall be studying these problems in which more than one component of the stress or strain occurs simultaneously. Thus, in this course we shall be generalizing these concepts and equations to facilitate three dimensional analysis of structures.

Before venturing into the generalization of these concepts and equations,

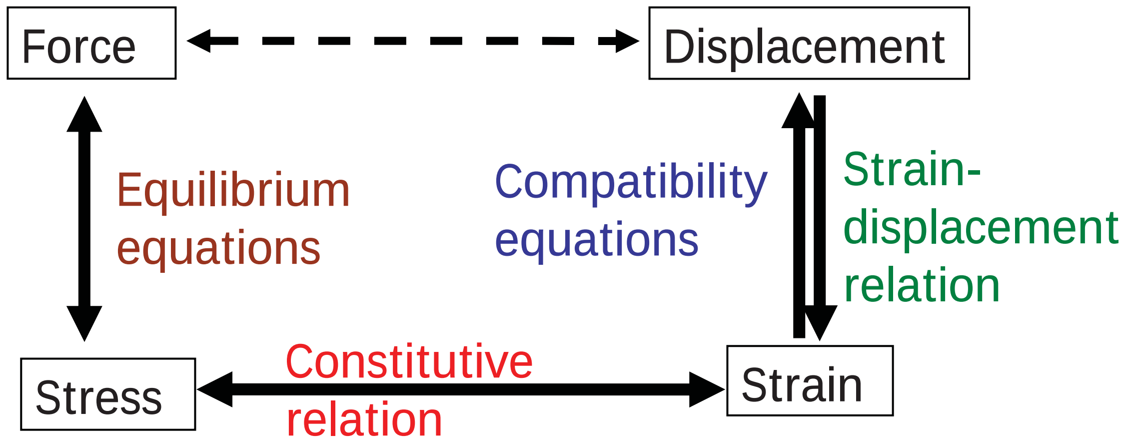



Figure 1.1: Basic concepts and equations in mechanics

a few drawbacks of the definitions and ideas that one might have acquired from the previous course needs to be highlighted and clarified. This we shall do in sections 1.1 and 1.2. Specifically, in section 1.1 we look at the four concepts in mechanics and in section 1.2 we look at the equations in mechanics. These sections also serve as a motivation for the mathematical tools that we would be developing in chapter 2. Then, in section 1.3 we look into various idealizations of the response of materials and the mathematical framework used to study them. However, in this course we shall be only focusing on the elastic response or more precisely, non-dissipative response of the materials. Finally, in section 1.4 we outline three ways by which we can solve problems in mechanics.

## 1.1 Basic Concepts in Mechanics

### 1.1.1 What is force?

Force is a mathematical idea to study the motion of bodies. It is not "real" as many think it to be. However, it can be associated with the twitching of the muscle, feeling of the burden of mass, linear translation of the motor, so on and so forth. Despite seeing only displacements we relate it to its cause the force, as the concept of force has now been ingrained.

Let us see why force is an idea that arises from mathematical need. Say, the position[1] ($\mathbf{x}_o$) and velocity ($\mathbf{v}_o$) of the body is known at some time, $t = t_o$, then one is interested in knowing where this body would be at a later time, $t = t_1$. It turns out that mathematically, if the acceleration ($\mathbf{a}$) of the body at any later instant in time is specified then the position of the body can be determined through Taylor's series. That is if

$$\mathbf{a} = \frac{d^2\mathbf{x}}{dt^2} = \mathbf{f}_a(t), \tag{1.1}$$

then from Taylor's series

$$\begin{aligned}\mathbf{x}_1 = \mathbf{x}(t_1) = \mathbf{x}(t_o) + \left.\frac{d\mathbf{x}}{dt}\right|_{t=t_o}(t_1 - t_o) + \left.\frac{d^2\mathbf{x}}{dt^2}\right|_{t=t_o}\frac{(t_1 - t_o)^2}{2!} \\ + \left.\frac{d^3\mathbf{x}}{dt^3}\right|_{t=t_o}\frac{(t_1 - t_o)^3}{3!} + \left.\frac{d^4\mathbf{x}}{dt^4}\right|_{t=t_o}\frac{(t_1 - t_o)^4}{4!} + \ldots,\end{aligned} \tag{1.2}$$

which when written in terms of $\mathbf{x}_o$, $\mathbf{v}_o$ and $\mathbf{a}$ reduces to[2]

$$\begin{aligned}\mathbf{x}_1 = \mathbf{x}_o + \mathbf{v}_o(t_1 - t_o) + \mathbf{f}_a(t_o)\frac{(t_1 - t_o)^2}{2!} + \left.\frac{d\mathbf{f}_a}{dt}\right|_{t=t_o}\frac{(t_1 - t_o)^3}{3!} \\ + \left.\frac{d^2\mathbf{f}_a}{dt^2}\right|_{t=t_o}\frac{(t_1 - t_o)^4}{4!} + \ldots.\end{aligned} \tag{1.3}$$

Thus, if the function $\mathbf{f}_a$ is known then the position of the body at any other instant in time can be determined. This function is nothing but force per unit mass[3], as per Newton's second law which gives a definition for the force. This shows that force is a function that one defines mathematically so that the position of the body at any later instance can be obtained from knowing its current position and velocity.

It is pertinent to point out that this function $\mathbf{f}_a$ could also be prescribed using the position, $\mathbf{x}$ and velocity, $\mathbf{v}$ of the body which are themselves function of time, $t$ and hence $\mathbf{f}_a$ would still be a function of time. Thus, $\mathbf{f}_a = \mathbf{g}(\mathbf{x}(t), \mathbf{v}(t), t)$. However, $\mathbf{f}_a$ could not arbitrarily depend on $t$, $\mathbf{x}$ and $\mathbf{v}$. At

[1] Any bold small case alphabet denotes a vector. Example $\mathbf{a}$, $\mathbf{x}$.

[2] Here it is pertinent to note that the subscripts denote the instant in time when position or velocity is determined. Thus, $\mathbf{x}_o$ denotes the position at time $t_o$ and $\mathbf{x}_1$ denotes the position at time $t_1$.

[3] Here the mass of the body is assumed to be a constant.

this point it suffices to say that the other two laws of Newton and certain objectivity requirements have to be met by this function. We shall see what these objectivity requirements are and how to prescribe functions that meet this requirement subsequently in chapter - 6.

Next, let us understand what kind of quantity is force. In other words is force a scalar or vector and why? Since, position is a vector and acceleration is second time derivative of position, it is also a vector. Then, it follows from equation (1.1) that $\mathbf{f}_a$ also has to be a vector. Therefore, force is a vector quantity. Numerous experiments also show that addition of forces follow vector addition law (or the parallelogram law of addition). In chapter 2 we shall see how the vector addition differs from scalar addition. In fact it is this addition rule that distinguishes a vector from a scalar and hence confirms that force is a vector.

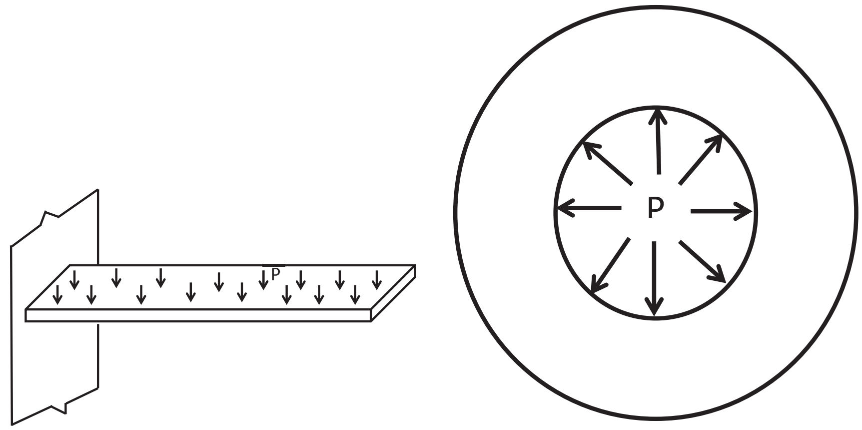
As a summary, we showed that force is a mathematical construct which is used to mathematically describe the motion of bodies.

### 1.1.2 What is stress?

As is evident from figure 1.1, stress is a quantity derived from force. The commonly stated definitions in an introductory course in mechanics for stress are:

1. Stress is the force acting per unit area
2. Stress is the resistance offered by the body to a force acting on it

While the first definition tells how to compute the stress from the force, this definition holds only for simple loading case. One can construct a number of examples where definition 1 does not hold. The following two cases are presented just as an example. Case -1: A cantilever beam of rectangular cross section with a uniform pressure, $p$, applied on the top surface, as shown in figure 1.2a. According to the definition 1 the stress in the beam should be $p$, but it is not. Case -2: An annular cylinder subjected to a pressure, $p$ at its inner surface, as shown in figure 1.2b. The net force acting on the cylinder is zero but the stresses are not zero at any location. Also, the stress is not $p$, anywhere in the interior of the cylinder. This being the state of the first definition, the second definition is of little use as it does not tell how to compute the stress. These definitions does not tell that there are various components of the stress nor whether the area over which the force




(a) A cantilever beam with uniform pressure applied on its top surface.

(b) An annular cylinder subjected to internal pressure.

Figure 1.2: Structures subjected to pressure loading

is considered to be distributed is the deformed or the undeformed. They do not distinguish between traction (or stress vector), $\mathbf{t}_{(\mathbf{n})}$ and stress tensor, $\boldsymbol{\sigma}$.

Traction is the distributed force acting per unit area of a cut surface or boundary of the body. This traction apart from varying spatially and temporally also depends on the plane of cut characterized by its normal. This quantity integrated over the cut surface gives the net force acting on that surface. Consequently, since force is a vector quantity this traction is also a vector quantity. The component of the traction along the normal direction[4], $\mathbf{n}$ is called as the normal stress ($\sigma_{(\mathbf{n})}$). The magnitude of the component of the traction[5] acting parallel to the plane is called as the shear stress ($\tau_{(\mathbf{n})}$).

If the force is distributed over the deformed area then the corresponding traction is called as the Cauchy traction ($\mathbf{t}_{(\mathbf{n})}$) and if the force is distributed over the undeformed or original area that traction is called as the Piola traction ($\mathbf{p}_{(\mathbf{n})}$). If the deformed area does not change significantly from the

[4] Here $\mathbf{n}$ is a unit vector.

[5] Recognize that there would be two components of traction acting on the plane of cut. Shear stress is neither of those components. For example, if traction on a plane whose normal is $\mathbf{e}_z$ is, $\mathbf{t}_{(\mathbf{e}_z)} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$, then the normal stress $\sigma_{(\mathbf{e}_z)} = a_z$ and the shear stress $\tau_{(\mathbf{e}_z)} = \sqrt{a_x^2 + a_y^2}$. See chapter 4 for more details.

original area, then both these traction would have nearly the same magnitude and direction. More details about these traction is presented in chapter 4.

The stress tensor, is a linear function (crudely, a matrix) that relates the normal vector, $\mathbf{n}$ to the traction acting on that plane whose normal is $\mathbf{n}$. The stress tensor could vary spatially and temporally but does not change with the plane of cut. Just like there is Cauchy and Piola traction, depending on over which area the force is distributed, there are two stress tensors. The Cauchy (or true) stress tensor, $\boldsymbol{\sigma}$ and the Piola-Kirchhoff stress tensor ($\mathbf{P}$). While these two tensors may nearly be the same when the deformed area is not significantly different from the original area, qualitatively these tensors are different. To satisfy the moment equilibrium in the absence of body couples, Cauchy stress tensor has to be symmetric tensor (crudely, symmetric matrix) and Piola-Kirchhoff stress tensor cannot be symmetric. In fact the transpose of the Piola-Kirchhoff stress tensor is called as the engineering stress or nominal stress. Moreover, there are many other stress measures obtained from the Cauchy stress and the gradient of the displacement which shall be studied in chapter 4.

### 1.1.3 What is displacement?

The difference between the position vectors of a material particle at two different instances of time is called as displacement. In general, the displacement of the material particle would depend on time; the instances between which the displacement is sought. It is also possible that different particles get displaced differently between the same two instances of time. Thus, displacement in general varies spatially and temporally. Displacement is what can be observed and measured. Forces, traction and stress tensors are introduced to explain (or mathematically capture) this displacement.

The displacement field is at least differentiable twice temporally so that acceleration could be computed. This stems from the observations that the location or velocity of the body does not change abruptly. Similarly, the basic tenant of continuum mechanics is that the displacement field is continuous spatially and is piecewise differentiable spatially at least twice. That is while the displacement field is required to be continuous over the entire body it is required to be twice differentiable not necessarily over the entire body but only on subsets of the body. Thus, in continuum mechanics interpenetration of two surfaces or separation and formation of new surfaces is precluded. The validity of the theory stops just before the body fractures. Notwithstanding this many attempt to use continuum mechanics concepts to understand the process of fracture.

A body is said to undergo rigid body displacement if the distance between any two particles that belongs to the body remains unchanged. That is in a rigid body displacement the particles that belong to a body do not move relative to each other. A body is said to be rigid if it always undergoes only rigid body displacement under action of any force. On the other hand, a body is said to be deformable if it allows relative displacement of its particles under the action of some force. Though, all real bodies are deformable, at times one could idealize a given body as rigid under the action of certain forces.

### 1.1.4 What is strain?

One observes that rigid body displacements of the body does not give raise to any stresses. Further, stresses are induced only when there is relative displacement of the material particles. Consequently, one requires a measure (or metric) for this relative displacement so that it can be related to the stress. The unique measure of relative displacement is the stretch ratio, $\lambda_{(\mathbf{A})}$, defined as the ratio of the deformed length to the original length of a material fiber along a given direction, $\mathbf{A}$. (Note that here $\mathbf{A}$ is a unit vector.) However, this measure has the drawback that when the body is not deformed the stretch ratio is 1 (by virtue of the deformed length being same as the original length) and hence inconvenient to write the constitutive relation of the form

$$\sigma_{(\mathbf{A})} = f(\lambda_{(\mathbf{A})}), \tag{1.4}$$

where $\sigma_{(\mathbf{A})}$ denotes the normal stress on a plane whose normal is $\mathbf{A}$. Since the stress is zero when the body is not deformed, the function $f$ should be such that $f(1) = 0$. Mathematical implementation of this condition that $f(1) = 0$ and that $f$ be a one to one function is thought to be difficult when $f$ is a nonlinear function of $\lambda_{(\mathbf{A})}$. Consequently, another measure of relative displacement is sought which would be 0 when the body is not deformed and less than zero when compressed and greater than zero when stretched. This measure is called as the strain, $\epsilon_{(\mathbf{A})}$. There is no unique way of obtaining the strain from the stretch ratio. The following functions satisfy the requirement of the strain:

$$\epsilon_{(\mathbf{A})} = \frac{\lambda_{(\mathbf{A})}^m - 1}{m}, \qquad \epsilon_{(\mathbf{A})} = \ln(\lambda_{(\mathbf{A})}), \tag{1.5}$$

where $m$ is some real number and ln stands for natural logarithm. Thus, if $m = 1$ in (1.5a) then the resulting strain is called as the engineering strain, if $m = -1$, it is called as the true strain, if $m = 2$ it is Cauchy-Green strain. The second function wherein $\epsilon_{(\mathbf{A})} = \ln(\lambda_{(\mathbf{A})})$, is called as the Hencky strain or the logarithmic strain.

Just like the traction and hence the normal stress changes with the orientation of the plane, the stretch ratio also changes with the orientation along which it is measured. We shall see in chapter 3 that a tensor called the Cauchy-Green deformation tensor carries all the information required to compute the stretch ratio along any direction. This is akin to the stress tensor which when known we could compute the traction or the normal stress in any plane.

## 1.2 Basic Equations in Mechanics

Having gained a superficial understanding of the four concepts in mechanics namely the force, stress, displacement and strain, let us look at the four equations that connect these concepts and the reasoning used to obtain them.

### 1.2.1 Equilibrium equations

Equilibrium equations are Newton's second law which states that the rate of change of linear momentum would be equal in magnitude and direction to the net applied force. Deformable bodies are subjected to two kinds of forces, namely, contact force and body force. As the name suggest the contact force arises by virtue of the body being in contact with its surroundings. Traction arises only due to these contact force and hence so does the stress tensor. The magnitude of the contact force depends on the contact area between the body and its surroundings. On the other hand, the body forces are action at a distance forces. Examples of body force are gravitational force, electromagnetic force. The magnitude of these body forces depend on the mass of the body and hence are generally expressed as per unit mass of the body and denoted by **b**.

On further assuming that the Newton's second law holds for any subpart of the body and that the stress field is continuously differentiable within the body the equilibrium equations can be written as:

$$div(\boldsymbol{\sigma}) + \rho\mathbf{b} = \rho\mathbf{a}, \tag{1.6}$$

where $\rho$ is the density, $\mathbf{a}$ is the acceleration and the mass is assumed to be conserved. Detail derivation of the above equation is given in chapter 5. The meaning of the operator $div(\cdot)$ can be found in chapter 2.

Also, the rate of change of angular momentum must be equal to the net applied moment on the body. Assuming that the moment is generated only by the contact forces and body forces, this condition requires that the Cauchy stress tensor to be symmetric. That is in the absence of body couples, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^t$, where the superscript $(\cdot)^t$ denotes the transpose. Here again the assumptions made to obtain the force equilibrium equation (1.6) should hold. See chapter 5 for detailed derivation.

### 1.2.2 Strain-Displacement relation

The relationship that connects the displacement field with the strain is called as the strain displacement relationship. As pointed out before there is no unique definition of the strain and hence there are various strain tensors. However, all these strains are some function of the gradient of the deformation field, $\mathbf{F}$; commonly called as the deformation gradient. The deformation field is a function that gives the position vector of any material particle that belongs to the body at any instance in time with the material particle identified by its location at some time $t_o$. Then, in chapter 3 we show that, the stretch ratio along a given direction $\mathbf{A}$ is,

$$\lambda_{(\mathbf{A})} = \sqrt{\mathbf{CA} \cdot \mathbf{A}}, \tag{1.7}$$

where $\mathbf{C} = \mathbf{F}^t\mathbf{F}$, is called as the right Cauchy-Green deformation tensor. When the body is undeformed, $\mathbf{F} = \mathbf{1}$ and hence, $\mathbf{C} = \mathbf{1}$ and $\lambda_{(\mathbf{A})} = 1$. Instead of looking at the deformation field, one can develop the expression for the stretch ratio, looking at the displacement field too. Now, the displacement field can be a function of the coordinates of the material particles in the reference or undeformed state or the coordinates in the current or deformed state. If the displacement is a function of the coordinates of the material particles in the reference configuration it is called as Lagrangian representation of the displacement field and the gradient of this Lagrangian displacement field is called as the Lagrangian displacement gradient and is denoted by $\mathbf{H}$. On the other hand if the displacement is a function of the coordinates of the material particle in the deformed state, such a representation of the displacement field is said to be Eulerian and the gradient of this

Eulerian displacement field is called as the Eulerian displacement gradient and is denoted by $\mathbf{h}$. Then it can be shown that (see chapter 3),

$$\mathbf{F} = \mathbf{H} + \mathbf{1}, \qquad \mathbf{F}^{-1} = \mathbf{1} - \mathbf{h}, \tag{1.8}$$

where, $\mathbf{1}$ stands for identity tensor (see chapter 2 for its definition). Now, the right Cauchy-Green deformation tensor can be written in terms of the Lagrangian displacement gradient as,

$$\mathbf{C} = \mathbf{1} + \mathbf{H} + \mathbf{H}^t + \mathbf{H}^t\mathbf{H}. \tag{1.9}$$

Note that the if the body is undeformed then $\mathbf{H} = \mathbf{0}$. Hence, if one cannot see the displacement of the body then it is likely that the components of the Lagrangian displacement gradient are going to be small, say of order $10^{-3}$. Then, the components of the tensor $\mathbf{H}^t\mathbf{H}$ are going to be of the order $10^{-6}$. Hence, the equation (1.9) for this case when the components of the Lagrangian displacement gradient is small can be approximately calculated as,

$$\mathbf{C} = \mathbf{1} + 2\boldsymbol{\epsilon}_L, \tag{1.10}$$

where

$$\boldsymbol{\epsilon}_L = \frac{1}{2}\left[\mathbf{H} + \mathbf{H}^t\right], \tag{1.11}$$

is called as the linearized Lagrangian strain. We shall see in chapter 3 that when the components of the Lagrangian displacement gradient is small, the stretch ratio (1.7) reduces to

$$\lambda_{\mathbf{A}} = 1 + \boldsymbol{\epsilon}_L\mathbf{A} \cdot \mathbf{A}. \tag{1.12}$$

Thus we find that $\boldsymbol{\epsilon}_L$ contains information about changes in length along any given direction, $\mathbf{A}$ when the components of the Lagrangian displacement gradient are small. Hence, it is called as the linearized Lagrangian strain. We shall in chapter 3 derive the various strain tensors corresponding to the various definition of strains given in equation (1.5).

Further, since $\mathbf{F}\mathbf{F}^{-1} = \mathbf{1}$, it follows from (1.8) that

$$\mathbf{H} = \mathbf{h} + \mathbf{H}\mathbf{h}, \tag{1.13}$$

which when the components of both the Lagrangian and Eulerian displacement gradient are small can be approximated as $\mathbf{H} = \mathbf{h}$. Thus, when the components of the Lagrangian and Eulerian displacement gradients are small these displacement gradients are the same. Hence, the Eulerian linearized strain defined as,

$$\boldsymbol{\epsilon}_E = \frac{1}{2}\left[\mathbf{h} + \mathbf{h}^t\right], \tag{1.14}$$

and the Lagrangian linearized strain, $\boldsymbol{\epsilon}_L$ would be the same when the components of the displacement gradients are small.

Equation (1.14) is the strain displacement relationship that we would use to solve boundary value problems in this course, as we limit ourselves to cases where the components of the Lagrangian and Eulerian displacement gradient is small.

### 1.2.3 Compatibility equation

It is evident from the definition of the linearized Lagrangian strain, (1.11) that it is a symmetric tensor. Hence, it has only 6 independent components. Now, one cannot prescribe arbitrarily these six components since a smooth differentiable displacement field should be obtainable from this six prescribed components. The restrictions placed on how this six components of the strain could vary spatially so that a smooth differentiable displacement field is obtainable is called as compatibility equation. Thus, the compatibility condition is

$$curl(curl(\boldsymbol{\epsilon})) = \mathbf{0}. \tag{1.15}$$

The derivation of this equation as well as the components of the $curl(\cdot)$ operator in Cartesian coordinates is presented in chapter 3.

It should also be mentioned that the compatibility condition in case of large deformations is yet to be obtained. That is if the components of the right Cauchy-Green deformation tensor, $\mathbf{C}$ is prescribed, the restrictions that have to be placed on these prescribed components so that a smooth differentiable deformation field could be obtained is unknown, except for some special cases.

### 1.2.4 Constitutive relation

Broadly constitutive relation is the equation that relates the stress (and stress rates) with the displacement gradient (and rate of displacement gradient). While the above three equations - Equilibrium equations, strain-displacement relation, compatibility equations - are independent of the material that the body is made up of and/or the process that the body is subjected to, the constitutive relation is dependent on the material and the process. Constitutive relation is required to bring in the dependance of the material in the response of the body and to have as many equations as there are unknowns, as will be shown in chapter 6.

The fidelity of the predictions, namely the likely displacement or stress for a given force depends only on the constitutive relation. This is so because the other three equations are the same irrespective of the material that the body is made up of. Consequently, a lot of research is being undertaken to arrive at better constitutive relations for materials.

It is difficult to have a constitutive relation that could describe the response of a material subjected to any process. Hence, usually constitutive relations are prescribed for a particular process that the material undergoes. The variables in the constitutive relation depends on the process that is being studied. The same material could undergo different processes depending on the stimuli; for example, the same material could respond elastically or plastically depending on say, the magnitude of the load or temperature. Hence it is only apt to qualify the process and not the material. However, it is customary to qualify the material instead of the process too. This we shall desist.

Traditionally, the constitutive relation is said to depend on whether the given material behaves like a solid or fluid and one elaborates on how to classify a given material as a solid or a fluid. A material that is not a solid is defined as a fluid. This means one has to define what a solid is. A couple of definitions of a solid are listed below:

1. Solid is one which can resist sustained shear forces without continuously deforming

2. Solid is one which does not take the shape of the container

Though these definitions are intuitive they are ambiguous. A class of materials called “viscoelastic solids”, neither take the shape of the container nor resist shear forces without continuously deforming. Also, the same material would behave like a solid, like a mixture of a solid and a fluid or like a fluid depending on say, the temperature and the mechanical stress it is being subjected to. These prompts us to say that a given material behaves in a solid-like or fluid-like manner. However, as we shall see, this classification of a given material as solid or fluid is immaterial. If one appeals to thermodynamics for the classification of the processes, the response of materials could be classified based on (1) Whether there is conversion of energy from one form to another during the process, and (2) Whether the process is thermodynamically equilibrated. Though, in the following section, we classify the response of materials based on thermodynamics, we also give the commonly stated definitions and discuss their shortcomings. In this course, as well as in all these classifications, it is assumed that there are no chemical changes occurring in the body and hence the composition of the body remains a constant.

## 1.3 Classification of the Response of Materials

First, it should be clarified that one should not get confused with the real body and its mathematical idealization. Modeling is all about idealizations that lead to predictions that are close to observations. To illustrate, the earth and the sun are assumed as point masses when one is interested in planetary motion. The same earth is assumed as a rigid sphere if one is interested in studying the eclipse. These assumptions are made to make the resulting problem tractable without losing on the required accuracy. In the same sprit, the all material responses, some amount of mechanical energy is converted into other forms of energy. However, in some cases, this loss in the mechanical energy is small that it can be idealized as having no loss, i.e., a non-dissipative process.

### 1.3.1 Non-dissipative response

A response is said to be non-dissipative if there is no conversion of mechanical energy to other forms of energy, namely heat energy. Commonly, a material responding in this fashion is said to be elastic. The common definitions of elastic response,

1. If the body's original size and shape can be recovered on unloading, the loading process is said to be elastic.

2. Processes in which the state of stress depends only on the current strain, is said to be elastic.

The first definition is of little use, because it requires one to do a complimentary process (unloading) to decide on whether the process that needs to be classified as being elastic. The second definition, though useful for deciding on the variables in the constitutive relation, it also requires one to do a complimentary process (unload and load again) to decide on whether the first process is elastic. The definition based on thermodynamics does not suffer from this drawback. In chapter 6 we provide examples where these three definitions are not equivalent. However, many processes (approximately) satisfy all the three definitions.

This class of processes also proceeds through thermodynamically equilibrated states. That is, if the body is isolated at any instant of loading (or displacement) then the stress, displacement, internal energy, entropy do not change with time.

Ideal gas, a fluid is the best example of a material that responds in a non-dissipative manner. Metals up to a certain stress level, called the yield stress, are also idealized as responding in a non dissipative manner. Thus, the notion that only solids respond in a non-dissipative manner is not correct.

Thus, for these non-dissipative, thermodynamically equilibrated processes the Cauchy stress and the deformation gradient can in general be related through an implicit function. That is, for isotropic materials (see chapter 6 for when a material is said to be isotropic), $\mathbf{f}(\boldsymbol{\sigma}, \mathbf{F}) = \mathbf{0}$. However, in classical elasticity it is customary to assume that Cauchy stress in a isotropic material is a function of the deformation gradient, $\boldsymbol{\sigma} = \hat{\mathbf{f}}(\mathbf{F})$. On requiring the restriction[6] due to objectivity and second law of thermodynamics to hold, it can be shown that if $\boldsymbol{\sigma} = \hat{\mathbf{f}}(\mathbf{F})$, then

$$\boldsymbol{\sigma} = \frac{\partial \psi_R}{\partial J_3}\mathbf{1} + \frac{2}{J_3}\left[\frac{\partial \psi_R}{\partial J_1}\mathbf{B} - \frac{\partial \psi_R}{\partial J_2}\mathbf{B}^{-1}\right], \tag{1.16}$$

where $\psi_R = \hat{\psi}_R(J_1, J_2, J_3)$ is the Helmoltz free energy defined per unit volume in the reference configuration, also called as the stored energy, $\mathbf{B} = \mathbf{F}\mathbf{F}^t$ and $J_1 = tr(\mathbf{B})$, $J_2 = tr(\mathbf{B}^{-1})$, $J_3 = \sqrt{\det(B)}$. When the components of the displacement gradient is small, then (1.16) reduces to,

$$\boldsymbol{\sigma} = tr(\boldsymbol{\epsilon})\lambda\mathbf{1} + 2\mu\boldsymbol{\epsilon}, \tag{1.17}$$

on neglecting the higher powers of the Lagrangian displacement gradient and where $\lambda$ and $\mu$ are called as the Lamè constants. The equation (1.17) is

[6] See chapter 6 for more details about these restrictions.

the famous Hooke's law for isotropic materials. In this course Hooke's law is the constitutive equation that we shall be using to solve boundary value problems.

Before concluding this section, another misnomer needs to be clarified. As can be seen from equation (1.16) the relationship between Cauchy stress and the displacement gradient can be nonlinear when the response is non-dissipative. Only sometimes as in the case of the material obeying Hooke's law is this relationship linear. It is also true that if the response is dissipative, the relationship between the stress and the displacement gradient is always nonlinear. However, nonlinear relationship between the stress and the displacement gradient does not mean that the response is dissipative. That is, nonlinear relationship between the stress and the displacement gradient is only a necessary condition for the response to be dissipative but not a sufficient condition.

### 1.3.2 Dissipative response

A response is said to be dissipative if there is conversion of mechanical energy to other forms of energy. A material responding in this fashion is popularly said to be inelastic. There are three types of dissipative response, which we shall see in some detail.

#### Plastic response

A material is said to deform plastically if the deformation process proceeds through thermodynamically equilibrated states but is dissipative. That is, if the body is isolated at any instant of loading (or displacement) then the stress, displacement, internal energy, entropy do not change with time. By virtue of the process being dissipative, the stress at an instant would depend on the history of the deformation. However, the stress does not depend on the rate of loading or displacement by virtue of the process proceeding through thermodynamically equilibrated states.

For plastic response, the classical constitutive relation is assumed to be of the form,

$$\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F}, \mathbf{F}^p, q_1, q_2), \tag{1.18}$$

where $\mathbf{F}^p$, $q_1$, $q_2$ are internal variables whose values could change with deformation and/or stress. For illustration, we have used two scalar internal variables and one second order tensor internal variable while there can be any number of tensor or scalar internal variables. In some theories the internal variables are given a physical interpretation but in general, these variable need not have any meaning and are proposed for mathematical modeling purpose only.

Thus, when a material deforms plastically, it does not return back to its original shape when unloaded; there would be a permanent deformation. Hence, the process is irreversible. The response does not depend on the rate of loading (or displacement). Metals like steel at room temperature respond plastically when stressed above a particular limit, called the yield stress.

### Viscoelastic response

If the dissipative process proceeds through states that are not in thermodynamic equilibrium[7], then it is said to be viscoelastic. Therefore, if a body is isolated at some instant of loading (or displacement) then the displacement (or the stress) continues to change with time. A viscoelastic material when subjected to constant stress would result in a deformation that changes with time which is called as creep. Also, when a viscoelastic material is subjected to a constant deformation field, its stress changes with time and this is called as stress relaxation. This is in contrary to a elastic or plastic material which when subjected to a constant stress would have a constant strain.

The constitutive relation for a viscoelastic response is of the form,

$$\mathbf{f}(\boldsymbol{\sigma}, \mathbf{F}, \dot{\boldsymbol{\sigma}}, \dot{\mathbf{F}}) = \mathbf{0}, \tag{1.19}$$

where $\dot{\boldsymbol{\sigma}}$ denotes the time derivative of stress and $\dot{\mathbf{F}}$ time derivative of the deformation gradient. Though here we have truncated to first order time derivatives, the general theory allows for higher order time derivatives too.

Thus, the response of a viscoelastic material depends on the rate at which it is loaded (or displaced) apart from the history of the loading (or displacement). The response of a viscoelastic material changes depending on whether load is controlled or displacement is controlled. This process too is irreversible and there would be unrecovered deformation immediately on removal of the load. The magnitude of unrecovered deformation after a long time (asymptotically) would tend to zero or remain the same constant value that it is immediately after the removal of load.

[7] A body is said to be in thermodynamic equilibrium if no quantity that describes its state changes when it is isolated from its surroundings. A body is said to be isolated when there is no mass or energy flux in to or out of the body.

Constitutive relations of the form,

$$\boldsymbol{\sigma} = \mathbf{f}(\dot{\mathbf{F}}), \tag{1.20}$$

which is a special case of the viscoelastic constitutive relation (1.19), is that of a viscous fluid.

In some treatments of the subject, a viscoelastic material would be said to be a combination of a viscous fluid and an elastic solid and the viscoelastic models are obtained by combining springs and dashpots. There are several philosophical problems associated with this viewpoint about which we cannot elaborate here.

### Viscoplastic response

This process too is dissipative and proceeds through states that are not in thermodynamic equilibrium. However, in order to model this class of response the constitutive relation has to be of the form,

$$\mathbf{f}(\boldsymbol{\sigma}, \mathbf{F}, \dot{\boldsymbol{\sigma}}, \dot{\mathbf{F}}, \mathbf{F}^p, q_1, q_2) = \mathbf{0}, \tag{1.21}$$

where $\mathbf{F}^p$, $q_1$, $q_2$ are the internal variables whose values could change with deformation and/or stress. Their significance is same as that discussed for plastic response. As can be easily seen the constitutive relation form for the viscoplastic response (1.21) encompasses viscoelastic, plastic and elastic response as a special case.

In this case, constant load causes a deformation that changes with time. Also, a constant deformation causes applied load to change with time. The response of the material depends on the rate of loading or displacement. The process is irreversible and there would be unrecovered deformation on removal of load. The magnitude of this unrecovered deformation varies with rate of loading, time and would tend to a value which is not zero. This dependance of the constant value that the unrecovered deformation tends on the rate of loading, could be taken as the characteristic of viscoplastic response.

Figure 1.3 shows the typical variation in the strain for various responses when the material is loaded, held at a constant load and unloaded, as discussed above. This kind of loading is called as the creep and recovery loading, helps one to distinguish various kinds of responses.

As mentioned already, in this course we shall focus on the elastic or non-dissipative response only.

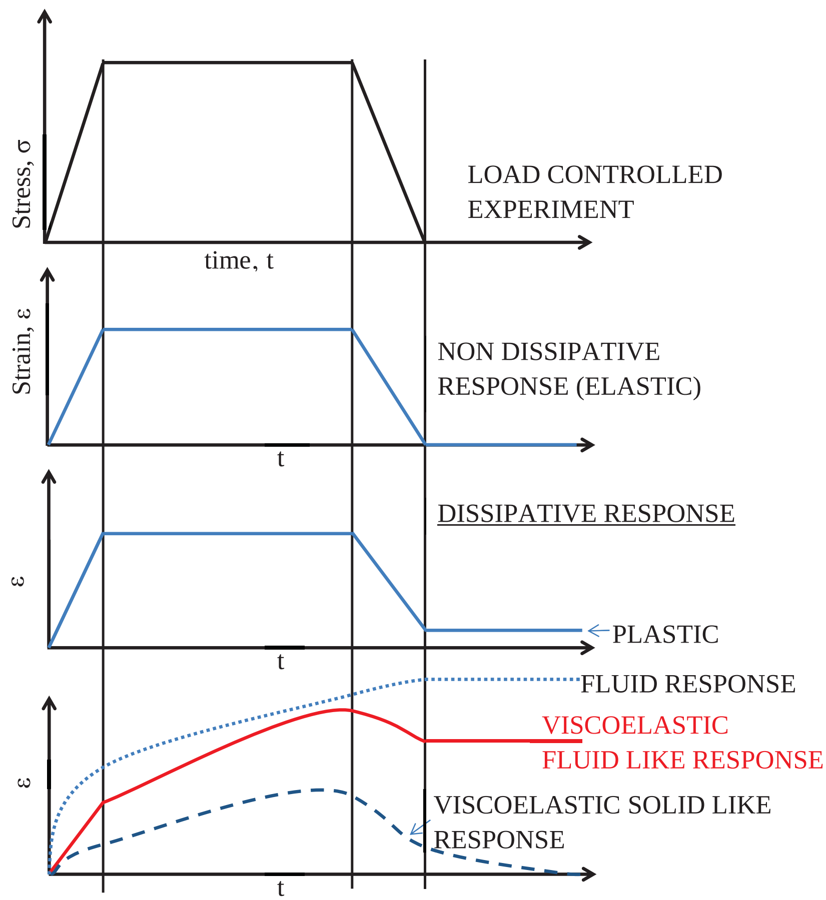



Figure 1.3: Schematic of the variation in the strain with time for various responses when the material is loaded and unloaded.

## 1.4 Solution to Boundary Value Problems

A boundary value problem is one in which we specify the traction applied on the surface of a body and/or displacement of the boundary of a body and are interested in finding the displacement and/or the stress at any interior point in the body or on part of the boundary where they were not specified. This specification of the boundary traction and/or displacement is called as boundary condition. The boundary condition is in a sense constitutive relation for the boundary. It tells how the body and its surroundings interact. Thus, in a boundary value problem one needs to prescribe the geometry of the body, the constitutive relation for the material that the body is made up of for the process it is going to be subjected to and the boundary condition. Using this information one needs to find the displacement and stress that the body is subjected to. The so found displacement and stress field should satisfy the equilibrium equations, constitutive relations, compatibility conditions and boundary conditions.

The purpose of formulating and solving a boundary value problem is to:

1. To ensure the stresses are within prescribed limits

2. To ensure that the displacements are within prescribed limits

3. To find the distribution of forces and moments on part of the boundary where displacements are specified

There are four type of boundary conditions. They are

1. **Displacement boundary condition:** Here the displacement of the entire boundary of the body alone is specified. This is also called as Dirichlet boundary condition

2. **Traction boundary condition:** Here the traction on the entire boundary of the body alone is specified. This is also called as Neumann boundary condition

3. **Mixed boundary condition:** Here the displacement is specified on part of the boundary and traction is specified on the remaining part of the boundary. Both traction as well as displacement are not specified over any part of boundary

4. **Robin boundary condition:** Here both the displacement and the traction are specified on the same part of the boundary.

There are three methods by which the displacement and stress field in the body can be found, satisfying all the required governing equations and the boundary conditions. Outline of these methods are presented next. The choice of a method depends on the type of boundary condition.

### 1.4.1 Displacement method

Here displacement field is taken as the basic unknown. Then, using the strain displacement relation, (1.14) the strain is computed. This strain in substituted in the constitutive relation, (1.17) to obtain the stress. The stress is then substituted in the equilibrium equation (1.6) to obtain 3 second order partial differential equations in terms of the components of the displacement field as,

$$(\lambda + \mu) grad(div(\mathbf{u})) + \mu \Delta \mathbf{u} + \rho \mathbf{b} = \rho \frac{d^2 \mathbf{u}}{dt^2}, \tag{1.22}$$

where $\Delta(\cdot)$ stands for the Laplace operator and $t$ denotes time. The detail derivation of this equation is given in chapter 7. Equation (1.22) is called the Navier-Lamè equations. Thus, in the displacement method equation (1.22) is solved along with the prescribed boundary condition.

If three dimensional solid elements are used for modeling the body in finite element programs, then the weakened form of equation (1.22) is solved for the specified boundary conditions.

### 1.4.2 Stress method

In this method, the stress field is assumed such that it satisfies the equilibrium equations as well as the prescribed traction boundary conditions. For example, in the absence of body forces and static equilibrium, it can be easily seen that if the Cartesian components of the stress are derived from a potential, $\phi = \tilde{\phi}(x, y, z)$ called as the Airy's stress potential as,

$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} & -\frac{\partial^2 \phi}{\partial x \partial y} & -\frac{\partial^2 \phi}{\partial x \partial z} \\ -\frac{\partial^2 \phi}{\partial x \partial y} & \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} & -\frac{\partial^2 \phi}{\partial y \partial z} \\ -\frac{\partial^2 \phi}{\partial x \partial z} & -\frac{\partial^2 \phi}{\partial y \partial z} & \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \end{pmatrix}, \tag{1.23}$$

then the equilibrium equations are satisfied. Having arrived at the stress, the strain is computed using

$$\boldsymbol{\epsilon} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda tr(\boldsymbol{\sigma})}{2\mu(3\lambda + 2\mu)}\mathbf{1}, \tag{1.24}$$

obtained by inverting the constitutive relation, (1.17). In order to be able to find a smooth displacement field from this strain, it has to satisfy compatibility condition (1.15). This procedure is formulated in chapter 7 and is followed to solve some boundary value problems in chapters 8 and 9.

### 1.4.3 Semi-inverse method

This method is used to solve problems when the constitutive relation is not given by Hooke's law (1.17). When the constitutive relation is not given by Hooke's law, displacement method results in three coupled nonlinear partial differential equations for the displacement components which are difficult to solve. Hence, simplifying assumptions are made for the displacement field, wherein a the displacement field is prescribed but for some constants and/or some functions. Except in cases where the constitutive relation is of the form (1.16), one has to make an assumption on the components of the stress which would be nonzero for this prescribed displacement field. Then, these nonzero components of the stress field is found in terms of the constants and unknown functions in the displacement field. On substituting these stress components in the equilibrium equations and boundary conditions, one obtains differential equations for the unknown functions and algebraic equations to find the unknown constants. The prescription of the displacement field is made in such a way that it results in ordinary differential equations governing the form of the unknown functions. Since part displacement and part stress are prescribed it is called semi-inverse method. This method of solving equations would not be illustrated in this course.

Finally, we say that the boundary value problem is well posed if (1) There exist a displacement and stress field that satisfies the boundary conditions and the governing equations (2) There exist only one such displacement and stress field (3) Small changes in the boundary conditions causes only small changes in the displacement and stress fields. The boundary value problem obtained when Hooke's law (1.17) is used for the constitutive relation is known to be well posed, as will be discussed in chapter 7.

## 1.5 Summary

Thus in this chapter we introduced the four concepts in mechanics, the four equations connecting these concepts as well as the methodologies used to solve boundary value problems. In the following chapters we elaborate on the same topics. It is not intended that in a first reading of this chapter, one would understand all the details. However, reading the same chapter at the end of this course, one should appreciate the details. This chapter summarizes the concepts that should be assimilated and digested during this course.

# Chapter 2

# Mathematical Preliminaries

## 2.1 Overview

In the introduction, we saw that some of the quantities like force is a vector or first order tensor, stress is a second order tensor or simply a tensor. The algebra and calculus associated with these quantities differs, in some aspects from that of scalar quantities. In this chapter, we shall study on how to manipulate equations involving vectors and tensors and how to take derivatives of scalar valued function of a tensor or tensor valued function of a tensor.

## 2.2 Algebra of vectors

A physical quantity, completely described by a single real number, such as temperature, density or mass is called a scalar. A vector is a directed line element in space used to model physical quantities such as force, velocity, acceleration which have both direction and magnitude. The vector could be denoted as $\mathbf{v}$ or $\underline{v}$. Here we denote vectors as $\mathbf{v}$. The length of the directed line segment is called as the magnitude of the vector and is denoted by $|\mathbf{v}|$. Two vectors are said to be equal if they have the same direction and magnitude. The point that vector is a geometric object, namely a directed line segment cannot be overemphasized. Thus, for us a set of numbers is not a vector.

The sum of two vectors yields a new vector, based on the parallelogram law of addition and has the following properties:

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u}, && (2.1)\\ (\mathbf{u}+\mathbf{v})+\mathbf{w} &= \mathbf{u}+(\mathbf{v}+\mathbf{w}), && (2.2)\\ \mathbf{u}+\mathbf{o} &= \mathbf{u}, && (2.3)\\ \mathbf{u}+(-\mathbf{u}) &= \mathbf{o}, && (2.4) \end{aligned}$$

where $\mathbf{o}$ denotes the zero vector with unspecified direction and zero length, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are any vectors. The parallelogram law of addition has been proposed for the vectors because that is how forces, velocities add up.

Then the scalar multiplication $\alpha\mathbf{u}$ produces a new vector with the same direction as $\mathbf{u}$ if $\alpha > 0$ or with the opposite direction to $\mathbf{u}$ if $\alpha < 0$ with the following properties:

$$\begin{aligned} (\alpha\beta)\mathbf{u} &= \alpha(\beta\mathbf{u}), && (2.5)\\ (\alpha+\beta)\mathbf{u} &= \alpha\mathbf{u}+\beta\mathbf{u}, && (2.6)\\ \alpha(\mathbf{u}+\mathbf{v}) &= \alpha\mathbf{u}+\alpha\mathbf{v}, && (2.7) \end{aligned}$$

where $\alpha$, $\beta$ are some scalars(real number).

The dot (or scalar or inner) product of two vectors $\mathbf{u}$ and $\mathbf{v}$ denoted by $\mathbf{u}\cdot\mathbf{v}$ assigns a real number to a given pair of vectors such that the following properties hold:

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \mathbf{v}\cdot\mathbf{u}, && (2.8)\\ \mathbf{u}\cdot\mathbf{o} &= 0, && (2.9)\\ \mathbf{u}\cdot(\alpha\mathbf{v}+\beta\mathbf{w}) &= \alpha(\mathbf{u}\cdot\mathbf{v})+\beta(\mathbf{u}\cdot\mathbf{w}), && (2.10)\\ \mathbf{u}\cdot\mathbf{u} &> 0, \quad \text{for all} \quad \mathbf{u}\neq\mathbf{o}, \quad \text{and} && \\ \mathbf{u}\cdot\mathbf{u} &= 0, \quad \text{if and only if} \quad \mathbf{u}=\mathbf{o}. && (2.11) \end{aligned}$$

The quantity $|\mathbf{u}|$ (or $\|\mathbf{u}\|$) is called the magnitude (or length or norm) of a vector $\mathbf{u}$ which is a non-negative real number is defined as

$$|\mathbf{u}| = (\mathbf{u}\cdot\mathbf{u})^{1/2} \geq 0. \qquad (2.12)$$

A vector $\mathbf{e}$ is called a unit vector if $|\mathbf{e}| = 1$. A nonzero vector $\mathbf{u}$ is said to be orthogonal (or perpendicular) to a nonzero vector $\mathbf{v}$ if: $\mathbf{u}\cdot\mathbf{v} = 0$. Then, the projection of a vector $\mathbf{u}$ along the direction of a vector $\mathbf{e}$ whose length is unity is given by: $\mathbf{u}\cdot\mathbf{e}$

So far algebra has been presented in symbolic (or direct or absolute) notation. It represents a very convenient and concise tool to manipulate most of the relations used in continuum mechanics. However, particularly in computational mechanics, it is essential to refer vector (and tensor) quantities to a basis. Also, for carrying out mathematical operations more intuitively it is often helpful to refer to components.

We introduce a fixed set of three basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ (sometimes introduced as $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) called a Cartesian basis, with properties:

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1, \tag{2.13}$$

so that any vector in three dimensional space can be written in terms of these three basis vectors with ease. However, in general, it is not required for the basis vectors to be fixed or to satisfy (2.13). Basis vectors that satisfy (2.13) are called as orthonormal basis vectors.

Any vector $\mathbf{u}$ in the three dimensional space is represented uniquely by a linear combination of the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, i.e.

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3, \tag{2.14}$$

where the three real numbers $u_1$, $u_2$, $u_3$ are the uniquely determined Cartesian components of vector $\mathbf{u}$ along the given directions $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, respectively. In other words, what we are doing here is representing any directed line segment as a linear combination of three directed line segments. This is akin to representing any real number using the ten arabic numerals.

If an orthonormal basis is used to represent the vector, then the components of the vector along the basis directions is nothing but the projection of the vector on to the basis directions. Thus,

$$u_1 = \mathbf{u} \cdot \mathbf{e}_1, \quad u_2 = \mathbf{u} \cdot \mathbf{e}_2, \quad u_3 = \mathbf{u} \cdot \mathbf{e}_3. \tag{2.15}$$

Using index (or subscript or suffix) notation relation (2.14) can be written as $\mathbf{u} = \sum_{i=1}^{3} u_i\mathbf{e}_i$ or, in an abbreviated form by leaving out the summation symbol, simply as

$$\mathbf{u} = u_i\mathbf{e}_i, \qquad \text{(sum over i = 1,2,3)}, \tag{2.16}$$

where we have adopted the summation convention, invented by Einstein. The summation convention says that whenever an index is repeated (only once) in the same term, then, a summation over the range of this index is implied unless otherwise indicated. The index $i$ that is summed over is said to be a dummy (or summation) index, since a replacement by any other symbol does not affect the value of the sum. An index that is not summed over in a given term is called a free (or live) index. Note that in the same equation an index is either dummy or free. Here we consider only the three dimensional space and denote the basis vectors by $\{\mathbf{e}_i\}_{i\in\{1,2,3\}}$ collectively.

In light of the above, relations (2.13) can be written in a more convenient form as

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \equiv \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j, \end{cases}, \tag{2.17}$$

where $\delta_{ij}$ is called the Kronecker delta. It is easy to deduce the following identities:

$$\delta_{ii} = 3, \qquad \delta_{ij}u_i = u_j, \qquad \delta_{ij}\delta_{jk} = \delta_{ik}. \tag{2.18}$$

The projection of a vector $\mathbf{u}$ onto the Cartesian basis vectors, $\mathbf{e}_i$ yields the $i^{th}$ component of $\mathbf{u}$. Thus, in index notation $\mathbf{u} \cdot \mathbf{e}_i = u_i$.

As already mentioned a set of numbers is not a vector. However, for ease of computations we represent the components of a vector, $\mathbf{u}$ obtained with respect to some basis as,

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \tag{2.19}$$

instead of writing as $\mathbf{u} = u_i\mathbf{e}_i$ using the summation notation introduced above. The numbers $u_1$, $u_2$ and $u_3$ have no meaning without the basis vectors which are present even though they are not mentioned explicitly.

If $u_i$ and $v_j$ represents the Cartesian components of vectors $\mathbf{u}$ and $\mathbf{v}$ respectively, then,

$$\mathbf{u} \cdot \mathbf{v} = u_i v_j \mathbf{e}_i \cdot \mathbf{e}_j = u_i v_j \delta_{ij} = u_i v_i, \tag{2.20}$$

$$|\mathbf{u}|^2 = u_i u_i = u_1^2 + u_2^2 + u_3^2. \tag{2.21}$$

Here we have used the replacement property of the Kronecker delta to write $v_j\delta_{ij}$ as $v_i$ which reflects the fact that only if $j = i$ is $\delta_{ij} = 1$ and otherwise $\delta_{ij} = 0$.

The cross (or vector) product of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \wedge \mathbf{v}$ produces a

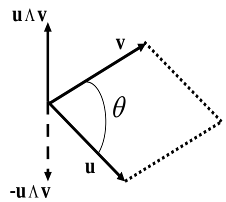



Figure 2.1: Cross product of two vectors **u** and **v**

new vector satisfying the following properties:

$$
\begin{aligned}
\mathbf{u} \wedge \mathbf{v} &= -(\mathbf{v} \wedge \mathbf{u}), && (2.22)\\
(\alpha \mathbf{u}) \wedge \mathbf{v} &= \mathbf{u} \wedge (\alpha \mathbf{v}) = \alpha(\mathbf{u} \wedge \mathbf{v}), && (2.23)\\
\mathbf{u} \wedge (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} \wedge \mathbf{v}) + (\mathbf{u} \wedge \mathbf{w}), && (2.24)\\
\mathbf{u} \wedge \mathbf{v} &= \mathbf{o}, \qquad \text{iff } \mathbf{u} \text{ and } \mathbf{v} \text{ are linearly dependent} && (2.25)
\end{aligned}
$$

when $\mathbf{u} \neq \mathbf{o}$ $\mathbf{v} \neq \mathbf{o}$. Two vectors **u** and **v** are said to be linearly dependent if $\mathbf{u} = \alpha \mathbf{v}$, for some constant $\alpha$. Note that because of the first property the cross product is not commutative.

The cross product characterizes the area of a parallelogram spanned by the vectors **u** and **v** given that

$$\mathbf{u} \wedge \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin(\theta)\mathbf{n}, \tag{2.26}$$

where $\theta$ is the angle between the vectors **u** and **v** and **n** is a unit vector normal to the plane spanned by **u** and **v**, as shown in figure 2.1.

In order to express the cross product in terms of components we introduce the permutation (or alternating or Levi-Civita) symbol $\epsilon_{ijk}$ which is defined as

$$\epsilon_{ijk} = \begin{cases} 1, & \text{for even permutations of } (i,j,k) \text{ (i.e. 123,231,312)}, \\ -1, & \text{for odd permutations of } (i,j,k) \text{ (i.e. 132,213,321)}, \\ 0, & \text{if there is repeated index}, \end{cases} \tag{2.27}$$

with the property $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$, $\epsilon_{ijk} = -\epsilon_{ikj}$ and $\epsilon_{ijk} = -\epsilon_{jik}$, respectively. Thus, for an orthonormal Cartesian basis, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $\mathbf{e}_i \wedge \mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k$

It could be verified that $\epsilon_{ijk}$ may be expressed as:

$$\epsilon_{ijk} = \delta_{i1}(\delta_{j2}\delta_{k3} - \delta_{j3}\delta_{k2}) + \delta_{i2}(\delta_{j3}\delta_{k1} - \delta_{j1}\delta_{k3}) + \delta_{i3}(\delta_{j1}\delta_{k2} - \delta_{j2}\delta_{k1}). \tag{2.28}$$

It could also be verified that the product of the permutation symbols $\epsilon_{ijk}\epsilon_{pqr}$ is related to the Kronecker delta by the relation

$$\epsilon_{ijk}\epsilon_{pqr} = \delta_{ip}(\delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}) + \delta_{iq}(\delta_{jr}\delta_{kp} - \delta_{jp}\delta_{kr}) + \delta_{ir}(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}). \quad (2.29)$$

We deduce from the above equation (2.29) the important relations:

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \qquad \epsilon_{ijk}\epsilon_{pjk} = 2\delta_{ip}, \qquad \epsilon_{ijk}\epsilon_{ijk} = 6. \quad (2.30)$$

The triple scalar (or box) product: $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ represents the volume V of a parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ forming a right handed triad. Thus, in index notation:

$$V = [\mathbf{u}, \mathbf{v}, \mathbf{w}] = (\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = \epsilon_{ijk} u_i v_j w_k. \quad (2.31)$$

Note that the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly dependent if and only if their scalar triple product vanishes, i.e., $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = 0$

The product $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$ is called the vector triple product and it may be verified that

$$\begin{aligned}
(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} &= \epsilon_{kqr}(\epsilon_{ijk} u_i v_j) w_q \mathbf{e}_r = \epsilon_{qrk}\epsilon_{ijk} u_i v_j w_q \mathbf{e}_r && (2.32)\\
&= (\delta_{qi}\delta_{rj} - \delta_{qj}\delta_{ri}) u_i v_j w_q \mathbf{e}_r && (2.33)\\
&= (u_q v_r w_q - u_r v_q w_q)\mathbf{e}_r && (2.34)\\
&= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}. && (2.35)
\end{aligned}$$

Similarly, it can be shown that

$$\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}. \quad (2.36)$$

Thus triple product, in general, is not associative, i.e., $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) \neq (\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$.

## 2.3 Algebra of second order tensors

A second order tensor $\mathbf{A}$, for our purposes here, may be thought of as a linear function that maps a directed line segment to another directed line segment. This we write as, $\mathbf{v} = \mathbf{A}\mathbf{u}$ where $\mathbf{A}$ is the linear function that assigns a vector $\mathbf{v}$ to each vector $\mathbf{u}$. Since $\mathbf{A}$ is a linear function,

$$\mathbf{A}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{A}\mathbf{u} + \beta\mathbf{A}\mathbf{v}, \quad (2.37)$$

for all vectors $\mathbf{u}$, $\mathbf{v}$ and all scalars $\alpha$, $\beta$.

If $\mathbf{A}$ and $\mathbf{B}$ are two second order tensors, we can define the sum $\mathbf{A}+\mathbf{B}$, the difference $\mathbf{A}-\mathbf{B}$ and the scalar multiplication $\alpha\mathbf{A}$ by the rules

$$(\mathbf{A}\pm\mathbf{B})\mathbf{u} = \mathbf{A}\mathbf{u}\pm\mathbf{B}\mathbf{u}, \tag{2.38}$$

$$(\alpha\mathbf{A})\mathbf{u} = \alpha(\mathbf{A}\mathbf{u}), \tag{2.39}$$

where $\mathbf{u}$ denotes an arbitrary vector. The important second order unit (or identity) tensor $\mathbf{1}$ and the second order zero tensor $\mathbf{0}$ are defined, respectively, by the relation $\mathbf{1u} = \mathbf{u}$ and $\mathbf{0u} = \mathbf{o}$, for all ($\forall$) vectors $\mathbf{u}$.

If the relation $\mathbf{v}\cdot\mathbf{A}\mathbf{v}\geq 0$ holds for all vectors, then $\mathbf{A}$ is said to be a positive semi-definite tensor. If the condition $\mathbf{v}\cdot\mathbf{A}\mathbf{v} > 0$ holds for all nonzero vectors $\mathbf{v}$, then $\mathbf{A}$ is said to be positive definite tensor. Tensor $\mathbf{A}$ is called negative semi-definite if $\mathbf{v}\cdot\mathbf{A}\mathbf{v}\leq 0$ and negative definite if $\mathbf{v}\cdot\mathbf{A}\mathbf{v} < 0$ for all vectors, $\mathbf{v}\neq\mathbf{o}$, respectively.

The tensor (or direct or matrix) product or the dyad of the vectors $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u}\otimes\mathbf{v}$. It is a second order tensor which linearly transforms a vector $\mathbf{w}$ into a vector with the direction of $\mathbf{u}$ following the rule

$$(\mathbf{u}\otimes\mathbf{v})\mathbf{w} = (\mathbf{v}\cdot\mathbf{w})\mathbf{u}. \tag{2.40}$$

The dyad satisfies the linearity property

$$(\mathbf{u}\otimes\mathbf{v})(\alpha\mathbf{w}+\beta\mathbf{x}) = \alpha(\mathbf{u}\otimes\mathbf{v})\mathbf{w}+\beta(\mathbf{u}\otimes\mathbf{v})\mathbf{x}. \tag{2.41}$$

The following relations are easy to establish:

$$\begin{aligned}
(\alpha\mathbf{u}+\beta\mathbf{v})\otimes\mathbf{w} &= \alpha(\mathbf{u}\otimes\mathbf{w})+\beta(\mathbf{v}\otimes\mathbf{w}), && (2.42)\\
(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x}) &= (\mathbf{v}\cdot\mathbf{w})(\mathbf{u}\otimes\mathbf{x}), && (2.43)\\
\mathbf{A}(\mathbf{u}\otimes\mathbf{v}) &= (\mathbf{A}\mathbf{u})\otimes\mathbf{v}, && (2.44)
\end{aligned}$$

where, $\mathbf{A}$ is an arbitrary second order tensor, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$ are arbitrary vectors and $\alpha$ and $\beta$ are arbitrary scalars. Dyad is not commutative, i.e., $\mathbf{u}\otimes\mathbf{v}\neq\mathbf{v}\otimes\mathbf{u}$ and $(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x})\neq(\mathbf{w}\otimes\mathbf{x})(\mathbf{u}\otimes\mathbf{v})$.

A dyadic is a linear combination of dyads with scalar coefficients, for example, $\alpha(\mathbf{u}\otimes\mathbf{v})+\beta(\mathbf{w}\otimes\mathbf{x})$. Any second-order tensor can be expressed as a dyadic. As an example, the second order tensor $\mathbf{A}$ may be represented by a linear combination of dyads formed by the Cartesian basis $\{\mathbf{e}_i\}$, i.e.,

$\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$. The nine Cartesian components of $\mathbf{A}$ with respect to $\{\mathbf{e}_i\}$, represented by $A_{ij}$ can be expressed as a matrix $[\mathbf{A}]$, i.e.,

$$[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}. \tag{2.45}$$

This is known as the matrix notation of tensor $\mathbf{A}$. We call $\mathbf{A}$, which is resolved along basis vectors that are orthonormal and fixed, a Cartesian tensor of order two. Then, the components of $\mathbf{A}$ with respect to a fixed, orthonormal basis vectors $\mathbf{e}_i$ is obtained as:

$$A_{ij} = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j. \tag{2.46}$$

The Cartesian components of the unit tensor $\mathbf{1}$ form the Kronecker delta symbol, thus $\mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_i$ and in matrix form

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.47}$$

Next we would like to derive the components of $\mathbf{u} \otimes \mathbf{v}$ along an orthonormal basis $\{\mathbf{e}_i\}$. Using the representation (2.46) we find that

$$\begin{aligned} (\mathbf{u} \otimes \mathbf{v})_{ij} &= \mathbf{e}_i \cdot (\mathbf{u} \otimes \mathbf{v})\mathbf{e}_j \\ &= (\mathbf{e}_i \cdot \mathbf{u})(\mathbf{e}_j \cdot \mathbf{v}) \\ &= u_i v_j, \end{aligned} \tag{2.48}$$

where $u_i$ and $v_j$ are the Cartesian components of the vectors $\mathbf{u}$ and $\mathbf{v}$ respectively. Writing the above equation in the convenient matrix notation we have

$$(\mathbf{u} \otimes \mathbf{v})_{ij} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix}. \tag{2.49}$$

The product of two second order tensors $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{AB}$, is again a second order tensor. It follows from the requirement $(\mathbf{AB})\mathbf{u} = \mathbf{A}(\mathbf{Bu})$, for all vectors $\mathbf{u}$.

Further, the product of second order tensors is not commutative, i.e., $\mathbf{AB} \neq \mathbf{BA}$. The components of the product $\mathbf{AB}$ along an orthonormal basis $\{\mathbf{e}_i\}$ is found to be:

$$\begin{aligned} (\mathbf{AB})_{ij} &= \mathbf{e}_i \cdot (\mathbf{AB})\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{A}(\mathbf{B}\mathbf{e}_j), && (2.50) \\ &= \mathbf{e}_i \cdot \mathbf{A}(B_{kj}\mathbf{e}_k) = (\mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_k)B_{kj}, && (2.51) \\ &= A_{ik}B_{kj}. && (2.52) \end{aligned}$$

The following properties hold for second order tensors:

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \mathbf{B} + \mathbf{A}, && (2.53) \\ \mathbf{A} + \mathbf{0} &= \mathbf{A}, && (2.54) \\ \mathbf{A} + (-\mathbf{A}) &= \mathbf{0}, && (2.55) \\ (\mathbf{A} + \mathbf{B}) + \mathbf{C} &= \mathbf{A} + (\mathbf{B} + \mathbf{C}), && (2.56) \\ (\mathbf{AB})\mathbf{C} &= \mathbf{A}(\mathbf{BC}) = \mathbf{ABC}, && (2.57) \\ \mathbf{A}^2 &= \mathbf{AA}, && (2.58) \\ (\mathbf{A} + \mathbf{B})\mathbf{C} &= \mathbf{AC} + \mathbf{BC}, && (2.59) \end{aligned}$$

Note that the relations $\mathbf{AB} = \mathbf{0}$ and $\mathbf{Au} = \mathbf{o}$ does not imply that $\mathbf{A}$ or $\mathbf{B}$ is $\mathbf{0}$ or $\mathbf{u} = \mathbf{o}$.

The unique transpose of a second order tensor $\mathbf{A}$ denoted by $\mathbf{A}^t$ is governed by the identity:

$$\mathbf{v} \cdot \mathbf{A}^t\mathbf{u} = \mathbf{u} \cdot \mathbf{A}\mathbf{v} \qquad (2.60)$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$.

Some useful properties of the transpose are

$$\begin{aligned} (\mathbf{A}^t)^t &= \mathbf{A}, && (2.61) \\ (\alpha\mathbf{A} + \beta\mathbf{B})^t &= \alpha\mathbf{A}^t + \beta\mathbf{B}^t, && (2.62) \\ (\mathbf{AB})^t &= \mathbf{B}^t\mathbf{A}^t, && (2.63) \\ (\mathbf{u} \otimes \mathbf{v})^t &= \mathbf{v} \otimes \mathbf{u}. && (2.64) \end{aligned}$$

From identity (2.60) we obtain $\mathbf{e}_i \cdot \mathbf{A}^t\mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{A}\mathbf{e}_i$, which gives, in regard to equation (2.46), the relation $(A^t)_{ij} = A_{ji}$.

The trace of a tensor $\mathbf{A}$ is a scalar denoted by $tr(\mathbf{A})$ and is defined as:

$$tr(\mathbf{A}) = \frac{[\mathbf{Au}, \mathbf{v}, \mathbf{w}] + [\mathbf{u}, \mathbf{Av}, \mathbf{w}] + [\mathbf{u}, \mathbf{v}, \mathbf{Aw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \qquad (2.65)$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are any vectors such that $[\mathbf{u}, \mathbf{v}, \mathbf{w}] \neq 0$, i.e., these vectors span the entire 3D vector space. Thus, $tr(\mathbf{m} \otimes \mathbf{n}) = \mathbf{m} \cdot \mathbf{n}$. Let us see how: Without loss of generality we can assume the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ to be $\mathbf{m}$, $\mathbf{n}$ and $(\mathbf{m} \wedge \mathbf{n})$ respectively. Then, (2.65) becomes

$$\begin{aligned} tr(\mathbf{m} \otimes \mathbf{n}) &= \frac{(\mathbf{n} \cdot \mathbf{m})[\mathbf{m}, \mathbf{n}, \mathbf{m} \wedge \mathbf{n}] + |\mathbf{n}|[\mathbf{m}, \mathbf{m}, \mathbf{m} \wedge \mathbf{n}] + [\mathbf{m}, \mathbf{n}, \mathbf{n}][\mathbf{m}, \mathbf{n}, \mathbf{m}]}{[\mathbf{m}, \mathbf{n}, \mathbf{m} \wedge \mathbf{n}]}, \\ &= \mathbf{n} \cdot \mathbf{m}. \end{aligned} \tag{2.66}$$

The following properties of trace is easy to establish from (2.65):

$$tr(\mathbf{A}^t) = tr(\mathbf{A}), \tag{2.67}$$
$$tr(\mathbf{AB}) = tr(\mathbf{BA}), \tag{2.68}$$
$$tr(\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha tr(\mathbf{A}) + \beta tr(\mathbf{B}). \tag{2.69}$$

Then, the trace of a tensor $\mathbf{A}$ with respect to the orthonormal basis $\{\mathbf{e}_i\}$ is given by

$$\begin{aligned} tr(\mathbf{A}) &= tr(A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) = A_{ij} tr(\mathbf{e}_i \otimes \mathbf{e}_j) \\ &= A_{ij}\mathbf{e}_i \cdot \mathbf{e}_j = A_{ij}\delta_{ij} = A_{ii} \\ &= A_{11} + A_{22} + A_{33} \end{aligned} \tag{2.70}$$

The dot product between two tensors denoted by $\mathbf{A} \cdot \mathbf{B}$, just like the dot product of two vectors yields a real value and is defined as

$$\mathbf{A} \cdot \mathbf{B} = tr(\mathbf{A}^t\mathbf{B}) = tr(\mathbf{B}^t\mathbf{A}) \tag{2.71}$$

Next, we record some useful properties of the dot operator:

$$\mathbf{1} \cdot \mathbf{A} = tr(\mathbf{A}) = \mathbf{A} \cdot \mathbf{1}, \tag{2.72}$$
$$\mathbf{A} \cdot (\mathbf{BC}) = \mathbf{B}^t\mathbf{A} \cdot \mathbf{C} = \mathbf{AC}^t \cdot \mathbf{B}, \tag{2.73}$$
$$\mathbf{A} \cdot (\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot \mathbf{Av}, \tag{2.74}$$
$$(\mathbf{u} \otimes \mathbf{v}) \cdot (\mathbf{w} \otimes \mathbf{x}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{x}). \tag{2.75}$$

Note that if we have the relation $\mathbf{A} \cdot \mathbf{B} = \mathbf{C} \cdot \mathbf{B}$, in general, we cannot conclude that $\mathbf{A}$ equals $\mathbf{C}$. $\mathbf{A} = \mathbf{C}$ only if the above equality holds for any arbitrary $\mathbf{B}$.

The norm of a tensor $\mathbf{A}$ is denoted by $|\mathbf{A}|$ (or $\|\mathbf{A}\|$). It is a non-negative real number and is defined by the square root of $\mathbf{A} \cdot \mathbf{A}$, i.e.,

$$|\mathbf{A}| = (\mathbf{A} \cdot \mathbf{A})^{1/2} = (A_{ij}A_{ij})^{1/2}. \tag{2.76}$$

The determinant of a tensor $\mathbf{A}$ is defined as:

$$\det(\mathbf{A}) = \frac{[\mathbf{Au}, \mathbf{Av}, \mathbf{Aw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]}, \tag{2.77}$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are any vectors such that $[\mathbf{u}, \mathbf{v}, \mathbf{w}] \neq 0$, i.e., these vectors span the entire 3D vector space. In index notation:

$$\det(\mathbf{A}) = \epsilon_{ijk} A_{i1} A_{j2} A_{k3}. \tag{2.78}$$

Then, it could be shown that

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B}), \tag{2.79}$$

$$\det(\mathbf{A}^t) = \det(\mathbf{A}). \tag{2.80}$$

A tensor $\mathbf{A}$ is said to be singular if and only if $\det(\mathbf{A}) = 0$. If $\mathbf{A}$ is a non-singular tensor i.e., $\det(\mathbf{A}) \neq 0$, then there exist a unique tensor $\mathbf{A}^{-1}$, called the inverse of $\mathbf{A}$ satisfying the relation

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{1}. \tag{2.81}$$

If tensors $\mathbf{A}$ and $\mathbf{B}$ are invertible (i.e., they are non-singular), then the properties

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}, \tag{2.82}$$

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}, \tag{2.83}$$

$$(\alpha\mathbf{A})^{-1} = \frac{1}{\alpha}\mathbf{A}^{-1}, \tag{2.84}$$

$$(\mathbf{A}^{-1})^t = (\mathbf{A}^t)^{-1} = \mathbf{A}^{-t}, \tag{2.85}$$

$$\mathbf{A}^{-2} = \mathbf{A}^{-1}\mathbf{A}^{-1}, \tag{2.86}$$

$$\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}, \tag{2.87}$$

hold.

Corresponding to an arbitrary tensor $\mathbf{A}$ there is a unique tensor $\mathbf{A}^*$, called the adjugate of $\mathbf{A}$, such that

$$\mathbf{A}^*(\mathbf{a} \wedge \mathbf{b}) = \mathbf{Aa} \wedge \mathbf{Ab}, \tag{2.88}$$

for any arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ in the vector space. Suppose that $\mathbf{A}$ is invertible and that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are arbitrary vectors, then

$$\begin{aligned} \mathbf{A}^{t*}(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} &= [\mathbf{A}^t\mathbf{a}, \mathbf{A}^t\mathbf{b}, \mathbf{A}^t\mathbf{A}^{-t}\mathbf{c}] \\ &= \det(\mathbf{A}^t)(\mathbf{a} \wedge \mathbf{b}) \cdot \{\mathbf{A}^{-t}\mathbf{c}\} \\ &= \det(\mathbf{A})\{\mathbf{A}^{-1}(\mathbf{a} \wedge \mathbf{b})\} \cdot \mathbf{c}, \end{aligned} \tag{2.89}$$

where successive use has been made of equations (2.88), (2.81), (2.63), (2.77) and (2.60). Because of the arbitrariness of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ there follows the connection

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{A}^{t*}, \tag{2.90}$$

between the inverse of $\mathbf{A}$ and the adjugate of $\mathbf{A}^t$.

### 2.3.1 Orthogonal tensor

An orthogonal tensor $\mathbf{Q}$ is a linear function satisfying the condition

$$\mathbf{Qu} \cdot \mathbf{Qv} = \mathbf{u} \cdot \mathbf{v}, \tag{2.91}$$

for all vectors $\mathbf{u}$ and $\mathbf{v}$. As can be seen, the dot product is invariant (its value does not change) due to the transformation of the vectors by the orthogonal tensor. The dot product being invariant means that both the angle, $\theta$ between the vectors $\mathbf{u}$ and $\mathbf{v}$ and the magnitude of the vectors $|\mathbf{u}|$, $|\mathbf{v}|$ are preserved. Consequently, the following properties of orthogonal tensor can be inferred:

$$\mathbf{QQ}^t = \mathbf{Q}^t\mathbf{Q} = \mathbf{1}, \tag{2.92}$$

$$\det(\mathbf{Q}) = \pm 1. \tag{2.93}$$

It follows from (2.92) that $\mathbf{Q}^{-1} = \mathbf{Q}^t$. If $\det(\mathbf{Q}) = 1$, $\mathbf{Q}$ is said to be proper orthogonal tensor and this transformation corresponds to a rotation. On the other hand, if $\det(\mathbf{Q}) = -1$, $\mathbf{Q}$ is said to be improper orthogonal tensor and this transformation corresponds to a reflection superposed on a rotation.

Figure 2.2 shows what happens to two directed line segments, $\mathbf{u}_1$ and $\mathbf{v}_1$ under the action the two kinds of orthogonal tensors. A proper orthogonal tensor corresponding to a rotation about the $\mathbf{e}_3$ basis, whose Cartesian coordinate components are given by

$$\mathbf{Q} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.94}$$

transforms, $\mathbf{u}_1$ to $\mathbf{u}_1^+$ and $\mathbf{v}_1$ to $\mathbf{v}_1^+$, maintaining their lengths and the angle between these line segments. Here $\alpha = \Phi - \Theta$. An improper orthogonal

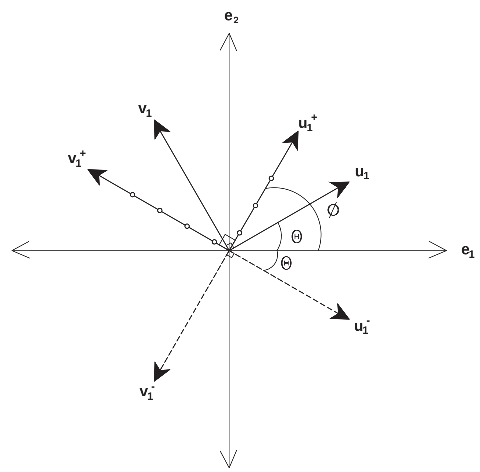



Figure 2.2: Schematic of transformation of directed line segments $\mathbf{u}_1$ and $\mathbf{v}_1$ under the action of proper orthogonal tensor to $\mathbf{u}_1^+$ and $\mathbf{v}_1^+$ as well as by improper orthogonal tensor to $\mathbf{u}_1^-$ and $\mathbf{v}_1^-$ respectively.

tensor corresponding to reflection about $\mathbf{e}_1$ axis, whose Cartesian coordinate components are given by

$$\mathbf{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.95}$$

transforms, $\mathbf{u}_1$ to $\mathbf{u}_1^-$ and $\mathbf{v}_1$ to $\mathbf{v}_1^-$, still maintaining their lengths and the angle between these line segments a constant.

### 2.3.2 Symmetric and skew tensors

A symmetric tensor, $\mathbf{S}$ and a skew symmetric tensor, $\mathbf{W}$ are such:

$$\mathbf{S} = \mathbf{S}^t \quad \text{or} \quad S_{ij} = S_{ji}, \quad \mathbf{W} = -\mathbf{W}^t \quad \text{or} \quad W_{ij} = -W_{ji}, \tag{2.96}$$

therefore the matrix components of these tensor reads

$$[\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}, \qquad [\mathbf{W}] = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix}, \tag{2.97}$$

thus, there are only six independent components in symmetric tensors and three independent components in skew tensors. Since, there are only three independent scalar quantities that in a skew tensor, it behaves like a vector with three components. Indeed, the relation holds:

$$\mathbf{W}\mathbf{u} = \boldsymbol{\omega} \wedge \mathbf{u}, \tag{2.98}$$

where $\mathbf{u}$ is any vector and $\boldsymbol{\omega}$ characterizes the axial (or dual) vector of the skew tensor $\mathbf{W}$, with the property $|\boldsymbol{\omega}| = |\mathbf{W}|/\sqrt{2}$ (proof is omitted). The relation between the Cartesian components of $\mathbf{W}$ and $\boldsymbol{\omega}$ is obtained as:

$$\begin{aligned} W_{ij} &= \mathbf{e}_i \cdot \mathbf{W}\mathbf{e}_j = \mathbf{e}_i \cdot (\boldsymbol{\omega} \wedge \mathbf{e}_j) = \mathbf{e}_i \cdot (\omega_k \mathbf{e}_k \wedge \mathbf{e}_j) \\ &= \mathbf{e}_i \cdot (\omega_k \epsilon_{kjl} \mathbf{e}_l) = \omega_k \epsilon_{kjl} \delta_{il} \\ &= \epsilon_{kji} \omega_k = -\epsilon_{ijk} \omega_k. \end{aligned} \tag{2.99}$$

Thus, we get

$$W_{12} = -\epsilon_{12k}\omega_k = -\omega_3, \tag{2.100}$$
$$W_{13} = -\epsilon_{13k}\omega_k = \omega_2, \tag{2.101}$$
$$W_{23} = -\epsilon_{23k}\omega_k = -\omega_1, \tag{2.102}$$

where the components $W_{12}$, $W_{13}$, $W_{23}$ form the entries of the matrix $[\mathbf{W}]$ as characterized in (2.97)b.

Any tensor $\mathbf{A}$ can always be uniquely decomposed into a symmetric tensor, denoted by $\mathbf{S}$ (or $sym(\mathbf{A})$), and a skew (or antisymmetric) tensor, denoted by $\mathbf{W}$ (or $skew(\mathbf{A})$). Hence, $\mathbf{A} = \mathbf{S} + \mathbf{W}$, where

$$\mathbf{S} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^t), \qquad \mathbf{W} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^t) \tag{2.103}$$

Next, we shall look at some properties of symmetric and skew tensors:

$$\mathbf{S} \cdot \mathbf{B} = \mathbf{S} \cdot \mathbf{B}^t = \mathbf{S} \cdot \frac{1}{2}(\mathbf{B} + \mathbf{B}^t), \tag{2.104}$$
$$\mathbf{W} \cdot \mathbf{B} = -\mathbf{W} \cdot \mathbf{B}^t = \mathbf{W} \cdot \frac{1}{2}(\mathbf{B} - \mathbf{B}^t), \tag{2.105}$$
$$\mathbf{S} \cdot \mathbf{W} = \mathbf{W} \cdot \mathbf{S} = 0, \tag{2.106}$$

where $\mathbf{B}$ denotes any second order tensor. The first of these equalities in the above equations is due to the property of the dot and trace operator, namely $\mathbf{A} \cdot \mathbf{B} = \mathbf{A}^t \cdot \mathbf{B}^t$ and that $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$.

### 2.3.3 Projection tensor

Consider any vector $\mathbf{u}$ and a unit vector $\mathbf{e}$. Then, we write $\mathbf{u} = \mathbf{u}_\parallel + \mathbf{u}_\perp$, with $\mathbf{u}_\parallel$ and $\mathbf{u}_\perp$ characterizing the projection of $\mathbf{u}$ onto the line spanned by $\mathbf{e}$ and onto the plane normal to $\mathbf{e}$ respectively. Using the definition of a tensor product (2.40) we deduce that

$$\begin{aligned} \mathbf{u}_\parallel &= (\mathbf{u}\cdot\mathbf{e})\mathbf{e} = (\mathbf{e}\otimes\mathbf{e})\mathbf{u} = \mathbf{P}^\parallel_\mathbf{e}\mathbf{u}, && (2.107)\\ \mathbf{u}_\perp &= \mathbf{u} - \mathbf{u}_\parallel = \mathbf{u} - (\mathbf{e}\otimes\mathbf{e})\mathbf{u} = (\mathbf{1} - \mathbf{e}\otimes\mathbf{e})\mathbf{u} = \mathbf{P}^\perp_\mathbf{e}\mathbf{u}, && (2.108)\end{aligned}$$

where

$$\begin{aligned} \mathbf{P}^\parallel_\mathbf{e} &= \mathbf{e}\otimes\mathbf{e}, && (2.109)\\ \mathbf{P}^\perp_\mathbf{e} &= \mathbf{1} - \mathbf{e}\otimes\mathbf{e}, && (2.110)\end{aligned}$$

are projection tensors of order two. A tensor $\mathbf{P}$ is a projection if $\mathbf{P}$ is symmetric and $\mathbf{P}^n = \mathbf{P}$ where $n$ is a positive integer, with properties:

$$\begin{aligned} \mathbf{P}^\parallel_\mathbf{e} + \mathbf{P}^\perp_\mathbf{e} &= \mathbf{1}, && (2.111)\\ \mathbf{P}^\parallel_\mathbf{e}\mathbf{P}^\parallel_\mathbf{e} &= \mathbf{P}^\parallel_\mathbf{e}, && (2.112)\\ \mathbf{P}^\perp_\mathbf{e}\mathbf{P}^\perp_\mathbf{e} &= \mathbf{P}^\perp_\mathbf{e}, && (2.113)\\ \mathbf{P}^\parallel_\mathbf{e}\mathbf{P}^\perp_\mathbf{e} &= \mathbf{0}. && (2.114)\end{aligned}$$

### 2.3.4 Spherical and deviatoric tensors

Any tensor of the form $\alpha\mathbf{1}$, with $\alpha$ denoting a scalar is known as a spherical tensor.

Every tensor $\mathbf{A}$ can be decomposed into its so called spherical part and its deviatoric part, i.e.,

$$\mathbf{A} = \alpha\mathbf{1} + dev(\mathbf{A}), \qquad \text{or} \qquad A_{ij} = \alpha\delta_{ij} + [dev(A)]_{ij}, \qquad (2.115)$$

where $\alpha = tr(\mathbf{A})/3 = A_{ii}/3$ and $dev(\mathbf{A})$ is known as a deviator of $\mathbf{A}$ or a deviatoric tensor and is defined as $dev(\mathbf{A}) = \mathbf{A} - (1/3)tr(\mathbf{A})\mathbf{1}$, or $[dev(A)]_{ij} = A_{ij} - (1/3)A_{kk}\delta_{ij}$. It then can be easily verified that $tr(dev(\mathbf{A})) = 0$, for any second order tensor $\mathbf{A}$.

### 2.3.5 Polar Decomposition theorem

Above we saw two additive decompositions of an arbitrary tensor $\mathbf{A}$. Of comparable significance in continuum mechanics are the multiplicative decompositions afforded by the polar decomposition theorem. This result states that $a$n arbitrary invertible tensor $\mathbf{A}$ can be uniquely expressed in the forms

$$\mathbf{A} = \mathbf{QU} = \mathbf{VQ}, \tag{2.116}$$

where $\mathbf{Q}$ is an orthogonal tensor and $\mathbf{U}$, $\mathbf{V}$ are positive definite symmetric tensors. It should be noted that $\mathbf{Q}$ is proper or improper orthogonal according as $\det(\mathbf{A})$ is positive or negative. (See, for example, Chadwick [1] for proof of the theorem.)

## 2.4 Algebra of fourth order tensors

A fourth order tensor $\mathbb{A}$ may be thought of as a linear function that maps second order tensor $\mathbf{A}$ into another second order tensor $\mathbf{B}$. While this is too narrow a viewpoint[1], it suffices for the study of mechanics. We write $\mathbf{B} = \mathbb{A} : \mathbf{A}$ which defines a linear transformation that assigns a second order tensor $\mathbf{B}$ to each second order tensor $\mathbf{A}$.

We can express any fourth order tensor, $\mathbb{A}$ in terms of the three Cartesian basis vectors as

$$\mathbb{A} = A_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \tag{2.117}$$

where $A_{ijkl}$ are the Cartesian components of $\mathbb{A}$. Thus, the fourth order tensor $\mathbb{A}$ has $3^4 = 81$ components. Remember that any repeated index has to be summed from one through three.

One example for a fourth order tensor is the tensor product of the four vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, denoted by $\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \otimes \mathbf{x}$. We have the useful property:

$$(\mathbf{u} \otimes \mathbf{v}) \otimes (\mathbf{w} \otimes \mathbf{x}) = \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \otimes \mathbf{x}. \tag{2.118}$$

If $\mathbb{A} = \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \otimes \mathbf{x}$, then it is easy to see that the Cartesian components of $\mathbb{A}$, $A_{ijkl} = u_i v_j w_k x_l$, where $u_i$, $v_j$, $w_k$, $x_l$ are the Cartesian components of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{x}$ respectively.

[1] A fourth order tensor can operate on a vector to yield third order tensor or can operate on a third order tensor to yield an vector.

Another example of a fourth order tensor, is the tensor obtained from the tensor product of two second order tensors, i.e., $\mathbb{D} = \mathbf{A} \otimes \mathbf{B}$, where $\mathbf{A}$, $\mathbf{B}$ are second order tensors and $\mathbb{D}$ is the fourth order tensor. In index notation we may write: $D_{ijkl} = A_{ij}B_{kl}$ where $D_{ijkl}$, $A_{ij}$, $B_{kl}$ are the Cartesian components of the respective tensors.

The double contraction of a fourth order tensor $\mathbb{A}$ with a second order tensor $\mathbf{B}$ results in a second order tensor, denoted by $\mathbb{A} : \mathbf{B}$ with the property that:

$$(\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \otimes \mathbf{x}) : (\mathbf{y} \otimes \mathbf{z}) = (\mathbf{w} \cdot \mathbf{y})(\mathbf{x} \cdot \mathbf{z})(\mathbf{u} \otimes \mathbf{v}), \tag{2.119}$$

$$(\mathbf{y} \otimes \mathbf{z}) : (\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \otimes \mathbf{x}) = (\mathbf{u} \cdot \mathbf{y})(\mathbf{v} \cdot \mathbf{z})(\mathbf{w} \otimes \mathbf{x}). \tag{2.120}$$

Hence, we can show that the components of $\mathbb{A}$, $A_{ijkl}$ can be expressed as

$$A_{ijkl} = (\mathbf{e}_i \otimes \mathbf{e}_j) \cdot \mathbb{A} : (\mathbf{e}_k \otimes \mathbf{e}_l) = (\mathbf{e}_i \otimes \mathbf{e}_j) : \mathbb{A} \cdot (\mathbf{e}_k \otimes \mathbf{e}_l) \tag{2.121}$$

Next, we compute the Cartesian components of $\mathbb{A} : \mathbf{B}$ to be:

$$\begin{aligned} \mathbb{A} : \mathbf{B} &= A_{ijkl}B_{mn}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) : (\mathbf{e}_m \otimes \mathbf{e}_n) \\ &= A_{ijkl}B_{mn}\delta_{km}\delta_{ln}\mathbf{e}_i \otimes \mathbf{e}_j \\ &= A_{ijkl}B_{kl}\mathbf{e}_i \otimes \mathbf{e}_j, \end{aligned} \tag{2.122}$$

where $A_{ijkl}$ and $B_{mn}$ are the Cartesian components of the tensors $\mathbb{A}$ and $\mathbf{B}$. Note that $\mathbf{B} : \mathbb{A} \neq \mathbb{A} : \mathbf{B}$.

Then, the following can be established:

$$(\mathbf{A} \otimes \mathbf{B}) : \mathbf{C} = (\mathbf{B} \cdot \mathbf{C})\mathbf{A}, \tag{2.123}$$

$$\mathbf{A} : (\mathbf{B} \otimes \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B})\mathbf{C}, \tag{2.124}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are second order tensors.

The unique transpose of a fourth order tensor $\mathbb{A}$ denoted by $\mathbb{A}^t$ is governed by the identity

$$\mathbf{B} \cdot \mathbb{A}^t : \mathbf{C} = \mathbf{C} \cdot \mathbb{A} : \mathbf{B} = \mathbb{A} : \mathbf{B} \cdot \mathbf{C} \tag{2.125}$$

for all the second order tensors $\mathbf{B}$ and $\mathbf{C}$. From the above identity we deduce the index relation $(A^t)_{ijkl} = A_{klij}$. The following properties of fourth order tensors can be established:

$$(\mathbb{A}^t)^t = \mathbb{A}, \tag{2.126}$$

$$(\mathbf{A} \otimes \mathbf{B})^t = \mathbf{B} \otimes \mathbf{A}. \tag{2.127}$$

Next, we define fourth order unit tensors $\mathbb{I}$ and $\bar{\mathbb{I}}$ so that

$$\mathbf{A} = \mathbb{I} : \mathbf{A}, \qquad \mathbf{A}^t = \bar{\mathbb{I}} : \mathbf{A}, \tag{2.128}$$

for any second order tensor $\mathbf{A}$. These fourth order unit tensors may be represented by

$$\mathbb{I} = \delta_{ik}\delta_{jl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j, \tag{2.129}$$

$$\bar{\mathbb{I}} = \delta_{il}\delta_{jk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \otimes \mathbf{e}_i, \tag{2.130}$$

where $(\mathbb{I})_{ijkl} = \delta_{ik}\delta_{jl}$ and $(\bar{\mathbb{I}})_{ijkl} = \delta_{il}\delta_{jk}$ define the Cartesian components of $\mathbb{I}$ and $\bar{\mathbb{I}}$, respectively. Note that $\bar{\mathbb{I}} \neq \mathbb{I}^t$.

The deviatoric part of a second order tensor $\mathbf{A}$ may be described by means of a fourth order projection tensor, $\mathbb{P}$ where

$$dev(\mathbf{A}) = \mathbb{P} : \mathbf{A}, \qquad \mathbb{P} = \mathbb{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}. \tag{2.131}$$

Thus the components of $dev(\mathbf{A})$ and $\mathbf{A}$ are related through the expression $[dev(\mathbf{A})]_{ij} = P_{ijkl}A_{kl}$, with $P_{ijkl} = \delta_{ik}\delta_{jl} - (1/3)\delta_{ij}\delta_{kl}$.

Similarly, the fourth order tensors $\mathbb{S}$ and $\mathbb{W}$ given by

$$\mathbb{S} = \frac{1}{2}(\mathbb{I} + \bar{\mathbb{I}}), \qquad \mathbb{W} = \frac{1}{2}(\mathbb{I} - \bar{\mathbb{I}}), \tag{2.132}$$

are such that for any second order tensor $\mathbf{A}$, they assign symmetric and skew part of $\mathbf{A}$ respectively, i.e.,

$$\mathbb{S} : \mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^t), \qquad \mathbb{W} : \mathbf{A} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^t). \tag{2.133}$$

### 2.4.1 Alternate representation for tensors

Till now, we represented second order tensor components in a matrix form, for the advantages that it offers in computing other quantities and defining other operators. However, when we want to study mapping of second order tensor on to another second order tensor, this representation seem to be inconvenient. For this purpose, we introduce this alternate representation. Now, we view the second order tensor as a column vector of nine components instead of a 3 by 3 matrix as introduced in (2.45). The order of these components is subjective. Keeping in mind the application of this is to study elasticity, we order the components of a general second order tensor, $\mathbf{A}$, as,

$$\{\mathbf{A}\} = \begin{Bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{12} \\ A_{13} \\ A_{21} \\ A_{23} \\ A_{31} \\ A_{32} \end{Bmatrix}. \tag{2.134}$$

In view of this, the fourth order tensor, which for us is a linear function that maps a second order tensor to another second order tensor, can be represented as a 9 by 9 matrix as,

$$\{\mathbf{B}\}_i = \sum_{j=1}^{9} [\mathbb{C}]_{ij} \{\mathbf{A}\}_j, \qquad i = 1 \ldots 9. \tag{2.135}$$

where $\mathbf{A}$ and $\mathbf{B}$ are second order tensors and $\mathbb{C}$ is a fourth order tensor. Note that as before the fourth order tensor has 81 (=9*9) components. Thus, now the fourth order tensor is a matrix which is the reason for representing the second order tensor as vector.

## 2.5 Eigenvalues, eigenvectors of tensors

The scalar $\lambda_i$ characterize eigenvalues (or principal values) of a tensor $\mathbf{A}$ if there exist corresponding nonzero normalized eigenvectors $\hat{\mathbf{n}}_i$ (or principal directions or principal axes) of $\mathbf{A}$, so that

$$\mathbf{A}\hat{\mathbf{n}}_i = \lambda_i \hat{\mathbf{n}}_i, \qquad (i = 1,2,3; \text{ no summation}). \tag{2.136}$$

To identify the eigenvectors of a tensor, we use subsequently a hat on the vector quantity concerned, for example, $\hat{\mathbf{n}}$.

Thus, a set of homogeneous algebraic equations for the unknown eigenvalues $\lambda_i$, $i = 1,2,3$, and the unknown eigenvectors $\hat{\mathbf{n}}_i$, $i = 1,2,3$, is

$$(\mathbf{A} - \lambda_i \mathbf{1})\hat{\mathbf{n}}_i = \mathbf{o}, \qquad (i = 1,2,3; \text{ no summation}). \tag{2.137}$$

For the above system to have a solution $\hat{\mathbf{n}}_i \neq \mathbf{o}$ the determinant of the system must vanish. Thus,

$$\det(\mathbf{A} - \lambda_i \mathbf{1}) = 0 \tag{2.138}$$

where

$$\det(\mathbf{A} - \lambda_i \mathbf{1}) = -\lambda_i^3 + I_1\lambda_i^2 - I_2\lambda_i + I_3. \tag{2.139}$$

This requires that we solve a cubic equation in $\lambda$, usually written as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0, \tag{2.140}$$

called the characteristic polynomial (or equation) for $\mathbf{A}$, the solutions of which are the eigenvalues $\lambda_i$, $i$ = 1,2,3. Here, $I_i$, $i$ = 1,2,3, are the so-called principal scalar invariants of $\mathbf{A}$ and are given by

$$\begin{aligned} I_1 &= tr(\mathbf{A}) = A_{ii} = \lambda_1 + \lambda_2 + \lambda_3, \\ I_2 &= \frac{1}{2}[(tr(\mathbf{A}))^2 - tr(\mathbf{A}^2)] = \frac{1}{2}(A_{ii}A_{jj} - A_{mn}A_{nm}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \\ I_3 &= \det(\mathbf{A}) = \epsilon_{ijk}A_{i1}A_{j2}A_{k3} = \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{2.141}$$

If $\mathbf{A}$ is invertible then we can compute $I_2$ using the expression $I_2 = tr(\mathbf{A}^{-1})$ $\det(\mathbf{A})$.

A repeated application of tensor $\mathbf{A}$ to equation (2.136) yields $\mathbf{A}^\alpha\hat{\mathbf{n}}_i = \lambda_i^\alpha\hat{\mathbf{n}}_i$, $i$ = 1,2,3, for any positive integer $\alpha$. (If $\mathbf{A}$ is invertible then $\alpha$ can be any integer; not necessarily positive.) Using this relation and (2.140) multiplied by $\hat{\mathbf{n}}_i$, we obtain the well-known Cayley-Hamilton equation:

$$\mathbf{A}^3 - I_1\mathbf{A}^2 + I_2\mathbf{A} - I_3\mathbf{1} = \mathbf{0}. \tag{2.142}$$

It states that every second order tensor $\mathbf{A}$ satisfies its own characteristic equation. As a consequence of Cayley-Hamilton equation, we can express $\mathbf{A}^\alpha$ in terms of $\mathbf{A}^2$, $\mathbf{A}$, $\mathbf{1}$ and principal invariants for positive integer, $\alpha >$ 2. (If $\mathbf{A}$ is invertible, the above holds for any integer value of $\alpha$ positive or negative provided $\alpha \neq \{0, 1, 2\}$.)

For a symmetric tensor $\mathbf{S}$ the characteristic equation (2.140) always has three real solutions and the set of eigenvectors form a orthonormal basis $\{\hat{\mathbf{n}}_i\}$ (the proof of this statement is omitted). Hence, for a positive definite symmetric tensor $\mathbf{A}$, all eigenvalues $\lambda_i$ are (real and) positive since, using (2.136), we have $\lambda_i = \hat{\mathbf{n}}_i \cdot \mathbf{A}\hat{\mathbf{n}}_i > 0$, $i$ = 1,2,3.

Any symmetric tensor $\mathbf{S}$ may be represented by its eigenvalues $\lambda_i$, $i =$ 1,2,3, and the corresponding eigenvectors of $\mathbf{S}$ forming an orthonormal basis $\{\hat{\mathbf{n}}_i\}$. Thus, $\mathbf{S}$ can be expressed as

$$\mathbf{S} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i, \tag{2.143}$$

called the spectral representation (or spectral decomposition) of $\mathbf{S}$. Thus, when orthonormal eigenvectors are used as the Cartesian basis to represent $\mathbf{S}$ then

$$[\mathbf{S}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{2.144}$$

The above holds when all the three eigenvalues are distinct. On the other hand, if there exists a pair of equal roots, i.e., $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$, with an unique eigenvector $\hat{\mathbf{n}}_3$ associated with $\lambda_3$, we deduce that

$$\mathbf{S} = \lambda_3(\hat{\mathbf{n}}_3 \otimes \hat{\mathbf{n}}_3) + \lambda[\mathbf{1} - (\hat{\mathbf{n}}_3 \otimes \hat{\mathbf{n}}_3)] = \lambda_3 \mathbf{P}^{\|}_{\hat{\mathbf{n}}_3} + \lambda \mathbf{P}^{\perp}_{\hat{\mathbf{n}}_3}, \tag{2.145}$$

where $\mathbf{P}^{\|}_{\hat{\mathbf{n}}_3}$ and $\mathbf{P}^{\perp}_{\hat{\mathbf{n}}_3}$ denote projection tensors introduced in (2.109) and (2.110) respectively. Finally, if all the three eigenvalues are equal, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then

$$\mathbf{S} = \lambda \mathbf{1}, \tag{2.146}$$

where every direction is a principal direction and every set of mutually orthogonal basis denotes principal axes.

It is important to recognize that eigenvalues characterize the physical nature of the tensor and that they do not depend on the coordinates chosen.

### 2.5.1 Square root theorem

Let $\mathbf{C}$ be symmetric and positive definite tensor. Then there is a unique positive definite, symmetric tensor $\mathbf{U}$ such that

$$\mathbf{U}^{\mathbf{2}} = \mathbf{C}. \tag{2.147}$$

We write $\sqrt{\mathbf{C}}$ for $\mathbf{U}$. If spectral representation of $\mathbf{C}$ is:

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i \hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i, \tag{2.148}$$

then the spectral representation for $\mathbf{U}$ is:

$$\mathbf{U} = \sum_{i=1}^{3} \sqrt{\lambda_i}\hat{\mathbf{n}}_i \otimes \hat{\mathbf{n}}_i, \tag{2.149}$$

where we have assumed that the eigenvalues of $\mathbf{C}$, $\lambda_i$ are distinct. On the other hand if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ then

$$\mathbf{U} = \sqrt{\lambda}\mathbf{1}. \tag{2.150}$$

If $\lambda_1 = \lambda_2 = \lambda \neq \lambda_3$, with an unique eigenvector $\hat{\mathbf{n}}_3$ associated with $\lambda_3$ then

$$\mathbf{U} = \sqrt{\lambda_3}(\hat{\mathbf{n}}_3 \otimes \hat{\mathbf{n}}_3) + \sqrt{\lambda}[\mathbf{1} - (\hat{\mathbf{n}}_3 \otimes \hat{\mathbf{n}}_3)] \tag{2.151}$$

(For proof of this theorem, see for example, Gurtin [2].)

## 2.6 Transformation laws

Consider two sets of mutually orthogonal basis vectors which share a common origin. They correspond to a 'new' and an 'old' (original) Cartesian coordinate system which we assume to be right-handed characterized by two sets of basis vectors $\{\tilde{\mathbf{e}}_i\}$ and $\{\mathbf{e}_i\}$, respectively. Hence, the new coordinate system could be obtained from the original one by a rotation of the basis vectors $\mathbf{e}_i$ about their origin. We then define the directional cosine matrix, $Q_{ij}$, as,

$$Q_{ij} = \mathbf{e}_i \cdot \tilde{\mathbf{e}}_j = \cos(\theta(\mathbf{e}_i, \tilde{\mathbf{e}}_j)). \tag{2.152}$$

Note that the first index on $Q_{ij}$ indicates the 'old' components whereas the second index holds for the 'new' components.

It is worthwhile to mention that vectors and tensors themselves remain invariant upon a change of basis - they are said to be independent of the coordinate system. However, their respective components do depend upon the coordinate system introduced. This is the reason why a set of numbers arranged as a 3 by 1 or 3 by 3 matrix is not a vector or a tensor.

### 2.6.1 Vectorial transformation law

Consider some vector $\mathbf{u}$ represented using the two sets of basis vectors $\{\mathbf{e}_i\}$ and $\{\tilde{\mathbf{e}}_i\}$, i.e.,

$$\mathbf{u} = u_i\mathbf{e}_i = \tilde{u}_j\tilde{\mathbf{e}}_j. \tag{2.153}$$

Recalling the method to find the components of the vector along the basis directions, (2.15),

$$\tilde{u}_j = \mathbf{u} \cdot \tilde{\mathbf{e}}_j = (u_i \mathbf{e}_i) \cdot \tilde{\mathbf{e}}_j = Q_{ij} u_i, \tag{2.154}$$

from the definition of the directional cosine matrix, (2.152). We assume that the relation between the basis vectors $\mathbf{e}_i$ and $\tilde{\mathbf{e}}_j$ is known and hence given the components of a vector in a basis, its components in another basis can be found using equation (2.154).

In an analogous manner, we find that

$$u_j = \mathbf{u} \cdot \mathbf{e}_j = (\tilde{u}_i \tilde{\mathbf{e}}_i) \cdot \mathbf{e}_j = Q_{ji} \tilde{u}_i. \tag{2.155}$$

The results of equations (2.154) and (2.155) could be cast in matrix notation as

$$[\tilde{\mathbf{u}}] = [\mathbf{Q}]^t [\mathbf{u}], \qquad \text{and} \qquad [\mathbf{u}] = [\mathbf{Q}][\tilde{\mathbf{u}}], \tag{2.156}$$

respectively. It is important to emphasize that the above equations are not identical to $\tilde{\mathbf{u}} = \mathbf{Q}^t \mathbf{u}$ and $\mathbf{u} = \mathbf{Q}\tilde{\mathbf{u}}$, respectively. In (2.156) $[\tilde{\mathbf{u}}]$ and $[\mathbf{u}]$ are column vectors characterizing components of the same vector in two different coordinate systems, whereas $\tilde{\mathbf{u}}$ and $\mathbf{u}$ are different vectors, in the later. Similarly, $[\mathbf{Q}]$ is a matrix of directional cosines, it is not a tensor even though it has the attributes of an orthogonal tensor as we will see next.

Combining equations (2.154) and (2.155), we obtain

$$u_j = Q_{ji} \tilde{u}_i = Q_{ji} Q_{ai} u_a, \quad \text{and} \quad \tilde{u}_j = Q_{ij} u_i = Q_{ij} Q_{ia} \tilde{u}_a \tag{2.157}$$

Hence, $(Q_{ai} Q_{ji} - \delta_{ja}) u_a = 0$ and $(Q_{ij} Q_{ia} - \delta_{ja}) \tilde{u}_a = 0$ for any vector $\mathbf{u}$. Therefore,

$$Q_{ai} Q_{ji} = \delta_{ja}, \qquad \text{or} \qquad [\mathbf{Q}][\mathbf{Q}]^t = [\mathbf{1}], \tag{2.158}$$

$$Q_{ia} Q_{ij} = \delta_{ja}, \qquad \text{or} \qquad [\mathbf{Q}]^t[\mathbf{Q}] = [\mathbf{1}]. \tag{2.159}$$

Thus, the transformation matrix, $Q_{ij}$ is sometimes called as orthogonal matrix but never as orthogonal tensor.

### 2.6.2 Tensorial transformation law

To determine the transformation laws for the Cartesian components of any second-order tensor $\mathbf{A}$, we proceed along the lines similar to that done for vectors. Since, we seek the components of the same tensor in two different basis,

$$\mathbf{A} = A_{ab}\mathbf{e}_a \otimes \mathbf{e}_b = \tilde{A}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{2.160}$$

Then it follows from (2.46) that,

$$\begin{aligned} \tilde{A}_{ij} &= \tilde{\mathbf{e}}_i \cdot \mathbf{A}\tilde{\mathbf{e}}_j = \tilde{\mathbf{e}}_i \cdot (A_{ab}\mathbf{e}_a \otimes \mathbf{e}_b)\tilde{\mathbf{e}}_j \\ &= A_{ab}(\mathbf{e}_b \cdot \tilde{\mathbf{e}}_j)(\mathbf{e}_a \cdot \tilde{\mathbf{e}}_i) = A_{ab}Q_{ai}Q_{bj}. \end{aligned} \tag{2.161}$$

In matrix notation, $[\tilde{\mathbf{A}}] = [\mathbf{Q}]^t[\mathbf{A}][\mathbf{Q}]$. In an analogous manner, we find that

$$A_{ij} = Q_{ik}Q_{jm}\tilde{A}_{km}, \qquad \text{or} \qquad [\mathbf{A}] = [\mathbf{Q}][\tilde{\mathbf{A}}][\mathbf{Q}]^t. \tag{2.162}$$

We emphasize again that these transformations relates the different matrices $[\tilde{\mathbf{A}}]$ and $[\mathbf{A}]$, which have the components of the same tensor $\mathbf{A}$ and the equations $[\tilde{\mathbf{A}}] = [\mathbf{Q}]^t[\mathbf{A}][\mathbf{Q}]$ and $[\mathbf{A}] = [\mathbf{Q}][\tilde{\mathbf{A}}][\mathbf{Q}]^t$ differ from the tensor equations $\tilde{\mathbf{A}} = \mathbf{Q}^t\mathbf{A}\mathbf{Q}$ and $\mathbf{A} = \mathbf{Q}\tilde{\mathbf{A}}\mathbf{Q}^t$, relating two different tensors, namely $\mathbf{A}$ and $\tilde{\mathbf{A}}$.

Finally, the $3^n$ components $A_{j_1j_2\ldots j_n}$ of a tensor of order $n$ (with $n$ indices $j_1$, $j_2$, $\ldots$, $j_n$) transform as

$$\tilde{A}_{i_1i_2\ldots i_n} = Q_{j_1i_1}Q_{j_2i_2}\ldots Q_{j_ni_n}A_{j_1j_2\ldots j_n}. \tag{2.163}$$

This tensorial transformation law relates the different components $\tilde{A}_{i_1i_2\ldots i_n}$ (along the directions $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$, $\tilde{\mathbf{e}}_3$) and $A_{j_1j_2\ldots j_n}$ (along the directions $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) of the same tensor of order $n$.

We note that, in general a second order tensor, $\mathbf{A}$ will be represented as $A_{ij}\mathbf{e}_i \otimes \mathbf{E}_j$, where $\{\mathbf{e}_i\}$ and $\{\mathbf{E}_j\}$ are different basis vectors spanning the same space. It is not necessary that the directional cosines $Q_{ij} = \mathbf{E}_i \cdot \tilde{\mathbf{E}}_i$ and $q_{ij} = \mathbf{e}_i \cdot \tilde{\mathbf{e}}_i$ be the same, where $\tilde{\mathbf{e}}_i$ and $\tilde{\mathbf{E}}_j$ are the 'new' basis vectors with respect to which the matrix components of $\mathbf{A}$ is sought. Thus, generalizing the above is straightforward; each directional cosine matrices can be different, contrary to the assumption made.

### 2.6.3 Isotropic tensors

A tensor $\mathbf{A}$ is said to be isotropic if its components are the same under arbitrary rotations of the basis vectors. The requirement is deduced from equation (2.161) as

$$A_{ij} = Q_{ki}Q_{mj}A_{km} \qquad \text{or} \qquad [\mathbf{A}] = [\mathbf{Q}]^t[\mathbf{A}][\mathbf{Q}]. \tag{2.164}$$

Of course here we assume that the components of the tensor are with respect to a single basis and not two or more independent basis.

Note that all scalars, zeroth order tensors are isotropic tensors. Also, zero tensors and unit tensors of all orders are isotropic. It can be easily verified that for second order tensors spherical tensor is also isotropic. The most general isotropic tensor of order four is of the form

$$\alpha(\mathbf{1} \otimes \mathbf{1}) + \beta\mathbb{I} + \gamma\bar{\mathbb{I}}, \tag{2.165}$$

where $\alpha$, $\beta$, $\gamma$ are scalars. The same in component form is given by: $\alpha\delta_{ij}\delta_{kl}$ + $\beta\delta_{ik}\delta_{jl}$ + $\gamma\delta_{il}\delta_{jk}$.

## 2.7 Scalar, vector, tensor functions

Having got an introduction to algebra of tensors we next focus on tensor calculus. Here we shall understand the meaning of a function and define what we mean by continuity and derivative of a function.

A function is a mathematical correspondence that assigns exactly one element of one set to each element of the same or another set. Thus, if we consider scalar functions of one scalar variable, then for each element (value) in the subset of the real line one associates an element in the real line or in the space of vectors or in the space of second order tensors. For example, $\Phi = \hat{\Phi}(t)$, $\mathbf{u} = \hat{\mathbf{u}}(t) = u_i(t)\mathbf{e}_i$, $\mathbf{A} = \hat{\mathbf{A}}(t) = A_{ij}(t)\mathbf{e}_i \otimes \mathbf{e}_j$ are scalar valued, vector valued, second order tensor valued scalar functions with a set of Cartesian basis vectors assumed to be fixed. The components $u_i(t)$ and $A_{ij}(t)$ are assumed to be real valued smooth functions of $t$ varying over a certain interval.

The first derivative of the scalar function $\Phi$ is simply $\dot{\Phi} = d\Phi/dt$. Recollecting from a first course in calculus, $d\Phi/dt$ stands for that unique value of the limit

$$\lim_{h \to 0} \frac{\hat{\Phi}(t+h) - \hat{\Phi}(t)}{h} \tag{2.166}$$

The first derivative of $\mathbf{u}$ and $\mathbf{A}$ with respect to $t$ (rate of change) denoted by $\dot{\mathbf{u}} = d\mathbf{u}/dt$ and $\dot{\mathbf{A}} = d\mathbf{A}/dt$, is given by the first derivative of their associated components. Since, $d\mathbf{e}_i/dt = \mathbf{o}$, $i = 1, 2, 3$, we have

$$\dot{\mathbf{u}} = \dot{u}_i(t)\mathbf{e}_i, \qquad \dot{\mathbf{A}} = \dot{A}_{ij}(t)\mathbf{e}_i \otimes \mathbf{e}_j. \tag{2.167}$$

In general, the $n^{th}$ derivative of $\mathbf{u}$ and $\mathbf{A}$ (for any desired $n$) denoted by $d^n\mathbf{u}/dt^n$ and $d^n\mathbf{A}/dt^n$, is a vector-valued and tensor-valued function whose components are $d^n u_i/dt^n$ and $d^n A_{ij}/dt^n$, respectively. Again we have assumed that $d\mathbf{e}_i/dt = \mathbf{o}$.

By applying the rules of differentiation, we obtain the identities:

$$\begin{aligned}
\dot{\overline{\mathbf{u} \pm \mathbf{v}}} &= \dot{\mathbf{u}} \pm \dot{\mathbf{v}}, && (2.168)\\
\dot{\overline{\Phi\mathbf{u}}} &= \dot{\Phi}\mathbf{u} + \Phi\dot{\mathbf{u}}, && (2.169)\\
\dot{\overline{\mathbf{u} \otimes \mathbf{v}}} &= \dot{\mathbf{u}} \otimes \mathbf{v} + \mathbf{u} \otimes \dot{\mathbf{v}}, && (2.170)\\
\dot{\overline{\mathbf{A} \pm \mathbf{B}}} &= \dot{\mathbf{A}} \pm \dot{\mathbf{B}}, && (2.171)\\
\dot{\overline{\mathbf{A}\mathbf{u}}} &= \dot{\mathbf{A}}\mathbf{u} + \mathbf{A}\dot{\mathbf{u}}, && (2.172)\\
\dot{\overline{\mathbf{A}^t}} &= \dot{\mathbf{A}}^t, && (2.173)\\
\dot{\overline{\mathbf{A} \cdot \mathbf{B}}} &= \dot{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \dot{\mathbf{B}}, && (2.174)\\
\dot{\overline{\mathbf{A}\mathbf{B}}} &= \dot{\mathbf{A}}\mathbf{B} + \mathbf{A}\dot{\mathbf{B}}, && (2.175)
\end{aligned}$$

where $\Phi$ is a scalar valued scalar function, $\mathbf{u}$ and $\mathbf{v}$ are vector valued scalar function, $\mathbf{A}$ and $\mathbf{B}$ are second order tensor valued scalar function. Here and elsewhere in this notes, the overbars cover the quantities to which the dot operations are applied.

Since, $\mathbf{A}\mathbf{A}^{-1} = \mathbf{1}$, $\dot{\overline{\mathbf{A}\mathbf{A}^{-1}}} = \mathbf{0}$. Hence, we compute:

$$\dot{\overline{\mathbf{A}^{-1}}} = -\mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{A}^{-1}. \tag{2.176}$$

A tensor function is a function whose arguments are one or more tensor variables and whose values are scalars, vectors or tensors. The functions $\Phi(\mathbf{B})$, $\mathbf{u}(\mathbf{B})$, and $\mathbf{A}(\mathbf{B})$ are examples of so-called scalar-valued, vector-valued and second order tensor-valued functions of one second order tensor variable $\mathbf{B}$, respectively. In an analogous manner, $\Phi(\mathbf{v})$, $\mathbf{u}(\mathbf{v})$, and $\mathbf{A}(\mathbf{v})$ are scalar-valued, vector-valued and second order tensor-valued functions of one vector variable $\mathbf{v}$, respectively.

Let $\mathfrak{D}$ denote the region of the vector space that is of interest. Then a scalar field $\Phi$, defined on a domain $\mathfrak{D}$, is said to be continuous if

$$\lim_{\alpha \to 0} |\Phi(\mathbf{v} + \alpha\mathbf{a}) - \Phi(\mathbf{v})| = 0, \quad \forall \mathbf{v} \in \mathfrak{D}, \mathbf{a} \in \mathfrak{V}, \tag{2.177}$$

where $\mathfrak{V}$ denote the set of all vectors in the vector space. The properties of continuity are attributed to a vector field $\mathbf{u}$ and a tensor field $\mathbf{A}$ defined on $\mathfrak{D}$, if they apply to the scalar field $\mathbf{u} \cdot \mathbf{a}$ and $\mathbf{a} \cdot \mathbf{A}\mathbf{b}$ $\forall$ $\mathbf{a}$ and $\mathbf{b} \in \mathfrak{V}$.

Consider a scalar valued function of one second order tensor variable $\mathbf{A}$, $\Phi(\mathbf{A})$. Assuming the function $\Phi$ is continuous in $\mathbf{A}$, the derivative of $\Phi$ with respect to $\mathbf{A}$, denoted by $\frac{\partial \Phi}{\partial \mathbf{A}}$, is the second order tensor that satisfies

$$\frac{\partial \Phi}{\partial \mathbf{A}} \cdot \mathbf{U} = \frac{d}{ds}\left[\Phi(\mathbf{A} + s\mathbf{U})\right]|_{s=0} = tr(\left(\frac{\partial \Phi}{\partial \mathbf{A}}\right)^t \mathbf{U}) = tr(\frac{\partial \Phi}{\partial \mathbf{A}}\mathbf{U}^t), \quad (2.178)$$

for any second order tensor $\mathbf{U}$.

**Example:** If $\Phi(\mathbf{A}) = \det(\mathbf{A})$, find $\frac{\partial \Phi}{\partial \mathbf{A}}$ both when $\mathbf{A}$ is invertible and when it is not.

Taking trace of the Cayley-Hamilton equation (2.142) and rearranging we get

$$\det(\mathbf{A}) = \frac{1}{6}[2tr(\mathbf{A}^3) - 3tr(\mathbf{A})tr(\mathbf{A}^2) + (tr(\mathbf{A}))^3]. \quad (2.179)$$

Hence,

$$\frac{\partial[\det(\mathbf{A})]}{\partial \mathbf{A}} = \frac{1}{6}[2\frac{\partial[tr(\mathbf{A}^3)]}{\partial \mathbf{A}} + 3\frac{\partial[tr(\mathbf{A})]}{\partial \mathbf{A}}[(tr(\mathbf{A}))^2 - tr(\mathbf{A}^2)] - 3tr(\mathbf{A})\frac{\partial[tr(\mathbf{A}^2)]}{\partial \mathbf{A}}]. \quad (2.180)$$

Using the properties of *trace* and dot product we find that

$$\begin{aligned} tr(\mathbf{A} + s\mathbf{U}) &= tr(\mathbf{A}) + s\mathbf{1} \cdot \mathbf{U}, & (2.181)\\ tr([\mathbf{A} + s\mathbf{U}]^2) &= tr(\mathbf{A}^2) + 2s\mathbf{A}^t \cdot \mathbf{U} + s^2 tr(\mathbf{U}^2), & (2.182)\\ tr([\mathbf{A} + s\mathbf{U}]^3) &= tr(\mathbf{A}^3) + 3s(\mathbf{A}^t)^2 \cdot \mathbf{U} + 3s^2\mathbf{A}^t \cdot \mathbf{U}^2 + s^3 tr(\mathbf{U}^3), & (2.183)\end{aligned}$$

from which, we deduce that

$$\begin{aligned} \frac{\partial[tr(\mathbf{A})]}{\partial \mathbf{A}} &= \mathbf{1}, & (2.184)\\ \frac{\partial[tr(\mathbf{A}^2)]}{\partial \mathbf{A}} &= 2\mathbf{A}^t, & (2.185)\\ \frac{\partial[tr(\mathbf{A}^3)]}{\partial \mathbf{A}} &= 3(\mathbf{A}^t)^2, & (2.186)\end{aligned}$$

using equation (2.178). Substituting equations (2.184) through (2.186) in (2.180) we get

$$\frac{\partial[\det(\mathbf{A})]}{\partial \mathbf{A}} = (\mathbf{A}^t)^2 + I_2\mathbf{1} - I_1\mathbf{A}^t. \quad (2.187)$$

If $\mathbf{A}$ is invertible, then multiplying (2.142) by $\mathbf{A}^{-t}$ and rearranging we get

$$(\mathbf{A}^t)^2 - I_1\mathbf{A}^t + I_2\mathbf{1} = I_3\mathbf{A}^{-t}. \quad (2.188)$$

In light of the above equation, (2.187) reduces to

$$\frac{\partial[\det(\mathbf{A})]}{\partial \mathbf{A}} = det(\mathbf{A})\mathbf{A}^{-t}. \tag{2.189}$$

Next, we consider a smooth tensor valued function of one second order tensor variable $\mathbf{A}$, $\mathbf{f}(\mathbf{A})$. As before, the derivative of $\mathbf{f}$ with respect to $\mathbf{A}$, denoted by $\frac{\partial \mathbf{f}}{\partial \mathbf{A}}$, is the fourth order tensor that satisfies

$$\frac{\partial \mathbf{f}}{\partial \mathbf{A}} : \mathbf{U} = \frac{d}{ds}\left[\mathbf{f}(\mathbf{A} + s\mathbf{U})\right]|_{s=0}, \tag{2.190}$$

for any second order tensor $\mathbf{U}$.

It is straight forward to see that when $\mathbf{f}(\mathbf{A}) = \mathbf{A}$, where $\mathbf{A}$ is any second order tensor, then (2.190) reduces to

$$\frac{\partial \mathbf{f}}{\partial \mathbf{A}} : \mathbf{U} = \mathbf{U}, \tag{2.191}$$

hence, $\frac{\partial \mathbf{f}}{\partial \mathbf{A}} = \mathbb{I}$, the fourth order unit tensor where we have used equation (2.128a).

If $\mathbf{f}(\mathbf{A}) = \mathbf{A}^t$ where $\mathbf{A}$ is any second order tensor, then using (2.190) it can be shown that $\frac{\partial \mathbf{f}}{\partial \mathbf{A}} = \bar{\mathbb{I}}$ obtained using equation (2.128b).

Now, say the tensor valued function $\mathbf{f}$ is defined only for symmetric tensors, and that $\mathbf{f}(\mathbf{S}) = \mathbf{S}$, where $\mathbf{S}$ is a symmetric second order tensor. Compute the derivative of $\mathbf{f}$ with respect to $\mathbf{S}$, i.e., $\frac{\partial \mathbf{f}}{\partial \mathbf{S}}$.

First, since the function $\mathbf{f}$ is defined only for symmetric tensors, we have to generalize the function for any tensor. This is required because $\mathbf{U}$ is any second order tensor. Hence, $\mathbf{f}(\mathbf{S}) = \mathbf{f}(\frac{1}{2}[\mathbf{A} + \mathbf{A}^t]) = \bar{\mathbf{f}}(\mathbf{A})$, where $\mathbf{A}$ is any second order tensor. Now, find the derivative of $\bar{\mathbf{f}}$ with respect to $\mathbf{A}$ and then rewrite the result in terms of $\mathbf{S}$. The resulting expression is the derivative of $\mathbf{f}$ with respect to $\mathbf{S}$. Hence,

$$\bar{\mathbf{f}}(\mathbf{A} + s\mathbf{U}) = \frac{1}{2}[\mathbf{A} + \mathbf{A}^t + s[\mathbf{U} + \mathbf{U}^t]]. \tag{2.192}$$

Substituting the above in equation (2.190) we get

$$\frac{\partial \bar{\mathbf{f}}}{\partial \mathbf{A}} : \mathbf{U} = \frac{1}{2}[\mathbf{U} + \mathbf{U}^t]. \tag{2.193}$$

Thus,

$$\frac{\partial \bar{\mathbf{f}}}{\partial \mathbf{A}} = \frac{1}{2}[\mathbb{I} + \bar{\mathbb{I}}]. \tag{2.194}$$

Consequently,

$$\frac{\partial \mathbf{f}}{\partial \mathbf{S}} = \frac{\partial \bar{\mathbf{f}}}{\partial \mathbf{A}} = \frac{1}{2}[\mathbb{I} + \bar{\mathbb{I}}]. \tag{2.195}$$

**Example - 1:** Assume that $\mathbf{A}$ is an invertible tensor and $\mathbf{f}(\mathbf{A}) = \mathbf{A}^{-1}$. Then show that

$$\frac{\partial \mathbf{f}}{\partial \mathbf{A}} : \mathbf{U} = -\mathbf{A}^{-1}\mathbf{U}\mathbf{A}^{-1}. \tag{2.196}$$

Recalling that all invertible tensors satisfy the relation $\mathbf{A}\mathbf{A}^{-1} = \mathbf{1}$, we obtain

$$(\mathbf{A} + s\mathbf{U})(\mathbf{A} + s\mathbf{U})^{-1} = \mathbf{1}. \tag{2.197}$$

Differentiating the above equation with respect to $s$

$$\mathbf{U}(\mathbf{A} + s\mathbf{U})^{-1} + (\mathbf{A} + s\mathbf{U})\frac{d}{ds}(\mathbf{A} + s\mathbf{U})^{-1} = \mathbf{0}. \tag{2.198}$$

Evaluating the above equation at $s = 0$, and rearranging

$$\frac{\partial \mathbf{f}}{\partial \mathbf{A}} : \mathbf{U} = \frac{d}{ds}(\mathbf{A} + s\mathbf{U})^{-1}|_{s=0} = -\mathbf{A}^{-1}\mathbf{U}\mathbf{A}^{-1}. \tag{2.199}$$

Hence proved.

**Example - 2:** Assume that $\mathbf{S}$ is an invertible symmetrical tensor and $\mathbf{f}(\mathbf{S}) = \mathbf{S}^{-1}$. Find $\frac{\partial \mathbf{f}}{\partial \mathbf{S}}$.

As before generalizing the given function, we have $\bar{\mathbf{f}}(\mathbf{A}) = \{\frac{1}{2}[\mathbf{A} + \mathbf{A}^t]\}^{-1}$. Following the same steps as in the previous example we obtain:

$$[\mathbf{A} + \mathbf{A}^t + s(\mathbf{U} + \mathbf{U}^t)][\mathbf{A} + \mathbf{A}^t + s(\mathbf{U} + \mathbf{U}^t)]^{-1} = \mathbf{1}. \tag{2.200}$$

Differentiating the above equation with respect to $s$ and evaluating at $s = 0$ and rearranging we get

$$\frac{\partial \bar{\mathbf{f}}}{\partial \mathbf{A}} : \mathbf{U} = 2\frac{d}{ds}[\mathbf{A} + \mathbf{A}^t + s(\mathbf{U} + \mathbf{U}^t)]^{-1}|_{s=0} = 2[\mathbf{A} + \mathbf{A}^t]^{-1}[\mathbf{U} + \mathbf{U}^t][\mathbf{A} + \mathbf{A}^t]^{-1}. \tag{2.201}$$

Therefore,

$$\frac{\partial \mathbf{f}}{\partial \mathbf{S}} : \mathbf{U} = -\frac{1}{2}\mathbf{S}^{-1}[\mathbf{U} + \mathbf{U}^t]\mathbf{S}^{-1}. \tag{2.202}$$

Let $\Phi$ be a smooth scalar-valued function and $\mathbf{A}$, $\mathbf{B}$ smooth tensor-valued functions of a tensor variable $\mathbf{C}$. Then, it can be shown that

$$\frac{\partial(\mathbf{A} \cdot \mathbf{B})}{\partial \mathbf{C}} = \mathbf{A} : \frac{\partial \mathbf{B}}{\partial \mathbf{C}} + \mathbf{B} : \frac{\partial \mathbf{A}}{\partial \mathbf{C}}, \tag{2.203}$$

$$\frac{\partial(\Phi \mathbf{A})}{\partial \mathbf{C}} = \mathbf{A} \otimes \frac{\partial \Phi}{\partial \mathbf{C}} + \Phi\frac{\partial \mathbf{A}}{\partial \mathbf{C}}. \tag{2.204}$$

## 2.8 Gradients and related operators

In this section, we consider scalar and vector-valued functions that assign a scalar and vector to each element in the subset of the set of position vectors for the points in 3D space. If $\mathbf{x}$ denotes the position vector of points in the 3D space, then $\hat{\Phi}(\mathbf{x})$ is a function that assigns a scalar $\Phi$ to each point in the 3D space of interest and $\Phi$ is called the scalar field. A few examples of scalar fields are temperature, density, energy. Similarly, the vector field $\hat{\mathbf{u}}(\mathbf{x})$ assigns a vector to each point in the 3D space of interest. Displacement, velocity, acceleration are a few examples of vector fields.

Let $\mathfrak{D}$ denote the region of the 3D space that is of interest, that is the set of position vectors of points that is of interest in the 3D space. Then a scalar field $\Phi$, defined on a domain $\mathfrak{D}$, is said to be continuous if

$$\lim_{\alpha\to 0} |\Phi(\mathbf{x}+\alpha\mathbf{a}) - \Phi(\mathbf{x})| = 0, \quad \forall \mathbf{x} \in \mathfrak{D}, \mathbf{a} \in \mathfrak{E}, \tag{2.205}$$

where $\mathfrak{E}$ denote the set of all position vectors of points in the 3D space and $\mathbf{a}$ is a constant vector. The scalar field $\Phi$ is said to be differentiable if there exist a vector field $\mathbf{w}$ such that

$$\lim_{\alpha\to 0} |\mathbf{w}(\mathbf{x})\cdot\mathbf{a} - \alpha^{-1}[\Phi(\mathbf{x}+\alpha\mathbf{a}) - \Phi(\mathbf{x})]| = 0, \quad \forall \mathbf{x} \in \mathfrak{D}, \mathbf{a} \in \mathfrak{E}. \tag{2.206}$$

It can be shown that there is at most one vector field, $\mathbf{w}$ satisfying the above equation (proof omitted). This unique vector field $\mathbf{w}$ is called the gradient of $\Phi$ and denoted by $grad(\Phi)$.

The properties of continuity and differentiability are attributed to a vector field $\mathbf{u}$ and a tensor field $\mathbf{T}$ defined on $\mathfrak{D}$, if they apply to the scalar field $\mathbf{u}\cdot\mathbf{a}$ and $\mathbf{a}\cdot\mathbf{T}\mathbf{b}$ $\forall$ $\mathbf{a}$ and $\mathbf{b} \in \mathfrak{E}$. Given that the vector field $\mathbf{u}$, is differentiable, the gradient of $\mathbf{u}$ denoted by $grad(\mathbf{u})$, is the tensor field defined by

$$\{grad(\mathbf{u})\}^t\mathbf{a} = grad(\mathbf{u}\cdot\mathbf{a}) \quad \forall \mathbf{a} \in \mathfrak{E} \tag{2.207}$$

The divergence and the curl of a vector $\mathbf{u}$ denoted as $div(\mathbf{u})$ and $curl(\mathbf{u})$, are respectively scalar valued and vector valued and are defined by

$$div(\mathbf{u}) = tr(grad(\mathbf{u})), \tag{2.208}$$

$$curl(\mathbf{u})\cdot\mathbf{a} = div(\mathbf{u}\wedge\mathbf{a}), \qquad \forall \mathbf{a} \in \mathfrak{E}. \tag{2.209}$$

When the tensor field $\mathbf{T}$ is differentiable, its divergence denoted by $div(\mathbf{T})$, is the vector field defined by

$$div(\mathbf{T})\cdot\mathbf{a} = div(\mathbf{T}\mathbf{a}), \qquad \forall \mathbf{a} \in \mathfrak{E}. \tag{2.210}$$

If $grad(\Phi)$ exist and is continuous, $\Phi$ is said to be continuously differentiable and this property extends to $\mathbf{u}$ and $\mathbf{T}$ if $grad(\mathbf{u}\cdot\mathbf{a})$ and $grad(\mathbf{a}\cdot\mathbf{Tb})$ exist and are continuous for all $\mathbf{a}$, $\mathbf{b}\in\mathfrak{E}$.

An important identity: For a continuously differentiable vector field, $\mathbf{u}$

$$div((grad(\mathbf{u}))^t) = grad(div(\mathbf{u})). \tag{2.211}$$

(Proof omitted)

Let $\Phi$ and $\mathbf{u}$ be some scalar and vector field respectively. Then, the Laplacian operator, denoted by $\Delta$ (or by $\nabla^2$), is defined as

$$\Delta(\Phi) = div(grad(\Phi)), \qquad \Delta(\mathbf{u}) = div(grad(\mathbf{u})). \tag{2.212}$$

The Hessian operator, denoted by $\boldsymbol{\nabla\nabla}$ is defined as

$$\boldsymbol{\nabla\nabla}(\Phi) = grad(grad(\Phi)) \tag{2.213}$$

If a vector field $\mathbf{u}$ is divergence free (i.e. $div(\mathbf{u}) = 0$) then it is called solenoidal. It is called irrotational if $curl(\mathbf{u}) = \mathbf{o}$. It can be established that

$$curl(curl(\mathbf{u})) = grad(div(\mathbf{u})) - \Delta(\mathbf{u}). \tag{2.214}$$

Consequently, if a vector field, $\mathbf{u}$ is both solenoidal and irrotational, then $\Delta(\mathbf{u}) = \mathbf{o}$ (follows from the above equation) and such a vector field is said to be harmonic. If a scalar field $\Phi$ satisfies $\Delta(\Phi) = 0$, then $\Phi$ is said to be harmonic.

**Potential theorem:** Let $\mathbf{u}$ be a smooth point field on an open or closed simply connected region $\mathcal{R}$ and let $curl(\mathbf{u}) = \mathbf{o}$. Then there is a continuously differentiable scalar field on $\mathcal{R}$ such that

$$\mathbf{u} = grad(\Phi) \tag{2.215}$$

From the definition of $grad$ (2.207), it can be seen that it is a linear operator. That is,

$$grad(\mathbf{u}_1 + \mathbf{u}_2) = grad(\mathbf{u}_1) + grad(\mathbf{u}_2). \tag{2.216}$$

Consequently, all other operators $div$, $\Delta$ are also linear operators.

Before concluding this section we collect a few identities that are useful subsequently. Here $\Phi$, $\Psi$ are scalar fields, $\mathbf{u}$, $\mathbf{v}$ are vector fields and $\mathbf{A}$ is second order tensor field.

$$
\begin{aligned}
div(\Phi\mathbf{u}) &= \Phi div(\mathbf{u}) + \mathbf{u}\cdot grad(\Phi), && (2.217)\\
div(\Phi\mathbf{A}) &= \Phi div(\mathbf{A}) + \mathbf{A}^t grad(\Phi), && (2.218)\\
div(\mathbf{A}^t\mathbf{u}) &= div(\mathbf{A})\cdot\mathbf{u} + \mathbf{A}\cdot grad(\mathbf{u}), && (2.219)\\
div(\mathbf{u}\wedge\mathbf{v}) &= \mathbf{v}\cdot curl(\mathbf{u}) - \mathbf{u}\cdot curl(\mathbf{v}), && (2.220)\\
div(\mathbf{u}\otimes\mathbf{v}) &= (grad(\mathbf{u}))\mathbf{v} + div(\mathbf{v})\mathbf{u}, && (2.221)\\
grad(\Phi\Psi) &= \Psi grad(\Phi) + \Phi grad(\Psi), && (2.222)\\
grad(\Phi\mathbf{u}) &= \mathbf{u}\otimes grad(\Phi) + \Phi grad(\mathbf{u}), && (2.223)\\
grad(\mathbf{u}\cdot\mathbf{v}) &= (grad(\mathbf{u}))^t\mathbf{v} + (grad(\mathbf{v}))^t\mathbf{u}, && (2.224)\\
curl(\Phi\mathbf{u}) &= grad(\Phi)\wedge\mathbf{u} + \Phi curl(\mathbf{u}), && (2.225)\\
curl(\mathbf{u}\wedge\mathbf{v}) &= \mathbf{u}div(\mathbf{v}) - \mathbf{v}div(\mathbf{u}) + (grad(\mathbf{u}))\mathbf{v} - (grad(\mathbf{v}))\mathbf{u}, && (2.226)\\
\Delta(\mathbf{u}\cdot\mathbf{v}) &= \mathbf{v}\cdot\Delta(\mathbf{u}) + 2grad(\mathbf{u})\cdot grad(\mathbf{v}) + \mathbf{u}\cdot\Delta(\mathbf{v}). && (2.227)
\end{aligned}
$$

Till now, in this section we defined all the quantities in closed form but we require them in component form to perform the actual calculations. This would be the focus in the reminder of this section. In the following, let $\mathbf{e}_i$ denote the Cartesian basis vectors and $x_i$ $i = 1, 2, 3$, the Cartesian coordinates.

Let $\Phi$ be differentiable scalar field, then it follows from equation (2.206), on replacing $\mathbf{a}$ by the base vectors $\mathbf{e}_i$ in turn, that the partial derivatives $\frac{\partial\Phi}{\partial x_i}$ exist in $\mathfrak{D}$ and that, moreover, $w_i = \frac{\partial\Phi}{\partial x_i}$. Hence

$$grad(\Phi) = \frac{\partial\Phi}{\partial x_i}\mathbf{e}_i. \tag{2.228}$$

Let $\mathbf{u}$ be differentiable vector field in $\mathfrak{D}$ and $\mathbf{a}$ some constant vector in $\mathfrak{E}$. Then, using (2.228) we compute

$$grad(\mathbf{u}\cdot\mathbf{a}) = \frac{\partial(\mathbf{u}\cdot\mathbf{a})}{\partial x_i}\mathbf{e}_i = \frac{\partial u_p}{\partial x_i}a_p\mathbf{e}_i = \frac{\partial u_p}{\partial x_i}\mathbf{e}_i\mathbf{a}\cdot\mathbf{e}_p = \left(\frac{\partial u_p}{\partial x_i}\mathbf{e}_i\otimes\mathbf{e}_p\right)\mathbf{a}, \tag{2.229}$$

where $a_p$ and $u_p$ are the Cartesian components of $\mathbf{a}$ and $\mathbf{u}$ respectively. Consequently, appealing to definition (2.207) we obtain

$$grad(\mathbf{u}) = \frac{\partial u_p}{\partial x_i}\mathbf{e}_p\otimes\mathbf{e}_i. \tag{2.230}$$

Then, according to the definition for divergence, (2.208)

$$div(\mathbf{u}) = \frac{\partial u_p}{\partial x_p}. \tag{2.231}$$

Next, we compute $div(\mathbf{u} \wedge \mathbf{a})$ to be

$$div(\mathbf{u} \wedge \mathbf{a}) = \pm \frac{\partial(\epsilon_{ijk} u_i a_j)}{\partial x_k} = \pm \epsilon_{ijk} \frac{\partial u_i}{\partial x_k} a_j = \pm \left( \epsilon_{ijk} \frac{\partial u_i}{\partial x_k} \mathbf{e}_j \right) \cdot (\mathbf{e}_p a_p), \tag{2.232}$$

using (2.231). Comparing the above result with the definition of *curl* in (2.209) we infer

$$curl(\mathbf{u}) = \pm \epsilon_{ijk} \frac{\partial u_i}{\partial x_k} \mathbf{e}_j \tag{2.233}$$

Finally, let $\mathbf{T}$ be a differentiable tensor field in $\mathfrak{D}$, then we compute

$$div(\mathbf{Ta}) = \frac{\partial T_{ij} a_j}{\partial x_i} = \frac{\partial T_{ij}}{\partial x_i} a_j = \left( \frac{\partial T_{ij}}{\partial x_i} \mathbf{e}_j \right) \cdot (a_p \mathbf{e}_p), \tag{2.234}$$

using equation (2.231). Then, we infer

$$div(\mathbf{T}) = \frac{\partial T_{ij}}{\partial x_i} \mathbf{e}_j, \tag{2.235}$$

from the definition (2.210).

The component form for the other operators, namely Laplacian and Hessian can be obtained as

$$\Delta(\Phi) = \frac{\partial^2 \Phi}{\partial x_i^2}, \qquad \Delta(\mathbf{u}) = \frac{\partial^2 u_j}{\partial x_i^2} \mathbf{e}_j, \qquad \text{sum } i \text{ from 1 to 3,} \tag{2.236}$$

$$\boldsymbol{\nabla}\boldsymbol{\nabla}(\Phi) = \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j, \tag{2.237}$$

respectively.

Above we obtained the components of the different operators in Cartesian coordinate system, we shall now proceed to obtain the same with respect to other coordinate systems. While using the method illustrated here we could obtain the components in any orthogonal curvilinear system, we choose the cylindrical polar coordinate system for illustration.

First, we outline a general procedure to find the basis vectors for a given orthogonal curvilinear coordinates. A differential vector $d\mathbf{p}$ in the 3D vector space is written as $d\mathbf{p} = dx_i\mathbf{e}_i$ relative to Cartesian coordinates. The same coordinate independent vector in orthogonal curvilinear coordinates is written as

$$d\mathbf{p} = \frac{\partial \mathbf{p}}{\partial x_i} dx_i = \frac{\partial \mathbf{p}}{\partial z_i} dz_i, \tag{2.238}$$

where $z_i$ denotes curvilinear coordinates. But

$$d\mathbf{p} = dx_i\mathbf{e}_i = dz_i\mathbf{g}_i, \tag{2.239}$$

where $\mathbf{g}_i$ is the basis vectors in the orthogonal curvilinear coordinates. Thus,

$$\mathbf{e}_i = \frac{\partial \mathbf{p}}{\partial x_i}, \qquad \text{and} \qquad \mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial z_i}. \tag{2.240}$$

Comparing equations (2.239) and (2.240) we obtain the desired transformation relation between the bases to be:

$$\mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial z_i} = \frac{\partial \mathbf{p}}{\partial x_k}\frac{\partial x_k}{\partial z_i} = \frac{\partial x_k}{\partial z_i}\mathbf{e}_k \tag{2.241}$$

The basis $\mathbf{g}_i$ so obtained will vary from point to point in the Euclidean space and will be orthogonal (by the choice of curvilinear coordinates) at each point but not orthonormal. Hence we define

$$(\mathbf{e}^c)_i = \mathbf{g}_i/|\mathbf{g}_i|, \qquad \text{(no summation on } i\text{)} \tag{2.242}$$

and use these as the basis vectors for the orthogonal curvilinear coordinates.

Let $(x, y, z)$ denote the coordinates of a typical point in Cartesian coordinate system and $(r, \theta, Z)$ the coordinates of a typical point in cylindrical polar coordinate system. Then, the coordinate transformation from cylindrical polar to Cartesian and vice versa is given by:

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad z = Z, \tag{2.243}$$

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right), \qquad Z = z, \tag{2.244}$$

respectively. From these we compute

$$\frac{\partial r}{\partial x} = \cos(\theta), \qquad \frac{\partial r}{\partial y} = \sin(\theta), \qquad \frac{\partial r}{\partial z} = 0, \tag{2.245}$$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{r}\sin(\theta), \qquad \frac{\partial \theta}{\partial y} = \frac{1}{r}\cos(\theta), \qquad \frac{\partial \theta}{\partial z} = 0, \tag{2.246}$$

$$\frac{\partial Z}{\partial x} = 0, \qquad \frac{\partial Z}{\partial y} = 0, \qquad \frac{\partial Z}{\partial z} = 1, \tag{2.247}$$

$$\frac{\partial x}{\partial r} = \cos(\theta), \qquad \frac{\partial x}{\partial \theta} = -r\sin(\theta), \qquad \frac{\partial x}{\partial Z} = 0, \tag{2.248}$$

$$\frac{\partial y}{\partial r} = \sin(\theta), \qquad \frac{\partial y}{\partial \theta} = r\cos(\theta), \qquad \frac{\partial y}{\partial Z} = 0, \tag{2.249}$$

$$\frac{\partial z}{\partial r} = 0, \qquad \frac{\partial z}{\partial \theta} = 0, \qquad \frac{\partial z}{\partial Z} = 1. \tag{2.250}$$

Consequently, the orthonormal cylindrical polar basis vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_Z)$ and Cartesian basis vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are related through the equations

$$\mathbf{e}_r = \cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2, \tag{2.251}$$

$$\mathbf{e}_\theta = -\sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_2, \tag{2.252}$$

$$\mathbf{e}_Z = \mathbf{e}_3, \tag{2.253}$$

obtained using equation (2.242) along with a change in notation: $((\mathbf{e}^c)_1, (\mathbf{e}^c)_2, (\mathbf{e}^c)_3) = (\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_Z)$.

Now, we can compute the components of $grad(\Phi)$ in cylindrical polar coordinates. Towards this, we obtain

$$\begin{aligned}
grad(\Phi) &= \frac{\partial \Phi}{\partial x}\mathbf{e}_1 + \frac{\partial \Phi}{\partial y}\mathbf{e}_2 + \frac{\partial \Phi}{\partial z}\mathbf{e}_3, \qquad \text{in Cartesian coordinate system} \\
&= \left(\frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \Phi}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial \Phi}{\partial Z}\frac{\partial Z}{\partial x}\right)\mathbf{e}_1 \\
&\qquad + \left(\frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \Phi}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial \Phi}{\partial Z}\frac{\partial Z}{\partial y}\right)\mathbf{e}_2 \\
&\qquad\quad + \left(\frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial \Phi}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial \Phi}{\partial Z}\frac{\partial Z}{\partial z}\right)\mathbf{e}_3 \\
&= \frac{\partial \Phi}{\partial r}(\cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2) + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}(-\sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_2) + \frac{\partial \Phi}{\partial Z}\mathbf{e}_3 \\
&= \frac{\partial \Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \Phi}{\partial Z}\mathbf{e}_Z,
\end{aligned} \tag{2.254}$$

using the chain rule for differentiation.

To obtain the components of $grad(\mathbf{u})$ one can follow the same procedure outlined above or can compute the same using the above result and the definition of $grad$ (2.207) as detailed below:

$$\begin{aligned} grad(\mathbf{u}\cdot\mathbf{a}) &= \left(\frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta + \frac{\partial}{\partial z}\mathbf{e}_z\right)(u_r a_r + u_\theta a_\theta + u_Z a_Z) \\ &= \left[\frac{\partial u_r}{\partial r}a_r + \frac{\partial u_\theta}{\partial r}a_\theta + \frac{\partial u_Z}{\partial r}a_Z\right]\mathbf{e}_r \\ &\quad +\frac{1}{r}\left[\frac{\partial u_r}{\partial\theta}a_r + u_r a_\theta + \frac{\partial u_\theta}{\partial\theta}a_\theta - u_\theta a_r + \frac{\partial u_Z}{\partial\theta}a_Z\right]\mathbf{e}_\theta \\ &\quad +\left[\frac{\partial u_r}{\partial Z}a_r + \frac{\partial u_\theta}{\partial Z}a_\theta + \frac{\partial u_Z}{\partial Z}a_Z\right]\mathbf{e}_Z, \end{aligned} \tag{2.255}$$

where we have used the following identities:

$$\frac{\partial a_r}{\partial r} = \frac{\partial a_\theta}{\partial r} = \frac{\partial a_Z}{\partial r} = 0, \tag{2.256}$$

$$\frac{\partial a_r}{\partial\theta} = a_\theta, \quad \frac{\partial a_\theta}{\partial\theta} = -a_r, \quad \frac{\partial a_Z}{\partial\theta} = 0, \tag{2.257}$$

$$\frac{\partial a_r}{\partial Z} = \frac{\partial a_\theta}{\partial Z} = \frac{\partial a_Z}{\partial Z} = 0, \tag{2.258}$$

obtained by recognizing that only the Cartesian components of $\mathbf{a}$ are to be treated as constants and not the cylindrical polar components, $a_r = a_x\cos(\theta)$ $+\ a_y\sin(\theta)$, $a_\theta = -a_x\sin(\theta) + a_y\cos(\theta)$, $a_Z = a_z$. From (2.255) and (2.207) the matrix components of $grad(\mathbf{u})$ in cylindrical polar coordinates are written as

$$grad(\mathbf{u}) = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r} & \frac{\partial u_r}{\partial Z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} & \frac{\partial u_\theta}{\partial Z} \\ \frac{\partial u_Z}{\partial r} & \frac{1}{r}\frac{\partial u_Z}{\partial\theta} & \frac{\partial u_Z}{\partial Z} \end{pmatrix}. \tag{2.259}$$

Using the techniques illustrated above the following identities can be established:

$$div(\mathbf{u}) = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_Z}{\partial Z}, \tag{2.260}$$

$$div(\mathbf{T}) = \left\{\begin{matrix} \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial\theta} + \frac{\partial T_{Zr}}{\partial Z} + \frac{T_{rr}-T_{\theta\theta}}{r} \\ \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{\partial T_{Z\theta}}{\partial Z} + \frac{T_{r\theta}+T_{\theta r}}{r} \\ \frac{\partial T_{rZ}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta Z}}{\partial\theta} + \frac{\partial T_{ZZ}}{\partial Z} + \frac{T_{rZ}}{r} \end{matrix}\right\}. \tag{2.261}$$

## 2.9 Integral theorems

### 2.9.1 Divergence theorem

Suppose $\mathbf{u}(\mathbf{x})$ is a vector field defined on some convex three dimensional region, $\mathcal{R}$ in physical space with volume $v$ and on a closed surface $a$ bounding this volume[2] and $\mathbf{u}(\mathbf{x})$ is continuous in $\mathcal{R}$ and continuously differentiable in the interior of $\mathcal{R}$. Then,

$$\int_a \mathbf{u} \cdot \mathbf{n} da = \int_v div(\mathbf{u}) dv, \tag{2.262}$$

where, $\mathbf{n}$ is the outward unit normal field acting along the surface $a$, $dv$ and $da$ are infinitesimal volume and surface element, respectively. Proof of this theorem is beyond the scope of these notes.

Using (2.262) the following identities can be established:

$$\int_a \mathbf{A}\mathbf{n} da = \int_v div(\mathbf{A}) dv, \tag{2.263}$$

$$\int_a \Phi \mathbf{n} da = \int_v grad(\Phi) dv, \tag{2.264}$$

$$\int_a \mathbf{n} \wedge \mathbf{u} da = \int_v curl(\mathbf{u}) dv, \tag{2.265}$$

$$\int_a \mathbf{u} \otimes \mathbf{n} da = \int_v grad(\mathbf{u}) dv, \tag{2.266}$$

$$\int_a \mathbf{u} \cdot \mathbf{A}\mathbf{n} da = \int_v div(\mathbf{A}^t \mathbf{u}) dv, \tag{2.267}$$

where $\Phi$, $\mathbf{u}$, $\mathbf{A}$ are continuously differentiable scalar, vector and tensor fields defined in $v$ and continuous in $v$ and $\mathbf{n}$ is the outward unit normal field acting along the surface $a$.

### 2.9.2 Stokes theorem

Stokes theorem relates a surface integral, which is valid over any open surface $a$, to a line integral around the bounding single simple closed curve $c$ in three dimensional space. Let $d\mathbf{x}$ denote the tangent vector to the curve, $c$ and $\mathbf{n}$

[2] To be more precise, it is sufficient if the domain is a regular region. See Kellogg [3] to understand what a regular region is.

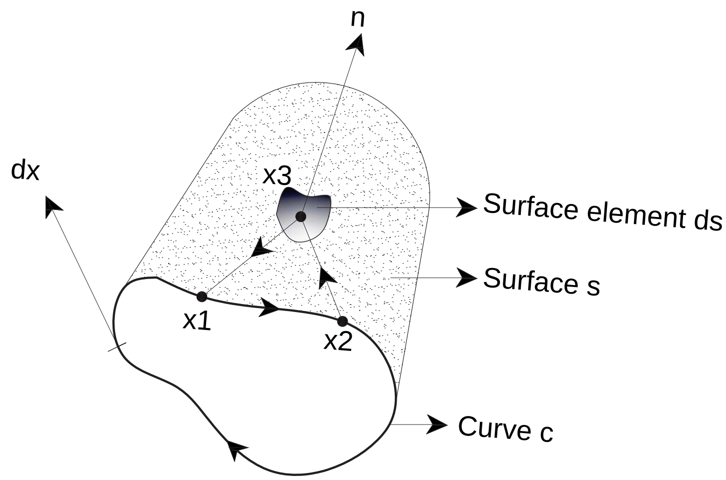



Figure 2.3: Open surface

denote the outward unit vector field $\mathbf{n}$ normal to $s$. The curve $c$ has positive orientation relative to $\mathbf{n}$ in the sense shown in figure 2.3. The indicated circuit with the adjacent points 1, 2, 3 (1, 2 on curve $c$ and 3 an interior point of the surface $s$) induced by the orientation of $c$ is related to the direction of $\mathbf{n}$ (i.e., a unit vector normal to $s$ at point 3) by right-hand screw rule. For a continuously differentiable vector field $\mathbf{u}$ defined on some region containing $a$, we have

$$\int_a curl(\mathbf{u}) \cdot \mathbf{n} da = \oint_c \mathbf{u} \cdot d\mathbf{x}. \tag{2.268}$$

Let $\Phi$ and $\mathbf{u}$ be continuously differentiable scalar and vector fields defined on $a$ and $c$ and $\mathbf{n}$ is the outward drawn normal to $a$. Then, the following can be shown:

$$\oint_c \Phi d\mathbf{x} = \int_a \mathbf{n} \wedge grad(\Phi) da, \tag{2.269}$$

$$\oint_c \mathbf{u} \wedge d\mathbf{x} = \int_a [div(\mathbf{u})\mathbf{n} - (grad(\mathbf{u}))^t \mathbf{n}] da. \tag{2.270}$$

### 2.9.3 Green's theorem

Finally, we record Greens's theorem. We do this first for simply connected domain and then for multiply connected domain. If any curve within the domain can be shrunk to a point in the domain without leaving the domain, then such a domain is said to be simply connected. A domain that is not simply connected is said to be multiply connected.

Applying Stoke's theorem, (2.268) to a planar simply connected domain with the vector field given as $\mathbf{u} = f(x,y)\mathbf{e}_1 + g(x,y)\mathbf{e}_2$,

$$\int_a \left( \frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \right) dxdy = \oint_c (fdx + gdy). \tag{2.271}$$

The above equation (2.271) when $f(x,y) = 0$ reduces to

$$\int_a \frac{\partial g}{\partial x} dxdy = \oint_c -gdy = \oint_c -g\sin(\theta)ds = \oint_c gn_x ds, \tag{2.272}$$

where $\theta$ is the angle made by the tangent to the curve $c$ with $\mathbf{e}_1$ and $n_x$ is the component of the unit normal $\mathbf{n}$ along $\mathbf{e}_1$ direction, refer to figure 2.4. Then, when $g(x,y) = 0$ equation (2.271) yields,

$$\int_a \frac{\partial f}{\partial y} dxdy = \oint_c fdx = \oint_c f\cos(\theta)ds = \oint_c fn_y ds, \tag{2.273}$$

where $n_y$ is the component of the unit normal along the $\mathbf{e}_2$ direction.

Similarly, applying Stoke's theorem (2.268) to a vector field, $\mathbf{u} = \hat{f}(y,z)\mathbf{e}_2 + \hat{g}(y,z)\mathbf{e}_3$ defined over a planar simply connected domain,

$$\int_a \left( \frac{\partial \hat{g}}{\partial y} - \frac{\partial \hat{f}}{\partial z} \right) dydz = \oint_c (\hat{f}dy + \hat{g}dz). \tag{2.274}$$

The above equation (2.274) when $\hat{f}(y,z) = 0$ reduces to

$$\int_a \frac{\partial \hat{g}}{\partial y} dydz = \oint_c \hat{g}n_y ds, \tag{2.275}$$

where $n_y$ is the component of the normal to the curve along the $\mathbf{e}_2$ direction and when $\hat{g}(y,z) = 0$ yields

$$\int_a \frac{\partial \hat{f}}{\partial z} dydz = \oint_c \hat{f}n_z ds, \tag{2.276}$$

where $n_z$ is the component of the normal to the curve along the $\mathbf{e}_3$ direction.

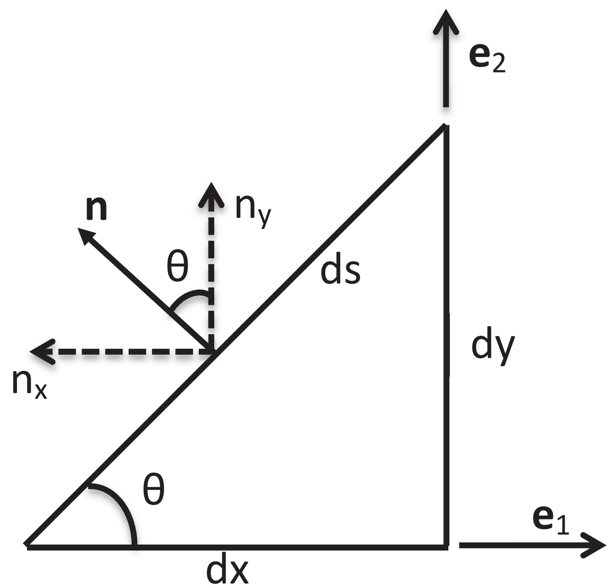



Figure 2.4: Relation between tangent and normal vectors to a curve in $xy$ plane

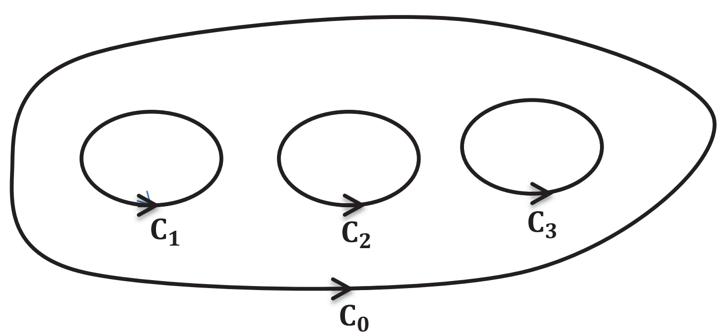



Figure 2.5: Illustration of a multiply connected region

In a simply connected domain, the region of interest has a single simple closed curve. In a multiply connected domain, the region of interest has several simple closed curves, like the one shown in figure 2.5. If the line integrals are computed by traversing in the direction shown in figure 2.5,

$$\int_a \left( \frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \right) dxdy = \oint_{C_0} (fdx + gdy) - \sum_{i=1}^{N} \oint_{C_i} (fdx + gdy), \qquad (2.277)$$

where the functions $f$ and $g$ are continuously differentiable in the domain, $C_0$ is the outer boundary of the domain, $C_i$'s are the boundary of the voids in the interior of the domain and we have assumed that there are $N$ such voids in the domain. We emphasize again our assumption that we transverse the interior voids in the same sense as the outer boundary.

The above equation (2.277) when $f(x, y) = 0$ reduces to

$$\int_a \frac{\partial g}{\partial x} dxdy = \oint_{C_0} g n_x^0 ds_0 - \sum_{i=1}^{N} \oint_{C_i} g n_x^i ds_i, \qquad (2.278)$$

where $n_x^i$ is the component of the unit normal $\mathbf{n}$ along $\mathbf{e}_1$ direction of the $i^{th}$ curve. Then, when $g(x, y) = 0$ equation (2.277) yields,

$$\int_a \frac{\partial f}{\partial y} dxdy = \oint_{C_0} f n_y^0 ds_0 - \sum_{i=1}^{N} \oint_{C_i} f n_y^i ds_i, \qquad (2.279)$$

where $n_y^i$ is the component of the unit normal along the $\mathbf{e}_2$ direction of the $i^{th}$ curve.

## 2.10 Summary

In this chapter, we looked at the mathematical tools required to embark on a study of mechanics. In particular, we began with a study on algebra of tensors and then graduated to study tensor calculus. Most of the mathematical results that we would be using is summarized in this chapter.

## 2.11 Self-Evaluation

1. Give reasons for the following: (a) $\delta_{ii} = 3$, (b) $\delta_{ij}\delta_{ij} = 3$, (c) $\delta_{ij}a_i = a_j$, (d) $\delta_{ij}a_{jk} = a_{ik}$, (e) $\delta_{ij}a_{kj} = a_{ki}$, (f) $\delta_{ij}a_{ij} = a_{ii}$, (g) $\delta_{ij}\delta_{jk}\delta_{ik} = 3$,

(h) $\delta_{ik}\delta_{jm}\delta_{ij} = \delta_{km}$ (i) $\epsilon_{kki} = 0$, (j) $\delta_{ij}\epsilon_{ijk} = 0$, (k) $\epsilon_{ijk}\epsilon_{ijk} = 6$, where $\delta_{ij}$ denotes the Kronecker delta and $\epsilon_{ijk}$ the permutation symbol

2. Show that $a_{ij} = a_{ji}$ when $\epsilon_{ijk}a_{jk} = 0$. $\epsilon_{ijk}$ is the permutation symbol

3. Expand $a_{ij}b_{ij}$

4. Expand $D_{ij}x_ix_j$. If possible simplify the expression $D_{ij}x_ix_j$ when (a) $D_{ij} = D_{ji}$ (b) $D_{ij} = -D_{ji}$

5. Express the expression $c_i = A_{ij}b_j - b_i$ as $c_i = B_{ij}b_j$ and thus obtain $B_{ij}$

6. Consider two vectors $\mathbf{a} = \mathbf{e}_1 + 4\mathbf{e}_2 + 6\mathbf{e}_3$ and $\mathbf{b} = 2\mathbf{e}_1 + 4\mathbf{e}_2$, where $\{\mathbf{e}_i\}$ are orthonormal Cartesian basis vectors. For these vectors compute,

   (a) Norm of $\mathbf{a}$
   (b) The angle between vectors $\mathbf{a}$ and $\mathbf{b}$
   (c) The area of the parallelogram bounded by the vectors $\mathbf{a}$ and $\mathbf{b}$
   (d) $\mathbf{b} \wedge \mathbf{a}$
   (e) The component of vector $\mathbf{a}$ in the direction $\mathbf{z} = \mathbf{e}_1 + \mathbf{e}_2$.

7. Show that $\mathbf{y} = m\mathbf{x}$ where $\mathbf{y}$ and $\mathbf{x}$ are vectors and $m$ is a scalar is a linear transformation. Find the second order tensor that maps the given vector $\mathbf{x}$ on to $\mathbf{y}$.

8. Show that $\mathbf{y} = m\mathbf{x} + \mathbf{c}$ where $\mathbf{y}$, $\mathbf{x}$ and $\mathbf{c}$ are vectors and $m$ is a scalar is not a linear transformation.

9. Establish the following $\epsilon_{ijk}T_{jk}\mathbf{e}_i = (T_{23} - T_{32})\mathbf{e}_1 + (T_{31} - T_{13})\mathbf{e}_2 + (T_{12} - T_{21})\mathbf{e}_3$ and use this to show that $\delta_{ij}\epsilon_{ijk}\mathbf{e}_k = \mathbf{o}$. Here $\{\mathbf{e}_i\}$ denotes the orthonormal Cartesian basis, $\delta_{ij}$ the Kronecker delta and $\epsilon_{ijk}$ the permutation symbol

10. Show that if tensor $\mathbf{A}$ is positive definite then $\det(\mathbf{A}) > 0$

11. Using Cayley-Hamilton theorem, deduce that

$$I_3 = \{[tr(\mathbf{A})]^3 - 3tr(\mathbf{A})tr(\mathbf{A}^2) + 2tr(\mathbf{A}^3)\}/6$$

for an arbitrary tensor $\mathbf{A}$.

12. Let $\mathbf{A}$ be an arbitrary tensor. Show that $\mathbf{A}^t\mathbf{A}$ and $\mathbf{A}\mathbf{A}^t$ are positive semi-definite symmetric tensors. If $\mathbf{A}$ is invertible, prove that these tensors - $\mathbf{A}^t\mathbf{A}$ and $\mathbf{A}\mathbf{A}^t$ - are positive definite.

13. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ be arbitrary vectors and $\mathbf{A}$ an arbitrary tensor. Show that

    (a) $(\mathbf{u} \otimes \mathbf{v})(\mathbf{w} \otimes \mathbf{x}) = (\mathbf{v} \cdot \mathbf{w})(\mathbf{u} \otimes \mathbf{x})$

    (b) $\mathbf{A}(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{A}\mathbf{u}) \otimes \mathbf{v}$

    (c) $(\mathbf{u} \otimes \mathbf{v})\mathbf{A} = \mathbf{u} \otimes (\mathbf{A}^t\mathbf{v})$

14. Let $\mathbf{Q}$ be a proper orthogonal tensor. If $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ form an orthonormal basis then a general representation for $\mathbf{Q}$ is: $\mathbf{Q} = \mathbf{p} \otimes \mathbf{p} + (\mathbf{q} \otimes \mathbf{q} + \mathbf{r} \otimes \mathbf{r})\cos(\theta) + (\mathbf{r} \otimes \mathbf{q} - \mathbf{q} \otimes \mathbf{r})\sin(\theta)$. Now, show that

    (a) $I_1 = I_2 = 1 + 2\cos(\theta)$

    (b) $I_3 = 1$

    (c) $\mathbf{Q}$ has only one real eigen value if $\theta \neq 0, \pi$

15. Let $\mathbf{A}$ be an invertible tensor which depends upon a real parameter, $t$. Assuming that $\frac{d\mathbf{A}}{dt}$ exists, prove that

$$\frac{d(\det(\mathbf{A}))}{dt} = \det(\mathbf{A}) tr\left(\frac{d\mathbf{A}}{dt}\mathbf{A}^{-1}\right)$$

16. Let $\phi(\mathbf{A}) = \det(\mathbf{A})$ and $\mathbf{A}$ be an invertible tensor. Then, show that $\frac{\partial\phi}{\partial\mathbf{A}} = I_3\mathbf{A}^{-1}$

17. Let $\phi$, $\mathbf{u}$ and $\mathbf{T}$ be differentiable scalar, vector and tensor fields. Using index notation verify the following for Cartesian coordinate system:

    (a) $grad(\phi\mathbf{u}) = \mathbf{u} \otimes grad(\phi) + \phi grad(\mathbf{u})$

    (b) $div\left(\frac{\mathbf{u}}{\phi}\right) = \frac{1}{\phi}div(\mathbf{u}) - \frac{1}{\phi^2}\mathbf{u} \cdot grad(\phi)$

    (c) $div(\phi\mathbf{T}) = \mathbf{T}^t grad(\phi) + \phi div(\mathbf{T})$

    (d) $curl(grad(\phi)) = \mathbf{o}$

    (e) $grad(\mathbf{u} \cdot \mathbf{u}) = 2grad(\mathbf{u})^t\mathbf{u}$

    (f) $div(\mathbf{T}^t\mathbf{u}) = \mathbf{T} \cdot grad(\mathbf{u}) + \mathbf{u} \cdot div(\mathbf{T})$

18. If $\mathbf{A}$ is an invertible tensor, then it can be decomposed uniquely as $\mathbf{A} = \mathbf{RU} = \mathbf{VR}$ where $\mathbf{R}$ is an orthogonal tensor, and $\mathbf{U}$, $\mathbf{V}$ are positive definite symmetric tensors. Then, show that $\mathbf{AA}^t = \mathbf{V}^2$ and $\mathbf{A}^t\mathbf{A} = \mathbf{U}^2$. Using these results prove that the eigenvalues of $\mathbf{A}^t\mathbf{A}$ and $\mathbf{AA}^t$ are the same and that their eigenvectors are related as: $\mathbf{p}_i = \mathbf{Rq}_i$, where $\mathbf{p}_i$ are the eigenvectors of $\mathbf{AA}^t$ and $\mathbf{q}_i$ are the eigenvectors of $\mathbf{A}^t\mathbf{A}$.

19. The matrix components of a tensor $\mathbf{A}$ when represented using Cartesian coordinate basis are
$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$
For this tensor find the following:
    (a) The spherical and deviatoric parts of $\mathbf{A}$
    (b) The eigenvalues of $\mathbf{A}$
    (c) The eigenvectors of $\mathbf{A}$

20. Let a new right-handed Cartesian coordinate system be represented by the set $\{\tilde{\mathbf{e}}_i\}$ of basis vectors with transformation law $\tilde{\mathbf{e}}_2 = -\sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_2$, $\tilde{\mathbf{e}}_3 = \mathbf{e}_3$. The origin of the new coordinate system coincides with the old origin.
    (a) Find $\tilde{\mathbf{e}}_1$ in terms of the old set $\{\mathbf{e}_i\}$ of basis vectors
    (b) Find the orthogonal matrix $[\mathbf{Q}]_{ij}$ that expresses the directional cosine of the new basis with respect to the old one
    (c) Express the vector $\mathbf{u} = -6\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3$ in terms of the new set $\{\tilde{\mathbf{e}}_i\}$ of basis vectors
    (d) Express the position vector, $\mathbf{r}$ of an arbitrary point using both $\{\mathbf{e}_i\}$ and $\{\tilde{\mathbf{e}}_i\}$ basis and establish the relationship between the components of $\mathbf{r}$ represented using $\{\mathbf{e}_i\}$ basis and $\{\tilde{\mathbf{e}}_i\}$ basis.

21. For the following components of a second order tensor:
$$\mathbf{T} = \begin{pmatrix} 10 & 4 & -6 \\ 4 & -6 & 8 \\ -6 & 8 & 14 \end{pmatrix},$$
relative to a Cartesian orthonormal basis, $\{\mathbf{e}_i\}$, determine

(a) The principal invariants

(b) The principal values, $p_1$, $p_2$, $p_3$

(c) The orientation of the principal directions with respect to the Cartesian basis, $\{\mathbf{e}\}$

(d) The orientation of the orthonormal basis, $\{\hat{\mathbf{e}}_i\}$ with respect to the basis, $\{\mathbf{e}_i\}$, so that the stress tensor has a matrix representation of the form, $\mathbf{T} = diag[p_1, p_2, p_3]$ when expressed with respect to the $\{\hat{\mathbf{e}}_i\}$ basis

(e) The transformation matrix, $[\mathbf{Q}]_{ij}$ which transforms the set of orthonormal basis $\{\mathbf{e}_i\}$ to $\{\hat{\mathbf{e}}_i\}$.

(f) The components of the same tensor $\mathbf{T}$ with respect to a basis $\{\tilde{\mathbf{e}}_i\}$ when $\tilde{\mathbf{e}}_1 = (\sqrt{3}\mathbf{e}_1 + \mathbf{e}_2)/2$, $\tilde{\mathbf{e}}_2 = (-\mathbf{e}_1 + \sqrt{3}\mathbf{e}_2)/2$, $\tilde{\mathbf{e}}_3 = \mathbf{e}_3$.

22. Let a tensor $\mathbf{A}$ be given by: $\mathbf{A} = \alpha[\mathbf{1} - \mathbf{e}_1 \otimes \mathbf{e}_1] + \beta[\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1]$, where $\alpha$ and $\beta$ are scalars and $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal unit vectors.

(a) Show that the eigenvalues $(\lambda_i)$ of $\mathbf{A}$ are

$$\lambda_1 = \alpha, \quad \lambda_2 = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \beta^2}, \quad \lambda_3 = \frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + \beta^2},$$

(b) Derive and show that the associated normalized eigenvectors $\mathbf{n}_i$ are given by:

$$\mathbf{n}_1 = \mathbf{e}_3, \quad \mathbf{n}_2 = \frac{\mathbf{e}_1 + (\lambda_2/\beta)\mathbf{e}_2}{\sqrt{1 + \lambda_2^2/\beta^2}}, \quad \mathbf{n}_3 = \frac{\mathbf{e}_1 + (\lambda_3/\beta)\mathbf{e}_2}{\sqrt{1 + \lambda_3^2/\beta^2}}.$$

23. Show that $\mathbf{u} \wedge \mathbf{v}$ is the axial vector for $\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}$. Using this establish the identity $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{b} \otimes \mathbf{c} - \mathbf{c} \otimes \mathbf{b})\mathbf{a}$, $\forall\ \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{V}$. Use could be made of the identity $[(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{u}] \cdot \mathbf{v} = (\mathbf{u} \wedge \mathbf{v}) \cdot (\mathbf{u} \wedge \mathbf{v})$.

24. Let $\mathcal{R}$ be a regular region and $\mathbf{T}$ a tensor field which is continuous in $\mathcal{R}$ and continuously differentiable in the interior of $\mathcal{R}$. Show that,

$$\int_{\partial\mathcal{R}} \mathbf{x} \wedge \mathbf{T}^t \mathbf{n} da = \int_{\mathcal{R}} (\mathbf{x} \wedge div(\mathbf{T}) - \boldsymbol{\tau}) dv,$$

where $\mathbf{x}$ is the position of a representative point of $\mathcal{R}$ and $\boldsymbol{\tau}$ is the axial vector of $\mathbf{T} - \mathbf{T}^t$. (Hint: Make use of the identity obtained in problem 23.)

25. Let $(r, \theta, \phi)$ denote the spherical coordinates of a typical point related to the Cartesian coordinates $(x, y, z)$ through $x = r\cos(\theta)\cos(\phi)$, $y = r\cos(\theta)\sin(\phi)$, $z = r\sin(\theta)$. Then express the spherical coordinate basis in terms of the Cartesian basis and show that

$$
\begin{aligned}
grad(\alpha) &= \frac{\partial \alpha}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \alpha}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\cos(\theta)}\frac{\partial \alpha}{\partial \phi}\mathbf{e}_\phi, \\
grad(\mathbf{v}) &= \begin{pmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{1}{r\cos(\theta)}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{1}{r\cos(\theta)}\frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{r}\tan(\theta) \\ \frac{\partial v_\phi}{\partial r} & \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} & \frac{1}{r\cos(\theta)}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} - \frac{v_\theta}{r}\tan(\theta) \end{pmatrix},
\end{aligned}
$$

$$
div(\mathbf{T}) =
\left\{ \begin{array}{c} \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{1}{r\cos(\theta)}\frac{\partial T_{\phi r}}{\partial \phi} + \frac{1}{r}[2T_{rr} - T_{\theta\theta} - T_{\phi\phi}] \\ \frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{1}{r\cos(\theta)}\frac{\partial T_{\phi\theta}}{\partial \phi} + \frac{1}{r}[2T_{r\theta} + T_{\theta r} + (T_{\theta\theta} - T_{\phi\phi})\tan(\theta)] \\ \frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\phi}}{\partial \theta} + \frac{1}{r\cos(\theta)}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{1}{r}[2T_{r\phi} + T_{\phi r} + (T_{\phi\theta} + T_{\theta\phi})\tan(\theta)] \end{array} \right\},
$$

where $\alpha$ is a scalar field, $\mathbf{v}$ is a vector field represented in spherical coordinates and $\mathbf{T}$ is a tensor field also represented using spherical coordinate basis vectors.

# Chapter 3

# Kinematics

## 3.1 Overview

Kinematics is the study of motion, regardless of what is causing it. All bodies in this universe are in motion. Hence, it becomes important to understand how the motion can be described or abstracted mathematically. First, we shall strive to develop an intuitive understanding of certain primitive concepts, namely, body, position and time. Then, we shall proceed to define motion, displacement and velocity fields. Here we shall see that there are two ways of mathematically representing these fields, called the material description and spatial description. Then, we shall dwell on kinematical considerations which are basic to solid mechanics.

## 3.2 Body

Intuitively we understand what a body is. We think that the body is made of a number of particles. Picking up some arbitrary particle we can talk about particles that are on top or below this particle, in front or behind this particle, and to the right or left of this particle. This relationship between the particles is maintained in all motions of the body, that is if two particles is on two different sides of another particle, these two particles stay on either side of the other particle for any motion of the body. Therefore, we say that the body is made up of ordered particles. If the number of particles is countless then the body is described as a continuum and when it is countable it is described to be discrete.

Now, we want to express these intuitive notions of the body in the language of mathematics. The mathematical analogue of the material particle is the point. Mathematically, by point we mean a collection of ‘n’ (real) numbers, for our purposes here. Assuming that we have a scale to measure the distance, then to identify other particles from a given particle, we need to give three distances, namely the distance on top, to the right and in the front of that particular particle. Hence, for each point we have to associate three numbers corresponding to the three distances mentioned above. Therefore the points are said to belong to a three dimensional space, since the value of $n$ is 3. If we use a linear scale, which is the usual practice, to measure the distance then we say that the points belong to a 3D Euclidean space or more specifically 3D Euclidean translation space, also sometimes referred to as 3D Euclidean point space. Further, in the language of mechanics, this mapping of the particles to points in the 3D Euclidean space is called a placer and is denoted by $\kappa$.

Though one might understand what a body is intuitively, some choices made here to mathematically represent it needs deliberation. For this we need to understand the mathematical notion of a “point”. Though we associate - $\cdot$ - it as a point. It is not. $\cdot$ is at best a collection of points. Point by definition is a limiting sphere obtained when the radius of the sphere tends to zero. Thus one cannot see a point; its volume is also zero. Stacking of points along one direction alone yields a line, stacking of lines along a direction perpendicular to the direction of the line yields a plane, stacking of planes along the normal direction to the plane yields a 3D cube which alone can be seen and has a volume. Thus, one should realize that by stacking countless number of points of zero volume we have created a finite volume, as a consequence of limiting process. Consequently, the same[1] countless number of points could occupy different volumes at different instances in time.

It should be clear from the above discussion that particle is not a chunk of material that many believe it to be. This is because the particle is mapped on to the mathematical idea of a point which as we saw above has zero volume.

Let $\mathcal{B}$ denote the abstract body, that is the set of particles that constitute the body. Assuming that the body is made of only 8 particles we write

$$\mathcal{B} = \{(X, Y, Z) | X \in \{0, 1\}, Y \in \{0, 1\}, Z \in \{0, 1\}\}, \tag{3.1}$$

[1]When the number of points is countless, using this adjective same is a oxymoron. But it is used to convey an idea.

meaning that the eight particles are mapped on to the points $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, $(0, 1, 1)$. The right hand side of the equation (3.1) means the following: Take an element each from the set X, Y and Z and form a new set called $\mathcal{B}$ which is the collection of these three elements. Mathematically, this is stated as: the set $\mathcal{B}$ is obtained by taking the Cartesian product[2] of the sets $X$, $Y$ and $Z$. The elements in the set X are 0 and 1 and that in the set Y and Z are also 0 and 1. The elements in the set need not be discrete they can be continuous also. In fact, they have to be continuous when the body is a continuum.

Thus, for the body made up of countless number of material particles, we write

$$\mathcal{B} = \{(X, Y, Z) | 0 \leq X \leq 1, 0 \leq Y \leq 1, 0 \leq Z \leq 1\}. \tag{3.2}$$

By this, we mean that each point within the unit cube is occupied by a material particle belonging to the body. Since the body occupies contiguous region in space it is apt that it is described as a continuum. Pictorially this configuration of the body is represented as shown in figure 3.1a.

It is just incidental that one of the vertex of the cube coincides with the point $(0, 0, 0)$. Another person can map the same body onto a different region of the Euclidean space and thus he might write

$$\mathcal{B} = \{(X, Y, Z) | X_0 \leq X \leq X_0+1, Y_0 \leq Y \leq Y_0+1, Z_0 \leq Z \leq Z_0+1\}, \tag{3.3}$$

where, $X_0$, $Y_0$, $Z_0$ are some constants. This representation of the unit cube is shown pictorially in figure 3.1b. In fact, it is not necessary that the cube be oriented such that the normal to its faces coincide with the Cartesian coordinate basis, it can make an angle with the coordinate basis too, as shown in figure 3.1c. This configuration of the body is analytically expressed as:

$$\begin{aligned} \mathcal{B} = \{(X\cos(\theta) + Y\sin(\theta) - X_0, -X\sin(\theta) + Y\cos(\theta) - Y_0, Z - Z_0) \\ | 0 \leq X \leq 1, 0 \leq Y \leq 1, 0 \leq Z \leq 1\}, \end{aligned} \tag{3.4}$$

where, $X_0$, $Y_0$, $Z_0$ and $\theta$ are some constants.

Before proceeding further, we recap the properties of the placers. First, the placers are one to one functions that is each material particle gets mapped

[2]The Cartesian product of two sets X and Y, denoted $X \times Y$, is the set of all possible ordered pairs whose first component is a member of X and whose second component is a member of Y.

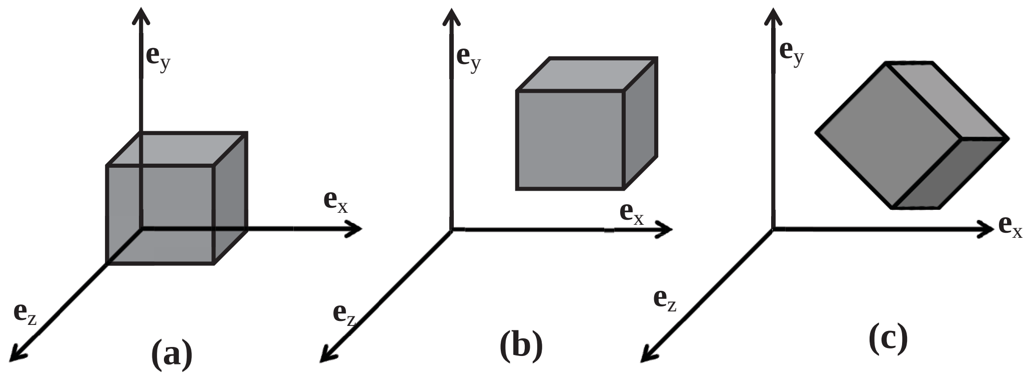



Figure 3.1: Schematic of cube oriented in space such that (a) One of the corners coincides with the origin and the normal to the faces of the cube are oriented parallel to the Cartesian basis (b) None of the corners coincides with the origin but the normal to the faces of the cube are oriented parallel to the Cartesian basis (c) None of the corners coincides with the origin and the normal to the faces of the cube are not oriented parallel to the Cartesian basis

on to a unique point in the 3D Euclidean space. However, there is no unique placer for a given body. Different persons can map a given body to different regions of Euclidean point space. We shall later see how this non-uniqueness of placers is built into the theory developed to describe the mechanical response of the bodies.

While defining bodies in equations (3.1) through (3.4) we tacitly assumed that the point space corresponded to Cartesian coordinates. This specification of the coordinate system is important to arrive at the formula to be used to compute the distance between two triplets. Recognize that if we are using the same formula to obtain the distance between two triplets in different coordinate systems, it is equivalent to using different scales to measure the distance.

Many a times, the choice of coordinate system is made so that we can define the body succinctly. For example, consider a body in the shape of an annular (or solid) right circular cylinder of length, $H$. Using cylindrical polar coordinates, $(R, \Theta, Z)$, this body is mathematically defined as:

$$\mathcal{B} = \{(R, \Theta, Z) | R_i \leq R \leq R_o, 0 \leq \Theta \leq 2\pi, 0 \leq Z \leq H\}, \tag{3.5}$$

where $R_i$ and $R_o$ are some positive constants, with $R_i = 0$ for a solid cylinder. Try defining this body using Cartesian coordinates.

Experience has shown that it is easier to do computations if we use vector space instead of point space. Naively, while in vector spaces we speak of position vectors of a point, in point space we speak about the coordinates of a point. Next, we would like to establish this relationship between the mathematical ideas of the vector space and point space. To do this, for each ordered pair of points $(a, b)$ in the Euclidean point space $\mathcal{E}$ there corresponds a unique vector in the Euclidean vector space, $\mathfrak{V}$, denoted by $\vec{ab}$, with the properties

(i) $\vec{ba} = -\vec{ab}, \qquad \forall\ a,\, b \in \mathcal{E}$.

(ii) $\vec{ab} = \vec{ac} + \vec{cb}, \qquad \forall\ a,\, b,\, c \in \mathcal{E}$.

(iii) Choosing arbitrarily a point $o$ from $\mathcal{E}$, and a Cartesian coordinate basis to span the vector space, $\mathfrak{V}$, there corresponds to each vector $\mathbf{a} \in \mathfrak{V}$ a unique point $a \in \mathcal{E}$ such that $\mathbf{a} = \vec{oa}$, the point $o$ is called the origin and $\mathbf{a}$ the position of $a$ relative to $o$.

In other words, property (iii) states that the components of any vector $\mathbf{a}$ $(a_1, a_2, a_3)$ relative to a chosen Cartesian coordinate basis correspond to the coordinates of some point $a$ in $\mathcal{E}$. The fact that this holds only for Cartesian coordinate system cannot be overemphasized. To see this, recall that in cylindrical polar coordinate system even though a triplet $(R, \Theta, Z)$ characterizes a point in $\mathcal{E}$, position vectors have components of the form $(R, 0, Z)$ only. That is the components of the position vector of a point is not same as the coordinates of the point.

The notions of distance and angle in $\mathcal{E}$ are derived from the scalar product on the supporting vector space $\mathfrak{V}$: the distance between the arbitrary points $a$ and $b$ is defined by $|\vec{ab}|$ and the angle, $\alpha$ subtended by $a$ and $b$ at a third arbitrary point $c$ by

$$\alpha = \cos^{-1}\left(\frac{\vec{ca}\cdot\vec{cb}}{|\vec{ca}||\vec{cb}|}\right), \quad (0 \le \phi \le \pi). \tag{3.6}$$

Thus, the choice of scale corresponds to the different expressions for the scalar product.

Considering the physical body to be made up of material particles, denoted by $P$, we have seen how it can be mapped to the 3D Euclidean point space and hence to the 3D Euclidean vector space. Thus, the region occupied by this body in the 3D Euclidean vector space is called as the configuration of the body and is denoted by $\mathfrak{B}$. Then, the placing function $\kappa$: $\mathcal{B} \to \mathfrak{B}$ and its inverse $\kappa^{-1}$: $\mathfrak{B} \to \mathcal{B}$ are defined as:

$$\mathbf{X} = \kappa(P), \qquad P = \kappa^{-1}(\mathbf{X}). \tag{3.7}$$

Recognize that the inverse exist because the mappings are one to one by definition. Henceforth, by placer we mean placing the body in the Euclidean vector space not the point space.

Next, let us understand what time is. Time concerns with the ordering of events. That is, it tells us whether an event occurred before or after a particular event. We shall assume that we cannot count the number of events. Hence, we map the events to the points on a real line. This assumption that the number of events is countless, is required so that we can take derivatives with respect to time. The part of the real line that is used to map a set of events is the prerogative of the person who establishes this correspondence which is similar to the mapping of the body on to the 3D Euclidean point space. The motion of a body is considered to be an event in mechanics. When a body moves, its configuration changes, that is the region occupied by the body in the 3D Euclidean vector space changes.

Let $t$ be a real variable denoting time such that $t \in \mathcal{I} \subseteq \mathcal{R}$, set of reals. If we could associate a configuration, $\mathfrak{B}_t$ for the body $\mathcal{B}$ for each instant of time in the interval of interest then the family of configurations $\{\mathfrak{B}_t : t \in \mathcal{I}\}$ is called motion of $\mathcal{B}$. Hence, we can define functions $\phi$: $\mathcal{B} \times \mathcal{I} \to \mathfrak{B}_t$ and $\phi^{-1}$: $\mathfrak{B}_t \times \mathcal{I} \to \mathcal{B}$ such that

$$\mathbf{x} = \boldsymbol{\phi}(P, t), \qquad P = \boldsymbol{\phi}^{-1}(\mathbf{x}, t). \tag{3.8}$$

In a motion of $\mathcal{B}$ a typical particle, $P$ occupies a succession of points which together form a curve in $\mathcal{E}$. This curve is called the path of $P$ and is given in a parametric manner by equation (3.8a). The velocity and acceleration of $P$ are defined as rate of change of position and velocity with time, respectively, as $P$ traverses its path.

While the definition of motion by (3.8a) is satisfactory, it would be useful if it were to be a function of the vectors instead of points. To achieve this we make use of the equation (3.7b). That is we choose a certain configuration of the body and identify the particles by the position vector of the point it occupies in this configuration, then

$$\mathbf{x} = \boldsymbol{\phi}(\kappa^{-1}(\mathbf{X}), t) = \boldsymbol{\chi}_\kappa(\mathbf{X}, t). \tag{3.9}$$

$\boldsymbol{\chi}_\kappa$ is called the motion field and the subscript $\kappa$ denotes that it depends on the configuration which is used to identify the particles. The configuration in which the particles are identified is called the reference configuration and is denoted by $\mathfrak{B}_r$. This definition does not require the reference configuration to be a configuration actually occupied by the body during its motion. If the reference configuration were to be a configuration actually occupied by the body during its motion at some time $t_o$, then (3.9) could be obtained from (3.8) as

$$\mathbf{x} = \boldsymbol{\phi}(\boldsymbol{\phi}^{-1}(\mathbf{X}, t_o), t) = \boldsymbol{\chi}_{t_o}(\mathbf{X}, t). \tag{3.10}$$

We call $\boldsymbol{\chi}_\kappa$ (or $\boldsymbol{\chi}_{t_o}$) the deformation field when it is independent of time or its dependence on time is irrelevant.

Now, we would like to make a few remarks. The definition of a body and motion is independent of whether we are concerned with rigid body or deformable body. However, certain structures or details impounded on the body depends on this choice. Till now, we have been talking about the body being a set of particles but to be precise we should have said material particles. This detail is important within the context of deformable bodies, as we shall see later. It should also be remembered that the distinction that a given particle is steel or wood is made right in the beginning. Thus, within the realms of classical mechanics we cannot model chemically reacting body where material particles of a particular type gets transformed into another. Because the body is considered to be a fixed set of material particles, they cannot die or be born or get transformed.

As we shall see later, an important kinematical quantity in the study of deformable bodies is the gradient of motion. For us to be able to find the gradient of motion, we require the motion field, (3.9) to be continuous on $\mathcal{B}$ and there should exist countless material particles of the same type (material) in spheres of radius, $r$ for any value of $r$ less than $\delta_o$ where $\delta_o$ is some constant. It is for this purpose, in the study of deformable bodies we require the body to be a continuum.

## 3.3 Deformation Gradient

The gradient of motion is generally called the deformation gradient and is denoted by $\mathbf{F}$. Thus

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \nabla(\mathbf{x}). \tag{3.11}$$

Since, $\boldsymbol{\chi}$ is a function of both $\mathbf{X}$ and $t$ we have used a partial derivative in the definition of the deformation gradient. Also, we haven't defined it as $Grad(\mathbf{x})$ because for the $Grad$ operator, as defined in chapter 2, the range of the function for which gradient is sought is any vector; not just position vectors. The difference becomes evident in curvilinear coordinate systems like the cylindrical polar coordinates.

Let $\{\mathbf{E}_i\}$ be the three Cartesian basis vectors in the reference configuration and $\{\mathbf{e}_i\}$ the basis vectors in the current configuration. Then, the deformation gradient is written as

$$\mathbf{F} = F_{ij}\mathbf{e}_i \otimes \mathbf{E}_j. \tag{3.12}$$

In general, the basis vectors $\mathbf{e}_i$ and $\mathbf{E}_j$ need not be the same. Since the deformation gradient depends on two sets of basis vectors, it is called a two-point tensor. It is pertinent here to point out that the *grad* operator as defined in chapter 2 (2.207), is not a two-point tensor either. The matrix components of the deformation gradient in Cartesian coordinate system is

$$(\mathbf{F})_{ij} = \begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{pmatrix}, \tag{3.13}$$

where $(X, Y, Z)$ and $(x, y, z)$ are the Cartesian coordinates of a typical material particle, $P$ in the reference and current configuration respectively. Similarly, the matrix components of the deformation gradient in cylindrical polar coordinate system is:

$$(\mathbf{F})_{ij} = \begin{pmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ r\frac{\partial \theta}{\partial R} & \frac{r}{R}\frac{\partial \theta}{\partial \Theta} & r\frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{pmatrix}, \tag{3.14}$$

where $(R, \Theta, Z)$ and $(r, \theta, z)$ are the cylindrical polar coordinates of a typical material particle, $P$ in the reference and current configuration respectively. Substituting

$$\begin{aligned} X = R\cos(\Theta), \qquad Y = R\sin(\Theta), \qquad Z = Z, \\ x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad z = z, \end{aligned} \tag{3.15}$$

in (3.13) we obtain

$$\left\{\begin{array}{c} (\mathbf{F})_{xX} \\ (\mathbf{F})_{xY} \\ (\mathbf{F})_{xZ} \\ (\mathbf{F})_{yX} \\ (\mathbf{F})_{yY} \\ (\mathbf{F})_{yZ} \\ (\mathbf{F})_{zX} \\ (\mathbf{F})_{zY} \\ (\mathbf{F})_{zZ} \end{array}\right\} = \left\{\begin{array}{c} \frac{\partial r}{\partial X}\cos(\theta) - r\sin(\theta)\frac{\partial \theta}{\partial X} \\ \frac{\partial r}{\partial Y}\cos(\theta) - r\sin(\theta)\frac{\partial \theta}{\partial Y} \\ \frac{\partial r}{\partial Z}\cos(\theta) - r\sin(\theta)\frac{\partial \theta}{\partial Z} \\ \frac{\partial r}{\partial X}\sin(\theta) + r\cos(\theta)\frac{\partial \theta}{\partial X} \\ \frac{\partial r}{\partial Y}\sin(\theta) + r\cos(\theta)\frac{\partial \theta}{\partial Y} \\ \frac{\partial r}{\partial Z}\sin(\theta) + r\cos(\theta)\frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial X} \\ \frac{\partial z}{\partial Y} \\ \frac{\partial z}{\partial Z} \end{array}\right\} \tag{3.16}$$

where

$$\frac{\partial(\cdot)}{\partial X} = \cos(\Theta)\frac{\partial(\cdot)}{\partial R} - \frac{\sin(\Theta)}{R}\frac{\partial(\cdot)}{\partial \Theta}, \tag{3.17}$$

$$\frac{\partial(\cdot)}{\partial Y} = \sin(\Theta)\frac{\partial(\cdot)}{\partial R} + \frac{\cos(\Theta)}{R}\frac{\partial(\cdot)}{\partial \Theta}, \tag{3.18}$$

$$\frac{\partial(\cdot)}{\partial Z} = \frac{\partial(\cdot)}{\partial Z}, \tag{3.19}$$

from which we obtain (3.14) recognizing that

$$\begin{array}{ll} \mathbf{E}_R = \cos(\Theta)\mathbf{E}_1 + \sin(\Theta)\mathbf{E}_2, & \mathbf{e}_r = \cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2, \\ \mathbf{E}_\Theta = -\sin(\Theta)\mathbf{E}_1 + \cos(\Theta)\mathbf{E}_2, & \mathbf{e}_\theta = -\sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_2, \\ \mathbf{E}_Z = \mathbf{E}_3, & \mathbf{e}_z = \mathbf{e}_3, \end{array}$$

where $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are the cylindrical polar coordinate basis vectors obtained from (2.242) using (3.15). Comparing equations (3.14) with (2.259) we see the difference between the *Grad* operator and $\nabla$ operator.

## 3.4 Lagrangian and Eulerian description

In the development of the basic principles of continuum mechanics, a body $\mathcal{B}$ is endowed with various physical properties which are represented by scalar or tensor fields, defined either on a reference configuration, $\mathfrak{B}_r$ or on the current configuration $\mathfrak{B}_t$. In the former case, the independent variables are the position vectors of the particles in the reference configuration, $\mathbf{X}$ and time, $t$. This characterization of the field with $\mathbf{X}$ and $t$ as independent variables is called Lagrangian (or material) description. In the latter case, the independent variables are the position vectors of the particles in the current configuration, $\mathbf{x}$ and time, $t$. The characterization of the field with $\mathbf{x}$ and $t$ as independent variables is called Eulerian (or spatial) description. Thus, density, displacement and stress are examples of scalar, vector and second order tensor fields respectively and can be represented as

$$\rho = \hat{\rho}(\mathbf{X}, t) = \tilde{\rho}(\mathbf{x}, t), \tag{3.20}$$

$$\mathbf{u} = \hat{\mathbf{u}}(\mathbf{X}, t) = \tilde{\mathbf{u}}(\mathbf{x}, t), \tag{3.21}$$

$$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{X}, t) = \tilde{\boldsymbol{\sigma}}(\mathbf{x}, t), \tag{3.22}$$

where the second equality is obtained by using the relation $\mathbf{X} = \boldsymbol{\chi}_\kappa^{-1}(\mathbf{x}, t)$, which is possible because the function $\boldsymbol{\chi}_\kappa$ in (3.9) is one to one. Here functions with a hat denote that the independent variables are $\mathbf{X}$ and $t$, while functions with a tilde denote that the independent variables are $\mathbf{x}$ and $t$.

To understand the difference between the Lagrangian description and Eulerian description, consider the flow of water through a pipe from a large tank. Now, if we seed the tank with micro-spheres and determine the velocity of these spheres as a function of time and their initial position in the tank then we get the Lagrangian description for the velocity. On the other hand, if we choose a point in the pipe and determine the velocity of the particles crossing that point as a function of time then we get the Eulerian description of velocity. While in fluid mechanics we use the Eulerian description, in solid mechanics we use Lagrangian description, mostly.

Next, we define what is called as the material time derivative or Lagrangian time derivative or total time derivative and spatial time derivative or Eulerian time derivative. It could be inferred from the above that various variables of interest are functions of time, $t$ and either $\mathbf{X}$ or $\mathbf{x}$. Hence, when we differentiate these variables with respect to time, we can hold either $\mathbf{X}$ a constant or $\mathbf{x}$ a constant. If we hold $\mathbf{X}$ a constant while differentiating with time, we call such a derivative total time derivative and denote it by $\frac{D(\cdot)}{Dt}$. On the other hand if we hold $\mathbf{x}$ a constant, we call it spatial time derivative and denote it by $\frac{d(\cdot)}{dt}$. Thus, for the scalar field defined by, say (3.20) we have

$$\frac{D\rho}{Dt} = \frac{\partial \hat{\rho}}{\partial t} = \frac{\partial \tilde{\rho}}{\partial t} + grad(\tilde{\rho}) \cdot \mathbf{v}, \tag{3.23}$$

$$\frac{d\rho}{dt} = \frac{\partial \tilde{\rho}}{\partial t} = \frac{\partial \hat{\rho}}{\partial t} + Grad(\hat{\rho}) \cdot \frac{d\mathbf{X}}{dt}, \tag{3.24}$$

where $grad(\cdot)$ stands for the gradient with respect to $\mathbf{x}$, $Grad(\cdot)$ stands for the gradient with respect to $\mathbf{X}$ and $\mathbf{v} = \frac{D\mathbf{x}}{Dt}$. The above equations are obtained by using the chain rule. Similarly, for a vector field defined by (3.21) we have

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \hat{\mathbf{u}}}{\partial t} = \frac{\partial \tilde{\mathbf{u}}}{\partial t} + grad(\tilde{\mathbf{u}})\mathbf{v}, \tag{3.25}$$

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \tilde{\mathbf{u}}}{\partial t} = \frac{\partial \hat{\mathbf{u}}}{\partial t} + Grad(\hat{\mathbf{u}})\frac{d\mathbf{X}}{dt}. \tag{3.26}$$

Recognize that $\frac{D\mathbf{X}}{Dt} = \mathbf{o}$ and $\frac{d\mathbf{x}}{dt} = \mathbf{o}$.

Finally, a note on the notation. If the starting letter is capitalized for any of the operators introduced in chapter 2 then it means that the derivative is with respect to the material coordinates, i.e. $\mathbf{X}$ otherwise, the derivative is with respect to the spatial coordinates, i.e. $\mathbf{x}$. In the above, we have already used this convention for the gradient operator.

## 3.5 Displacement, velocity and acceleration

The vector field, $\mathbf{u}$, defined as

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \tag{3.27}$$

represents the displacement field, that is the displacement of the material particles initially at $\mathbf{X}$ or currently at $\mathbf{x}$ depending on whether Lagrangian or Eulerian description is used. Thus,

$$\hat{\mathbf{u}}(\mathbf{X}, t) = \boldsymbol{\chi}_\kappa(\mathbf{X}, t) - \mathbf{X}, \tag{3.28}$$

$$\tilde{\mathbf{u}}(\mathbf{x}, t) = \mathbf{x} - \boldsymbol{\chi}_\kappa^{-1}(\mathbf{x}, t). \tag{3.29}$$

Figure 3.2 schematically shows the displacement vector. As shown in the figure, displacement vector is a straight line segment joining the location of a material particle in the current and reference configuration. While, in the schematic we have used the same coordinate system to describe all the three directed line segments namely, the position vector of a particle in the reference configuration, current configuration and the displacement vector; this is not necessary. In fact, if one uses cylindrical polar coordinate basis, the position vector of the material particle in the current and reference configuration, in general would be different.

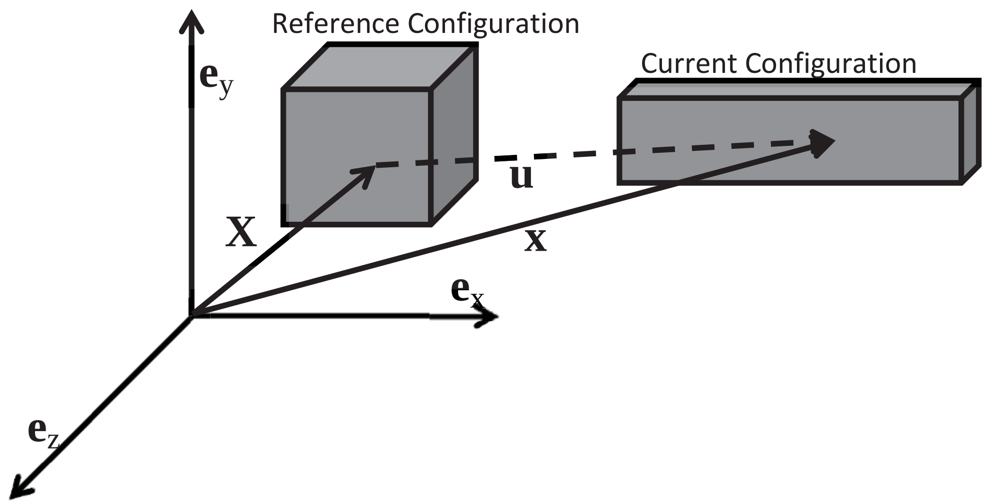



Figure 3.2: Schematic of deformation of a cube ($\mathbf{X}$ position vector of a typical material particle in the reference configuration, $\mathbf{x}$ position vector of the same material particle whose position vector in the reference configuration is $\mathbf{X}$ in the current configuration, $\mathbf{u}$ displacement vector of this material particle).

The first and second total time derivatives of the motion field is called the velocity and acceleration respectively. Thus

$$\mathbf{v} = \frac{D\boldsymbol{\chi}}{Dt}, \qquad \mathbf{a} = \frac{D^2\boldsymbol{\chi}}{Dt^2} = \frac{D\mathbf{v}}{Dt}. \tag{3.30}$$

We note that both the velocity and acceleration can be expressed as a Lagrangian field or Eulerian field. From the definition of the displacement (3.27) it can be seen that the velocity and acceleration can be equivalently written as,

$$\mathbf{v} = \frac{D\mathbf{u}}{Dt}, \qquad \mathbf{a} = \frac{D^2\mathbf{u}}{Dt^2}. \tag{3.31}$$

### 3.5.1 Gradient of displacement

As we have seen before, the displacement field can be expressed as a Lagrangian field or Eulerian field. Thus, we can have a Lagrangian displacement gradient, $\mathbf{H}$ and a Eulerian displacement gradient, $\mathbf{h}$ defined as

$$\mathbf{H} = Grad(\hat{\mathbf{u}}), \qquad \mathbf{h} = grad(\tilde{\mathbf{u}}). \tag{3.32}$$

Recognize that these gradients are not the same. To see this, we compute the displacement gradient in terms of the deformation gradient as

$$\mathbf{H} = Grad(\hat{\mathbf{u}}) = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} - \frac{\partial \mathbf{X}}{\partial \mathbf{X}} = \mathbf{F} - \mathbf{1}, \tag{3.33}$$

$$\mathbf{h} = grad(\tilde{\mathbf{u}}) = \frac{\partial \mathbf{x}}{\partial \mathbf{x}} - \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \mathbf{1} - \mathbf{F}^{-1}, \tag{3.34}$$

### 3.5.2 Example

A certain motion of a continuum body in the material description is given in the form:

$$x_1 = X_1 - \exp(-t)X_2, \qquad x_2 = \exp(-t)X_1 + X_2, \qquad x_3 = X_3, \tag{3.35}$$

for $t > 0$. Find the displacement, velocity and acceleration components in terms of the material and spatial coordinates and time. Also find the deformation and displacement. Assume that the same Cartesian coordinate basis and origin is used to describe the body both in the current and the reference configuration.

It follows from (3.27) that

$$\hat{\mathbf{u}} = -\exp(-t)X_2\mathbf{E}_1 + \exp(-t)X_1\mathbf{E}_2. \tag{3.36}$$

Similarly, from (3.30) we obtain

$$\hat{\mathbf{v}} = \exp(-t)X_2\mathbf{E}_1 - \exp(-t)X_1\mathbf{E}_2, \tag{3.37}$$

$$\hat{\mathbf{a}} = -\exp(-t)X_2\mathbf{E}_1 + \exp(-t)X_1\mathbf{E}_2, . \tag{3.38}$$

Inverting the motion field (3.35) we obtain

$$X_1 = \frac{[x_1 + \exp(-t)x_2]}{[1 + \exp(-2t)]}, \qquad X_2 = \frac{[x_2 - \exp(-t)x_1]}{[1 + \exp(-2t)]}, \qquad X_3 = x_3. \tag{3.39}$$

Substituting (3.39) in (3.36), (3.37) and (3.38) we obtain the Eulerian form of the displacement, velocity and acceleration fields as

$$\begin{aligned}
\tilde{\mathbf{u}} &= -\frac{[x_2 - \exp(-t)x_1]}{[1 + \exp(-2t)]}\exp(-t)\mathbf{E}_1 + \frac{[x_1 + \exp(-t)x_2]}{[1 + \exp(-2t)]}\exp(-t)\mathbf{E}_2, \\
\tilde{\mathbf{v}} &= \frac{[x_2 - \exp(-t)x_1]}{[1 + \exp(-2t)]}\exp(-t)\mathbf{E}_1 - \frac{[x_1 + \exp(-t)x_2]}{[1 + \exp(-2t)]}\exp(-t)\mathbf{E}_2, \\
\tilde{\mathbf{a}} &= -\frac{[x_2 - \exp(-t)x_1]}{[1 + \exp(-2t)]}\exp(-t)\mathbf{E}_1 + \frac{[x_1 + \exp(-t)x_2]}{[1 + \exp(-2t)]}\exp(-t)\mathbf{E}_2.
\end{aligned}$$

Here we have made use of the fact that the same Cartesian basis vectors are used to define both the current and the reference configuration.

Next, we compute the deformation gradient to be

$$\mathbf{F} = \begin{pmatrix} 1 & -\exp(-t) & 0 \\ \exp(-t) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.40}$$

and the Lagrangian and Eulerian displacement gradient as

$$\mathbf{H} = \begin{pmatrix} 0 & -\exp(-t) & 0 \\ \exp(-t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.41}$$

$$\mathbf{h} = \begin{pmatrix} \frac{\exp(-2t)}{[1+\exp(-2t)]} & -\frac{\exp(-t)}{[1+\exp(-2t)]} & 0 \\ \frac{\exp(-t)}{[1+\exp(-2t)]} & \frac{\exp(-2t)}{[1+\exp(-2t)]} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.42}$$

Notice that at $t = 0$, $\mathbf{F} \neq \mathbf{1}$, but $\mathbf{F}$ tends to $\mathbf{1}$ as time, $t$ tends to $\infty$. This just indicates that the configuration chosen as reference is not the one at time $t = 0$, but at some other time and this is permissible.

## 3.6 Transformation of curves, surfaces and volume

Before rigorously deriving the general expressions for transformation of curves, surfaces and volumes, let us obtain these expressions after various simplifying assumptions.

Let us assume the following:

1. The same Cartesian coordinate basis and origin is used to describe the body in both its reference and current configuration.

2. The body is subjected to displacement $\mathbf{u}$ and the Lagrangian description of the displacement field is known and is sufficiently smooth for the required derivatives to exist. That is

$$\mathbf{u} = u_x(X,Y,Z)\mathbf{e}_x + u_y(X,Y,Z)\mathbf{e}_y + +u_z(X,Y,Z)\mathbf{e}_z, \tag{3.43}$$

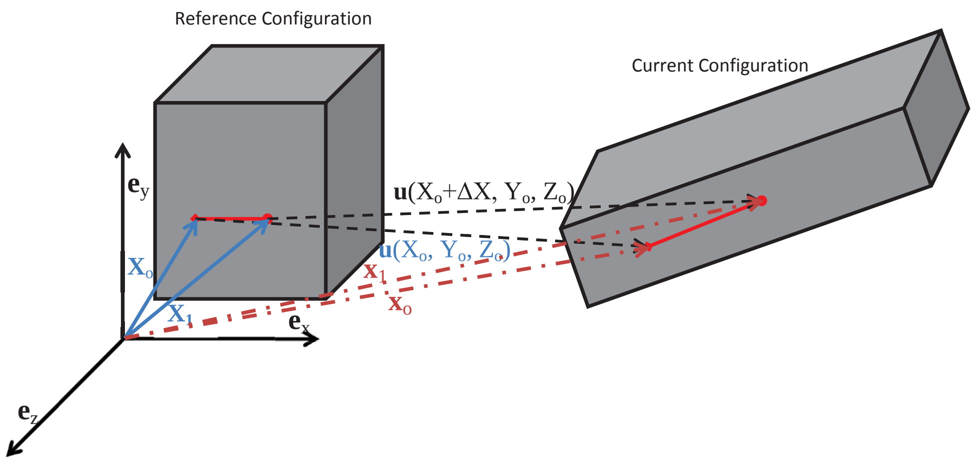



Figure 3.3: Schematic of the deformation of a line oriented along $\mathbf{e}_x$ of length $\Delta X$.

where $u_x(X,Y,Z)$, $u_y(X,Y,Z)$ and $u_z(X,Y,Z)$ are some known functions of the $X$, $Y$ and $Z$ the coordinates of the material particles in the reference configuration.

3. It is assumed that the components of the displacement can be approximated as

$$
\begin{aligned}
&u_x(X_o+\Delta X, Y_o+\Delta Y, Z_o+\Delta Z) \approx u_x(X_o,Y_o,Z_o)\\
&\qquad + \left.\frac{\partial u_x}{\partial X}\right|_{(X_o,Y_o,Z_o)} \Delta X + \left.\frac{\partial u_x}{\partial Y}\right|_{(X_o,Y_o,Z_o)} \Delta Y + \left.\frac{\partial u_x}{\partial Z}\right|_{(X_o,Y_o,Z_o)} \Delta Z,\\
&u_y(X_o+\Delta X, Y_o+\Delta Y, Z_o+\Delta Z) \approx u_y(X_o,Y_o,Z_o)\\
&\qquad + \left.\frac{\partial u_y}{\partial X}\right|_{(X_o,Y_o,Z_o)} \Delta X + \left.\frac{\partial u_y}{\partial Y}\right|_{(X_o,Y_o,Z_o)} \Delta Y + \left.\frac{\partial u_y}{\partial Z}\right|_{(X_o,Y_o,Z_o)} \Delta Z,\\
&u_z(X_o+\Delta X, Y_o+\Delta Y, Z_o+\Delta Z) \approx u_z(X_o,Y_o,Z_o)\\
&\qquad + \left.\frac{\partial u_z}{\partial X}\right|_{(X_o,Y_o,Z_o)} \Delta X + \left.\frac{\partial u_z}{\partial Y}\right|_{(X_o,Y_o,Z_o)} \Delta Y + \left.\frac{\partial u_z}{\partial Z}\right|_{(X_o,Y_o,Z_o)} \Delta Z,
\end{aligned}
\tag{3.44}
$$

where the displacement of the material particle occupying the point $\mathbf{X}_o$ is $\mathbf{u}(X_o, Y_o, Z_o)$ and that displacement of the material particle that is occupying, $\mathbf{X}_1$ is $\mathbf{u}(X_o + \Delta X, Y_o + \Delta Y, Z_o + \Delta Z)$. Here it is pertinent to point out that the above is a truncated Taylor's series. Since, we are interested in arbitrarily small values of $\Delta X$, $\Delta Y$, $\Delta Z$, we have truncated the Taylor's series after the first term. However, this is not a valid approximation for many functions. For example, functions of the form $aX^{3/2}$, where $a$ is a constant defined over the domain, say $0.01 \leq X \leq 0.1$ the value of the derivatives greater then second order would be more than the first order derivative. Hence, the expression derived based on this assumption, though intuitive has its limitations.

First, we are interested in finding how the length of the straight line of length $\Delta X$ oriented along the $\mathbf{e}_x$ direction in the reference configuration of the body, as shown in the figure 3.3 has changed. With the above assumptions, the deformed length of the straight line of length $\Delta X$ oriented along the $\mathbf{e}_x$ direction in the reference configuration is given by,

$$\begin{aligned} \|\mathbf{x}_1 - \mathbf{x}_o\| &= \|\mathbf{X}_1 - \mathbf{X}_o + \mathbf{u}(\mathbf{X}_1) - \mathbf{u}(\mathbf{X}_o)\| \\ &= \Delta X \sqrt{\left[1 + \frac{\partial u_x}{\partial X}\right]^2 + \left[\frac{\partial u_y}{\partial X}\right]^2 + \left[\frac{\partial u_z}{\partial X}\right]^2}, \end{aligned} \tag{3.45}$$

where $\mathbf{X}_1 = \mathbf{X}_o + \Delta X \mathbf{e}_x$, $\mathbf{x}_o$ denotes the current position vector of the material particle whose position in the reference configuration is $\mathbf{X}_o$, $\mathbf{x}_1$ denotes the current position vector of the material particle whose position in the reference configuration is $\mathbf{X}_1$ and use is made of equation (3.44). Now, if the magnitude of the components of the gradient of the displacement are small (say of the magnitude of $10^{-3}$), then equation (3.45) could be approximately calculated as,

$$\|\mathbf{x}_1 - \mathbf{x}_o\| = \Delta X \left[1 + \frac{\partial u_x}{\partial X}\right]. \tag{3.46}$$

Hence, the stretch ratio defined as the ratio of the current length to the undeformed length for a line element oriented along the $\mathbf{e}_x$ direction is,

$$\begin{aligned} \Lambda_{(\mathbf{e}_x)} &= \frac{\|\mathbf{x}_1 - \mathbf{x}_o\|}{\|\mathbf{X}_1 - \mathbf{X}_o\|} \\ &= \sqrt{\left[1 + \frac{\partial u_x}{\partial X}\right]^2 + \left[\frac{\partial u_y}{\partial X}\right]^2 + \left[\frac{\partial u_z}{\partial X}\right]^2} \approx \left[1 + \frac{\partial u_x}{\partial X}\right]. \end{aligned} \tag{3.47}$$

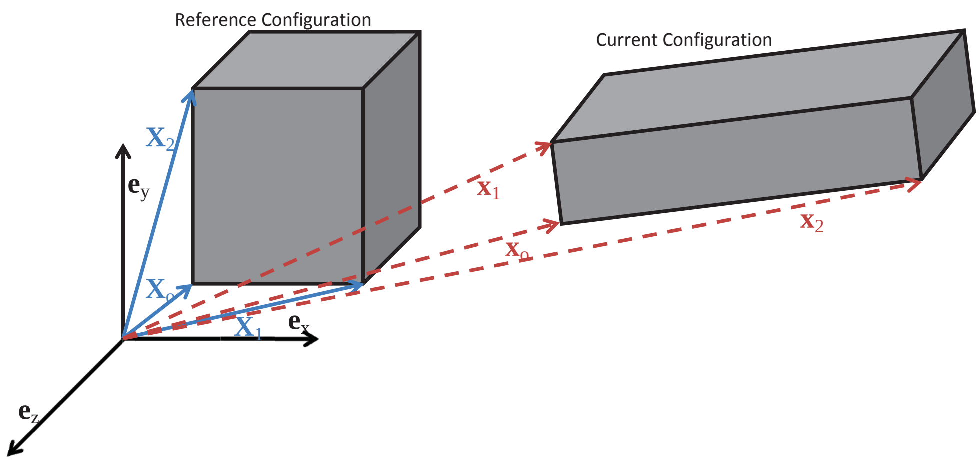



Figure 3.4: Schematic of the deformation of a face of the cube.

which can be approximately computed as,

$$\Lambda_{(\mathbf{e}_x)} \approx \left[1 + \frac{\partial u_x}{\partial X}\right]. \tag{3.48}$$

when the components of the displacement gradient are small. Following a similar procedure, one can determine how the length of line elements oriented along any direction changes. We shall derive a general expression for the same later in section 3.6.1.

Next we are interested in finding how the area of a face of a small cuboid changes due to deformation. Let us assume that the face of the cuboid whose normal coincides with the $\mathbf{e}_z$ basis is of interest and the sides of this face of the cuboid are of length $\Delta X$ along the $\mathbf{e}_x$ direction and $\Delta Y$ along the $\mathbf{e}_y$ direction. Thus, as shown in figure 3.4, $\mathbf{X}_o$ $(= X_o\mathbf{e}_x + Y_o\mathbf{e}_y + Z_o\mathbf{e}_z)$, $\mathbf{X}_1$ $(= (X_o+\Delta X)\mathbf{e}_x+Y_o\mathbf{e}_y+Z_o\mathbf{e}_z)$ and $\mathbf{X}_2$ $(= X_o\mathbf{e}_x+(Y_o+\Delta Y)\mathbf{e}_y+Z_o\mathbf{e}_z)$ denote the position vector of the three corners of the face of the cuboid whose normal is $\mathbf{e}_z$ in the reference configuration and $\mathbf{x}_o$, $\mathbf{x}_1$ and $\mathbf{x}_2$ denote the position of the same three corners of the face of the cuboid in the current configuration. For the same three assumptions listed above, the deformed area of this face of the cuboid is given by,

$$\begin{aligned} a &= \|(\mathbf{x}_1 - \mathbf{x}_o) \wedge (\mathbf{x}_2 - \mathbf{x}_o)\| \\ &= \|(\mathbf{X}_1 - \mathbf{X}_o + \mathbf{u}(\mathbf{X}_1) - \mathbf{u}(\mathbf{X}_o)) \wedge (\mathbf{X}_2 - \mathbf{X}_o + \mathbf{u}(\mathbf{X}_2) - \mathbf{u}(\mathbf{X}_o))\| \\ &= (\Delta X)(\Delta Y) \left| \left\{ \left[ \frac{\partial u_y}{\partial X}\frac{\partial u_z}{\partial Y} - \frac{\partial u_z}{\partial X}\left(1 + \frac{\partial u_y}{\partial Y}\right)\right] \mathbf{e}_x \right.\right. \\ &\quad + \left[\frac{\partial u_z}{\partial X}\frac{\partial u_x}{\partial Y} - \left(1 + \frac{\partial u_x}{\partial X}\right)\frac{\partial u_z}{\partial Y}\right]\mathbf{e}_y \\ &\quad \left.\left. + \left[\left(1 + \frac{\partial u_x}{\partial X}\right)\left(1 + \frac{\partial u_y}{\partial Y}\right) - \frac{\partial u_y}{\partial X}\frac{\partial u_x}{\partial Y}\right]\mathbf{e}_z \right\}\right|, \end{aligned} \tag{3.49}$$

and the orientation of the normal to this deformed face is computed as,

$$\begin{aligned} \mathbf{n} &= \frac{(\mathbf{x}_1 - \mathbf{x}_o) \wedge (\mathbf{x}_2 - \mathbf{x}_o)}{\|(\mathbf{x}_1 - \mathbf{x}_o) \wedge (\mathbf{x}_2 - \mathbf{x}_o)\|} \\ &= \frac{(\mathbf{X}_1 - \mathbf{X}_o + \mathbf{u}(\mathbf{X}_1) - \mathbf{u}(\mathbf{X}_o)) \wedge (\mathbf{X}_2 - \mathbf{X}_o + \mathbf{u}(\mathbf{X}_2) - \mathbf{u}(\mathbf{X}_o))}{\|(\mathbf{X}_1 - \mathbf{X}_o + \mathbf{u}(\mathbf{X}_1) - \mathbf{u}(\mathbf{X}_o)) \wedge (\mathbf{X}_2 - \mathbf{X}_o + \mathbf{u}(\mathbf{X}_2) - \mathbf{u}(\mathbf{X}_o))\|}. \end{aligned} \tag{3.50}$$

Recollect that in chapter 2 we mentioned that the cross product of two vectors characterizes the area of the parallelogram spanned by them. We have made use of this to obtain the above expressions.

Finally, we find the volume of the deformed cuboid. Again recollecting from chapter 2, the box product of three vectors yields the volume of the parallelepiped spanned by them, the deformed volume of the cuboid is given by,

$$\begin{aligned} v &= [\mathbf{x}_1 - \mathbf{x}_o, \mathbf{x}_2 - \mathbf{x}_o, \mathbf{x}_3 - \mathbf{x}_o] \\ &= (\Delta X)(\Delta Y)(\Delta Z) \left\{ \left[\left(1 + \frac{\partial u_z}{\partial Z}\right)\left(1 + \frac{\partial u_y}{\partial Y}\right) - \frac{\partial u_y}{\partial Z}\frac{\partial u_z}{\partial Y}\right]\left(1 + \frac{\partial u_x}{\partial X}\right) \right. \\ &\quad + \left[\frac{\partial u_z}{\partial Y}\frac{\partial u_x}{\partial Z} - \left(1 + \frac{\partial u_z}{\partial Z}\right)\frac{\partial u_x}{\partial Y}\right]\frac{\partial u_y}{\partial X} \\ &\quad \left. + \left[\frac{\partial u_x}{\partial Y}\frac{\partial u_y}{\partial Z} - \left(1 + \frac{\partial u_y}{\partial Y}\right)\frac{\partial u_x}{\partial Z}\right]\frac{\partial u_z}{\partial X} \right\}, \end{aligned} \tag{3.51}$$

where $\mathbf{x}_o$, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ are the position vectors of the material particles in the current configuration corresponding to whose position vector in the reference configuration are $\mathbf{X}_o$ $(= X_o\mathbf{e}_x + Y_o\mathbf{e}_y + Z_o\mathbf{e}_z)$, $\mathbf{X}_1$ $(= (X_o + \Delta X)\mathbf{e}_x +$

$Y_o\mathbf{e}_y + Z_o\mathbf{e}_z$), $\mathbf{X}_2$ ($= X_o\mathbf{e}_x + (Y_o + \Delta Y)\mathbf{e}_y + Z_o\mathbf{e}_z$) and $\mathbf{X}_3$ ($= X_o\mathbf{e}_x + Y_o\mathbf{e}_y + (Z_o + \Delta Z)\mathbf{e}_z$) respectively.

When the magnitude of the components of the displacement gradient are small, then the deformed volume of the cuboid could be computed from equation (3.51) as,

$$v \approx (\Delta X)(\Delta Y)(\Delta Z)\left[1 + \frac{\partial u_x}{\partial X} + \frac{\partial u_y}{\partial Y} + \frac{\partial u_z}{\partial Z}\right], \tag{3.52}$$

by neglecting the quadratic and cubic terms in equation (3.51).

### 3.6.1 Transformation of curves

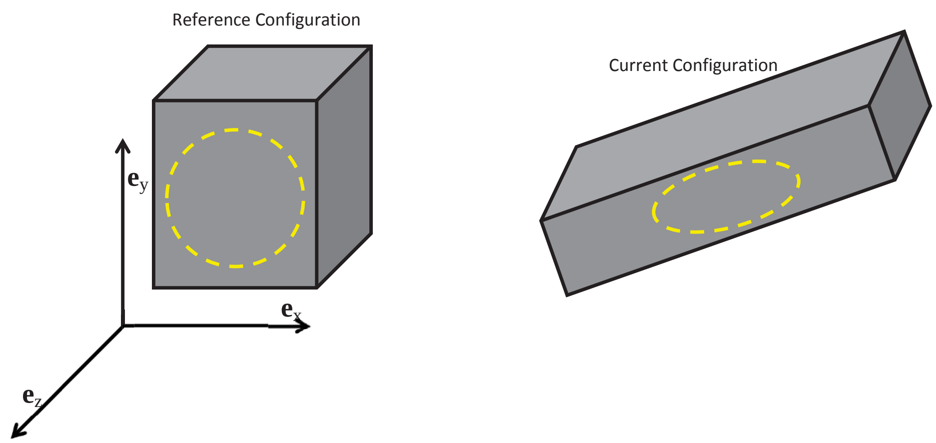
We know the position vector of any particle belonging to the body at various instances of time, $t$. But now we are interested in finding how a set of contiguous points forming a curve changes its shape. That is, we are interested in finding how a circle inscribed in the reference configuration changes its shape, say into an ellipse, in the current configuration. As we have required the deformation field to be one to one, closed curves like circle, ellipse, will remain as closed curves and open curves like straight line, parabola remains open. The position vectors of the particles that occupy a curve can be described using a single variable, say $\xi$. For example, a circle of radius $R$ in the plane whose normal coincides with $\mathbf{e}_z$, as shown in figure 3.5a would be described as $(R\cos(\xi), R\sin(\xi), Z_o)$, where $R$ and $Z_o$ are constants and $0 \leq \xi \leq 2\pi$.

Consider a material curve (or a curve in the reference configuration), $\mathbf{X} = \mathbf{\Gamma}(\xi) \subset \mathfrak{B}_r$, where $\xi$ denotes a parametrization. The material curve by virtue of it being defined in the reference configuration is not a function of time. During a certain motion, the material curve deforms into another curve called the spatial curve, $\mathbf{x} = \boldsymbol{\gamma}(\xi, t) \subset \mathfrak{B}_t$, (see figure 3.5b). The spatial curve at a fixed time $t$ is then defined by the parametric equation

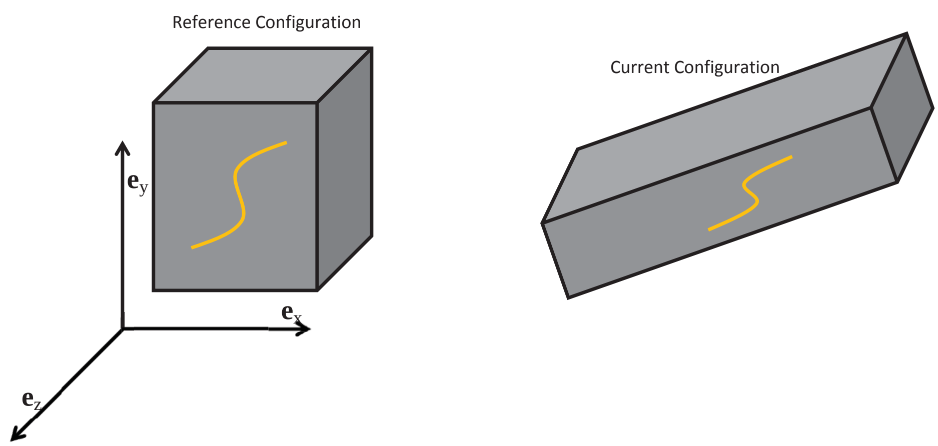
$$\mathbf{x} = \boldsymbol{\gamma}(\xi, t) = \boldsymbol{\chi}(\mathbf{\Gamma}(\xi), t). \tag{3.53}$$

We denote the tangent vector to the material curve as $\Delta\mathbf{X}$ and the tangent vector to the spatial curve as $\Delta\mathbf{x}$ and are defined by

$$\Delta\mathbf{X} = \frac{d\mathbf{\Gamma}}{d\xi}\Delta\xi, \qquad \Delta\mathbf{x} = \frac{\partial\gamma}{\partial\xi}\Delta\xi \tag{3.54}$$




(a) Deformation of a closed curve




(b) Deformation of a open curve

Figure 3.5: Curves in the reference configuration deforming into another curve in the current configuration

By using (3.53) and the chain rule we find that

$$\frac{\partial \gamma}{\partial \xi} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}} \frac{d\mathbf{X}}{d\xi} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}} \frac{d\mathbf{\Gamma}}{d\xi}. \tag{3.55}$$

Hence, from equation (3.54) and the definition of the deformation gradient, (3.11) we find that

$$\Delta \mathbf{x} = \mathbf{F} \Delta \mathbf{X}. \tag{3.56}$$

Expression (3.56) clearly defines a linear transformation which generates a vector $\Delta\mathbf{x}$ by the action of the second-order tensor $\mathbf{F}$ on the vector $\Delta\mathbf{X}$. In summary: material tangent vectors map into spatial tangent vectors via the deformation gradient. This is the physical significance of the deformation gradient.

In the literature, the tangent vectors $\Delta\mathbf{x}$ and $\Delta\mathbf{X}$, in the current and reference configuration are often referred to as the spatial line element and the material line element, respectively. This is correct only when the curve in the reference and current configuration is a straight line.

Next, we introduce the concept of stretch ratio which is defined as the ratio between the current length to its original length. When we say original length, we mean undeformed length, that is the length of the fiber when it is not subjected to any force. It should be pointed out at the outset that many a times the undeformed length would not be available and in these cases it is approximated as the length in the reference configuration.

In general, the stretch ratio depends on the location and orientation of the material fiber for which it is computed. Thus, we can fix the orientation of the material fiber in the reference configuration or in the current configuration. Corresponding to the configuration in which the orientation of the fiber is fixed we have two stretch measures. Recognize that we can fix the orientation of the fibers in only one configuration because its orientation in others is determined by the motion.

First, we consider deformations in which any straight line segments in the reference configuration gets mapped on to a straight line segment in the current configuration. Such a deformation field which maps straight line segments in the reference configuration to straight line segments in the current configuration is called homogeneous deformation. We shall in section 3.10, illustrate that when the deformation is homogeneous, the Cartesian components of the deformation gradient would be a constant. As a consequence of assuming the curve to be a straight line, the tangent and secant to the curve is the same. Hence, saying that we are considering a material fiber[3] of length $\Delta L$ initially oriented along $\mathbf{A}$, (where $\mathbf{A}$ is a unit vector) and located at $\mathbf{P}$, is same as saying that we are studying the influence of the deformation on the tangent vector, $\Delta L\mathbf{A}$ at $\mathbf{P}$. Due to some motion of the body, the material particle that occupied the point $\mathbf{P}$ will occupy the point $\mathbf{p} \in \mathfrak{B}_t$ and the tangent vector $\Delta L\mathbf{A}$ is going to be mapped on to a tangent vector with length, say $\Delta l$ and orientation $\mathbf{a}$. From (3.56) we have

$$\Delta l\mathbf{a} = \Delta L\mathbf{F}\mathbf{A}. \tag{3.57}$$

Dotting the above equation with $\Delta l\mathbf{a}$, we get

$$(\Delta l)^2 = (\Delta L)^2\mathbf{F}\mathbf{A} \cdot \mathbf{F}\mathbf{A} = (\Delta L)^2\mathbf{F}^t\mathbf{F}\mathbf{A} \cdot \mathbf{A}, \tag{3.58}$$

where we have used the fact that $\mathbf{a}$ is a unit vector, the relation (3.57) and the definition of transpose. Defining the right Cauchy-Green deformation tensor, $\mathbf{C}$ as

$$\mathbf{C} = \mathbf{F}^t\mathbf{F}, \tag{3.59}$$

and using equation (3.58) we obtain

$$\Lambda_{\mathbf{A}} = \frac{\Delta l}{\Delta L} = \sqrt{\mathbf{C}\mathbf{A} \cdot \mathbf{A}}, \tag{3.60}$$

where $\Lambda_{\mathbf{A}}$ represents the stretch ratio of a line segment initially oriented along $\mathbf{A}$. The stretch ratio, $\Lambda_{\mathbf{A}}$ is the one that could be directly determined in an experiment. However, there is no reason why we should restrict our studies to line segments which are initially oriented along a given direction. We can also study about material fibers that are finally oriented along a given direction. Towards this, we invert equation (3.57) to get

$$\Delta L\mathbf{A} = \Delta l\mathbf{F}^{-1}\mathbf{a}. \tag{3.61}$$

Dotting the above equation with $\Delta L\mathbf{A}$, we get

$$(\Delta L)^2 = (\Delta l)^2\mathbf{F}^{-1}\mathbf{a} \cdot \mathbf{F}^{-1}\mathbf{a} = (\Delta l)^2\mathbf{F}^{-t}\mathbf{F}^{-1}\mathbf{a} \cdot \mathbf{a}, \tag{3.62}$$

using arguments similar to that used to get (3.58) from (3.57). Defining the left Cauchy-Green deformation tensor or the Finger deformation tensor, $\mathbf{B}$ as

$$\mathbf{B} = \mathbf{F}\mathbf{F}^t, \tag{3.63}$$

[3]A material fiber is also sometimes called as infinitesimal line elements.

and using equation (3.62) we obtain

$$\Lambda_{\mathbf{a}} = \frac{\Delta l}{\Delta L} = \frac{1}{\sqrt{\mathbf{B}^{-1}\mathbf{a}\cdot\mathbf{a}}}, \tag{3.64}$$

where $\Lambda_{\mathbf{a}}$ represents the stretch ratio of a fiber finally oriented along $\mathbf{a}$.

Next, we would like to compute the final angle between two two straight line segments initially oriented along $\mathbf{A}_1$ and $\mathbf{A}_2$. Recognizing that the angle between two straight line segments, $\alpha$ is computed using the expression:

$$\alpha = \cos^{-1}\left(\frac{\mathbf{A}_1\cdot\mathbf{A}_2}{|\mathbf{A}_1||\mathbf{A}_2|}\right). \tag{3.65}$$

Using (3.65) and similar arguments as that used to obtain the stretch ratio, it can be shown that the final angle, $\alpha_f$ between the straight line segments initially oriented along $\mathbf{A}_1$ and $\mathbf{A}_2$ is given by

$$\alpha_f = \cos^{-1}\left(\frac{\mathbf{C}\mathbf{A}_1\cdot\mathbf{A}_2}{\sqrt{\mathbf{C}\mathbf{A}_1\cdot\mathbf{A}_1}\sqrt{\mathbf{C}\mathbf{A}_2\cdot\mathbf{A}_2}}\right). \tag{3.66}$$

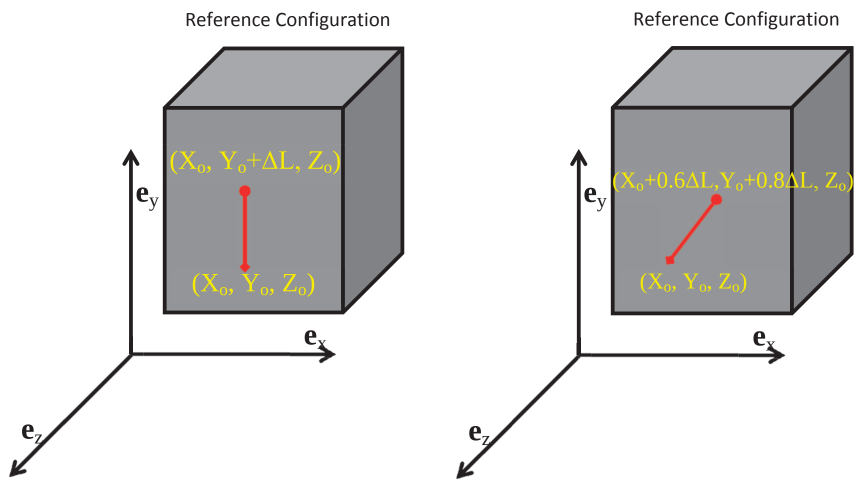
Just, as in the case of stretch ratio, we can also study the initial angle, $\alpha_i$ between straight line segments that are finally oriented along $\mathbf{a}_1$ and $\mathbf{a}_2$. For this case, following the same steps outlined above we compute

$$\alpha_i = \cos^{-1}\left(\frac{\mathbf{B}^{-1}\mathbf{a}_1\cdot\mathbf{a}_2}{\sqrt{\mathbf{B}^{-1}\mathbf{a}_1\cdot\mathbf{a}_1}\sqrt{\mathbf{B}^{-1}\mathbf{a}_2\cdot\mathbf{a}_2}}\right). \tag{3.67}$$

We like to record that as in the case of stretch ratio, $\alpha_i$ and $\alpha_f$ depends on the orientation of the line segments.

Next, let us see what happens when the deformation is inhomogeneous. For inhomogeneous deformation, the Cartesian components of the deformation gradient would vary spatially and the straight line segment in the reference configuration would have become a curve in the current configuration. Hence, distinction between the tangent and the secant has to be made. Now, the deformed length of the curve, $\Delta l$ corresponding to that of a straight line of length $\Delta L$ oriented along $\mathbf{A}$ is obtained from equation (3.56) as,

$$\begin{aligned}\Delta l &= \int_0^1 \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2 + \left(\frac{dz}{d\xi}\right)^2} d\xi \\ &= \int_0^1 \sqrt{\mathbf{C}\mathbf{A}\cdot\mathbf{A}\left[\left(\frac{dX}{d\xi}\right)^2 + \left(\frac{dY}{d\xi}\right)^2 + \left(\frac{dZ}{d\xi}\right)^2\right]} d\xi,\end{aligned} \tag{3.68}$$




(a) Straight line oriented along $\mathbf{E}_y$ at $(X_o, Y_o, Z_o)$.

(b) Straight line oriented along $(3\mathbf{E}_x + 4\mathbf{E}_y)/5$ at $(X_o, Y_o, Z_o)$.

Figure 3.6: Schematic of straight lines in a body

where $\mathbf{A}$ is a constant vector, $\mathbf{C} = \hat{\mathbf{C}}(X(\xi), Y(\xi), Z(\xi))$, $(X(\xi), Y(\xi), Z(\xi))$ denotes the coordinates of the material points that constitutes the straight line and $\xi$ varies between 0 and 1 to parameterize the straight line. Thus, for a line oriented along say $\mathbf{E}_y$ direction of length $\Delta L$, starting from a point $(X_o, Y_o, Z_o)$, (see figure 3.6a) $X(\xi) = X_o$, $Y(\xi) = Y_o + \xi(\Delta L)$, $Z(\xi) = Z_o$. For a line oriented along say $(3\mathbf{E}_x + 4\mathbf{E}_Y)/5$ of length $\Delta L$, starting from $(X_o, Y_o, Z_o)$, (see figure 3.6b) $X(\xi) = X_o + 3\xi(\Delta L)/5$, $Y(\xi) = Y_o + 4\xi(\Delta L)/5$, $Z(\xi) = Z_o$. In fact, if one relaxes the assumption that $\mathbf{A}$ is a constant in equation (3.68) then the straight line in the reference configuration could be a curve and the deformed length could still be calculated from (3.68).

Hence, the stretch ratio for the case of inhomogeneous deformation is,

$$\Lambda_{\mathbf{A}} = \frac{\Delta l}{\Delta L} = \frac{\int_{\xi_o}^{\xi_1} \sqrt{\mathbf{CA}\cdot\mathbf{A}}\sqrt{\left(\frac{dX}{d\xi}\right)^2 + \left(\frac{dY}{d\xi}\right)^2 + \left(\frac{dZ}{d\xi}\right)^2} d\xi}{\int_{\xi_o}^{\xi_1} \sqrt{\left(\frac{dX}{d\xi}\right)^2 + \left(\frac{dY}{d\xi}\right)^2 + \left(\frac{dZ}{d\xi}\right)^2} d\xi}. \tag{3.69}$$

### 3.6.2 Transformation of areas

Having seen how points, curves, tangent vectors in the reference configuration gets mapped to points, curves and tangent vectors in the current configuration, we are now in a position to look at how surfaces get mapped. Of interest, is how a unit vector, $\mathbf{N}$ normal to an infinitesimal material surface element $\Delta A$ map on to a unit vector $\mathbf{n}$ normal to the associated infinitesimal spatial surface element $\Delta a$.

Let $S_r$ denote the material surface in $\mathfrak{B}_r$ that is of interest. Then, an element of area $\Delta A$ at a point $\mathbf{P}$ on $S_r$ is defined through the vectors $\Delta\mathbf{X}$ and $\Delta\mathbf{Y}$, each tangent to $S_r$ at $\mathbf{P}$, as $\Delta A\mathbf{N} = \Delta\mathbf{X} \wedge \Delta\mathbf{Y}$ where $\mathbf{N}$ is a unit vector normal to $S_r$ at $\mathbf{P}$. Due to some motion of the body, the material surface $S_r$ gets mapped on to another material surface $S_t$ in $\mathfrak{B}_t$, the material particle that occupied the point $\mathbf{P}$ on $S_r$ gets mapped onto a point $\mathbf{p}$ on $S_t$ and the tangent vectors $\Delta\mathbf{X}$ and $\Delta\mathbf{Y}$ gets mapped on to another pair of tangent vectors at $\mathbf{p}$ denoted by $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$. Thus, we have

$$\Delta A\mathbf{N} = \Delta\mathbf{X} \wedge \Delta\mathbf{Y}, \qquad \Delta a\mathbf{n} = \Delta\mathbf{x} \wedge \Delta\mathbf{y}. \tag{3.70}$$

Since, $\Delta\mathbf{X}$ and $\Delta\mathbf{Y}$ are tangent vectors we have $\Delta\mathbf{x} = \mathbf{F}\Delta\mathbf{X}$ and $\Delta\mathbf{y} = \mathbf{F}\Delta\mathbf{Y}$. Therefore,

$$\Delta a\mathbf{n} = (\mathbf{F}\Delta\mathbf{X}) \wedge (\mathbf{F}\Delta\mathbf{Y}) = \mathbf{F}^*(\Delta\mathbf{X} \wedge \Delta\mathbf{Y}) = \det(\mathbf{F})\mathbf{F}^{-t}\mathbf{N}\Delta A, \tag{3.71}$$

where we have made use of (2.88) and (2.90). Thus, the relation

$$\Delta a\mathbf{n} = \det(\mathbf{F})\mathbf{F}^{-t}\mathbf{N}\Delta A, \tag{3.72}$$

called Nanson's formula is an often used expression in continuum mechanics.

Next, we are interested in computing the deformed area, $a$. For this case, from Nanson's formula we obtain,

$$(\Delta a)\mathbf{n} \cdot (\Delta a)\mathbf{n} = \det(\mathbf{F})\mathbf{F}^{-t}\mathbf{N}(\Delta A) \cdot \det(\mathbf{F})\mathbf{F}^{-t}\mathbf{N}(\Delta A), \tag{3.73}$$

which simplifies to

$$a = \int_A \det(\mathbf{F})\sqrt{\mathbf{C}^{-1}\mathbf{N} \cdot \mathbf{N}}dA. \tag{3.74}$$

on using the equation (3.59) and the fact that $\mathbf{n}$ is a unit vector. Notice that, once again the deformed area is determined by the right Cauchy-Green deformation tensor.

### 3.6.3 Transformation of volumes

Next, we seek to obtain the relation between elemental volumes in the reference and current configuration. An element of volume $\Delta V$ at a interior point $\mathbf{P}$ in reference configuration is given by the scalar triple product $[\Delta\mathbf{X}, \Delta\mathbf{Y}, \Delta\mathbf{Z}]$ where $\Delta\mathbf{X}$, $\Delta\mathbf{Y}$ and $\Delta\mathbf{Z}$ are vectors oriented along the sides of an infinitesimal parallelepiped. Since, the point under consideration is interior, we can always find three material curves such that they pass through the point $\mathbf{P}$ and $\Delta\mathbf{X}$, $\Delta\mathbf{Y}$ and $\Delta\mathbf{Z}$ are its tangent vectors at $\mathbf{P}$. Now say due to some motion of the body, the material particle that occupied the point $\mathbf{P}$, now occupies the point $\mathbf{p}$, then the curves under investigation would pass through $\mathbf{p}$ and the tangent vectors of these curves at this point $\mathbf{p}$ would now be $\Delta\mathbf{x}$, $\Delta\mathbf{y}$, $\Delta\mathbf{z}$. Hence,

$$\begin{aligned}\Delta v = [\Delta\mathbf{x}, \Delta\mathbf{y}, \Delta\mathbf{z}] = [\mathbf{F}\Delta\mathbf{X}, \mathbf{F}\Delta\mathbf{Y}, \mathbf{F}\Delta\mathbf{Z}] &= \det(\mathbf{F})[\Delta\mathbf{X}, \Delta\mathbf{Y}, \Delta\mathbf{Z}] \\ &= \det(\mathbf{F})\Delta V, \end{aligned} \quad (3.75)$$

where we have used the definition of the determinant (2.77). Since, the coordinate basis vectors for the reference and current configuration have the same handedness, $\det(\mathbf{F}) \geq 0$. Further since, we require the mapping of the body from one configuration to the other be one to one, material particles in a parallelepiped cannot be mapped on to a surface. Hence, $\det(\mathbf{F}) \neq 0$. Therefore $\det(\mathbf{F}) > 0$. Thus, for a vector field to be a deformation field, $\det(\mathbf{F}) > 0$.

## 3.7 Properties of the deformation tensors

Since $\det(\mathbf{F}) > 0$, using the polar decomposition theorem (2.116) the deformation gradient can be represented as:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \quad (3.76)$$

where $\mathbf{R}$ is a proper orthogonal tensor, $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors respectively. (These stretch tensors are called right and left because they are on the right and left of the orthogonal tensor, $\mathbf{R}$.) These stretch tensors are unique, positive definite and symmetric.

Substituting (3.76) in (3.59) and (3.63) we obtain

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{R}^t\mathbf{V}^2\mathbf{R}, \qquad \mathbf{B} = \mathbf{V}^2 = \mathbf{R}\mathbf{U}^2\mathbf{R}^t. \quad (3.77)$$

Next, we record certain properties of the right and left Cauchy-Green deformation tensors.

1. The right Cauchy-Green deformation tensor, $\mathbf{C}$ depends only on the coordinate basis used in the reference configuration, i.e., $\mathbf{C} = C_{ij}\mathbf{E}_i \otimes \mathbf{E}_j = F_{ai}F_{aj}\mathbf{E}_i \otimes \mathbf{E}_j$

2. The left Cauchy-Green deformation tensor depends only on the coordinate basis used in the current configuration, i.e., $\mathbf{B} = B_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = F_{ia}F_{ja}\mathbf{e}_i \otimes \mathbf{e}_j$

3. It is easy to see that both the tensors $\mathbf{C}$ and $\mathbf{B}$ are symmetric and positive definite.

4. Since the deformation tensors are symmetric, they have three real principal values and their principal directions are orthonormal.

5. Both these tensors have the same eigen or principal values but different eigen or principal directions. To see this, let $\mathbf{N}_a$ be the principal direction of the tensor $\mathbf{U}$ and $\varepsilon_a$ its principal value that is

$$\mathbf{U}\mathbf{N}_a = \varepsilon_a \mathbf{N}_a, \tag{3.78}$$

then

$$\mathbf{C}\mathbf{N}_a = \mathbf{U}^2\mathbf{N}_a = \varepsilon_a^2\mathbf{N}_a. \tag{3.79}$$

Now let $\mathbf{n}_a = \mathbf{R}\mathbf{N}_a$. Then

$$\mathbf{B}\mathbf{n}_a = \mathbf{R}\mathbf{U}^2\mathbf{R}^t\mathbf{R}\mathbf{N}_a = \varepsilon_a^2\mathbf{R}\mathbf{N}_a = \varepsilon_a^2\mathbf{n}_a. \tag{3.80}$$

Thus, we have shown that $\varepsilon_a^2$ is the principal value for both $\mathbf{C}$ and $\mathbf{B}$ tensors and $\mathbf{N}_a$ and $\mathbf{R}\mathbf{N}_a$ are their principal directions respectively.

## 3.8 Strain Tensors

Now, we shall look at the concept of strain. This is some quantity defined by us and hence there are various definitions of the same. Also, as we shall see later, we can develop the constitutive relation independent of these definitions. Experimental observations show that relative displacement of particles alone gives raise to stress. A measure of this relative displacement is the stretch ratio. However, this measure has the drawback that when the body is not deformed the stretch ratio is 1 (by virtue of the deformed length being same as the original length) and hence thought to be inconvenient to formulate constitutive relations. Consequently, another measure of relative displacement is sought which would be 0 when the body is not deformed and less than zero when compressed and greater than zero when stretched. This measure is called as the strain, $\epsilon_{(\mathbf{A})}$. There is no unique way of obtaining the strain from the stretch ratio. The following functions satisfy the requirement of the strain:

$$\epsilon_{(\mathbf{A})} = \frac{\Lambda^m_{(\mathbf{A})} - 1}{m}, \qquad \epsilon_{(\mathbf{A})} = \ln(\Lambda_{(\mathbf{A})}), \tag{3.81}$$

where $m$ is some real number and ln stands for natural logarithm. Thus, if $m = 1$ in (1.5a) then the resulting strain is called as the engineering strain, if $m = -1$, it is called as the true strain, if $m = 2$ it is Cauchy-Green strain. The second function wherein $\epsilon_{(\mathbf{A})} = \ln(\lambda_{(\mathbf{A})})$, is called as the Hencky strain or the logarithmic strain.

Let us start by looking at the case when $m = 2$ in equation (3.81), that is the case,

$$\epsilon^{CG}_{(\mathbf{A})} = \frac{\Lambda^2_{(\mathbf{A})} - 1}{2}. \tag{3.82}$$

Substituting (3.60) in (3.82) we obtain

$$\epsilon^{CG}_{(\mathbf{A})} = \frac{1}{2}(\mathbf{CA}\cdot\mathbf{A} - 1) = \frac{1}{2}(\mathbf{C} - \mathbf{1})\mathbf{A}\cdot\mathbf{A} = \mathbf{EA}\cdot\mathbf{A}, \tag{3.83}$$

where, $\mathbf{E} = 0.5[\mathbf{C} - \mathbf{1}]$, is called the Cauchy-Green strain tensor. Thus, Cauchy-Green strain tensor carries information about the strain in the material fibers initially oriented along a given direction. Substituting Lagrangian displacement gradient, defined in (3.33), for deformation gradient in the expression for $\mathbf{E}$, we obtain

$$\mathbf{E} = \frac{1}{2}[\mathbf{H} + \mathbf{H}^t + \mathbf{H}^t\mathbf{H}]. \tag{3.84}$$

If $tr(\mathbf{HH}^t) << 1$, i.e., each of the components of $\mathbf{H}$ is close to zero[4], then we can compute $\mathbf{E}$ approximately as

$$\mathbf{E} \approx \frac{1}{2}[\mathbf{H} + \mathbf{H}^t] = \boldsymbol{\epsilon}_L. \tag{3.85}$$

[4]Realize that $tr(\mathbf{HH}^t) = H^2_{11} + H^2_{12} + H^2_{13} + H^2_{21} + H^2_{22} + H^2_{23} + H^2_{31} + H^2_{32} + H^2_{33}$

$\boldsymbol{\epsilon}_L$ is called as the Lagrangian linearized strain and will be used extensively while studying linearized elasticity.

Next, we examine the case when $m = 1$, in equation (3.81), that is the case,

$$\epsilon_{(\mathbf{A})}^{LS} = \Lambda_{(\mathbf{A})} - 1. \tag{3.86}$$

Substituting (3.60) in (3.86) we obtain

$$\epsilon_{(\mathbf{A})}^{LS} = \sqrt{\mathbf{CA} \cdot \mathbf{A}} - 1. \tag{3.87}$$

Substituting Lagrangian displacement gradient, defined in (3.33), for deformation gradient in the expression for $\mathbf{C}$ and using the definition (3.85) for Lagrangian linearized strain, equation (3.87) evaluates to

$$\epsilon_{(\mathbf{A})}^{LS} = \sqrt{1 + 2\boldsymbol{\epsilon}_L\mathbf{A} \cdot \mathbf{A} + (\mathbf{H}^t\mathbf{H})\mathbf{A} \cdot \mathbf{A}} - 1, \tag{3.88}$$

which when the components of the Lagrangian displacement gradient are small could be approximately computed as

$$\epsilon_{(\mathbf{A})}^{LS} = \sqrt{1 + 2\boldsymbol{\epsilon}_L\mathbf{A} \cdot \mathbf{A}} - 1 \approx \boldsymbol{\epsilon}_L\mathbf{A} \cdot \mathbf{A}, \tag{3.89}$$

where we have approximately evaluated the square root using Taylor's series[5]. Thus, Lagrangian linearized strain carries the information on changes in length of material fibers initially oriented along a given direction when the changes in length are small.

Instead of studying the strain in fibers oriented initially along a given direction, one can also study strains in fibers finally oriented along a given direction, $\mathbf{a}$. Now, again the following functions satisfy the requirement of the strain:

$$\epsilon_{(\mathbf{a})} = \frac{\Lambda_{(\mathbf{a})}^m - 1}{m}, \qquad \epsilon_{(\mathbf{a})} = \ln(\Lambda_{(\mathbf{a})}), \tag{3.90}$$

where $m$ is some real number and ln stands for natural logarithm. We study the case when $m = -2$ in equation (3.90), that is the case,

$$\epsilon_{(\mathbf{a})}^{AH} = \frac{1 - \Lambda_{(\mathbf{a})}^{-2}}{2}. \tag{3.91}$$

Substituting (3.64) in (3.91) and rearranging we obtain

$$\epsilon_{(\mathbf{A})}^{AH} = \frac{1}{2}(1 - \mathbf{B}^{-1}\mathbf{a} \cdot \mathbf{a}) = \frac{1}{2}(\mathbf{1} - \mathbf{B}^{-1})\mathbf{a} \cdot \mathbf{a} = \mathbf{ea} \cdot \mathbf{a}. \tag{3.92}$$

[5] $\sqrt{1 + 2x} \approx 1 + x$, when $x$ is small.

where, $\mathbf{e} = 0.5[\mathbf{1} - \mathbf{B}^{-1}]$, is called the Almansi-Hamel strain tensor and carries information about the strain in the material fibers finally oriented along a given direction. Substituting Eulerian displacement gradient, defined in (3.34), for $\mathbf{F}^{-1}$ in the expression for $\mathbf{e}$, we obtain

$$\mathbf{e} = \frac{1}{2}[\mathbf{h} + \mathbf{h}^t - \mathbf{h}^t\mathbf{h}]. \tag{3.93}$$

If $tr(\mathbf{h}\mathbf{h}^t) << 1$, then we can compute $\mathbf{e}$ approximately as

$$\mathbf{e} \approx \frac{1}{2}[\mathbf{h} + \mathbf{h}^t] = \boldsymbol{\epsilon}_E. \tag{3.94}$$

$\boldsymbol{\epsilon}_E$ is called as the Eulerian linearized strain.

The Lagrangian and Eulerian displacement gradients are related through the equation

$$\mathbf{H} = \mathbf{h} + \mathbf{h}\mathbf{H}, \tag{3.95}$$

obtained from the requirement that $\mathbf{F}\mathbf{F}^{-1} = \mathbf{1}$ and substituting the expressions (3.33) and (3.34) for $\mathbf{F}$ and $\mathbf{F}^{-1}$ respectively. It immediately transpires that when each of the components of $\mathbf{H}$ and $\mathbf{h}$ are close to zero then $\mathbf{H} \approx \mathbf{h}$ and hence both the Lagrangian and Eulerian linearized strain are numerically the same. Hence, when the distinction is not important the subscript, $_L$ or $_E$ would be dropped and the linearized strain denoted as $\boldsymbol{\epsilon}$

There are other strain measures called the Hencky strain tensor in the Lagrangian and Eulerian form defined as $\ln(\mathbf{U})$ and $\ln(\mathbf{V})$ respectively. This strain tensor corresponds the case where the strain along a given direction is defined as the natural logarithm of the corresponding stretch ratio. Hencky strain is also sometimes called as the logarithmic strain.

## 3.9 Normal and shear strain

From now on we shall in this course, study the response of bodies only for cases when the value of the components of both the Lagrangian and Eulerian displacement gradient are small. As we have shown above, this means that not only does the changes in length be small but the rotations also needs to be small. Under these severe but practically reasonable assumptions, the strain-displacement relation is

$$\boldsymbol{\epsilon} = \frac{1}{2}\left[\mathbf{h} + \mathbf{h}^t\right], \tag{3.96}$$

where we stop distinguishing between Lagrangian and Eulerian gradient, as they would be approximately the same. Thus, the Cartesian components of the linearized strain are,

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right] & \frac{1}{2}\left[\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right] \\ \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right] & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left[\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right] \\ \frac{1}{2}\left[\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right] & \frac{1}{2}\left[\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right] & \frac{\partial u_z}{\partial z} \end{pmatrix} \tag{3.97}$$

where if the displacement field is given a Lagrangian description we would use a Lagrangian gradient and if the displacement field is given a Eulerian description Eulerian gradient is used. Strictly speaking, we should only use the Eulerian gradient since this strain is to be related to the Cauchy stress.

Similarly the cylindrical polar components of the linearized strain are,

$$\begin{aligned} \boldsymbol{\epsilon} &= \begin{pmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{rz} \\ \epsilon_{r\theta} & \epsilon_{\theta\theta} & \epsilon_{\theta z} \\ \epsilon_{rz} & \epsilon_{\theta z} & \epsilon_{zz} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right] & \frac{1}{2}\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right] \\ \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right] & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{1}{2}\left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right] \\ \frac{1}{2}\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right] & \frac{1}{2}\left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right] & \frac{\partial u_z}{\partial z} \end{pmatrix} \end{aligned} \tag{3.98}$$

where we have made use of the expression (2.259) for finding the gradient of the displacement field in cylindrical coordinates.

### 3.9.1 Normal strain

The component of the strain that represents the changes in length along a given direction is called normal strain. From equation (3.89), the change in length along a given direction $\mathbf{A}$ is,

$$\frac{(\Delta l - \Delta L)}{\Delta L} = \boldsymbol{\epsilon}\mathbf{A} \cdot \mathbf{A}. \tag{3.99}$$

Thus, the change in length along coordinate basis directions - $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ - is given by the components - $\epsilon_{xx}$, $\epsilon_{yy}$, $\epsilon_{zz}$ - respectively. Therefore these components are called as the normal strain components.

### 3.9.2 Principal strain

Of interest is the maximum change in length that occurs in the body subjected to a given force and the direction in which this maximum change in length occurs. Thus, we need to find the vector $\mathbf{A}$ such that $\|\mathbf{A}\| = 1$ and it maximizes, $\epsilon_{(\mathbf{A})}$ defined as,

$$\epsilon_{(\mathbf{A})} = \boldsymbol{\epsilon}\mathbf{A} \cdot \mathbf{A}. \tag{3.100}$$

This constrained optimization is done by what is called as the Lagrange-multiplier method. Towards this, we introduce the function

$$\mathcal{L}(\mathbf{A}, \alpha^*) = \mathbf{A} \cdot \boldsymbol{\epsilon}\mathbf{A} - \alpha^*[\|\mathbf{A}\|^2 - 1], \tag{3.101}$$

where $\alpha^*$ is the Lagrange multiplier and the condition $\|\mathbf{A}\|^2 - 1 = 0$ characterizes the constraint condition. At locations where the extremal values of $\mathcal{L}$ occurs, the derivatives $\frac{\partial \mathcal{L}}{\partial \mathbf{A}}$ and $\frac{\partial \mathcal{L}}{\partial \alpha^*}$ must vanish, i.e.,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{A}} = 2(\boldsymbol{\epsilon}\mathbf{A} - \alpha^*\mathbf{A}) = \mathbf{o}, \tag{3.102}$$

$$\frac{\partial \mathcal{L}}{\partial \alpha^*} = \|\mathbf{A}\|^2 - 1 = 0. \tag{3.103}$$

To obtain these equations we have made use of the fact that linearized strain tensor is symmetric. Thus, we have to find three $\hat{\mathbf{A}}_a$ and $\alpha_a^*$'s such that

$$[\boldsymbol{\epsilon} - \alpha_a^*\mathbf{1}]\hat{\mathbf{A}}_a = \mathbf{o}, \qquad \|\hat{\mathbf{A}}_a\|^2 = 1, \tag{3.104}$$

($a = 1, 2, 3$; no summation) which is nothing but the eigenvalue problem involving the tensor $\boldsymbol{\epsilon}$ with the Lagrange multiplier being identified as the eigenvalue. Hence, the results of section 2.5 follows. In particular, for (3.104a) to have a non-trivial solution

$$(\alpha_a^*)^3 - I_1(\alpha_a^*)^2 + I_2\alpha_a^* - I_3 = 0, \tag{3.105}$$

where

$$I_1 = tr(\boldsymbol{\epsilon}), \qquad I_2 = \frac{1}{2}[I_1^2 - tr(\boldsymbol{\epsilon}^2)], \qquad I_3 = \det(\boldsymbol{\epsilon}), \tag{3.106}$$

the principal invariants of the strain $\boldsymbol{\epsilon}$. As stated in section 2.5, equation (3.105 has three real roots, since the linearized strain tensor is symmetric.

These roots ($\alpha_a^*$) will henceforth be denoted by $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ and are called principal strains. The principal strains include both the maximum and minimum normal strains among all material fibers passing through a given $\mathbf{x}$.

The corresponding three orthonormal eigenvectors $\hat{\mathbf{A}}_a$, which are then characterized through the relation (3.104) are called the principal directions of $\boldsymbol{\epsilon}$. Further, these eigenvectors form a mutually orthogonal basis since the strain tensor $\boldsymbol{\epsilon}$ is symmetric. This property of the strain tensor also allows us to represent $\boldsymbol{\epsilon}$ in the spectral form

$$\boldsymbol{\epsilon} = \sum_{a=1}^{3} \epsilon_a \hat{\mathbf{A}}_a \otimes \hat{\mathbf{A}}_a. \tag{3.107}$$

### 3.9.3 Shear strain

As we did to obtain equation (3.89) when the components of the displacement gradient are small, the right Cauchy-Green deformation tensor could be approximately computed as, $\mathbf{C} \approx \mathbf{1} + 2\boldsymbol{\epsilon}$. For this approximation, the deformed angle between two line segments initially oriented along $\mathbf{A}_1$ and $\mathbf{A}_2$ directions given in (3.66) simplifies to,

$$\cos(\alpha_f) = \mathbf{A}_1 \cdot \mathbf{A}_2 + 2\boldsymbol{\epsilon}\mathbf{A}_1 \cdot \mathbf{A}_2, \tag{3.108}$$

where we have approximated $(1 + 2\boldsymbol{\epsilon}\mathbf{A} \cdot \mathbf{A})$ as 1, since the components of $\boldsymbol{\epsilon}$ is much less than 1 (of the order of $10^{-3}$) and the components of $\mathbf{A}$ is less than or equal to 1. If $\alpha$ is the angle between the line elements in the reference configuration, then $\cos(\alpha) = \mathbf{A}_1 \cdot \mathbf{A}_2$. Using the trigonometric relation, $\cos(A) - \cos(B) = 2\sin((A+B)/2)\sin((B-A)/2)$ and the assumption that the change in angle, $\alpha_f - \alpha$, is small, equation (3.108) could be written as,

$$\alpha_f - \alpha = \frac{2}{\sin(\alpha)}\boldsymbol{\epsilon}\mathbf{A}_1 \cdot \mathbf{A}_2. \tag{3.109}$$

When $\mathbf{A}_1$ and $\mathbf{A}_2$ are identified with the orthonormal coordinate basis, then $\sin(\alpha) = 1$ and $\boldsymbol{\epsilon}\mathbf{A}_1 \cdot \mathbf{A}_2$ represents the off diagonal terms in the matrix representation of the components of the strain tensor. Hence, the off diagonal terms are the shear strains which represents half of the change in angle between orthogonal line elements oriented along the coordinate basis directions.

Therefore, we conclude that the diagonal elements in the matrix representation of the components of the strain tensor are related to the changes

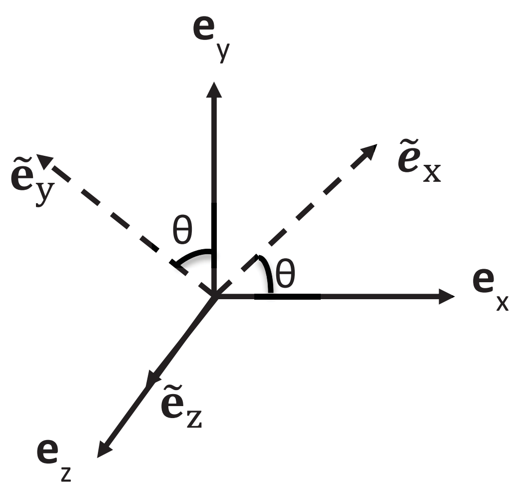



Figure 3.7: Transformation of basis vectors

in length of line elements oriented along the direction of the orthonormal coordinate basis and the off diagonal elements are related to the changes in angle of these line elements oriented along the direction of the orthonormal basis. We also infer that there is no change in angle between line elements along which the change in length is a extremum. This is a consequence of the observation that the off diagonal elements are zero in the equation (3.107) representing the strain using the principal directions as the coordinate basis.

### 3.9.4 Transformation of linearized strain tensor

Now, we are interested in finding how the components of the linearized strain change due to transformation of the coordinate basis. By virtue of the linearized strain being a second order tensor, the transformation laws (2.161) derived in section 2.6.2 hold. Thus, if $Q_{ij}$ $(= \mathbf{e}_i \cdot \tilde{\mathbf{e}}_j)$ represents the directional cosine matrix of the transformation from coordinate basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$, then the strain components in the $\tilde{\mathbf{e}}_i$ basis, $\tilde{\epsilon}_{ij}$ could be obtained from the strain components in the $\mathbf{e}_i$ basis, $\epsilon_{ij}$, through the equation,

$$\tilde{\epsilon}_{ij} = \epsilon_{ab} Q_{ai} Q_{bj}. \tag{3.110}$$

Let us specialize to a state of strain wherein the strain tensor has a representation,

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{xy} & \epsilon_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.111}$$

This state of strain is called as plane strain. A state of strain is said to be plane strain if at least one of its principal values is zero which means that $\det(\boldsymbol{\epsilon}) = 0$. Let us further assume that $\tilde{\mathbf{e}}_j$ is related to $\mathbf{e}_i$ through,

$$\tilde{\mathbf{e}}_1 = \cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2, \quad \tilde{\mathbf{e}}_2 = -\sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_2, \quad \tilde{\mathbf{e}}_3 = \mathbf{e}_3. \tag{3.112}$$

This represents a anticlockwise rotation of the $\mathbf{e}_i$ basis about $\mathbf{e}_3$ basis to obtain the new basis vectors (see figure 3.7). Substituting equations (3.111) and (3.112) in (3.110) we obtain,

$$\begin{aligned} \tilde{\epsilon}_{xx} &= \epsilon_{xx}\cos^2(\theta) + \epsilon_{xy}\sin(2\theta) + \epsilon_{yy}\sin^2(\theta) \\ &= \frac{[\epsilon_{xx} + \epsilon_{yy}]}{2} + \frac{[\epsilon_{xx} - \epsilon_{yy}]}{2}\cos(2\theta) + \epsilon_{xy}\sin(2\theta), \end{aligned} \tag{3.113}$$

$$\begin{aligned} \tilde{\epsilon}_{xy} &= -\epsilon_{xx}\sin(\theta)\cos(\theta) + \epsilon_{xy}[\cos^2(\theta) - \sin^2(\theta)] + \epsilon_{yy}\sin(\theta)\cos(\theta) \\ &= -\frac{[\epsilon_{xx} - \epsilon_{yy}]}{2}\sin(2\theta) - \epsilon_{xy}\cos(2\theta), \end{aligned} \tag{3.114}$$

$$\begin{aligned} \tilde{\epsilon}_{yy} &= \epsilon_{xx}\sin^2(\theta) - \epsilon_{xy}\sin(2\theta) + \epsilon_{yy}\cos^2(\theta) \\ &= \frac{[\epsilon_{xx} + \epsilon_{yy}]}{2} - \frac{[\epsilon_{xx} - \epsilon_{yy}]}{2}\cos(2\theta) - \epsilon_{xy}\sin(2\theta). \end{aligned} \tag{3.115}$$

These set of equations is what is popularly called as Mohr's equations. The fact that this is for a special state of strain and special transformation of the coordinate basis cannot be overemphasized.

## 3.10 Homogeneous Motions

Let us now understand what we mean when we say the motion to be homogeneous. Say, in the reference configuration, we have marked straight lines with different slopes on each of the three mutually perpendicular planes (material curves). Even, if only some of the line segments transform into curves the motion is inhomogeneous. Examples of such motions are plenty. A beam subjected to a pure bending moment[6] is an example. Thus, we can show that for a homogeneous motion the matrix components of the deformation gradient, $\mathbf{F}$ with respect to a Cartesian basis can depend only on time and hence any homogeneous motion can be written in the form:

$$\mathbf{x} = \mathbf{F}(t)\mathbf{X} + \mathbf{c}(t), \tag{3.116}$$

where $\mathbf{c}$ is some vector which depends on time, $\mathbf{X}$ and $\mathbf{x}$ are the position vectors of the same material particle in the reference and current configuration, respectively.

Recognize that, in general, the matrix components of the deformation gradient, $\mathbf{F}$ with respect to a curvilinear basis (say cylindrical polar basis) need not be a constant for the deformation to be homogeneous nor would the deformation be homogeneous if the matrix components with respect to a curvilinear basis were to be a constant. For example, consider a deformation of the form:

$$r = R/g(\Theta), \qquad \theta = f(\Theta), \qquad z = Z,$$

where

$$\begin{aligned} f &= 2\tan^{-1}\left(\frac{\sqrt{a_0^2 - a_1^2 - a_2^2}}{2(a_0 - a_1)}\tan\left(\frac{\Theta}{2}\right) - \frac{a_2}{a_0 - a_1}\right), \\ g &= \sqrt{\frac{a_0 + a_1\cos(2\theta) + a_2\sin(2\theta)}{a_0^2 - a_1^2 - a_2^2}}, \end{aligned}$$

$(R, \Theta, Z)$ and $(r, \theta, z)$ are coordinates of the same material particle in cylindrical polar coordinates before and after deformation, $a_i$'s are constants. A straightforward computation will show that the matrix components of deformation gradient tensor in cylindrical polar coordinates depends on $\Theta$ but when the same tensor is represented using Cartesian coordinates is independent of $\Theta$.

[6] A long beam bends in such a way the plane sections normal to the axis of the beam remain plane. Hence, straight lines contained in planes perpendicular to the axis of the beam transform into straight lines. However, lines parallel to the axis of the beam transform into curves.

### 3.10.1 Rigid body Motion

An example, of homogeneous motion is rigid body motion. Here apart from straight lines deforming into straight lines, the distance between any two points in the body remain the same. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be the position vectors of two material points in the reference configuration and let the position vector of the same material particle in the current configuration be denoted by $\mathbf{x}_1$ and $\mathbf{x}_2$, then, if the body undergoes rigid body motion:

$$\mid \mathbf{x}_1 - \mathbf{x}_2 \mid = \mid \mathbf{X}_1 - \mathbf{X}_2 \mid . \tag{3.117}$$

Since, this motion is homogeneous

$$\mid \mathbf{x}_1 - \mathbf{x}_2 \mid = \mid \mathbf{F}(\mathbf{X}_1 - \mathbf{X}_2) \mid . \tag{3.118}$$

Combining equations (3.117) and (3.118) we obtain

$$(\mathbf{F}^t\mathbf{F} - \mathbf{1})(\mathbf{X}_1 - \mathbf{X}_2) = \mathbf{o}. \tag{3.119}$$

Since $\mathbf{X}_1 \neq \mathbf{X}_2$ and (3.119) has to hold for any pair of material particles, i.e., $\mathbf{X}_1$ and $\mathbf{X}_2$ are some arbitrary but distinct vectors, we require that

$$\mathbf{F}^t\mathbf{F} = \mathbf{1}, \tag{3.120}$$

for a rigid body motion. Recalling the definition of an orthogonal tensor, (2.92) and comparing it with (3.120), we immediately recognize that for a rigid body motion, the deformation gradient has to be an orthogonal tensor.

For example, consider, a motion given by

$$\mathbf{x} = \mathbf{Q}(t)[\mathbf{X} - \mathbf{X}_o] + \mathbf{c}(t), \tag{3.121}$$

where $\mathbf{Q}$ is an orthogonal tensor which is a function of time, $\mathbf{X}_o$ is a constant vector and $\mathbf{c}$ is a vector function of time.

Straight forward computation from the definition of the various quantities would show that the deformation gradient, $\mathbf{F}$, the right and left Cauchy-Green deformation tensors, $\mathbf{C}$ and $\mathbf{B}$ respectively, the Cauchy-Green strain tensor, $\mathbf{E}$ and the Almansi-Hamel strain tensor, $\mathbf{e}$ for this rigid body deformation are,

$$\mathbf{F} = \mathbf{Q}, \qquad \mathbf{C} = \mathbf{1}, \qquad \mathbf{B} = \mathbf{1}, \qquad \mathbf{E} = \mathbf{0}, \qquad \mathbf{e} = \mathbf{0},$$

The Lagrangian displacement gradient, $\mathbf{H}$ and the Eulerian displacement gradient, $\mathbf{h}$ are:

$$\mathbf{H} = \mathbf{Q} - \mathbf{1}, \qquad \mathbf{h} = \mathbf{1} - \mathbf{Q}^t,$$

then, the Lagrangian linearized strain tensor, $\boldsymbol{\epsilon}_L$ and the Eulerian linearized strain tensor, $\boldsymbol{\epsilon}_E$ are evaluated to be,

$$\boldsymbol{\epsilon}_L = \frac{1}{2}[\mathbf{Q} + \mathbf{Q}^t - 2\mathbf{1}], \qquad \boldsymbol{\epsilon}_E = \frac{1}{2}[2\mathbf{1} - \mathbf{Q} - \mathbf{Q}^t],$$

Observe that the given motion corresponds to rigid body rotation of the body about $\mathbf{X}_o$ and a translation. For this motion, we expect the strains to be zero because the length between two points do not change in a rigid body deformation. While the Cauchy-Green strain tensor and Almansi-Hamel strain tensor meets these requirements the linearized strain tensors don't. However, recognize that these measures are valid only for cases when $tr(\mathbf{H}\mathbf{H}^t) = tr(\mathbf{h}\mathbf{h}^t) = tr(2\mathbf{1} - \mathbf{Q} - \mathbf{Q}^t) << 1$. That is to use linearized strain measures the rigid body rotation has to be small. Hence, linearized strain measures can be used only when the change in length is small and the angle of rotation of the line segment is small too.

### 3.10.2 Uniaxial or equi-biaxial motion

An uniaxial or equi-biaxial motion has the form:

$$x = \lambda_1 X, \qquad y = \lambda_2 Y, \qquad z = \lambda_2 Z, \tag{3.122}$$

where $\lambda_1$ and $\lambda_2$ are functions of time and $(X, Y, Z)$ denotes the Cartesian coordinates of a typical material particle in the reference configuration and $(x, y, z)$ denote the Cartesian coordinates of the same material particle in the current configuration. If this motion is effected by the application of the force (traction) along just one direction (in this case along $\mathbf{e}_x$), then it is called uniaxial motion. On the other hand if the deformation is effected by the application of force along two directions(in this case along $\mathbf{e}_y$ and $\mathbf{e}_z$), then it is called equi-biaxial motion.

For the assumed motion field (3.122), the deformation gradient, $\mathbf{F}$, the right and left Cauchy-Green deformation tensors, $\mathbf{C}$ and $\mathbf{B}$ respectively, the Cauchy-Green strain tensor, $\mathbf{E}$ and the Almansi-Hamel strain tensor, $\mathbf{e}$ are computed to be,

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{pmatrix},$$

$$\mathbf{E} = \frac{1}{2}\begin{pmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_2^2 - 1 \end{pmatrix}, \qquad \mathbf{e} = \frac{1}{2}\begin{pmatrix} 1 - \lambda_1^{-2} & 0 & 0 \\ 0 & 1 - \lambda_2^{-2} & 0 \\ 0 & 0 & 1 - \lambda_2^{-2} \end{pmatrix},$$

The Lagrangian displacement gradient, $\mathbf{H}$ and the Eulerian displacement gradient, $\mathbf{h}$ are:

$$\mathbf{H} = \begin{pmatrix} \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_2 - 1 & 0 \\ 0 & 0 & \lambda_2 - 1 \end{pmatrix}, \qquad \mathbf{h} = \begin{pmatrix} 1 - \frac{1}{\lambda_1} & 0 & 0 \\ 0 & 1 - \frac{1}{\lambda_2} & 0 \\ 0 & 0 & 1 - \frac{1}{\lambda_2} \end{pmatrix},$$

then, the Lagrangian linearized strain tensor, $\boldsymbol{\epsilon}_L$ and the Eulerian linearized strain tensor, $\boldsymbol{\epsilon}_E$ are evaluated to be,

$$\boldsymbol{\epsilon}_L = \begin{pmatrix} \lambda_1 - 1 & 0 & 0 \\ 0 & \lambda_2 - 1 & 0 \\ 0 & 0 & \lambda_2 - 1 \end{pmatrix}, \qquad \boldsymbol{\epsilon}_E = \begin{pmatrix} 1 - \frac{1}{\lambda_1} & 0 & 0 \\ 0 & 1 - \frac{1}{\lambda_2} & 0 \\ 0 & 0 & 1 - \frac{1}{\lambda_2} \end{pmatrix},$$

Though the form of the Cauchy-Green, Almansi-Hamel, Lagrangian linearized and Eulerian linearized strain look different it can be easily verified that when $\lambda_i$ is close to 1, numerically their values would also be close to each other.

Now, say a unit cube oriented in space such that $\mathcal{B} = \{(X, Y, Z) | 0 \leq X \leq 1, 0 \leq Y \leq 1, 0 \leq Z \leq 1\}$ is subjected to a deformation of the form (3.122). Then, the deformed volume of the cube, $v$ as given by (3.75) is $v = \det(\mathbf{F}) = \lambda_1\lambda_2^2$. Notice that the cube being of unit dimensions its original volume is 1. Similarly, the deformed surface area, $a$ of the face whose normal coincides with $\mathbf{e}_x$ in the reference configuration is computed from (3.74) as, $a = \det(\mathbf{F})\sqrt{\mathbf{C}^{-1}\mathbf{e}_x \cdot \mathbf{e}_x} = \lambda_2^2$ and the deformed normal direction is found using Nanson's formula (3.72) as, $\mathbf{n} = \det(\mathbf{F})\mathbf{F}^{-t}\mathbf{e}_x(1/a) = \mathbf{e}_x$. The deformed length of a line element oriented along $(\mathbf{e}_x + \mathbf{e}_y)/\sqrt{2}$ and of unit length is $\sqrt{(\lambda_1^2 + \lambda_2^2)/2}$ obtained by using (3.60). Similarly, the deformed angle between line segments oriented along $\mathbf{e}_x$ and $(\mathbf{e}_x + \mathbf{e}_y)/\sqrt{2}$ is, $\alpha_f = \cos^{-1}(\lambda_1/\sqrt{(\lambda_1^2 + \lambda_2^2)})$ which is found using (3.66).

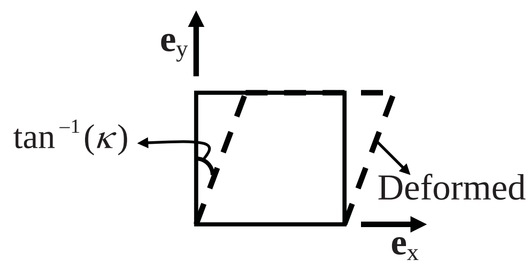



Figure 3.8: Schematic of simple shear deformation in the x-y plane

### 3.10.3 Isochoric motions

Motions in which the volume of a body does not change are called isochoric motions. Thus it follows from (3.75) that $\det(\mathbf{F}) = 1$ for these motions. In case of motions for which the magnitude of the components of the displacement gradient are small, then $\det(\mathbf{F})$ can be approximately computed as $1 + tr(\boldsymbol{\epsilon})$ and thus for this deformation to be isochoric it suffices if $tr(\boldsymbol{\epsilon})$ $= 0$. Isochoric motion can be homogeneous or inhomogeneous. However, the examples that we shall consider here are homogeneous motions.

An isochoric uniaxial or equi-biaxial motion takes the form:

$$x = \lambda X, \qquad y = \frac{Y}{\sqrt{\lambda}}, \qquad z = \frac{Z}{\sqrt{\lambda}}, \tag{3.123}$$

where $\lambda$ is a function of time and as before $(X, Y, Z)$ denotes the Cartesian coordinates of a typical material particle in the reference configuration and $(x, y, z)$ denote the Cartesian coordinates of the same material particle in the current configuration. It is easy to check that $\det(\mathbf{F})$ for this deformation is 1.

A simple shear motion has the form:

$$x = X + \kappa Y, \qquad y = Y, \qquad z = Z, \tag{3.124}$$

where $\kappa$ is only a function of time. It is easy to verify that this motion is isochoric. In this case, the body is assumed to shear in the $X - Y$ plane that is the angle between line segments initially oriented along the $\mathbf{e}_x$ and $\mathbf{e}_y$ direction change (see figure 3.8) but the angle between line segments initially oriented along $\mathbf{e}_y$ and $\mathbf{e}_z$ direction or $\mathbf{e}_z$ and $\mathbf{e}_x$ direction does not change.

For the pure shear motion field (3.124), the deformation gradient, $\mathbf{F}$, the right and left Cauchy-Green deformation tensors, $\mathbf{C}$ and $\mathbf{B}$ respectively, the Cauchy-Green strain tensor, $\mathbf{E}$ and the Almansi-Hamel strain tensor, $\mathbf{e}$ are computed to be,

$$\mathbf{F} = \begin{pmatrix} 1 & \kappa & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & \kappa & 0 \\ \kappa & 1+\kappa^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1+\kappa^2 & \kappa & 0 \\ \kappa & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\mathbf{E} = \frac{1}{2}\begin{pmatrix} 0 & \kappa & 0 \\ \kappa & \kappa^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{e} = \frac{1}{2}\begin{pmatrix} 0 & \kappa & 0 \\ \kappa & -\kappa^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

The Lagrangian displacement gradient, $\mathbf{H}$ and the Eulerian displacement gradient, $\mathbf{h}$ are:

$$\mathbf{H} = \begin{pmatrix} 0 & \kappa & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 0 & \kappa & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then, the Lagrangian linearized strain tensor, $\boldsymbol{\epsilon}_L$ and the Eulerian linearized strain tensor, $\boldsymbol{\epsilon}_E$ are evaluated to be,

$$\boldsymbol{\epsilon}_L = \begin{pmatrix} 0 & \frac{\kappa}{2} & 0 \\ \frac{\kappa}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\epsilon}_E = \begin{pmatrix} 0 & \frac{\kappa}{2} & 0 \\ \frac{\kappa}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

It can be seen from the above that while the Cauchy-Green and Almansi-Hamel tells that the length of line segments along the $\mathbf{e}_y$ direction would change apart from a change in angle of line segments oriented along the $\mathbf{e}_x$ and $\mathbf{e}_y$ directions, Lagrangian linearized and Eulerian linearized strain does not tell that the length of the line segments along $\mathbf{e}_y$ changes. This is because, the change in length along the $\mathbf{e}_y$ direction is of order $\kappa^2$, terms that we neglected to obtain linearized strain. Careful experiments on steel wires of circular cross section subjected to torsion[7] shows axial elongation, akin to the development of the normal strain along $\mathbf{e}_y$ direction. It is this observation that proved that linearized strain is an approximation of the actual strain.

[7]We shall see in chapter 8 that torsion of circular cross section is a pure shear deformation.

Now, say a unit cube oriented in space such that $\mathcal{B} = \{(X, Y, Z)|0 \leq X \leq 1, 0 \leq Y \leq 1, 0 \leq Z \leq 1\}$ is subjected to a deformation of the form (3.124). Then, the deformed volume of the cube, $v$ as given by (3.75) is $v = \det(\mathbf{F}) = 1$. Similarly, the deformed surface area, $a$ of the face whose normal coincides with $\mathbf{e}_x$ in the reference configuration is computed from (3.74) as, $a = \det(\mathbf{F})\sqrt{\mathbf{C}^{-1}\mathbf{e}_x \cdot \mathbf{e}_x} = 1 + \kappa^2$ and the deformed normal direction is found using Nanson's formula (3.72) as, $\mathbf{n} = \det(\mathbf{F})\mathbf{F}^{-t}\mathbf{e}_x(1/a) = [\mathbf{e}_x - \kappa\mathbf{e}_y]/[1 + \kappa^2]$. The deformed length of a line element oriented along $(\mathbf{e}_x + \mathbf{e}_y)/\sqrt{2}$ and of unit length is $\sqrt{1 + \kappa + \kappa^2/2}$ obtained by using (3.60). Similarly, the deformed angle between line segments oriented along $\mathbf{e}_x$ and $(\mathbf{e}_x + \mathbf{e}_y)/\sqrt{2}$ is, $\alpha_f = \cos^{-1}([1 + \kappa]/\sqrt{2 + 2\kappa + \kappa^2})$ which is found using (3.66). In the above computations we have not assumed that the value of $\kappa$ is small; if we do the final expressions would be much simpler.

## 3.11 Compatibility condition

Till now we studied on ways to find the strain given a displacement field. However, there arises a need while solving boundary value problems wherein we would need to find the displacement given the strain field. This problem involves finding the 3 components of the displacement from the 6 independent components of the strain (only six components since strain is a symmetric tensor) that has been prescribed. To be able to find a continuous displacement field from the prescribed 6 components of the linearized strain, all these 6 components cannot be prescribed arbitrarily. The restrictions that these prescribed 6 components of the strain should satisfy is given by the compatibility condition. These restrictions are obtained from the requirement that the sequence of differentiation is immaterial when the first derivative of the multivariate function is continuous, i.e., $\frac{\partial^2 u_x}{\partial x \partial y} = \frac{\partial^2 u_x}{\partial y \partial x}$. Working with index notation, the Cartesian components of the linearized strain is,

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right], \tag{3.125}$$

where $u_i$ are the Cartesian components of the displacement vector, $x_i$ are the Cartesian coordinates of a point. Then, computing

$$\begin{aligned}
\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} &= \frac{1}{2}\left[\frac{\partial^3 u_i}{\partial x_k \partial x_l \partial x_j} + \frac{\partial u_j}{\partial x_k \partial x_l \partial x_i}\right], && (3.126)\\
\frac{\partial^2 \epsilon_{kl}}{\partial x_i \partial x_j} &= \frac{1}{2}\left[\frac{\partial^3 u_k}{\partial x_i \partial x_j \partial x_l} + \frac{\partial u_l}{\partial x_i \partial x_j \partial x_k}\right], && (3.127)\\
\frac{\partial^2 \epsilon_{jl}}{\partial x_i \partial x_k} &= \frac{1}{2}\left[\frac{\partial^3 u_j}{\partial x_i \partial x_k \partial x_l} + \frac{\partial u_l}{\partial x_i \partial x_k \partial x_j}\right], && (3.128)\\
\frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_l} &= \frac{1}{2}\left[\frac{\partial^3 u_i}{\partial x_j \partial x_l \partial x_k} + \frac{\partial u_k}{\partial x_j \partial x_l \partial x_i}\right], && (3.129)
\end{aligned}$$

we find that

$$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \epsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \epsilon_{jl}}{\partial x_i \partial x_k} - \frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_l} = 0, \qquad (3.130)$$

since the order of differentiation is immaterial. Equation (3.130) is called Saint Venant compatibility equations. It can be also seen that equation (3.130) has 81 individual equations as each of the index - $i$, $j$, $k$, $l$ - takes values 1, 2 and 3. It has been shown that of these 81 equations most are either simple identities or repetitions and only 6 are meaningful. These six equations involving the 6 independent Cartesian components of the linearized strain are:

$$\begin{aligned}
\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} &= 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} && (3.131)\\
\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} &= 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z} && (3.132)\\
\frac{\partial^2 \epsilon_{xx}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial x^2} &= 2\frac{\partial^2 \epsilon_{xz}}{\partial x \partial z} && (3.133)\\
\frac{\partial}{\partial y}\left(\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{xz}}{\partial y}\right) &= \frac{\partial^2 \epsilon_{yy}}{\partial z \partial x} && (3.134)\\
\frac{\partial}{\partial z}\left(\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{xz}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial z}\right) &= \frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} && (3.135)\\
\frac{\partial}{\partial x}\left(\frac{\partial \epsilon_{xz}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} - \frac{\partial \epsilon_{yz}}{\partial x}\right) &= \frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} && (3.136)
\end{aligned}$$

Next, we show that these 6 equations are necessary conditions. For this assume $\epsilon_{xy} = xy$ and all other component of the strains are 0. For this strain it can be shown that equation (3.131) alone is violated and that no smooth displacement field would result in this state of strain. Similarly, on assuming

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & yz \\ 0 & yz & 0 \end{pmatrix}, \qquad \boldsymbol{\epsilon} = \begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ xz & 0 & 0 \end{pmatrix},$$

we find that for each of these state of strain only one of the compatibility equations is violated; equation (3.132) for the first strain and (3.133) for the other strain. In the same fashion, on assuming

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\epsilon} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & xy \end{pmatrix}, \qquad \boldsymbol{\epsilon} = \begin{pmatrix} yz & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we find that for each of these state of strain only one of the compatibility equations is violated; equation (3.134) for the first strain, (3.135) for the second strain and (3.136) for the last strain. Thus, we find that all the six equations are required to ensure the existence of a smooth displacement field given a strain field.

However, there are claims that the number of independent compatibility equations is only three. This is not correct for the following reasons. On differentiating the strain field four times instead of twice as done in case of compatibility condition, the six compatibility equations (3.131) through (3.136) can be shown to be equivalent to the following three equations:

$$\frac{\partial^4 \epsilon_{xx}}{\partial y^2 \partial z^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left[-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{xz}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\right], \tag{3.137}$$

$$\frac{\partial^4 \epsilon_{yy}}{\partial z^2 \partial x^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left[\frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{xz}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z}\right], \tag{3.138}$$

$$\frac{\partial^4 \epsilon_{zz}}{\partial x^2 \partial y^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left[\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{xz}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial z}\right]. \tag{3.139}$$

This assumes existence of non-zero higher derivatives of strain field which may not be always true, as in the case of the strain field assumed above. Hence, only if higher order derivatives of strain field exist and is different from zero can one replace the 6 compatibility conditions (3.131) through (3.136) by (3.137) through (3.139).

Further, it is not easy to show that the above compatibility conditions (3.131) through (3.136) are sufficient to find a smooth displacement field given a strain field (see for example Sadd [4] to establish sufficiency of these conditions). In fact the sufficiency of these conditions in multiply connected[8] bodies is yet to be established.

It should be mentioned that the form of the compatibility equation depends on the strain measure that is prescribed. For Cauchy-Green strain it is given in terms of the Riemannian tensor for simply connected bodies[9]. However, it is still not clear what these compatibility conditions have to be for other classes of bodies when Cauchy-Green strain is used. Development of compatibility conditions for other strain measures is beyond the scope of this course.

## 3.12 Summary

In this chapter, we saw how to describe the motion of a body. Then, we focused on finding how curves, surfaces and volumes get transformed due to this motion. These transformations seem to depend on right Cauchy-Green deformation tensor, $\mathbf{C}$, related to the gradient of the deformation field, $\mathbf{F}$ through

$$\mathbf{C} = \mathbf{F}^t\mathbf{F}. \tag{3.140}$$

This was followed by a discussion on a need for a quantity called strain and various definitions of the strain. Associated with each definition of strain, we found a strain tensor that carried all the information required to compute the strain along any direction. Here we showed that when the components of the gradient of the displacement field, $\mathbf{h}$, is small then the term $\mathbf{h}^t\mathbf{h}$ in the definition of the strain can be ignored to obtain, what is called as the linearized strain tensor,

$$\boldsymbol{\epsilon} = \frac{1}{2}[\mathbf{h} + \mathbf{h}^t]. \tag{3.141}$$

[8]Bodies where simple closed curves cannot be shrunk to a point without going outside the body are called multiply connected. Annular cylinders, annular spheres (tennis ball) are examples of multiply connected bodies.

[9]Bodies where simple closed curves can be shrunk to a point without going outside the body are called simply connected. Cuboid, solid cylinders, solid spheres are examples of simply connected bodies.

By virtue of the linearized strain being symmetric, there are only 6 independent components. Since, these 6 independent components are to be obtained from a smooth displacement field, they cannot be prescribed arbitrarily. It was shown that they have to satisfy the compatibility condition.

This completes our study on the motion of bodies. Since, in this course, we are interested in understanding the response of solid like materials undergoing non-dissipative response, we did not study in detail about velocity and its gradients. A course on fluid mechanics would focus on velocity field and its gradient. In the following chapter we shall discuss the concept of stress.

## 3.13 Self-Evaluation

1. A body in the form of a cube, $\mathcal{B} = \{(X, Y, Z) | 0 \leq X \leq 1cm, 0 \leq Y \leq 1cm, 0 \leq Z \leq 1cm\}$ in the reference configuration, is subjected to the following deformation field: $x = X$, $y = Y + A * Z$, $z = Z + A * Y$, where $A$ is a constant and $(X, Y, Z)$ are the Cartesian coordinates of a material point before deformation and $(x, y, z)$ are the Cartesian coordinates of the same material point after deformation. For the specified deformation field:

   (a) Determine the displacement vector components in both the material and spatial forms.

   (b) Determine the location of the particle originally at Cartesian coordinates $(1, 0, 1)$

   (c) Determine the location of the particle in the reference configuration, if its current Cartesian coordinates are $(1, 0, 1)$

   (d) Determine the displacement of the particle originally at Cartesian coordinates $(1, 0, 1)$

   (e) Determine the displacement of the particle currently at Cartesian coordinates $(1, 0, 1)$

   (f) Determine the deformation gradient and Eulerian and Lagrangian displacement gradients

   (g) Calculate the right Cauchy-Green deformation tensor

   (h) Calculate the linearized Lagrangian strain and linearized Eulerian strain. Compare and comment on the value of the strain measures

(i) Calculate the change in the angle between two line segments initially oriented along $\mathbf{E}_y$ and $\mathbf{E}_z$ directions in the reference configuration

(j) Calculate the change in the volume of the cube

(k) Calculate the deformed surface area and its orientation for each of the six faces of a cube

(l) Calculate the change in length of the straight line segments of length 1 mm initially oriented along (i) $\mathbf{E}_y$ (ii) $\mathbf{E}_z$ (iii) $(\mathbf{E}_y + \mathbf{E}_z)/\sqrt{2}$

(m) Is the motion field realizable for any value of A in the cube? Justify. If not, find the values that A can take.

(n) Determine the displaced location of the material particles which originally comprise

   (i) The plane circular surface $X = 0$, $Y^2 + Z^2 = 0.25$,
   (ii) The plane elliptical surface $X = 0$, $9Y^2 + 4Z^2 = 1$.
   (iii) The plane elliptical surface $X = 0$, $4Y^2 + 9Z^2 = 1$.

(o) Sketch the displaced configurations for (i), (ii) and (iii) in the above problem if $A = 0.1$.

(p) Sketch the deformed configuration of the cube assuming $A = 0.1$.

2. Rework the parts (a) to (p) in the above problem if the cube is subjected to a displacement field of the form, $\mathbf{u} = (A * Y + 2A * Z)\mathbf{e}_y + (3A * Y - A * Z)\mathbf{e}_z$, where $(X, Y, Z)$ denotes the coordinates of a typical material particle in the reference configuration and $\{\mathbf{e}_i\}$ the Cartesian coordinate basis.

3. Which of the following displacement fields of a cube is homogeneous?

   (a) $\mathbf{u} = A * Z\mathbf{e}_y + A * Z\mathbf{e}_z$, where A is a constant
   (b) $\mathbf{u} = [(\cos(\theta) - 1)X + \sin(\theta)Y]\mathbf{e}_x + [-\sin(\theta)X + (\cos(\theta) - 1)Y]\mathbf{e}_y$, where $\theta$ is some constant
   (c) $\mathbf{u} = 3XY^2\mathbf{e}_x + 2XZ\mathbf{e}_y + (Z^2 - XY)\mathbf{e}_z$

   Here as usual $(X, Y, Z)$ denotes the coordinates of a typical material particle in the reference configuration and $\{\mathbf{e}_i\}$ the Cartesian coordinate basis. For these deformation fields find parts (a) to (l) in problem 1.

4. A body in the form of an annular cylinder, $\mathcal{B} = \{(R, \Theta, Z)|0.5 \leq R \leq 1cm, 0 \leq \Theta \leq 2\pi, 0 \leq Z \leq 10cm\}$ is subjected to the following deformation field: $r = \sqrt{r_o^2 - \frac{1}{\Lambda} + \frac{R^2}{\Lambda}}$, $\theta = \Theta + \Omega Z$, $z = \Lambda Z$, where $(R, \Theta, Z)$ denote the coordinates of a typical material particle in the reference configuration, $(r, \theta, z)$ denote the coordinates of the same material particle in the current configuration, $r_o$, $\Omega$ and $\Lambda$ are constants. For this deformation field:

   (a) Determine the displacement vector components in both the material and spatial forms.

   (b) Determine the location of the particle originally at cylindrical polar coordinates $(1, 0, 5)$

   (c) Determine the location of the particle in the reference configuration, if its current cylindrical polar coordinates are $(1, 0, 5)$

   (d) Determine the displacement of the particle originally at cylindrical polar coordinates $(1, 0, 5)$

   (e) Determine the displacement of the particle currently at cylindrical polar coordinates $(1, 0, 5)$

   (f) Determine the deformation gradient and Eulerian and Lagrangian displacement gradients

   (g) Calculate the right Cauchy-Green deformation tensor

   (h) Calculate the linearized Lagrangian strain and linearized Eulerian strain. Compare and comment on the value of the strain measures

   (i) Sketch the deformed shape of the annular cylinder assuming $r_o = 1.1$, $\Lambda = 1.2$ and $\Omega = 0.1$

   (j) Calculate the change in the volume of the annular cylinder

   (k) Calculate the deformed surface area and its orientation for each of the two lateral faces and top and bottom surfaces of the cylinder. Assume $r_o = 1.1$, $\Lambda = 1.2$ and $\Omega = 0.1$

   (l) Calculate the change in the angle between two line segments oriented along $\mathbf{E}_R$ and $\mathbf{E}_Z$ directions in the reference configuration at $(0.5, 0, 5)$. Assume $r_o = 1.1$, $\Lambda = 1.2$ and $\Omega = 0.1$

   (m) Calculate the change in length of the straight line segments of 1 mm length located at $(1, 0, 5)$ and oriented along (i) $\mathbf{E}_R$ (ii) $\mathbf{E}_z$ (iii) $(\mathbf{E}_R + \mathbf{E}_\Theta)/\sqrt{2}$. Assume $r_o = 1.1$, $\Lambda = 1.2$ and $\Omega = 0.1$

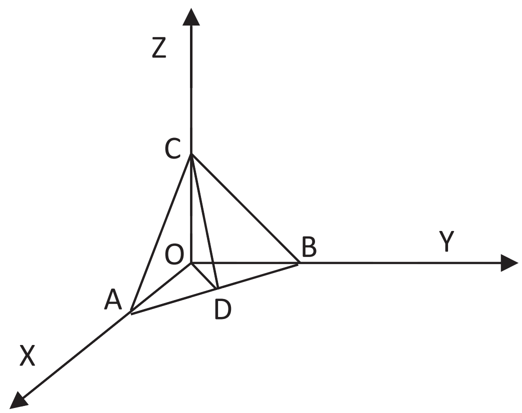



Figure 3.9: Figure for problem 5

(n) Is the motion field realizable for any value of the constants - $r_o$, $\Lambda$ and $\Omega$ - in the cylinder? Justify. If not, find the values that these constants can take.

(o) Find the conditions when this deformation would be homogeneous.

5. Consider the following homogeneous deformation: $x = a_1X + k_1Y$, $y = a_2Y + k_2Z$, $z = a_3Z$, where $a_i$ and $k_i$ are constants and $(X, Y, Z)$ are the Cartesian coordinates of a material particle in the reference configuration and $(x, y, z)$ are the Cartesian coordinates of the same material particle in the current configuration. For this deformation field and the tetrahedron $OABC$ shown in figure 3.9 such that $OA = OB = OC$ and $AD = DB$, compute

   a. The change in length of the line segment $AB$

   b. The change in angle between line segment $AB$ and $CD$

   c. The deformed surface area of the faces $ABC$, $OAB$ and $OBC$

6. A body in the form of a unit cube, $\mathcal{B} = \{(X, Y, Z)|0 \leq X \leq 1, 0 \leq Y \leq 1, 0 \leq Z \leq 1\}$ in the reference configuration, is subjected to the following deformation field: $x = X$, $y = Y + A*X$, $z = Z$, where $A$ is a constant and $(X, Y, Z)$ are the Cartesian coordinates of a material point before deformation and $(x, y, z)$ are the Cartesian coordinates of the

same material point after deformation. For the specified deformation field:

(a) Compute the right Cauchy-Green deformation tensor

(b) Compute the linearized Lagrangian strain tensor

(c) Compute the eigenvalues and eigenvectors for right Cauchy-Green deformation tensor

(d) Compute the eigenvalues and eigenvectors for linearized strain tensor

(e) Find the direction of the line segments in the reference configuration along which the maximum extension or shortening occurs and its value

(f) Find the directions of the line segments in the reference configuration along which no extension or shortening takes place in the cube

(g) Find the maximum change in angle between line segments and the orientation of the line segments for which this occurs.

(h) Find the directions orthogonal to planes in which no change of area occurs

(i) Find the change in volume of the cube

(j) Decompose deformation gradient, $\mathbf{F}$ as $\mathbf{F} = \mathbf{RU}$, where $\mathbf{R}$ is the proper orthogonal tensor and $\mathbf{U}$ is a symmetric positive definite tensor.

7. A body in the form of a unit cube, $\mathcal{B} = \{(X, Y, Z)|0 \leq X \leq 1, 0 \leq Y \leq 1, 0 \leq Z \leq 1\}$ in the reference configuration, is subjected to the following linearized strain field:

$$\boldsymbol{\epsilon} = \begin{pmatrix} AY^3 + BX^2 & CXY(X+Y) & 0 \\ CXY(X+Y) & AX^3 + DY & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.142}$$

where $A$, $B$, $C$, $D$ are constants, find conditions, if any, on the constants if this strain field is to be obtained from a smooth displacement field of the cube. For this value of the constants, find the smooth displacement field that gives raise to the above linearized strain field in a cube. Assume the coordinates of the point $(0,0,0)$ after deformation is $(10,0,0)$ and that of the point $(0,0,1)$ after deformation is $(10,0,1)$ to determine the unknown constants in the displacement field. For what magnitude of these constants is the use of linearized strain justified.

8. A cube of side 10 cm, is subjected to a uniform plane state of strain whose Cartesian components are:

$$\boldsymbol{\epsilon} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} * 10^{-4}. \tag{3.143}$$

For this constant strain field, find

(a) Eigenvalues and eigenvectors of this strain tensor

(b) Maximum change in length and the direction of the material fiber along which this occurs

(c) Maximum change in angle and the orientation of the material fibers along which this occurs

(d) Whether the deformation is isochoric

(e) Change in length of a material fiber oriented along $(\mathbf{e}_x + \mathbf{e}_y)/\sqrt{2}$ and of length 1 mm

(f) Change in angle between line elements oriented along $(\mathbf{e}_x+\mathbf{e}_y)/\sqrt{2}$ and $(\mathbf{e}_x - \mathbf{e}_y)/\sqrt{2}$

9. A rosette strain gauge is an electromechanical device that can measure relative surface elongations in three directions. Bonding such a device to the surface of a structure allows determination of elongation along the direction in which the gauge is located. Figure 3.10 shows the orientation of gauges in one such rosette along with the coordinate system used to study the problem. For a particular loading, these gauges measured the strain along their directions as $\epsilon_a = 0.001$, $\epsilon_b = 0.002$ and $\epsilon_c = 0.004$. Assuming the state of strain at the point of measurement is plane, find the Cartesian components of the strain: $\epsilon_{xx}$, $\epsilon_{yy}$ and $\epsilon_{xy}$ for the orientation of the basis also shown in the figure 3.10.

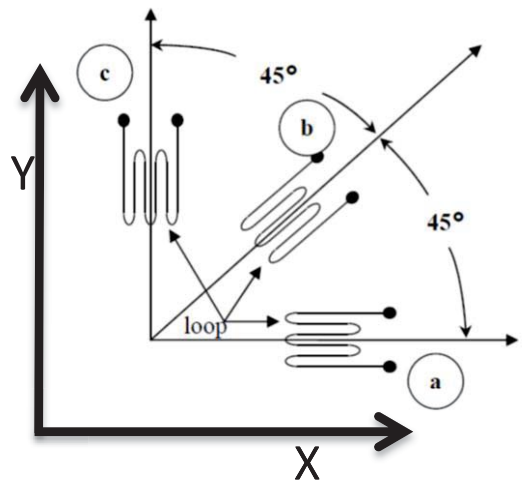



Figure 3.10: Figure for problem 9: Schematic of a strain rosette

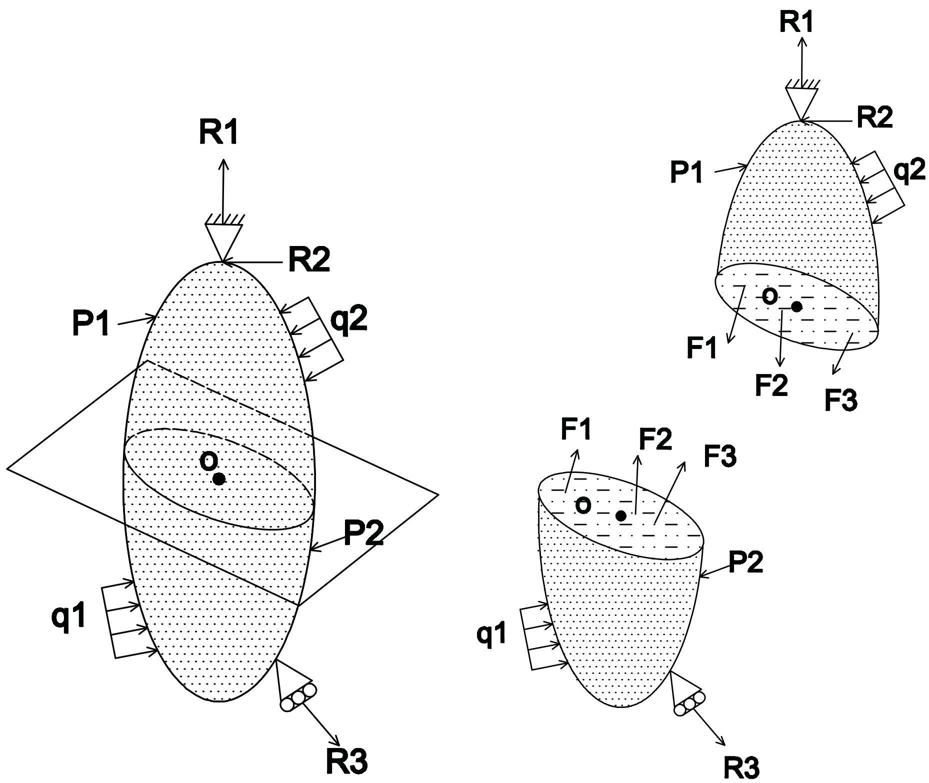
10. It is commonly stated that the rigid body rotation undergone by a body for a given deformation when the components of the displacement gradient are small is the skew-symmetric part of the displacement gradient, i.e. $\boldsymbol{\omega} = [grad(\mathbf{u}) - grad(\mathbf{u})^t]/2$. Show that this is not the case by computing $\mathbf{R}$, the orthogonal tensor in the polar decomposition of the deformation gradient, $\mathbf{F}$ and $\boldsymbol{\omega}$ for the following displacement field of a cube with sides 10 cm: $\mathbf{u} = (AX + BY)\mathbf{e}_x + CY\mathbf{e}_y + DZ\mathbf{e}_z$, where $A$, $B$, $C$, $D$ are constants, $(X, Y, Z)$ are the Cartesian coordinates of a typical material particle in the reference configuration and $\mathbf{e}_i$ the Cartesian coordinate basis. Recollect that $\mathbf{R}$ represents the true rigid body rotation component of the deformation.

# Chapter 4

# Traction and Stress

## 4.1 Overview

Mechanics is the branch of science that describes or predicts the state of rest or of motion of bodies subjected to some forces. In the last chapter, we studied about mathematical descriptors of the state of rest or of motion of bodies. Now, we shall focus on the cause for motion or change of geometry of the body; the force. While in rigid body mechanics, the concept of force is sufficient to describe or predict the motion of the body, in deformable bodies it is not. For example, two bars made of the same material and of same length but with different cross sectional area, will undergo different amount of elongation when subjected to the same pulling force acting parallel to the axis of the bar. It was then found that if we define a quantity called strain which is defined as the ratio between the change in length to the original length of the bar along the direction of the applied load and a quantity called stress which is defined as the force acting per unit area then the stress and strain could be related through an equation that is independent of the geometry of the body but depend only on the material that the body is made up of. While these definitions of strain and stress are adequate to study homogeneous deformations resulting from uniform stress states, these concepts have to be generalized to study the motion of bodies subjected to a non-uniform distribution of forces and/or couples. In this chapter, we shall generalize the concept of stress having already generalized the concept of strain in the last chapter.




(a) External loads acting on a solid in equilibrium

(b) Internal loads acting on a plane passing by O inside a solid

Figure 4.1: Free body diagram showing the forces acting on a body

## 4.2 Traction vectors and stress tensors

Here we focus attention on a deformable continuum body $\mathcal{B}$ occupying an arbitrary region $\mathfrak{B}_t$ of the Euclidean vector space with boundary surface $\partial\mathfrak{B}_t$ at time $t$, as shown in figure 4.1a. We consider a general case where in arbitrary forces act on parts or the whole of the boundary surface called the external forces or loads. Recognizing that these external forces arose because we isolated the body from its surroundings, to find the internal forces we have to section the body. Thus, to find the internal forces on a internal surface passing through $O$, as indicated in the figure 4.1a, we have to section the body as shown in the figure 4.1b, with the cutting plane coinciding with the

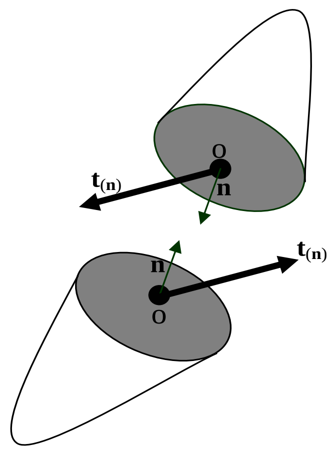



Figure 4.2: Traction vectors acting on infinitesimal surface elements with outward unit normal, $\mathbf{n}$

internal surface. Now, we focus our attention on an infinitesimal part of the body centered around $O$, see figure 4.2. Let $\mathbf{x}$ be the position vector of the point $O$, $\mathbf{n}$ a unit vector directed along the outward normal to an infinitesimal spatial surface element $\Delta s$, at $\mathbf{x}$ and $\Delta\mathbf{f}$ denote the infinitesimal resultant force acting on the surface element $\Delta s$. Then, we claim that

$$\Delta\mathbf{f} = \int_{\Delta s} \mathbf{t}_{(\mathbf{n})} da, \tag{4.1}$$

where $\mathbf{t}_{(\mathbf{n})} = \tilde{\mathbf{t}}(\mathbf{x}, t, \mathbf{n})$ represents the Cauchy (or true) traction vector and the integration here denotes an area integral. Thus, Cauchy traction vector is the force per unit surface area defined in the current configuration acting at a given location. The Cauchy traction vector and hence the infinitesimal force at a given location depends also on the orientation of the cutting plane, i.e., the unit normal $\mathbf{n}$. This means that the traction and hence the infinitesimal force at the point $O$, for a vertical cutting plane could be different from that of a horizontal cutting plane. However, the traction on the two pieces of the cut body would be such that they are equal in magnitude but opposite in direction, in order to satisfy Newton's third law of motion. Hence,

$$\mathbf{t}_{(\mathbf{n})} = -\mathbf{t}_{(-\mathbf{n})}. \tag{4.2}$$

This requires: $\tilde{\mathbf{t}}(\mathbf{x}, t, \mathbf{n}) = -\tilde{\mathbf{t}}(\mathbf{x}, t, -\mathbf{n})$.

Relationship (4.1) is referred to as Cauchy's postulate. It is worthwhile to mention that in any experiment we infer only these traction vectors. In literature, the traction vector is also called as stress vector because they have the units of stress, i.e., force per unit area. However, here we shall not use this terminology and for us stress is always a tensor, as defined next.

### 4.2.1 Cauchy stress theorem

There exist unique second order tensor field $\boldsymbol{\sigma}$, called the Cauchy (or true) stress tensor, so that

$$\mathbf{t}_{(\mathbf{n})} = \tilde{\mathbf{t}}(\mathbf{x}, t, \mathbf{n}) = \boldsymbol{\sigma}^t \mathbf{n}. \tag{4.3}$$

where $\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}(\mathbf{x}, t) = \hat{\boldsymbol{\sigma}}(\mathbf{X}, t)$, $\mathbf{X} \in \mathfrak{B}_r$, is the position vector of the material particle in the reference configuration. It is easy to verify that the requirement (4.2) is met by (4.3). It will be shown in the next chapter that the Cauchy stress tensor, $\boldsymbol{\sigma}$ has to be a symmetric tensor. However, the proof of Cauchy stress theorem is beyond the scope of these lecture notes.

### 4.2.2 Components of Cauchy stress

It is recalled that only traction vector can be determined or inferred in the experiments and hence the components of the stress tensor is estimated from finding the traction vector on three (mutually perpendicular) planes. Let us see how.

The components of the Cauchy stress tensor, $\boldsymbol{\sigma}$ with respect to an orthonormal basis $\{\mathbf{e}_a\}$ is given by:

$$\sigma_{ab} = \mathbf{e}_a \cdot \boldsymbol{\sigma} \mathbf{e}_b = \boldsymbol{\sigma}^t \mathbf{e}_a \cdot \mathbf{e}_b = \mathbf{t}_{(\mathbf{e}_a)} \cdot \mathbf{e}_b, \tag{4.4}$$

In view of Cauchy's theorem $\mathbf{t}_{(\mathbf{e}_a)} = \boldsymbol{\sigma}^t \mathbf{e}_a$, $a = 1, 2, 3$, characterize the three traction vectors acting on the surface elements whose outward normals point in the directions $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, respectively. Then, the components of these traction vectors along $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ gives the various components of the stress tensor. Thus, for each stress component $\sigma_{ab}$ we adopt the mathematically logical convention that the index $b$ characterizes the component of the traction vector, $\mathbf{t}_{(\mathbf{e}_a)}$, at a point $\mathbf{x}$ in the direction of the associated base vector $\mathbf{e}_b$ and the index $a$ characterizes the orientation of the area element on which $\mathbf{t}_{(\mathbf{n})}$ is acting.

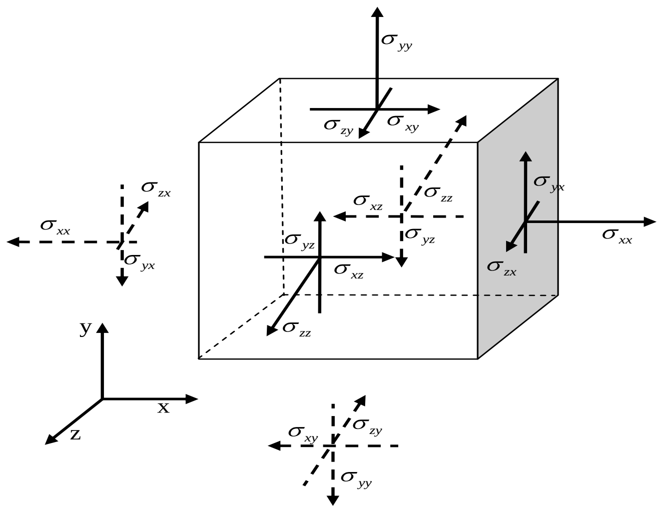



Figure 4.3: Cartesian components of the stress tensor acting on the faces of a cube

It is important to note that some authors reverse this convention by identifying the index $b$ with the orientation of the normal to the plane of cut and the index $a$ with the component of the traction along the direction $\mathbf{e}_a$. Irrespective of the convention adopted the end results will be the same because of two reasons: (i) The definition of divergence is suitably modified with a transpose (ii) the Cauchy stress tensor is anyway symmetric. In other words, to find the components of the Cauchy stress tensor, at a given location, we isolate an infinitesimal cube with the location of interest at its center. The cube is oriented such that the outward normal to its sides is along or opposite to the direction of the basis vectors. Hence, for general curvilinear coordinates, the sides of the cube need not be plane nor make right angles with each other. Compare the stress cube for Cartesian basis (figure 4.3) and cylindrical polar basis (figure 4.4). Then, we determine the traction that is acting on each of the six faces of the cube. Due to equation (4.2) only three of these six traction vectors are independent. The components of the three traction vectors along the the three basis vectors gives the nine components of the stress tensor.

If the component of the traction is along the direction of the basis vectors on planes whose outward normal coincides with the direction of the basis

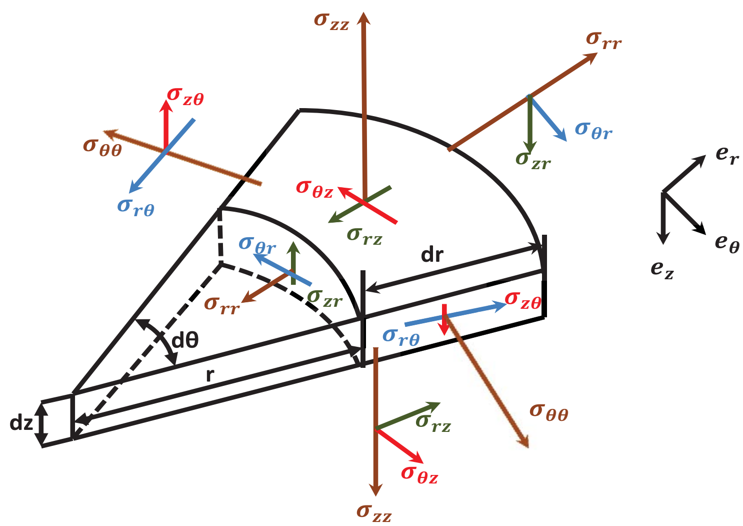



Figure 4.4: Cylindrical polar basis components of the stress tensor acting on the faces of a cube

vectors it is considered to be positive and negative otherwise. Consistently, if the component of the traction is opposite to the direction of the basis vectors on faces of the cube whose outward normal is opposite in direction to the basis vectors, it is considered to be positive and negative otherwise. Thus, one should not be confused that the same stress component points in opposite directions on opposite planes. This is necessary for the cube under consideration to be in equilibrium. It should also be appreciated that the sign of some of the stress component does change when the direction of the coordinate basis is reversed even if the right handedness of the basis vectors is maintained. This is so, because the sign of the stress component depends both on the direction of coordinate basis as well as the outward normal. Thus, figure 4.3 portrays the positive components of the stress tensor when Cartesian coordinate basis is used and it is called as the stress cube.

The positive cylindrical polar components of the stress tensor is depicted on a cylindrical wedge in figure 4.4. Recognize that the direction of the cylindrical polar components of the stress changes with the location unlike the Cartesian coordinate components whose directions are fixed. This happens because the direction of the cylindrical polar coordinate basis vectors depends on the location. Here $\sigma_{rr}$ component of the stress is called as the radial stress,

$\sigma_{\theta\theta}$ component of the stress is known as the hoop or the circumferential stress, $\sigma_{zz}$ component of the stress is the axial stress.

## 4.3 Normal and shear stresses

Let the traction vector $\mathbf{t}_{(\mathbf{n})}$ for a given current position $\mathbf{x}$ at time $t$ act on an arbitrarily oriented surface element characterized by an outward unit normal vector $\mathbf{n}$.

The traction vector $\mathbf{t}_{(\mathbf{n})}$ may be resolved into the sum of a vector along the normal $\mathbf{n}$ to the plane, denoted by $\mathbf{t}^{\|}_{(\mathbf{n})}$ and a vector perpendicular to $\mathbf{n}$ denoted by $\mathbf{t}^{\perp}_{(\mathbf{n})}$, i.e., $\mathbf{t}_{(\mathbf{n})} = \mathbf{t}^{\|}_{(\mathbf{n})} + \mathbf{t}^{\perp}_{(\mathbf{n})}$. From the results in section 2.3.3, it could be seen that

$$\mathbf{t}^{\|}_{(\mathbf{n})} = [\mathbf{n}\cdot\mathbf{t}_{(\mathbf{n})}]\mathbf{n} = [\mathbf{n}\cdot\boldsymbol{\sigma}\mathbf{n}]\mathbf{n} = \sigma_{\mathbf{n}}\mathbf{n}, \tag{4.5}$$

$$\mathbf{t}^{\perp}_{(\mathbf{n})} = \mathbf{t}_{(\mathbf{n})} - [\mathbf{n}\cdot\mathbf{t}_{(\mathbf{n})}]\mathbf{n} = [\mathbf{1} - \mathbf{n}\otimes\mathbf{n}]\mathbf{t}_{(\mathbf{n})} = \tau_{\mathbf{n}}\mathbf{m}, \tag{4.6}$$

where

$$\tau_{\mathbf{n}} = \mid \mathbf{t}_{(\mathbf{n})} - [\mathbf{n}\cdot\mathbf{t}_{(\mathbf{n})}]\mathbf{n} \mid = \mid \boldsymbol{\sigma}\mathbf{n} - [\mathbf{n}\cdot\boldsymbol{\sigma}\mathbf{n}]\mathbf{n} \mid, \tag{4.7}$$

$$\mathbf{m} = \frac{1}{\tau_{\mathbf{n}}}\{\mathbf{t}_{(\mathbf{n})} - [\mathbf{n}\cdot\mathbf{t}_{(\mathbf{n})}]\mathbf{n}\}. \tag{4.8}$$

It could easily be verified that $\mathbf{n}\cdot\mathbf{m} = 0$. This means $\mathbf{m}$ is a vector embedded in the surface. Then, $\sigma_{\mathbf{n}}$ is called the normal traction and $\tau_{\mathbf{n}}$ is called the shear traction. As the names suggest, normal traction acts perpendicular to the surface and shear traction acts tangential or parallel to the surface.

Since, $\mathbf{t}_{(\mathbf{n})} = \mathbf{t}^{\|}_{(\mathbf{n})} + \mathbf{t}^{\perp}_{(\mathbf{n})}$, we obtain the useful relation

$$\mid \mathbf{t}_{(\mathbf{n})} \mid^2 = \sigma_{\mathbf{n}}^2 + \tau_{\mathbf{n}}^2, \tag{4.9}$$

where we have made use of the property that $\mathbf{n}\cdot\mathbf{m} = 0$.

It could be seen from figure 4.3 that the stress components corresponding to Cartesian basis, $\sigma_{xx}$, $\sigma_{yy}$ and $\sigma_{zz}$ act normal to their respective surfaces and hence are called normal stresses and the remaining independent components, $\sigma_{xy}$, $\sigma_{yz}$ and $\sigma_{zx}$ act parallel to the surface and hence are called shear stresses. (Remember that Cauchy stress is a symmetric tensor.) When the stress components are determined with respect to any coordinate basis vectors, there will be three normal stresses corresponding to $\sigma_{ii}$ and three shear stresses, $\sigma_{ij}$, $i \neq j$. Here to call certain components of the stresses as normal and others as shear, we have exploited the relationship between the components of the stress tensor and the traction vector.

## 4.4 Principal stresses and directions

At a given location in the body the magnitude of the normal and shear traction depend on the orientation of the plane. An isotropic body can fail along any plane and hence we require to find the maximum magnitude of these normal and shear traction and the planes for which these occur. As we shall see on planes where the maximum or minimum normal stresses occur the shear stresses are zero. However, in the plane on which maximum shear stress occurs, the normal stresses will exist.

### 4.4.1 Maximum and minimum normal traction

In order to obtain the maximum and minimum values of $\sigma_{\mathbf{n}}$, the normal traction, we have to find the orientation of the plane in which this occurs. This is done by maximizing (4.5) subject to the constraint that $\mid \mathbf{n} \mid = 1$. This constrained optimization is done by what is called as the Lagrange-multiplier method. Towards this, we introduce the function

$$\mathcal{L}(\mathbf{n}, \lambda^*) = \mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n} - \lambda^*[\mid \mathbf{n} \mid^2 - 1], \tag{4.10}$$

where $\lambda^*$ is the Lagrange multiplier and the condition $\mid \mathbf{n} \mid^2 - 1 = 0$ characterizes the constraint condition. At locations where the extremal values of $\mathcal{L}$ occurs, the derivatives $\frac{\partial \mathcal{L}}{\partial \mathbf{n}}$ and $\frac{\partial \mathcal{L}}{\partial \lambda^*}$ must vanish, i.e.,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{n}} = 2(\boldsymbol{\sigma}\mathbf{n} - \lambda^*\mathbf{n}) = \mathbf{o}, \tag{4.11}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda^*} = \mid \mathbf{n} \mid^2 - 1 = 0. \tag{4.12}$$

To obtain these equations we have made use of the fact that Cauchy stress tensor is symmetric. Thus, we have to find three $\hat{\mathbf{n}}_a$ and $\lambda^*_a$'s such that

$$(\boldsymbol{\sigma} - \lambda^*_a \mathbf{1})\hat{\mathbf{n}}_a = \mathbf{o}, \qquad \mid \hat{\mathbf{n}}_a \mid^2 = 1, \tag{4.13}$$

($a$ = 1, 2, 3; no summation) which is nothing but the eigenvalue problem involving the tensor $\boldsymbol{\sigma}$ with the Lagrange multiplier being identified as the eigenvalue. Hence, the results of section 2.5 follows. In particular, for (4.13a) to have a non-trivial solution

$$\lambda_a^{*3} - K_1\lambda_a^{*2} + K_2\lambda_a^* - K_3 = 0, \tag{4.14}$$

where

$$K_1 = tr(\boldsymbol{\sigma}), \qquad K_2 = \frac{1}{2}[K_1^2 - tr(\boldsymbol{\sigma}^2)], \qquad K_3 = \det(\boldsymbol{\sigma}), \tag{4.15}$$

the principal invariants of the stress $\boldsymbol{\sigma}$. As stated in section 2.5, equation (4.14) has three real roots, since the Cauchy stress tensor is symmetric. These roots $(\lambda_a^*)$ will henceforth be denoted by $\sigma_1$, $\sigma_2$ and $\sigma_3$ and are called principal stresses. The principal stresses include both the maximum and minimum normal stresses among all planes passing through a given $\mathbf{x}$.

The corresponding three orthonormal eigenvectors $\hat{\mathbf{n}}_a$, which are then characterized through the relation (4.13) are called the principal directions of $\boldsymbol{\sigma}$. The planes for which these eigenvectors are normal are called principal planes. Further, these eigenvectors form a mutually orthogonal basis since the stress tensor $\boldsymbol{\sigma}$ is symmetric. This property of the stress tensor also allows us to represent $\boldsymbol{\sigma}$ in the spectral form

$$\boldsymbol{\sigma} = \sum_{a=1}^{3} \sigma_a \hat{\mathbf{n}}_a \otimes \hat{\mathbf{n}}_a. \tag{4.16}$$

It is immediately apparent that the shear stresses in the principal planes are zero. Thus, principal planes can also be defined as those planes in which the shear stresses vanish. Consequently, $\sigma_a$'s are normal stresses.

### 4.4.2 Maximum and minimum shear traction

Next, we are interested in finding the direction of the unit vector $\mathbf{n}$ at $\mathbf{x}$ that gives the maximum and minimum values of the shear traction, $\tau_{\mathbf{p}}$. This is important because in many metals the failure is due to sliding of planes resulting due to the shear traction exceeding a critical value along some plane.

In the following we choose the eigenvectors $\{\hat{\mathbf{n}}_a\}$ of $\boldsymbol{\sigma}$ as the set of basis vectors. Then, according to the spectral decomposition (4.16), all the non-diagonal matrix components of the Cauchy stress vanish. Then, the traction vector $\mathbf{t}_{(\mathbf{p})}$ on an arbitrary plane with normal $\mathbf{p}$ could simply be written as

$$\mathbf{t}_{(\mathbf{p})} = \boldsymbol{\sigma}\mathbf{p} = \sigma_1 p_1 \hat{\mathbf{n}}_1 + \sigma_2 p_2 \hat{\mathbf{n}}_2 + \sigma_3 p_3 \hat{\mathbf{n}}_3, \tag{4.17}$$

where $\mathbf{p} = p_1\hat{\mathbf{n}}_1 + p_2\hat{\mathbf{n}}_2 + p_3\hat{\mathbf{n}}_3$. It then follows from (4.9) that

$$\begin{aligned} \tau_{\mathbf{p}}^2 &= \mid \mathbf{t}_{(\mathbf{p})} \mid^2 - \sigma_{\mathbf{p}}^2 \\ &= \sigma_1^2 p_1^2 + \sigma_2^2 p_2^2 + \sigma_3^2 p_3^2 - (\sigma_1 p_1^2 + \sigma_2 p_2^2 + \sigma_3 p_3^2)^2, \end{aligned} \tag{4.18}$$

where $\sigma_{\mathbf{p}}$ is the normal traction on the plane whose normal is $\mathbf{p}$.

With the constraint condition $\mid \mathbf{p} \mid^2 = 1$ we can eliminate $p_3$ from the equation (4.18). Then, since the principal stresses, $\sigma_1$, $\sigma_2$, $\sigma_3$ are known, $\tau_{\mathbf{p}}^2$ is a function of only $p_1$ and $p_2$. Therefore to obtain the extremal values of $\tau_{\mathbf{p}}^2$ we differentiate $\tau_{\mathbf{p}}^2$ with respect to $p_1$ and $p_2$ and equate it to zero[1]:

$$\frac{\partial \tau_{\mathbf{p}}^2}{\partial p_1} = 2p_1[\sigma_1 - \sigma_3]\{\sigma_1 - \sigma_3 - 2[(\sigma_1 - \sigma_3)p_1^2 + (\sigma_2 - \sigma_3)p_2^2]\} = 0, \tag{4.19}$$

$$\frac{\partial \tau_{\mathbf{p}}^2}{\partial p_2} = 2p_2[\sigma_2 - \sigma_3]\{\sigma_2 - \sigma_3 - 2[(\sigma_1 - \sigma_3)p_1^2 + (\sigma_2 - \sigma_3)p_2^2]\} = 0. \tag{4.20}$$

The above set of equations has three classes of solutions.

- Set - 1: $p_1 = p_2 = 0$ and hence $p_3 = \pm 1$, obtained from the condition that $\mid \mathbf{p} \mid^2 = 1$.

- Set - 2: $p_1 = 0$, $p_2 = \pm 1/\sqrt{2}$ and hence $p_3 = \pm 1/\sqrt{2}$.

- Set - 3: $p_1 = \pm 1/\sqrt{2}$, $p_2 = 0$ and hence $p_3 = \pm 1/\sqrt{2}$.

Instead of eliminating $p_3$ we could eliminate $p_2$ or $p_1$ initially and find the remaining unknowns by adopting a procedure similar to the above. Then, we shall find the extremal values of $\tau_{\mathbf{p}}$ could also occur when

- Set - 4: $p_1 = p_3 = 0$ and hence $p_2 = \pm 1$.

- Set - 5: $p_2 = p_3 = 0$ and hence $p_1 = \pm 1$.

- Set - 6: $p_3 = 0$, $p_1 = \pm 1/\sqrt{2}$ and hence $p_2 = \pm 1/\sqrt{2}$.

[1]Recognize that the extremal values of both $\tau_{\mathbf{p}}$ and $\tau_{\mathbf{p}}^2$ occur at the same location.

Substituting these solutions in (4.18) we find the extremal values of $\tau_{\mathbf{p}}$. Thus, $\tau_{\mathbf{p}} = 0$ when $\mathbf{p} = \pm\hat{\mathbf{n}}_1$ or $\mathbf{p} = \pm\hat{\mathbf{n}}_2$ or $\mathbf{p} = \pm\hat{\mathbf{n}}_3$ and

$$\mathbf{p} = \pm\frac{1}{\sqrt{2}}\hat{\mathbf{n}}_2 \pm \frac{1}{\sqrt{2}}\hat{\mathbf{n}}_3, \qquad \tau_{\mathbf{p}}^2 = \frac{1}{4}(\sigma_2 - \sigma_3)^2, \tag{4.21}$$

$$\mathbf{p} = \pm\frac{1}{\sqrt{2}}\hat{\mathbf{n}}_1 \pm \frac{1}{\sqrt{2}}\hat{\mathbf{n}}_3, \qquad \tau_{\mathbf{p}}^2 = \frac{1}{4}(\sigma_1 - \sigma_3)^2, \tag{4.22}$$

$$\mathbf{p} = \pm\frac{1}{\sqrt{2}}\hat{\mathbf{n}}_1 \pm \frac{1}{\sqrt{2}}\hat{\mathbf{n}}_2, \qquad \tau_{\mathbf{p}}^2 = \frac{1}{4}(\sigma_1 - \sigma_2)^2. \tag{4.23}$$

Consequently, the maximum magnitude of the shear traction denoted by $\tau_{max}$ is given by the largest of the three values of (4.21b) - (4.23b). Thus, we obtain

$$\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min}), \tag{4.24}$$

where $\sigma_{max}$ and $\sigma_{min}$ denote the maximum and minimum magnitudes of principal stresses, respectively. Recognize that the maximum shear stress acts on a plane that is shifted about an angle of $\pm 45$ degrees to the principal plane in which the maximum and minimum principal stresses act. In addition, we can show that the normal traction $\sigma_{\mathbf{p}}$ to the plane in which $\tau_{max}$ occurs has the value $\sigma_{\mathbf{p}} = (\sigma_{max} + \sigma_{min})/2$.

## 4.5 Stresses on a Octahedral plane

Consider now a tetrahedron element similar to that of figure 4.5 with a plane equally inclined to the principal axes. Hence, its normal is given by

$$\mathbf{p} = \pm\frac{1}{\sqrt{3}}\hat{\mathbf{n}}_1 \pm \frac{1}{\sqrt{3}}\hat{\mathbf{n}}_2 \pm \frac{1}{\sqrt{3}}\hat{\mathbf{n}}_3, \tag{4.25}$$

where $\hat{\mathbf{n}}_a$ are the three principal directions of the Cauchy stress tensor. Then, the normal traction, $\sigma_{oct}$, on this plane is

$$\sigma_{oct} = \frac{1}{3}[\sigma_1 + \sigma_2 + \sigma_3], \tag{4.26}$$

obtained using (4.5) and the spectral representation for the Cauchy stress (4.16). Similarly, the shear traction, $\tau_{oct}$ on this plane is

$$\tau_{oct}^2 = \frac{1}{3}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2] - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2. \tag{4.27}$$

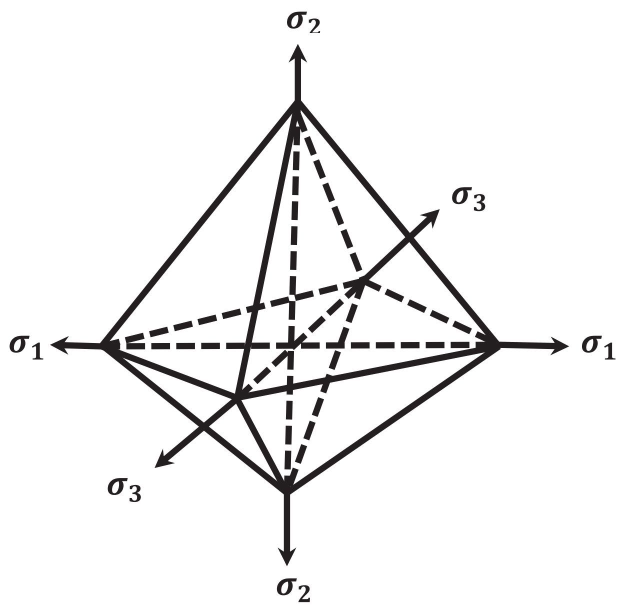



Figure 4.5: Eight octahedral planes equally inclined to the principal axes.

From the definition of principal invariants for stress (4.15) it is easy to verify that

$$\sigma_{oct} = \frac{K_1}{3}, \qquad \tau_{oct} = \frac{2}{9}K_1^2 - \frac{2}{3}K_2. \tag{4.28}$$

Using the above formula one can compute the octahedral normal and shear traction for stress tensor represented using some arbitrary basis.

## 4.6 Examples of state of stress

Specifying a state of stress means providing sufficient information to compute the components of the stress tensor with respect to some basis. As discussed before, knowing $\mathbf{t}_{(\mathbf{n})}$ for three independent pairs of $\{\mathbf{t}_{(\mathbf{n})}, \mathbf{n}\}$ we can construct the stress tensor $\boldsymbol{\sigma}$. Hence, specifying the set of pairs $\{(\mathbf{t}_{(\mathbf{n})}, \mathbf{n})\}$ for three independent normal vectors, $\mathbf{n}$, at a given point, so that the stress tensor could be uniquely determined tantamount to prescribing the state of stress. Next, we shall look at some states of stress.

The state of stress is said to be uniform if the stress tensor does not depend on the space coordinates at each time $t$, when the stress tensor is represented using Cartesian basis vectors.

If the stress tensor has a representation

$$\boldsymbol{\sigma} = \sigma \mathbf{n} \otimes \mathbf{n}, \tag{4.29}$$

at some point, where $\mathbf{n}$ is a unit vector, we say it is in a pure normal stress state. Post-multiplying (4.29) with the unit vector $\mathbf{n}$ we find that

$$\boldsymbol{\sigma}\mathbf{n} = \sigma(\mathbf{n} \otimes \mathbf{n})\mathbf{n} = \sigma(\mathbf{n} \cdot \mathbf{n})\mathbf{n} = \sigma\mathbf{n} = \mathbf{t}_{\mathbf{n}}^{\parallel}. \tag{4.30}$$

Evidently, the traction is along (or opposite to) $\mathbf{n}$. This stress $\sigma$ characterizes either pure tension (if $\sigma > 0$) or pure compression (if $\sigma < 0$).

If we have a uniform stress state and the stress tensor when represented using a Cartesian basis is such that $\sigma_{xx} = \sigma = const$ and all other stress components are zero, then such a stress state is referred to as uniaxial tension or uniaxial compression depending on whether $\sigma$ is positive or negative respectively. This may be imagined as the stress in a rod with uniform cross-section generated by forces applied to its plane ends in the $x-$ direction. However, recognize that this is not the only system of forces that would result in the above state of stress; all that we require is that the resultant of a system of forces should be oriented along the $\mathbf{e}_x$ direction.

If the stress tensor has a representation

$$\boldsymbol{\sigma} = \sigma(\mathbf{n} \otimes \mathbf{n} + \mathbf{m} \otimes \mathbf{m}), \tag{4.31}$$

at any point, where $\mathbf{n}$ and $\mathbf{m}$ are unit vectors such that $\mathbf{n} \cdot \mathbf{m} = 0$, then it is said to be in equibiaxial stress state.

On the other hand, if the stress tensor has a representation

$$\boldsymbol{\sigma} = \tau(\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n}), \tag{4.32}$$

at any point, where $\mathbf{n}$ and $\mathbf{m}$ are unit vectors such that $\mathbf{n} \cdot \mathbf{m} = 0$, then it is said to be in pure shear stress state. Post-multiplying (4.32) with the unit vector $\mathbf{n}$ we obtain

$$\begin{aligned} \boldsymbol{\sigma}\mathbf{n} &= \tau(\mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n})\mathbf{n} \\ &= \tau[(\mathbf{m} \cdot \mathbf{n})\mathbf{n} + (\mathbf{n} \cdot \mathbf{n})\mathbf{m}] = \tau\mathbf{m} = \mathbf{t}_{\mathbf{m}}^{\perp}. \end{aligned} \tag{4.33}$$

Evidently, $\mathbf{t}_{\mathbf{m}}^{\perp}$ is tangential to the surface whose outward unit normal is along (or opposite to) $\mathbf{n}$.

More generally, if the stress tensor has a representation

$$\boldsymbol{\sigma} = \sigma_1 \mathbf{n} \otimes \mathbf{n} + \sigma_2 \mathbf{m} \otimes \mathbf{m}, \tag{4.34}$$

at any point, where $\mathbf{n}$ and $\mathbf{m}$ are unit vectors such that $\mathbf{n} \cdot \mathbf{m} = 0$, then it is said to be in plane or biaxial state of stress. That is in this case one of the principal stresses is zero. A general matrix representation for the stress tensor corresponding to a plane stress state is:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.35}$$

Here we have assumed that $\mathbf{e}_3$ is a principal direction and that there exist no stress components along this direction. We could have assumed the same with respect to any one of the other basis vectors. A plane stress state occurs at any unloaded surface in a continuum body and is of practical interest.

Next, we consider 3D stress states. Analogous to the equibiaxial stress state in 2D, if the stress tensor has a representation

$$\boldsymbol{\sigma} = -p\mathbf{1}, \tag{4.36}$$

at some point, we say that it is in a hydrostatic state of stress and $p$ is called as hydrostatic pressure. It is just customary to consider compressive hydrostatic pressure to be positive and hence the negative sign. Post-multiplying (4.36) by some unit vector $\mathbf{n}$, we obtain

$$\boldsymbol{\sigma}\mathbf{n} = (-p\mathbf{1})\mathbf{n} = -p\mathbf{n} = \mathbf{t}_{\mathbf{n}}^{\|}. \tag{4.37}$$

Thus, on any surface only normal traction acts, which is characteristic of (elastic) fluids at rest that is not able to sustain shear stresses. Hence, this stress is called hydrostatic.

Any other state of stress is called to be triaxial stress state.

Many a times the stress is uniquely additively decomposed into two parts namely an hydrostatic component and a deviatoric component, that is

$$\boldsymbol{\sigma} = \frac{1}{3} tr(\boldsymbol{\sigma})\mathbf{1} + \boldsymbol{\sigma}^{dev}. \tag{4.38}$$

Thus, the deviatoric stress is by definition,

$$\boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma} - \frac{1}{3} tr(\boldsymbol{\sigma})\mathbf{1}, \tag{4.39}$$

and has the property that $tr(\boldsymbol{\sigma}^{dev}) = 0$. Physically the hydrostatic component of the stress is supposed to cause volume changes in the body and the deviatoric component cause distortion in the body. (Shear deformation is a kind of distortional deformation.)

## 4.7 Other stress measures

Till now, we have been looking only at the Cauchy (or true) stress. As its name suggest this is the "true" measure of stress. However, there are many other definition of stresses but all of these are propounded just to facilitate easy algebra. Except for the first Piola-Kirchhoff stress, other stress measures do not have a physical interpretation as well.

### 4.7.1 Piola-Kirchhoff stress tensors

Let $\partial\Omega_t$ denote some surface within or on the boundary of the body in the current configuration. Then, the net force acting on this surface is given by

$$\mathbf{f} = \int_{\partial\Omega_t} \mathbf{t}_{(\mathbf{n})} da = \int_{\partial\Omega_t} \boldsymbol{\sigma}\mathbf{n} da. \tag{4.40}$$

It is easy to compute the above integral when the Cauchy stress is expressed as a function of $\mathbf{x}$, the position vector of a typical material point in the current configuration, i.e. $\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}(\mathbf{x}, t)$. On the other hand computing the above integral becomes difficult when the stress is a function of $\mathbf{X}$, the position vector of a typical material point in the reference configuration, i.e. $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{X}, t)$. To facilitate the computation of the above integral in the later case, we appeal to the Nanson's formula (3.72) to obtain

$$\mathbf{f} = \int_{\partial\Omega_t} \mathbf{t}_{(\mathbf{n})} da = \int_{\partial\Omega_t} \boldsymbol{\sigma}\mathbf{n} da = \int_{\partial\Omega_r} \det(\mathbf{F})\boldsymbol{\sigma}\mathbf{F}^{-t}\mathbf{N} dA, \tag{4.41}$$

where $\partial\Omega_r$ is the surface in the reference configuration, formed by the same material particles that formed the surface $\partial\Omega_t$ in the current configuration. Then, the first Piola-Kirchhoff stress, $\mathbf{P}$ is defined as

$$\mathbf{P} = \det(\mathbf{F})\boldsymbol{\sigma}\mathbf{F}^{-t}. \tag{4.42}$$

Using this definition of Piola-Kirchhoff stress, equation (4.41) can be written as

$$\mathbf{f} = \int_{\partial\Omega_t} \boldsymbol{\sigma}\mathbf{n} da = \int_{\partial\Omega_r} \mathbf{P}\mathbf{N} dA. \tag{4.43}$$

Assuming, the stress to be uniform over the surface of interest and the state of stress corresponds to pure normal stress, it is easy to see that while Cauchy stress is force per unit area in the current configuration, Piola-Kirchhoff stress is force per unit area in the reference configuration. This is the essential difference between the Piola-Kirchhoff stress and Cauchy stress. Recognize that in both the cases the force acts only in the current configuration. In lieu of this Piola-Kirchhoff stress is a two-point tensor, similar to the deformation gradient, in that it is a linear transformation that relates the unit normal to the surface in the reference configuration to the traction acting in the current configuration. Thus, if $\{\mathbf{E}_a\}$ denote the three Cartesian basis vectors used to describe the reference configuration and $\{\mathbf{e}_a\}$ denote the Cartesian basis vectors used to describe the current configuration, then the Cauchy stress and Piola-Kirchhoff stress are represented as

$$\boldsymbol{\sigma} = \sigma_{ab}\mathbf{e}_a \otimes \mathbf{e}_b, \qquad \mathbf{P} = P_{aB}\mathbf{e}_a \otimes \mathbf{E}_B. \tag{4.44}$$

We had mentioned earlier that the Cauchy stress tensor is symmetric for reasons to be discussed in the next chapter. Since, $\boldsymbol{\sigma} = \mathbf{P}\mathbf{F}^t/\det(\mathbf{F})$, the symmetric restriction on Cauchy stress tensor requires the Piola-Kirchhoff stress to satisfy the relation

$$\mathbf{P}\mathbf{F}^t = \mathbf{F}\mathbf{P}^t. \tag{4.45}$$

Consequently, in general $\mathbf{P}$ is not symmetric and has nine independent components.

During an experiment, it is easy to find the surface areas in the reference configuration and determine the traction or forces acting on these surfaces as the experiment progresses. Thus, Piola-Kirchhoff stress is what could be directly determined. Also, it is the stress that is reported, in many cases. The Cauchy stress is then obtained using the equation (4.42) from the estimate of the Piola-Kirchhoff stress and the deformation of the respective surface.

Further, since in solid mechanics, the coordinates of the material particles in the reference configuration is used as independent variable, first Piola-Kirchhoff stress plays an important role in formulating the boundary value problem as we shall see in the ensuing chapters.

The transpose of the Piola-Kirchhoff stress, $\mathbf{P}^t$, is called as engineering stress or nominal stress.

Another stress measure called the second Piola-Kirchhoff stress, $\mathbf{S}$ defined as

$$\mathbf{S} = \det(\mathbf{F})\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-t} = \mathbf{F}^{-1}\mathbf{P} \tag{4.46}$$

is used in some studies. Note that this tensor is symmetric and hence its utility.

### 4.7.2 Kirchhoff, Biot and Mandel stress measures

Next, we define a few stress measures popular in literature. It is easy to establish their inter-relationship and properties which we leave it as an exercise.

The Kirchhoff stress tensor, $\boldsymbol{\tau}$ is defined as: $\boldsymbol{\tau} = \det(\mathbf{F})\boldsymbol{\sigma}$.

The Biot stress tensor, $\mathbf{T}_B$ also called material stress tensor is defined as: $\mathbf{T}_B = \mathbf{R}^t\mathbf{P}$, where $\mathbf{R}$ is the orthogonal tensor obtained during polar decomposition of $\mathbf{F}$.

The co-rotated Cauchy stress tensor $\boldsymbol{\sigma}_u$, introduced by Green and Naghdi is defined as: $\boldsymbol{\sigma}_u = \mathbf{R}^t\boldsymbol{\sigma}\mathbf{R}$.

The Mandel stress tensor, used often to describe inelastic response of materials is defined as $\boldsymbol{\Sigma} = \mathbf{CS}$, where $\mathbf{C}$ is the right Cauchy-Green stretch tensor introduced in the previous chapter.

## 4.8 Summary

In this chapter, we introduced the concept of traction and stress tensor. If the forces acting in the body are assumed to be distributed over the surfaces in the current configuration of the body, then the corresponding traction is called as Cauchy traction and the stress tensor associated with this traction is called as the Cauchy stress tensor, $\boldsymbol{\sigma}$. Similarly, if the forces acting in the body is distributed over the surface in the reference configuration, then this traction is called as Piola traction and the stress tensor associated with this traction the Piola-Kirchhoff stress, $\mathbf{P}$. We also defined various types of stresses like normal stress, shear stress, hydrostatic stress, octahedral stress. Having grasped the concepts of strain and stress we shall proceed to find the relation between the applied force and realized displacement in the following chapters.

## 4.9 Self-Evaluation

1. Cauchy stress state corresponding to an orthonormal Cartesian basis ($\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$) is as given below:

$$\boldsymbol{\sigma} = \begin{pmatrix} -10 & 5 & 0 \\ 5 & 10 & 0 \\ 0 & 0 & 0 \end{pmatrix} MPa.$$

   For this stress state,

   (a) Draw the stress cube

   (b) Identify whether this stress states correspond to the plane stress

   (c) Find the normal and shear stress on a plane whose normal is oriented along the $\mathbf{e}_x$ direction.

   (d) Find the normal and shear stress on a plane whose normal makes equal angles with all the three basis vectors, i.e., $\mathbf{n} = (\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)/\sqrt{3}$.

   (e) Find the components of the stress tensor in the new basis $\{\tilde{\mathbf{e}}_x, \tilde{\mathbf{e}}_y, \tilde{\mathbf{e}}_z\}$. The new basis is obtained by rotating an angle 30 degrees in the clockwise direction about $\mathbf{e}_z$ axis.

   (f) Find the principal invariants of the stress

   (g) Find the principal stresses

   (h) Find the maximum shear stress

   (i) Find the plane on which the maximum normal stresses occurs

   (j) Find the plane on which the maximum shear stress occurs

   (k) Find the normal stress on the plane on which the maximum shear stress occurs

   (l) Find the shear stress on the plane on which the maximum normal stress occurs.

   (m) Find the normal and shear stresses on the octahedral plane

   (n) Find the hydrostatic and deviatoric component of the stress

2. Cauchy stress state corresponding to an orthonormal Cartesian basis ($\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$) is as given below:

$$\boldsymbol{\sigma} = \begin{pmatrix} -10 & 0 & 5 \\ 0 & 10 & 0 \\ 5 & 0 & 10 \end{pmatrix} MPa.$$

For this stress state solve parts (a) through (n) in problem 1.

3. For the plane Cauchy stress state shown in figure 4.6, write the stress tensor and solve parts (b) through (n) in problem 1.

4. For the Cauchy stress cube shown in figure 4.7, write the stress tensor and solve parts (b) through (n) in problem 1.

5. An annular circular cylinder with deformed outer diameter 25 mm is subjected to a tensile force of 1200 N as shown in figure 4.8. Experiments revealed that a uniform axial true stress of 3.8 MPa has developed due to this applied force. Using the above information, determine the deformed inner diameter of the cylinder.

6. Three pieces of wood having 3.75 cm x 3.75 cm square cross-sections are glued together and to the foundation as shown in figure 4.9. If the horizontal force $P = 3,000$ N what is the average shearing engineering stress in each of the glued joints?

7. Find the maximum permissible value of load $P$ for the bolted joint shown in the figure 4.10, if the allowable shear stress in the bolt material is 100 MPa. Bolts are 20 mm in diameter. List the assumptions in your calculation, if any.

8. Two wooden planks each 25 mm thick and 150 mm wide are joined by the glued mortise joint shown in figure 4.11. Knowing that the joint will fail when the average engineering shearing stress in the glue exceeds 0.9 MPa, determine the smallest allowable length $d$ of the cuts if the joint is to withstand an axial load $P$ that the planks would otherwise resist if there were no joint. Assume that the wooden plank fails if the engineering axial stresses exceed 1.5 MPa.

9. Two wooden members of 90 x 140 mm uniform rectangular cross section are joined by the simple glued scarf splice shown in figure 4.12.

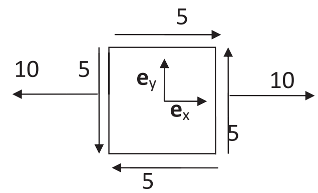



Figure 4.6: Plane Cauchy stress state for problem 3. The unit for stresses shown in the figure is MPa

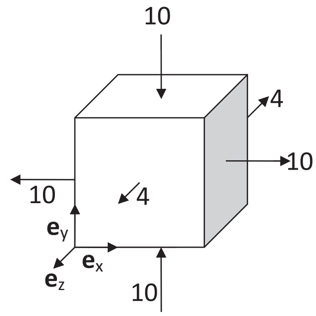



Figure 4.7: Cauchy Stress cube for problem 4. The unit for stresses shown in the figure is MPa

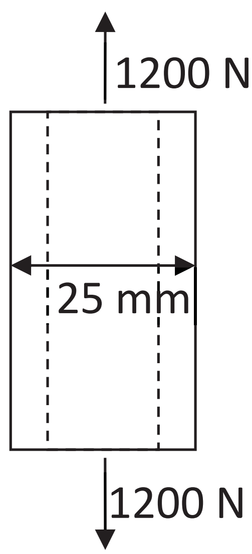



Figure 4.8: Figure for problem 5

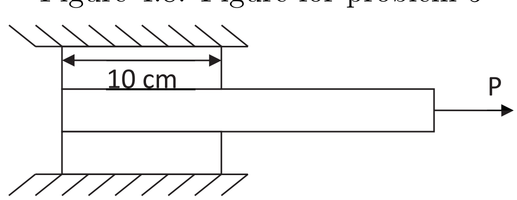



Figure 4.9: Figure for problem 6

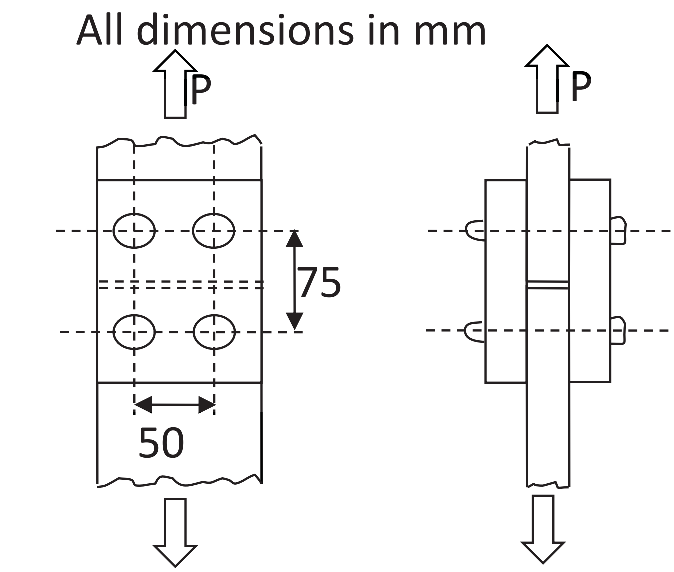



Figure 4.10: Figure for problem 7

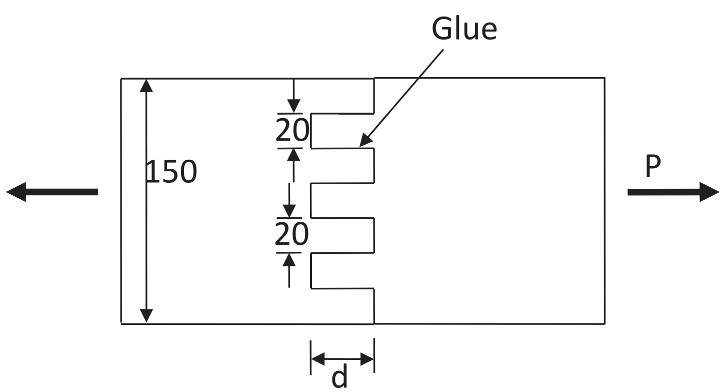



Figure 4.11: Figure for problem 8

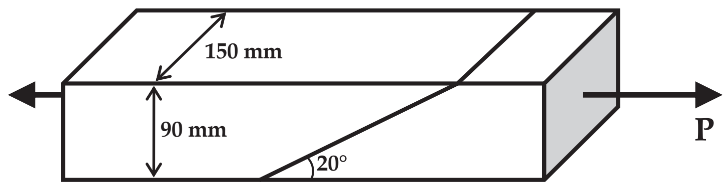



Figure 4.12: Figure for problem 9

Knowing that the maximum allowable engineering shearing stress in the glued splice is 0.75 MPa, determine the largest axial load $P$ that can be safely applied as shown in figure 4.12.

10. A plate has been subjected to the following state of Cauchy stress:

$$\boldsymbol{\sigma} = \begin{pmatrix} -5 & 0 & 5 \\ 0 & 0 & 0 \\ 5 & 0 & 10 \end{pmatrix} MPa,$$

where the components of the stress is with respect to an orthonormal Cartesian basis ($\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$). For this state of stress, answer the following:

(a) Does this state of stress correspond to plane stress? Justify

(b) If the plate were to be made of reinforced concrete, find the direction(s) in which reinforcement needs to be provided. Recollect that reinforcement is provided in concrete to resist tensile stresses.

(c) Find the additional $\sigma_{zz}$ component of the stress that can be applied, to the above plate, so that the maximum normal stress is 230 MPa

(d) If the plate were to be made of wood, in which the grains orientation, $\mathbf{m} = 0.61\mathbf{e}_x + 0.50\mathbf{e}_y + 0.61\mathbf{e}_z$ and if the maximum shear stress that can be sustained along the orientation of the grains is 20 MPa, determine if the wooden plate will fail for the above state of stress.

(e) Now say, hydrostatic state of stress is applied in addition to the above state of stress, find the hydrostatic pressure at which the wooden plate will fail, if it will.

# Chapter 5

# Balance Laws

## 5.1 Overview

The balance laws, discussed in this chapter, has to be satisfied for all bodies, irrespective of the material that they are made up of. The fundamental balance laws that would be discussed here are conservation of mass, linear momentum, and angular momentum. However, the particular form that these equations take would depend on the processes that one is interested in. Here our interest is in what is called as purely mechanical processes and hence we are not concerned about conservation of energy or electric charge or magnetic flux. These general balance equations in themselves do not suffice to determine the deformation or motion of a body subject to given loading. Also, bodies identical in geometry but made up of different material undergo different deformation or motion when subjected to the same boundary traction. Hence, to formulate a determinate problem, it is usually necessary to specify the material which the body is made. In continuum mechanics, such specification is stated by constitutive equations, which relate, for example, the stress tensor and the stretch tensor. We shall see more about these constitutive relations in the subsequent chapter while we shall focus on balance laws in this chapter.

## 5.2 System

We define a system as a particular collection of matter or a particular region of space. The complement of a system, i.e., the matter or region outside the system, is called the surroundings. The surface that separates the system from its surroundings is called the boundary or wall of the system.

A closed system consists of a fixed amount of matter in a region $\Omega$ in space with boundary surface $\partial\Omega$ which depends on time $t$. While no matter can cross the boundary of a closed system, energy in the form of work or heat can cross the boundary. The volume of a closed system is not necessarily fixed. If the energy does not cross the boundary of the system, then we say that the boundary is insulated and such a system is said to be mechanically and thermally isolated. An isolated system is an idealization for no physical system is truly isolated, for there are always electromagnetic radiation into and out of the system.

An open system (or control volume) focuses in on a region in space, $\Omega_c$ which is independent of time $t$. The enclosing boundary of the open system, over which both matter and energy can cross is called a control surface, which we denote by $\partial\Omega_c$.

## 5.3 Conservation of Mass

At the intuitive level mass is perceived to be a measure of the amount of material contained in an arbitrary portion of the body. As such it is a non-negative scalar quantity independent of time and not generally determined by the size of the configuration occupied by the arbitrary sub-body. It is not a count of the number of material particles in the body or its sub-parts. However, the mass of a body is the sum of the masses of its parts. These statements can be formalized mathematically by characterizing mass as a set function with certain properties and we proceed on the basis of the following definitions.

Let $\mathfrak{B}_t$ be an arbitrary configuration of a body $\mathcal{B}$ and let $\mathfrak{A}$ be a set of points in $\mathfrak{B}_t$ occupied by the particles in an arbitrary subset $\mathcal{A}$ of $\mathcal{B}$. If with $\mathcal{A}$ there is associated a non-negative real number $m(\mathcal{A})$ having a physical dimension independent of time and distance and such that

(i) $m(\mathcal{A}_1 \cup \mathcal{A}_2) = m(\mathcal{A}_1) + m(\mathcal{A}_2)$ for all pairs $\mathcal{A}_1$, $\mathcal{A}_2$ of disjoint subsets of $\mathcal{B}$, and

(ii) $m(\mathcal{A}) \to 0$ as volume of $A$ tends to zero,

$\mathcal{B}$ is said to be a material body with mass function $m$. The mass content of $\mathfrak{A}$, denoted by $m(\mathfrak{A})$, is identified with the mass $m(\mathcal{A})$ of $\mathcal{A}$. Property two is a consequence of our assumption that the body is a continuum, precluding the presence of point masses. Properties (i) and (ii) imply the existence of a scalar field $\rho$, defined on $\mathcal{B}$, such that

$$m(\mathcal{A}) = m(\mathfrak{A}) = \int_{\mathfrak{A}} \rho dv. \tag{5.1}$$

$\rho$ is called the mass density, or simply the density, of the material with which the body is made up of.

In non-relativistic mechanics mass cannot be produced or destroyed, so the mass of a body is a conserved quantity. Hence, if a body has a certain mass in the reference configuration it must stay the same during a motion. Hence, we write

$$m(\mathfrak{B}_r) = m(\mathfrak{B}_t) > 0, \tag{5.2}$$

for all times $t$, where $\mathfrak{B}_r$ denotes the region occupied by the body in the reference configuration and $\mathfrak{B}_t$ denotes the region occupied by the body in the current configuration.

In the case of a material body executing a motion $\{\mathfrak{B}_t : t \in I\}$, the density $\rho$ is defined as a scalar field on the configurations $\{\mathfrak{B}_t\}$ and the mass content $m(\mathfrak{A}_t)$ of an arbitrary region in the current configuration $\mathfrak{B}_t$ is equal to the mass $m(\mathcal{A})$ of $\mathcal{A}$. Since, $m(\mathcal{A})$ does not depend upon $t$ we deduce directly from (5.1) the equation of mass balance

$$\frac{D}{Dt}\int_{\mathfrak{A}_t} \rho dv = 0. \tag{5.3}$$

$\rho = \hat{\rho}(\mathbf{x}, t)$, the density is assumed to be continuously differentiable jointly in the position and time variables on which it depends. Since, we are interested in the total time derivative and the current volume of the body changes with time, the differentiation and integration operations cannot be interchanged. To be able to change the order of the integration and differentiation, we have to convert it to an integral over the reference configuration. This is accomplished by using (3.75). Hence, we obtain

$$\frac{D}{Dt}\int_{\mathfrak{A}_t} \rho dv = \frac{D}{Dt}\int_{\mathfrak{A}_r} \rho \det(\mathbf{F}) dV = \int_{\mathfrak{A}_r} \frac{D}{Dt}(\rho \det(\mathbf{F})) dV = 0. \tag{5.4}$$

Expanding the above equation we obtain

$$\int_{\mathfrak{A}_r} [\frac{D\rho}{Dt}\det(\mathbf{F}) + \rho\frac{D\det(\mathbf{F})}{Dt}] dV = 0. \tag{5.5}$$

Now, we compute $\frac{D\det(\mathbf{F})}{Dt}$. Towards this, using the chain rule for differentiation,

$$\frac{D\det(\mathbf{F})}{Dt} = \frac{\partial\det(\mathbf{F})}{\partial\mathbf{F}} \cdot \frac{D\mathbf{F}}{Dt}. \tag{5.6}$$

Using the result from equation (2.187),

$$\frac{D\det(\mathbf{F})}{Dt} = \det(\mathbf{F})\mathbf{F}^{-t} \cdot \frac{D\mathbf{F}}{Dt} = \det(\mathbf{F})tr(\frac{D\mathbf{F}}{Dt}\mathbf{F}^{-1}), \tag{5.7}$$

where to obtain the last expression we have made use of the definition of a dot product of two tensors, (2.71) and the property of the trace operator, (2.68). Defining, $\mathbf{l} = grad(\mathbf{v})$, it can be seen that

$$\mathbf{l} = \frac{\partial^2\boldsymbol{\chi}}{\partial\mathbf{x}\partial t} = \frac{\partial^2\boldsymbol{\chi}}{\partial\mathbf{X}\partial t}\frac{\partial\mathbf{X}}{\partial\mathbf{x}} = \frac{D}{Dt}\left(\frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}}\right)\mathbf{F}^{-1} = \frac{D\mathbf{F}}{Dt}\mathbf{F}^{-1}, \tag{5.8}$$

obtained using the chain rule and by interchanging the order of the spatial and temporal derivatives. Hence, equation (5.7) reduces to,

$$\frac{D\det(\mathbf{F})}{Dt} = \det(\mathbf{F})tr(\mathbf{l}) = \det(\mathbf{F})tr(grad(\mathbf{v})) = \det(\mathbf{F})div(\mathbf{v}), \tag{5.9}$$

on using the definition of the divergence, (2.208).

Substituting (5.9) in (5.5) we obtain

$$\int_{\mathfrak{A}_r}[\frac{D\rho}{Dt} + \rho div(\mathbf{v})]\det(\mathbf{F})dV = 0. \tag{5.10}$$

Again using (3.75) we obtain

$$\int_{\mathfrak{A}_t}[\frac{D\rho}{Dt} + \rho div(\mathbf{v})]dv = 0, \tag{5.11}$$

wherein the integrand is continuous in $\mathfrak{B}_t$ and the range of integration is an arbitrary subregion of $\mathfrak{B}_t$. Since, the integral vanishes in $\mathfrak{B}_t$ and in any arbitrary subregion of $\mathfrak{B}_t$, the integrand should vanish, i.e.,

$$\frac{D\rho}{Dt} + \rho div(\mathbf{v}) = 0. \tag{5.12}$$

Subject to the presumed smoothness of $\rho$, equations (5.3) and (5.12) are equivalent expressions of the conservation of mass.

Further, since (5.4) has to hold for any arbitrary subpart of $\mathfrak{A}_r$, we get

$$\frac{D}{Dt}(\rho \det(\mathbf{F})) = 0, \qquad \text{hence,} \qquad \rho \det(\mathbf{F}) = \rho_r, \tag{5.13}$$

where $\rho_r$ is the density in the reference configuration, $\mathfrak{B}_r$. Note that, whenever the body occupies $\mathfrak{B}_r$, $\det(\mathbf{F}) = 1$ and hence, $\rho = \rho_r$, giving $\rho_r$ as the constant value of $\rho \det(\mathbf{F})$ and hence the referential equation of conservation of mass

$$\rho = \frac{\rho_r}{\det(\mathbf{F})}. \tag{5.14}$$

A body which is able to undergo only isochoric motions is said to be composed of incompressible material. Since, $\det(\mathbf{F}) = 1$ for isochoric motions, we see from equation (5.14) that the density does not change with time, $t$. Consequently, for a body made up of an incompressible material, if the density is uniform in some configuration it has to be the same uniform value in every configuration which the body can occupy. On the other hand if a body is made up of compressible material this is not so, in general.

Let $\phi$ be a scalar field and $\mathbf{u}$ a vector field representing properties of a moving material body, $\mathcal{B}$, and let $\mathfrak{A}_t$ be an arbitrary material region in the current configuration of $\mathcal{B}$. Then show that:

$$\frac{D}{Dt}\int_{\mathfrak{A}_t} \rho\phi dv = \int_{\mathfrak{A}_t} \rho\frac{D\phi}{Dt} dv, \tag{5.15}$$

$$\frac{D}{Dt}\int_{\mathfrak{A}_t} \rho\mathbf{u} dv = \int_{\mathfrak{A}_t} \rho\frac{D\mathbf{u}}{Dt} dv. \tag{5.16}$$

Using similar arguments as above

$$\begin{aligned}
\frac{D}{Dt}\int_{\mathfrak{A}_t} \rho\phi dv &= \int_{\mathfrak{A}_r} \frac{D}{Dt}(\rho\phi \det(\mathbf{F}))dV, \\
&= \int_{\mathfrak{A}_r} \left(\rho\frac{D\phi}{Dt} + \left[\frac{D\rho}{Dt} + \rho div(\mathbf{v})\right]\phi\right)\det(\mathbf{F})dV \\
&= \int_{\mathfrak{A}_t} \left(\rho\frac{D\phi}{Dt} + \left[\frac{D\rho}{Dt} + \rho div(\mathbf{v})\right]\phi\right) dv, \qquad (5.17)
\end{aligned}$$

and on using equation (5.12) we obtain the first of the required results. Following the same steps as outlined above, we can show equation (5.16) is true.

## 5.4 Conservation of momentum

There are two kinds of forces that act in a body. They are:

1. Contact forces
2. Body forces

Contact forces arise due to contact between two bodies. Body forces are action at a distance forces like gravitational force. Similarly, while contact forces act per unit area of the boundary of the body, body forces act per unit mass of the body. Both these forces result in the generation of stresses.

Before deriving these conservation laws of momentum rigorously, we obtain the same using using an approximate analysis, as we did for the transformation of curves, areas and volumes. Moreover, we obtain these for a 2D body. Consider an 2D infinitesimal element in the current configuration with dimensions $\Delta x$ along the $x$ direction and $\Delta y$ along the $y$ direction, as shown in figure 5.1. We assume that the center of the infinitesimal element, $O$ is located at $(x + 0.5\Delta x, y + 0.5\Delta y)$.

We assume that the Cauchy stress, $\boldsymbol{\sigma}$ varies over the infinitesimal element in the current configuration, $\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}(x, y)$. That is we have assumed Eulerian description for the stress. Since, we are considering only 2D state of stress, the relevant Cartesian components of the stress are $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{xy}$ and $\sigma_{yx}$. Expanding the Cartesian components of the stresses using Taylor's series up to first order we obtain:

$$\sigma_{xx}(x + \Delta x, y) = \sigma_{xx}(x, y) + \frac{\partial \sigma_{xx}}{\partial x}|_{(x,y)}\Delta x, \tag{5.18}$$

$$\sigma_{xy}(x + \Delta x, y) = \sigma_{xy}(x, y) + \frac{\partial \sigma_{xy}}{\partial x}|_{(x,y)}\Delta x, \tag{5.19}$$

$$\sigma_{yx}(x, y + \Delta y) = \sigma_{yx}(x, y) + \frac{\partial \sigma_{yx}}{\partial y}|_{(x,y)}\Delta y, \tag{5.20}$$

$$\sigma_{yy}(x, y + \Delta y) = \sigma_{yy}(x, y) + \frac{\partial \sigma_{yy}}{\partial y}|_{(x,y)}\Delta y. \tag{5.21}$$

Also seen in figure 5.1 are the Cartesian components of the Cauchy stress acting on various faces of the infinitesimal element, obtained using the equations (5.18) through (5.21).

Now, we are interested in the equilibrium of the infinitesimal element in the deformed configuration under the action of these stresses and the body

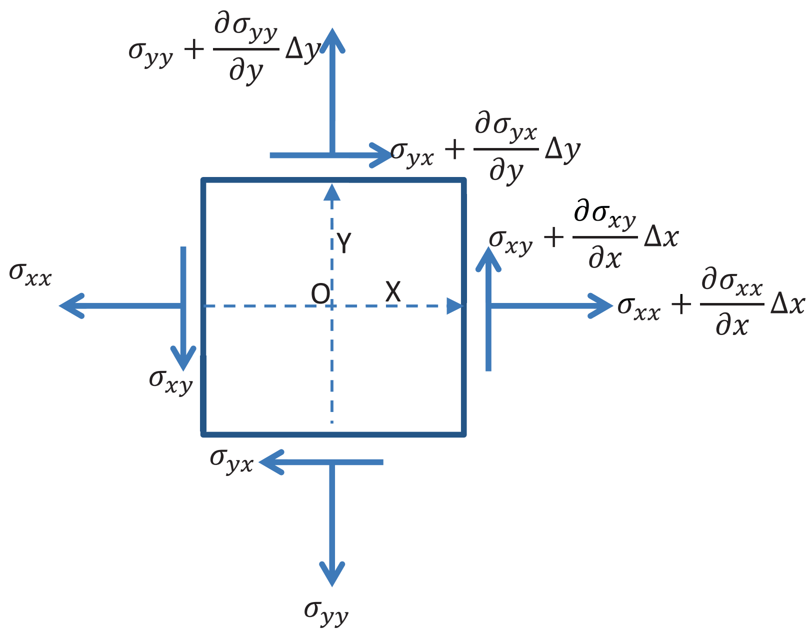



Figure 5.1: 2D Infinitesimal element showing the variation of Cartesian components of the stresses on the various faces.

force, $\mathbf{b}$ with Cartesian components $b_x$ and $b_y$ along the $x$ and $y$ direction respectively. Also, we shall assume that the infinitesimal element is accelerating with acceleration, $\mathbf{a}$ with Cartesian components $a_x$ and $a_y$ along the $x$ and $y$ direction respectively. Thus, force equilibrium along the $x$ direction requires:

$$\begin{aligned}\rho(\Delta x)(\Delta y)(1)a_x &= \left[\sigma_{xx}(x,y) + \frac{\partial \sigma_{xx}}{\partial x}|_{(x,y)}\Delta x - \sigma_{xx}(x,y)\right](\Delta y)(1) \\ &\quad + \left[\sigma_{yx}(x,y) + \frac{\partial \sigma_{yx}}{\partial y}|_{(x,y)}\Delta y - \sigma_{yx}(x,y)\right](\Delta x)(1) \\ &\quad + \rho(\Delta x)(\Delta y)(1)b_x, \end{aligned} \tag{5.22}$$

where $\rho$ is the density and the element is assumed to have a unit thickness. Note that here $\rho(\Delta x)(\Delta y)(1)$ gives the infinitesimal mass of the element and the stresses are multiplied by the respective areas over which they act to get the forces; since only the forces should satisfy the equilibrium equations. Simplifying, (5.22) we obtain:

$$\rho a_x = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \rho b_x. \tag{5.23}$$

Similarly, writing the force equilibrium equation along the $y$ direction:

$$\begin{aligned}\rho(\Delta x)(\Delta y)(1)a_y &= \left[\sigma_{xy}(x,y) + \frac{\partial \sigma_{xy}}{\partial x}|_{(x,y)}\Delta x - \sigma_{xy}(x,y)\right](\Delta y)(1) \\ &\quad + \left[\sigma_{yy}(x,y) + \frac{\partial \sigma_{yy}}{\partial y}|_{(x,y)}\Delta y - \sigma_{yy}(x,y)\right](\Delta x)(1) \\ &\quad + \rho(\Delta x)(\Delta y)(1)b_y. \end{aligned} \tag{5.24}$$

The above equation simplifies to:

$$\rho a_y = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho b_y. \tag{5.25}$$

Finally, we appeal to the moment equilibrium. Here we assume that there are no body couples or contact couples acting in the infinitesimal element. Since, we can take moment equilibrium about any point, for convenience, we do it about the point $O$, marked in figure 5.1, to obtain:

$$\begin{aligned}&\left[\sigma_{xy}(x,y) + \sigma_{xy}(x,y) + \frac{\partial \sigma_{xy}}{\partial x}|_{(x,y)}\Delta x\right]\frac{(\Delta x)}{2}(\Delta y)(1) \\ &\quad = \left[\sigma_{yx}(x,y) + \sigma_{yx}(x,y) + \frac{\partial \sigma_{yx}}{\partial y}|_{(x,y)}\Delta y\right]\frac{(\Delta y)}{2}(\Delta x)(1). \end{aligned} \tag{5.26}$$

Simplifying the above equation we obtain:

$$\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x}\frac{\Delta x}{2} = \sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}\frac{\Delta y}{2}, \tag{5.27}$$

which in the limit $\Delta x$ and $\Delta y$ tending to zero, the limit that we are interested in, reduces to requiring

$$\sigma_{xy} = \sigma_{yx}. \tag{5.28}$$

Thus, a plane Cauchy stress field should satisfy the equations (5.23), (5.25) and (5.28) for it to be admissible.

Generalizing it for 3D, the Cauchy stress field should satisfy:

$$\rho a_x = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + \rho b_x, \tag{5.29}$$

$$\rho a_y = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + \rho b_y, \tag{5.30}$$

$$\rho a_z = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z, \tag{5.31}$$

and

$$\sigma_{xy} = \sigma_{yx}, \quad \sigma_{xz} = \sigma_{zx}, \quad \sigma_{yz} = \sigma_{zy}. \tag{5.32}$$

The equations (5.29) through (5.31) are written succinctly as

$$\rho \mathbf{a} = div(\boldsymbol{\sigma}) + \rho \mathbf{b}, \tag{5.33}$$

and equations (5.32) tells us that the Cauchy stress should be symmetric.

Similarly, requiring a sector of cylindrical shell to be in equilibrium, under the action of spatially varying stresses, it can be shown that

$$\rho a_r = \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho b_r, \tag{5.34}$$

$$\rho a_\theta = \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{\sigma_{r\theta} + \sigma_{\theta r}}{r} + \rho b_\theta, \tag{5.35}$$

$$\rho a_z = \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho b_z, \tag{5.36}$$

and

$$\sigma_{r\theta} = \sigma_{\theta r}, \quad \sigma_{rz} = \sigma_{zr}, \quad \sigma_{\theta z} = \sigma_{z\theta}. \tag{5.37}$$

has to hold.

### 5.4.1 Conservation of linear momentum

Let $\mathfrak{A}_t$ be an arbitrary sub-region in the configuration $\mathfrak{B}_t$ of a body $\mathcal{B}$. The total linear momentum, $\mathbf{\Gamma}(\mathfrak{A}_t)$, of the material particles occupying $\mathfrak{A}_t$ is defined by

$$\mathbf{\Gamma}(\mathfrak{A}_t) = \int_{\mathfrak{A}_t} \rho \mathbf{v} dv. \tag{5.38}$$

Next, we find the forces acting on the body. Since, the part $\mathfrak{A}_t$ has been isolated from its surroundings, traction $\mathbf{t}_{(\mathbf{n})}(\mathbf{x})$, introduced in the last chapter, acts on the boundary of the $\mathfrak{A}_t$. In addition to traction, which arises between parts of the body that are in contact, there exist forces like gravity which act on the material particles not through contact and are called body force. The body forces are denoted by $\mathbf{b}$ and are defined per unit mass. Hence, the resultant force, $\mathcal{F}$ acting on $\mathfrak{A}_t$ is

$$\mathcal{F}(\mathfrak{A}_t) = \int_{\partial\mathfrak{A}_t} \mathbf{t}_{(\mathbf{n})} da + \int_{\mathfrak{A}_t} \rho \mathbf{b} dv. \tag{5.39}$$

Now, using Newton's second law of motion, which states that the rate of change of linear momentum of the body must equal the resultant force that acts on the body in both magnitude and direction we obtain

$$\frac{D\mathbf{\Gamma}}{Dt} = \frac{D}{Dt}\int_{\mathfrak{A}_t} \rho \mathbf{v} dv = \int_{\partial\mathfrak{A}_t} \mathbf{t}_{(\mathbf{n})} da + \int_{\mathfrak{A}_t} \rho \mathbf{b} dv = \mathcal{F}. \tag{5.40}$$

Using (3.30), definition of acceleration, $\mathbf{a}$ and (5.16) the above equation can be written as

$$\int_{\mathfrak{A}_t} \rho \frac{D\mathbf{v}}{Dt} dv = \int_{\mathfrak{A}_t} \rho \mathbf{a} dv = \int_{\partial\mathfrak{A}_t} \mathbf{t}_{(\mathbf{n})} da + \int_{\mathfrak{A}_t} \rho \mathbf{b} dv. \tag{5.41}$$

Next, using Cauchy's stress theorem (4.3) and the divergence theorem (2.263) the above equation further can be reduced to

$$\int_{\mathfrak{A}_t} \rho \mathbf{a} dv = \int_{\partial\mathfrak{A}_t} \boldsymbol{\sigma}\mathbf{n} da + \int_{\mathfrak{A}_t} \rho \mathbf{b} dv = \int_{\mathfrak{A}_t} \left[div(\boldsymbol{\sigma}) + \rho \mathbf{b}\right] dv. \tag{5.42}$$

Since, the above equation has to hold for $\mathfrak{B}_t$ and any subset, $\mathfrak{A}_t$, of it

$$\rho \mathbf{a} = div(\boldsymbol{\sigma}) + \rho \mathbf{b}. \tag{5.43}$$

Many a times, we are interested in cases for which $\mathbf{a} = \mathbf{o}$, then (5.43) becomes

$$div(\boldsymbol{\sigma}) + \rho\mathbf{b} = \mathbf{o}, \tag{5.44}$$

which is referred to as Cauchy's equation of equilibrium. Further, there arises scenarios where the body forces can be neglected and in those cases $div(\boldsymbol{\sigma}) = \mathbf{o}$. Thus, divergence free stress fields are called self equilibrated stress fields.

Equation (5.43) is in spatial form, that is it assumes that $\boldsymbol{\sigma} = \tilde{\boldsymbol{\sigma}}(\mathbf{x}, t)$, $\mathbf{b} = \tilde{\mathbf{b}}(\mathbf{x}, t)$, $\mathbf{a} = \tilde{\mathbf{a}}(\mathbf{x}, t)$, $\rho = \tilde{\rho}(\mathbf{x}, t)$ are known. But on most occasions, especially in solid mechanics, we know only the material form of these fields that is $\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{X}, t)$, $\mathbf{b} = \hat{\mathbf{b}}(\mathbf{X}, t)$, $\mathbf{a} = \hat{\mathbf{a}}(\mathbf{X}, t)$, $\rho = \hat{\rho}(\mathbf{X}, t)$ and in those occasions it is difficult to obtain the spatial divergence of these fields. To make it easier at these instances, we seek a statement of balance of linear momentum in material form. For this, equation (5.42a) has to be modified. Towards this, we obtain

$$\begin{aligned}
\int_{\mathfrak{A}_t} \rho\mathbf{a}dv &= \int_{\mathfrak{A}_r} \rho\mathbf{a}\det(\mathbf{F})dV = \int_{\mathfrak{A}_r} \rho_r\mathbf{a}dV \\
&= \int_{\partial\mathfrak{A}_t} \boldsymbol{\sigma}\mathbf{n}da + \int_{\mathfrak{A}_t} \rho\mathbf{b}dv \\
&= \int_{\partial\mathfrak{A}_r} \det(\mathbf{F})\boldsymbol{\sigma}\mathbf{F}^{-t}\mathbf{N}dA + \int_{\mathfrak{A}_r} \rho\mathbf{b}\det(\mathbf{F})dV \\
&= \int_{\mathfrak{A}_r} Div(\mathbf{P})dV + \int_{\mathfrak{A}_r} \rho_r\mathbf{b}dV,
\end{aligned} \tag{5.45}$$

for which we have successively used equation (3.75), (5.14), (3.72) and definition of Piola-Kirchhoff stress (4.42) and the divergence theorem. Since, (5.45) has to hold for any arbitrary subpart of the reference configuration, we obtain the material differential form of the balance of linear momentum as

$$\rho_r\mathbf{a} = Div(\mathbf{P}) + \rho_r\mathbf{b}. \tag{5.46}$$

It is worthwhile to emphasize that irrespective of the choice of the independent variable, the equilibrium of the forces is established only in the current configuration. In other words, whether we use material or spatial description for the stress, the forces and moments have to be equilibrated in the deformed configuration and not the reference configuration.

### 5.4.2 Conservation of angular momentum

Let $\mathfrak{A}_t$ be an arbitrary sub-region in the configuration $\mathfrak{B}_t$ of a body $\mathcal{B}$. The total angular momentum, $\mathbf{\Omega}(\mathfrak{A}_t)$, of the material particles occupying $\mathfrak{A}_t$ is defined by

$$\mathbf{\Omega}(\mathfrak{A}_t) = \int_{\mathfrak{A}_t} [(\mathbf{x} - \mathbf{x}_o) \wedge \rho \mathbf{v} + \mathbf{p}] dv, \tag{5.47}$$

where $\mathbf{x}_o$ is the point about which the moment is taken and $\mathbf{p}$ is the intrinsic angular momentum per unit volume in the current configuration.

Next, we find the moment due to the forces acting on the body. As discussed before there are two kind of forces, surface or contact forces and body forces act on the body which contribute to the moment. Apart from moment arising due to applied forces, there could exist couples distributed per unit surface area, $\mathbf{m}$ and body couples $\mathbf{c}$, defined per unit mass. Thus, $\mathbf{m} = \tilde{\mathbf{m}}(\mathbf{x}, t, \mathbf{n})$ and $\mathbf{c} = \tilde{\mathbf{c}}(\mathbf{x}, t)$. Hence, the resultant moment, $\boldsymbol{\omega}$ acting on $\mathfrak{A}_t$ is

$$\boldsymbol{\omega}(\mathfrak{A}_t) = \int_{\partial \mathfrak{A}_t} [(\mathbf{x} - \mathbf{x}_o) \wedge \mathbf{t}_{(\mathbf{n})} + \mathbf{m}] da + \int_{\mathfrak{A}_t} [\rho (\mathbf{x} - \mathbf{x}_o) \wedge \mathbf{b} + \mathbf{c}] dv. \tag{5.48}$$

However, in non-polar bodies that are the subject matter of the study here, there exist no body couples or couples distributed per unit surface area or intrinsic angular momentum, i.e., $\mathbf{c} = \mathbf{m} = \mathbf{p} = \mathbf{o}$.

The balance of angular momentum states that the rate of change of the angular momentum must equal the applied momentum in both direction and magnitude. Hence,

$$\frac{D\mathbf{\Omega}}{Dt} = \boldsymbol{\omega}. \tag{5.49}$$

Further simplification of this balance principle involves some tensor algebra. First, we begin by calculating the left hand side of the above equation:

$$\begin{aligned}
\frac{D\mathbf{\Omega}}{Dt} &= \frac{D}{Dt}\left[\int_{\mathfrak{A}_t} [(\mathbf{x} - \mathbf{x}_o) \wedge \rho \mathbf{v} + \mathbf{p}] dv\right] \\
&= \int_{\mathfrak{A}_r} \frac{D}{Dt} \{[(\mathbf{x} - \mathbf{x}_o) \wedge \rho \mathbf{v} + \mathbf{p}] \det(\mathbf{F})\} \, dV \\
&= \int_{\mathfrak{A}_r} \mathbf{v} \wedge \rho \mathbf{v} \det(\mathbf{F}) + (\mathbf{x} - \mathbf{x}_o) \wedge \left\{\frac{D}{Dt}(\rho \det(\mathbf{F}))\mathbf{v} + \rho \mathbf{a} \det(\mathbf{F})\right\} dV \\
&\quad + \int_{\mathfrak{A}_r} \frac{D}{Dt} (\mathbf{p} \det(\mathbf{F})) \, dV.
\end{aligned} \tag{5.50}$$

Since, $\mathbf{v} \wedge \mathbf{v} = \mathbf{o}$ and using equations (5.13) and (5.9), (5.50) reduces to

$$\begin{aligned} \frac{D\boldsymbol{\Omega}}{Dt} &= \int_{\mathfrak{A}_r} [(\mathbf{x} - \mathbf{x}_o) \wedge \rho\mathbf{a}] \det(\mathbf{F}) dV + \int_{\mathfrak{A}_r} \left[\frac{D\mathbf{p}}{Dt} + \mathbf{p} div(\mathbf{v})\right] \det(\mathbf{F}) dV \\ &= \int_{\mathfrak{A}_t} \left\{[(\mathbf{x} - \mathbf{x}_o) \wedge \rho\mathbf{a}] + \frac{D\mathbf{p}}{Dt} + \mathbf{p} div(\mathbf{v})\right\} dv. \end{aligned} \tag{5.51}$$

Next, we simplify the right hand side of the equation (5.49). Towards that, we first show that

$$\int_{\partial\mathfrak{A}_t} (\mathbf{x} - \mathbf{x}_o) \wedge \boldsymbol{\sigma}^t \mathbf{n} da = \int_{\mathfrak{A}_t} [(\mathbf{x} - \mathbf{x}_o) \wedge div(\boldsymbol{\sigma}) - \boldsymbol{\tau}]\, dv, \tag{5.52}$$

where $\boldsymbol{\tau}$ is the axial vector of $(\boldsymbol{\sigma} - \boldsymbol{\sigma}^t)$. We begin by establishing a few vector identities which enable us to establish (5.52).

**Identity - 1:** If $\mathbf{u}$ and $\mathbf{v}$ are arbitrary vectors then $\mathbf{u} \wedge \mathbf{v}$ is the axial vector of the skew-symmetric tensor $\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}$.

**Proof:** It is straightforward to verify that $\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}$ is skew-symmetric. Then we observe that

$$(\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v})(\mathbf{u} \wedge \mathbf{v}) = \{\mathbf{u} \cdot (\mathbf{u} \wedge \mathbf{v})\}\mathbf{v} - \{\mathbf{v} \cdot (\mathbf{u} \wedge \mathbf{v})\}\mathbf{u} = \mathbf{o}, \tag{5.53}$$

hence, the axial vector of $\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}$ is a scalar multiple of $\mathbf{u} \wedge \mathbf{v}$. Accordingly,

$$(\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v})\mathbf{a} = (\mathbf{a} \cdot \mathbf{u})\mathbf{v} - (\mathbf{a} \cdot \mathbf{v})\mathbf{u} = \alpha(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{a}, \tag{5.54}$$

for any vector $\mathbf{a}$. Now, we have to show that $\alpha = 1$. Setting $\mathbf{a} = \mathbf{u}$ in the above equation and then forming a scalar product on each side with $\mathbf{v}$ we have

$$(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})^2 = \alpha\{(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{u}\} \cdot \mathbf{v} \tag{5.55}$$

Using (2.36) we find that

$$\alpha\{(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{u}\} \cdot \mathbf{v} = \alpha[(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})^2]. \tag{5.56}$$

Comparing equations (5.55) and (5.56) we find that $\alpha = 1$.

**Identity - 2:** For any vector $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in the vector space

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{b} \otimes \mathbf{c} - \mathbf{c} \otimes \mathbf{b})\mathbf{a}. \tag{5.57}$$

This identity follows immediately from the above identity.

**Identity - 3:** Let $\mathfrak{A}_t$ be a regular region with boundary $\partial\mathfrak{A}_t$, let $\mathbf{n}$ be the outward unit normal to $\partial\mathfrak{A}_t$ and let $\mathbf{u}$ be a vector field and $\boldsymbol{\sigma}$ a tensor field, each continuous in $\mathfrak{A}_t$ and continuously differentiable in the interior of $\mathfrak{A}_t$. Then

$$\int_{\partial\mathfrak{A}_t} \mathbf{u} \otimes \boldsymbol{\sigma}\mathbf{n} da = \int_{\mathfrak{A}_t} \{\mathbf{u} \otimes div(\boldsymbol{\sigma}) + grad(\mathbf{u})\boldsymbol{\sigma}\} dv. \tag{5.58}$$

Towards proving this identity, note that

$$\begin{aligned} \int_{\partial\mathfrak{A}_t} (\boldsymbol{\sigma}\mathbf{n} \otimes \mathbf{u})\mathbf{a} da &= \int_{\partial\mathfrak{A}_t} (\mathbf{u} \cdot \mathbf{a})\boldsymbol{\sigma}\mathbf{n} da = \int_{\mathfrak{A}_t} div((\mathbf{u} \cdot \mathbf{a})\boldsymbol{\sigma}) dv \\ &= \int_{\mathfrak{A}_t} [(\mathbf{u} \cdot \mathbf{a}) div(\boldsymbol{\sigma}) + \boldsymbol{\sigma}^t grad(\mathbf{u} \cdot \mathbf{a}) dv \\ &= \int_{\mathfrak{A}_t} [div(\boldsymbol{\sigma}) \otimes \mathbf{u} + \boldsymbol{\sigma}^t \{grad(\mathbf{u})\}^t] \mathbf{a} dv \end{aligned} \tag{5.59}$$

where $\mathbf{a}$ is an arbitrary constant vector. To obtain the above we have made use of the identities

$$\begin{aligned} \int_{\partial v} \mathbf{T}\mathbf{n} da &= \int_v div(\mathbf{T}) dv, \\ div(\Phi\mathbf{T}) &= \Phi div(\mathbf{T}) + \mathbf{T}^t grad(\Phi), \end{aligned} \quad \begin{aligned} (5.60) \\ (5.61) \end{aligned}$$

and the definition of the gradient. Further, since $\mathbf{a}$ is arbitrary in (5.59) we have the identity

$$\int_{\partial\mathfrak{A}_t} \boldsymbol{\sigma}\mathbf{n} \otimes \mathbf{u} da = \int_{\mathfrak{A}_t} [div(\boldsymbol{\sigma}) \otimes \mathbf{u} + \boldsymbol{\sigma}^t \{grad(\mathbf{u})\}^t] dv \tag{5.62}$$

Taking the transpose of the above identity, we obtain the required vector identity (5.58).

Now, we are in a position to prove (5.52). For this we replace $\mathbf{b}$ by $(\mathbf{x}-\mathbf{x}_o)$ and $\mathbf{c}$ by $\boldsymbol{\sigma}\mathbf{n}$ in (5.57) and then integrating each side over the surface $\partial\mathfrak{A}_t$ we obtain

$$\mathbf{a} \wedge \int_{\partial\mathfrak{A}_t} (\mathbf{x}-\mathbf{x}_o) \wedge (\boldsymbol{\sigma}\mathbf{n}) da = \left( \int_{\partial\mathfrak{A}_t} \{(\mathbf{x} - \mathbf{x}_o) \otimes (\boldsymbol{\sigma}\mathbf{n}) - (\boldsymbol{\sigma}\mathbf{n}) \otimes (\mathbf{x} - \mathbf{x}_o)\} da \right) \mathbf{a}, \tag{5.63}$$

$\mathbf{a}$ being an arbitrary vector. From (5.58) it follows that

$$\int_{\partial\mathfrak{A}_t} (\mathbf{x} - \mathbf{x}_o) \otimes (\boldsymbol{\sigma}\mathbf{n}) da = \int_{\mathfrak{A}_t} \{(\mathbf{x} - \mathbf{x}_o) \otimes div(\boldsymbol{\sigma}) + \boldsymbol{\sigma}\} dv, \quad (5.64)$$

$$\int_{\partial\mathfrak{A}_t} (\boldsymbol{\sigma}\mathbf{n}) \otimes (\mathbf{x} - \mathbf{x}_o) da = \int_{\mathfrak{A}_t} \{div(\boldsymbol{\sigma}) \otimes (\mathbf{x} - \mathbf{x}_o) + \boldsymbol{\sigma}^t\} dv, \quad (5.65)$$

Substituting the above equations in (5.63), we obtain

$$\begin{aligned}\mathbf{a} \wedge \int_{\partial\mathfrak{A}_t} (\mathbf{x} - \mathbf{x}_o) \wedge (\boldsymbol{\sigma}\mathbf{n}) da &= \int_{\mathfrak{A}_t} \{[(\mathbf{x} - \mathbf{x}_o) \otimes div(\boldsymbol{\sigma}) - div(\boldsymbol{\sigma}) \otimes (\mathbf{x} - \mathbf{x}_o)]\mathbf{a} \\ &\quad + (\boldsymbol{\sigma} - \boldsymbol{\sigma}^t)\mathbf{a}\} dv \quad (5.66)\end{aligned}$$

Again using (5.57) and the definition of axial vector to a skew-symmetric tensor (2.98) the above equations can be reduced to

$$\mathbf{a} \wedge \int_{\partial\mathfrak{A}_t} (\mathbf{x} - \mathbf{x}_o) \wedge (\boldsymbol{\sigma}\mathbf{n}) da = \mathbf{a} \wedge \int_{\mathfrak{A}_t} [(\mathbf{x} - \mathbf{x}_o) \wedge div(\boldsymbol{\sigma}) - \boldsymbol{\tau}] dv, \quad (5.67)$$

and hence the required identity (5.52), since $\mathbf{a}$ is an arbitrary vector.

Substituting (5.51) and (5.52) in (5.49) yields

$$\begin{aligned}\int_{\partial\mathfrak{A}_t} \mathbf{m} da &= \int_{\mathfrak{A}_t} [(\mathbf{x} - \mathbf{x}_o) \wedge (\rho\mathbf{a} - div(\boldsymbol{\sigma}) - \rho\mathbf{b})] dv \\ &\quad + \int_{\mathfrak{A}_t} [\frac{D\mathbf{p}}{Dt} + \mathbf{p} div(\mathbf{v}) - \mathbf{c} + \boldsymbol{\tau}] dv. \quad (5.68)\end{aligned}$$

Since, the body also satisfies the balance of linear momentum (5.43), the first term on the RHS is zero. Hence, the balance of angular momentum requires

$$\int_{\partial\mathfrak{A}_t} \mathbf{m} da = \int_{\mathfrak{A}_t} [\frac{D\mathbf{p}}{Dt} + \mathbf{p} div(\mathbf{v}) - \mathbf{c} + \boldsymbol{\tau}] dv. \quad (5.69)$$

Here we are interested only in non-polar bodies and hence, $\mathbf{p} = \mathbf{c} = \mathbf{m} = \mathbf{o}$. Therefore, (5.69) simplifies to requiring

$$\int_{\mathfrak{A}_t} \boldsymbol{\tau} dv = \mathbf{o}. \quad (5.70)$$

Since, this has to hold for any arbitrary subparts of the body, $\mathfrak{B}_t$, $\boldsymbol{\tau} = \mathbf{o}$. Consequently,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^t, \quad (5.71)$$

that is the Cauchy stress tensor has to be symmetric in non-polar bodies.

## 5.5 Summary

Since, in this course we are interested only in purely mechanical process, we studied only conservation of mass and momentum. There are other quantities like energy, charge which is also conserved. In courses on thermodynamics or electromagnetism, one would study the other conservation laws. The one equation that would be used frequently in the following chapters is the final expression for the conservation of linear momentum, also called as equilibrium equations,

$$\rho\mathbf{a} = div(\boldsymbol{\sigma}) + \rho\mathbf{b}, \tag{5.72}$$

where $\rho$ is the density, $\mathbf{b}$ is the body force per unit mass, $\boldsymbol{\sigma}$ is the Cauchy stress tensor.

## 5.6 Self-Evaluation

1. Determine which of the following Cauchy stress fields are possible within a body at rest assuming that there are no body forces acting on it:

$$\text{(a)} \qquad \boldsymbol{\sigma} = \begin{pmatrix} -\frac{3}{2}x^2y^2 & xy^3 & 0 \\ xy^3 & -\frac{1}{4}y^4 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\text{(b)} \qquad \boldsymbol{\sigma} = \begin{pmatrix} 3yz & z^2 & 5y^2 \\ z^2 & 7xz & 2x^2 \\ 5y^2 & 2x^2 & 9xy \end{pmatrix},$$

$$\text{(c)} \qquad \boldsymbol{\sigma} = \begin{pmatrix} 3x+5y & 7x-3y & 0 \\ 7x-3y & 2x-7y & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\text{(d)} \qquad \boldsymbol{\sigma} = \begin{pmatrix} 3x & -3y & 0 \\ -7x & 7y & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\text{(e)} \qquad \boldsymbol{\sigma} = \begin{pmatrix} 7x & -3x & 0 \\ 7y & 3y & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the components of the stress are with respect to orthonormal Cartesian basis and $(x, y, z)$ denote the Cartesian coordinates of a typical material particle in the current configuration of the body.

2. The Cartesian components of the Cauchy stress tensor in a plate at rest is

$$\boldsymbol{\sigma} = \begin{pmatrix} -2x^2 & -7+4xy+z & 1+x-3y \\ -7+4xy+z & 3x^2-2y^2+5z & 0 \\ 1+x-3y & 0 & -5+x+3y+3z \end{pmatrix}.$$

   Find the body force that should act on the plate so that this state of stress is realizable in the body.

3. A body at rest is subjected to a plane state of stress such that the non-zero Cartesian components of the Cauchy stress are $\sigma_{xx}$, $\sigma_{xz}$ and $\sigma_{zz}$. Derive the equilibrium equations for this special case. Then, show that in the absence of body forces the equilibrium equations hold if,

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial z^2}, \quad \sigma_{xz} = -\frac{\partial^2\phi}{\partial x\partial z}, \quad \sigma_{zz} = \frac{\partial^2\phi}{\partial x^2},$$

   where $\phi = \bar{\phi}(x,y)$, for any choice of $\phi$. $\phi$ is called the Airy's stress potential.

4. Derive form first principles and show that for the plane stress problem in cylindrical polar coordinates with the non-zero cylindrical polar components of the Cauchy stress being $\sigma_{rr}$, $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$, as shown in figure 5.2, the equilibrium equations in the absence of body forces with the body in static equilibrium are

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0, \qquad \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2\sigma_{r\theta}}{r} = 0.$$

5. Is any stress field that satisfies the equilibrium equations realizable in a given body? Discuss.

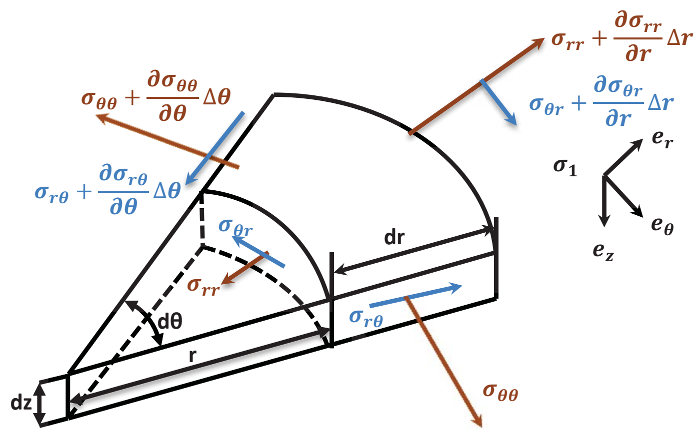



Figure 5.2: Figure for problem 4. Variation of the cylindrical polar components of the Cauchy stress over an infinitesimal cylindrical wedge, in plane stress state

# Chapter 6

# Constitutive Relations

## 6.1 Overview

The concepts and equations introduced in chapters 3 to 5, within the framework of non-relativistic mechanics, are essential to characterize kinematics, stresses and balance principles and as mentioned already these concepts and equations hold for any body undergoing a purely mechanical process. However, they do not distinguish one material from another which is contradictory with the experiments. Also, for a non-polar body while there are only seven equations[1] there are thirteen[2] scalar fields that needs to be determined. Thus, there are more unknowns in the balance laws than available equations and hence all of the unknown fields cannot be determined from the balance laws alone. In other words, balance laws alone are incapable of determining the response (displacement of the body due to an applied force) of deformable bodies[3]. They must be augmented by additional equations, called constitutive relations or equations of state, which depends on the material that the body is made up of. A constitutive relation approximates the observed physical behavior of a material under specific conditions of interest. To summarize, constitutive relations are required for two reasons:

- To bring in the material dependence in the force displacement relation.

[1] As we saw in the last chapter, there are 1 equation due to balance of mass, 3 equations due to balance of linear momentum and 3 equations due to balance of angular momentum.

[2] The thirteen scalar fields are: density, three components for body force and nine components of stress. Here we assume that the body is non-polar.

[3] If we were to specify a body as rigid, then balance laws alone is sufficient to determine its motion. However, specifying a body as rigid is a constitutive specification.

- To bridge the gap between the number of unknowns and the available equations while trying to develop a force displacement relation.

Thus, we may constitutively prescribe the six independent components of Cauchy stress. Now, for a non-polar body, we have as many equations as there are variables. Then, the balance of mass is used to obtain the density and balance of linear momentum to obtain the body force. By prescribing the constitutive relation only for the six components of the stress field, we have already made use of the balance of angular momentum as applicable for a non-polar body. We hasten to emphasize the arbitrariness in the choice of variables that are constitutively prescribed and those that are found from the balance laws.

Quantities characterizing a system at a certain time are called state variables. The state variables are quantities like stress, density. Contrary to some presentations of thermodynamics, we do not consider kinematical quantities as state variables. We treat them as independent variables, because they are the directly measurable quantities in an experiment. On the other hand, none of the state variables can be measured directly in an experiment. However, we require the state variables to depend (explicitly or implicitly) on the kinematical quantities and the function that establishes this relation between kinematic variables and state variables is called constitutive relation. Reemphasizing, these constitutive relation depend on the material that the body is made up of and of course, the process that is being studied.

Naturally, it is at this stage where the major subdivisions of the subject, such as the theories of viscous flow, elasticity branch out. These subdivisions are nothing but assumptions made on which kinematic variables determine the Cauchy stress. For example, if Cauchy stress is assumed to be prescribed through a function of the Eulerian gradient of the velocity field, i.e., $\boldsymbol{\sigma} = \mathbf{g}(grad(\mathbf{v}))$, then it can describe the flow of a viscous material. On the other hand, if Cauchy stress is assumed to be depend explicitly on the gradient of the deformation field, i.e., $\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F})$, then this can describe the elastic deformation of a solid. As this course is concerned about elastic response of the body, let us understand what we mean by elastic response?

## 6.2 Definition of elastic process

All bodies deform under the application of load. Depending on the characteristics of this deformation, the process or the body is classified as elastic or inelastic. The characteristics of this deformation depends on the material, temperature, magnitude of the applied load and among many other factors. Now, let us understand the characteristics of a elastic process.

Some of the definitions of an elastic process in the literature are:

1. The processes in which the original size and shape can be recovered is termed as elasticity.

2. Processes in which, the value of state variables in a given configuration are independent of how it was reached is called elastic.

3. A non-dissipative process is called a elastic process.

The first two definitions are popular in the literature. However, they are of little use because they cannot tell whether a process that the body is currently being subjected to is elastic. Such a conclusion can be arrived at only after subjecting the body to a complementary process. However, the last definition does not have such a drawback.

Further, it turns out that these various definitions are not equivalent. To see this, consider the uniaxial stress versus stretch ratio plot shown in figure 6.1a. Such a response, possible in shape memory alloys, would qualify as elastic only if the first definition is used. Since, the stress corresponding to a given stretch ratio depends on whether it is being loaded or unloaded it is not elastic according to the second definition. Also, since the loading and unloading path are different, there will be dissipation (wherein the mechanical energy is converted into thermal energy). Hence, it cannot be elastic by the last definition either.

Consider a axial stress vs. stretch response as shown in figure 6.1b which will be characterized as elastic according to definitions one and three. But the value of state variable stress could be anything corresponding to a stretch ratio of $\Lambda_o$. Hence, it is not elastic according to definition two. Of course, such a stress versus stretch response occurs for an idealized system made of a spring and an inextensible chord as shown in figure 6.1c. But then elasticity is an idealized process too and biological soft tissues response can be idealized as shown in figure 6.1b.

For us, a process is elastic only if it is consistent with all three definitions.

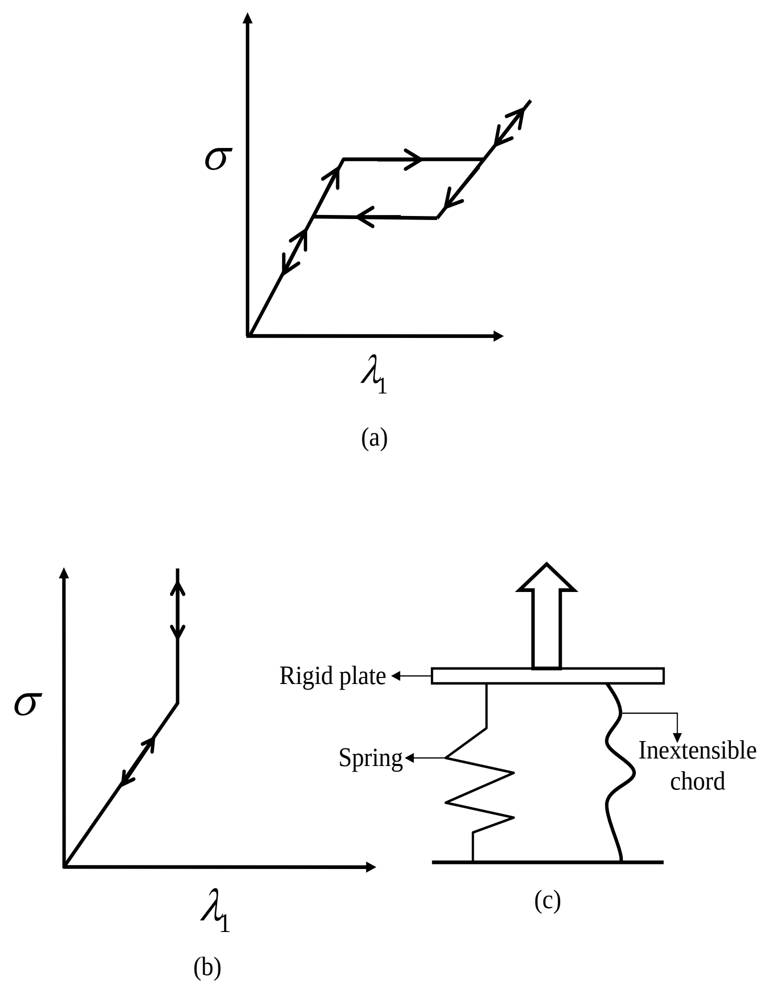



Figure 6.1: Uniaxial stress, $\sigma$ versus axial stretch ratio, $\lambda_1$ plot for (a) superelastic process (b) mechanical model shown in (c).

Before proceeding further a few words of caution is necessary. In reality, no process is elastic but some are close to being elastic. It is common in the literature, as we also did in the first paragraph, to call a material or a body to be elastic. This is incorrect, in a strict sense. It is the process that the body is being subjected to which is elastic. After all, the same body under different circumstances deforms inelastically.

Based on the above definition of elasticity, it can be shown that the Cauchy stress, $\boldsymbol{\sigma}$, in an elastic process would at most depend on the deformation gradient, $\mathbf{F}$, three material unit vectors, $\mathbf{M}_i$ and the state of Cauchy stress in the reference configuration[4], $\boldsymbol{\sigma}_R$ and thus,

$$\mathbf{g}(\boldsymbol{\sigma}, \mathbf{F}, \boldsymbol{\sigma}_R, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3) = \mathbf{0}. \tag{6.1}$$

In other words, equation (6.1) is an assumption on how the state variable, stress, varies with the motion of the body. Since, by definition 2 for an elastic process, the value of stress cannot depend on the history of the motion field that the body has experienced, we are assured that there is an implicit function that relates the Cauchy stress and the deformation gradient.

We shall assume $\boldsymbol{\sigma}_R = \mathbf{0}$, that is the reference configuration is stress free and that the Cauchy stress is related *explicitly* to the deformation gradient, so that equation (6.1) could be simplified to,

$$\boldsymbol{\sigma} = \mathbf{h}(\mathbf{F}, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3). \tag{6.2}$$

Further if we assume that the material is isotropic, that is the response of the material is same in all directions (see section 6.3.2 for more detailed discussion), the Cauchy stress would not depend on the three unit material vectors, $\mathbf{M}_i$. For this case, the Cauchy stress depends explicitly only on the deformation gradient, i.e.,

$$\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F}). \tag{6.3}$$

Thus, we have to find the relation between the six independent components of the stress tensor and the nine independent components of the deformation gradient. To find this relation in general through experimentation alone is daunting, as we illustrate now. Let us say the relation between the components of the Cauchy stress ($\sigma_{ij}$) and deformation gradient ($F_{kl}$) is linear and of the form,

$$\sigma_{ij} = C_{ijkl}(F_{kl} - \delta_{kl}), \tag{6.4}$$

[4]Do not confuse the stress in the reference configuration with Piola-Kirchhoff stresses. While $\boldsymbol{\sigma}_R = \mathbf{f}(\mathbf{1}, \boldsymbol{\sigma}_R)$, Piola-Kirchhoff stress, $\mathbf{P} = \det(\mathbf{F})\mathbf{f}(\mathbf{F}, \boldsymbol{\sigma}_R)\mathbf{F}^{-t}$.

where $C_{ijkl}$ is the components of a constant fourth order tensor and $\delta_{kl}$ is the Kronecher delta. Even for this case, we have to find 6*9 = 54 constants which means we require 54 independent measurements. This is too many. While for metals this relationship is linear for polymers it is nonlinear. It is easy to see that the number of constants required in case of nonlinear relationship would be much higher than the linear case. Hence, if there is some way by which we can reduce the number of unknown functions from 6 and the variables that it depends on from 9 in equation (6.3) it would be useful. It turns out that this is possible and this is what we shall see next.

## 6.3 Restrictions on constitutive relation

There are certain restrictions, guidelines and principles that are common which the constitutive relations for different materials and under different conditions of interest, should satisfy. Thus, we require the constitutive relation to ensure that the value of the state variables for a given state of the body are independent of the placement of the body in the Euclidean space and to honor the coordinate transformation rules. There is also, the concept of material symmetry which to a large extent determines the variables in the constitutive relation. The following section is devoted towards understanding these concepts.

### 6.3.1 Restrictions due to objectivity

While the physical body was given a mathematical representation, we made two arbitrary choices regarding:

1. The region in the Euclidean point space to which the body is mapped.

2. The orientation of the basis vectors with respect to which the position, velocity, acceleration, etc. of the material particles are defined.

Since these choices are arbitrary, the state of the body cannot depend on these choices. (Clearly, a body subjected to a particular load cannot fail just because, the body is mapped on to a different region of Euclidean point space or a different set of basis vectors is used.) Hence, the constitutive relations that relate these state variables with the independent variables, i.e., kinematic variables, should also not depend to these choices.

The value of state variables variables have to be invariant with respect to the above choices because they mathematically describe the state of the body. We emphasize that the value of the state variables have to be invariant and not same. Thus, while the value of scalar valued state variables like density have to be the same, the components of tensor valued state variables like stress will change with the choice of the basis according to the transformation rules discussed in section 2.6. However, as pointed out in 2.6 just because the components of a tensor is different does not mean they are different tensors. Only, if the value of at least one of the certain scalar valued functions of the components of the tensor are different[5], the two tensors are said to be different.

### Restriction due to non-uniqueness of placers

In this section, we shall explore the restrictions that needs to be placed on the constitutive relations so that they remain invariant for the different equivalent mappings of the body on to the Euclidean point space. From a physical standpoint, the same restriction could be viewed as requiring the constitutive relation of the body be the same irrespective of where the body is. That is, say, the deflection of the beam to a given set of loads should remain the same whether it is tested in Chennai or Delhi or Washington or London.

It should be realized that in the different mappings of the body onto the Euclidean space, it is the point space that changes and not the vector space. If the vector space were to change, it means that we are changing the basis vectors. Further, it is assumed that the scale used to measure distances between two points does not change. Consequently, the distance between any two points in the body does not change due to these different mappings of the body. Hence, this restriction, as we shall see, tantamount to requiring that the state variables be invariant to superposed rigid body rotation of the current or the reference configuration.

Let $\mathbf{x}$ and $\mathbf{x}^+$ denote the position vector of a material particle at time $t$, mapped on to different regions of the Euclidean point space. Then, they are

[5]For example, if $\mathbf{A}$ is a second order tensor then one of the following: $tr(\mathbf{A})$, $tr(\mathbf{A}^{-1})$, $\det(\mathbf{A})$, must be different. There is nothing unique about the choice of the scalar valued functions of $\mathbf{A}$. One could have chosen $tr(\mathbf{A}^2)$ instead of $\det(\mathbf{A})$ or more generally, $tr(\mathbf{A}^m)$ where $m$ takes some integer value.

related by the equation:

$$\mathbf{x}^+ = \mathbf{Q}\mathbf{x} + \mathbf{c}, \tag{6.5}$$

where $\mathbf{Q}$ and $\mathbf{c}$ are functions of time, $t$ and $\mathbf{Q}$ is an orthogonal tensor, not necessarily a proper orthogonal tensor[6]. While it is easy to see that (6.5) preserves the distance between any two points, the converse can be proved by following the same steps as that outlined in section 3.10.1.

Since, similar non-uniqueness exist with respect to placers for the reference configuration

$$\mathbf{X}^+ = \mathbf{Q}_o\mathbf{X} + \mathbf{c}_o, \tag{6.6}$$

where $\mathbf{X}$ and $\mathbf{X}^+$ are the position vectors of same material particle mapped on to the different regions of the Euclidean point space, $\mathbf{Q}_o$ is a constant orthogonal tensor and $\mathbf{c}_o$ is a constant vector.

Next, we like to compute the standard kinematical quantities and see how they change due to these equivalent mappings of the body on to the Euclidean point space. This is a prerequisite to find the restrictions on the constitutive relations due to this requirement.

We begin by considering the transformations of the motion field, $\boldsymbol{\chi}$. Let

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \qquad \mathbf{x}^+ = \boldsymbol{\chi}^+(\mathbf{X}^+, t) \tag{6.7}$$

then using equations (6.5) and (6.6) we obtain

$$\mathbf{x}^+ = \boldsymbol{\chi}^+(\mathbf{X}^+, t) = \mathbf{Q}\boldsymbol{\chi}(\mathbf{X}, t) + \mathbf{c} = \mathbf{Q}\boldsymbol{\chi}(\mathbf{Q}_o^t(\mathbf{X}^+ - \mathbf{c}_o), t) + \mathbf{c}, \tag{6.8}$$

Now, we can find the relation between the deformation gradients, $\mathbf{F}$ and $\mathbf{F}^+$ defined as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \qquad \mathbf{F}^+ = \frac{\partial \mathbf{x}^+}{\partial \mathbf{X}^+}, \tag{6.9}$$

using (6.6), (6.8) and chain rule for differentiation to be

$$\mathbf{F}^+ = \mathbf{Q}\mathbf{F}\frac{\partial \mathbf{X}}{\partial \mathbf{X}^+} = \mathbf{Q}\mathbf{F}\mathbf{Q}_o^t. \tag{6.10}$$

Next, we find the relation between the velocity fields, $\mathbf{v}$ and $\mathbf{v}^+$ defined as

$$\mathbf{v} = \frac{D\mathbf{x}}{Dt}, \qquad \mathbf{v}^+ = \frac{D\mathbf{x}^+}{Dt}, \tag{6.11}$$

[6] We allow improper orthogonal tensors because what is considered as top or bottom, left or right is subjective.

using equation (6.5) as

$$\mathbf{v}^+ = \mathbf{Q}\mathbf{v} + \frac{D\mathbf{Q}}{Dt}\mathbf{x} + \frac{D\mathbf{c}}{Dt} = \mathbf{Q}\mathbf{v} + \frac{D\mathbf{Q}}{Dt}\mathbf{Q}^t(\mathbf{x}^+ - \mathbf{c}) + \frac{D\mathbf{c}}{Dt}. \tag{6.12}$$

Similarly, we obtain the relationship between accelerations $\mathbf{a}$ and $\mathbf{a}^+$ as

$$\mathbf{a}^+ = \mathbf{Q}\mathbf{a} + 2\frac{D\mathbf{Q}}{Dt}\mathbf{v} + \frac{D^2\mathbf{Q}}{Dt^2}\mathbf{x} + \frac{D^2\mathbf{c}}{Dt^2}. \tag{6.13}$$

In order to simplify equations (6.12) and (6.13) we introduce, the skew tensor

$$\boldsymbol{\Omega} = \frac{D\mathbf{Q}}{Dt}\mathbf{Q}^t. \tag{6.14}$$

To show that $\boldsymbol{\Omega}$ is a skew tensor, taking the total time derivative of the relation $\mathbf{Q}\mathbf{Q}^t = \mathbf{1}$, we obtain

$$\frac{D\mathbf{Q}}{Dt}\mathbf{Q}^t + \mathbf{Q}\frac{D\mathbf{Q}^t}{Dt} = \mathbf{0}, \tag{6.15}$$

from which the required result follows. In lieu of definition (6.14), equations (6.12) and (6.13) can be written as

$$\mathbf{v}^+ = \mathbf{Q}\mathbf{v} + \frac{D\mathbf{c}}{Dt} + \boldsymbol{\Omega}(\mathbf{x}^+ - \mathbf{c}), \tag{6.16}$$

$$\mathbf{a}^+ = \mathbf{Q}\mathbf{a} + \frac{D^2\mathbf{c}}{Dt^2} + 2\boldsymbol{\Omega}\left(\mathbf{v}^+ - \frac{D\mathbf{c}}{Dt}\right) + \left(\frac{D\boldsymbol{\Omega}}{Dt} - \boldsymbol{\Omega}^2\right)(\mathbf{x}^+ - \mathbf{c}). \tag{6.17}$$

where we have made use of the identity

$$\frac{D\mathbf{Q}}{Dt}\mathbf{v} = \boldsymbol{\Omega}\left(\mathbf{v}^+ - \frac{D\mathbf{c}}{Dt}\right) - \boldsymbol{\Omega}^2(\mathbf{x}^+ - \mathbf{c}),$$

to obtain (6.17).

Now, we would like to see how the displacement transforms due to different placements of the body in the Euclidean point space. Recalling that the displacement field is defined as:

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \mathbf{u}^+ = \mathbf{x}^+ - \mathbf{X}^+. \tag{6.18}$$

The Eulerian displacement gradient transforms like

$$
\begin{aligned}
\mathbf{h} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} &= \mathbf{1} - \mathbf{F}^{-1}, \\
\mathbf{h}^+ = \frac{\partial \mathbf{u}^+}{\partial \mathbf{x}^+} &= \mathbf{1} - (\mathbf{F}^+)^{-1} \\
&= \mathbf{1} - \mathbf{Q}_o\mathbf{F}^{-1}\mathbf{Q}^t = \mathbf{1} - \mathbf{Q}_o\mathbf{Q}^t + \mathbf{Q}_o\mathbf{h}\mathbf{Q}^t, \qquad (6.19)
\end{aligned}
$$

where we have used (6.10) and (3.34). Similarly, the Lagrangian displacement gradient, is computed to be

$$
\begin{aligned}
\mathbf{H} &= \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{F} - \mathbf{1}, \\
\mathbf{H}^+ &= \frac{\partial \mathbf{u}^+}{\partial \mathbf{X}^+} = \mathbf{F}^+ - \mathbf{1} = \mathbf{Q}\mathbf{H}\mathbf{Q}_o^t + \mathbf{Q}\mathbf{Q}_o^t - \mathbf{1}, \qquad (6.20)
\end{aligned}
$$

using (6.10) and (3.33). It is apparent from the above equations that displacement gradient transforms as $\mathbf{Q}\mathbf{h}\mathbf{Q}^t$ (or $\mathbf{Q}_o\mathbf{H}\mathbf{Q}_o^t$) only if $\mathbf{Q} = \mathbf{Q}_o$.

Then, it is straightforward to show that the right and left Cauchy-Green deformation tensor transform as:

$$
\mathbf{C}^+ = \mathbf{Q}_o\mathbf{C}\mathbf{Q}_o^t, \qquad \mathbf{B}^+ = \mathbf{Q}\mathbf{B}\mathbf{Q}^t. \qquad (6.21)
$$

Next, we examine what happens to stress tensors due to different placements of the body in the current configuration. Recalling traction is related to the Cauchy stress, by the relation $\mathbf{t}_{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n}$, where $\mathbf{n}$ is the outward unit normal at a point $\mathbf{x}$ on the boundary surface $\partial\mathfrak{B}_t$ of an arbitrary region $\mathfrak{B}_t$.

As evidenced from the figure (6.2) a different placer of the body in the Euclidean space, in general, alters the orientation of the normal with respect to a fixed set of basis vectors. Let $\mathbf{n}$ and $\mathbf{n}^+$ denote the outward unit normals to the same material surface $\partial\mathcal{B}$ and at the same material particle, $P$, in two different placements of the body in the Euclidean space, $\mathfrak{B}_t$ and $\mathfrak{B}_t^+$. Since, we are interested in the same material surface, we can appeal to Nanson's formula (3.72) to relate the unit normals as

$$
\mathbf{n}^+ = \mathbf{Q}\mathbf{n}. \qquad (6.22)
$$

where we have made use of the equation (6.5) which provides the relationship between the placements $\mathfrak{B}_t$ and $\mathfrak{B}_t^+$.

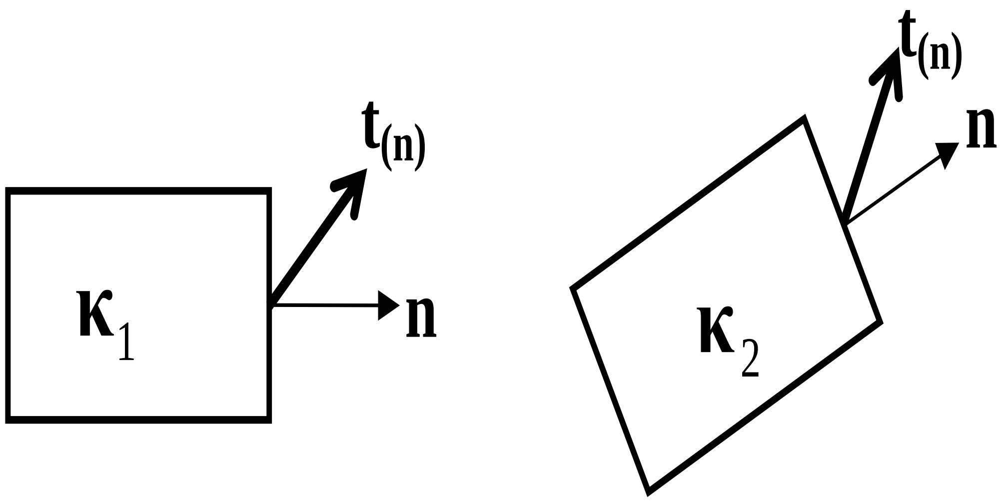



Figure 6.2: Two placements ($\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$) of the same body in the same state along with a unit normal ($\mathbf{n}$) and traction ($\mathbf{t}_{(\mathbf{n})}$).

Similarly, the traction vector $\mathbf{t}_{(\mathbf{n})}$ and $\mathbf{t}^+_{(\mathbf{n}^+)}$ acting on the same material surface, $\partial\mathcal{B}$ and at the same material particle $P$, in two different placements of the body, $\mathfrak{B}_t$ and $\mathfrak{B}_t^+$ are related through:

$$\mathbf{t}^+_{(\mathbf{n}^+)} = \mathbf{Q}\mathbf{t}_{(\mathbf{n})}. \tag{6.23}$$

To obtain the above equation we have made use of the fact that the normal and shear traction acting on a material surface in two different placements of the body in the current configuration has to be same, i.e., $\mathbf{t}^+_{(\mathbf{n}^+)} \cdot \mathbf{n}^+ = \mathbf{t}_{(\mathbf{n})} \cdot \mathbf{n}$. Then, (6.23) is obtained by recognizing that $(\mathbf{Q}^t\mathbf{t}^+_{(\mathbf{n}^+)} - \mathbf{t}_{(\mathbf{n})}) \cdot \mathbf{n} = 0$ for any choice of $\mathbf{n}$ where we have made use of (6.22).

Since, $\mathbf{t}^+_{(\mathbf{n}^+)} = \boldsymbol{\sigma}^+\mathbf{n}^+$ and $\mathbf{t}_{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n}$, we obtain

$$\mathbf{Q}\boldsymbol{\sigma}\mathbf{n} = \boldsymbol{\sigma}^+\mathbf{Q}\mathbf{n}, \tag{6.24}$$

using equations (6.22) and (6.23) and the above equation has to hold for any choice of $\mathbf{n}$. Therefore, the Cauchy stress will transform as

$$\boldsymbol{\sigma}^+ = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^t. \tag{6.25}$$

Since, the definition of Cauchy stress is independent of the choice of reference configuration, different placers of the reference configuration, does not cause any change in the value of the Cauchy stress.

Next, let us see how body forces transform. These forces are essentially action at a distance forces acting along a straight line joining the material particles in the body under investigation to another point in space belonging to some other body. Thus, if $\mathbf{r}$ is the position vector of the point belonging to another body, then the direction along which $\mathbf{b}$ acts, is given by $(\mathbf{x}-\mathbf{r})/(\mid \mathbf{x}-\mathbf{r} \mid)$. Then, in a different placement of the body the body force acts along a direction $(\mathbf{x}^+ - \mathbf{r}^+)/(\mid \mathbf{x}^+ - \mathbf{r}^+ \mid)$. Because the new placement of the body is related to its original placement by equation (6.5), we obtain

$$\mathbf{b}^+ = \hat{b}(\mid \mathbf{x}^+ - \mathbf{r}^+ \mid)\frac{\mathbf{x}^+ - \mathbf{r}^+}{\mid \mathbf{x}^+ - \mathbf{r}^+ \mid} = \hat{b}(\mid \mathbf{Qx} - \mathbf{Qr} \mid)\frac{\mathbf{Qx} - \mathbf{Qr}}{\mid \mathbf{Qx} - \mathbf{Qr} \mid}$$
$$= \hat{b}(\mid \mathbf{x} - \mathbf{r} \mid)\frac{\mathbf{Q}(\mathbf{x} - \mathbf{r})}{\mid \mathbf{x} - \mathbf{r} \mid} = \mathbf{Qb}, \quad (6.26)$$

where $\hat{b}(\cdot)$ denotes a function of the distance between two points of interest. Here $\mathbf{r}^+ = \mathbf{Qr} + \mathbf{c}$ because the transformation rule is applicable not only to points that the body under study occupies but to its neighboring bodies as well. This is required so that the relationship between the body under investigation and its neighbors are preserved.

Finally, for future reference, $div(\boldsymbol{\sigma})$ transforms due to equivalent placements of the body as,

$$div^+(\boldsymbol{\sigma}^+) = div(\boldsymbol{\sigma}^+)\mathbf{Q} = div(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^t)\mathbf{Q} = \mathbf{Q}div(\boldsymbol{\sigma}), \quad (6.27)$$

where $div^+$ denotes the divergence with respect to $\mathbf{x}^+$.

Till now, we have just seen how various quantities transform between different placements of the same body in the same state. Using this we are in a position to state what restriction the requirement that the state variables be invariant to the different placers of the body in the same state places on the constitutive relation, given it depends on certain kinematic quantities and other state variables. For example, as we saw above for elastic process the stress is a function of deformation gradient, i.e.,

$$\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F}). \quad (6.28)$$

Now, due to different placement of the body in the reference and current configurations,

$$\boldsymbol{\sigma}^+ = \mathbf{f}(\mathbf{F}^+). \quad (6.29)$$

Note that the function does not change and this is what restricts the nature of the function. Using equations (6.25) and (6.10), (6.29) can be written as

$$\boldsymbol{\sigma}^+ = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^t = \mathbf{f}(\mathbf{F}^+) = \mathbf{f}(\mathbf{Q}\mathbf{F}\mathbf{Q}_o^t). \tag{6.30}$$

Substituting for $\boldsymbol{\sigma}$ from equation (6.28) we obtain a restriction of the form of the function $\mathbf{f}$ as:

$$\mathbf{Q}\mathbf{f}(\mathbf{F})\mathbf{Q}^t = \mathbf{f}(\mathbf{Q}\mathbf{F}\mathbf{Q}_o^t), \tag{6.31}$$

for any orthogonal tensors $\mathbf{Q}$ and $\mathbf{Q}_o$ and $\mathbf{F} \in \mathcal{D} \subseteq Lin^+$, the set of all linear transformations whose determinant is positive.

The question on how to obtain the form of the function, given the restriction (6.31), is dealt in detail next. To understand how this restriction (6.31) fixes the form of the tensor-valued tensor function, let us look at a similar requirement on scalar-valued scalar function, $g(x)$. Say the restriction on this scalar function is: $g(m * x) = m * g(x)$, where $m$ is some arbitrary constant. It then immediately tells that the function is a linear function, i.e., $g(x) = a * x$, where $a$ is some constant. Similarly, it transpires that the restriction (6.31) requires,

$$\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F}) = \alpha_0\mathbf{1} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^{-1}, \tag{6.32}$$

where $\alpha_i = \hat{\alpha}_i(J_1, J_2, J_3)$,

$$J_1 = tr(\mathbf{B}), \quad J_2 = tr(\mathbf{B}^{-1}), \quad J_3 = [\det(\mathbf{B})]^{1/2} = \det(\mathbf{F}), \tag{6.33}$$

are the invariants of $\mathbf{B}$.

Now, we show how equation (6.32) is obtained from the restriction (6.31). Towards this, first we set $\mathbf{Q} = \mathbf{1}$ to obtain,

$$\mathbf{f}(\mathbf{F}) = \mathbf{f}(\mathbf{F}\mathbf{Q}_o^t) = \mathbf{f}(\mathbf{V}\mathbf{R}) = \mathbf{f}(\mathbf{V}\mathbf{R}\mathbf{Q}_o^t), \tag{6.34}$$

where we have made use of the polar decomposition theorem (2.116). Then, let us pick $\mathbf{Q}_o = \mathbf{R}$, since $\mathbf{R}$ is also an orthogonal tensor. With this choice equation (6.34) yields

$$\mathbf{f}(\mathbf{F}) = \mathbf{f}(\mathbf{V}) = \bar{\mathbf{f}}(\mathbf{B}). \tag{6.35}$$

The last equality arises because $\mathbf{B} = \mathbf{V}^2$ and square-root theorem, (2.147) ensures the existence of an unique $\mathbf{V}$ such that $\mathbf{V} = \sqrt{\mathbf{B}}$. Next, we shall again appeal to equation (6.31) but now we shall not set $\mathbf{Q}$ to be $\mathbf{1}$ to obtain

$$\mathbf{Q}\bar{\mathbf{f}}(\mathbf{B})\mathbf{Q}^t = \bar{\mathbf{f}}(\mathbf{Q}\mathbf{B}\mathbf{Q}^t), \tag{6.36}$$

$\forall$ $\mathbf{Q} \in \mathcal{O}$. The above equation can be rewritten as

$$\bar{\mathbf{f}}(\mathbf{B}) = \mathbf{Q}^t \bar{\mathbf{f}}(\mathbf{Q}\mathbf{B}\mathbf{Q}^t)\mathbf{Q}, \tag{6.37}$$

$\forall$ $\mathbf{Q} \in \mathcal{O}$.

**Theorem 7.1:** A symmetric second order tensor valued function $\bar{\mathbf{f}}$ defined over the space of symmetric second order tensors, satisfies (6.37) if and only if it has a representation

$$\boldsymbol{\sigma} = \bar{\mathbf{f}}(\mathbf{B}) = \alpha_0 \mathbf{1} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^2 \tag{6.38}$$

where $\alpha_0$, $\alpha_1$, $\alpha_2$ are functions of principal invariants of $\mathbf{B}$, $I_i$, i.e.,

$$\alpha_i = \bar{\alpha}_i(I_1, I_2, I_3). \tag{6.39}$$

*Proof:* [7] Before proving the above theorem we prove the following theorem:

**Theorem 7.2:** Let $\alpha$ be a scalar function defined over the space of symmetric positive definite second order tensors. Then $\alpha(\mathbf{Q}\mathbf{B}\mathbf{Q}^t) = \alpha(\mathbf{B})$ $\forall$ $\mathbf{Q} \in \mathcal{O}$ if and only if there exist a function $\overline{\alpha}$, defined on $\mathcal{R}^+ \times \mathcal{R}^+ \times \mathcal{R}^+$, such that $\alpha(\mathbf{B}) = \overline{\alpha}(\lambda_1, \lambda_2, \lambda_3)$, where $\overline{\alpha}(\lambda_1, \lambda_2, \lambda_3)$ is insensitive to permutations of $\lambda_i$ and $\lambda_1$, $\lambda_2$, $\lambda_3$ are the eigenvalues of $\mathbf{B}$. Hence, $\alpha(\mathbf{B}) = \hat{\alpha}(I_1, I_2, I_3)$.

*Proof:* [8] Writing $\mathbf{B}$ in the spectral form

$$\mathbf{B} = \lambda_1 \mathbf{b}_1 \otimes \mathbf{b}_1 + \lambda_2 \mathbf{b}_2 \otimes \mathbf{b}_2 + \lambda_3 \mathbf{b}_3 \otimes \mathbf{b}_3, \tag{6.40}$$

where $\mathbf{b}_i$'s are the ortho-normal eigenvectors of $\mathbf{B}$ and hence

$$\mathbf{Q}\mathbf{B}\mathbf{Q}^t = \lambda_1 \mathbf{Q}\mathbf{b}_1 \otimes \mathbf{Q}\mathbf{b}_1 + \lambda_2 \mathbf{Q}\mathbf{b}_2 \otimes \mathbf{Q}\mathbf{b}_2 + \lambda_3 \mathbf{Q}\mathbf{b}_3 \otimes \mathbf{Q}\mathbf{b}_3. \tag{6.41}$$

Since, $\alpha(\mathbf{Q}\mathbf{B}\mathbf{Q}^t) = \alpha(\mathbf{B})$ $\forall$ $\mathbf{Q} \in \mathcal{O}$, $\alpha(\mathbf{B})$ must be independent of the orientation of the principal directions of $\mathbf{B}$ and must depend on $\mathbf{B}$ only through its eigenvalues, $\lambda_1$, $\lambda_2$, $\lambda_3$.

Next, choose $\mathbf{Q}$ to be a rotation of $\pi/2$ about $\mathbf{b}_3$ so that $\mathbf{Q}\mathbf{b}_1 = \mathbf{b}_2$, $\mathbf{Q}\mathbf{b}_2 = -\mathbf{b}_1$ and $\mathbf{Q}\mathbf{b}_3 = \mathbf{b}_3$. Hence,

$$\alpha(\mathbf{B}) = \alpha(\lambda_2 \mathbf{b}_1 \otimes \mathbf{b}_1 + \lambda_1 \mathbf{b}_2 \otimes \mathbf{b}_2 + \lambda_3 \mathbf{b}_3 \otimes \mathbf{b}_3) \tag{6.42}$$

[7] Adapted from Serrin [5] and Ogden [6]. The original proof is due to Rivlin and Ericksen [7]

[8] Adapted from Ogden [6]

from which we deduce that $\overline{\alpha}(\lambda_1, \lambda_2, \lambda_3) = \overline{\alpha}(\lambda_2, \lambda_1, \lambda_3)$. In similar fashion it can be shown that $\overline{\alpha}$ is insensitive to other permutations of $\lambda_i$'s.

Finally, recalling that the eigenvalues are the solutions of the characteristic equation

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0, \tag{6.43}$$

in principal, $\lambda_i$'s can be expressed uniquely in terms of the principal invariants and hence $\alpha(\mathbf{B}) = \hat{\alpha}(I_1, I_2, I_3)$.

The converse of the theorem 7.2 is proved easily from the property of trace and determinants. Hence, we have proved theorem 7.2.

**Theorem 7.3:** If $\bar{\mathbf{f}}$ satisfies (6.37) then the eigenvalues of $\bar{\mathbf{f}}(\mathbf{B})$ are functions of the principal invariants of $\mathbf{B}$.

*Proof:* Let $\gamma(\mathbf{B})$ be the eigenvalue of $\bar{\mathbf{f}}(\mathbf{B})$. Then,

$$\det[\bar{\mathbf{f}}(\mathbf{B}) - \gamma(\mathbf{B})\mathbf{1}] = 0. \tag{6.44}$$

The corresponding eigenvalue of $\bar{\mathbf{f}}(\mathbf{QBQ}^t)$ is $\gamma(\mathbf{QBQ}^t)$ and hence

$$\det[\bar{\mathbf{f}}(\mathbf{QBQ}^t) - \gamma(\mathbf{QBQ}^t)\mathbf{1}] = 0. \tag{6.45}$$

This can be written as

$$\det[\mathbf{Q}\{\bar{\mathbf{f}}(\mathbf{B}) - \gamma(\mathbf{QBQ}^t)\mathbf{1}\}\mathbf{Q}^t] = 0, \tag{6.46}$$

by using (6.37) and the relation $\mathbf{QQ}^t = \mathbf{1}$. Using the property of determinants the above equation reduces to

$$\det[\bar{\mathbf{f}}(\mathbf{B}) - \gamma(\mathbf{QBQ}^t)\mathbf{1}] = 0, \tag{6.47}$$

which has to hold for all $\mathbf{Q} \in \mathcal{O}$. Comparing (6.44) and (6.47)

$$\gamma(\mathbf{QBQ}^t) = \gamma(\mathbf{B}), \tag{6.48}$$

for all $\mathbf{Q} \in \mathcal{O}$, which by theorem 2.2 implies that $\gamma(\mathbf{B}) = \hat{\alpha}(I_1, I_2, I_3)$

**Theorem 7.4:** If $\boldsymbol{\sigma} = \bar{\mathbf{f}}(\mathbf{B})$ satisfies (6.37) then $\bar{\mathbf{f}}(\mathbf{B})$ is coaxial with $\mathbf{B}$, i.e., the principal directions of $\mathbf{B}$ and $\bar{\mathbf{f}}(\mathbf{B})$ would be the same.

*Proof:* Consider an eigenvector $\mathbf{b}_1$ of $\mathbf{B}$ and define an orthogonal transformation $\mathbf{Q}$ by

$$\mathbf{Qb}_1 = -\mathbf{b}_1, \qquad \mathbf{Qb}_j = \mathbf{b}_j \qquad \text{if} \qquad \mathbf{b}_1 \cdot \mathbf{b}_j = 0, \tag{6.49}$$

i.e. $\mathbf{Q}$ is a reflection on the plane normal to $\mathbf{b}_1$. Now, $\mathbf{QBQ}^t = \mathbf{B}$ and hence, by (6.37), $\mathbf{Q}\boldsymbol{\sigma} = \boldsymbol{\sigma}\mathbf{Q}$. We therefore have

$$\mathbf{Q}(\boldsymbol{\sigma}\mathbf{b}_1) = \boldsymbol{\sigma}(\mathbf{Q}\mathbf{b}_1) = -\boldsymbol{\sigma}\mathbf{b}_1, \tag{6.50}$$

and we see that $\mathbf{Q}$ transforms the vector $\boldsymbol{\sigma}\mathbf{b}_1$ into its opposite. Since, the only vectors transformed by the reflection $\mathbf{Q}$ into their opposites are the multiples of $\mathbf{b}_1$, it follows that $\mathbf{b}_1$ is an eigenvector of $\boldsymbol{\sigma}$. Similarly, it can be shown that every eigenvector of $\mathbf{B}$ is also an eigenvector of $\boldsymbol{\sigma}$. Hence, $\overline{\mathbf{f}}(\mathbf{B})$ is coaxial with $\mathbf{B}$.

Now we prove theorem 7.1.

Clearly, if (6.38) along with (6.39) holds then (6.37) is satisfied and hence we have to prove only the converse.

It follows from theorem 7.3 and theorem 7.4 that $\overline{\mathbf{f}}(\mathbf{B})$ is coaxial with $\mathbf{B}$ and its eigenvalues are functions of the principal invariants of $\mathbf{B}$. Let $\lambda_1$,$\lambda_2$,$\lambda_3$ and $f_1$,$f_2$,$f_3$ be the eigenvalues of $\mathbf{B}$ and $\overline{\mathbf{f}}(\mathbf{B})$ respectively and consider the equations

$$\alpha_0 + \alpha_1\lambda_i + \alpha_2\lambda_i^2 = f_i, \qquad (i = 1, 2, 3) \tag{6.51}$$

for the three unknowns $\alpha_0$, $\alpha_1$, $\alpha_2$. Assuming the $\lambda_i$ and $f_i$ are given and that $\lambda_i$'s are distinct it follows that $\alpha_i$'s are determined uniquely in terms of $\lambda_i$ and $f_i$ which are themselves determined uniquely by the principal invariants of $\mathbf{B}$. Thus, since $\mathbf{B}$ is coaxial with $\overline{\mathbf{f}}(\mathbf{B})$ and $\alpha_i$ are functions of the principal invariants of $\mathbf{B}$; equation (6.38) follows from (6.51) provided the eigenvalues of $\mathbf{B}$ are distinct, of course. When the eigenvalues of $\mathbf{B}$ are not distinct $\alpha_2$ or $\alpha_1$ and $\alpha_2$ could be chosen arbitrarily, depending on whether the algebraic multiplicity of the eigenvalues is 2 or 3 respectively. However, this choice may cause some $\alpha_i$ to become discontinuous even when $\overline{\mathbf{f}}(\mathbf{B})$ remains continuous, Truesdell and Noll [8] and Serrin [5] provide example of such cases.

Finally, from Cayley-Hamilton theorem, (2.142) we obtain

$$\mathbf{B}^2 = I_1\mathbf{B} - I_2\mathbf{1} + I_3\mathbf{B}^{-1}. \tag{6.52}$$

Then, observing that the principal invariants of the positive definite, $\mathbf{B}$, are related bijectively to the invariants $J_1$, $J_2$ and $J_3$, as defined in (6.33), we note that

$$\alpha_i = \hat{\alpha}_i(I_1, I_2, I_3) = \overline{\alpha}_i(J_1, J_2, J_3) \tag{6.53}$$

for $i = (0, 1, 2)$. Substituting (6.52) and (6.53) in (6.38) we obtain

$$\boldsymbol{\sigma} = \alpha_0\mathbf{1} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^{-1}. \tag{6.54}$$

Before concluding this section let us investigate a couple of issues. The first issue is whether the balance laws should also be invariant to different placements of the body in the same state. A straight forward calculation will show that balance of mass is invariant to different placements of the same body in the same state. However, balance of linear momentum and hence angular momentum are not. In fact, due to equivalent placements of the body, the balance of linear momentum equation transforms as

$$\mathbf{Q}div(\boldsymbol{\sigma}) + \rho\mathbf{Q}\mathbf{b} = \rho\left(\mathbf{Q}\mathbf{a} + 2\frac{D\mathbf{Q}}{Dt}\mathbf{v} + \frac{D^2\mathbf{Q}}{Dt^2}\mathbf{x} + \frac{D^2\mathbf{c}}{Dt^2}\right). \tag{6.55}$$

But for the above equation to be consistent, it is required that

$$2\frac{D\mathbf{Q}}{Dt}\mathbf{v} + \frac{D^2\mathbf{Q}}{Dt^2}\mathbf{x} + \frac{D^2\mathbf{c}}{Dt^2} = \mathbf{o}, \tag{6.56}$$

since, $div(\boldsymbol{\sigma}) + \rho\mathbf{b} = \rho\mathbf{a}$. The above equation holds only for placements for which $\frac{D^2\mathbf{c}}{Dt^2} = \mathbf{o}$ and $\mathbf{Q}(t) = \mathbf{Q}^c$, a constant. Thus, a school of thought requires the constitutive relations describing the state variables to be invariant only for placements of the body related as:

$$\mathbf{x}^+ = \mathbf{Q}^c\mathbf{x} + \mathbf{c}_t t + \mathbf{c}_o, \tag{6.57}$$

where $\mathbf{Q}^c$ is a constant orthogonal tensor, $\mathbf{c}_t$ and $\mathbf{c}_o$ are constant vectors. Equation (6.57) is called Galilean transformation.

The other school of thought, argues that equivalent placements of the body in a particular state is a thought experiment to obtain some restriction on the constitutive relations describing the state variables which is not applicable for balance laws. The rational behind their argument is that for a given body force, the balance laws will hold only for certain motions of the body[9]. On the other hand, there are no such restrictions for the constitutive relations. Hence, there is no inconsistency if motions that are not admissible according to the balance law are used to find restrictions on the constitutive relations. We find merit in this school of thought.

The second issue is on the existence of unique placer for the reference configuration. Some researchers are of the opinion that the placer to the reference

[9] As we shall see, within the realms of finite elasticity, Ericksen [9] has showed that in the absence of body forces, deformations of the form: $r = r(R)$, $\theta = \Theta$, $z = Z$, where $(R, \Theta, Z)$ and $(r, \theta, z)$ are the coordinates of a material particle in the reference and current configuration respectively, is not possible in some compressible materials.

configuration has to be unique for agreement on the material symmetry of the body. In section 6.3.2, we show that the opinion of these researchers is incorrect.

### Restrictions due to non-uniqueness of the basis vectors

We have been writing equations in direct form, that is independent of the choice of basis vectors. To beginners it is natural to ask the question, if we were to continue to write equations in direct form what restriction can this choice of basis vectors place? To answer this question let us look at an example. Say, some person investigating the response of some body subjected to some process finds that Cauchy stress, $\boldsymbol{\sigma}$ is related to the deformation gradient, $\mathbf{F}$, as $\boldsymbol{\sigma} = \alpha(\mathbf{F} - \mathbf{1})$, where $\alpha$ is a constant. Even though we have written in direct notation, say, we change the basis vectors used to describe the reference configuration, then the matrix components of the Cauchy stress will not change. However, the matrix components of the deformation gradient will change resulting in a contradiction and inadmissability of the proposed relation.

To elaborate, recall that $\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ and $\mathbf{F} = F_{ij}\mathbf{e}_i \otimes \mathbf{E}_j$, where $\mathbf{e}_i$ is the basis vectors used to represent the current configuration and $\mathbf{E}_j$ is the basis vectors used in the reference configuration. Then, let $[\bar{\boldsymbol{\sigma}}]$ and $[\bar{\mathbf{F}}]$ represent the matrix components of the Cauchy stress and deformation gradient with respect to the $\{\bar{\mathbf{E}}_i\}$ basis. Similarly, let $[\boldsymbol{\sigma}]$ and $[\mathbf{F}]$ denote the matrix components of the Cauchy stress and deformation gradient with respect to the $\{\mathbf{E}_i\}$ basis. It follows from the transformation laws 2.6 that

$$[\bar{\boldsymbol{\sigma}}] = [\boldsymbol{\sigma}], \qquad [\bar{\mathbf{F}}] = [\mathbf{F}][\mathbf{Q}], \tag{6.58}$$

where $Q_{ij} = \mathbf{E}_i \cdot \bar{\mathbf{E}}_j$. Immediately, the contradiction in the constitutive relation, $\boldsymbol{\sigma} = \alpha(\mathbf{F} - \mathbf{1})$ is apparent.

Let us see what restriction is placed on the form of the function (or functional) to ensure that it obeys the necessary transformation rules. Thus, say, we postulate that: $\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F})$. Let $Q_{ij} = \mathbf{e}_i \cdot \bar{\mathbf{e}}_j$ and $Q^o_{ij} = \mathbf{E}_i \cdot \bar{\mathbf{E}}_j$. Due to this transformation of the basis vectors in the current and the reference configuration, the stress transforms as $[\bar{\boldsymbol{\sigma}}] = [\mathbf{Q}]^t[\boldsymbol{\sigma}][\mathbf{Q}]$ and the deformation gradient transforms as $[\bar{\mathbf{F}}] = [\mathbf{Q}]^t[\mathbf{F}][\mathbf{Q}^o]$. Then, we require that $[\bar{\boldsymbol{\sigma}}] = [\mathbf{f}([\bar{\mathbf{F}}])]$. Hence, the function $\mathbf{f}(\cdot)$ should be such that

$$[\mathbf{Q}]^t[\mathbf{f}([\mathbf{F}])][\mathbf{Q}] = [\mathbf{f}([\mathbf{Q}]^t[\mathbf{F}][\mathbf{Q}^o])] \tag{6.59}$$

for any $[\mathbf{Q}]$ and $[\mathbf{Q}^o]$ such that $[\mathbf{Q}][\mathbf{Q}]^t = [\mathbf{Q}]^t[\mathbf{Q}] = [\mathbf{Q}^o][\mathbf{Q}^o]^t = [\mathbf{Q}^o]^t[\mathbf{Q}^o] = [\mathbf{1}]$ and for all $\mathbf{F} \in \mathcal{D} \subseteq Lin^+$.

While it is non-trivial to deduce the form of the function $\mathbf{f}(\mathbf{F})$ satisfying the condition (6.59), it is easy to verify that

$$\mathbf{f}(\mathbf{F}) = \alpha_0\mathbf{1} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^{-1}, \tag{6.60}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^t$, $\alpha_i = \hat{\alpha}_i(tr(\mathbf{B}), tr(\mathbf{B}^{-1}), \det(\mathbf{F}))$ upholds the condition (6.59).

Looking at restriction (6.31) and (6.59) one might wonder on the need for (6.59). Because if (6.31) were to hold then (6.59) will always hold. However, recognize that (6.59) is a different requirement than (6.31) and these differences would be apparent for cases where the constitutive relation depends on velocity or its derivatives. To see this, a change of basis used to describe the current configuration results in the velocity vector transforming as $[\mathbf{Q}][\mathbf{v}]$, which is different from (6.16) and hence the difference between the restrictions (6.59) and (6.31). To clarify, even if the basis were to change continuously with time, the velocity will transform as $[\mathbf{Q}(t)][\mathbf{v}]$ because the choice of the basis determines only the components of the velocity vector and not the velocity of the particle which depends only on the relative motion of the particle and the observer.

### 6.3.2 Restrictions due to Material Symmetry

Before looking at the concept of material symmetry, we have to understand what we mean by the body being homogeneous or inhomogeneous. Intuitively, we think that a body is homogeneous if it is made up of the same material. However, there is no particular variable in our formulation that uniquely characterizes the material and only the material. But the constitutive relation (or equation of state) which relates the state variables and the kinematic variable depends on the material. However, since the value of the kinematic variable, say the displacement or the deformation gradient, depends on the configuration used as reference, the constitutive relation also depends on the reference configuration or more particularly on the value of the state variables in the configuration used as reference. Hence, just because the constitutive relation for two particles are different does not mean that they are different materials, the difference can arise due to the use of different configurations being used as reference.

Consequently, mathematically, we say that two particles in a body belong to the same material, if there exist a configuration in which the density and temperature of these particles are same and with respect to which the constitutive equations are also same. In other words, what we are looking at is if the value of the state variables evolve in the same manner when two particles along with their neighborhood are subjected to identical motion fields from some reference configuration in which the value of the state variables are the same.

A body that is made up of particles that belong to the same material is called homogeneous. If a body is not homogeneous it is inhomogeneous. Now, say we have a body, in which different subsets of the body have the same constitutive relation only when different configurations are used as reference, i.e., any configuration that the body can take without breaking its integrity would result in different constitutive relations for different material particles, then the issue is how to classify such a body. Any body with residual stresses[10] like shrink fitted shafts, biological bodies are a couple of examples of bodies that fall in this category. One school of thought is to classify these bodies also as inhomogeneous, we subscribe to this definition simply for mathematical convenience.

Having seen what a homogeneous body is we are now in a position to understand what an isotropic material is. Consider an experimentalist who has mathematically represented the reference configuration of a homogeneous body, i.e., identify the region of the Euclidean point space that this body occupies, has found the spatial variation of the state variables. Now, say without the knowledge of the experimentalist, this reference configuration of the body is deformed (or rotated). Then, the question is will this deformation (or rotation) be recognized by the experimentalist? Theoretically, if the experimentalist cannot identify the deformation (or rotation), then the functional form of the constitutive relations should be the same for this deformed and initial reference configuration. This set of indistinguishable deformation or rotation forms a group called the symmetry group and it depends on the material as well as the configuration that it is in. If the symmetry group contains all the elements in the orthogonal group[11] then the material in that configuration is said to possess isotropic material symmetry. If the symmetry group does not contain all the elements in the orthogonal group, the material in that configuration is said to be anisotropic. There are various classes of

[10]The non-uniform stresses field in a body free of boundary traction is called residual stress.

[11]The set of all linear transformations, $\mathbf{Q}$, such that $\mathbf{Q}\mathbf{Q}^t = \mathbf{Q}^t\mathbf{Q} = \mathbf{1}$ form a group called the orthogonal group.

anisotropy like transversely isotropic, orthorhombic, etc., depending on the elements contained in the symmetry group.

Here we like to emphasize on some subtleties. Firstly, we emphasize that the symmetry group of a material depends on the configuration in which it is assessed. Thus, the material in a stress free configuration could be isotropic but the same material in uniaxially stressed state will not be isotropic. Secondly, we allow the body to be deformed because it has been shown [10] that certain deformations superposed on an uniaxially extended body does not alter the state of the body. Hence, it should be recognized that the symmetry group, $\mathcal{G} \subseteq \mathcal{H}$, the unimodular group[12], since the volume cannot change in an equivalent placement of the body. In fact, for a perfect gas the symmetry group, $\mathcal{G} = \mathcal{H}$. Thirdly, unlike in the restriction due to objectivity, in this case only the body is rotated or deformed virtually, not its surroundings. Even though like in the restriction due to objectivity the rotation or deformation is virtual, it has to maintain the integrity of the body and satisfy the balance laws, i.e., be statically admissible because the rotated or deformed configuration is also in equilibrium.

Now, let us mathematically investigate the restriction material symmetry imposes on the constitutive relation. Continuing with our example, $\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F})$ we examine the restriction on $\mathbf{f}(\mathbf{F})$ due to material symmetry. For any $\mathbf{G} \in \mathcal{G}$, the restriction due to material symmetry requires that

$$\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F}) = \mathbf{f}(\mathbf{FG}), \tag{6.61}$$

$\forall\, \mathbf{F} \in \mathcal{D}$. Of course, for this case the restriction is similar to that obtained for objectivity (6.31), except that now, $\mathbf{Q} = \mathbf{1}$ and $\mathbf{Q}_o$ is not necessarily limited to orthogonal tensors. However, for isotropic material the symmetry group is the set of all orthogonal tensors only.

To further elucidate the difference between the restriction due to objectivity and material symmetry, consider a material whose response is different along a direction $\mathbf{M}$ identified in the reference configuration. Then, $\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F}, \mathbf{M})$. Now due to objectivity we require

$$\mathbf{Q}\mathbf{f}(\mathbf{F}, \mathbf{M})\mathbf{Q}^t = \mathbf{f}(\mathbf{Q}\mathbf{F}\mathbf{Q}_o^t, \mathbf{Q}_o\mathbf{M}), \tag{6.62}$$

for any orthogonal tensors $\mathbf{Q}$ and $\mathbf{Q}_o$ and for any $\mathbf{F} \in \mathcal{D}$. Due to material symmetry we require

$$\mathbf{f}(\mathbf{F}, \mathbf{M}) = \mathbf{f}(\mathbf{FG}, \mathbf{M}), \tag{6.63}$$

[12] The set of all linear transformations such that their determinant is positive and equal to one, form a group called the unimodular group.

$\forall$ $\mathbf{F} \in \mathcal{D}$ and $\mathbf{G} \in \mathcal{G}$. In (6.63) the right hand side is not $\mathbf{GM}$ because, the rotation (or deformation) $\mathbf{G}$, of the reference configuration is not distinguishable. In other words, $\mathbf{G}$ belongs to $\mathcal{G}$ only if $\mathbf{GM} = \mathbf{M}$. Thus, it immediately transpires that restriction due to material symmetry (6.63) is not same as that due to objectivity (6.62).

It is also instructive to note that if $\mathbf{GM} = \mathbf{M}$ holds for any an orthogonal tensor, $\mathbf{G}$, then $\mathbf{M} = \mathbf{o}$. Thus, the result that for isotropic materials the response would be same in all directions and consequently there are no preferred directions. If the material response along one direction is different, then it is called as transversely isotropic and if its response along three directions are different, it is called orthotropic.

## 6.4 Isotropic Hooke's law

Having established for isotropic materials if Cauchy stress is explicitly related to the deformation gradient, then this relationship would be of the form,

$$\boldsymbol{\sigma} = \mathbf{f}(\mathbf{F}) = \alpha_0 \mathbf{1} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^{-1}, \tag{6.64}$$

where $\alpha_i = \hat{\alpha}_i(J_1, J_2, J_3)$, called the material response functions and needs to be determined from experiments. Thus, restriction due to objectivity, has reduced the number of variables in the function from 9 to 3 and the number of unknown functions from 6 to 3. Next, let us see if we can further reduce the number of variables that the function depends upon or the number of functions themselves. Since, in an elastic process there is no dissipation of energy, this reduces the number of unknown functions to be determined to just one and thus Cauchy stress is given by[13],

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial J_3}\mathbf{1} + \frac{2}{J_3}\left[\frac{\partial W}{\partial J_1}\mathbf{B} - \frac{\partial W}{\partial J_2}\mathbf{B}^{-1}\right], \tag{6.65}$$

where $W = \hat{W}(J_1, J_2, J_3)$, is called as the stored (or strain) energy per unit volume of the reference configuration. Notice that here the stored energy function is the only function that needs to be determined through experimentation.

[13]Derivation of this equation is beyond the scope of this course.

In practice, the materials undergo a non-dissipative process only when the relative displacements are small, resulting in the components of the displacement gradient being small. Hence, it is of interest to see the implications of this approximation on a general representation for Cauchy stress (6.64).

In chapter 3 (section (3.5.1), we saw that the deformation gradient is related to the Lagrangian and Eulerian displacement gradient as:

$$\mathbf{F} = \mathbf{H} + \mathbf{1}, \qquad \mathbf{F}^{-1} = \mathbf{1} - \mathbf{h}. \tag{6.66}$$

Hence, the left Cauchy-Green deformation tensor is given by

$$\mathbf{B} = \mathbf{H} + \mathbf{H}^t + \mathbf{H}\mathbf{H}^t + \mathbf{1}, \qquad \mathbf{B}^{-1} = \mathbf{1} - \mathbf{h} - \mathbf{h}^t + \mathbf{h}^t\mathbf{h}. \tag{6.67}$$

If $tr(\mathbf{H}\mathbf{H}^t) << 1$ and $tr(\mathbf{h}\mathbf{h}^t) << 1$, then we could approximately calculate the left Cauchy-Green deformation tensor as

$$\mathbf{B} \approx \mathbf{H} + \mathbf{H}^t + \mathbf{1}, \qquad \mathbf{B}^{-1} \approx \mathbf{1} - \mathbf{h} - \mathbf{h}^t. \tag{6.68}$$

We also saw in chapter 3 (section 3.8) that when $tr(\mathbf{H}\mathbf{H}^t) << 1$ and $tr(\mathbf{h}\mathbf{h}^t) << 1$, $\mathbf{H} \approx \mathbf{h}$. Therefore we need not distinguish between the Lagrangian and Eulerian linearized strain. By virtue of these approximations the expressions in equation (6.68) can be written as:

$$\mathbf{B} \approx 2\boldsymbol{\epsilon} + \mathbf{1}, \qquad \mathbf{B}^{-1} \approx \mathbf{1} - 2\boldsymbol{\epsilon}, \tag{6.69}$$

where $\boldsymbol{\epsilon}$ denotes the (Lagrangian or Eulerian) linearized strain tensor. Consequently, the invariants are calculated approximately as[14]

$$J_1 \approx 3 + 2tr(\boldsymbol{\epsilon}), \qquad J_2 \approx 3 - 2tr(\boldsymbol{\epsilon}), \qquad J_3 \approx 1 + tr(\boldsymbol{\epsilon}). \tag{6.70}$$

Substituting (6.69) in a general representation for stress in a isotropic material, (6.64) we obtain

$$\boldsymbol{\sigma} = \alpha_0^l \mathbf{1} + \alpha_1^l \boldsymbol{\epsilon}. \tag{6.71}$$

[14] The last expression is obtained by substituting (6.69) in $J_3 = \sqrt{I_2/J_2}$ and using Taylor's series expansion about $tr(\boldsymbol{\epsilon}) = 0$, to obtain

$$J_3 = \sqrt{\frac{I_2}{J_2}} = \sqrt{\frac{J_1^2 - tr(\mathbf{B}^2)}{2J_2}} \approx \sqrt{\frac{3 + 4tr(\boldsymbol{\epsilon})}{3 - 2tr(\boldsymbol{\epsilon})}} \approx \sqrt{1 + 2tr(\boldsymbol{\epsilon})} \approx 1 + tr(\boldsymbol{\epsilon}).$$

where

$$\alpha_0^l = \alpha_0 + \alpha_1 + \alpha_2, \qquad \alpha_1^l = 2(\alpha_1 - \alpha_2), \tag{6.72}$$

are functions of $tr(\boldsymbol{\epsilon})$ by virtue of (6.70). It is necessary that $\alpha_0^l$ should be a linear function of $tr(\boldsymbol{\epsilon})$ and $\alpha_1^l$ a constant; because we neglected the higher order terms of $\boldsymbol{\epsilon}$ to obtain the above expression. Consequently, the expression for Cauchy stress (6.71) simplifies to,

$$\boldsymbol{\sigma} = tr(\boldsymbol{\epsilon})\lambda\mathbf{1} + 2\mu\boldsymbol{\epsilon}, \tag{6.73}$$

where $\lambda$ and $\mu$ are Lamè constants and these constants alone need to be determined from experiments. The relation (6.73) is called the Hooke's law for isotropic materials.

Since, the relation between the Cauchy stress and linearized strain is linear we can invert the constitutive relation (6.73) and write the linearized strain in terms of the stress as,

$$\boldsymbol{\epsilon} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}tr(\boldsymbol{\sigma})\mathbf{1}. \tag{6.74}$$

Taking trace on both sides of equation (6.73) we obtain,

$$tr(\boldsymbol{\sigma}) = tr(\boldsymbol{\epsilon})[3\lambda + 2\mu], \tag{6.75}$$

on noting that $tr(\cdot)$ is a linear operator and that $tr(\mathbf{1}) = 3$. Thus, equation (6.74) is obtained by rearranging equation (6.73) and using equation (6.75).

Lamè constants while allows one to write the constitutive relations succinctly, their physical meaning and the methodology for their experimental determination is not obvious. Hence, we define various material parameters, such as Young's modulus, shear modulus, bulk modulus, Poisson's ratio, which have a physical meaning and is easy to determine experimentally.

Before proceeding further, a few comments on the Hooke's law has to be made. From the derivation, it is clear that (6.73) is an approximation to the correct and more general (6.64). Consequently, Hooke's law does not have some of the characteristics one would expect a robust constitutive relation to possess. The first drawback is that contrary to the observations rigid body rotations induces stresses in the body. To see how this happens, recall from chapter 3 (section 3.10.1) that linearized strain is not zero when the body is subjected to rigid body rotations. Since, strain is not zero, it follows from (6.73) that stress would be present. The second drawback is that the

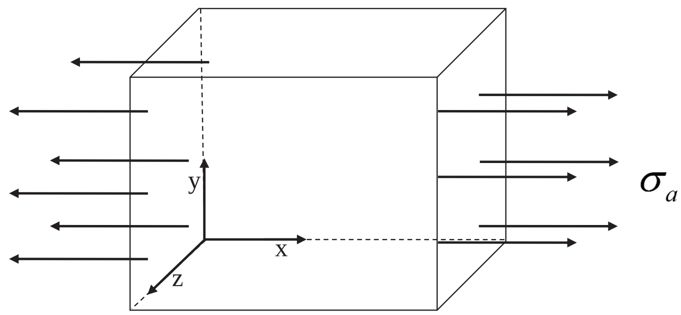



Figure 6.3: Uniaxial loading of a cuboid

constitutive relation (6.73) does not by itself satisfy the restriction due to objectivity. In fact it is not even Galilean invariant. Given that, due to equivalent placement of the body the displacement gradient transforms as given in equation (6.19) and Cauchy stress transforms as given in equation (6.25), it is evident that the restriction due to non-uniqueness of placers is not met. Despite these drawbacks, it is useful and gives good engineering estimates of the stresses and displacements under applied loads in certain classes of bodies.

## 6.5 Material parameters

In this section, we define the various material parameters and relate them to the Lamè constants. We do this by defining the stress that is applied on the body in the shape of a cuboid. Since, the body is assumed to obey Hooke's law, the state of strain gets fixed once the state of stress is specified because of the relation (6.74). Consequently, these parameters can be defined by prescribing the state of strain in the body as well.

### 6.5.1 Young's modulus and Poisson's ratio

Consider a cuboid being subjected to a uniform normal traction on two of its faces as shown in figure 6.3. Assuming the stress field is uniform, that is spatially constant, the Cartesian components of the stress at any point in the body is,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.76}$$

Substituting the above in equation (6.74) for the stress we obtain the strain as

$$\boldsymbol{\epsilon} = \begin{pmatrix} \frac{1}{2\mu}\left[1 - \frac{\lambda}{3\lambda+2\mu}\right]\sigma_a & 0 & 0 \\ 0 & -\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_a & 0 \\ 0 & 0 & -\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_a \end{pmatrix} \tag{6.77}$$

Now, Young's modulus, $E$, is defined as the ratio of the uniaxial stress to the component of the linearized strain along the direction of the applied uniaxial stress, i.e.,

$$E = \frac{\sigma_{xx}}{\epsilon_{xx}} = \frac{2\mu}{1 - \frac{\lambda}{(3\lambda+2\mu)}} = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}. \tag{6.78}$$

The Poisson's ratio , $\nu$ is defined as the negative of the ratio of the component of the strain along a direction perpendicular to the axis of loading, called the transverse strain to the component of the strain along the axis of loading, called the axial strain, i.e.,

$$\nu = -\frac{\epsilon_{yy}}{\epsilon_{xx}} = \frac{\lambda}{2(\lambda + \mu)}. \tag{6.79}$$

In equations (6.78) and (6.79) we expressed the Young's Modulus and Poisson's ratio in terms of the Lamè constants. This relation can be inverted to express Lamè constants in terms of $E$ and $\nu$ as,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{6.80}$$

Finally, substituting equation (6.80) in equation (6.73) and (6.74), the constitutive relations can be written in terms of the Young's modulus and Poisson's ratio as,

$$\boldsymbol{\sigma} = \frac{E\nu}{(1+\nu)(1-2\nu)} tr(\boldsymbol{\epsilon})\mathbf{1} + \frac{E}{(1+\nu)}\boldsymbol{\epsilon}, \tag{6.81}$$

$$\boldsymbol{\epsilon} = \frac{(1+\nu)}{E}\boldsymbol{\sigma} - \frac{\nu}{E} tr(\boldsymbol{\sigma})\mathbf{1}. \tag{6.82}$$

### 6.5.2 Shear Modulus

Consider a cuboid being subjected to uniform pure shear stress of the form

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.83}$$

Substituting the above state of stress in (6.74), the state of strain is obtained as,

$$\boldsymbol{\epsilon} = \frac{1}{2\mu} \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.84}$$

In chapter 3, section 3.10.3, we showed that if the angle change between two line segments oriented along $X$ and $Y$ direction is $\kappa$, then this simple shearing deformation results in,

$$\boldsymbol{\epsilon} = \frac{1}{2} \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.85}$$

Then, the shear modulus, $G$ is defined as the ratio of the shear stress ($\tau$) to change in angle ($\kappa$) due to this applied shear stress between two orthogonal line elements in the plane of shear i.e.,

$$G = \frac{\tau}{\kappa} = \mu. \tag{6.86}$$

Using equation (6.80b), the shear modulus can be written in terms of the Young's modulus and Poisson's ratio as,

$$G = \frac{E}{(1+\nu)}. \tag{6.87}$$

### 6.5.3 Bulk Modulus

Consider a cuboid being subjected to uniform pure hydrostatic stress,

$$\boldsymbol{\sigma} = p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.88}$$

Using (6.74) it could be seen that the above state of stress result in the strain tensor being,

$$\boldsymbol{\epsilon} = \frac{p}{(3\lambda + 2\mu)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.89}$$

The bulk modulus, $K$, is defined as the ratio of the mean hydrostatic stress to the volumetric strain when the body is subjected to pure hydrostatic stress. Thus,

$$K = \frac{tr(\boldsymbol{\sigma})}{3tr(\boldsymbol{\epsilon})} = \frac{1}{3}(3\lambda + 2\mu). \tag{6.90}$$

Now, we would like to express the bulk modulus in terms of Young's modulus and Poisson's ratio. Towards this, we substitute equation (6.80) in (6.90) to obtain

$$K = \frac{E}{3(1 - 2\nu)}. \tag{6.91}$$

## 6.6 Restriction on material parameters

Next, we would like to limit the range of values that these material parameters can take so that the response predicted by using these constitutive relations confirm with the observations.

Let us now see why Young's and shear modulus cannot be $\infty$. From the definition of the Young's modulus we find that the axial displacement due to an applied axial load has to be zero if $E = \infty$. Similarly, if $G = \infty$ the change in angle due to an applied shear stress is zero. These would happen only if the body is rigid since, no strain develops despite stress being applied. However, the focus of the study here is deformable bodies. Hence, we obtain the condition that $E < \infty$, $G < \infty$.

However, the bulk modulus can be $\infty$. From the definition of bulk modulus, it is clear that if $K = \infty$ for some material, then the volumetric strain developed due to applied hydrostatic pressure, for these materials has to be zero. This means that the volume of the body made of this material does not change, the material is incompressible. Some materials like rubber, polymers are known to be nearly incompressible. Moreover, it is also known that the volume of these materials do not change in any deformation. This means that these materials are capable of undergoing only isochoric deformations. Constitutive relations for such incompressible materials are obtained in section

6.7.1.

Since, we expect that tensile stress produce elongation and compressive stresses produce shortening, the three modulus - Young's, shear and bulk - should be positive. Since, $E$ is positive, for $G$ to be positive and finite, equation (6.87) requires $(1 + \nu) > 0$. Thus,

$$\nu > -1. \tag{6.92}$$

Note that, this is a strict inequality because if $(1 + \nu) = 0$, $G = \infty$, (since $0 < E < \infty$) which is not permissible.

Similarly, for $K$ to be positive, it transpires from equation (6.91) that $(1 - 2\nu) \geq 0$, since $E$ is positive. Hence,

$$\nu \leq 0.5. \tag{6.93}$$

Here we allow for equality because $K$ can be $\infty$.

Combining these both restrictions (6.92) and (6.93) on the Poisson's ratio,

$$-1 < \nu \leq 0.5. \tag{6.94}$$

Thus, Poisson's ratio can be negative, and it has been measured to be negative for certain foams. What this means is that as the body is stretched along a particular direction, the cross sectional area over which the load is distributed can increase. However, for most materials, especially metals, this cross sectional area decreases and therefore the Poisson's ratio is positive.

Finally, we show that the three modulus values cannot be zero. From the definition of these modulus, they being zero means that, any amount of strain can develop even when no stress is applied. This means that there can be displacement without the force, when the modulus value is zero. Since, there has to be force for displacement, this means that the modulus cannot be zero.

In table 6.1 the restrictions on various parameters are summarized. The point to note is that one of the Lamè constants, $\lambda$ has no restrictions. From equation (6.80), it can be seen that if $0 \leq \nu \leq 0.5$, then from the restrictions on $E$ and $\nu$ it can be said that $\lambda \geq 0$. However, $\lambda < 0$ for certain foams whose Poisson's ratio is negative. Therefore $\lambda$ has no restrictions.

## 6.7 Internally constraint materials

Till now we have been focusing on materials that have no internal constraint, that is, if the required body force and boundary traction can be applied, any

Table 6.1: Restrictions on material parameters

| Material Parameter | Symbol | Restriction |
|---|---|---|
| Young's modulus | $E$ | $0 < E < \infty$ |
| Shear modulus | $G$ | $0 < G < \infty$ |
| Bulk modulus | $K$ | $0 < K \leq \infty$ |
| Poisson's ratio | $\nu$ | $-1 < \nu \leq 0.5$ |
| Lamè constant | $\mu$ | $0 < \mu < \infty$ |
| | $\lambda$ | $-\infty < \lambda < \infty$ |

smooth displacement field can be realized in bodies made of these materials. However, in some materials this is not true. Only smooth displacement fields that satisfy certain constraints are realizable. The most common constraint on the displacement field is that it be volume preserving. Contrary to the reality, beginners tend to think that in all materials volume is preserved in all feasible deformations just because the cross sectional area reduces when uniaxially stretched. However, this is not true. We show this next. Recollecting from chapter 3 (section 3.6) that the ratio of the deformed volume, $v$ to the original volume, $V$ in case of homogeneous deformation is given by,

$$\frac{v}{V} = \det(\mathbf{F}) \approx 1 + tr(\boldsymbol{\epsilon}) \tag{6.95}$$

where we have used equation (6.70c) to approximately compute $\det(\mathbf{F})$ when the components of the displacement gradient are small. Using equation (6.77) that gives the state of strain in the cuboid subjected to uniaxial stress, $\sigma_a$, the change in its volume is computed to be

$$v - V = V tr(\boldsymbol{\epsilon}) = V\frac{\sigma_a}{(3\lambda + 2\mu)} = V\frac{\sigma_a}{3K} = V\frac{\sigma_a}{E}(1 - 2\nu), \tag{6.96}$$

where the last two equalities are obtained using equation (6.90) and (6.91). Thus, it is apparent that the volume of the cuboid changes as the magnitude of the uniaxial stress changes when $\nu \neq 0.5$. For physically possible values of Poisson's ratio, other than 0.5, the volume increases when uniaxially stretched and decreases when compressed.

In this section, we outline general principle to generate constitutive relations for internally constrained materials. Assuming that the materials internal constraint can be given by a equation of the form,

$$\bar{\zeta}(\mathbf{F}) = 0, \tag{6.97}$$

where $\bar{\zeta}$ is a scalar valued function. The requirement that the internal constraint (6.97) be objective necessitates

$$\bar{\zeta}(\mathbf{F}) = \bar{\zeta}(\mathbf{Q}\mathbf{F}\mathbf{Q}_o^t), \qquad \forall \quad \text{orthogonal} \qquad \mathbf{Q}\text{and} \quad \mathbf{Q}_o. \tag{6.98}$$

For (6.98) to hold:

$$\bar{\zeta}(\mathbf{F}) = \breve{\zeta}(\mathbf{B}) = \breve{\zeta}(J_1, J_2, J_3) = 0, \tag{6.99}$$

where $J_i$ are the invariants of $\mathbf{B}$ as defined in (6.33). The proof for this is left as an exercise; the steps leading to this is similar to that used to obtain the constitutive representation for Cauchy stress.

In order to accommodate its motion to an internal constraint a material body must be able to bring appropriate contact forces into play and the constitutive equation governing its stress response must be such as to allow these forces to act. Thus, the stress at a point $\mathbf{x}$ and time $t$ is uniquely determined by the value of the deformation gradient to within a symmetric tensor $\mathbf{A}$ which is not determined by the motion of the body and which does no work in any motion compatible with the constraints, i.e.,

$$\boldsymbol{\sigma} = \alpha_0 \mathbf{1} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^{-1} + \mathbf{A}, \tag{6.100}$$

where $\alpha_i = \hat{\alpha}_i(J_1, J_2, J_3)$ are the same material response functions as defined before in (6.32) and $J_i$'s are invariants of $\mathbf{B}$ defined in (6.33).

It can be shown that[15], the rate at which the applied stresses does work is given by the expression $\boldsymbol{\sigma} \cdot \mathbf{l}$, where $\mathbf{l} = grad(\mathbf{v})$, the Eulerian gradient of the velocity field. Hence, the requirement that $\mathbf{A}$ do no work requires that

$$\mathbf{A} \cdot \mathbf{l} = 0, \tag{6.101}$$

[15] Derivation of this can be found in standard text books in continuum mechanics like [1, 2]. For understanding these derivation a lot more concepts needs to be grasped which is beyond the scope of this course. In fact, the following derivation here is also not easy to follow; but made as simple as possible. To proceed further in this course, it suffices to understand that there is a formal procedure to obtain constitutive relations for materials with constraints.

for all allowable motions. To relate $\mathbf{A}$ with $\breve{\zeta}$ we take material time derivative of (6.99) to obtain

$$\frac{\partial \zeta}{\partial \mathbf{B}}\mathbf{B} \cdot \mathbf{l} = 0. \tag{6.102}$$

Now let

$$\mathbf{A} = p\frac{\partial \zeta}{\partial \mathbf{B}}\mathbf{B}, \tag{6.103}$$

where $p$ is an arbitrary scalar to be determined from boundary conditions and/or equilibrium equations. While it is easy to show that, the above choice for the constraint stress satisfies the requirement (6.101), it is difficult to show that this is the only choice and is beyond the scope of this course.

Hence, a general representation for stress with kinematic constraint (6.99) is given by

$$\boldsymbol{\sigma} = \alpha_0\mathbf{1} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^{-1} + p\frac{\partial \zeta}{\partial \mathbf{B}}\mathbf{B}, \tag{6.104}$$

with a undetermined scalar $p$ to be found using equilibrium equations and/or boundary conditions.

### 6.7.1 Incompressible materials

As already discussed, material that can undergo only isochoric motions is called incompressible material. As was shown in chapter - 3 for isochoric motions, $\det(\mathbf{F}) = J_3 = 1$. Hence,

$$\breve{\zeta}(J_1, J_2, J_3) = J_3 - 1 = 0. \tag{6.105}$$

Substituting (6.105) in (6.103) and using (2.189) we obtain

$$\mathbf{A} = p\frac{J_3}{2}\mathbf{B}^{-1}\mathbf{B} = -p^*\mathbf{1}, \tag{6.106}$$

where $p^* = -p/2$. Consequently, a general representation for Cauchy stress (6.104) for incompressibility constraint reduces to,

$$\boldsymbol{\sigma} = (\alpha_0 - p^*)\mathbf{1} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^{-1}, \tag{6.107}$$

where now, $\alpha_i = \overline{\alpha}_i(J_1, J_2)$ since $J_3$ is identically 1. Now introducing, $p^+ = -\alpha_0 + p^*$, we obtain

$$\boldsymbol{\sigma} = -p^+\mathbf{1} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^{-1}, \tag{6.108}$$

where without any loss of generality we assume $p^+$ to be some arbitrary scalar to be determined from boundary conditions and/or equilibrium equations. For convenience and brevity in notation we shall drop the superscript + in $p^+$ and write

$$\boldsymbol{\sigma} = -p\mathbf{1} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^{-1}, \tag{6.109}$$

which we consider as the most general representation for Cauchy stress in an incompressible material being subjected to elastic deformation.

As before (see section 6.4), it can be shown that when the components of the displacement gradient is small, the kinematical constraint equation (6.105) reduces to requiring,

$$tr(\boldsymbol{\epsilon}) = 0, \tag{6.110}$$

and consequently the equation (6.109) can be approximated as,

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu_{inc}\boldsymbol{\epsilon}, \tag{6.111}$$

where $\mu_{inc}$ is a constant material parameter and $p$ is an arbitrary scalar to be determined from equilibrium equations and/or boundary condition. Equation (6.111) is the constitutive relation for an incompressible material undergoing elastic deformation such that the components of the displacement gradient are small.

Before concluding this section, we would like to find the value of all the material parameters defined in section 6.5 for this incompressible material.

In case of the cuboid being subjected to uniaxial stress, the state of stress and strain are:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix}. \tag{6.112}$$

The incompressibility condition (6.110) requires that

$$\epsilon_1 + 2\epsilon_2 = 0. \tag{6.113}$$

Hence, the incompressible materials Poisson's ratio,

$$\nu_{inc} = -\frac{\epsilon_2}{\epsilon_1} = 0.5. \tag{6.114}$$

Substituting (6.112) along with the requirement that $\epsilon_2 = -\epsilon_1/2$ obtained from (6.113) for stress and strain in (6.111) we obtain:

$$\begin{pmatrix} \sigma_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -p + 2\mu_{inc}\epsilon_1 & 0 & 0 \\ 0 & -p - \mu_{inc}\epsilon_1 & 0 \\ 0 & 0 & -p - \mu_{inc}\epsilon_1 \end{pmatrix}. \tag{6.115}$$

For the above equation to hold, $p = -\mu_{inc}\epsilon_1$ and hence, $\sigma_a = 3\mu_{inc}\epsilon_1$. Thus, the incompressible materials Young's modulus,

$$E_{inc} = \frac{\sigma_a}{\epsilon_1} = 3\mu_{inc}. \tag{6.116}$$

If the cuboid is being subjected to pure shear state of stress and strain,

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\epsilon} = \begin{pmatrix} 0 & \kappa/2 & 0 \\ \kappa/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.117}$$

then it is straightforward to verify that for this state of strain $tr(\boldsymbol{\epsilon}) = 0$ and therefore the incompressibility condition (6.110) is met. Substituting, (6.117) in (6.111) we obtain:

$$\begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -p & \mu_{inc}\kappa & 0 \\ \mu_{inc}\kappa & -p & 0 \\ 0 & 0 & -p \end{pmatrix}. \tag{6.118}$$

For this equation to hold, $p = 0$ and $\tau = \mu_{inc}\kappa$. Now, the incompressible shear modulus is,

$$G_{inc} = \frac{\tau}{\kappa} = \mu_{inc}. \tag{6.119}$$

Finally, if a incompressible cuboid is subjected to hydrostatic pressure, $\boldsymbol{\sigma} = -p\mathbf{1}$ and $\boldsymbol{\epsilon} = \mathbf{0}$, for the incompressibility constraint (6.110) and the constitutive relation (6.111) to hold. Thus, incompressible materials bulk modulus, $K_{inc} = \infty$.

Understandably there is only one material parameter in (6.111). The incompressibility constraint (6.110) requires that $\nu_{inc} = 0.5$ or equivalently, $K_{inc} = \infty$ there by fixing one of the two independent material parameters in the isotropic Hooke's law.

As you would have noticed, for incompressible materials, both the state of stress and strain are prescribed and found necessary to solve certain boundary value problems. On the other hand for unconstrained materials only stress (or strain) needs to be specified for the same boundary value problems. Hence, the solution techniques used for solving unconstrained materials is different from that of constrained materials. In the remainder of this course we focus on the unconstrained materials.

## 6.8 Orthotropic Hooke's law

From the above exposition it would have been clear that isotropic Hooke's law is an approximation to a more general theory. We could proceed in a similar fashion to obtain orthotropic Hooke's law. On the other hand, as is done in many textbooks on linearized elasticity, we can begin by stating that the Cauchy stress is linearly related to the linearized strain, impose the restriction due to material symmetry on this constitutive relation and obtain isotropic or orthotropic Hooke's law. However, there are some differences in the orthotropic Hooke's law obtained by these two approaches. Without develing into these differences, in this section we obtain the orthotropic Hooke's law by the second approach.

Since, we assume that the second order tensor stress is linearly related to another second order tensor, the strain, we say that the stress is obtained by the action of a fourth order tensor on strain,

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\epsilon}, \tag{6.120}$$

where $\mathbb{C}$ is the fourth order elasticity tensor. Equivalently, one may invert the equation (6.120) to write strain in terms of stress as,

$$\boldsymbol{\epsilon} = \mathbb{D} : \boldsymbol{\sigma}, \tag{6.121}$$

where $\mathbb{D}$ is the fourth order elastic compliance tensor.

Recalling from chapter 2 (section 2.4) the fourth order tensor has 81 components. As discussed in section 2.4.1, for the following discussion it is easy to view second order tensors as column vectors with 9 components and fourth order tensors as 9 by 9 matrix. By virtue of the stress and strain being symmetric tensors, with only six independent components equation (6.120) can be written as,

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{xz} \\ \epsilon_{yz} \end{Bmatrix}. \tag{6.122}$$

Thus, of the 81 components only 36 are independent.

As stated before, since elastic process is non-dissipative, the stress is derivable from a potential called the stored (or strain) energy, $W$ as,

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\epsilon}}, \tag{6.123}$$

where $W = \hat{W}(\boldsymbol{\epsilon})$. The above equation in index notation is given by

$$\sigma_i = \frac{\partial W}{\partial \epsilon_i}, \tag{6.124}$$

when the stress and strain tensors are represented as column vectors with six components. Now, $\sigma_1 = \sigma_{xx}$, $\sigma_2 = \sigma_{yy}$, $\sigma_3 = \sigma_{zz}$, $\sigma_4 = \sigma_{xy}$, $\sigma_5 = \sigma_{xz}$, $\sigma_6 = \sigma_{yz}$ and $\epsilon_1 = \epsilon_{xx}$, $\epsilon_2 = \epsilon_{yy}$, $\epsilon_3 = \epsilon_{zz}$, $\epsilon_4 = \epsilon_{xy}$, $\epsilon_5 = \epsilon_{xz}$, $\epsilon_6 = \epsilon_{yz}$. This is just a change in notation.

It follows from (6.120) that the elasticity tensor could be obtained from

$$\mathbb{C} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\epsilon}}, \quad \text{in index notation} \quad (\mathbb{C})_{ij} = \frac{\partial \sigma_i}{\partial \epsilon_j}. \tag{6.125}$$

Substituting (6.124) in the above equation,

$$\mathbb{C}_{ij} = \frac{\partial^2 W}{\partial \epsilon_j \partial \epsilon i}. \tag{6.126}$$

Thus, if $W$ is a smooth function of strain, which it is, then the order of differentiation would not matter. Hence, elasticity tensor would be a symmetric fourth order tensor. This means that the number of independent components in $\mathbb{C}$ reduces from 36 to 21 components.

Further reduction in the number of independent components of the elasticity tensor cannot be done by restriction due to non-uniqueness of placers argument. Given that, due to equivalent placement of the body the displacement gradient transforms as given in equation (6.19) and Cauchy stress transforms as given in equation (6.25), it is evident that the restriction due to non-uniqueness of placers is not met by the constitutive relation (6.120). Then, due to changes in the coordinate basis, the elasticity tensor should transforms as

$$\tilde{\mathbb{C}}_{ijkl} = Q_{ia}Q_{jb}Q_{kc}Q_{ld}\mathbb{C}_{abcd}, \tag{6.127}$$

according to the transformation law for fourth order tensor (see section 2.6). Ideally we should require that the components of the elasticity tensor be the same irrespective of the choice of the basis. In other words, this means, as discussed in section 2.6.3, that the elasticity tensor should be the isotropic fourth order tensor. Since, the elasticity fourth order tensor is symmetric, in the general representation for the isotropic fourth order tensor (2.165), $\gamma = \beta = \mu$ and thus,

$$\mathbb{C} = \lambda\mathbf{1}\otimes\mathbf{1} + \mu[\mathbb{I} + \bar{\mathbb{I}}], \tag{6.128}$$

where we have replaced the constant $\alpha$ with $\lambda$. This $\lambda$ and $\mu$ are the same Lamè constants. Writing (6.128) in matrix form,

$$\mathbb{C} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{pmatrix}. \tag{6.129}$$

Thus, we have obtained the isotropic form of the Hooke's law.

Contrary to this ideal requirement, it is required that the components of the elasticity tensor be the same only for certain equivalent choices of the basis vectors. A material with three mutually perpendicular planes of symmetry is called *orthotropic*. This means that 180 degree rotation about each of the coordinate basis should not change the components of the elasticity tensor. Thus, for an orthotropic material the components of $\tilde{\mathbb{C}}$ must be same as that of $\mathbb{C}$ in (6.127) for the following choices of $Q_{pq}$:

$$[\mathbf{Q}]_{pq} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [\mathbf{Q}]_{pq} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad [\mathbf{Q}]_{pq} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{6.130}$$

It can be seen that as a result of this condition, the number of independent components in the elasticity tensor reduces from 21 to 9 for an orthotropic material and hence $\mathbb{C}$ for an orthotropic material is given by:

$$\mathbb{C} = \begin{pmatrix} C_1 & C_2 & C_3 & 0 & 0 & 0 \\ C_2 & C_4 & C_5 & 0 & 0 & 0 \\ C_3 & C_4 & C_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_9 \end{pmatrix}, \tag{6.131}$$

where $C_1$, $C_2$, ..., $C_9$ are material parameters.

Similarly, for other equivalent choices of the basis vector, we obtain different forms for the elasticity tensor.

One can now define material parameters like Young's modulus, Poisson's ratio, shear modulus and elaborate on the experiments that needs to be done to estimate the 9 constants characterizing the orthotropic material. Then, one can look at restrictions on them. Instead of doing these, we now focus on solving boundary value problems involving isotropic materials that obey Hooke's law.

## 6.9 Summary

In this chapter, we saw what a constitutive relation is? why do we need constitutive relations? Then we understood what we mean by an elastic response. This was followed by a discussion on the general restrictions that the constitutive relation has to satisfy and its application to get a general representation for Cauchy stress in an isotropic material undergoing elastic process. When the components of the displacement gradient are small, we showed that this general representation reduces to isotropic Hooke's law. Then, we introduced the various material parameters used to express the Hooke's law and established the relationship between them. We also found physically reasonable range of values that these parameters can take. While all these were dealt with in detail, we looked briefly on how to obtain constitutive relations for materials with constraints. As an illustration of this methodology, we obtained the constitutive relation for incompressible materials, materials which undergo only isochoric deformations. Finally, we also derived the Hooke's law for orthotropic materials. The one equation that should be remembered from this chapter is:

$$\boldsymbol{\sigma} = \lambda tr(\boldsymbol{\epsilon})\mathbf{1} + 2\mu\boldsymbol{\epsilon}. \tag{6.132}$$

## 6.10 Self-Evaluation

1. A body in the form of a cube of side 10 cm is subjected to a deformation of the form: $x = (1+a_1)X + a_2Y$, $y = (1+a_3)Y$, $z = (1+a_4)Z$, where $a_i$'s are constants and $(X, Y, Z)$ are the Cartesian coordinates of a material point before deformation and $(x, y, z)$ are the Cartesian coordinates of the same material point after deformation. If the Cauchy stress, $\boldsymbol{\sigma}$ is related to the deformation gradient, $\mathbf{F}$, through $\boldsymbol{\sigma} = \mu[\mathbf{1} - \mathbf{F}\mathbf{F}^t]$, where $\mu$ is a constant. Find the (a) Cauchy stress, (b) First Piola-Krichhoff stress, and (c) Second Piola-Krichhoff stress, for the above deformations. Then, show that all these three stresses are nearly the same when $a_i$'s are small.

2. Show that a plane stress state will result in a plane strain state only when $tr(\boldsymbol{\sigma}) = 0$ in a material that obeys isotropic Hooke's law. Recollect that this state of stress wherein $tr(\boldsymbol{\sigma}) = 0$ is called pure shear state of stress.

3. Show that a plane strain state will result in a plane stress state only when $tr(\boldsymbol{\epsilon}) = 0$ in a material that obeys isotropic Hooke's law.

4. Find the relationship between shear modulus, bulk modulus and Poisson's ratio.

5. A cube of side 1 cm, defined by $\mathcal{B} = \{(X, Y, Z) | 0 \leq X \leq 1cm, 0 \leq Y \leq 1cm, 0 \leq Z \leq 1cm\}$ obeying isotropic Hooke's law with Young's modulus, $E = 200$ GPa and Poisson's ratio, $\nu = 0.3$ is subjected to an uniform Cauchy stress,

$$\boldsymbol{\sigma} = \begin{pmatrix} -100 & 50 & 0 \\ 50 & 100 & 0 \\ 0 & 0 & 0 \end{pmatrix} MPa,$$

corresponding to an orthonormal Cartesian basis ($\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$). For this cube:

(a) Determine the linearized strain, $\boldsymbol{\epsilon}$

(b) Determine the displacement vector given that the displacement of the point $(0,0,0)$, $\mathbf{u}(0,0,0) = \mathbf{o}$ and that of the point $(0,0,1)$ is $\mathbf{u}(0,0,1) = \mathbf{o}$ .

(c) Determine the location of the particle originally at Cartesian coordinates $(1,0,1)$

(d) Determine the location of the particle in the reference configuration, if its current Cartesian coordinates are $(1,0,1)$

(e) Determine the displacement of the particle originally at Cartesian coordinates $(1,0,1)$

(f) Determine the displacement of the particle currently at Cartesian coordinates $(1,0,1)$

(g) Calculate the change in the angle between two line segments initially oriented along $\mathbf{e}_y$ and $\mathbf{e}_z$ directions in the reference configuration

(h) Calculate the change in the volume of the cube

(i) Calculate the deformed surface area and its orientation for each of the six faces of a cube

(j) Calculate the change in length of the straight line segments of length 1 mm oriented initially along (i) $\mathbf{e}_x$ (ii) $\mathbf{e}_y$ (iii) $(\mathbf{e}_x+\mathbf{e}_y)/\sqrt{2}$

(k) Determine the displaced location of the material particles which originally comprise

   (i) The plane circular surface $Z = 0$, $X^2 + Y^2 = 0.25$,

   (ii) The plane elliptical surface $Z = 0$, $9X^2 + 4Y^2 = 1$.

   (iii) The plane elliptical surface $Z = 0$, $4X^2 + 9Y^2 = 1$.

(l) Sketch the displaced configurations for (i), (ii) and (iii) in the above problem.

(m) Sketch the deformed configuration of the cube.

6. A cube of side 1 cm, defined by $\mathcal{B} = \{(X,Y,Z)|0 \le X \le 1cm, 0 \le Y \le 1cm, 0 \le Z \le 1cm\}$ obeying isotropic Hooke's law with Young's modulus, $E = 200$ GPa and Poisson's ratio, $\nu = 0.3$ is subjected to an

uniform Cauchy stress,

$$\boldsymbol{\sigma} = \begin{pmatrix} -100 & 50 & 0 \\ 50 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix} MPa,$$

corresponding to an orthonormal Cartesian basis ($\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$). For this cube find parts (a) through (m) in problem 5.

7. A cube of side 1 cm is subjected to a displacement field of the form, $\mathbf{u} = (A * y + 2A * z)\mathbf{e}_y + (3A * y - A * z)\mathbf{e}_z$, where $(x, y, z)$ denotes the coordinates of a typical material particle in the current configuration, $\{\mathbf{e}_i\}$ the Cartesian coordinate basis and $A = 10^{-4}$, a constant. If the cube is made up of a material that obeys isotropic Hooke's law with Young's modulus, $E = 200$ GPa and Poisson's ratio, $\nu = 0.3$,

   (a) Find the Cartesian components of the Cauchy stress with respect to the $\{\mathbf{e}_i\}$ basis.

   (b) Identify whether this stress states correspond to the plane stress

   (c) Find the normal and shear stress on a plane whose normal is oriented along the $\mathbf{e}_x$ direction.

   (d) Find the normal and shear stress on a plane whose normal makes equal angles with all the three basis vectors, i.e., $\mathbf{n} = (\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)/\sqrt{3}$.

   (e) Find the components of the stress tensor in the new basis $\{\tilde{\mathbf{e}}_x, \tilde{\mathbf{e}}_y, \tilde{\mathbf{e}}_z\}$. The new basis is obtained by rotating an angle 30 degrees in the clockwise direction about $\mathbf{e}_z$ axis.

   (f) Find the principal invariants of the stress

   (g) Find the principal stresses

   (h) Find the maximum shear stress

   (i) Find the plane on which the maximum normal stresses occurs

   (j) Find the plane on which the maximum shear stress occurs

   (k) Find the normal stress on the plane on which the maximum shear stress occurs

   (l) Find the shear stress on the plane on which the maximum normal stress occurs.

(m) Find the normal and shear stresses on the octahedral plane

(n) Find the hydrostatic and deviatoric component of the stress

8. A cube of side 10 cm and made up of a material that obeys isotropic Hooke's law with Young's modulus, $E = 200$ GPa and Poisson's ratio, $\nu = 0.3$, is subjected to a uniform plane state of strain whose Cartesian components are:

$$\boldsymbol{\epsilon} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} * 10^{-4}. \tag{6.133}$$

For this constant strain field, solve parts (a) through (n) in problem 7.

9. A body in the form of a unit cube, $\mathcal{B} = \{(X, Y, Z) | 0 \leq X \leq 1, 0 \leq Y \leq 1, 0 \leq Z \leq 1\}$ in the reference configuration, is subjected to the following linearized strain field:

$$\boldsymbol{\epsilon} = \begin{pmatrix} AY^3 + BX^2 & CXY(X+Y) & 0 \\ CXY(X+Y) & AX^3 + DY & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.134}$$

where $A$, $B$, $C$, $D$ are constants. Find conditions, if any, on the constants if this strain field is to be obtained from a smooth displacement field of the cube. For this value of the constants, find the stress field if the cube obeys isotropic Hooke's law with Young's modulus, $E = 200$ GPa and Poisson's ratio, $\nu = 0.3$ and verify if this stress field satisfies the equilibrium equations assuming that there are no body forces acting on the cube and that the cube is in static equilibrium.

10. Determine which of the following Cauchy stress fields are realizable in a body that obeys isotropic Hooke's law and which is at rest assuming that there are no body forces acting on it:

$$\text{(a)} \qquad \boldsymbol{\sigma} = \begin{pmatrix} -\frac{3}{2}x^2y^2 & xy^3 & 0 \\ xy^3 & -\frac{1}{4}y^4 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\text{(b)} \qquad \boldsymbol{\sigma} = \begin{pmatrix} 3yz & z^2 & 5y^2 \\ z^2 & 7xz & 2x^2 \\ 5y^2 & 2x^2 & 9xy \end{pmatrix},$$

(c) $$\boldsymbol{\sigma} = \begin{pmatrix} 3x+5y & 7x-3y & 0 \\ 7x-3y & 2x-7y & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(d) $$\boldsymbol{\sigma} = \begin{pmatrix} 3x & -3y & 0 \\ -7x & 7y & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the components of the stress are with respect to orthonormal Cartesian basis and $(x, y, z)$ denote the Cartesian coordinates of a typical material particle in the current configuration of the body. Recollect that for a stress field to be realizable in a body it should not only satisfy the equilibrium equations but also the compatibility conditions.

# Chapter 7

# Boundary Value Problem: Formulation

## 7.1 Overview

In the previous chapters, we mastered the basic concepts of strain and stress along with the four basic equations, namely, the strain-displacement relation, compatibility condition, constitutive relation and the equilibrium equations. We are now in a position to find the stresses and displacement throughout the body - in the interior as well as in its boundary - given the displacement over some part of the boundary of the body and the traction on some other part of the boundary. Usually the part of the boundary where traction is specified, displacement would not be given and vice versa. In this course we shall formulate the boundary value problem for a body made up of isotropic material obeying Hooke's law and deforming in such a manner that the relative displacement between its particles is small[1]. Then, we discuss three strategies to solve this boundary value problem.

Before we endeavor on the formulation of the boundary value problem and discuss strategies to solve it, we recollect and record the basic equations. If $\mathbf{u}$ represents the displacement that the particles undergo from a stress free reference configuration to the deformed current configuration with the deformation taking place due to application of the traction on the boundary,

[1] This condition is to ensure that the components of the displacement gradient is small.

then we define the linearized strain as,

$$\boldsymbol{\epsilon} = \frac{1}{2}[\mathbf{h} + \mathbf{h}^t], \tag{7.1}$$

where, $\mathbf{h} = grad(\mathbf{u})$. The equation (7.1) is called as the strain displacement relation. The constitutive relation that relates the stress to the strain that is to be used in this course is the Hooke's law. The various forms of the same Hooke's law that would be used is recorded:

$$\boldsymbol{\sigma} = \lambda tr(\boldsymbol{\epsilon})\mathbf{1} + 2\mu\boldsymbol{\epsilon}, \tag{7.2}$$

$$\boldsymbol{\sigma} = \frac{E}{(1+\nu)}\left[\frac{\nu}{(1-2\nu)}tr(\boldsymbol{\epsilon})\mathbf{1} + \boldsymbol{\epsilon}\right], \tag{7.3}$$

$$\boldsymbol{\epsilon} = \frac{(1+\nu)}{E}\boldsymbol{\sigma} - \frac{\nu}{E}tr(\boldsymbol{\sigma})\mathbf{1}, \tag{7.4}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress, $\lambda$ and $\mu$ are Lamè constants and $E$ is the Young's modulus and $\nu$ is the Poisson's ratio. The conservation of linear momentum equation,

$$div(\boldsymbol{\sigma}) + \rho(\mathbf{b} - \mathbf{a}) = \mathbf{o}, \tag{7.5}$$

where $\rho$ is the density, $\mathbf{b}$ is the body force per unit mass and $\mathbf{a}$ is the acceleration, for our purposes here reduces to:

$$div(\boldsymbol{\sigma}) = \mathbf{o}, \tag{7.6}$$

since we have ignored the body forces and look at configurations that are in static equilibrium under the action of the applied static traction. That is the displacement is assumed not to depend on time. It is appropriate that we discuss these assumptions in some detail. By ignoring the body forces we are ignoring the stresses that arise in the body due to its own mass. Since, these stresses are expected to be much small in comparison to the stresses induced in the body due to traction acting on its boundary and these stresses practically do not vary over the surface of the earth, ignoring the body forces is justifiable. In the same spirit, all that we require is that the magnitude of acceleration be small, if not zero.

Finally, we document the compatibility conditions; the conditions that ensures the existence of a smooth displacement field given a strain field:

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} \tag{7.7}$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \epsilon_{yz}}{\partial y \partial z} \tag{7.8}$$

$$\frac{\partial^2 \epsilon_{xx}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial x^2} = 2\frac{\partial^2 \epsilon_{xz}}{\partial x \partial z} \tag{7.9}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x} - \frac{\partial \epsilon_{xz}}{\partial y}\right) = \frac{\partial^2 \epsilon_{yy}}{\partial z \partial x} \tag{7.10}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{xz}}{\partial y} - \frac{\partial \epsilon_{xy}}{\partial z}\right) = \frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} \tag{7.11}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \epsilon_{xz}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} - \frac{\partial \epsilon_{yz}}{\partial x}\right) = \frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} \tag{7.12}$$

## 7.2 Formulation of boundary value problem

Formulation of the boundary value problem involves specification of the geometry of the body, the constitutive relation and the boundary conditions. To elaborate, we have to define the region of the Euclidean point space the body occupies in the reference configuration, which is denoted by $\mathcal{B}$. For example, the body might occupy a region that is a unit cube, then using Cartesian coordinates the body is defined as $\mathcal{B} = \{(X, Y, Z) | 0 \leq X \leq 1, 0 \leq Y \leq 1, 0 \leq Z \leq 1\}$. Alternatively, the body might be the annular region between two annular cylinders of radius $R_o$ and $R_i$ and height $H$, then using cylindrical polar coordinates the body may be defined as $\mathcal{B} = \{(R, \Theta, Z) | R_i \leq R \leq R_o, 0 \leq \Theta \leq 2\pi, 0 \leq Z \leq H\}$. The boundary of the body, denoted as $\partial\mathcal{B}$ is a surface that encloses the body. For illustration, in the cube example, the surface is composite of 6 different planes, defined by $X = 0$, $X = 1$, $Y = 0$, $Y = 1$, $Z = 0$ and $Z = 1$ respectively and in the annular cylinder example the surface is composed of 4 different planes, defined by $R = R_i$, $R = R_o$, $Z = 0$, $Z = H$.

In this course, at least, the constitutive relation is known; it is Hooke's law for isotropic materials. In real life problem, the weakest link in the formulation of the boundary value problem is the constitutive relation.

Then finally one needs to prescribe boundary conditions. These are specifications of the displacement or traction on the surface of the body. Depending on what is prescribed on the surface, there are four type of boundary conditions. They are

1. **Displacement boundary condition:** Here the displacement is specified on the entire boundary of the body. This is also called as Dirichlet boundary condition

2. **Traction boundary condition:** Here the traction is specified on the entire boundary of the body. This is also called as Neumann boundary condition

3. **Mixed boundary condition:** Here the displacement is specified on part of the boundary and traction is specified on the remaining part of the boundary. Both traction as well as displacement are not specified over any part of boundary

4. **Robin boundary condition:** Here both the displacement and the traction are specified on the same part of the boundary.

Once, the geometry of the body, constitutive relations and boundary conditions are prescribed then finding the Cauchy stress and displacement over the entire region of the body such that the displacement is continuous and differentiable over the entire region occupied by the body and the stress computed using this displacement field from the constitutive relation satisfies the equilibrium equations is called as solving the boundary value problem. If for a given geometry of the body, constitutive relations and boundary conditions, there exists only one displacement and stress field as a solution to the boundary value problem then the solution to the boundary value problem is said to be unique.

Now it is appropriate to make a few comments regarding the choice of the independent variable for the boundary value problem. That is, when we say the region occupied by the body, we should have been more specific and said whether this is the region occupied by the body in the reference or current configuration. As described in section 3.4 of chapter 3, the displacement and the stresses can be given an Eulerian or Lagrangian description, i.e.,

$$\mathbf{u} = \hat{\mathbf{u}}(\mathbf{X}) = \tilde{\mathbf{u}}(\mathbf{x}), \qquad \boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\mathbf{X}) = \tilde{\boldsymbol{\sigma}}(\mathbf{x}). \tag{7.13}$$

In this course, to be specific, we follow the Eulerian description of these fields. That is we use the region occupied by the body in the current configuration as the spatial domain of the boundary value problem. However, this domain is not known a priori. Therefore, we approximate the region occupied by the body in the current configuration with the region occupied by the body in the reference configuration. This approximation is justified, since we are interested only in problems where the components of the gradient of the displacement field are small and the magnitude of the displacement is also small. Then, one might ponder as to why we cannot say we are following the Lagrangian description of these fields. In chapter 5 section 5.4.1, we showed that the conservation of linear momentum takes different forms depending on the description of the stress field. The equation used here (7.5) is derived assuming Eulerian or spatial description of the stress field. If we use (5.46) instead of (7.5) then we would have said we are following Lagrangian description of these fields. Following Lagrangian description also, we would obtain the same governing equations if we assume that the components of the gradient of the deformation field are small and the magnitude of the displacement is small; but we have to do a little more algebra than saying upfront that we are following an Eulerian or spatial description.

## 7.3 Techniques to solve boundary value problems

Depending on the boundary condition specified the solution can be found using one of the following two techniques. Outline of these methods is presented next.

### 7.3.1 Displacement method

Here we take the displacement field as the basic unknown that need to be determined. Then using this displacement field we find the strain using the strain displacement relation (7.1). The so computed strain is substituted in the constitutive relation written using Lamè constants (7.2) to obtain

$$\boldsymbol{\sigma} = \lambda div(\mathbf{u})\mathbf{1} + \mu \left[grad(\mathbf{u}) + grad(\mathbf{u})^t\right], \tag{7.14}$$

where we have used the definition of divergence operator, (2.208) and the property of the trace operator (2.67). Substituting (7.14) in the reduced equilibrium equations, under the assumption that body forces can be ignored and the body is in static equilibrium, (7.5) we obtain

$$(\lambda+\mu)grad(div(\mathbf{u}))+\mu\Delta\mathbf{u}+div(\mathbf{u})grad(\lambda)+2\boldsymbol{\epsilon}grad(\mu)+\rho\mathbf{b} = \rho\frac{D^2\mathbf{u}}{Dt^2}, \quad (7.15)$$

where we have used equation (3.31) to write the acceleration in terms of the displacement and $\frac{D(\cdot)}{Dt}E$ denotes the total time derivative. In addition, to obtain the equation (7.15), we have used the following identities:

1. Since divergence is a linear operator,

$$div(\lambda div(\mathbf{u})\mathbf{1} + 2\mu\boldsymbol{\epsilon}) = div(\lambda div(\mathbf{u})\mathbf{1}) + div(2\mu\boldsymbol{\epsilon}), \quad (7.16)$$

2. It follows from equation (2.218) that

$$\begin{aligned} div(\lambda div(\mathbf{u})\mathbf{1}) &= grad(\lambda div(\mathbf{u})), & (7.17)\\ div(2\mu\boldsymbol{\epsilon}) &= 2\boldsymbol{\epsilon}grad(\mu) + \mu\left[div(grad(\mathbf{u})) + div((grad(\mathbf{u}))^t)\right], & (7.18) \end{aligned}$$

   where we have used the fact that $div$ is a linear operator and $\mathbf{1m} = \mathbf{m}$ when $\mathbf{m}$ is any vector and $\mathbf{1}$ is the second order identity tensor.

3. From the identity (2.222) it follows that,

$$grad(\lambda div(\mathbf{u})) = grad(\lambda)div(\mathbf{u}) + \lambda grad(div(\mathbf{u})). \quad (7.19)$$

4. Using definition of the Laplace operator (2.212)

$$div(grad(\mathbf{u})) = \Delta\mathbf{u}. \quad (7.20)$$

5. Using the identity (2.211) we note that

$$div((grad(\mathbf{u}))^t) = grad(div(\mathbf{u})). \quad (7.21)$$

Thus, we obtain (7.15) by substituting successive substitution of equations (7.17) through (7.21) in (7.16).

In order to simplify equation (7.15), we make the following assumptions:

1. The body is homogeneous. Hence $\lambda$ and $\mu$ are constants

2. Body forces can be ignored

3. Body is in static equilibrium under the applied traction

In lieu of these assumptions, equation (7.15) reduces to

$$(\lambda + \mu) grad(div(\mathbf{u})) + \mu \Delta \mathbf{u} = \mathbf{o}. \tag{7.22}$$

If body forces cannot be ignored but the other two assumptions hold, then (7.15) reduces to

$$(\lambda + \mu) grad(div(\mathbf{u})) + \mu \Delta \mathbf{u} + \rho \mathbf{b} = \mathbf{o}. \tag{7.23}$$

In this course, we attempt to find the displacement field that satisfies (7.22) along with the prescribed boundary conditions. We compute the stress field corresponding to the determined displacement field, using equation (7.2) where the strain is related to the displacement field through equation (7.1). We illustrate this method in section 7.4.1.

### 7.3.2 Stress method

Here we use stress as the basic unknown that needs to be determined. This method is applicable only for cases when the inertial forces ($\rho\mathbf{a}$) can be neglected. Since, we have assumed stress as the basic unknown, we want to express the compatibility conditions (7.7) through (7.12) in terms of the stresses. For this we compute the strains in terms of the stresses using the constitutive relation (7.4) and substitute in the compatibility conditions to obtain the following 6 equations:

$$(1-\nu)\left[\frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2}\right] - \nu\left[\frac{\partial^2 \sigma_{zz}}{\partial x^2} + \frac{\partial^2 \sigma_{zz}}{\partial y^2}\right] = 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}, \tag{7.24}$$

$$(1-\nu)\left[\frac{\partial^2 \sigma_{yy}}{\partial z^2} + \frac{\partial^2 \sigma_{zz}}{\partial y^2}\right] - \nu\left[\frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{xx}}{\partial z^2}\right] = 2(1+\nu)\frac{\partial^2 \sigma_{yz}}{\partial y \partial z}, \tag{7.25}$$

$$(1-\nu)\left[\frac{\partial^2 \sigma_{xx}}{\partial z^2} + \frac{\partial^2 \sigma_{zz}}{\partial x^2}\right] - \nu\left[\frac{\partial^2 \sigma_{yy}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial z^2}\right] = 2(1+\nu)\frac{\partial^2 \sigma_{xz}}{\partial x \partial z}, \tag{7.26}$$

$$2(1+\nu)\frac{\partial}{\partial y}\left(\frac{\partial \sigma_{xy}}{\partial z} + \frac{\partial \sigma_{yz}}{\partial x} - \frac{\partial \sigma_{xz}}{\partial y}\right) = \frac{\partial^2}{\partial z \partial x}\left(\sigma_{yy} - \nu[\sigma_{xx} + \sigma_{zz}]\right), \tag{7.27}$$

$$2(1+\nu)\frac{\partial}{\partial z}\left(\frac{\partial \sigma_{yz}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial y} - \frac{\partial \sigma_{xy}}{\partial z}\right) = \frac{\partial^2}{\partial x \partial y}\left(\sigma_{zz} - \nu[\sigma_{xx} + \sigma_{yy}]\right), \tag{7.28}$$

$$2(1+\nu)\frac{\partial}{\partial x}\left(\frac{\partial \sigma_{xy}}{\partial z} + \frac{\partial \sigma_{xz}}{\partial y} - \frac{\partial \sigma_{yz}}{\partial x}\right) = \frac{\partial^2}{\partial y \partial z}\left(\sigma_{xx} - \nu[\sigma_{yy} + \sigma_{zz}]\right), \tag{7.29}$$

where we have assumed that the body is homogeneous and hence Young's modulus, $E$ and Poisson's ratio, $\nu$ do not vary spatially. Now, we have to find the 6 components of the stress such that the 6 equations (7.24) through (7.29) holds along with the three equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho b_x = 0, \tag{7.30}$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho b_y = 0, \tag{7.31}$$
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z = 0, \tag{7.32}$$

where $b_i$'s are the Cartesian components of the body forces. The above equilibrium equations (7.30) through (7.32) are obtained from (7.5) by setting $\mathbf{a} = \mathbf{o}$.

If the body forces could be obtained from a potential, $\beta = \tilde{\beta}(x, y, z)$ called as the load potential, as

$$\mathbf{b} = -\frac{1}{\rho} grad(\beta), \tag{7.33}$$

then the Cartesian components of the Cauchy stress could be obtained from a potential, $\phi = \tilde{\phi}(x, y, z)$ called as the Airy's stress potential and the load potential, $\beta$ as,

$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \beta & -\frac{\partial^2 \phi}{\partial x \partial y} & -\frac{\partial^2 \phi}{\partial x \partial z} \\ -\frac{\partial^2 \phi}{\partial x \partial y} & \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + \beta & -\frac{\partial^2 \phi}{\partial y \partial z} \\ -\frac{\partial^2 \phi}{\partial x \partial z} & -\frac{\partial^2 \phi}{\partial y \partial z} & \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \beta \end{pmatrix}, \tag{7.34}$$

so that the equilibrium equations (7.30) through (7.32) is satisfied for any choice of $\phi$. Substituting for the Cartesian components of the stress from equation (7.34) in the compatibility equations (7.24) through (7.29) and simplifying we obtain:

$$\frac{(1-2\nu)}{(\nu-1)} \left[ \frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} + \frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2} \right] = \frac{\partial^2}{\partial z^2} \left[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right], \tag{7.35}$$
$$\frac{(1-2\nu)}{(\nu-1)} \left[ \frac{\partial^4 \phi}{\partial y^4} + \frac{\partial^4 \phi}{\partial y^2 \partial z^2} + \frac{\partial^4 \phi}{\partial z^4} + \frac{\partial^2 \beta}{\partial y^2} + \frac{\partial^2 \beta}{\partial z^2} \right] = \frac{\partial^2}{\partial x^2} \left[ \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right], \tag{7.36}$$
$$\frac{(1-2\nu)}{(\nu-1)} \left[ \frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial x^2 \partial z^2} + \frac{\partial^4 \phi}{\partial z^4} + \frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial z^2} \right] = \frac{\partial^2}{\partial y^2} \left[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right], \tag{7.37}$$

$$\frac{\partial^2}{\partial x \partial z}\left[2\frac{\partial^2 \phi}{\partial y^2} + (1-\nu)\left\{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right\} + (1-2\nu)\beta\right] = 0, \tag{7.38}$$

$$\frac{\partial^2}{\partial x \partial y}\left[2\frac{\partial^2 \phi}{\partial z^2} + (1-\nu)\left\{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right\} + (1-2\nu)\beta\right] = 0, \tag{7.39}$$

$$\frac{\partial^2}{\partial y \partial z}\left[2\frac{\partial^2 \phi}{\partial x^2} + (1-\nu)\left\{\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right\} + (1-2\nu)\beta\right] = 0. \tag{7.40}$$

Thus, a potential that satisfies equations (7.35) through (7.40) and the prescribed boundary conditions is said to be the solution to the given boundary value problem. Once the Airy's stress potential is obtained, the stress field could be computed using (7.34). Using this stress field the strain field is computed using the constitutive relation (7.4). From this strain field, the smooth displacement field is obtained by integrating the strain displacement relation (7.1).

### Plane stress formulation

Next, we specialize the above stress formulation for the plane stress case. Without loss of generality, let us assume that the Cartesian components of this plane stress state is

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.41}$$

Further, let us assume that body forces are absent and that the Airy's stress function depends on only $x$ and $y$. Thus, $\beta = 0$ and $\phi = \bar{\phi}(x, y)$. Note that this assumption for the Airy's stress function does not ensure $\sigma_{zz} = 0$, whenever $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \neq 0$. Hence, plane stress formulation is not a specialization of the general 3D problem. Therefore, we have to derive the governing equations again following the same procedure.

Since, we assume that there are no body forces and the Airy's stress function depends only on $x$ and $y$, the Cartesian components of the stress are related to the Airy's stress function as,

$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial^2 \phi}{\partial y^2} & -\frac{\partial^2 \phi}{\partial x \partial y} & 0 \\ -\frac{\partial^2 \phi}{\partial x \partial y} & \frac{\partial^2 \phi}{\partial x^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.42}$$

Substituting for stress from equation (7.42) in the constitutive relation (7.4) we obtain,

$$\boldsymbol{\epsilon} = \frac{1}{E}\begin{pmatrix} \frac{\partial^2\phi}{\partial y^2} - \nu\frac{\partial^2\phi}{\partial x^2} & -(1+\nu)\frac{\partial^2\phi}{\partial x\partial y} & 0 \\ -(1+\nu)\frac{\partial^2\phi}{\partial x\partial y} & \frac{\partial^2\phi}{\partial x^2} - \nu\frac{\partial^2\phi}{\partial y^2} & 0 \\ 0 & 0 & -\nu\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right] \end{pmatrix}. \tag{7.43}$$

Substituting for strain from equation (7.43) in the compatibility condition (7.7) through (7.12), the non-trivial equations are

$$\frac{\partial^4\phi}{\partial y^4} + \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} = 0, \tag{7.44}$$

$$\frac{\partial^2}{\partial y^2}\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right] = 0, \tag{7.45}$$

$$\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right] = 0, \tag{7.46}$$

$$\frac{\partial^2}{\partial x\partial y}\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right] = 0, \tag{7.47}$$

Now, for equations (7.45) through (7.47) to hold,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \alpha_1 x + \alpha_2 y + \alpha_3, \tag{7.48}$$

where $\alpha_i$'s are constants. Differentiating equation (7.48) with respect to $x$ twice and adding to this the result of differentiation of equation (7.48) with respect to $y$ twice, we obtain equation (7.44). Thus, for equations (7.44) through (7.47) to hold it suffices that $\phi$ satisfy equation (7.48) along with the prescribed boundary conditions. Comparing the expression for $\epsilon_{zz}$ in equation (7.43) and the requirement (7.48) arising from compatibility equations (7.8), (7.9) and (7.11) is a restriction on how the out of plane normal strain can vary, i.e., $\epsilon_{zz} = \bar{\alpha}_1 x + \bar{\alpha}_2 y + \bar{\alpha}_3$, where $\bar{\alpha}_i$'s are some constants. Hence, this requirement that $\phi$ satisfy equation (7.48) does not lead to solution of a variety of boundary value problems. Due to Poisson's effect plane stress does not lead to plane strain and vice versa, resulting in the present difficulty. To overcome this difficulty, it has been suggested that for plane problems one should use the 2 dimensional constitutive relations, instead of 3 dimensional constitutive relations that we have been using till now.

Since, the constitutive relation is 2 dimensional, plane stress implies plane strain and the three independent Cartesian components of the plane stress and strain are related as,

$$\epsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right], \quad \epsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right], \quad \epsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy}. \tag{7.49}$$

Inverting the above equations we obtain

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left[\epsilon_{xx} + \nu\epsilon_{yy}\right], \quad \sigma_{yy} = \frac{E}{1-\nu^2}\left[\epsilon_{yy} + \nu\epsilon_{xx}\right], \quad \sigma_{xy} = \frac{E}{(1+\nu)}\epsilon_{xy}. \tag{7.50}$$

By virtue of using (7.49) to compute the strain for the plane state of stress given in equation (7.42), the only non-trivial restriction from compatibility condition is (7.7) which requires that

$$\frac{\partial^4\phi}{\partial y^4} + \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} = 0, \tag{7.51}$$

Equation (7.51) is called as the bi-harmonic equation. Thus, for two dimensional problems formulated using stress, one has to find the Airy's stress potential that satisfies the boundary conditions and the bi-harmonic equation (7.51). Then using this stress potential, the stresses are computed using (7.42). Having estimated the stress, the strain are found from the two dimensional constitutive relation (7.49). Finally, using this estimated strain, the strain displacement relation (7.1) is integrated to obtain the smooth displacement field. We study bending problems in chapter 8 using this approach.

Recognize that equation (7.51) is nothing but $\Delta(\Delta(\phi)) = 0$, where $\Delta(\cdot)$ is the Laplacian operator.

Next, we would like to formulate the plane stress problem in cylindrical polar coordinates. Let us assume that the cylindrical polar components of this plane stress state is

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & 0 \\ \sigma_{r\theta} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.52}$$

Further, let us assume that body forces are absent and that the Airy's stress function depends on only $r$ and $\theta$, i.e. $\phi = \hat{\phi}(r,\theta)$. For this case, the Cauchy stress cylindrical polar components are assumed to be

$$\boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} & -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) & 0 \\ -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) & \frac{\partial^2\phi}{\partial r^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.53}$$

so that it satisfies the static equilibrium equations in the absence of body forces, equation (7.6). Then, using a 2 dimensional constitutive relation, the cylindrical polar components of the strain are related to the cylindrical polar components of the stress through,

$$\epsilon_{rr} = \frac{1}{E}\left[\sigma_{rr} - \nu\sigma_{\theta\theta}\right], \quad \epsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\sigma_{rr}\right], \quad \epsilon_{r\theta} = \frac{(1+\nu)}{E}\sigma_{r\theta}. \tag{7.54}$$

As shown before, the only non-trivial restriction from compatibility condition in 2 dimensions is (7.7) and this in cylindrical polar coordinates takes the form,

$$\frac{\partial^2\epsilon_{\theta\theta}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\epsilon_{rr}}{\partial\theta^2} + \frac{2}{r^2}[\epsilon_{rr} - \epsilon_{\theta\theta}] = \frac{2}{r}\frac{\partial^2\epsilon_{r\theta}}{\partial r\partial\theta}. \tag{7.55}$$

Substituting equation (7.54) and (7.53) in (7.55) and simplifying we obtain,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0. \tag{7.56}$$

A general periodic solution to the bi-harmonic equation in cylindrical polar coordinates, (7.56) is

$$\begin{aligned} \phi = {} & a_{01} + a_{02}\ln(r) + a_{03}r^2 + a_{04}r^2\ln(r) + \left[b_{01} + b_{02}\ln(r) + b_{03}r^2 + b_{04}r^2\ln(r)\right]\theta \\ & + \left[a_{11}r + a_{12}r\ln(r) + \frac{a_{13}}{r} + a_{14}r^3 + a_{15}r\theta + a_{16}r\theta\ln(r)\right]\cos(\theta) \\ & + \left[b_{11}r + b_{12}r\ln(r) + \frac{b_{13}}{r} + b_{14}r^3 + b_{15}r\theta + b_{16}r\theta\ln(r)\right]\sin(\theta) \\ & + \sum_{n=2}^{\infty}\left[a_{n1}r^n + a_{n2}r^{2+n} + a_{n3}r^{-n} + a_{n4}r^{2-n}\right]\cos(n\theta) \\ & + \sum_{n=2}^{\infty}\left[b_{n1}r^n + b_{n2}r^{2+n} + b_{n3}r^{-n} + b_{n4}r^{2-n}\right]\sin(n\theta), \end{aligned} \tag{7.57}$$

where $a_{nm}$ and $b_{nm}$ are constants to be determined from boundary conditions. We illustrate the use of this solution in section 7.4.2.

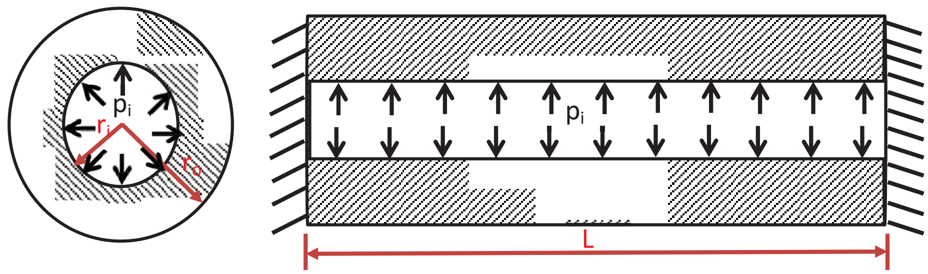



Figure 7.1: Schematic of inflation of an annular cylinder

## 7.4 Illustrative example

Having formulated the boundary value problem and seen at techniques to solve it, in this section we illustrate the same by solving some standard boundary value problems.

### 7.4.1 Inflation of an annular cylinder

In the first boundary value problem that is of interest, the body is in the form of an annular cylinder as shown in the figure 7.1. We use cylindrical polar coordinates to study this problem. Consequently, the body in the reference configuration is assumed to occupy a region in the Euclidean point space defined by, $\mathcal{B} = \{(r, \theta, z) | r_i \leq r \leq r_o, 0 \leq \theta \leq 2\pi, 0 \leq z \leq L\}$ that is the region between two coaxial right circular cylinders of radius $r_i$ and $r_o$ respectively and of length $L$.

The boundary conditions that is of interest are also illustrated in figure 7.1. Thus, the cylinder is held fixed at constant length. Consequently, there is no axial displacement of the planes defined by $z = 0$ and $z = L$ of the cylinder,

$$u_z(r, \theta, 0) = u_z(r, \theta, L) = 0, \tag{7.58}$$

where $u_z$ represents the axial component of the displacement field. Then, the outer surface defined by $r = r_o$, of the cylinder is traction free, i.e.,

$$\mathbf{t}_{(\mathbf{e}_r)}(r_o, \theta, z) = \mathbf{o}. \tag{7.59}$$

On the remaining surface, the inner surface defined by $r = r_i$, of the cylinder only radial stress acts as shown in figure 7.1. Therefore,

$$\mathbf{t}_{(\mathbf{e}_r)}(r_i, \theta, z) = -p_i \mathbf{e}_r, \tag{7.60}$$

where $p_i$ is some positive constant. By virtue of the boundary conditions being independent of time and the constitutive relation is that of an elastic[2] response, we are justified in assuming that the body is in static equilibrium under the action of boundary traction and hence $\mathbf{a} = \mathbf{o}$. Strictly, the traction boundary condition has to be applied on the deformed surface and not on the original surface. However, as discussed in section 7.2, we approximate the deformed surface with the original surface since they are close, in lieu of our assumptions that the magnitude of the displacement is small and that the magnitude of the components of the displacement gradient is also small.

To proceed further one has to decide whether to use displacement or stress as the basic unknown. Here we use displacement as the basic unknown. In general, one needs to solve the three partial differential equations involving the components of the displacement field, (7.22). Note by using (7.22) instead of (7.15), we have ignored the body forces and as per the discussion above we have assumed the body to be in static equilibrium.

Next, in order to avoid solving partial differential equations, appropriate assumptions are made for the displacement field so that the governing equation (7.22) reduces to a ordinary differential equation. This is possible because, for mixed boundary condition boundary value problems for bodies in static equilibrium and in the absence of body forces, it is shown in section 7.5.1 that there exist a unique solution for a body made up of a material that obeys isotropic Hooke's law. In light of the boundary condition (7.58) we assume $u_z = 0$. Further, we expect the cross section of the cylinder to deform as shown in the figure 7.2. That is we expect the circular cross section to remain circular but with a different radius and initially straight radial lines on the cross section to remain straight after deformation. Further we assume that there is no axial variation in the displacement field, that is any section along the axis of the cylinder deforms in the same fashion. These suggest that there is no circumferential component of the displacement field, i.e., $u_\theta = 0$ and that the radial component of the displacement field vary radially only. Thus,

$$\mathbf{u} = u_r(r)\mathbf{e}_r. \tag{7.61}$$

[2] As discussed in chapter 1, if the response is viscoelastic or viscoplastic, a constant stress can cause a time dependent variation in the displacement or vice versa. Consequently, we cannot drop time in the formulation for the viscoelastic or viscoplastic response.

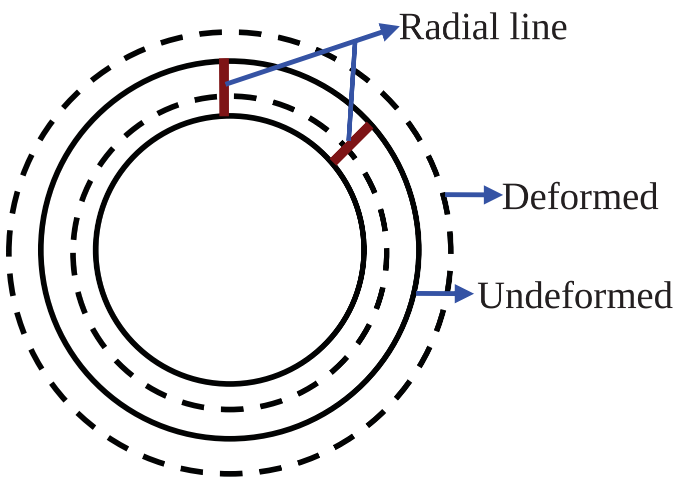



Figure 7.2: Schematic of deformation of the cross section of a right circular annular cylinder

Substituting equation (7.61) for displacement field in (7.22) and using equations (2.259), (2.260) and (2.261) we obtain

$$(\lambda + 2\mu)\left[\frac{d^2u_r}{dr^2} + \frac{du_r}{dr}\frac{1}{r} - \frac{u_r}{r^2}\right] = 0. \tag{7.62}$$

Since, $(\lambda + 2\mu) \neq 0$, for equation (7.62) to hold we require

$$\frac{d^2u_r}{dr^2} + \frac{du_r}{dr}\frac{1}{r} - \frac{u_r}{r^2} = \frac{d^2u_r}{dr^2} + \frac{d}{dr}\left(\frac{u_r}{r}\right) = 0. \tag{7.63}$$

Solving the ordinary differential equation (7.63) we get

$$u_r = \frac{C_1}{2}r + \frac{C_2}{r}, \tag{7.64}$$

where $C_1$ and $C_2$ are integration constants to be found from the boundary conditions (7.59) and (7.60). Having found the unknown function in the displacement field (7.61), using (2.259) the cylindrical polar coordinate components of linearized strain can be computed as,

$$\boldsymbol{\epsilon} = \begin{pmatrix} \frac{du_r}{dr} & 0 & 0 \\ 0 & \frac{u_r}{r} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{C_1}{2} - \frac{C_2}{r^2} & 0 & 0 \\ 0 & \frac{C_1}{2} + \frac{C_2}{r^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.65}$$

Substituting the above strain in the constitutive relation (7.2) the cylindrical polar coordinate components of the Cauchy stress is obtained as

$$\boldsymbol{\sigma} = \begin{pmatrix} (\lambda+\mu)C_1 - \mu\frac{C_2}{r^2} & 0 & 0 \\ 0 & (\lambda+\mu)C_1 + \mu\frac{C_2}{r^2} & 0 \\ 0 & 0 & \lambda C_1 \end{pmatrix}. \tag{7.66}$$

Recognizing that for the above state of stress, $\mathbf{t}_{(\mathbf{e}_r)} = [(\lambda+\mu)C_1 - \mu C_2/r^2]\mathbf{e}_r$, boundary conditions (7.59) and (7.60) yield

$$\begin{pmatrix} 1 & -\frac{1}{r_o^2} \\ 1 & -\frac{1}{r_i^2} \end{pmatrix} \begin{Bmatrix} (\lambda+\mu)C_1 \\ \mu C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -p_i \end{Bmatrix}. \tag{7.67}$$

Solving the above equations we obtain:

$$C_1 = \frac{r_i^2}{(r_o^2 - r_i^2)} \frac{p_i}{(\lambda+\mu)}, \quad C_2 = \frac{r_i^2 r_o^2}{(r_o^2 - r_i^2)} \frac{p_i}{\mu}. \tag{7.68}$$

Substituting equation (7.68) in (7.66),

$$\boldsymbol{\sigma} = \begin{pmatrix} p_i \frac{r_i^2}{(r_o^2 - r_i^2)} \left[1 - \frac{r_o^2}{r^2}\right] & 0 & 0 \\ 0 & p_i \frac{r_i^2}{(r_o^2 - r_i^2)} \left[1 + \frac{r_o^2}{r^2}\right] & 0 \\ 0 & 0 & 2\nu p_i \frac{r_i^2}{(r_o^2 - r_i^2)} \end{pmatrix}, \tag{7.69}$$

where we have used (6.79) to deduce that $2\nu = \lambda/(\lambda+\mu)$. Substituting equation (7.68) in (7.64),

$$u_r = p_i \frac{r_i^2}{(r_o^2 - r_i^2)} \left[\frac{r}{(\lambda+\mu)} + \frac{r_o^2}{r\mu}\right]. \tag{7.70}$$

Figure 7.3 plots the variation of the radial ($\sigma_{rr}$) and hoop ($\sigma_{\theta\theta}$) stresses with respect to $r$ as given in equation (7.69) when $r_i = r_o/2$. It is also clear from equation (7.69) that the axial stresses do not vary across the cross section. When $r_i = r_o/2$, the constant axial stress value is $2p_i\nu/3$. Thus, we find that due to inflation of the annular cylinder, tensile hoop stresses develop and the radial stresses are always compressive. The magnitude of the hoop stress is greater than radial stress. The magnitude of the axial stress is nearly one-tenth of the maximum hoop stress and it is tensile when $\nu >$

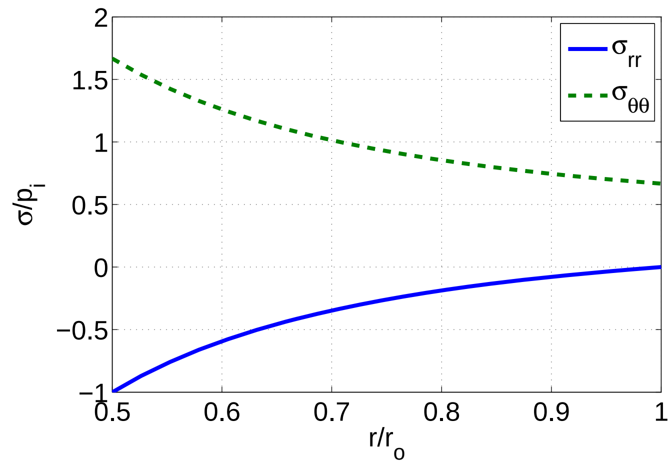



Figure 7.3: Plot of the radial ($\sigma_{rr}$) and hoop ($\sigma_{\theta\theta}$) stresses given in equation (7.69) when $r_o = 1$ and $r_i = r_o/2$.

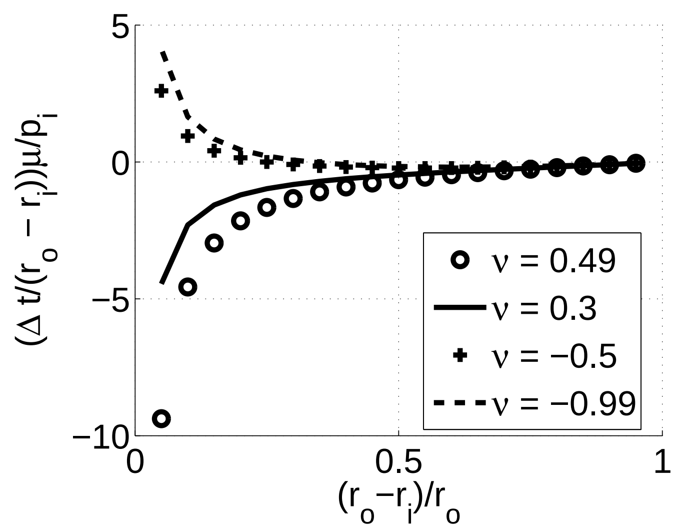



Figure 7.4: Plot of the variation in the ratio of change in thickness ($\Delta t$) of the cylinder to its original thickness ($r_o - r_i$) as a function of the thickness of the cylinder for various possible values of Poisson's ratio, $\nu$. $\mu$ is the shear modulus and $p_i$ is the radial component of the normal stress at the inner surface.

0 and compressive when $\nu < 0$ and is zero when $\nu = 0$. Next, we physically explain this variation of the axial stresses with the Poisson's ratio. The hoop stresses by virtue of being tensile in nature will cause a reduction in the axial length due to Poisson's effect[3] if the Poisson's ratio, $\nu > 0$ and increase in length if $\nu < 0$ and no change if $\nu = 0$. Similarly, the radial stress being compressive in nature will cause an increase in the axial length when $\nu > 0$, decrease in axial length when $\nu < 0$ and no change in axial length when $\nu = 0$. However, the actual change in axial length would be a sum of both the change in length due to hoop and radial stress. Since, the hoop stress is more than the radial stress, if no axial force is applied, the axial length would reduce when $\nu > 0$, will not change when $\nu = 0$ and will increase when $\nu < 0$. Thus, if the length is to remain unchanged, a tensile axial force has to be applied to counter the reduction in length when $\nu > 0$ and a compressive axial force when $\nu < 0$ and no axial force is required when $\nu = 0$. It can be seen from the expression for axial stress in equation (7.69) that the expression for the axial stresses is consistent with the expectation that it be positive when $\nu > 0$, negative when $\nu < 0$ and zero when $\nu = 0$.

Now, we study the changes that occur to the thickness of the annular cylinder. Towards this, the ratio of change in thickness of the cylinder, $\Delta t$ to its original thickness is obtained as

$$\frac{\Delta t}{t} = \frac{u_r(r_o) - u_r(r_i)}{r_o - r_i} = \frac{p_i}{\mu} \frac{r_o}{r_o - r_i} \frac{r_i^2}{(r_o^2 - r_i^2)} \left\{ 1 - \frac{r_o}{r_i} + \left[ 1 - \frac{r_i}{r_o} \right] \frac{1 - 2\nu}{(1 - \nu)} \right\}, \tag{7.71}$$

where we have used (6.80) to write Lamè constants in terms of the Poisson's ratio, $\nu$ and Young's modulus $E$. Figure 7.4 plots the variation in the ratio of change in thickness of the cylinder to its original thickness as a function of the thickness of the cylinder for various possible values of Poisson's ratio for a given radial component of the normal stress at the inner surface, $p_i$. It can be seen from the figure that while the thickness decreases when $\nu \geq 0$, thickness increases when $\nu < 0$ for thickness less than a critical value. By virtue of the radial component of the normal stress being compressive in nature one would expect a reduction in the thickness of the cylinder. However, the hoop stress being in tension, due to Poisson's effect there would be reduction in thickness when $\nu > 0$ and an increase when $\nu < 0$. The actual change in the thickness is the sum of both the reduction due to radial stress and alteration

[3]Line elements on a plane whose normal coincides with the applied tensile stress shorten if $\nu > 0$ and elongate if $\nu < 0$. This effect is called as the Poisson's effect.

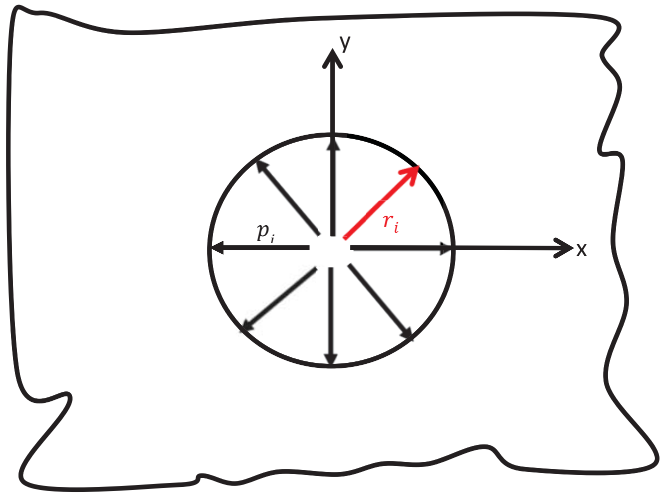



Figure 7.5: Schematic of pressurized hole in an infinite medium

due to hoop stress. Hence, the thickness decreases when $\nu \geq 0$ and increases when $\nu < 0$ for thickness less than a critical value.

Finally, we would like to show that due to inflation the inner as well as outer radius of the cylinder increases. Noting that the deformed inner radius of the cylinder is $r_i + u_r(r_i)$ and that of the deformed outer radius is $r_o + u_r(r_o)$, for these radius to increase we need to show that $u_r(r_i) > 0$ and $u_r(r_o) > 0$. Towards this, using equation (7.70) and (6.80) we obtain,

$$u_r(r) = r\frac{p_i}{\mu}\frac{r_i^2}{(r_o^2 - r_i^2)}\left[\frac{(1-2\nu)}{(1-\nu)} + \frac{r_o^2}{r^2}\right]. \tag{7.72}$$

Recollecting from table 6.1, that the physically possible values for the Poisson's ratio is: $-1 < \nu \leq 0.5$, it is straightforward to see that $(1-2\nu)/(1-\nu)$ $\geq 0$ for these possible range of values for $\nu$. Also, it can be seen from table 6.1 that $\mu > 0$. Further, by definition $r_o > r_i$ and $p_i > 0$. Therefore, $u_r(r)$ $> 0$ for any $r$, since each term in the expression for $u_r$ is positive. Hence, in particular $u_r(r_i) > 0$ and $u_r(r_o) > 0$ and hence the radius of the cylinder increases due to inflation, as one would expect.

**Pressurized hole in an infinite medium**

As a limiting case of the above solution, we would like to study the problem of a hole subjected to uniform radial component of the normal stress in an infinite medium as shown in figure 7.5. We continue to use cylindrical polar coordinates. Therefore the body in the reference configuration is assumed to occupy the region of the Euclidean point space defined by $\mathcal{B} = \{(r, \theta, z) | r_i \leq r < \infty, 0 \leq \theta \leq 2\pi, -\infty < z < \infty\}$. The boundary conditions for this problem is essentially same as that in the inflation of an annular cylinder, except that now $r_o$ tends to $\infty$. For completeness the boundary conditions for the present problem is:

$$\mathbf{t}_{(\mathbf{e}_r)}(\infty, \theta, z) = \mathbf{o}, \tag{7.73}$$

$$\mathbf{t}_{(\mathbf{e}_r)}(r_i, \theta, z) = -p_i \mathbf{e}_r, \tag{7.74}$$

$$u_z(r, \theta, \pm\infty) = 0. \tag{7.75}$$

Hence, tending $r_o$ to $\infty$, in equations (7.69) and (7.70) we obtain the stress and displacement field for the present problem as,

$$\boldsymbol{\sigma} = \begin{pmatrix} -p_i \frac{r_i^2}{r^2} & 0 & 0 \\ 0 & p_i \frac{r_i^2}{r^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.76}$$

$$u_r = \frac{p_i}{\mu} \frac{r_i^2}{r}. \tag{7.77}$$

Thus, we find that both the stress and the displacement tend to zero as $r$ tends to $\infty$. This means that the effect of pressurized hole is not felt at a distance far away from the hole.

Having studied the problem of pressurized hole, in the following section we shall study the influence of the hole to uniaxial tensile load applied to the infinite medium with the hole.

### 7.4.2 Uniaxial tensile loading of a plate with a hole

In the second boundary value problem that we study, the body is an infinite medium with a circular stress free hole subjected to a far field tension along the $x-$direction as shown in the figure 7.6. We envisage to solve this boundary value problem using the stress formulation. In particular we assume

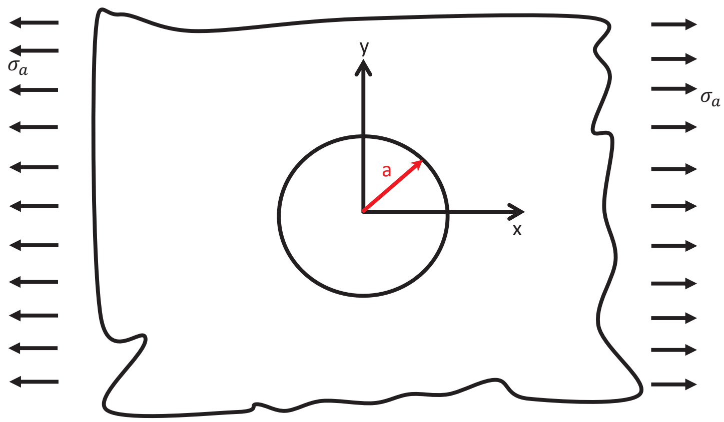



Figure 7.6: Stress free hole in an infinite medium subjected to a uniform far field tension along $x-$direction

plane stress state and the body to be two dimensional and use cylindrical polar coordinates to study this problem. Thus, the body in the reference configuration is assumed to occupy a region in the Euclidean point space defined by $\mathcal{B} = \{(r, \theta) | a \leq r < \infty, 0 \leq \theta \leq 2\pi\}$, where $a$ is a constant characterizing the size of the hole in the infinite medium. The boundary conditions that is of interest are

$$\mathbf{t}_{(\mathbf{e}_r)}(a, \theta) = \mathbf{o}, \tag{7.78}$$

$$\mathbf{t}_{(\cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta)}(\infty, \theta) = \sigma_a \left[\cos(\theta)\mathbf{e}_r - \sin(\theta)\mathbf{e}_\theta\right], \tag{7.79}$$

$$\mathbf{t}_{(\sin(\theta)\mathbf{e}_r + \cos(\theta)\mathbf{e}_\theta)}(\infty, \theta) = \mathbf{o}, \tag{7.80}$$

This boundary condition needs some explanation. The boundary condition (7.78) tells that the boundary of the hole is traction free i.e.

$$\sigma_{rr}(a, \theta) = 0, \tag{7.81}$$

$$\sigma_{r\theta}(a, \theta) = 0. \tag{7.82}$$

The second boundary condition (7.79) tells that the traction far away from the center of the hole, on a surface whose normal coincides with $\mathbf{e}_x$ is $\sigma_a \mathbf{e}_x$. Similarly, boundary condition (7.80) tells that a surface whose normal coincides with $\mathbf{e}_y$ and is far away from the hole is traction free. In the equations (7.79) and (7.80), we have represented Cartesian basis $\mathbf{e}_x$ and $\mathbf{e}_y$ in cylindrical polar coordinate basis. From equations (7.79) and (7.80) we obtain,

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{Bmatrix} \sigma_{rr}^* \\ \sigma_{r\theta}^* \\ \sigma_{\theta\theta}^* \end{Bmatrix} = \begin{Bmatrix} \sigma_a \cos(\theta) \\ -\sigma_a \sin(\theta) \\ 0 \\ 0 \end{Bmatrix}, \tag{7.83}$$

where $\sigma_{ij}^*$ represents the value of the stress at $(\infty, \theta)$. Of the four equations only 3 are independent. Picking any three equations and solving for the cylindrical polar components of the stress, we obtain

$$\sigma_{rr}^* = \sigma_{rr}(\infty, \theta) = \sigma_a \frac{1 + \cos(2\theta)}{2}, \tag{7.84}$$

$$\sigma_{r\theta}^* = \sigma_{r\theta}(\infty, \theta) = -\sigma_a \frac{\sin(2\theta)}{2}, \tag{7.85}$$

$$\sigma_{\theta\theta}^* = \sigma_{\theta\theta}(\infty, \theta) = \sigma_a \frac{1 - \cos(2\theta)}{2}. \tag{7.86}$$

Thus, the boundary conditions translate into five conditions on the components of the stress given by equations, (7.81), (7.82), (7.84), (7.85) and (7.86).

Based on the observation that the far field stresses are a function of $\cos(2\theta)$, we infer that the Airy's stress function, $\phi$ also should contain $\cos(2\theta)$ term. Then, since the solution at $\theta$ equal to 0 and $2\pi$ should be the same, the Airy's stress function can depend on $\theta$ only through the trigonometric functions present in the general solution (7.57). We assume Airy's stress function with only five constants from the general solution (7.57) as,

$$\phi = a_{02} \ln(r) + a_{03} r^2 + \left[a_{21} r^2 + a_{23} r^{-2} + a_{24}\right] \cos(2\theta), \tag{7.87}$$

and examine whether the prescribed boundary conditions can be met. Consideration that the stress at $\infty$ be finite required dropping of the terms $r^2 \ln(r)$ and $r^4$. The constant $a_{01}$ does not enter the expressions for stress and hence indeterminate. So we assumed it to be zero. For the assumption of Airy's stress function (7.87), the cylindrical polar components of the stress is computed using (7.53) to be

$$\sigma_{rr} = \frac{a_{02}}{r^2} + 2a_{03} - \left[2a_{21} + \frac{6a_{23}}{r^4} + \frac{4a_{24}}{r^2}\right]\cos(2\theta), \tag{7.88}$$

$$\sigma_{\theta\theta} = -\frac{a_{02}}{r^2} + 2a_{03} + \left[2a_{21} + \frac{6a_{23}}{r^4}\right]\cos(2\theta), \tag{7.89}$$

$$\sigma_{r\theta} = \left[2a_{21} - \frac{6a_{23}}{r^4} - \frac{2a_{24}}{r^2}\right]\sin(2\theta), \tag{7.90}$$

Now, the boundary conditions: (7.81), (7.82) and (7.84) yield:

$$\begin{pmatrix} \frac{1}{a^2} & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & \frac{6}{a^4} & \frac{4}{a^2} \\ 0 & 0 & 2 & -\frac{6}{a^4} & -\frac{2}{a^2} \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{pmatrix} \begin{Bmatrix} a_{02} \\ a_{03} \\ a_{21} \\ a_{23} \\ a_{24} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{\sigma_a}{2} \\ \frac{\sigma_a}{2} \end{Bmatrix}, \tag{7.91}$$

where we have equated the coefficients of $\cos(2\theta)$, $\sin(2\theta)$ and the constant to the values of the right hand side of these equations. It can be seen that the equations (7.85) and (7.86) yield the same last two equations in (7.91). Solving the linear equations (7.91) for $a_{ij}$'s, we obtain

$$a_{02} = -\frac{a^2\sigma_a}{4}, \quad a_{03} = \frac{\sigma_a}{4}, \quad a_{21} = -\frac{\sigma_a}{4}, \quad a_{23} = -\frac{a^4\sigma_a}{4}, \quad a_{24} = \frac{a^2\sigma_a}{2}. \tag{7.92}$$

Since, we are able to meet the required boundary conditions with the assumed form of the Airy's stress function, this is the required stress function.

Substituting the constants (7.92) in the equations (7.88) through (7.90) we obtain

$$\sigma_{rr} = \frac{\sigma_a}{2}\left\{\left[1 - \frac{a^2}{r^2}\right] + \left[1 + 3\frac{a^4}{r^4} - 4\frac{a^2}{r^2}\right]\cos(2\theta)\right\}, \tag{7.93}$$

$$\sigma_{\theta\theta} = \frac{\sigma_a}{2}\left\{\left[1 + \frac{a^2}{r^2}\right] - \left[1 + 3\frac{a^4}{r^4}\right]\cos(2\theta)\right\}, \tag{7.94}$$

$$\sigma_{r\theta} = -\frac{\sigma_a}{2}\left[1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{r^2}\right]\sin(2\theta). \tag{7.95}$$

Having obtained the stress, we compute the strains from these stresses using the constitutive relation (7.54) as

$$\begin{aligned}\epsilon_{rr} &= \frac{\partial u_r}{\partial r} \\ &= \frac{\sigma_a}{2}\left\{\frac{(1-\nu)}{E} + \frac{(1+\nu)}{E}\left[-\frac{a^2}{r^2} + \left[1 + 3\frac{a^4}{r^4}\right]\cos(2\theta)\right] - \frac{4}{E}\frac{a^2}{r^2}\cos(2\theta)\right\},\end{aligned} \tag{7.96}$$

$$\begin{aligned}\epsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\ &= \frac{\sigma_a}{2}\left\{\frac{(1-\nu)}{E} + \frac{(1+\nu)}{E}\left[\frac{a^2}{r^2} - \left[1 + 3\frac{a^4}{r^4}\right]\cos(2\theta)\right] + \frac{4\nu}{E}\frac{a^2}{r^2}\cos(2\theta)\right\},\end{aligned} \tag{7.97}$$

$$\begin{aligned}\epsilon_{r\theta} &= \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right] \\ &= -\frac{(1+\nu)}{E}\frac{\sigma_a}{2}\left[1 - 3\frac{a^4}{r^4} + 2\frac{a^2}{r^2}\right]\sin(2\theta).\end{aligned} \tag{7.98}$$

Integrating equation (7.96) we obtain,

$$u_r = \frac{\sigma_a}{2}\left\{\frac{(1-\nu)}{E}r + \frac{(1+\nu)}{E}\left[\frac{a^2}{r} + \left[r - \frac{a^4}{r^3}\right]\cos(2\theta)\right] + \frac{4}{E}\frac{a^2}{r}\cos(2\theta)\right\} + \frac{df}{d\theta}, \tag{7.99}$$

where $f = \hat{f}(\theta)$ is a function of $\theta$. Substituting (7.99) in equation (7.97) and integrating we find

$$u_\theta = -\frac{\sigma_a}{2}\left\{\frac{(1+\nu)}{E}\left[r + \frac{a^4}{r^3}\right] + 2\frac{(1-\nu)}{E}\frac{a^2}{r}\right\}\sin(2\theta) + f(\theta) + g(r), \tag{7.100}$$

where $g(r)$ is some function of $r$. Substituting equations (7.99) and (7.100) in equation (7.98) and simplifying we obtain

$$r\frac{dg}{dr} - g - f + \frac{d^2 f}{d\theta^2} = 0. \tag{7.101}$$

Since $g$ is a function of only $r$ and $f$ only of $\theta$, we require

$$r\frac{dg}{dr} - g = f - \frac{d^2 f}{d\theta^2} = C_0, \tag{7.102}$$

where $C_0$ is a constant. Solving the linear ordinary differential equations (7.102) we obtain

$$g = C_1 r - C_0, \qquad f = C_0 + C_2 \exp(\theta) + C_3 \exp(-\theta), \tag{7.103}$$

where $C_i$'s are constants. To ensure that the tangential displacement of the ray $\theta = 0$ and $\theta = 2\pi$ to be the same and there be no rigid body displacement we require that,

$$u_\theta(r, 0) = u_\theta(r, 2\pi) = 0. \tag{7.104}$$

The condition (7.104) translates into requiring $C_1 = C_2 = C_3 = 0$. Thus, the displacement field is computed to be,

$$u_r = \frac{\sigma_a}{2}\left\{\frac{(1-\nu)}{E} r + \frac{(1+\nu)}{E}\left[\frac{a^2}{r} + \left[r - \frac{a^4}{r^3}\right]\cos(2\theta)\right] + \frac{4}{E}\frac{a^2}{r}\cos(2\theta)\right\}, \tag{7.105}$$

$$u_\theta = -\frac{\sigma_a}{2}\left\{\frac{(1+\nu)}{E}\left[r + \frac{a^4}{r^3}\right] + 2\frac{(1-\nu)}{E}\frac{a^2}{r}\right\}\sin(2\theta). \tag{7.106}$$

Thus, we have found the displacement and stress field and hence have solved this boundary value problem. We would like to find the maximum stress that occurs and its location for this boundary value problem.

First we begin with hoop stress, $\sigma_{\theta\theta}$ given in equation (7.94). Recollecting that the extremum value of a function could occur either at the boundary points or at points where the derivative of the function goes to zero, the extremum hoop stress could occur at a radial location, $r = a\sqrt{6\cos(2\theta)}$, when $\cos(2\theta) > 0$ or at the boundary points, $r = a$ and $r$ tending to $\infty$. In figure 7.7 we plot the hoop stress as a function of $r$ for various values of $\theta$. As expected, the function is monotonically decreasing function of $r$ when $\cos(2\theta)$ $< 0$ and therefore the maximum value occurs at $r = a$ for this case. When $\cos(2\theta) > 0$, the minimum occurs at $r = a$, then the maximum occurs when $r = a\sqrt{6\cos(2\theta)}$ and then the stress decreases and asymptotically reaches the value of $\sigma_a[1 - \cos(2\theta)]/2$. Therefore, at $r = a$ an extremum of the hoop stress occurs. The circumferential variation of the hoop stress at $r = a$ is given by $\sigma_a[1 - 2\cos(2\theta)]$. Hence, the maximum hoop stress occurs at $r = a$ and $\theta = \pi/2$ (or $\theta = 3\pi/2$) and the value of the maximum hoop stress is $3\sigma_a$. Thus, the stress concentration factor, defined as the ratio of the maximum stress in the structure to the far field stress is 3 ($= \sigma_{\theta\theta}^{max}/\sigma_a$).

Next, we study the circumferential shearing stress, $\sigma_{r\theta}$ given in equation (7.95). The extremum value of this stress occurs at a radial location $r = a\sqrt{3}$

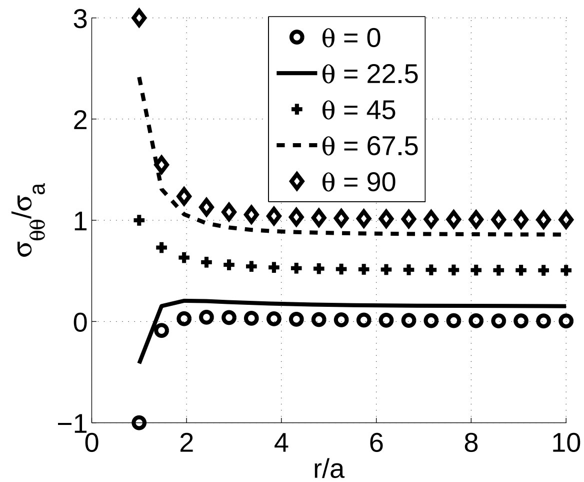



Figure 7.7: Variation of hoop stress, $\sigma_{\theta\theta}$, with radial location, $r$ for various orientation of the ray, $\theta$ when an infinite medium with a hole of radius $a$ is subjected to a uniaxial tensile stress $\sigma_a$

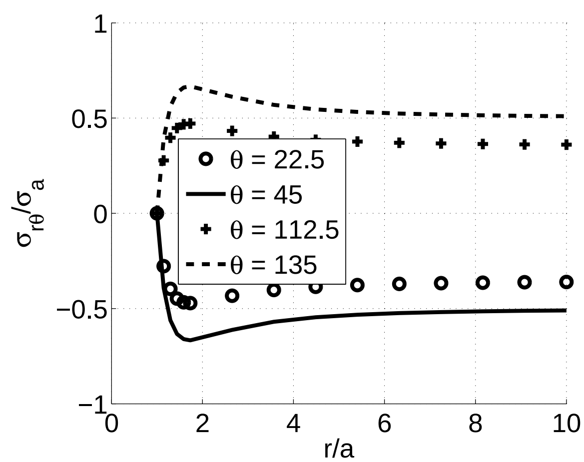
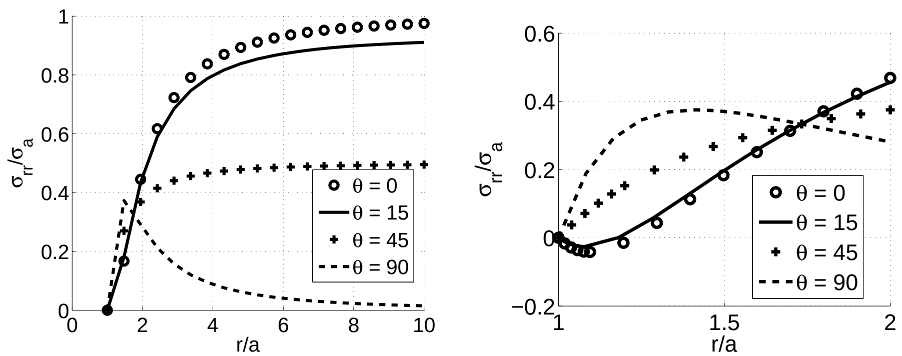



Figure 7.8: Variation of circumferential shearing stress, $\sigma_{r\theta}$, with radial location, $r$ for various orientation of the ray, $\theta$ when an infinite medium with a hole of radius $a$ is subjected to a uniaxial tensile stress $\sigma_a$




(a) When $r$ varies between 1 and 10 (b) When $r$ varies between 1 and 2

Figure 7.9: Variation of radial stress, $\sigma_{rr}$, with radial location, $r$ for various orientation of the ray, $\theta$ when an infinite medium with a hole of radius $a$ is subjected to a uniaxial tensile stress $\sigma_a$

and the value of the circumferential shearing stress at this radial location is $-2\sigma_a \sin(2\theta)/3$. At the boundary points the circumferential shearing stress takes values 0 at $r = a$ and $-\sigma_a \sin(2\theta)/2$. The variation of $\sigma_{r\theta}$ with the radial location, $r$ and the orientation of the ray, $\theta$ is shown in figure 7.8. Thus, it can be seen that the absolute maximum value of the stress, $\sigma_{r\theta}$ occurs at $r = a\sqrt{3}$, $\theta = m\pi/4$, where $m$ takes one of the values from the set $\{1, 3, 5, 7\}$ and this maximum value is $2\sigma_a/3$.

Finally, we examine the radial stress, $\sigma_{rr}$ given in equation (7.93). An extremum value of this stress occurs at

$$r_{crit} = a\sqrt{6\cos(2\theta)/[1 + 4\cos(2\theta)]} \tag{7.107}$$

when[4] $0 \le \theta \le \pi/6$ or $2\pi/6 \le \theta \le 4\pi/6$ or $5\pi/6 \le \theta \le 7\pi/6$ or $8\pi/6 \le \theta \le 10\pi/6$ or $11\pi/6 \le \theta \le 2\pi$. This extremum value of the radial stress is $-\sigma_a\{11\cos(2\theta)/9 + 1/9 + 5/[36\cos(2\theta)]\}/2$. At the boundary the radial stresses takes the value 0 at $r = a$ and $\sigma_a[1 + \cos(2\theta)]/2$ as $r$ tends to $\infty$. Figure 7.9 portrays the variation of the radial stress with radial location, $r$ for various orientations of the ray, $\theta$. It can be seen from the figure that for certain orientations (say, $\theta = 90$ degrees) the maximum value of the radial stress occurs at $r = a\sqrt{6\cos(2\theta)/[1 + 4\cos(2\theta)]}$ and for some other

[4]The restriction on $\theta$ is obtained by the requirement that $\sqrt{6\cos(2\theta)/[1 + 4\cos(2\theta)]} \ge 1$.

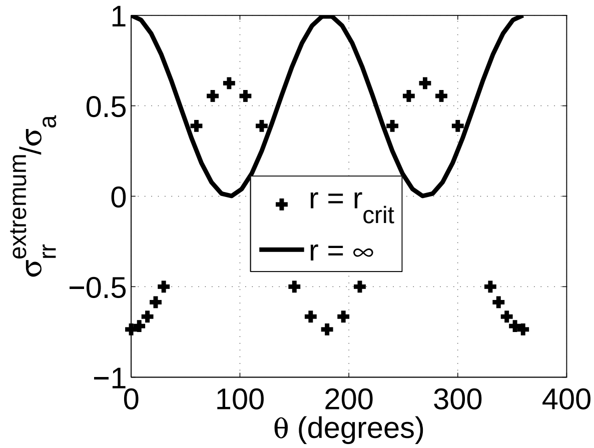



Figure 7.10: Variation of two extremum radial stress $\sigma_{rr}$ at $r_{crit}$ [given in equation (7.107)] and as $r$ tends to $\infty$, with the orientation of the ray, $\theta$ when an infinite medium with a hole of radius $a$ is subjected to a uniaxial tensile stress $\sigma_a$

orientations (say, $\theta = 15$ degrees) the maximum value occurs when $r$ tends to $\infty$. To understand this, in figure 7.10 we plot the two extremum radial stresses - one at $r = r_{crit}$ and the other value that occurs when $r$ tends to $\infty$. It can be seen from the figure that when $\pi/3 \leq \theta \leq 2\pi/3$ and $8\pi/3 \leq \theta \leq 10\pi/3$ the maximum radial stress occurs at $r_{crit}$ and for all other values of $\theta$ it occurs as $r$ tends to $\infty$. In any case the maximum value is always less than $\sigma_a$.

This concludes our illustration of the two techniques to solve boundary value problems in this chapter. However, in the remaining chapters we shall see employment of these techniques to solve more boundary value problems of interest in engineering.

## 7.5 General results

In this section, we record two results that are useful while solving boundary value problems. One result tells when one can expect an unique solution to a given boundary value problem. The other allows us to construct solutions to complex boundary conditions from simple cases.

### 7.5.1 Uniqueness of solution

Now, we are interested in showing that there could at most be one solution that could satisfy the prescribed displacement or mixed boundary conditions, in a given body made of a material that obeys Hooke's law and in static equilibrium with no body forces acting on it. If traction boundary condition is specified, we shall see that the stress is uniquely determined but the displacement is not for the bodies made of a material that obeys Hooke's law and in static equilibrium with no body forces acting on it.

Towards this, we consider a general setting with $\mathcal{B}$ being some region occupied by the body and $\partial\mathcal{B}$ the boundary of the body. Let us also assume that displacement is prescribed over some part of the boundary $\partial\mathcal{B}_u$ and traction specified on the remaining part of the boundary, $\partial\mathcal{B}_\sigma$. Let $(\mathbf{u}_1, \boldsymbol{\sigma}_1)$ and $(\mathbf{u}_2, \boldsymbol{\sigma}_2)$ be the two distinct solutions to a given boundary value problem. Let us define

$$\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2. \tag{7.108}$$

By virtue of $(\mathbf{u}_1, \boldsymbol{\sigma}_1)$ and $(\mathbf{u}_2, \boldsymbol{\sigma}_2)$ satisfying the specified boundary conditions,

$$\mathbf{u}(\mathbf{x}) = \mathbf{o}, \quad \text{for} \quad \mathbf{x} \in \partial\mathcal{B}_u, \tag{7.109}$$

$$\mathbf{t}_{(\mathbf{n})}(\mathbf{x}) = \boldsymbol{\sigma}\mathbf{n} = \mathbf{o}, \quad \text{for} \quad \mathbf{x} \in \partial\mathcal{B}_\sigma. \tag{7.110}$$

First, we examine the term,

$$W = \boldsymbol{\sigma} \cdot grad(\mathbf{u}) = \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}, \tag{7.111}$$

here to obtain the last equality we have used (2.104) and the fact that the Cauchy stress $\boldsymbol{\sigma}$ is a symmetric tensor. Since, we are interested in materials that obey isotropic Hooke's law, we substitute for the stress from[5]

$$\boldsymbol{\sigma} = \left[K - \frac{2}{3}G\right] tr(\boldsymbol{\epsilon})\mathbf{1} + 2G\boldsymbol{\epsilon}, \tag{7.112}$$

in (7.111) to obtain

$$W = K[tr(\boldsymbol{\epsilon})]^2 + 2G\left\{tr(\boldsymbol{\epsilon}^2) - \frac{1}{3}[tr(\boldsymbol{\epsilon})]^2\right\}. \tag{7.113}$$

[5]This equation is obtained by writing Lamè constants in terms of bulk modulus, $K$ and shear modulus, $G$ and substituting in equation (7.2).

As recorded in table 6.1 $G > 0$ and $K > 0$. Further recognizing that irrespective of the sign of the Cartesian components of the strain,

$$\begin{aligned} & tr(\boldsymbol{\epsilon}^2) - \frac{1}{3}[tr(\boldsymbol{\epsilon})]^2 \\ & \quad = \frac{1}{3}\left[(\epsilon_{xx} - \epsilon_{yy})^2 + (\epsilon_{yy} - \epsilon_{zz})^2 + (\epsilon_{zz} - \epsilon_{xx})^2\right] + 2\epsilon_{xy}^2 + 2\epsilon_{yz}^2 + 2\epsilon_{xz}^2 > 0, \end{aligned} \tag{7.114}$$

on assuming that $\boldsymbol{\epsilon} \neq \mathbf{0}$. Thus, since each term in equation (7.113) is positive we infer that

$$W > 0, \tag{7.115}$$

as long as $\boldsymbol{\epsilon} \neq \mathbf{0}$.

Next, we want to express

$$\int_{\mathcal{B}} W dv = \int_{\mathcal{B}} \boldsymbol{\sigma} \cdot grad(\mathbf{u}) dv, \tag{7.116}$$

in terms of the boundary conditions alone. Towards this, using the result in equation (2.219) we write

$$div(\boldsymbol{\sigma}\mathbf{u}) = div(\boldsymbol{\sigma}) \cdot \mathbf{u} + \boldsymbol{\sigma} \cdot grad(\mathbf{u}). \tag{7.117}$$

Since the body is in static equilibrium and has no body forces acting on it, from equation (7.6) $div(\boldsymbol{\sigma}) = \mathbf{o}$. Using $div(\boldsymbol{\sigma}) = \mathbf{o}$ in equation (7.117) and substituting the result in (7.116) we get

$$\int_{\mathcal{B}} W dv = \int_{\mathcal{B}} div(\boldsymbol{\sigma}\mathbf{u}) dv = \int_{\partial\mathcal{B}} \mathbf{u} \cdot \boldsymbol{\sigma}^t \mathbf{n} da, \tag{7.118}$$

where to obtain the last equality we have used the result (2.267). Substituting the conditions (7.109) and (7.110) on the displacement and stress, in (7.118) that

$$\int_{\mathcal{B}} W dv = 0. \tag{7.119}$$

Since, from equation (7.115) the integrand in the equation (7.119) is positive and $W = 0$ only if $\boldsymbol{\epsilon} = \mathbf{0}$, it follows that $\boldsymbol{\epsilon} = \mathbf{0}$ everywhere in the body. Then, it is straight forward to see that $\boldsymbol{\sigma} = \mathbf{0}$, everywhere in the body. It can then be shown that, we do this next for a special case, the conditions

$\boldsymbol{\epsilon} = \mathbf{0}$ everywhere in the body and (7.109) imply $\mathbf{u} = \mathbf{o}$ everywhere in the body.

Integrating the first order differential equations on the Cartesian components of the displacement field when $\boldsymbol{\epsilon} = \mathbf{0}$, we obtain

$$\mathbf{u} = (K_1 y + K_2 z + K_4)\mathbf{e}_x + (-K_1 x + K_3 z + K_5)\mathbf{e}_y - (K_2 x + K_3 y - K_6)\mathbf{e}_z, \quad (7.120)$$

where $K_i$'s are constants. Knowing the displacement of 3 points, say $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ and $(x_3, y_3, z_3)$, that are not collinear to be zero, the constants $K_i$'s could be uniquely determined by solving the following system of equations:

$$\begin{pmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{pmatrix} \begin{Bmatrix} K_1 \\ K_2 \\ K_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{pmatrix} -x_1 & z_1 & 1 \\ -x_2 & z_2 & 1 \\ -x_3 & z_3 & 1 \end{pmatrix} \begin{Bmatrix} K_1 \\ K_3 \\ K_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$
$$\begin{pmatrix} -x_1 & -y_1 & 1 \\ -x_2 & -y_2 & 1 \\ -x_3 & -y_3 & 1 \end{pmatrix} \begin{Bmatrix} K_2 \\ K_3 \\ K_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \quad (7.121)$$

As long as the three points are not collinear, it can be seen that the above set of equations are independent and hence, $K_i = 0$. However, if the displacement is not known at 3 points, as in the case of traction boundary condition being specified, the displacement field cannot be uniquely determined.

Hence, we conclude that the solution to the boundary value problem involving a body in static equilibrium, under the absence of body forces and made of a material that obeys isotropic Hooke's law is unique except in cases where only traction boundary condition is specified. As a consequence of this theorem, if **a** solution has been found for a given boundary conditions it is **the** solution for a body in static equilibrium, under the absence of body forces and made of a material that obeys isotropic Hooke's law.

### 7.5.2 Principle of superposition

This principle states that $F$or a given body made up of a material that obeys isotropic Hooke's law, in static equilibrium and whose magnitude of displacement is small, if $\{\mathbf{u}^{(1)}, \boldsymbol{\sigma}^{(1)}\}$ is a solution to the prescribed body forces, $\mathbf{b}^{(1)}$ and boundary conditions, $\{\mathbf{u}_b^{(1)}, \mathbf{t}_{(\mathbf{n})}^{(1)}\}$ and $\{\mathbf{u}^{(2)}, \boldsymbol{\sigma}^{(2)}\}$ is a solution to the prescribed body forces, $\mathbf{b}^{(2)}$ and boundary conditions, $\{\mathbf{u}_b^{(2)}, \mathbf{t}_{(\mathbf{n})}^{(2)}\}$ then

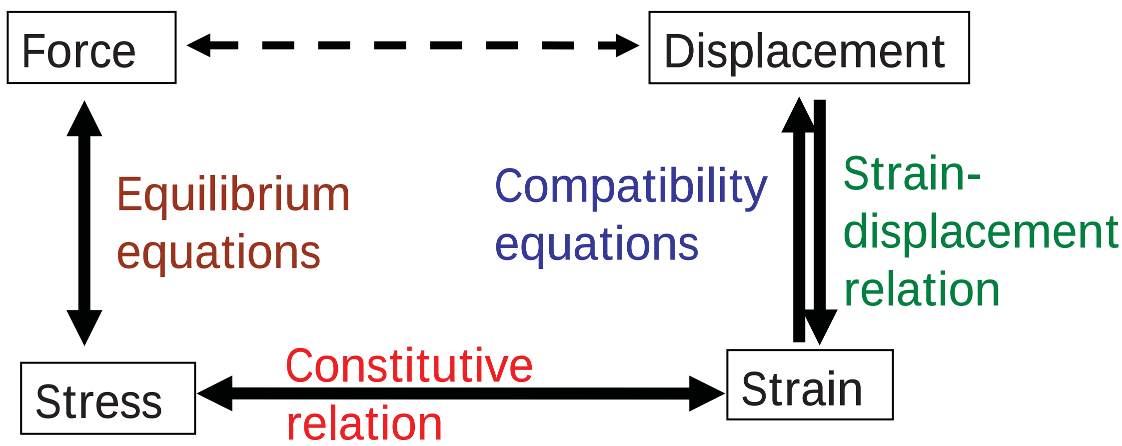



Figure 7.11: Basic variables and equations in mechanics

$\{\mathbf{u}^{(1)} + \mathbf{u}^{(2)}, \boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}\}$ will be a solution to the problem with body forces, $\mathbf{b}^{(1)} + \mathbf{b}^{(2)}$ and boundary conditions, $\{\mathbf{u}_b^{(1)} + \mathbf{u}_b^{(2)}, \mathbf{t}_{(\mathbf{n})}^{(1)} + \mathbf{t}_{(\mathbf{n})}^{(2)}\}$

The proof of the above statement follows immediately from the fact that equation (7.23) is linear. That is,

$$
\begin{aligned}
\mathbf{o} &= (\lambda+\mu)grad(div(\mathbf{u}^{(1)} + \mathbf{u}^{(2)})) + \mu\Delta(\mathbf{u}^{(1)} + \mathbf{u}^{(2)}) + \rho(\mathbf{b}^{(1)} + \mathbf{b}^{(2)}) \\
&= (\lambda+\mu)grad(div(\mathbf{u}^{(1)})) + \mu\Delta\mathbf{u}^{(1)} + \rho\mathbf{b}^{(1)} \\
&\qquad +(\lambda+\mu)grad(div(\mathbf{u}^{(2)})) + \mu\Delta\mathbf{u}^{(2)} + \rho\mathbf{b}^{(2)} \\
&= \mathbf{o} + \mathbf{o},
\end{aligned}
$$

where we have used the linearity property of the *grad*, *div* and $\Delta$ operators (see section 2.8).

This is one of the most often used principles to solve problems in engineering. We shall illustrate the use of this principle in chapter 11.

## 7.6 Summary

In this chapter, we formulated the boundary value problem for an isotropic material undergoing small elastic deformations obeying Hooke's law. Two techniques were outlined to solve this problem. These two techniques are summarized in figure 7.11. Thus, in the displacement approach, one starts

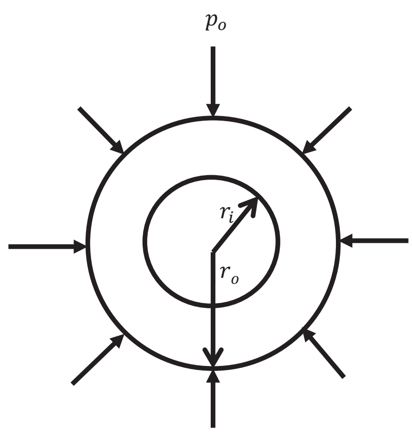



Figure 7.12: An annular cylinder pressurized from outside. (Figure for Problem 1)

with the displacement use the strain displacement relation to compute the strain and then the constitutive relation to find the stress and finally use the equilibrium equations to get the governing equation that the displacement field has to satisfy along with the prescribed boundary conditions. In the stress approach, stress field is assumed in terms of a potential, such that equilibrium equations hold, then the strain is computed using the constitutive relation. In order to be able to obtain a smooth displacement field using this strain field, it is required that the stress potential satisfy the compatibility conditions along with the prescribed boundary conditions. We then illustrated these techniques by solving two boundary value problems, that of inflation of an annular cylinder and that of a plate with a hole subjected to uniaxial tension far away from the hole.

## 7.7 Self-Evaluation

1. A body in the form of an annular cylinder is pressurized from outside as shown in figure 7.12. Obtain the stress and displacement field for this boundary value problem and show that the developed hoop stresses are compressive in nature and that the thickness of the annular cylinder increases due to the applied external pressure. Assume that the body is homogeneous and is made of a material that obeys isotropic Hooke's

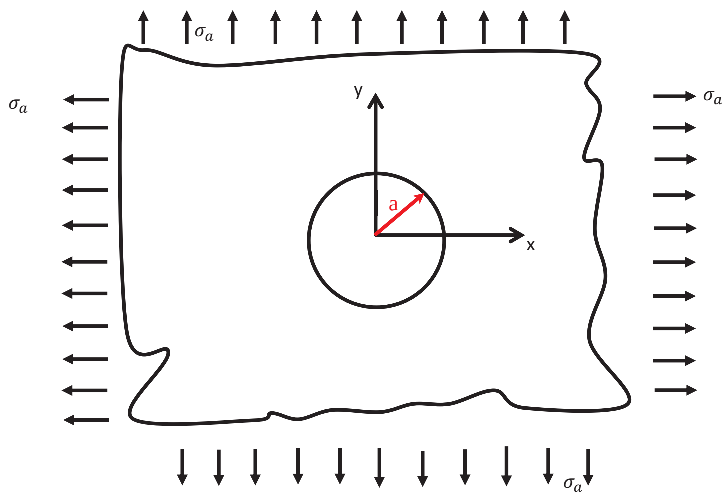



Figure 7.13: Stress free hole in an infinite medium subjected to equal biaxial loading. (Figure for Problem 2)

law.

2. Using the solution obtained in problem 1, study the problem of a stress free hole in an infinite medium under equal biaxial loading at infinity, as shown in figure 7.13. Recognize that this solution can be obtained by tending $r_o$ to $\infty$ in the solutions to problem 1. Then show that the hoop stresses developed at the boundary of the hole is twice the far field equal biaxial stress applied.

3. The annular cylinder shown in figure 7.14 is pressurized from both inside and outside. Assuming the cylinder to be homogeneous and made of a material that obeys isotropic Hooke's law with Young's modulus, $E = 200$ GPa and Poisson's ratio, $\nu = 0.3$,

   (a) Obtain the location and magnitude of the maximum and minimum hoop stress and radial stress.

   (b) Obtain the maximum principal stress, its location and magnitude.

   (c) Use thin cylinder approximation to obtain the hoop and radial stress. Use this to obtain the maximum principal stress.

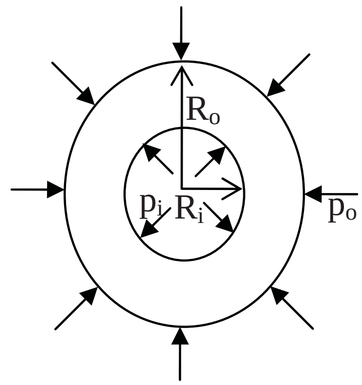



Figure 7.14: An annular cylinder pressurized from inside and outside. (Figure for Problems 3, 5 and 6)

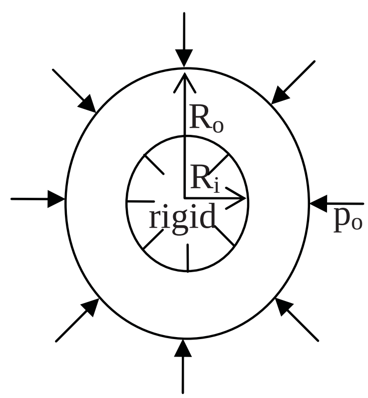



Figure 7.15: An annular cylinder pressurized from outside with its inner surface prevented from radially displacing. (Figure for Problem 4)

(d) Plot the ratio of the maximum principal stress obtained in part b to that in part c as a function of $(1 - R_i/R_o) - 1$ for $R_i/R_o = 0.05$ to 0.99. Plot the same for $(p_i, p_o) = \{(1, 0), (0, 1), (1, 0.5), (0.5, 1)\}$. Using the above plots verify the reasonableness of the following classification: $d/t < 20$ thick cylinder, $d/t > 40$ thin cylinder, where $d$ is the mean diameter of the cylinder and $t$ is its thickness.

4. Assuming the annular cylinder shown in figure 7.15, to be homogeneous and made of a material that obeys isotropic Hooke's law with Young's modulus, $E = 200$ GPa and Poisson's ratio, $\nu = 0.3$ repeat parts (a) and (b) in problem 3. Here the inner boundary is fixed, i.e., $u_r(r_i, \theta, z) = 0$. Find the hoop stress and radial stress as $R_i/R_o$ tends to 0 and comment on the significance of this limit.

5. Now assume the annular cylinder shown in figure 7.14 is inhomogeneous and is composed of two annular cylinders made of aluminium and steel respectively. Solve parts (a) to (d) in problem 3 assuming

   (a) Aluminium cylinder occupies the region $\mathcal{B} = \{(r, \theta, z) | R_i \leq r \leq (R_i + R_o)/2, 0 \leq \theta \leq 2\pi, 0 \leq z \leq H\}$ and steel cylinder occupies the region $\mathcal{B} = \{(r, \theta, z) | (R_i + R_o)/2 \leq r \leq R_o, 0 \leq \theta \leq 2\pi, 0 \leq z \leq H\}$.

   (b) Steel cylinder occupies the region $\mathcal{B} = \{(r, \theta, z) | R_i \leq r \leq (R_i + R_o)/2, 0 \leq \theta \leq 2\pi, 0 \leq z \leq H\}$ and aluminium cylinder occupies the region $\mathcal{B} = \{(r, \theta, z) | (R_i + R_o)/2 \leq r \leq R_o, 0 \leq \theta \leq 2\pi, 0 \leq z \leq H\}$.

   Take Young's modulus of steel and aluminium to be $E_{steel} = 200$ GPa and $E_{aluminium} = 70$ GPa respectively and Poisson's ratio of these materials to be $\nu_{steel} = 0.3$ and $\nu_{aluminium} = 0.27$.

6. Assume that an aluminium annular cylinder occupying the region $\mathcal{B} = \{(r, \theta, z) | R_i \leq r \leq (\delta + (R_i + R_o)/2), 0 \leq \theta \leq 2\pi, 0 \leq z \leq H\}$ in the stress free state is shrunk and fitted into a steel annular cylinder occupying the region $\mathcal{B} = \{(r, \theta, z) | (R_i + R_o)/2 \leq r \leq R_o, 0 \leq \theta \leq 2\pi, 0 \leq z \leq H\}$ in the stress free state. Then this composite shrink fitted cylinder is pressurized as shown in figure 7.14. For this case solve parts (a) to (d) in problem 3. Assume $\delta = 0.0001 R_o$. Take Young's modulus of steel and aluminium to be $E_{steel} = 200$ GPa and $E_{aluminium}$

= 70 GPa respectively and Poisson's ratio of these materials to be $\nu_{steel}$ = 0.3 and $\nu_{aluminium}$ = 0.27.

# Chapter 8

# Bending of Prismatic Straight Beams

## 8.1 Overview

In this chapter, we formulate and study the bending of straight and prismatic beams. First we study the bending of these beams when the loading is in the plane of symmetry of the beams cross section. Here we derive again the strength of materials solution and compare it with the solution obtained from two dimensional elasticity formulation for two loading cases, namely pure bending of a beam and uniform loading of a simply supported beam. Then, we study the bending of beams when the loading does not pass through the plane of symmetry. We do this by assuming the loading causes only bending and no torsion. We will then show that for this to happen the load should be applied at a point called the shear center of the cross-section. We conclude by presenting a method to compute this shear center.

Before proceeding further, let us begin by understanding what a beam is. A beam is a structural member whose length along one direction, called the longitudinal axis, is larger than its dimensions on the plane perpendicular to it and is subjected to only transverse loads (i.e., the loads acting perpendicular to the longitudinal axis). A typical beam with rectangular cross section is shown in figure 8.1. Thus, this beam occupies the points in the Euclidean point space defined by $\mathcal{B} = \{(x, y, z)| - l \leq x \leq l, -c \leq y \leq c, -b \leq z \leq b\}$ where $l$, $c$ and $b$ are constants, such that $l > c \geq b$ with $l/c$ typically in the range 10 to 20.

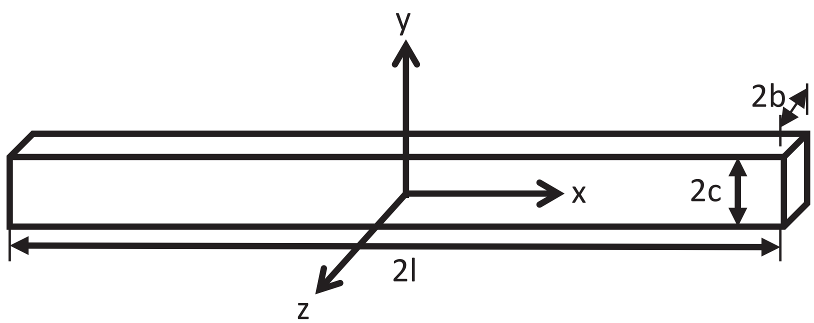



Figure 8.1: Schematic of a beam

Next let us understand which moment is a bending moment and how it is related to the components of the stress. The component of the moment parallel to the longitudinal axis of the beam is called as torsional moment and the remaining two component of the moments are called bending moments. Thus, for the beam with the axis oriented as shown in figure 8.1, the $M_x$ component of the moment is called the torsional moment and the remaining two components, $M_y$ and $M_z$ the bending moments. To relate these moments to the components of the stress tensor, first we find the traction that is acting on a plane defined by $x = x_o$, a constant. This traction is computed for a general state of stress as

$$\mathbf{t}_{(\mathbf{e}_x)} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \sigma_{xx}\mathbf{e}_x + \sigma_{xy}\mathbf{e}_y + \sigma_{xz}\mathbf{e}_z. \tag{8.1}$$

Now we are interested in the net force acting at this section, i.e.,

$$\mathbf{F} = P\mathbf{e}_x + V_y\mathbf{e}_y + V_z\mathbf{e}_z = \int_a \mathbf{t}_{(\mathbf{e}_x)}(x_o, y, z)dydz = \int_a (\sigma_{xx}\mathbf{e}_x + \sigma_{xy}\mathbf{e}_y + \sigma_{xz}\mathbf{e}_z)dydz, \tag{8.2}$$

where $P$ is called the axial force, $V_y$ and $V_z$ the shear force. Equating the components in equation (8.2) we obtain

$$P(x_o) = \int_a \sigma_{xx}(x_o, y, z)dydz, \tag{8.3}$$

$$V_y(x_o) = \int_a \sigma_{xy}(x_o, y, z)dydz, \tag{8.4}$$

$$V_z(x_o) = \int_a \sigma_{xz}(x_o, y, z)dydz, \tag{8.5}$$

where the axial force and the shear force could vary along the longitudinal axis of the beam. Then, we are also interested in the net moment acting about the point $(x_o, y, z)$ due to the traction $\mathbf{t}_{(\mathbf{e}_x)}$, i.e.,

$$\mathbf{M} = \int_a (y\mathbf{e}_y + z\mathbf{e}_z) \wedge (\sigma_{xx}\mathbf{e}_x + \sigma_{xy}\mathbf{e}_y + \sigma_{xz}\mathbf{e}_z)dydz$$
$$= \int_a \left[(\sigma_{xz}y - \sigma_{xy}z)\mathbf{e}_x + \sigma_{xx}z\mathbf{e}_y - \sigma_{xx}y\right]dydz. \tag{8.6}$$

Thus, the torsional moment is given by

$$M_x(x_o) = \int_a (\sigma_{xz}y - \sigma_{xy}z)dydz, \tag{8.7}$$

and the two bending moments by

$$M_y(x_o) = \int_a \sigma_{xx}zdydz, \tag{8.8}$$
$$M_z(x_o) = -\int_a \sigma_{xx}ydydz. \tag{8.9}$$

The two shear forces and bending moments are also called as the force resultants or the stress resultants in the analysis of beams. These stress resultants vary only along the longitudinal axis of the beam.

Next we want to recast the static equilibrium equations in the absence of body forces, (7.6) in terms of the stress resultants. Towards this, we record the differential form of the equilibrium equations in Cartesian coordinates:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \tag{8.10}$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0, \tag{8.11}$$
$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \tag{8.12}$$

Multiply equation (8.10) with $y$ and integrate over the cross sectional area to obtain

$$\int_a \left[\frac{\partial(y\sigma_{xx})}{\partial x} + \left(\frac{\partial(y\sigma_{xy})}{\partial y} - \sigma_{xy}\right) + \frac{\partial(y\sigma_{xz})}{\partial z}\right]dydz = 0, \tag{8.13}$$

Recognizing that in the first term the integration is with respect to $y$ and $z$ but the differentiation is with respect to another independent variable $x$, the order can be interchanged to obtain

$$\frac{\partial}{\partial x}\left(\int_a \sigma_{xx} y dydz\right) = -\frac{dM_z}{dx}, \tag{8.14}$$

where we have used equation (8.9). To evaluate the second term in equation (8.13) we appeal to the Green's theorem (2.275) and find that

$$\int_a \frac{\partial(y\sigma_{xy})}{\partial y} dydz = \oint_c y\sigma_{xy} n_y ds = 0, \tag{8.15}$$

The last equality is because there are no shear stresses applied on the outer lateral surfaces of the beam. For the same reason the last term in equation (8.13) also evaluates to be zero:

$$\int_a \frac{\partial(y\sigma_{xz})}{\partial z} dydz = \oint_c y\sigma_{xz} n_z ds = 0. \tag{8.16}$$

It follows from equation (8.4) that the third term in equation (8.13) is $V_y$, i.e.,

$$\int_a \sigma_{xy} dydz = V_y. \tag{8.17}$$

Substituting equations (8.14) through (8.17) in (8.13) and simplifying we obtain

$$\frac{dM_z}{dx} + V_y = 0. \tag{8.18}$$

On multiplying equation (8.10) with $z$ and integrating over the cross sectional area we obtain

$$\int_a \left[\frac{\partial(z\sigma_{xx})}{\partial x} + \frac{\partial(z\sigma_{xy})}{\partial y} + \left(\frac{\partial(z\sigma_{xz})}{\partial z} - \sigma_{xz}\right)\right] dydz = 0. \tag{8.19}$$

Using the equations (8.8) and (8.5) and Green's theorem and following the same steps as above, it can be shown that equation (8.19) evaluates to requiring

$$\frac{dM_y}{dx} - V_z = 0. \tag{8.20}$$

Integrating equation (8.11) over the cross sectional area we obtain

$$\int_a \left[ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \right] dydz = 0. \tag{8.21}$$

Using equation (8.4), the first term in equation (8.21) evaluates to

$$\int_a \frac{\partial \sigma_{xy}}{\partial x} dydz = \frac{dV_y}{dx}, \tag{8.22}$$

where by virtue of the differentiation and integration being on different independent variables, we have changed their order. Appealing to Green's theorem (2.275), the second term in the equation (8.21) evaluates to

$$\int_a \frac{\partial \sigma_{yy}}{\partial y} dydz = \oint_c \sigma_{yy} n_y ds = q_y, \tag{8.23}$$

where $q_y$ is the transverse loading per unit length applied on the beam. The last term in equation (8.21) evaluates as

$$\int_a \frac{\partial \sigma_{yz}}{\partial z} dydz = \oint_c \sigma_{yz} n_z ds = 0, \tag{8.24}$$

wherein we have again appealed to Green's theorem (2.276) and the fact that no shear stresses are applied on the lateral surface of the beam. Substituting equations (8.22) through (8.24) in equation (8.21) it simplifies to

$$\frac{dV_y}{dx} + q_y = 0. \tag{8.25}$$

Integrating equation (8.12) over the cross sectional area we obtain

$$\int_a \left[ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right] dydz = 0. \tag{8.26}$$

Using the equation (8.5) and the Green's theorem and following the procedure similar to that described above, it can be shown that equation (8.26) simplifies to

$$\frac{dV_z}{dx} + q_z = 0, \tag{8.27}$$

where

$$q_z = \oint_c \sigma_{zz} n_z ds. \tag{8.28}$$

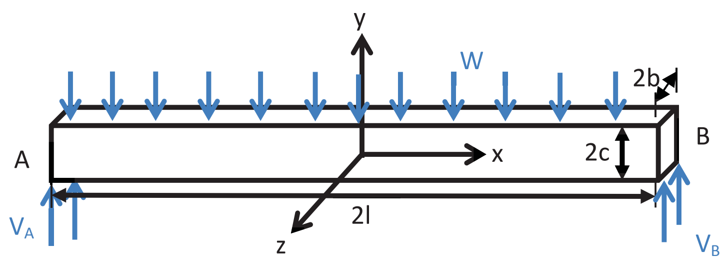



Figure 8.2: Schematic of a simply supported beam subjected to uniformly distributed load ($W$) on the top surface

Thus, we find that the three equilibrium equations reduces to four ordinary differential equations (8.18), (8.20), (8.25) and (8.27) in terms of the stress resultants in a beam.

Now, we are in a position to find the displacements and stresses in a beam subjected to some transverse loading. However, we shall initially confine ourselves to loading along the plane of symmetry of the cross section that too only along one direction.

## 8.2 Symmetrical bending

Here we assume that the beam is loaded along a single symmetric plane of the cross section. Without loss of generality we assume that this symmetric loading plane is the $xy$ plane of the beam. For illustration, let us assume that the cross section of the beam is rectangular with depth $2c$ and width $2b$ and length $2l$ as shown in figure 8.2. Further, let us assume that the beam is simply supported at the ends $A$ and $B$ and is subjected to a uniform pressure loading $W$ on its top surface as pictorially represented in figure 8.2. We derive the strength of materials solution before obtaining the 2 dimensional elasticity solution for this problem. While the strength of materials solution is generic, in that it is applicable for any cross section, elasticity solution is specific for rectangular cross section alone.

### 8.2.1 Strength of materials solution

Here the displacement approach is used to obtain the solution. Hence, the main assumption here is regarding the displacement field. The assumption

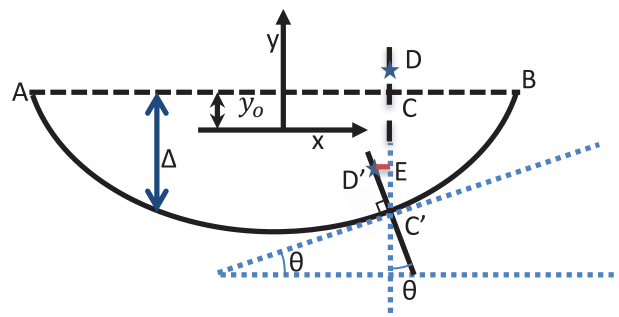



Figure 8.3: Schematic of deformation of a beam. $ACB$ beam before deformation, $AC'B$ beam after deformation.

is that sections that are plane and perpendicular to the neutral axis (to be defined shortly) of the beam remain plane and perpendicular to the deformed neutral axis of the beam, as shown in figure 8.3. Hence, the displacement field is,

$$\mathbf{u} = -(y - y_o)\frac{d\Delta}{dx}\mathbf{e}_x + \Delta(x)\mathbf{e}_y, \tag{8.29}$$

where $\Delta(x)$ is a function of $x$ denotes the displacement of the neutral axis of the beam along the $\mathbf{e}_y$ direction and $y_o$ is a constant. $y_o$ is the $y$ coordinate of the neutral axis before the deformation in the chosen coordinate system and is a constant because the beam is straight. Before proceeding further let us see why the $x$ component of the displacement is as given. The line $D'E$ in figure 8.3 denotes the magnitude of the $x$ component of the displacement. $D'E$ $= C'D'$ $\sin(\theta)$ which is approximately computed as $D'E$ = CD$\theta$, assuming small rotations and that there is no shortening of line segments along the $\mathbf{e}_y$ direction so that $C'D'$ = CD. Now if $y$ coordinate of $C$ is $y_o$ and that of $D$ is $y$, then the length of line segment $CD = (y - y_o)$. Similarly, from the assumption that the plane sections perpendicular to the neutral axis remain perpendicular to the deformed neutral axis, $\theta = \frac{d\Delta}{dx}$, the slope of the tangent of the deformed neutral axis, as shown in the figure 8.3. Therefore $D'E =$ $(y - y_o)\frac{d\Delta}{dx}$ and the $x$ component of the displacement is $-D'E$ since, it is in a direction opposite to the $\mathbf{e}_x$ direction.

Now, the gradient of the displacement field for the assumed displacement

(8.29) is

$$\mathbf{h} = \begin{pmatrix} -[y-y_o]\frac{d^2\Delta}{dx^2} & -\frac{d\Delta}{dx} & 0 \\ \frac{d\Delta}{dx} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{8.30}$$

and therefore the linearized strain is

$$\boldsymbol{\epsilon} = \frac{1}{2}[\mathbf{h} + \mathbf{h}^t] = \begin{pmatrix} -[y-y_o]\frac{d^2\Delta}{dx^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.31}$$

Using the one dimensional constitutive relation, $\sigma_{(\mathbf{n})} = E\epsilon_{(\mathbf{n})}$, where $\sigma_{(\mathbf{n})}$ and $\epsilon_{(\mathbf{n})}$ are the normal stress and strain along the direction $\mathbf{n}$, we obtain

$$\sigma_{xx} = -E[y-y_o]\frac{d^2\Delta}{dx^2}, \tag{8.32}$$

where we have equated the normal stress and strain along the $\mathbf{e}_x$ direction. Substituting equation (8.32) in the equation (8.3) we obtain

$$P = \int_a E[y-y_o]\frac{d^2\Delta}{dx^2}da. \tag{8.33}$$

Since, in a beam there would be no net applied axial load $P = 0$. Solving equation (8.33) for $y_o$ under the assumption that no net axial load is applied,

$$y_o = \frac{\int_a Eyda}{\int_a Eda}. \tag{8.34}$$

If the beam is also homogeneous then Young's modulus, $E$ is a constant and therefore, $y_o = (\int_a yda)/(\int_a da)$, centroid of the cross section. Since, we have assumed that there is no net applied axial load, i.e.,

$$\int_a \sigma_{xx}da = 0, \tag{8.35}$$

equation (8.9) can be written as,

$$M_z = -\int_a y\sigma_{xx}da = -\int_a y\sigma_{xx}da + \int_a y_o\sigma_{xx}da = -\int_a [y-y_o]\sigma_{xx}da, \tag{8.36}$$

where $y_o$ is a constant given in equation (8.34). Substituting equation (8.32) in equation (8.36) we obtain,

$$M_z = \int_a [y - y_o]^2 E \frac{d^2\Delta}{dx^2} da = \frac{d^2\Delta}{dx^2} \int_a [y - y_o]^2 E da. \tag{8.37}$$

If the cross section of the beam is homogeneous, the above equation can be written as,

$$M_z = \frac{d^2\Delta}{dx^2} E \int_a [y - y_o]^2 da = \frac{d^2\Delta}{dx^2} E I_{zz}, \tag{8.38}$$

where,

$$I_{zz} = \int_a [y - y_o]^2 da, \tag{8.39}$$

is the moment of inertia about the $z$ axis.

Combining equations (8.37) and (8.32) we obtain for inhomogeneous beams

$$-\frac{\sigma_{xx}}{E(y - y_o)} = \frac{d^2\Delta}{dx^2} = \frac{M_z}{\int_a [y - y_o]^2 E da}, \tag{8.40}$$

where $y_o$ is as given in equation (8.34). Combining equations (8.38) and (8.32) we obtain for homogeneous beams

$$-\frac{\sigma_{xx}}{(y - y_o)} = E\frac{d^2\Delta}{dx^2} = \frac{M_z}{I_{zz}}, \tag{8.41}$$

where $y_o$ is the $y$ coordinate of the centroid of the cross section which can be taken as 0 without loss of generality provided the origin of the coordinate system used is located at the centroid of the cross section.

Next, we would like to define neutral axis. Neutral axis is defined as the line of intersection of the plane on which the bending stress is zero ($y = y_o$) and the plane along which the resultant load acts.

Equations (8.40) and (8.41) relate the bending stresses and displacement to the bending moment in an inhomogeneous and homogeneous beam respectively and is called as the bending equation. While these equations are sufficient to find all the stresses in a beam subjected to a constant bending moment, one further needs to relate the shear stresses to the shear force that arises when the beam is subjected to a bending moment that varies along the longitudinal axis of the beam. This we shall do next.

Consider a section of the beam, $pqrs$ as shown in figure 8.4 when the beam is subjected to a bending moment that varies along the longitudinal

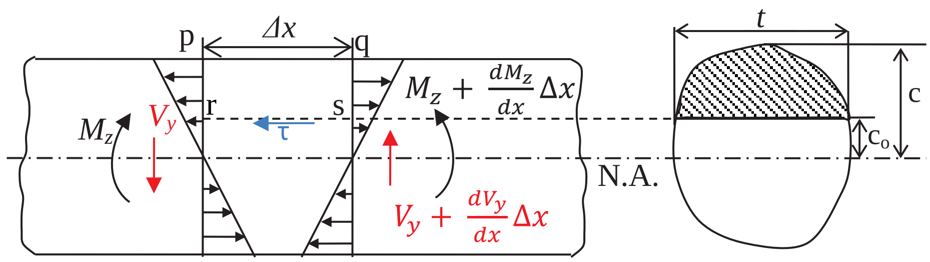



Figure 8.4: Schematic of stresses acting on a beam subjected to varying bending moment

axis of the beam. For the force equilibrium of section $pqrs$, the shear stress, $\tau$, as indicated in the figure 8.4 should be

$$\tau(\Delta X)t = \int_a \left\{ [y - y_o]\frac{1}{I_{zz}}\left[M_z + \frac{dM_z}{dx}\Delta x\right] - [y - y_o]\frac{M_z}{I_{zz}} \right\} da$$
$$= \frac{dM_z}{dx}\frac{\Delta X}{I_{zz}} \int_a [y - y_o] da, \quad (8.42)$$

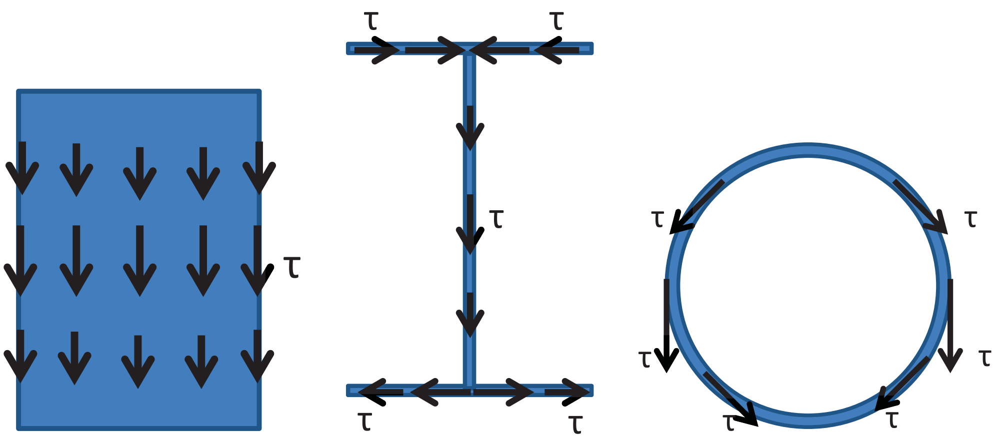
where the integration is over the cross sectional area above the section of interest, i.e., the shaded area shown in figure 8.4 and $t$ is the width of the cross section at the section of interest. Substituting equation (8.18) in (8.42) and simplifying we obtain,

$$\tau = -\frac{V_y}{I_{zz}t}\int_a [y - y_o] da. \quad (8.43)$$

Having obtained the magnitude of this shear stress, next we discuss the direction along which this acts. For thick walled sections with say, $l/b < 20$, as in the case of well proportioned rectangular beams, this shear stress is the $\sigma_{xy}$ (and the complimentary $\sigma_{yx}$) component. For thin walled sections, this shear stress would act tangential to the profile of the cross section as shown in the figure 8.5b and figure 8.5c. Thus it has both the $\sigma_{xy}$ and $\sigma_{xz}$ shear components acting on these sections.

### 8.2.2 2D Elasticity solution

While the strength of material solution is generic in that it can be used for any type of loading and cross sections of any shape as long as it is loaded in




(a) Thick walled rectangular section. (b) Thin walled I section. (c) Thin walled circular section.

Figure 8.5: Direction of the shear stresses acting on a cross section of a beam subjected to varying bending moment

its plane of symmetry, 2D elasticity solution has to be developed for specific cross sections and loading scenarios. Consequently, we assume that the cross section is rectangular in shape with width $2b$ and depth $2c$. Thus, the body is assumed to occupy the region in the Euclidean point space defined by $\mathcal{B} = \{(x, y, z) | -l \leq x \leq l, -c \leq y \leq c, -b \leq z \leq b\}$ before the application of the load. Using the stress formulation introduced in chapter 7, we study the response of this body subjected to three types of load.

### Pure bending of a simply supported beam

Consider the case of a straight beam subject to end moments as shown in figure 8.6. It can be seen from the figure that the top and bottom surfaces are free of traction, i.e.

$$\mathbf{t}_{(\mathbf{e}_y)}(x, c) = \mathbf{o}, \quad \mathbf{t}_{(-\mathbf{e}_y)}(x, -c) = \mathbf{o}. \tag{8.44}$$

The surfaces defined by $x = \pm l$ has traction on its face such that the net force is zero but it results in a bending moment, $M_z$. Representing these

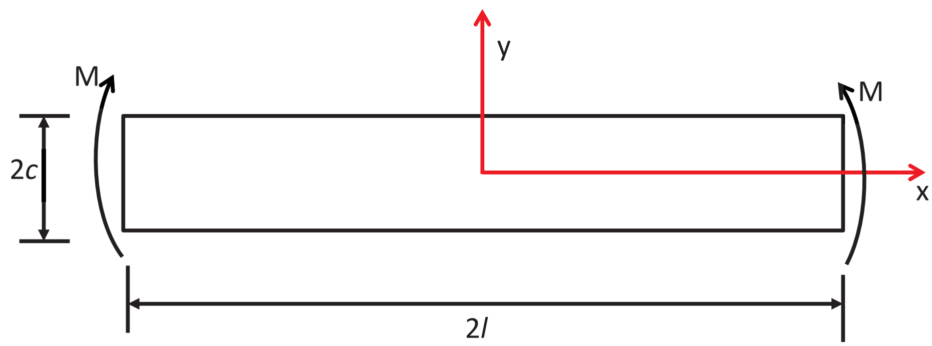



Figure 8.6: Beam subjected to end moments

conditions mathematically,

$$\int_a \mathbf{t}_{(\mathbf{e}_x)}(l,y)dydz = \mathbf{o}, \quad \int_a \mathbf{t}_{(-\mathbf{e}_x)}(-l,y)dydz = \mathbf{o},$$
$$\int_a (y\mathbf{e}_y + z\mathbf{e}_z) \wedge \mathbf{t}_{(\mathbf{e}_x)}(l,y)dydz = M\mathbf{e}_z,$$
$$\int_a (y\mathbf{e}_y + z\mathbf{e}_z) \wedge \mathbf{t}_{(-\mathbf{e}_x)}(-l,y)dydz = -M\mathbf{e}_z. \quad (8.45)$$

Thus, the exact point wise loading on the ends is not considered and only the statically equivalent effect is modeled. Consequently, the boundary conditions on the ends of the beam have been relaxed, and only the statically equivalent condition will be satisfied. This fact leads to a solution that is not necessarily valid at the ends of the beam and could result in more than one stress field satisfying the prescribed conditions.

Assuming the state of stress to be plane, such that

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (8.46)$$

the boundary condition (8.44) translates into requiring,

$$\sigma_{yy}(x, \pm c) = 0, \quad (8.47)$$
$$\sigma_{xy}(x, \pm c) = 0. \quad (8.48)$$

For the assumed state of stress (8.46) the boundary condition (8.45) requires,

$$\begin{aligned}
\int_{-c}^{c} 2b\sigma_{xy}(\pm l, y)dy &= 0, && (8.49)\\
\int_{-b}^{b} zdz \int_{-c}^{c} \sigma_{xy}(\pm l, y)dy &= 0, && (8.50)\\
\int_{-c}^{c} 2b\sigma_{xx}(\pm l, y)dy &= 0, && (8.51)\\
\int_{-c}^{c} 2b\sigma_{xx}(\pm l, y)ydy &= -M. && (8.52)
\end{aligned}$$

Recognize that condition (8.50) holds irrespective of what the variation of the shear stress $\sigma_{xy}$ is.

As discussed in detail in chapter 7 (section 7.3.2), for stress formulation, we assume that the Cartesian components of stress are obtained from a potential, $\phi = \hat{\phi}(x, y)$, called the Airy's stress function through

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \tag{8.53}$$

Then, we have to find the potential such that it satisfies the boundary conditions (8.47) through (8.52) and the bi-harmonic equation, $\Delta(\Delta(\phi)) = 0$.

We express $\phi$ as a power series:

$$\phi(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} x^m y^n, \tag{8.54}$$

where $A_{mn}$ are constant coefficients to be determined from the boundary conditions and the requirement that it satisfy the bi-harmonic equation.

The choice of the stress function is based on the fact that a third order Airy's stress function will give rise to a linear stress field, and this linear boundary loading on the ends $x = \pm l$, will satisfy the requirements (8.49) through (8.52). Based on this observation, we choose the Airy's stress function as

$$\phi = A_{03} y^3 + A_{30} x^3 + A_{21} x^2 y + A_{12} x y^2. \tag{8.55}$$

Then, the stress field takes the form

$$\sigma_{xx} = 6A_{03}y + 2A_{12}x, \quad \sigma_{yy} = 6A_{30}x + 2A_{21}y, \quad \sigma_{xy} = 2[A_{12}y + A_{21}x]. \tag{8.56}$$

Substituting the above stress in boundary conditions (8.47) and (8.48) we find that $A_{30} = A_{21} = A_{12} = 0$. Thus, the Airy's stress function reduces to,

$$\phi = A_{03} y^3, \tag{8.57}$$

and the stress field becomes

$$\sigma_{xx} = 6A_{03} y, \quad \sigma_{yy} = \sigma_{xy} = 0. \tag{8.58}$$

Substituting (8.60) in the boundary conditions (8.49) through (8.52), we find that (8.49) through (8.51) are satisfied identically and (8.52) requires,

$$A_{03} = -\frac{1}{8}\frac{M}{c^3 b}. \tag{8.59}$$

It can be verified that the Airy's stress function (8.57) satisfies the biharmonic equation trivially.

Substituting (8.59) in (8.58) we obtain,

$$\sigma_{xx} = \frac{3M}{4c^3 b} y, \quad \sigma_{yy} = \sigma_{xy} = 0. \tag{8.60}$$

Using the 2 dimensional Hooke's law (7.49), the strain field corresponding to the stress field (8.60) is computed to be

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} = -\frac{3M}{4Ec^3 b} y, \tag{8.61}$$

$$\epsilon_{yy} = \frac{\partial u_y}{\partial y} = \nu \frac{3M}{4Ec^3 b} y, \tag{8.62}$$

$$\epsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right] = 0. \tag{8.63}$$

Integrating (8.61) we obtain

$$u_x = -\frac{3M}{4Ec^3 b} xy + f(y), \tag{8.64}$$

where $f$ is an arbitrary function of $y$. Similarly, integrating (8.62) we obtain

$$u_y = \frac{3M\nu}{8Ec^3 b} y^2 + g(x), \tag{8.65}$$

where $g$ is an arbitrary function of $x$. Substituting equations (8.64) and (8.65) in equation (8.63) and simplifying we obtain

$$-\frac{3M}{4Ec^3b}x + \frac{df}{dy} + \frac{dg}{dx} = 0. \tag{8.66}$$

For equation (8.66) to hold,

$$\frac{df}{dy} = -C_0, \quad \frac{dg}{dx} = C_0 + \frac{3M}{4Ec^3b}x, \tag{8.67}$$

where $C_0$ is a constant. Integrating (8.67) we obtain

$$f(y) = -C_0 y + C_1, \quad g(x) = C_0 x + \frac{3M}{8Ec^3b}x^2 + C_2, \tag{8.68}$$

where $C_1$ and $C_2$ are integration constants. Substituting (8.68) in equations (8.64) and (8.65) we obtain

$$u_x = -\frac{3M}{4Ec^3b}xy - C_0 y + C_1, \quad u_y = \frac{3M}{8Ec^3b}[x^2 + \nu y^2] + C_0 x + C_2. \tag{8.69}$$

The constants $C_i$'s are to be evaluated from displacement boundary conditions. Assuming the beam to be simply supported at the ends $A$ and $B$, we require

$$u_y(\pm l, 0) = 0, \quad \text{and} \quad u_x(-l, 0) = 0, \tag{8.70}$$

where we have assumed the left side support to be hinged (i.e., both the vertical and horizontal displacement is not possible) and the right side support to be a roller (i.e. only vertical displacement is restrained). Substituting (8.69) in (8.70) we obtain

$$C_0 l + C_2 = -\frac{3Ml^2}{8Ec^3b}, \tag{8.71}$$

$$-C_0 l + C_2 = -\frac{3Ml^2}{8Ec^3b}, \tag{8.72}$$

$$C_1 = 0. \tag{8.73}$$

Solving the equations (8.71) and (8.72) for $C_0$ and $C_2$ we obtain

$$C_0 = 0, \quad C_2 = -\frac{3Ml^2}{8Ec^3b}. \tag{8.74}$$

Substituting, equations (8.73) and (8.74) in the equation (8.69) we obtain the displacement field as,

$$\mathbf{u} = -\frac{3M}{4Ec^3b}xy\mathbf{e}_x + \frac{3M}{8Ec^3b}[x^2 + \nu y^2 - l^2]\mathbf{e}_y. \tag{8.75}$$

If the displacement boundary condition is different, then the requirement (8.70) will change and hence the displacement field.

We now wish to compare this elasticity solution with that obtained by strength of materials approach. The bending equation (8.41), for rectangular cross section being studied and the constant moment case reduces to,

$$-\frac{\sigma_{xx}}{y} = \frac{3M}{4c^3b} = E\frac{d^2\Delta}{dx^2}, \tag{8.76}$$

where we have used the fact that for a rectangular cross section of depth $2c$ and width $2b$, $I_{zz} = 4c^3b/3$ and that $y_o = 0$ as the origin is at the centroid of the cross section. Using the first equality in equation (8.76) we obtain the stress field as,

$$\sigma_{xx} = -\frac{3M}{4c^3b}y, \quad \sigma_{yy} = \sigma_{xy} = 0, \tag{8.77}$$

Comparing equations (8.60) and (8.77) we find that the stress field is the same in both the approaches. Then, using the last equality in equation (8.76) we obtain,

$$\Delta = \frac{3M}{4c^3b}\frac{x^2}{2} + D_1x + D_2, \tag{8.78}$$

where $D_1$ and $D_2$ are integration constants to be found from the displacement boundary condition (8.70). This simply supported boundary condition requires that $\Delta(\pm l) = 0$, i.e.,

$$D_1l + D_2 = -\frac{3M}{4c^3b}\frac{l^2}{2}, \tag{8.79}$$

$$-D_1l + D_2 = -\frac{3M}{4c^3b}\frac{l^2}{2}. \tag{8.80}$$

Solving the above equations we obtain,

$$D_1 = 0, \quad D_2 = -\frac{3M}{4c^3b}\frac{l^2}{2}. \tag{8.81}$$

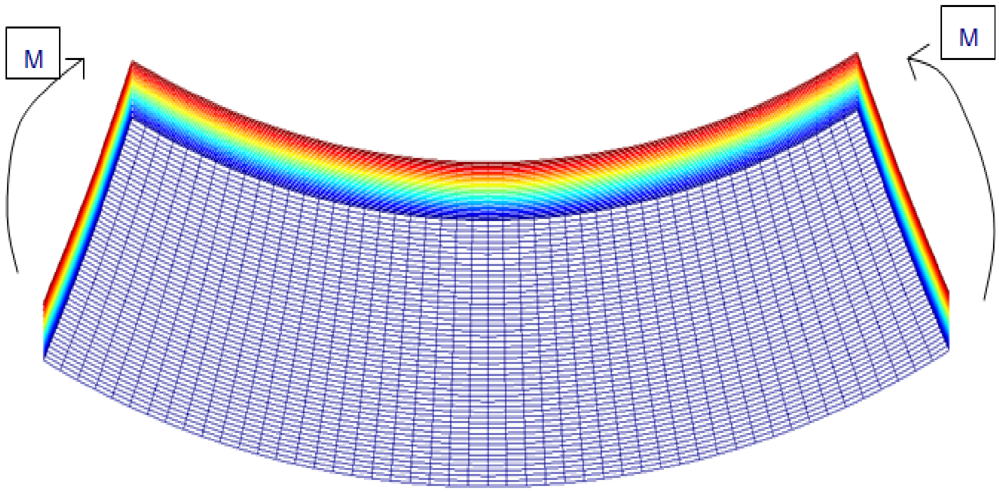



Figure 8.7: Deformed shape of a beam subjected to pure bending as obtained from the elasticity solution

Substituting (8.81) in (8.78) and the resulting equation in (8.29) we obtain

$$\mathbf{u} = -\frac{3M}{4Ec^3b}xy\mathbf{e}_x + \frac{3M}{8c^3b}[x^2 - l^2]\mathbf{e}_y. \tag{8.82}$$

Comparing the strength of materials displacement field (8.82) with that of the elasticity solution (8.75) we find that the $x$ component of the displacement field is the same in both the cases. However, while the $y$ component of the displacement is in agreement with the displacement of the neutral axis, i.e., when $y = 0$, it is not in other cases. This is understandable, as in the strength of material solution we ignored the Poisson's effect and used only a 1D constitutive relation. This means that the length of the filaments oriented along the $y$ direction changes in the elasticity solution, which is explicitly assumed to be zero in the strength of materials solution. Figure 8.7 plots the deformed shape of a beam subjected to pure bending as obtained from the elasticity solution.

We shall find that this near agreement of the elasticity and strength of materials solution for pure bending of the beam does not hold for other loadings, as we shall see next.

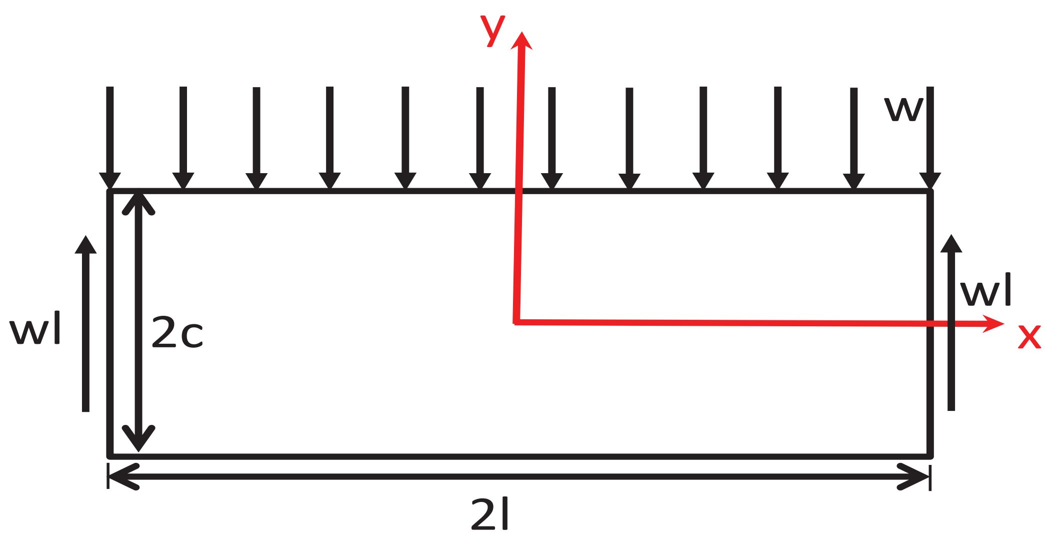



Figure 8.8: Beam carrying uniformly transverse loading.

**Simply supported beam subjected to uniformly distributed transverse loading**

The next problem that we solve is that of a beam carrying a uniformly distributed transverse loading $w$ along its top surface,as shown in Figure 8.8. As before the traction boundary conditions for this problem are

$$\mathbf{t}_{(\mathbf{e}_y)}(x, c) = -\frac{w}{2b}\mathbf{e}_y, \quad (8.83)$$

$$\mathbf{t}_{(-\mathbf{e}_y)}(x, -c) = \mathbf{o}, \quad (8.84)$$

$$\int_{-c}^{c} 2b\mathbf{t}_{(\mathbf{e}_x)}(l, y)dy = wl\mathbf{e}_y, \quad (8.85)$$

$$\int_{-c}^{c} 2b\mathbf{t}_{(-\mathbf{e}_x)}(-l, y)dy = wl\mathbf{e}_y, \quad (8.86)$$

$$\int_{-c}^{c} 2b(y\mathbf{e}_y + z\mathbf{e}_z) \wedge \mathbf{t}_{(\mathbf{e}_x)}(l, y)dy = \mathbf{o}, \quad (8.87)$$

$$\int_{-c}^{c} 2b(y\mathbf{e}_y + z\mathbf{e}_z) \wedge \mathbf{t}_{(-\mathbf{e}_x)}(-l, y)dy = \mathbf{o}. \quad (8.88)$$

While exact point wise conditions are specified on the top and bottom surfaces, at the right and left surfaces the resultant horizontal axial force and moment are set to zero and the resultant vertical shear force is specified such that it satisfies the overall equilibrium. As before assuming plane stress conditions, (8.46), the boundary conditions (8.83) through (8.88) evaluates to,

$$\sigma_{xy}(x, \pm c) = 0, \tag{8.89}$$
$$\sigma_{yy}(x, -c) = 0, \tag{8.90}$$
$$\sigma_{yy}(x, c) = -\frac{w}{2b}, \tag{8.91}$$
$$\int_{-c}^{c} \sigma_{xx}(\pm l, y) dy = 0, \tag{8.92}$$
$$\int_{-c}^{c} \sigma_{xy}(l, y) dy = wl, \tag{8.93}$$
$$\int_{-c}^{c} \sigma_{xy}(-l, y) dy = -wl, \tag{8.94}$$
$$\int_{-c}^{c} \sigma_{xx}(\pm l, y) y dy = 0. \tag{8.95}$$

Thus, we have to chose Airy's stress function such that conditions (8.89) through (8.95) holds along with the bi-harmonic equation.

Seeking Airy's stress function in the form of the polynomial (8.54), we try the following form,

$$\phi = A_{20}x^2 + A_{21}x^2y + A_{03}y^3 + A_{23}x^2y^3 + A_{05}y^5, \tag{8.96}$$

where $A_{ij}$'s are constants. For this polynomial to satisfy the bi-harmonic equation,

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0, \tag{8.97}$$

it is required that

$$A_{05} = -\frac{A_{23}}{5}. \tag{8.98}$$

For this assumed form for the Airy's stress function (8.96), the stress field found using equation 8.53 is,

$$\sigma_{xx} = 6A_{03}y + 6A_{23}(x^2y - \frac{2}{3}y^3), \tag{8.99}$$
$$\sigma_{yy} = 2A_{20} + 2A_{21}y + 2A_{23}y^3, \tag{8.100}$$
$$\sigma_{xy} = -2A_{21}x - 6A_{23}xy^2, \tag{8.101}$$

where we have used (8.98).

Substituting equation (8.101) in the boundary condition (8.89) we obtain,

$$A_{21} + 3A_{23}c^2 = 0. \tag{8.102}$$

Then, using equation (8.100) in the boundary conditions (8.90),(8.91) we obtain,

$$2A_{20} - 2A_{21}c - 2A_{23}c^3 = 0, \tag{8.103}$$
$$2A_{20} + 2A_{21}c + 2A_{23}c^3 = -\frac{w}{2b}. \tag{8.104}$$

Solving the above three equations for the constants $A_{20}$, $A_{21}$ and $A_{23}$, we obtain

$$A_{20} = -\frac{w}{8b}, \quad A_{21} = -\frac{3w}{16bc}, \quad A_{23} = \frac{w}{16bc^3}. \tag{8.105}$$

Next, substituting equation (8.99) in the boundary condition (8.92) we find that it holds for any choice of the remaining constant, $A_{03}$. Similarly, when the constants, $A_{ij}$ are as given in (8.105), the shear stress (8.101) satisfies the boundary conditions (8.93) and (8.94). Thus, substituting equation (8.99) in the boundary condition (8.95) we obtain

$$A_{03} = -A_{23}(l^2 - \frac{2}{5}c^2) = -\frac{w}{16bc}\left[\frac{l^2}{c^2} - \frac{2}{5}\right]. \tag{8.106}$$

Thus, the assumed form for the Airy's stress function (8.96) with the appropriate choice for the constants, satisfies the required boundary conditions and the bi-harmonic equation and therefore is a solution to the given boundary value problem.

Substituting the values for the constants from equations (8.105) and (8.106) in equations (8.99) through (8.101), the resulting stress field can be written as,

$$\sigma_{xx} = -\frac{3w}{8bc}\left[\frac{l^2}{c^2} - \frac{2}{5}\right] y + \frac{3w}{8bc^3}\left[x^2 y - \frac{2}{3}y^3\right], \tag{8.107}$$

$$\sigma_{yy} = -\frac{w}{4b}\left[1 + \frac{3}{2}\frac{y}{c} - \frac{1}{2}\frac{y^3}{c^3}\right], \tag{8.108}$$

$$\sigma_{xy} = \frac{3wx}{8bc}\left[1 - \frac{y^2}{c^2}\right]. \tag{8.109}$$

Having computed the stress fields, next we determine the displacement field. As usual, we use the 2 dimensional constitutive relation (7.49) to obtain the strain field as

$$\begin{aligned}\epsilon_{xx} &= \frac{\partial u_x}{\partial x} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} \\ &= \frac{w}{4Eb}\left\{\frac{3y}{2c^3}\left[x^2 - \frac{2}{3}y^2\right] - \frac{3y}{2c}\left[\frac{l^2}{c^2} - \frac{2}{5}\right] + \nu\left[1 + \frac{3}{2}\frac{y}{c} - \frac{1}{2}\frac{y^3}{c^3}\right]\right\},\end{aligned} \tag{8.110}$$

$$\begin{aligned}\epsilon_{yy} &= \frac{\partial u_y}{\partial y} = \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{xx}}{E} \\ &= \frac{w}{4Eb}\left\{-\left[1 + \frac{3}{2}\frac{y}{c} - \frac{1}{2}\frac{y^3}{c^3}\right] - \frac{3y\nu}{2c^3}\left[x^2 - \frac{2}{3}y^2\right] + \frac{3y\nu}{2c}\left[\frac{l^2}{c^2} - \frac{2}{5}\right]\right\},\end{aligned} \tag{8.111}$$

$$\epsilon_{xy} = \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right] = \sigma_{xy}\frac{(1+\nu)}{E} = \frac{(1+\nu)}{E}\frac{3wx}{8bc}\left[1 - \frac{y^2}{c^2}\right]. \tag{8.112}$$

Integrating (8.110) we obtain

$$u_x = \frac{w}{4Eb}\left\{\frac{yx}{2c^3}\left[x^2 - 2y^2 - 3l^2 + \frac{3}{5}c^2\right] + \nu x\left[1 + \frac{3}{2}\frac{y}{c} - \frac{1}{2}\frac{y^3}{c^3}\right] + f(y)\right\}, \tag{8.113}$$

where $f(y)$ is a yet to be determined function of $y$. Similarly integrating (8.111),

$$\begin{aligned}u_y = -\frac{w}{4Eb}\Bigg\{y + \frac{3}{4}\frac{y^2}{c} - \frac{1}{8}\frac{y^4}{c^3} + \frac{3y^2\nu}{4c^3}\left[x^2 - \frac{1}{3}y^2\right] - \frac{3y^2\nu}{4c}\left[\frac{l^2}{c^2} - \frac{2}{5}\right] \\ - g(x)\Bigg\},\end{aligned} \tag{8.114}$$

where $g(x)$ is a yet to be determined function of $x$.

Substituting equations (8.113) and (8.114) in (8.112) and simplifying we obtain

$$\frac{x^3}{2c^3} - \frac{27}{10}\frac{x}{c} - \frac{3\nu}{2}\frac{x}{c} - \frac{3}{2}\frac{l^2 x}{c^3} + \frac{df}{dy} + \frac{dg}{dx} = 0. \tag{8.115}$$

For equation (8.115) to hold,

$$\frac{df}{dy} = C_0, \quad \frac{dg}{dx} = -\frac{x^3}{2c^3} + \frac{27}{10}\frac{x}{c} + \frac{3\nu}{2}\frac{x}{c} + \frac{3}{2}\frac{l^2 x}{c^3} + C_0, \tag{8.116}$$

where $C_0$ is a constant. Integrating the differential equation (8.115) we obtain

$$f(y) = C_0 y + C_1, \tag{8.117}$$

$$g(x) = \frac{x^2}{2c}\left[\frac{27}{10} + \frac{3\nu}{2} + \frac{3l^2}{2c^2}\right] - \frac{x^4}{8c^3} + C_0 x + C_2, \tag{8.118}$$

$$\tag{8.119}$$

where $C_1$ and $C_2$ are constants to be determined from the displacement boundary condition. Assuming the beam to be simply supported at the ends $A$ and $B$, we require

$$u_y(\pm l, 0) = 0, \quad \text{and} \quad u_x(-l, 0) = 0, \tag{8.120}$$

where we have assumed the left side support to be hinged (i.e., both the vertical and horizontal displacement is not possible) and the right side support to be a roller (i.e. only vertical displacement is restrained). Substituting (8.114) and (8.118) in (8.120a) we obtain

$$C_0 l + C_2 = \frac{l^4}{8c^3} - \frac{l^2}{2c}\left[\frac{27}{10} + \frac{3\nu}{2} + \frac{3l^2}{2c^2}\right], \tag{8.121}$$

$$-C_0 l + C_2 = \frac{l^4}{8c^3} - \frac{l^2}{2c}\left[\frac{27}{10} + \frac{3\nu}{2} + \frac{3l^2}{2c^2}\right]. \tag{8.122}$$

Solving equations (8.121) and (8.122) for $C_0$ and $C_2$,

$$C_0 = 0, \quad C_2 = \frac{l^4}{8c^3} - \frac{l^2}{2c}\left[\frac{27}{10} + \frac{3\nu}{2} + \frac{3l^2}{2c^2}\right]. \tag{8.123}$$

Substituting equations (8.113), (8.117) and (8.123) in (8.120b), we obtain

$$C_1 = -\nu l. \tag{8.124}$$

Thus, the final form of the displacements is given by

$$u_x = \frac{3w}{8Ebc^3}\left\{\frac{yx}{3}\left[x^2 - 2y^2 - 3l^2 + \frac{3}{5}c^2\right] + \frac{2c^3\nu}{3}x\left[1 - l + \frac{3}{2}\frac{y}{c} - \frac{1}{2}\frac{y^3}{c^3}\right]\right\}, \tag{8.125}$$

$$u_y = -\frac{3w}{8Ebc^3}\left\{\left[y + \frac{3}{4}\frac{y^2}{c} - \frac{1}{8}\frac{y^4}{c^3}\right]\frac{2c^3}{3} + \frac{y^2\nu}{2}\left[x^2 - \frac{1}{3}y^2\right] - \frac{y^2\nu}{2c^2}\left[\frac{l^2}{c^2} - \frac{2}{5}\right]\right.$$
$$\left. - \frac{x^2c^2}{3}\left[\frac{27}{10} + \frac{3\nu}{2} + \frac{3l^2}{2c^2}\right] + \frac{x^4}{12} - \frac{l^4}{12} + \frac{l^2c^2}{3}\left[\frac{27}{10} + \frac{3\nu}{2} + \frac{3l^2}{2c^2}\right]\right\}. \tag{8.126}$$

In order to facilitate the comparison of this elasticity solution with that obtained from the strength of materials approach, we rewrite the stress and displacements field obtained using the elasticity approach in terms of the moment of inertia of the rectangular cross section of depth $2c$ and width $2b$, $I_{zz} = 4bc^3/3$, as

$$\sigma_{xx} = \frac{w}{2I_{zz}}\left[x^2 - l^2 + \frac{2c^2}{5} - \frac{2y^2}{3}\right]y, \tag{8.127}$$

$$\sigma_{yy} = \frac{w}{2I_{zz}}\left(\frac{y^3}{3} - c^2y - \frac{2}{3}c^3\right), \tag{8.128}$$

$$\sigma_{xy} = \frac{w}{2I_{zz}}x\left(c^2 - y^2\right). \tag{8.129}$$

$$u_x = \frac{w}{2EI_{zz}}\left\{\frac{yx}{3}\left[x^2 - 2y^2 - 3l^2 + \frac{3}{5}c^2\right] + \frac{2c^3\nu}{3}x\left[1 - l + \frac{3}{2}\frac{y}{c} - \frac{1}{2}\frac{y^3}{c^3}\right]\right\}, \tag{8.130}$$

$$u_y = -\frac{w}{2EI_{zz}}\left\{\left[y + \frac{3}{4}\frac{y^2}{c} - \frac{1}{8}\frac{y^4}{c^3}\right]\frac{2c^3}{3} + \frac{y^2\nu}{2}\left[x^2 - \frac{1}{3}y^2\right] - \frac{y^2\nu}{2c^2}\left[\frac{l^2}{c^2} - \frac{2}{5}\right]\right.$$
$$\left. - \frac{x^2c^2}{3}\left[\frac{27}{10} + \frac{3\nu}{2} + \frac{3l^2}{2c^2}\right] + \frac{x^4}{12} - \frac{l^4}{12} + \frac{l^2c^2}{3}\left[\frac{27}{10} + \frac{3\nu}{2} + \frac{3l^2}{2c^2}\right]\right\}. \tag{8.131}$$

Now, we obtain the stress and displacement field from the strength of materials approach. The bending equation (8.41) for this boundary value problem reduces to

$$-\frac{\sigma_{xx}}{y} = \frac{w}{2I_{zz}}\left[l^2 - x^2\right] = E\frac{d^2\Delta}{dx^2}, \tag{8.132}$$

where we have taken $y_o = 0$ as the origin of the coordinate system coincides with the centroid of the cross section and substituted for bending moment,

$$M_z = wl(l+x) - \frac{w}{2}(l+x)^2 = \frac{w}{2}\left[l^2 - x^2\right]. \tag{8.133}$$

From the first equality in equation (8.132) we obtain,

$$\sigma_{xx} = -\frac{w}{2I_{zz}}\left[l^2 - x^2\right] y, \tag{8.134}$$

Using the equation (8.43) the shear stress, $\sigma_{xy}$ for this cross section and loading is estimated as,

$$\sigma_{xy} = -\frac{V_y}{I_{zz}2b}\int_{-b}^{b} dz \int_{y}^{c} y dy, \tag{8.135}$$

where we have used $y_o = 0$. Noting that $V_y = wl - w(l+x) = -wx$, equation (8.135) simplifies to

$$\sigma_{xy} = \frac{w}{2I_{zz}} x\left[c^2 - y^2\right], \tag{8.136}$$

In strength of materials solution we do not account for the variation of the $\sigma_{yy}$ component of the stress. Hence,

$$\sigma_{yy} = 0, \tag{8.137}$$

From solving the ordinary differential equation in the last equality in equation (8.132) we obtain,

$$\Delta = \frac{w}{4EI_{zz}}\left[l^2x^2 - \frac{x^4}{6} + D_1 x + D_2\right], \tag{8.138}$$

where $D_1$ and $D_2$ are constants to be found from the displacement boundary condition (8.120). The boundary condition (8.120a) requires that

$$D_1 l + D_2 = -\frac{5}{6}l^4, \tag{8.139}$$

$$-D_1 l + D_2 = -\frac{5}{6}l^4. \tag{8.140}$$

Solving equations (8.139) and (8.140) for the constants $D_1$ and $D_2$ we obtain

$$D_1 = 0, \qquad D_2 = -\frac{5}{6}l^4. \tag{8.141}$$

Substituting (8.141) in equation (8.138) we obtain

$$\Delta = \frac{w}{4EI_{zz}}\left[l^2x^2 - \frac{x^4}{6} - \frac{5}{6}l^4\right]. \tag{8.142}$$

Hence the displacement field in a simply supported beam subjected to transverse loading obtained from strength of materials approach is,

$$\mathbf{u} = -y\frac{w}{4EI_{zz}}\left[2l^2x - \frac{2x^3}{3}\right]\mathbf{e}_x + \frac{w}{4EI_{zz}}\left[l^2x^2 - \frac{x^4}{6} - \frac{5}{6}l^4\right]\mathbf{e}_y. \tag{8.143}$$

Before concluding this section let us compare the 2 dimensional elasticity solution with the strength of material solution. Comparing equations (8.136) with equation (8.129), we find that identical shear stress, $\sigma_{xy}$ variation is obtained in both the approaches. However, comparing equations (8.134) and (8.127) we find that the expression for the bending stress, $\sigma_{xx}$ obtained by both these approaches are different. First observe that in strength of materials solution $\sigma_{xx}(\pm l, y) = 0$. However, in the elasticity solution, $\sigma_{xx}(\pm l, y) = wy[c^2/5 - y^2/3]/I_{zz}$. Figure 8.9 plots the variation of $\sigma_{xx}(\pm l, y)2b/w$ with respect to $y/c$. Moreover, at any section we find that the bending stress, $\sigma_{xx}$ varies nonlinearly with respect to $y$ in the elasticity solution. To understand how different the elasticity solution is from the strength of materials solution, in figure 8.10 we plot both the variation of the bending normal stress, $\sigma_{xx}(0, y)2b/w$ as a function of $y/c$ for various values of $l/c$. It can be seen from the figure that for values of $l/c \leq 1$ the differences are significant but as the value of $l/c$ tends to get larger the differences diminishes. Also notice that the maximum bending stress $\max(\sigma_{xx}(0, y))$ varies quadratically as a function of $l/c$. In figure 8.11 we plot the variation of the stress $\sigma_{yy}2b/w$ with $y/c$ to find that its magnitude is less than 1. Thus, for typical beams with $l/c > 10$, the bending stresses $\sigma_{xx}$ is 100 times more than these other stresses that they can be ignored, as done in strength of materials solution.

Having examined the difference in the stresses let us now examine the displacements. The maximum deflection of the neutral axis of the beam in the elasticity solution is

$$u_y^{max} = u_y(0, 0) = -\frac{5wl^4}{24EI_{zz}}\left\{1 + \frac{c^2}{l^2}\left[\frac{54}{25} + \frac{6\nu}{5}\right]\right\}, \tag{8.144}$$

obtained from equation (8.131). The corresponding value calculated from strength of materials solution (8.142) is

$$\Delta_{max} = \Delta(0) = -\frac{5wl^4}{24EI}. \tag{8.145}$$

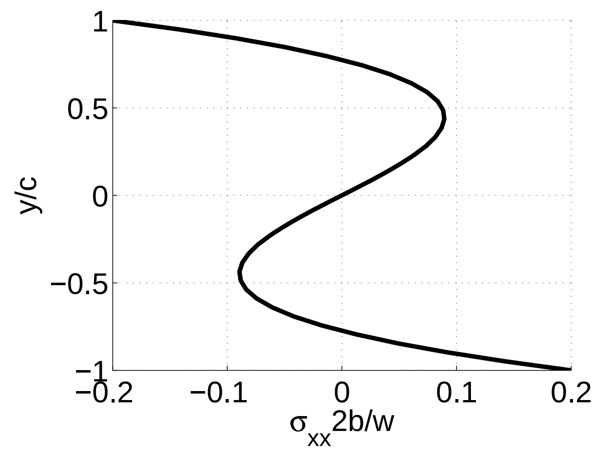
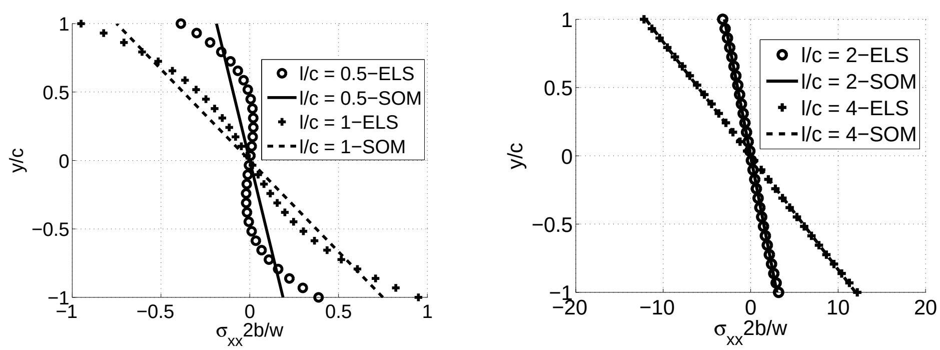



Figure 8.9: Variation of the stress $\sigma_{xx}$ at the supports along the depth of the simply supported beam subjected to transverse loading




(a) For values of $l/c \leq 1$, large differences between elasticity and strength of materials solution.

(b) For values of $l/c > 1$, negligible differences between elasticity and strength of materials solution.

Figure 8.10: Variation of the stress $\sigma_{xx}$ at mid span of a simply supported beam subjected to transverse loading

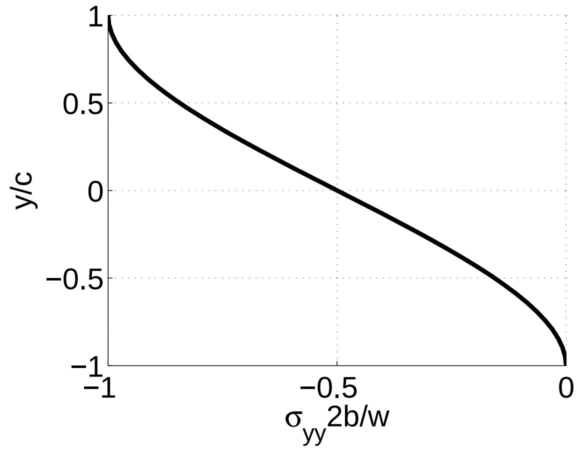



Figure 8.11: Variation of the stress $\sigma_{yy}$ along the depth of the simply supported beam subjected to transverse loading

Comparing equations (8.144) and (8.145) it can be seen that when $l/c \gg 1$ the results are approximately the same. Thus, again we find that for long beams with $l/c > 10$ the strength of materials solution is close to the elasticity solution. Note that from equation 8.125, the $x$ component of displacement indicates that plane sections do not remain plane. However, for well proportioned beams, i.e. beams with $l/c > 10$, the deviation from being plane is insignificant.

**Simply supported beam subjected to sinusoidal loading**

Finally, we consider a simply supported rectangular cross section beam subjected to sinusoidally varying transverse load along its top edge as shown in Fig 8.12. Now, we shift the coordinate origin to the left end surface of the beam. Consequently, the beam in its initial state is assumed to occupy a region in the Euclidean point space defined by $\mathcal{B} = \{(x, y, z)|0 \leq x \leq 2l, -c \leq y \leq c, -b \leq z \leq b\}$.

The traction boundary conditions for this problem are

$$\mathbf{t}_{(\mathbf{e}_y)}(x, c) = -\frac{q_o}{2b}\sin\left(\pi\frac{x}{l}\right)\mathbf{e}_y, \tag{8.146}$$

$$\mathbf{t}_{(-\mathbf{e}_y)}(x, -c) = \mathbf{o}, \tag{8.147}$$

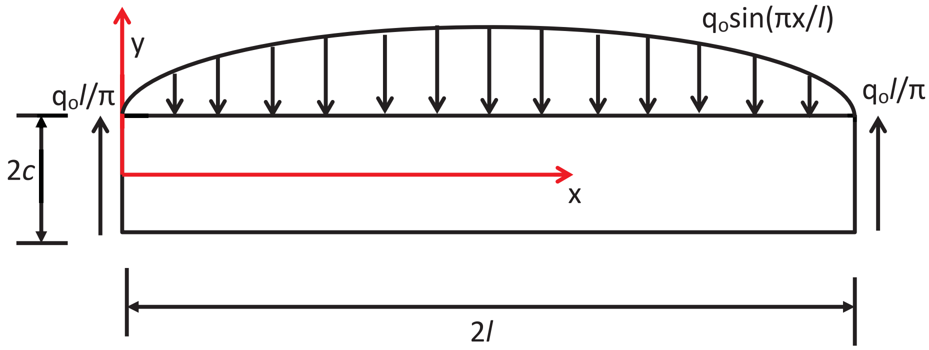



Figure 8.12: Simply supported beam subjected to sinusoidally varying transverse load

$$\int_{-c}^{c} 2b\mathbf{t}_{(\mathbf{e}_x)}(l, y)dy = \frac{q_o}{\pi} l\mathbf{e}_y, \tag{8.148}$$

$$\int_{-c}^{c} 2b\mathbf{t}_{(-\mathbf{e}_x)}(0, y)dy = \frac{q_o}{\pi} l\mathbf{e}_y, \tag{8.149}$$

$$\int_{-c}^{c} 2b(y\mathbf{e}_y + z\mathbf{e}_z) \wedge \mathbf{t}_{(\mathbf{e}_x)}(l, y)dy = \mathbf{o}, \tag{8.150}$$

$$\int_{-c}^{c} 2b(y\mathbf{e}_y + z\mathbf{e}_z) \wedge \mathbf{t}_{(-\mathbf{e}_x)}(0, y)dy = \mathbf{o}. \tag{8.151}$$

The conditions (8.146) through (8.151) on assuming that the beam is subjected to a plane state of stress, translates into requiring

$$\sigma_{xy}(x, \pm c) = 0, \tag{8.152}$$

$$\sigma_{yy}(x, c) = -\frac{q_0}{2b}\sin(\pi x/l), \tag{8.153}$$

$$\sigma_{yy}(x, -c) = 0, \tag{8.154}$$

$$\int_{-c}^{c} 2b\sigma_{xy}(0, y)dy = -q_0 l/\pi, \tag{8.155}$$

$$\int_{-c}^{c} 2b\sigma_{xy}(l, y)dy = q_0 l/\pi, \tag{8.156}$$

$$\int_{-c}^{c} 2b\sigma_{xx}(0,y)dy = 0, \tag{8.157}$$
$$\int_{-c}^{c} 2b\sigma_{xx}(l,y)dy = 0, \tag{8.158}$$

$$\int_{-c}^{c} 2b\sigma_{xx}(0,y)ydy = 0, \tag{8.159}$$
$$\int_{-c}^{c} 2b\sigma_{xx}(l,y)ydy = 0, \tag{8.160}$$
$$\tag{8.161}$$

As before, we solve this boundary value problem using stress approach. Towards this, we assume the following form for the Airy's stress function

$$\phi = \sin(\beta x)\left\{[A_1 + A_3\beta y]\sinh(\beta y) + [A_2 + A_4\beta y]\cosh(\beta y)\right\}, \tag{8.162}$$

where $A_i$'s are constants to be determined from boundary conditions. It is straightforward to verify that the above choice of Airy's stress function, (8.162) satisfies the bi-harmonic equation.

Then, the Cartesian components of the stress for the assumed Airy's stress function (8.162) is

$$\sigma_{xx} = \beta^2\sin(\beta x)\{A_1\sinh(\beta y) + A_3\left[y\beta\sinh(\beta y) + 2\cosh(\beta y)\right] + A_2\cosh(\beta y) + A_4\left[\beta y\cosh(\beta y) + 2\sinh(\beta y)\right]\}, \tag{8.163}$$

$$\sigma_{yy} = -\beta^2\sin(\beta x)\left\{[A_1 + A_3\beta y]\sinh(\beta y) + [A_2 + A_4\beta y]\cosh(\beta y)\right\}, \tag{8.164}$$

$$\sigma_{xy} = \beta^2\cos(\beta x)\{A_1\cosh(\beta y) + A_3\left[\beta y\cosh(\beta y) + \sinh(\beta y)\right] + A_2\sinh(\beta y) + A_4\left[\beta y\sinh(\beta y) + \cosh(\beta y)\right]\}. \tag{8.165}$$

Now applying the boundary condition we evaluate the constants, $A_i$'s. The condition 8.152 implies that

$$A_1\cosh(\beta c) + A_3\left[\beta c\cosh(\beta c) + \sinh(\beta c)\right] + A_2\sinh(\beta c) + A_4\left[\beta c\sinh(\beta c) + \cosh(\beta c)\right] = 0. \tag{8.166}$$

$$A_1 \cosh(\beta c) - A_3 \left[\beta c \cosh(\beta c) + \sinh(\beta c)\right] - A_2 \sinh(\beta c) \\ + A_4 \left[\beta c \sinh(\beta c) + \cosh(\beta c)\right] = 0. \quad (8.167)$$

Adding equations (8.166) and (8.167) we obtain,

$$A_1 \cosh(\beta c) + A_4 \left[\beta c \sinh(\beta c) + \cosh(\beta c)\right] = 0, \quad (8.168)$$

Subtracting equations (8.166) and (8.167) we obtain,

$$A_2 \sinh(\beta c) + A_3 \left[\beta c \cosh(\beta c) + \sinh(\beta c)\right] = 0. \quad (8.169)$$

We write $A_1$ in terms of $A_4$ using equation (8.168) as

$$A_1 = -A_4[\beta c \tanh(\beta c) + 1]. \quad (8.170)$$

Similarly, we write $A_2$ in terms of $A_3$ using equation (8.169) as

$$A_2 = -A_3[\beta c \coth(\beta c) + 1]. \quad (8.171)$$

Substituting equations (8.170) and (8.171) in (8.164)

$$\sigma_{yy} = -\beta^2 \sin(\beta x)\{A_4 \left[\beta y \cosh(\beta y) - (\beta c \tanh(\beta c) + 1) \sinh(\beta y)\right] \\ + A_3 \left[\beta y \sinh(\beta y) - (\beta c \coth(\beta c) + 1) \cosh(\beta y)\right]\}. \quad (8.172)$$

Applying boundary condition (8.154) we obtain a relation between $A_3$ and $A_4$ as

$$A_3 = -A_4 \tanh(\beta c) \frac{\beta c - \sinh(\beta c) \cosh(\beta c)}{\beta c + \sinh(\beta c) \cosh(\beta c)}. \quad (8.173)$$

The boundary condition (8.153) requires

$$q_0 \sin(\frac{\pi x}{l}) = 2\beta^2 \sin(\beta x) \left[\frac{\beta c - \sinh(\beta c) \cosh(\beta c)}{\cosh(\beta c)}\right] A_4. \quad (8.174)$$

In order for equation 8.174 to be true for all $x$, $\beta = \pi/l$, and so $A_4$ is determined as,

$$A_4 = \frac{q_0 \cosh\left(\frac{\pi c}{l}\right)}{4b\frac{\pi^2}{l^2}\left[\frac{\pi c}{l} - \sinh\left(\frac{\pi c}{l}\right) \cosh\left(\frac{\pi c}{l}\right)\right]}. \quad (8.175)$$

It can be verified that for these choice of constants, boundary conditions (8.155) through (8.160) is satisfied. Thus, we have found a stress function that satisfies the bi-harmonic equation and the traction boundary conditions and therefore is a solution to the boundary value problem.

The displacements are determined through integration of the strain displacement relations. Since the steps in its computation is same as in the above two examples, only the final results are recorded here

$$\begin{aligned} u_x = -\frac{\beta}{E}\cos(\beta x)\{&A_1[1+\nu]\sinh(\beta y) + A_2[1+\nu]\cosh(\beta y) \\ &+ A_3\left[(1+\nu)\beta y\sinh(\beta y) + 2\cosh(\beta y)\right] \\ &+ A_4\left[(1+\nu)\beta y\cosh(\beta y) + 2\sinh(\beta y)\right]\} - C_0 y + C_1, \end{aligned} \quad (8.176)$$

$$\begin{aligned} u_y = -\frac{\beta}{E}\sin(\beta x)\{&A_1[1+\nu]\cosh(\beta y) + A_2[1+\nu]\sinh(\beta y) \\ &+ A_3\left[(1+\nu)\beta y\cosh(\beta y) - (1-\nu)\sinh(\beta y)\right] \\ &+ A_4\left[(1+\nu)\beta y\sinh(\beta y) - (1-\nu)\cosh(\beta y)\right]\} + C_0 x + C_2, \end{aligned} \quad (8.177)$$

where $C_i$'s are constants to be determined from displacement boundary conditions. As before, to model a simply supported beam, we choose displacement boundary conditions as

$$u_x(0,0) = u_y(0,0) = u_y(l,0) = 0. \quad (8.178)$$

The constant $C_i$'s determined using the above displacement conditions is

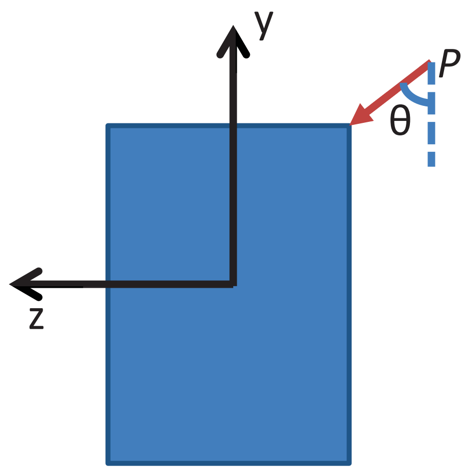
$$C_0 = C_2 = 0, \quad C_1 = \frac{\beta}{E}\left[A_2\,(1+\nu) + 2A_3\right]. \quad (8.179)$$

In order to facilitate comparison of this displacement field with that obtained from strength of materials approach, the vertical displacement of the centerline is determined from equation (8.177) as

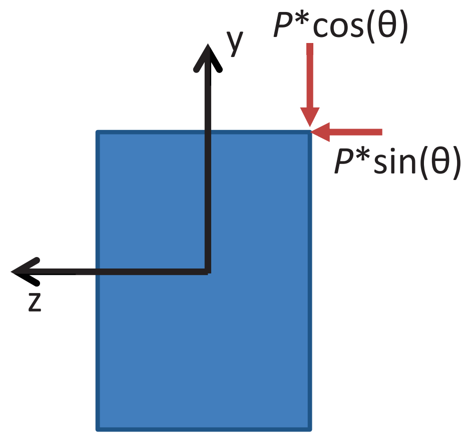
$$u_y(x,0) = \frac{A_4\beta}{E}\sin(\beta x)\left[2 + (1+\nu)\beta c\tanh(\beta c)\right]. \quad (8.180)$$

For the case when $l \gg c$, we approximately compute $A_4$ as $A_4 \approx -3q_0 l^5/8bc^3\pi^5$, and so the equation (8.180) becomes

$$u_y(x,0) = -\frac{3q_0 l^4}{4bc^3\pi^4 E}\sin\left(\frac{\pi x}{l}\right)\left[1 + \frac{1+\nu}{2}\frac{\pi c}{l}\tanh\left(\frac{\pi c}{l}\right)\right]. \quad (8.181)$$




(a) Actual loading.




(b) Load resolved along the axis of symmetry.

Figure 8.13: Schematic of transverse loading of a beam with rectangular cross section

Without going into the details, the vertical deflection of the beam computed using strength of materials approach is,

$$\Delta = -\frac{3q_0l^4}{4bc^3\pi^4E}\sin\left(\frac{\pi x}{l}\right). \tag{8.182}$$

When $l/c > 10$ it can be seen that both the elasticity and strength of materials solution is in agreement as expected.

## 8.3 Asymmetrical bending

Having shown that for the symmetrical bending the strength of material approximation is robust when $l/c > 10$, we proceed to use the same approximation for asymmetrical bending. In the case of asymmetrical bending the loading plane does not coincide with the plane of symmetry. As an illustration, consider the rectangular section, loaded as shown in the figure 8.13a. The applied load can be resolved along the symmetry plane, as shown in figure 8.13b. Thus, the beam bends in both the $xy$ plane due to loading along $y$ direction and the $xz$ plane due to loading along the $z$ direction. As per the strength of materials assumption that the plane section before deformation remain plane and that the sections normal to the neutral axis, remain normal after the deformation, the displacement field for this case is,

$$\mathbf{u} = -\left\{[y - y_o]\frac{d\Delta_y}{dx} + [z - z_o]\frac{d\Delta_z}{dx}\right\}\mathbf{e}_x + \Delta_y(x)\mathbf{e}_y + \Delta_z(x)\mathbf{e}_z, \qquad (8.183)$$

where $\Delta_y(x)$ and $\Delta_z(x)$ are yet to be determined functions of $x$, $y_o$ and $z_o$ are constants. The above displacement field is obtained by superposing the bending displacement due to transverse loading along one symmetric plane, say $xy$ plane, (8.29) and the displacement filed due to loading along another symmetric plane say $xz$ (obtained by substituting $z$ in place of $y$ in equation (8.29)). Recollect from section 7.5.2 that for the linear elastic material that we are studying, we can superpose solutions as long as the displacements are small.

As required in the displacement approach, we next compute the linearized strain corresponding to the displacement field, (8.183) as

$$\boldsymbol{\epsilon} = \begin{pmatrix} -\left\{[y - y_o]\frac{d^2\Delta_y}{dx^2} + [z - z_o]\frac{d^2\Delta_z}{dx^2}\right\} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (8.184)$$

Then, using the 1 dimensional constitutive relation, the stress is obtained as

$$\sigma_{xx} = -E\left\{[y - y_o]\frac{d^2\Delta_y}{dx^2} + [z - z_o]\frac{d^2\Delta_z}{dx^2}\right\}. \qquad (8.185)$$

Substituting (8.185) in equation (8.3) and using the condition that no axial load is applied we obtain

$$\iint_a E\left\{[y - y_o]\frac{d^2\Delta_y}{dx^2} + [z - z_o]\frac{d^2\Delta_z}{dx^2}\right\}dydz = 0. \qquad (8.186)$$

For equation (8.186) to hold we require that

$$\iint_a E[y - y_o]dydz = 0, \qquad \iint_a E[z - z_o]dydz = 0, \qquad (8.187)$$

since in equation (8.186) $\Delta_y$ and $\Delta_z$ are independent functions of $x$ and in particular $\Delta_y \neq -k\Delta_z$, where $k$ is a constant. From equation (8.187) we obtain,

$$y_o = \frac{\iint_a Eydydz}{\iint_a Edydz}, \qquad z_o = \frac{\iint_a Ezdydz}{\iint_a Edydz}. \qquad (8.188)$$

If the beam is also homogeneous then Young's modulus, $E$ is a constant and therefore,

$$y_o = (\iint_a ydydz)/(\iint_a dydz), \qquad z_o = (\iint_a zdydz)/(\iint_a dydz), \quad (8.189)$$

the $y$ and $z$ coordinates of the centroid of the cross section. Without loss of generality the origin of the coordinate system can be assumed to be located at the centroid of the cross section and hence $y_o = z_o = 0$.

Since, we have assumed that there is no net applied axial load, i.e.,

$$\int_a \sigma_{xx} da = 0, \quad (8.190)$$

equation (8.9) and (8.8) can be written as,

$$M_z = -\int_a y\sigma_{xx}da = -\int_a y\sigma_{xx}da + \int_a y_o\sigma_{xx}da = -\int_a [y - y_o]\sigma_{xx}da, \quad (8.191)$$

$$M_y = \int_a z\sigma_{xx}da = \int_a z\sigma_{xx}da - \int_a z_o\sigma_{xx}da = \int_a [z - z_o]\sigma_{xx}da, \quad (8.192)$$

where $y_o$ and $z_o$ are constants as given in equation (8.188). Substituting equation (8.185) in equation (8.191) we obtain,

$$\begin{aligned} M_z &= \iint_a [y - y_o]^2 E\frac{d^2\Delta_y}{dx^2}dydz + \iint_a [y - y_o][z - z_o]E\frac{d^2\Delta_z}{dx^2}dydz \\ &= \frac{d^2\Delta_y}{dx^2}\iint_a [y - y_o]^2 Edydz + \frac{d^2\Delta_z}{dx^2}\iint_a [y - y_o][z - z_o]Edydz. \quad (8.193) \end{aligned}$$

If the cross section of the beam is homogeneous, the above equation can be written as,

$$\begin{aligned} M_z &= \frac{d^2\Delta_y}{dx^2}E\iint_a [y - y_o]^2dydz + \frac{d^2\Delta_z}{dx^2}E\iint_a [y - y_o][z - z_o]dydz \\ &= \frac{d^2\Delta_y}{dx^2}EI_{zz} + \frac{d^2\Delta_z}{dx^2}EI_{yz}, \quad (8.194) \end{aligned}$$

where,

$$I_{zz} = \iint\limits_a [y - y_o]^2 dydz, \qquad I_{yz} = \iint\limits_a [y - y_o][z - z_o]dydz, \tag{8.195}$$

are the moment of inertia about the $z$ axis, the axis about which the applied forces produces a moment, $M_z$ and the product moment of inertia. Substituting equation (8.185) in equation (8.192) we obtain,

$$\begin{aligned} M_y &= -\iint\limits_a [y - y_o][z - z_o]E\frac{d^2\Delta_y}{dx^2}dydz - \iint\limits_a [z - z_o]^2 E\frac{d^2\Delta_z}{dx^2}dydz \\ &= -\frac{d^2\Delta_y}{dx^2}\iint\limits_a [y - y_o][z - z_o]Edydz - \frac{d^2\Delta_z}{dx^2}\iint\limits_a [z - z_o]^2 Edydz. \end{aligned} \tag{8.196}$$

If the cross section of the beam is homogeneous, the above equation can be written as,

$$\begin{aligned} M_y = -\frac{d^2\Delta_y}{dx^2}E\iint\limits_a [y - y_o][z - z_o]dydz - \frac{d^2\Delta_z}{dx^2}E\iint\limits_a [z - z_o]^2 dydz \\ = -\frac{d^2\Delta_y}{dx^2}EI_{yz} - \frac{d^2\Delta_z}{dx^2}EI_{yy}, \end{aligned} \tag{8.197}$$

where,

$$I_{yy} = \iint\limits_a [z - z_o]^2 dydz, \tag{8.198}$$

is the moment of inertia about the $y$ axis, the axis about which the applied forces produces a moment, $M_y$. Solving equations (8.194) and (8.197) for $\Delta_y$ and $\Delta_z$ we obtain,

$$\frac{d^2\Delta_y}{dx^2} = \frac{I_{yz}M_y + I_{yy}M_z}{E[I_{yy}I_{zz} - I_{yz}^2]}, \quad \frac{d^2\Delta_z}{dx^2} = -\frac{I_{yz}M_z + I_{zz}M_y}{E[I_{yy}I_{zz} - I_{yz}^2]}. \tag{8.199}$$

Substituting (8.199) in (8.185) we obtain

$$\sigma_{xx} = -[y - y_o]\frac{I_{yz}M_y + I_{yy}M_z}{I_{yy}I_{zz} - I_{yz}^2} + [z - z_o]\frac{I_{yz}M_z + I_{zz}M_y}{I_{yy}I_{zz} - I_{yz}^2}, \tag{8.200}$$

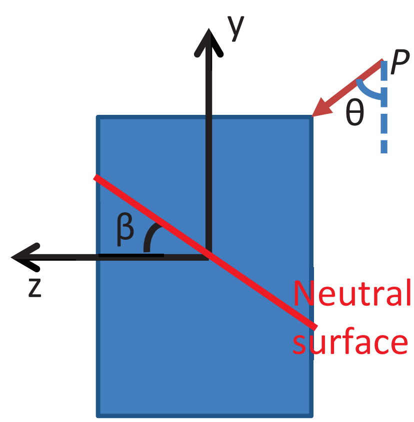



Figure 8.14: Schematic showing the neutral surface for a beam with rectangular cross section subjected to transverse loading

where $y_o$ and $z_o$ are as given in equation (8.189). Equation (8.200) gives the bending normal stress when a cross section is subjected to both $M_y$ and $M_z$ bending moments or when the cross section is subjected to a bending moment about an axis for which the product moment of inertia is not 0, i.e., loading is not along a plane of symmetry.

Before proceeding further, a few observations on equation (8.200) have to be made. It is clear from the equation (8.200) that the bending normal stress varies linearly over the cross sectional surface. As in the case of symmetrical bending, there exist a surface which has zero bending normal stress. This zero bending normal stress surface, called as the neutral surface is defined by

$$\tan(\beta) = \frac{y - y_o}{z - z_o} = \frac{I_{yz}M_z + I_{zz}M_y}{I_{yz}M_y + I_{yy}M_z}. \tag{8.201}$$

Figure 8.14 shows a typical neutral surface for a beam with rectangular cross section subjected to transverse loading not in the plane of symmetry of the cross section.

Having found the bending normal stress, next we find the bending shear stress. Towards this, we consider the equilibrium of a cuboid $pqrstuvw$ as shown in figure 8.15, taken from a beam subjected to asymmetric bending moment that varies along the longitudinal axis of the beam. Note that on this cuboid, bending normal stress $\sigma_{xx}^-$ acts on the face $prtv$ and $\sigma_{xx}^+$ acts on the face $qsuw$, shear stress, $\sigma_{xy}$ acts in plane $rsvw$ and shear stress, $\sigma_{xz}$ acts

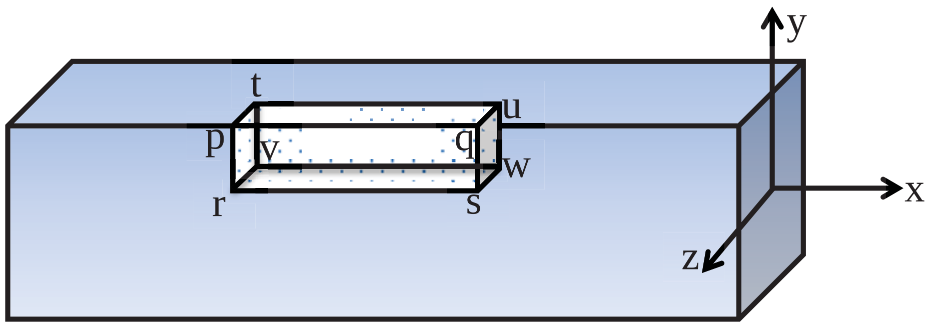



Figure 8.15: Schematic of a section of a beam with rectangular cross section to find the shear stresses due to asymmetric loading

in plane $tuvw$. Now, the force balance along the $x$ direction requires that

$$\sigma_{xy}(\Delta x)(b-z)+\sigma_{xz}(\Delta x)(c-y)+\iint_a[\sigma_{xx}^{+}-\sigma_{xx}^{-}]dzdy=0, \tag{8.202}$$

where we have assumed that constant shear stresses act on faces $rsvw$ and $pqrs$ and that the bending normal stress varies linearly over the faces $prtv$ and $qsuw$ as indicated in equation (8.200). Appealing to Taylor's series, we write,

$$\sigma_{xx}^{+}=\sigma_{xx}^{-}+\frac{d\sigma_{xx}}{dx}(\Delta x), \tag{8.203}$$

truncating the series after first order term, since our interest is in the limit $(\Delta x)$ tending to zero. Differentiating (8.200) with respect to $x$ we obtain,

$$\begin{aligned}\frac{d\sigma_{xx}}{dx}=-\frac{[y-y_o]}{[I_{zz}I_{yy}-I_{yz}^2]}&\left[I_{yy}\frac{dM_z}{dx}+I_{yz}\frac{dM_y}{dx}\right]\\&+\frac{[z-z_o]}{[I_{zz}I_{yy}-I_{yz}^2]}\left[I_{yz}\frac{dM_z}{dx}+I_{zz}\frac{dM_y}{dx}\right]\end{aligned} \tag{8.204}$$

Substituting equations (8.18) and (8.20) in (8.204),

$$\frac{d\sigma_{xx}}{dx}=-\frac{[y-y_o]}{[I_{zz}I_{yy}-I_{yz}^2]}\left[-I_{yy}V_y+I_{yz}V_z\right]+\frac{[z-z_o]}{[I_{zz}I_{yy}-I_{yz}^2]}\left[-I_{yz}V_y+I_{zz}V_z\right]. \tag{8.205}$$

Substituting equation (8.205) in (8.203) and using the resulting equation in (8.202) we obtain

$$\sigma_{xy}(b-z)+\sigma_{xz}(c-y) = -\frac{[I_{yy}V_y - I_{yz}V_z]}{[I_{zz}I_{yy}-I_{yz}^2]}\int_y^c \left\{[y-y_o]\int_z^b dz\right\} dy - \frac{[I_{zz}V_z - I_{yz}V_y]}{[I_{zz}I_{yy}-I_{yz}^2]}\int_z^b \left\{[z-z_o]\int_y^c dy\right\} dz, \quad (8.206)$$

where the limits of the integration have been arrived at for the rectangular cross section shown in figure 8.15. Thus, if the section is thick walled, equation (8.206) is insufficient to determine all the shear stresses as such. Hence, strength of materials approach cannot yield the shear stresses in a thick walled section subjected to asymmetrical loading. However, it should be mentioned that the shear stresses in a well proportioned thick walled section, with $l/c > 10$, for which the strength of materials solution is applicable, will develop shear stresses which are an order of magnitude, at least, less than that of the bending stresses.

On the other hand since the direction of the shear stress is well defined in thin walled sections. It is going to be tangential to the cross section at the point of interest, as in the case of symmetric bending. Further, by virtue of the section being thin walled, we assume the shear stresses to be uniform across its thickness, $t$. Now, the shear stress, $\tau$ acting as shown in figure 8.16 balances the imbalance created due to the variation of the bending normal stresses along the longitudinal axis of the beam. Following the same steps as in the case of thick walled cross sections detailed above, it can be shown that

$$\tau = -\frac{1}{t}\left\{\frac{[I_{yy}V_y - I_{yz}V_z]}{[I_{zz}I_{yy}-I_{yz}^2]}\int_a [y-y_o]dydz + \frac{[I_{zz}V_z - I_{yz}V_y]}{[I_{zz}I_{yy}-I_{yz}^2]}\int_a [z-z_o]dydz\right\}, \quad (8.207)$$

where the area over which the integration is to be performed is the shaded region shown in figure 8.16. If one is to use polar coordinates to describe the cross section and defining $r_o = \sqrt{y_o^2+z_o^2}$ and $\beta = \tan^{-1}(y_o/z_o)$, the equation

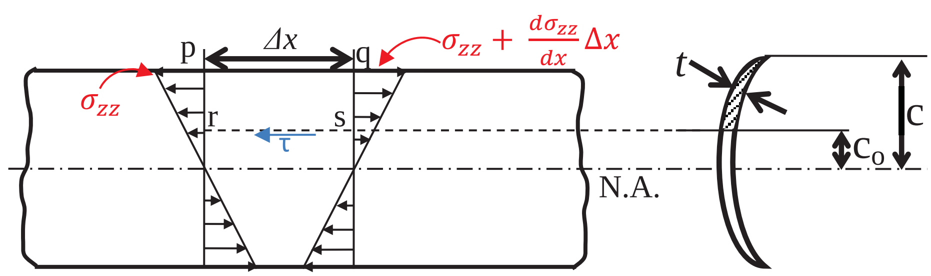



Figure 8.16: Stresses acting on a thin walled cross section beam subjected to asymmetric loading

(8.207) can be written as,

$$\tau = -\frac{[I_{yy}V_y - I_{yz}V_z]}{[I_{zz}I_{yy} - I_{yz}^2]}\int_\phi^\alpha r^2\left[\sin(\theta) - \frac{r_o}{r}\sin(\beta)\right]d\theta - \frac{[I_{zz}V_z - I_{yz}V_y]}{[I_{zz}I_{yy} - I_{yz}^2]}\int_\phi^\alpha r^2\left[\cos(\theta) - \frac{r_o}{r}\cos(\beta)\right]d\theta, \quad (8.208)$$

where $r$ could be a function of $\theta$ but $r_o$ and $\beta$ are constants. Thus, in equation (8.208) the shear stress, $\tau$ is a function of $\phi$ only.

While this representation is convenient in cases where the section is not made up of straight line segments, a representation using the perimeter length, $s$, of the cross section is useful when the section is made up of straight line segments. Thus, the expression for the shear stress in terms of the perimeter length as shown in figure 8.16 is,

$$\tau = -\frac{[I_{yy}V_y - I_{yz}V_z]}{[I_{zz}I_{yy} - I_{yz}^2]}\int_0^s [r\sin(\theta) - y_o]ds - \frac{[I_{zz}V_z - I_{yz}V_y]}{[I_{zz}I_{yy} - I_{yz}^2]}\int_0^s [r\cos(\theta) - z_o]ds, \quad (8.209)$$

where now $r$ and $\theta$ have to be expressed as a function of $s$. Here we have used the relation, $ds = rd\theta$ to obtain (8.209) from (8.208).

Thus, integrating the ordinary differential equations (8.199) we obtain the deflections of the beam along the $y$ and $z$ directions and hence the displacement field for the beam, (8.183) can be computed. Using equation (8.200) the bending normal stress is evaluated. While these calculations are the same for thin or thick walled sections, the shear stress estimation is different. We could not find the shear stresses in case of thick walled sections using the strength of materials approach. However, equation (8.207) gives the shear stresses in thin walled sections. This completes the solution to asymmetrical bending problem.

## 8.4 Shear center

Having found the solution to symmetrical and asymmetrical bending, in this section we find where the load has to be applied so that it produces no torsion.

Shear center is defined as the point about which the external load has to be applied so that it produces no twisting moment.

Recall from equation (8.7) the torsional moment due to the shear force $\sigma_{xy}$ and $\sigma_{xz}$ about the origin is,

$$M_x = \int_a [\sigma_{xz} y - \sigma_{xy} z] dydz. \tag{8.210}$$

Since, $\int_a \sigma_{xz} dydz = V_z$ and $\int_a \sigma_{xy} dydz = V_y$, the moment about some other point $(y_{sc}, z_{sc})$ would be,

$$M_x^{sc} = \int_a [\sigma_{xz} y - \sigma_{xy} z] dydz - V_z y_{sc} + V_y z_{sc}. \tag{8.211}$$

If this point $(y_{sc}, z_{sc})$ is the shear center, then $M_x^{sc} = 0$. Thus, we have to find $y_{sc}$ and $z_{sc}$ such that,

$$\int_a [\sigma_{xz} y - \sigma_{xy} z] dydz - V_z y_{sc} + V_y z_{sc} = 0, \tag{8.212}$$

holds. We have two unknowns but only one equation. Hence, we cannot find $y_{sc}$ and $z_{sc}$ uniquely, in general. If the loading is such that only shear force $V_y$ is present, then

$$z_{sc} = \frac{1}{V_y} \int_a [\sigma_{xy} z - \sigma_{xz} y] dydz. \tag{8.213}$$

Similarly, if $V_y = 0$,

$$y_{sc} = \frac{1}{V_z} \int_a [\sigma_{xz} y - \sigma_{xy} z] dydz. \tag{8.214}$$

Equations (8.213) and (8.214) are used to find the coordinates of the shear center with respect to the chosen origin of the coordinate system, which for homogeneous sections is usually taken as the centroid of the cross section. Thus, the point that $(y_{sc}, z_{sc})$ are the coordinates of the shear center from the origin of the chosen coordinate system which in many cases would be the centroid of the section cannot be overemphasized. In the case of thin walled sections which develop shear stresses tangential to the cross section, $\sigma_{xy} = -\tau \sin(\theta)$ and $\sigma_{xz} = \tau \cos(\theta)$, where $\tau$ is the magnitude of the shear stress and $\theta$ is the angle the tangent to the cross section makes with the $z$ direction.

By virtue of the shear stress depending linearly on the shear force (see equations (8.43) and (8.207)), it can be seen that the coordinates of the shear center is a geometric property of the section.

### 8.4.1 Illustrative examples

Next, to illustrate the use of equations (8.213) and (8.214) we find the shear center for some shapes.

#### Example 1: Rectangular section

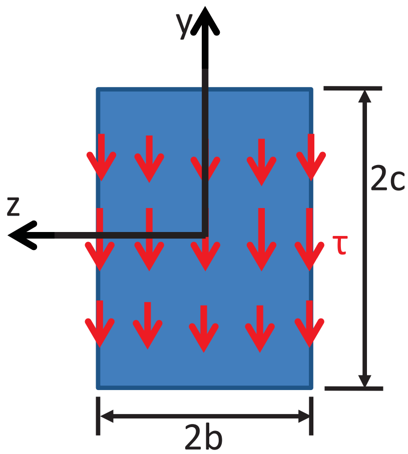
The first section that we consider is a thick walled rectangular section as shown in figure 8.17 having a depth $2c$ and width $2b$. The chosen coordinate basis coincides with the two axis of symmetry that this section has and the origin is at the centroid of the cross section.

First, we shall compute the $z$ coordinate of the shear center $z_{sc}$. For this only shear force $V_y$ should act on the cross section. Shear force $V_y$ would be caused due to loading along the $xy$ plane, a plane of symmetry for the cross section. Therefore the shear stress, $\sigma_{xy}$ is computed using (8.43) as

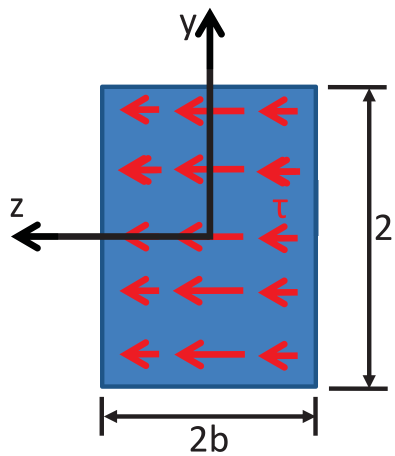
$$\sigma_{xy} = -\frac{V_y}{2b(2b(2c)^3/12)}(2b)\int_y^c y dy = \frac{3V_y}{8bc}\left[1 - \frac{y^2}{c^2}\right], \tag{8.215}$$

where we have used the fact that $y_o = 0$, since the origin is located at the centroid of the cross section. Further, for this loading $\sigma_{xz} = 0$. Substituting (8.215) and $\sigma_{xz} = 0$ in (8.213) we obtain

$$z_{sc} = \frac{3}{8bc}\int_{-b}^{b} z dz \int_{-c}^{c}\left[1 - \frac{y^2}{c^2}\right] dy = 0. \tag{8.216}$$




(a) Rectangular cross section subjected to a shear force $V_y$.




(b) Rectangular cross section subjected to a shear force $V_z$.

Figure 8.17: Schematic of a rectangular cross section subjected to a shear force along one direction

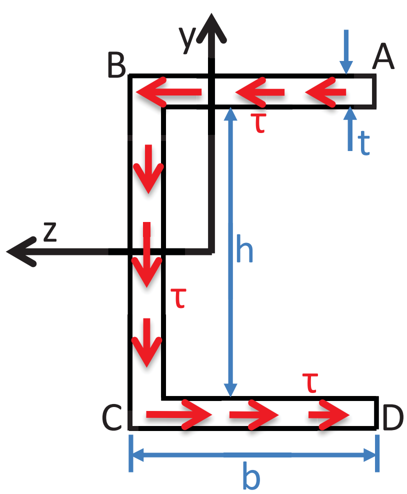
Next, we shall compute the $y$ coordinate of the shear center $y_{sc}$. Now, only shear force $V_z$ should act. This shear force would be produced by loading along the $xz$ plane, also a plane of symmetry for the cross section. This loading produces a shear stress as shown in figure 8.17b whose magnitude is again computed using (8.43) as

$$\sigma_{xz} = -\frac{V_z}{2c(2c(2b)^3/12)}(2c)\int_z^b zdz = \frac{3V_z}{8bc}\left[1 - \frac{z^2}{b^2}\right]. \tag{8.217}$$

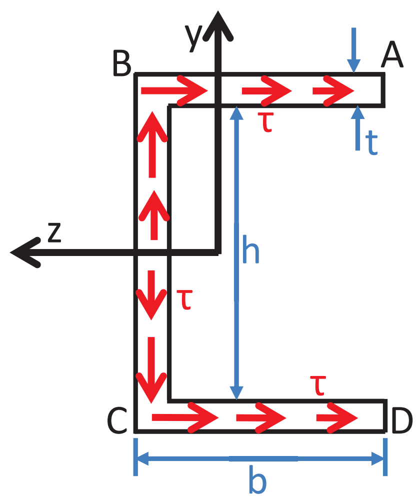
and $\sigma_{xy} = 0$. Substituting (8.217) in (8.214) we obtain

$$y_{sc} = \frac{3}{8bc}\int_{-c}^{c} ydy \int_{-b}^{b}\left[1 - \frac{z^2}{c^2}\right]dz = 0. \tag{8.218}$$

Thus, for the rectangular cross section, the shear center is located at the origin of the coordinate system, which in turn is the centroid of the cross section. Hence, the shear center coincides with the centroid of the cross section.




(a) Channel cross section subjected to a shear force $V_y$.




(b) Channel cross section subjected to a shear force $V_z$.

Figure 8.18: Schematic of a channel cross section subjected to a shear force along one direction

**Example 2: Channel section**

The next section that we study is the channel section with orientation and dimensions as shown in figure 8.18. The flange and web thickness of the channel is the same. Before proceeding to compute the shear center the other geometric properties, the centroid and the moment of inertia's for the cross section is computed. The origin of the coordinate system being used is at the centroid of the cross section. Then, the distance from the centroid of the cross section to the top most fiber of the cross section $AB$ is $y_{AB} = c = h/2 + t$. Similarly, the distance of the left most fiber in the web of the cross section, $BC$ is $z_{BC} = (ht + 2b^2)/(2(h + 2b))$. Now,

$$\begin{aligned} I_{yz} &= 0, && (8.219)\\ I_{zz} &= \frac{1}{12}\left[th^3 + 2bt^3\right] + \frac{1}{2}bt(h+t)^2, && (8.220)\end{aligned}$$

$$\begin{aligned} I_{yy} &= \frac{1}{12}\left[ht^3 + 2tb^3\right] + ht\left(z_{BC} - \frac{t}{2}\right)^2 + 2bt\left(\frac{b}{2} - z_{BC}\right)^2 \\ &= \frac{1}{12}\left[ht^3 + 2tb^3\right] + \frac{ht}{4}\left(\frac{b(b-2t)}{(h+2b)}\right)^2 + \frac{bt}{2}\left(\frac{h(b-t)}{(h+2b)}\right)^2. \quad (8.221)\end{aligned}$$

Towards computing the location of the shear center along the $z$ direction, we first compute the shear stress acting on the cross section due to a shear force $V_y$ alone. The magnitude of the shear stress, $\tau$ is found using (8.209) as

$$\tau = \begin{cases} -\frac{V_y}{I_{zz}}\frac{(h+t)}{2}\int_0^s ds & 0 \le s \le \left(b - \frac{t}{2}\right) \\ -\frac{V_y}{I_{zz}}\left[\frac{(h+t)}{2}\int_0^{b-t/2} ds \right. & \\ \quad \left. + \int_{b-t/2}^{s}\left[\frac{(h+t)}{2} - \left(s - b + \frac{t}{2}\right)\right] ds\right] & \left(b - \frac{t}{2}\right) \le s \le \left(b + h + \frac{t}{2}\right) \\ -\frac{V_y}{I_{zz}}\left[\frac{(h+t)}{2}\int_0^{b-t/2} ds - \frac{(h+t)}{2}\int_{b+h+t/2}^{s} ds \right. & \\ \quad \left. + \int_{b-t/2}^{b+h+t/2}\left[\frac{(h+t)}{2} - \left(s - b + \frac{t}{2}\right)\right] ds\right] & \left(b + h + \frac{t}{2}\right) \le s \le (2b + h) \end{cases}. \quad (8.222)$$

Evaluating the integrals and simplification yields,

$$\tau = \begin{cases} -\frac{V_y}{2I_{zz}}(h+t)s & 0 \leq s \leq \left(b - \frac{t}{2}\right) \\ -\frac{V_y}{2I_{zz}}\left[(h+t)\left(b-\frac{t}{2}\right) - s^2 + \left(b-\frac{t}{2}\right)^2\right. & \\ \qquad \left. + (h+2b)\left(s - b + \frac{t}{2}\right)\right] & \left(b-\frac{t}{2}\right) \leq s \leq \left(b+h+\frac{t}{2}\right) \\ \frac{V_y}{2I_{zz}}(h+t)(2b+h-s) & \left(b+h+\frac{t}{2}\right) \leq s \leq (2b+h) \end{cases}. \tag{8.223}$$

Since this shear stress has to be tangential to the cross section, it would be $\sigma_{xz}$ component in the flanges and $\sigma_{xy}$ component in the web. Rewriting equation (8.213) in terms of integration over the perimeter length,

$$z_{sc} = \frac{t}{V_y}\int_0^{S_o} \left[\sigma_{xy} r\cos(\theta) - \sigma_{xz} r\sin(\theta)\right] ds, \tag{8.224}$$

where $S_o$ is the total length of the perimeter of the cross section. Evaluating the above equation, (8.224) for the channel section yields,

$$\begin{aligned} z_{sc} = \frac{t}{2I_{zz}}\Bigg\{ &\int_0^{b-t/2} (h+t)s\frac{(h+t)}{2}ds \\ &+ \int_{b+h+t/2}^{2b+h} (h+t)(2b+h-s)\frac{(h+t)}{2}ds \\ &- \left(z_{BC} - \frac{t}{2}\right)\int_{b-t/2}^{b+h+t/2} \left[(h+t)\left(b-\frac{t}{2}\right) - s^2 + \left(b-\frac{t}{2}\right)^2\right. \\ &\left. + (h+2b)\left(s-b+\frac{t}{2}\right)\right] ds \Bigg\}. \end{aligned} \tag{8.225}$$

Evaluating the integrals in (8.225) and simplifying we obtain,

$$z_{sc} = \frac{t(h+t)^2(2b-t)^2}{16I_{zz}} + \left[z_{BC} - \frac{t}{2}\right](h+6b-2t)\frac{t(h+t)^2}{12I_{zz}}. \tag{8.226}$$

For computing the location of the shear center along the $y$ direction, we next compute the shear stress acting on the cross section due to a shear force

$V_z$ alone. The magnitude of the shear stress, $\tau$ is found using (8.209) as

$$\tau = \begin{cases} -\frac{V_z}{I_{yy}} \int_0^s (s + z_{BC} - b) ds & 0 \leq s \leq \left(b - \frac{t}{2}\right) \\ -\frac{V_z}{I_{yy}} \left[ \int_0^{b-t/2} (s + z_{BC} - b) ds \right. & \\ \quad \left. + \left(z_{BC} - \frac{t}{2}\right) \int_{b-t/2}^{s} ds \right] & \left(b - \frac{t}{2}\right) \leq s \leq \left(b + h + \frac{t}{2}\right) \\ -\frac{V_z}{I_{yy}} \left[ \int_0^{b-t/2} (s + z_{BC} - b) ds \right. & \\ \quad + \left(z_{BC} - \frac{t}{2}\right) \int_{b-t/2}^{b+h+t/2} ds & \\ \quad \left. + \int_{b+h+t/2}^{s} (h - s + z_{BC} + b) ds \right] & \left(b + h + \frac{t}{2}\right) \leq s \leq (2b + h) \end{cases} . \tag{8.227}$$

Integrating the above equation we obtain,

$$\tau = \begin{cases} -\frac{V_z}{I_{yy}} \left[ \frac{s^2}{2} + (z_{BC} - b)s \right] & 0 \leq s \leq \left(b - \frac{t}{2}\right) \\ -\frac{V_z}{I_{yy}} \left[ \frac{(2b-t)^2}{8} + (z_{BC} - b)\frac{(2b-t)}{2} \right. & \\ \quad \left. + \left(z_{BC} - \frac{t}{2}\right) \left(s - \frac{(2b-t)}{2}\right) \right] & \left(b - \frac{t}{2}\right) \leq s \leq \left(b + h + \frac{t}{2}\right) \\ -\frac{V_z}{I_{yy}} \left[ \frac{(2b-t)^2}{8} + (z_{BC} - b)\frac{(2b-t)}{2} \right. & \\ \quad + \left(z_{BC} - \frac{t}{2}\right)(h + t) - \frac{s^2}{2} & \\ \quad + \frac{(b+h+t/2)^2}{2} & \\ \quad \left. + (z_{BC} + b + h)\left(s - b - h - \frac{t}{2}\right) \right] & \left(b + h + \frac{t}{2}\right) \leq s \leq (2b + h) \end{cases} . \tag{8.228}$$

Since this shear stress has to be tangential to the cross section, as before, it would be $\sigma_{xz}$ component in the flanges and $\sigma_{xy}$ component in the web. Rewriting equation (8.214) in terms of integration over the perimeter length,

$$y_{sc} = -\frac{t}{V_z} \int_0^{S_o} \left[\sigma_{xy} r \cos(\theta) - \sigma_{xz} r \sin(\theta)\right] ds, \tag{8.229}$$

where $S_o$ is the total length of the perimeter of the cross section. Evaluating the above equation, (8.229) for the channel section yields,

$$
\begin{aligned}
y_{sc} = -\frac{t}{I_{yy}} \Bigg\{ & \frac{(h+t)}{2} \int_0^{b-t/2} \left[ \frac{s^2}{2} + (z_{BC} - b)s \right] ds \\
& - \left( z_{BC} - \frac{t}{2} \right) \int_{b-t/2}^{b+h+t/2} \left[ \frac{(2b-t)^2}{8} + (z_{BC} - b)\frac{(2b-t)}{2} \right. \\
& \qquad \left. + \left( z_{BC} - \frac{t}{2} \right) \left( s - \frac{(2b-t)}{2} \right) \right] ds \\
& - \frac{(h+t)}{2} \int_{b+h+t/2}^{2b+h} \left[ \frac{(2b-t)^2}{8} + (z_{BC} - b)\frac{(2b-t)}{2} + \left( z_{BC} - \frac{t}{2} \right)(h+t) - \frac{s^2}{2} \right. \\
& \qquad \left. + \frac{(b+h+t/2)^2}{2} + (z_{BC} + b + h)\left( s - b - h - \frac{t}{2} \right) \right] ds \Bigg\} \quad (8.230)
\end{aligned}
$$

Evaluating the integrals and simplifying, we find that $y_{sc} = 0$!

**Example 3: Circular arc**

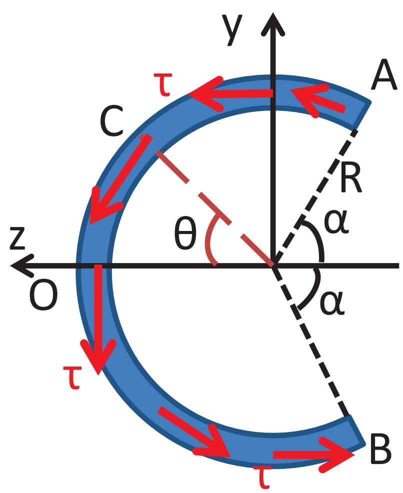
The final section that we use to illustrate the procedure to find the shear center is an arc of a circular section with radius $R$ and a uniform thickness $t$. The arc is assumed to span from $-(\pi - \alpha) \leq \theta \leq (\pi - \alpha)$. Thus, it is symmetrical about the $z$ direction. For convenience, we assume the origin of the coordinate system to be located at the center of the circle. First, we compute the centroid of the cross section,

$$
y_o = \frac{\int_a y da}{\int_a da} = \frac{\int_{-(\pi-\alpha)}^{\pi-\alpha} R\sin(\theta) t(Rd\theta)}{\int_{-(\pi-\alpha)}^{\pi-\alpha} t(Rd\theta)} = 0, \quad (8.231)
$$

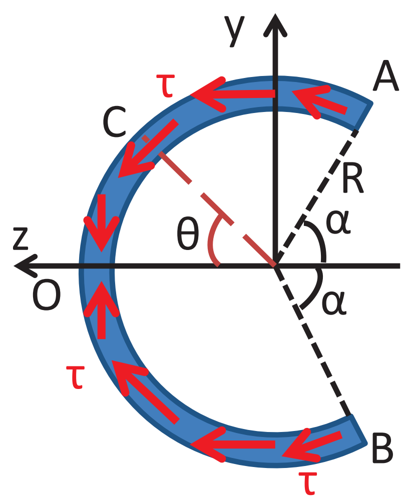
$$
z_o = \frac{\int_a z da}{\int_a da} = \frac{\int_{-(\pi-\alpha)}^{\pi-\alpha} R\cos(\theta) t(Rd\theta)}{\int_{-(\pi-\alpha)}^{\pi-\alpha} t(Rd\theta)} = R\frac{\sin(\alpha)}{(\pi-\alpha)}. \quad (8.232)
$$

Here we have identified $y_o$ and $z_o$ with the coordinates of the centroid of the cross section, since it is homogeneous. Next, we compute the moment of inertias about the centroid,

$$
\begin{aligned}
I_{zz} = \int_a [y - y_o]^2 da = & \int_{-(\pi-\alpha)}^{\pi-\alpha} [R\sin(\theta) - 0]^2 t(Rd\theta) \\
& = R^3 t \left[ \pi - \alpha + \frac{1}{2}\sin(2\alpha) \right], \quad (8.233)
\end{aligned}
$$




(a) Circular arc section subjected to a shear force $V_y$.




(b) Circular arc section subjected to a shear force $V_z$.

Figure 8.19: Schematic of a circular arc section subjected to a shear force along one direction

$$I_{yz} = \int_a [y - y_o][z - z_o]da = \int_{-(\pi-\alpha)}^{\pi-\alpha} [R\sin(\theta) - 0][R\cos(\theta) - z_o]t(Rd\theta) = 0. \tag{8.234}$$

$$\begin{aligned} I_{yy} = \int_a [z - z_o]^2 da &= \int_{-(\pi-\alpha)}^{\pi-\alpha} [R\cos(\theta) - z_o]^2 t(Rd\theta) \\ &= R^3 t \left[ \left(1 + 2\frac{z_o^2}{R^2}\right)(\pi - \alpha) - \frac{1}{2}\sin(2\alpha) - 4\frac{z_o}{R}\sin(\alpha) \right] \\ &= R^3 t \left[ \pi - \alpha - 2\frac{\sin^2(\alpha)}{\pi - \alpha} - \frac{1}{2}\sin(2\alpha) \right], \end{aligned} \tag{8.235}$$

where the last equality is obtained on substituting for $z_o$ from (8.232).

Towards computing the $z$ coordinate of the shear center, we compute the shear stress distribution in the circular arc when only shear force $V_y$ is acting on the cross section. The magnitude of the shear stress, $\tau$ is found using (8.208) as,

$$\tau = -\frac{V_y}{I_{zz}} \int_\phi^{\pi-\alpha} R^2 \sin(\theta)d\theta = -\frac{V_y}{I_{zz}} R^2[\cos(\alpha) + \cos(\phi)]. \tag{8.236}$$

This shear stress would act tangential to the cross section at every location as indicated in figure 8.19a. Therefore, $\sigma_{xy} = -\tau\cos(\phi)$ and $\sigma_{xz} = \tau\sin(\phi)$, where $\phi$ is the angle the tangent makes with the $y$ axis. Appealing to equation (8.213) we obtain,

$$\begin{aligned} z_{sc} &= -\frac{1}{V_y} \int_{-(\pi-\alpha)}^{\pi-\alpha} [\tau\cos(\phi)(R\cos(\phi)) + \tau\sin(\phi)(R\sin(\phi))]\, tRd\phi \\ &= \frac{R^4 t}{I_{zz}} \int_{-(\pi-\alpha)}^{\pi-\alpha} [\cos(\alpha) + \cos(\phi)]d\phi = R\frac{4[(\pi - \alpha)\cos(\alpha) + \sin(\alpha)]}{[2(\pi - \alpha) + \sin(2\alpha)]}, \end{aligned} \tag{8.237}$$

where we have used equations (8.236) and (8.233) respectively. It can be seen that when $0 < \alpha \leq \pi/2$, $z_{sc} > 0$ and in fact $R < z_{sc} \leq 4R/\pi$.

Next, for computing the $y$ coordinate of the shear center, we compute the shear stress distribution in the circular arc when only shear force $V_z$ is acting on the cross section. The magnitude of the shear stress, $\tau$ is found using (8.208) as,

$$\tau = -\frac{V_z}{I_{yy}} \int_{\phi}^{\pi-\alpha} R^2 \left[ \cos(\theta) - \frac{\sin(\alpha)}{(\pi-\alpha)} \right] d\theta$$
$$= -\frac{V_z}{I_{yy}} R^2 \left[ \sin(\alpha) \frac{\phi}{(\pi-\alpha)} - \sin(\phi) \right]. \quad (8.238)$$

This shear stress would act tangential to the cross section at every location as indicated in figure 8.19b. Hence, as before, $\sigma_{xy} = -\tau\cos(\phi)$ and $\sigma_{xz} = \tau\sin(\phi)$, where $\phi$ is the angle the tangent makes with the $y$ axis. Appealing to equation (8.214) we obtain,

$$y_{sc} = \frac{1}{V_z} \int_{-(\pi-\alpha)}^{\pi-\alpha} [\tau\cos(\phi)(R\cos(\phi)) + \tau\sin(\phi)(R\sin(\phi))]\, tRd\phi$$
$$= -\frac{R^4 t}{I_{yy}} \int_{-(\pi-\alpha)}^{\pi-\alpha} \left[ \sin(\alpha)\frac{\phi}{(\pi-\alpha)} - \sin(\phi) \right] d\phi = 0, \quad (8.239)$$

where we have used equations (8.238) and (8.235) respectively. Thus, the shear center is located along the $z$ axis.

By virtue of the $z_{sc}$ being greater than $R$, the shear center is located outside the cross section. Hence, for the loading to pass through the shear center similar issues as discussed in the channel section exist.

## 8.5 Summary

In this chapter we solved boundary value problems corresponding to bending of straight, prismatic members. We obtained the solution - the stress and displacement field - by assuming the displacement field in the strength of materials approach and the stress field in the elasticity approach. We also compared the solutions and found good agreement between the solution obtained by both these approaches, when the length to depth ratio of these members are greater than 10, for the case when the loading passes through a plane of symmetry. Then, we studied asymmetric bending and obtained the solution by starting with an assumption on the displacement field. Though we did not obtain the elasticity solution for this case, it can be obtained by using principle of superposition and leave it as an exercise to the student to work the details. Then we introduced an concept called the shear center. It

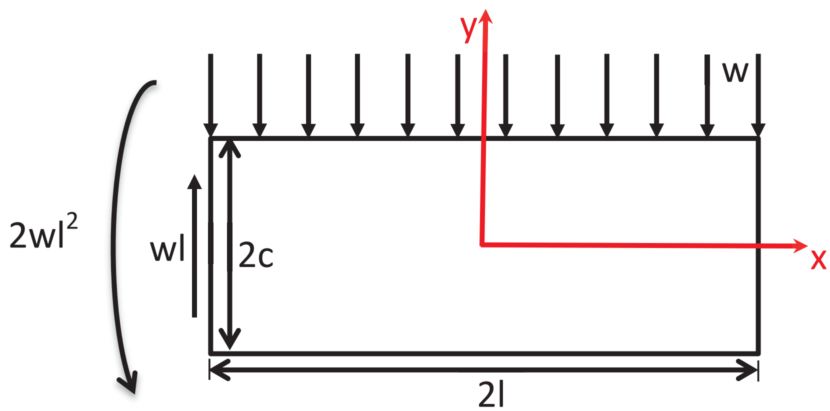



Figure 8.20: Cantilever beam subjected to uniformly distributed load on top surface. Figure for problem 2

is the point about which the external load has to be applied so that there is no twisting of cross section. We outlined a method to find this shear center and illustrated the same for three sections.

## 8.6 Self-Evaluation

1. Find the displacement and stress fields in a cantilever beam with rectangular cross section of length $2l$, depth $2c$ and width $2b$ subjected to a pure bending moment by assuming that plane section remain plane and normal to the neutral axis. Solve the same problem by assuming a cubic polynomial for the Airy's stress function. Compare the solutions and draw inferences.

2. Find the displacement and stress fields in a cantilever beam with rectangular cross section of length $2l$, depth $2c$ and width $2b$ subjected to a uniformly distributed load on the top surface, as shown in figure 8.20, by assuming that plane section remain plane and normal to the neutral axis. Solve the same problem by assuming a suitable polynomial for the Airy's stress function. Compare the solutions and draw inferences.

3. A simply supported wood beam of rectangular cross section carries a

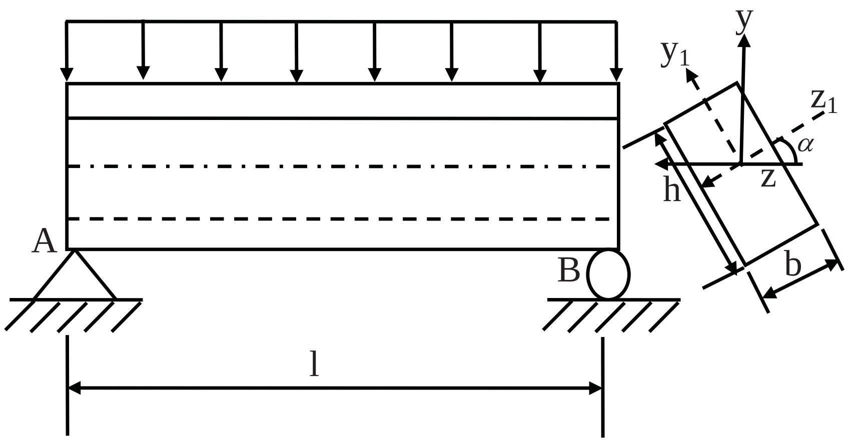



Figure 8.21: Simply supported beam subjected to uniformly distributed load on top surface with the load not acting in the plane of symmetry. Figure for problem 3

uniform load of intensity, $w$ as shown in figure 8.21. The plane of symmetry of the beam, $x_1y_1$ plane, is inclined to the vertical $xy$-plane of loading by an angle $\alpha$ as shown. Calculate the maximum bending stresses if $l = 3$ m, $w = 3$ kN/m, $b = 0.15$ m, $h = 0.2$ m and $\tan(\alpha) = 0.5$.

4. For the T-section shown in figure 8.22 find the shear center. The dimensions in the figure are in millimeters.

5. For the homogeneous trapezoidal cross section shown in figure 8.23 show that the neutral surface is located at a distance, $c_2 = h(b_2 + 2b_1)/(3(b_2 + b_1))$, from the bottom of the section. Also calculate the location of the shear center.

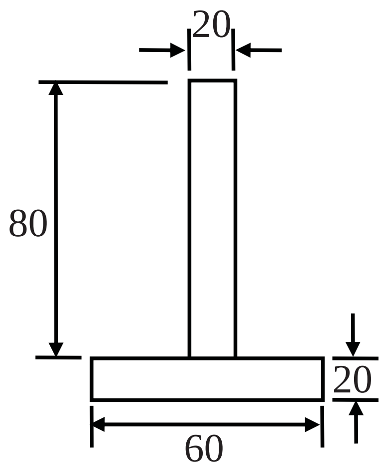



Figure 8.22: T-section. Dimensions in mm. Figure for problem 4

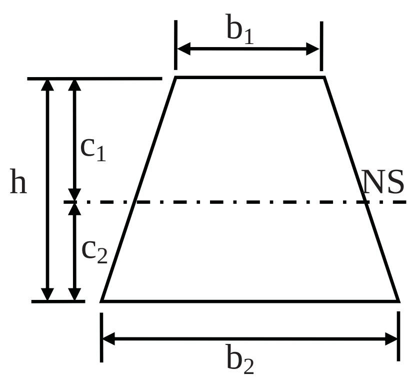



Figure 8.23: Trapezoidal section. NS stands for neutral surface. Figure for problem 5

# Chapter 9

# End Torsion of Prismatic Bars

## 9.1 Overview

In the last chapter, we studied the response of straight prismatic beams subjected to bending moments. In this chapter, we study the stresses that develop and the displacement that these members undergo when subjected to twisting moment. While in the analysis of beams subjected to bending moments we maintained some generality with the loading, the study of torsion is focused on a single loading case - one end fixed and the other end free to twist, as shown in figure 9.1. Here the double arrow indicates the twisting moment. However, we study this boundary value problem for a variety of cross sections. This analysis for torsion depends on whether the section is thick walled or thin walled as in the case of bending moments. It also depends on whether the section is open or closed. We shall discuss in detail as to when a section is closed subsequently.


Figure 9.1: Schematic of a straight prismatic member subjected to end torsion

Before proceeding further, we would like to point out the change in the orientation of the coordinate system, from that used in the study of the beams. This is necessitated because for some problems we would be using cylindrical polar coordinates, and want the cross section to be in the $xy$ plane. We use cylindrical polar coordinates with this orientation so that the boundary of the cross section can be defined easily. Since, the value of displacement or stress in a material particle is independent of the coordinate system used, it is expected that the analyst pick coordinate systems that is convenient for a problem and not that which he is comfortable with. Hence, the change.

For this orientation of the coordinate system, following the same steps as discussed in detail in chapter 8, it can be shown that the torsional moment, $M_z$ is given by,

$$M_z = \int_a (\sigma_{yz}x - \sigma_{xz}y)dxdy, \tag{9.1}$$

and for completeness, the other two components of the moment, which are bending moments are,

$$\begin{aligned} M_y &= -\int_a \sigma_{zz}xdxdy, & (9.2) \\ M_x &= \int_a \sigma_{zz}ydxdy. & (9.3) \end{aligned}$$

Comparing equations (9.1) with (9.2) and (9.3) it can be seen that while the bending moments are due to normal stresses, torsional moment is due to shear stresses. This is the major difference between the bending and twisting moment. While the bending moment gives raise to normal stresses predominantly, twisting moment gives raise to shear stresses predominantly.

Now let us see how to classify the sections. As in case of bending, a cross section would be classified as thin walled if the thickness of the cross section is such that it is much less than the characteristic dimensions of the cross section. Typically, if the ratio of the thickness of the cross section to the length of the member is less than 0.1, the section is classified as thin. If the section is not thin, it is considered to be thick.

If a section when twisted can deform such that plane sections before deformation remain plane after twisting is called closed section. Sections which do not deform in the above manner are called open sections. Solid Circular cross section is an example of closed section and solid rectangular

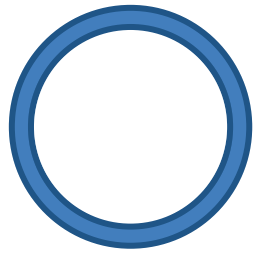


Figure 9.2: Section classified as closed sections when subjected to torsion

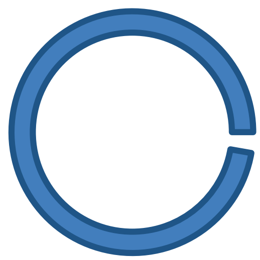


Figure 9.3: Section classified as open sections when subjected to torsion

cross section is an example of open sections. Similarly, a thin walled annular cylinder (see figure 9.2) is an example of closed section and if the same annular cylinder has a longitudinal slit (see figure 9.3), it is an example of open section. All closed section allows for continuous variation of shear stresses such that it is no where zero except at the centroid of the cross section. A section which requires the shear stresses to be zero at locations apart from the centroid of the cross section, like at the corners of a rectangular cross section, is an open section. It should be pointed out that in elliptical shaped cross section also the shear stress is zero only at the centroid of the cross section but is an open section as plane sections do not remain plane after twisting. Thus, the requirement on the shear stress is just necessary but not sufficient to classify a given section as closed.

In the following section, we find the displacement field and stress field for twisting of a thick walled sections and then focus on thin walled sections.

## 9.2 Twisting of thick walled closed section

First, we study the twisting of thick walled closed section and then thin walled closed section. This problem we study using the displacement approach. We also use cylindrical polar coordinates to formulate and study the problem. While the solution that we obtain does not require that the cross section to be closed section be circular, we shall assume this for illustration purpose, mainly to fix the geometry of the body. Thus, we assume the body to be an right circular annular (or solid) cylinder occupying a region in the Euclidean point space given by: $\mathcal{B} = \{(r, \theta, z) | R_i \leq r \leq R_o, 0 \leq \theta \leq 2\pi, 0 \leq z \leq L\}$, where $R_i$, $R_o$ and $L$ are constants with $R_i = 0$ in case of solid cylinder. Since, we are interested in this body being subjected to pure torsional moment, the traction boundary conditions are

$$\mathbf{t}_{(\mathbf{e}_r)}(R_o, \theta, z) = \mathbf{o}, \qquad \mathbf{t}_{(\mathbf{e}_r)}(R_i, \theta, z) = \mathbf{o}, \tag{9.4}$$

$$\int_a \mathbf{t}_{(-\mathbf{e}_z)}(r, \theta, 0) da = \mathbf{o}, \qquad \int_a \mathbf{t}_{(\mathbf{e}_z)}(r, \theta, L) da = \mathbf{o}, \tag{9.5}$$

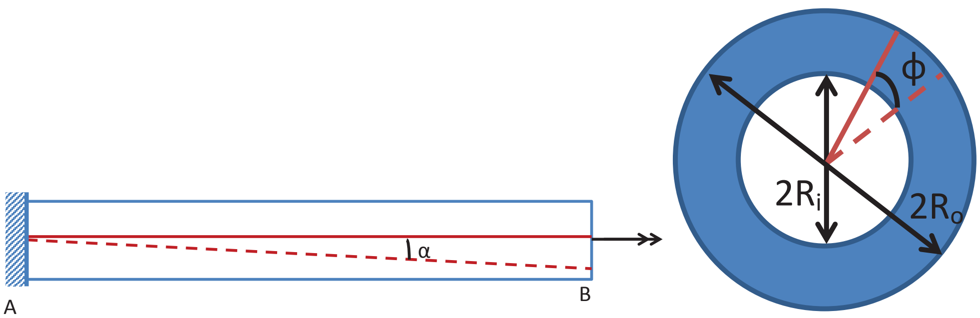
$$\int_a r\mathbf{e}_r \wedge \mathbf{t}_{(-\mathbf{e}_z)}(r, \theta, 0) da = -T\mathbf{e}_z, \qquad \int_a r\mathbf{e}_r \wedge \mathbf{t}_{(\mathbf{e}_z)}(r, \theta, L) da = T\mathbf{e}_z, \tag{9.6}$$

where $T$ is a constant. Further, since one end of this body is assumed to be fixed and the other end free to displace radially and circumferentially, the displacement boundary conditions for this problem is,

$$\mathbf{u}(r, \theta, 0) = \mathbf{o}, \qquad u_z(r, \theta, L) = 0, \tag{9.7}$$

where $u_z$ denotes the $z$ component of the displacement field and we have assumed that the surface of the body defined by $z = 0$ is fixed and the other surfaces are free to displace radially and circumferentially.

Since, we are going to solve the problem using the displacement approach, we have to assume the displacement field. We shall assume that plane sections normal to the axis of the member, $\mathbf{e}_z$ remain plane and perpendicular to the axis. This means that the $z$ component of the displacement field is only a function of $z$, i.e., $u_z = \hat{u}_z(z)$. Further, the boundary condition (9.7) requires that $u_z(0) = u_z(L) = 0$. Consistent with these requirements, we assume that the $z$ component of the displacement, $u_z = 0$. We could have also assumed $u_z = C \sin(m\pi z/L)$, where $m$ is an integer and $C$ is a constant. However, it can be shown that such an assumption to satisfy balance of linear momentum would require that $C = 0$. Then, we shall assume that




(a) Schematic of deformation in the $yz$ plane

(b) Schematic of deformation in the $xy$ plane

Figure 9.4: Schematic of the deformation of a straight prismatic member subjected to end torsion

there is no radial component of the displacement, i.e., $u_r = 0$. Since, straight radial lines before deformation remain as straight radial lines after deformation with their length unchanged, we are justified in making the assumption that $u_r = 0$. Further, we assume that the straight line along the axis of the member remains straight but rotates by an angle $\theta$ and that the radial lines rotate by an angle $\phi$ as shown in the figure, 9.4 when subjected to twisting moment. Consequently, the circumferential component of the displacement, $u_\theta = \Omega r z$, where $\Omega$ is called as the angle of twist per unit length. The slope of the straight line along the axis of the member located at a radial distance, $r^*$ after deformation, $\alpha = \Omega r^*$. Similarly, the rotation undergone by radial lines in a plane defined by $z = z^*$, where $z^*$ is a constant, is $\phi = \Omega z^*$. Thus, the displacement field is taken as,

$$\mathbf{u} = \Omega r z \mathbf{e}_\theta. \tag{9.8}$$

It can then be verified that this displacement field satisfies the displacement boundary conditions (9.7). Then, the strain field corresponding to the displacement field (9.8) is

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\Omega r}{2} \\ 0 & \frac{\Omega r}{2} & 0 \end{pmatrix}. \tag{9.9}$$

Using isotropic Hooke's law, (7.2) the stress corresponding to the strain (9.9)

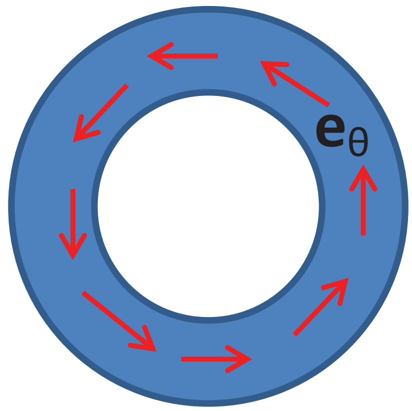



Figure 9.5: Schematic of variation of $\mathbf{e}_\theta$ over the circumference of the cylinder.

is computed as

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu\Omega r \\ 0 & \mu\Omega r & 0 \end{pmatrix}. \tag{9.10}$$

It can then be verified that the above state of stress, satisfies the static equilibrium equations in the absence of body forces, (7.6) trivially. It can also be verified that the stress given in equation (9.10) satisfies the boundary condition (9.4) trivially. For the stress state (9.10) the boundary condition (9.5) evaluates to

$$\int_a \mathbf{t}_{(-\mathbf{e}_z)}(r,\theta,0)da = -\int_a \mu\Omega r\mathbf{e}_\theta da, \quad \int_a \mathbf{t}_{(\mathbf{e}_z)}(r,\theta,0)da = \int_a \mu\Omega r\mathbf{e}_\theta da, \tag{9.11}$$

Since, the direction of $\mathbf{e}_\theta$ changes with the location as shown in figure 9.5, we write it in terms of the fixed Cartesian basis and obtain

$$\pm\int_{R_i}^{R_o} \mu\Omega r^2 dr \int_0^{2\pi} [-\sin(\theta)\mathbf{e}_x + \cos(\theta)\mathbf{e}_y]d\theta = \mathbf{o}. \tag{9.12}$$

Thus, the boundary condition (9.5) holds for the stress state (9.10).

Then, using the boundary condition (9.6) we obtain the relationship between the applied torque, $T$ and the angle of twist per unit length, $\Omega$ as,

$$\int_a r\mathbf{e}_r \wedge \mathbf{t}_{(\mathbf{e}_z)}(r,\theta,L)da = \int_a r^2\mu\Omega da\mathbf{e}_z = T\mathbf{e}_z. \tag{9.13}$$

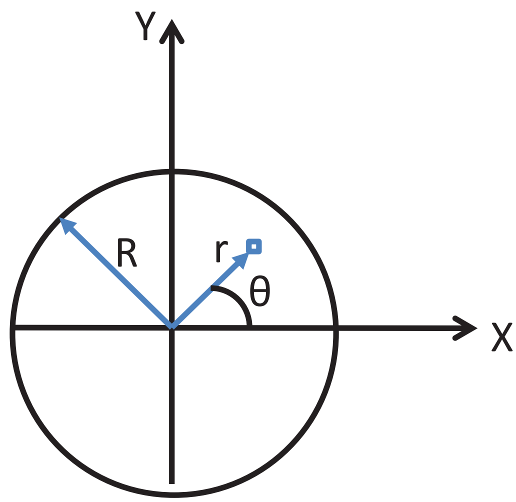



Figure 9.6: Solid circular cross section of a bar

Thus,

$$\Omega = \frac{T}{\int_a r^2 \mu da}, \tag{9.14}$$

which for homogeneous sections reduces to

$$\mu\Omega = \frac{T}{J}, \tag{9.15}$$

where

$$J = \int_a r^2 da, \tag{9.16}$$

is the polar moment of inertia. Combining equations (9.10) and (9.15) we obtain

$$\frac{\sigma_{\theta z}}{r} = \mu\Omega = \frac{T}{J}. \tag{9.17}$$

This is called the torsion equation.

### 9.2.1 Circular bar

As discussed, for illustration, we consider a bar with solid circular cross section of radius $R$, as shown in figure 9.6. From the general expression for

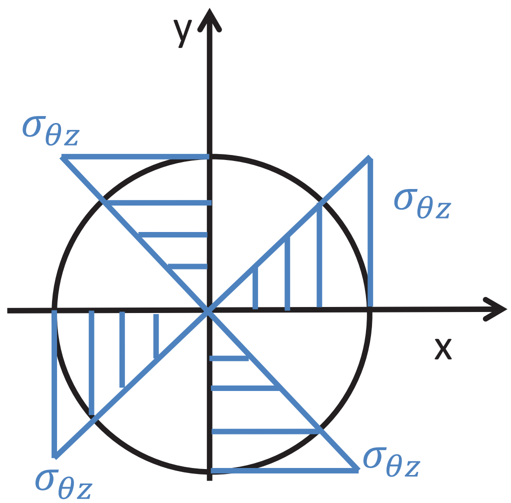



Figure 9.7: Shear stresses in a bar with solid circular cross section subjected to end torsion

torsion of a thick walled close section (9.17), we find that the shear stresses in the bar is given by,

$$\sigma_{\theta z} = r\frac{2T}{\pi R^4}, \tag{9.18}$$

where we have computed $J = \int_0^{2\pi} d\theta \int_0^R r^3 dr$. Thus, we find that the shear stress varies linearly with the radial distance, as shown in figure 9.7.

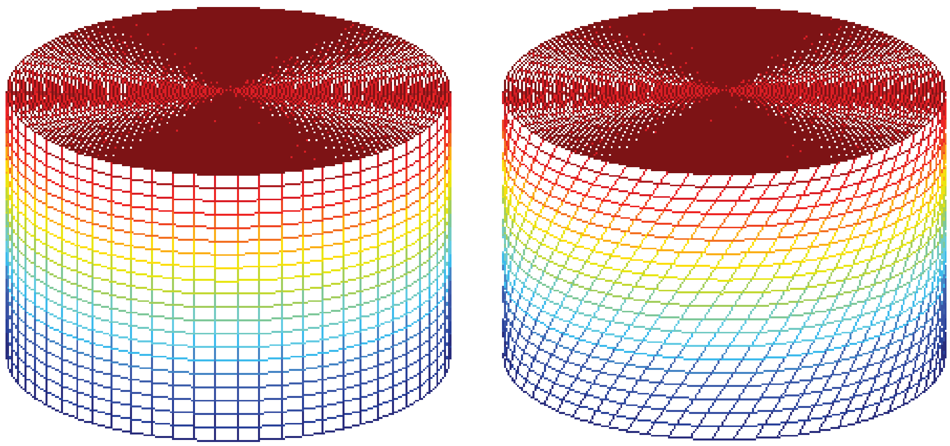
Then, the angle of twist per unit length, $\Omega$ is also obtained from (9.17) as,

$$\Omega = \frac{2T}{\mu\pi R^4}. \tag{9.19}$$

Having determined the angle of twist per unit length, the entire displacement field corresponding to an applied torque, $T$ given by equation (9.8) is known. The same is plotted in figure 9.8.

## 9.3 Twisting of solid open section

In this section, as in the previous, we consider a bar subjected to twisting moments at its ends. The bar axis is straight and the shape of the cross section



(a) Undeformed circular bar (b) Deformed circular bar

Figure 9.8: Deformation of a bar with circular cross section subjected to end torsion

is constant along the axis. The bar is assumed to have a solid cross section, i.e, a simply connected domain. A domain is said to be simply connected if any closed curve in the domain can be shrunk to a point in the domain itself, without leaving the domain. A domain which is not simply connected is said to be multiply connected. We shall analyze multiply connected domains in the next section.

Though the problem formulation does not assume any specific shape for the cross section, the displacement and stress field obtained are cross section specific. Let us assume that the boundary of the cross section is defined by a function, $f(x, y) = 0$. This function for a ellipse centered about the origin and major and minor axis oriented about the $\mathbf{e}_x$ and $\mathbf{e}_y$ directions would be, $f(x, y) = x^2/a^2 + y^2/b^2 - 1$. Consequently, for the body is assumed to occupy a region in Euclidean point space, defined by $\mathcal{B} = \{(x, y, z)|f(x, y) \leq 0, 0 \leq Z \leq L\}$, where $L$ is a constant. Here we are assuming that the bar is a solid cross section with the cross section having only one surface, $f(x, y) = 0$ denoted by $\partial\mathcal{A}$. To be more precise, the cross section is simply connected. Since, we are interested in the case where in the bar is subjected to pure end torsional moment, the traction boundary conditions are:

$$\mathbf{t}_{(\mathbf{e}_n)}(\partial\mathcal{A}, z) = \mathbf{o}, \tag{9.20}$$

$$\int_a \mathbf{t}_{(-\mathbf{e}_z)}(x, y, 0)da = \mathbf{o}, \tag{9.21}$$

$$\int_a \mathbf{t}_{(\mathbf{e}_z)}(x, y, L)da = \mathbf{o}, \tag{9.22}$$

$$\int_a (x\mathbf{e}_x + y\mathbf{e}_y) \wedge \mathbf{t}_{(-\mathbf{e}_z)}(z, y, 0)da = -T\mathbf{e}_z, \tag{9.23}$$

$$\int_a (x\mathbf{e}_x + y\mathbf{e}_y) \wedge \mathbf{t}_{(\mathbf{e}_z)}(x, y, L)da = T\mathbf{e}_z, \tag{9.24}$$

where

$$\mathbf{e}_n = \frac{[\frac{\partial f}{\partial y}\mathbf{e}_x - \frac{\partial f}{\partial x}\mathbf{e}_y]}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}, \tag{9.25}$$

and $T$ is a constant. Here $\mathbf{e}_n$ is obtained such that it is in the $xy$ plane, perpendicular to $grad(f)$ and is a unit vector. Recognize that $grad(f)$ gives the tangent vector to the cross section for any point in $\partial\mathcal{A}$. The orientation of $\mathbf{e}_n$ and $grad(f)$ is as shown in figure 9.9. Further, since one end of this body is assumed to be fixed against twisting but free to displace axially and the other end free to displace in all directions, the displacement boundary conditions for this problem is,

$$\mathbf{u}(x, y, 0) = u_z(x, y)\mathbf{e}_z, \tag{9.26}$$

where $u_z$ is any function of $x$ and $y$. Here we have assumed that the surface of the body defined by $z = 0$ is fixed against twisting but free to displace axially and the other surfaces are free to displace in all directions.

In a solid cross section, as a result of a twist, each cross section undergoes a rotational displacement about the $z$ axis. For any two cross sections the relative angle of rotation is called the angle of twist between the two sections. Let $\Delta\beta$ be the angle of twist for two cross sections at distance $\Delta z$ and $\Omega$ be the angle of twist per unit length of the bar, then

$$\Delta\beta = \Omega(\Delta z). \tag{9.27}$$

Since the conditions are same for all cross sections, the above equation applies to any two cross sections at a distance $\Delta z$ along the bar length. Thus, $\Omega$ is

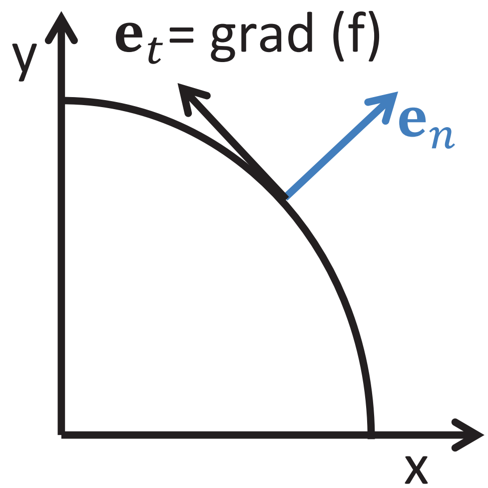



Figure 9.9: Boundary conditions at the surface of a bar subjected to end torsion

a constant along the bar length. Also, since at $z = 0$, $\beta = 0$, by virtue of the surface defined by $z = 0$ being fixed against rotation, the angle of twist, $\beta$ for a cross sections at distance $z$ from the surface defined by $z = 0$ is

$$\beta = \Omega z. \tag{9.28}$$

Consider a cross section at a distance $z$ from the fixed end. In that section consider a point $A$ with coordinates $(x, y, z)$. Assuming the origin of the coordinate system to be located at the axis of twisting, let $r$ denote the radial distance of the point $A$ from the origin and let $\theta$ denote the angle that this radial line makes with the $\mathbf{e}_x$, as shown in the figure 9.10. Let $A$ move to $A'$ due to twisting of the bar, as shown in the figure 9.10. The arc length $AA'$ would be equal to $r\beta$, where $\beta$ is the angle of twist at the cross section and is related to the angle of twist per unit length through (9.28). Hence, the arc length $AA'$ would be,

$$\widehat{AA'} = r\beta = r\Omega z. \tag{9.29}$$

Since, the angle of twist $\beta$ is small, we approximate the secant length, i.e., the length of the straight line between $AA'$ with its arc length. Consequently, the $x$ component of the displacement of $A$ is given by,

$$u_x = -AA' \sin(\theta) = -\widehat{AA'} \sin(\theta) = -r\Omega z \sin(\theta) = -\Omega z y, \tag{9.30}$$

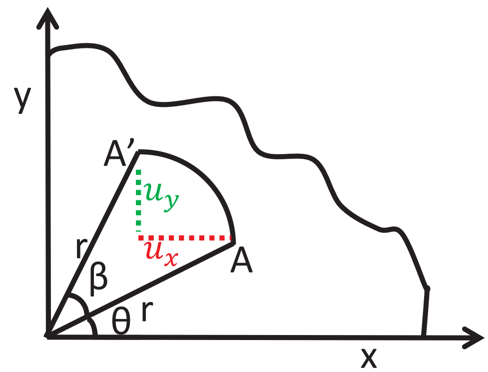



Figure 9.10: Displacements in the cross section of a bar subjected to end twisting moment

where we have used (9.29) and the relation that $y = r\sin(\theta)$. Similarly, the $y$ component of the displacement of $A$ is,

$$u_x = AA'\cos(\theta) = \widehat{AA'}\cos(\theta) = r\Omega z\cos(\theta) = \Omega zx, \tag{9.31}$$

where as before we used (9.29) and the relation $x = r\cos(\theta)$. Since, there is no restraint against displacement along $\mathbf{e}_z$ direction at any section, all cross sections is assumed to undergo the same displacement along the $z$ direction and hence

$$u_z = \psi(x, y), \tag{9.32}$$

where $\psi$ is a function of $x$ and $y$ only. That is, we have assumed that the $z$ component of the displacement is independent of the axial location of the section. Presence of this $z$ component of the displacement is the characteristic of open sections. It is due to this $z$ component of the displacement the plane section distorts and is said to warp. Consequently, $\psi$ is called Saint-Venant's warping function. Thus, the displacement field for a bar subjected to twisting moments at the end and is free to warp is given by,

$$\mathbf{u} = -\Omega zy\mathbf{e}_x + \Omega zx\mathbf{e}_y + \psi(x, y)\mathbf{e}_z. \tag{9.33}$$

It can be easily verified that the above displacement field (9.33) satisfies the displacement boundary condition (9.26).

Substituting equation (9.33) in the strain-displacement equation (7.1), the Cartesian components of the strain are computed to be

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 0 & -\Omega y + \frac{\partial\psi}{\partial x} \\ 0 & 0 & \Omega x + \frac{\partial\psi}{\partial y} \\ -\Omega y + \frac{\partial\psi}{\partial x} & \Omega x + \frac{\partial\psi}{\partial y} & 0 \end{pmatrix}. \tag{9.34}$$

For this state of strain, the corresponding Cartesian components of the stress are obtained from Hooke's law (7.2) as

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \mu\left(-\Omega y + \frac{\partial\psi}{\partial x}\right) \\ 0 & 0 & \mu\left(\Omega x + \frac{\partial\psi}{\partial y}\right) \\ \mu\left(-\Omega y + \frac{\partial\psi}{\partial x}\right) & \mu\left(\Omega x + \frac{\partial\psi}{\partial y}\right) & 0 \end{pmatrix}. \tag{9.35}$$

For this stress state, (9.35) to be possible in a body in static equilibrium, without any body force acting on it, it should satisfy the balance of linear momentum equations (7.6) which requires that,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0. \tag{9.36}$$

Now, we have to find $\psi$ such that equation (9.36) holds along with the prescribed traction boundary conditions (9.20) through (9.24).

Boundary condition (9.20) requires

$$\mu\left[-\Omega y + \frac{\partial\psi}{\partial x}\right]\frac{\partial f}{\partial y} - \mu\left[\Omega x + \frac{\partial\psi}{\partial y}\right]\frac{\partial f}{\partial x} = 0, \tag{9.37}$$

for $(x, y) \in \partial\mathcal{A}$. Both the boundary conditions (9.21) and (9.22) yield the following restriction on $\psi$,

$$\int_a \mu\left[-\Omega y + \frac{\partial\psi}{\partial x}\right] da = 0, \quad \int_a \mu\left[\Omega x + \frac{\partial\psi}{\partial y}\right] da = 0, \tag{9.38}$$

On assuming that the bar is homogeneous and appealing to Green's theorem (section 2.9.3), equation (9.38) reduces to,

$$\int_a \mu\left[-\Omega y + \frac{\partial\psi}{\partial x}\right] da = -\mu\Omega y_c A + \oint_c \psi \frac{\frac{\partial f}{\partial y}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} ds = 0, \tag{9.39}$$

$$\int_a \mu\left[\Omega x + \frac{\partial\psi}{\partial y}\right] da = \mu\Omega x_c A - \oint_c \psi \frac{\frac{\partial f}{\partial x}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}} ds = 0, \tag{9.40}$$

where $(x_c, y_c)$ is the coordinates of the centroid of the cross section. On assuming that the axis of twisting coincides with the centroid and further requiring that the origin of the coordinate system coincide with the centroid of the cross section, equations (9.39) and (9.40) requires,

$$\oint_c \psi \frac{\partial f}{\partial y} ds = 0, \quad \oint_c \psi \frac{\partial f}{\partial x} ds = 0. \tag{9.41}$$

Finally, the boundary conditions (9.23) and (9.24) require that

$$\int_a (x\sigma_{yz} - y\sigma_{xz}) da = \int_a \mu\Omega(x^2 + y^2) da + \int_a \mu \left( x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x} \right) da = T. \tag{9.42}$$

Recognizing that $\int_a (x^2 + y^2) da = J$, polar moment of inertia equation (9.42) can be simplified as,

$$T = \mu\Omega J + \mu \int_a \left( x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x} \right) da, \tag{9.43}$$

where we have used the assumption already made that the bar is homogeneous. For many cross sections, $\int_a \left( x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x} \right) da < 0$. Thus, the torsional stiffness[1] which would be $\mu J$, in the absence of warping (i.e., $\psi = 0$), decreases due to warping (i.e., when $\psi \neq 0$). Hence, warping is said to decrease the torsional stiffness of the cross section. Equation (9.43) is used find $\Omega$.

Experience has shown that it is difficult to find $\psi$ that satisfies the governing equation (9.36) along with the boundary conditions (9.37) and (9.41). Hence, we recast the problem using Prandtl stress function.

### Stress function formulation

Defining a differentiable function, $\phi = \hat{\phi}(x, y)$ called the Prandtl stress function, we relate the components of the stress to this stress function as,

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \frac{\partial \phi}{\partial y} \\ 0 & 0 & -\frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} & -\frac{\partial \phi}{\partial x} & 0 \end{pmatrix}, \tag{9.44}$$

[1] Torsional stiffness is defined as the torque required to cause a unit angle of twist per unit length.

so that the equilibrium equations (7.6) are satisfied for any choice of $\phi$.

Now, one can proceed in one of two ways. Follow the standard approach and find the strain corresponding to the stress state (9.44), substitute the same in the compatibility conditions and find the governing equation that $\phi$ should satisfy. Here we obtain the same governing equation through an alternate approach.

Equating the stress states (9.35) and (9.44) as it represents for the same boundary value problem, we obtain

$$\frac{\partial \phi}{\partial y} = \mu\left[-\Omega y + \frac{\partial \psi}{\partial x}\right], \tag{9.45}$$

$$\frac{\partial \phi}{\partial x} = -\mu\left[\Omega x + \frac{\partial \psi}{\partial y}\right]. \tag{9.46}$$

Differentiating equation (9.45) with respect to $y$ and equation (9.46) with respect to $x$ and adding the resulting equations, we obtain

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2\mu\Omega, \tag{9.47}$$

where we have made use of the requirement that $\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$. By virtue of the stress function being smooth enough so that $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$, the governing equation for $\psi$ (9.36) is trivially satisfied.

The boundary condition given in equation (9.37) in terms of the warping function, $\psi$ is rewritten in terms of the Prandtl stress function, $\phi$ using equations (9.45) and (9.46) as,

$$\frac{\partial \phi}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial x} = grad(\phi) \cdot grad(f) = 0, \tag{9.48}$$

for $(x, y) \in \partial\mathcal{A}$. Since, $grad(f) \neq \mathbf{o}$ and its direction changes with the location on the boundary of the cross section, $grad(\phi) = \mathbf{o}$ on the boundary of the cross section. This implies that

$$\phi(x, y) = C_0, \quad \text{for } (x, y) \in \partial\mathcal{A}, \tag{9.49}$$

where $C_0$ is a constant. Since, the stress and displacement fields depend only in the derivative of the stress potential and not its value at a location, it suffices to find this stress function up to a constant[2]. Therefore, we arbitrarily set $C_0 = 0$, understanding that it can take any value and that the stress and displacement field would not change because of this. Hence, equation (9.49) reduces to requiring,

$$\phi(x, y) = 0, \quad \text{for } (x, y) \in \partial\mathcal{A}. \tag{9.50}$$

The boundary condition (9.21) and (9.22) for the stress state (9.44) require

$$\int_a \frac{\partial \phi}{\partial y} da = 0, \qquad \int_a \frac{\partial \phi}{\partial x} da = 0. \tag{9.51}$$

Appealing to Green's theorem for simply connected domains, the above equation (9.51) reduces to

$$\int_a \frac{\partial \phi}{\partial y} da = \oint_c \phi \frac{\partial f}{\partial x} ds = C_0 \oint_c \frac{\partial f}{\partial x} ds = 0, \tag{9.52}$$

$$\int_a \frac{\partial \phi}{\partial x} da = \oint_c \phi \frac{\partial f}{\partial y} ds = C_0 \oint_c \frac{\partial f}{\partial y} ds = 0, \tag{9.53}$$

where we have used (9.49) and the fact that $\oint_c \frac{\partial f}{\partial x} ds = 0$ and $\oint_c \frac{\partial f}{\partial y} ds = 0$ for closed curves.

Finally, the boundary conditions (9.23) and (9.24) for the stress state (9.44) yields,

$$-\int_a \left[ x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \right] da = T. \tag{9.54}$$

Noting that

$$\int_a x \frac{\partial \phi}{\partial x} = \int_a \left[ \frac{\partial (x\phi)}{\partial x} - \phi \right] da = \oint_c x\phi \frac{\partial f}{\partial y} ds - \int_a \phi da = -\int_a \phi da, \tag{9.55}$$

$$\int_a y \frac{\partial \phi}{\partial y} = \int_a \left[ \frac{\partial (y\phi)}{\partial y} - \phi \right] da = \oint_c y\phi \frac{\partial f}{\partial x} ds - \int_a \phi da = -\int_a \phi da, \tag{9.56}$$

where we have used the Green's theorem for simply connected domain and equation (9.50). Substituting the above equations in (9.54) we obtain,

$$T = 2 \int_a \phi da. \tag{9.57}$$

[2]This means that if $\phi$ is a stress function for the given boundary value problem then $\phi + C_0$, where $C_0$ is some constant, will also be an admissible stress function for the same boundary value problem.

Thus, we have to find $\phi$ such that the governing equation (9.47) has to hold along with the boundary condition (9.50). Then, we use equation (9.57) to find the torsional moment required to realize a given angle of twist per unit length $\Omega$.

**Membrane analogy**

The governing equation (9.47) along with the boundary condition (9.50) and (9.57) is identical with those governing the static deflection under uniform pressure of an elastic membrane in the same shape as that of the member subjected to torsion. This fact creates an analogy between the torsion problem and elastic membrane under uniform pressure. This analogy is exploited to get the qualitative features that the stress function should posses and this aids in developing approximate solutions. Obtaining the governing equation for the static deflection of an elastic membrane under uniform pressure and showing that it is identical to (9.47) is beyond the scope of this lecture notes. One may refer Sadd [4] for the same.

In the following three sections, we apply the above general framework to solve problems of bars with specific cross section shapes subjected to end torsion.

### 9.3.1 Solid elliptical section

The first cross section shape that we consider is that of an ellipse. That is we study a bar with elliptical cross section, whose boundary is defined by the function,

$$f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \tag{9.58}$$

where $a$ and $b$ are constants and we have assumed that the major and minor axis of the ellipse coincides with the coordinate basis, as shown in the figure 9.11.

Choosing the Prandtl stress function to be,

$$\phi = C\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right], \tag{9.59}$$

where $C$ is a constant, we find that it satisfies the boundary condition (9.50).

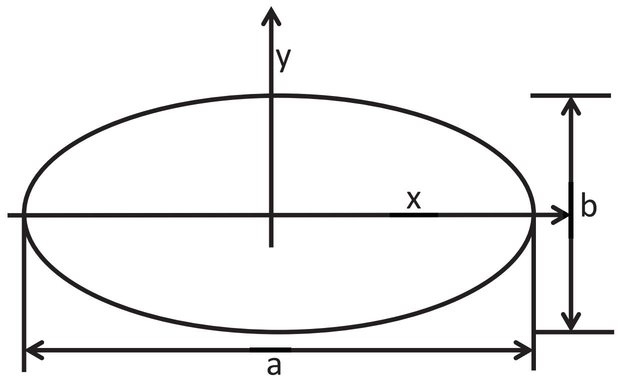



Figure 9.11: Elliptical cross section of a bar

Substituting the stress function, (9.59) in the equation (9.47) we obtain

$$C = -\mu\Omega\frac{a^2b^2}{a^2+b^2}. \tag{9.60}$$

Using (9.59) and (9.60) in (9.57), twisting moment is obtained as

$$T = \mu\Omega\pi\frac{a^3b^3}{a^2+b^2}. \tag{9.61}$$

Corresponding to the stress function (9.59) the shear stresses are computed to be

$$\sigma_{xz} = \frac{\partial\phi}{\partial y} = \frac{2C}{b^2}y = -\frac{T}{2I_{xx}}y, \sigma_{yz} = -\frac{\partial\phi}{\partial x} = \frac{2C}{a^2}x = \frac{T}{2I_{yy}}x, \tag{9.62}$$

where $I_{xx} = \pi ab^3/4$, $I_{yy} = ba^3/4$ and we have substituted for the constant, $C$ from equation (9.60). The variation of these shear stresses over the cross section is indicated in figure 9.12.

The resultant shear stress in the $xy$ plane is given by,

$$\tau = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2} = \frac{2T}{\pi ab}\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}. \tag{9.63}$$

It is clear from equation (9.63) that the extremum shear stress occurs at $(0,0)$ or the boundary of the cross section. It can be seen that at $(0,0)$ minimum

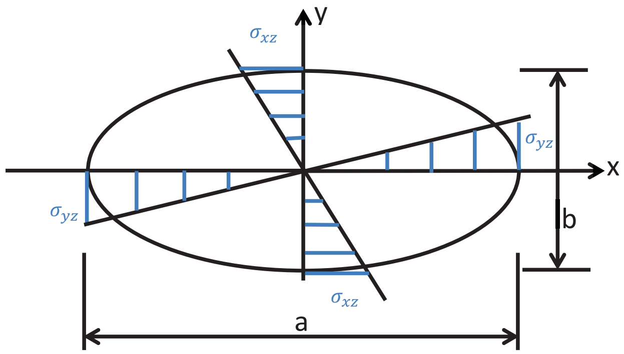



Figure 9.12: Shear stresses in a bar with elliptical cross section subjected to end torsion

shear stress occurs, as $\tau = 0$ at this location. At the boundary the extremum shear stresses would occur at $(0, \pm b)$ and $(\pm a, 0)$. Thus, the extremum shear stresses are $\tau_{ext} = 2T/(\pi ab^2)$ at $(0, \pm b)$ and $\tau_{ext} = 2T/(\pi ba^2)$ at $(\pm a, 0)$. If $a > b$, then the maximum shear stress in the elliptical cross section is,

$$\tau_{max} = \frac{2T}{\pi ab^2}. \tag{9.64}$$

Substituting (9.59) in equations (9.45) and (9.46) and rearranging we obtain

$$\frac{\partial \psi}{\partial x} = \Omega y + \frac{2C}{\mu b^2} y, \quad \frac{\partial \psi}{\partial y} = -\left[\Omega x + \frac{2C}{\mu a^2} x\right]. \tag{9.65}$$

Solving the differential equations (9.65) we obtain

$$\psi(x, y) = T \frac{b^2 - a^2}{\pi a^3 b^3 \mu} xy + D_0, \tag{9.66}$$

where $D_0$ is a constant. Since, there is no rigid body translation, we require that $\psi(0, 0) = 0$. Hence, $D_0 = 0$. Thus, we find that the section warps into a diagonally symmetric surface as shown in figure 9.13.

Rearranging the equation (9.61) we can get $\Omega$ as a function of torque, $T$ as

$$\Omega = \frac{T[a^2 + b^2]}{\mu \pi a^3 b^3} \tag{9.67}$$

The entire displacement field, (9.33) for a bar with elliptical cross section is now known that we have found the warping function (9.66). The deformation

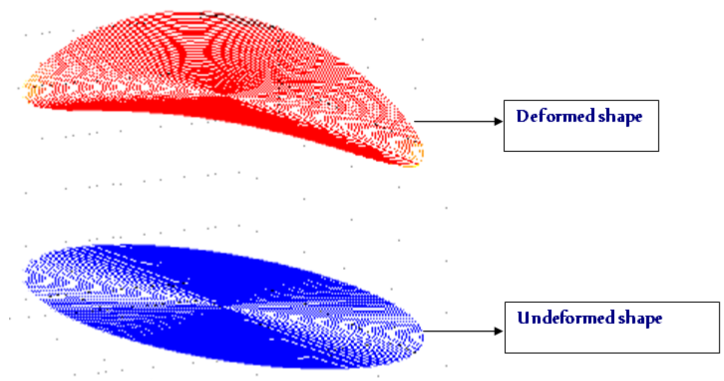



Figure 9.13: Warping deformation of a elliptical cross section due to end torsion

of a bar with the elliptical cross section computed using this displacement field is shown in figure 9.14.

Before concluding this section let us see the error that would have been made if one analyzes the elliptical section as a closed section. The polar moment of inertia for the elliptical section could be computed and shown to be $J = \pi ab(a^2 + b^2)/4$. Then, the angle of twist per unit length computed from (9.17) would be,

$$\Omega_{cl} = \frac{4T}{\mu\pi ab[a^2 + b^2]}. \tag{9.68}$$

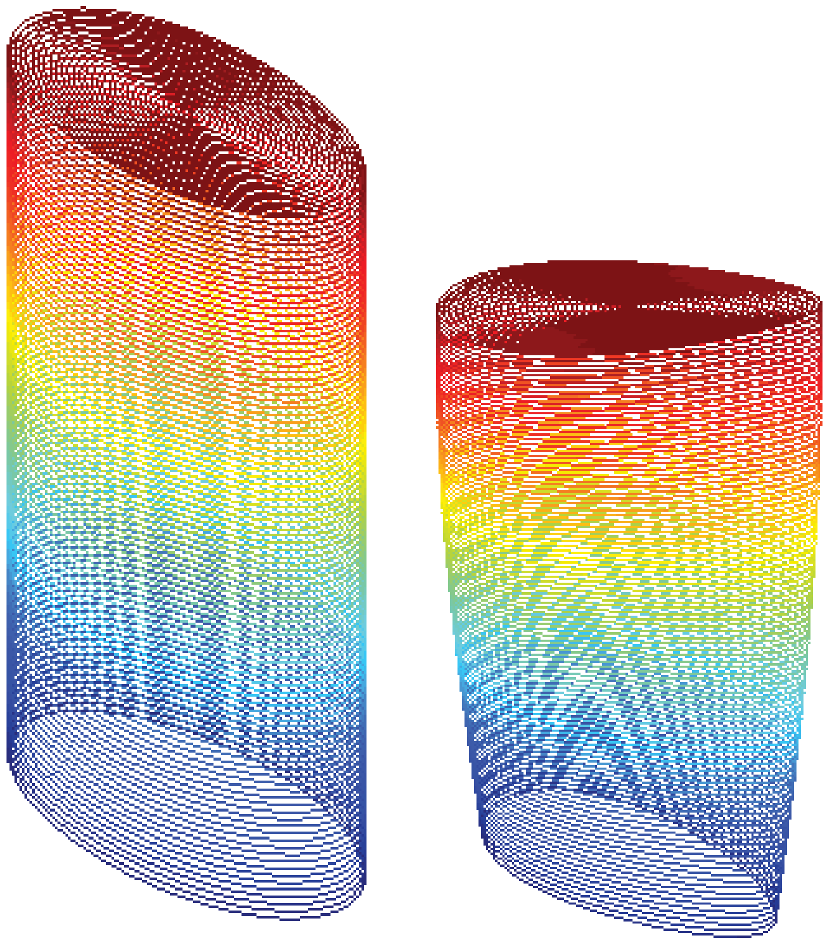
The ratio of $\Omega_{cl}/\Omega$, computed from equations (9.68) and (9.67) is

$$\frac{\Omega_{cl}}{\Omega} = \frac{4a^2b^2}{(a^2 + b^2)^2} < 1. \tag{9.69}$$

Thus, for a given torque open section twists more.

Next, we examine the shear stresses. If one uses (9.17) to compute the shear stresses, they would erroneously conclude that the maximum shear stress occurs at $(\pm a, 0)$, by virtue of these points being farthest from the center of twist and the value of this maximum shear stress would be,

$$\tau^{cl}_{max} = \frac{4T}{\pi b[a^2 + b^2]} \tag{9.70}$$



(a) Undeformed elliptical bar (b) Deformed elliptical bar

Figure 9.14: A bar with elliptical cross section subjected to end twisting moment

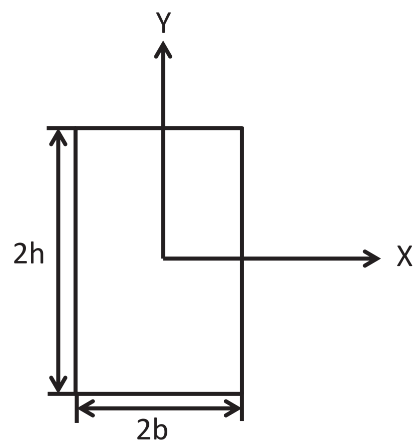



Figure 9.15: Rectangular cross section of a bar

Comparing equations (9.64) and (9.70) we find that

$$\frac{\tau_{max}^{cl}}{\tau_{max}} = 2\frac{ab}{a^2+b^2} < 1. \tag{9.71}$$

Thus, closed section underestimates the stresses in the bar.

Hence, from both strength and serviceability point of view assuming the elliptical cross section to be closed would result in an unsafe design.

### 9.3.2 Solid rectangular section

Next, we assume that the bar has a rectangular cross section of width $2b$ and depth $2h$, as shown in figure 9.15. For this section the boundary is defined by the function,

$$f(x,y) = (b^2 - x^2)(h^2 - y^2). \tag{9.72}$$

It can be verified that choosing the Prandtl stress function, $\phi$ as $Cf(x,y)$ would not satisfy the governing equation (9.47). Hence, we assume a stress function of the form,

$$\phi = \mu\Omega[V(x,y) + b^2 - x^2], \tag{9.73}$$

where $V$ is a function of $(x,y)$.

Substituting equation (9.73) in equation (9.47) and simplifying we obtain,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{9.74}$$

Then, the boundary condition (9.50) requires that

$$V(\pm b, y) = 0, \quad V(x, \pm h) = x^2 - b^2. \tag{9.75}$$

Thus, we need to solve the differential equation (9.74) subject to the boundary condition (9.75).

Towards this, we seek solution of the form

$$V = V_x(x)V_y(y), \tag{9.76}$$

where $V_x$ is a function of $x$ and $V_y$ is a function of $y$ alone. Substituting the assumed special form for $V$, (9.76) in equation (9.74) and simplifying we obtain,

$$\frac{1}{V_x}\frac{d^2V_x}{dx^2} = -\frac{1}{V_y}\frac{d^2V_y}{dy^2} = k^2, \tag{9.77}$$

where $k$ is any constant. Recognize that the left hand side of the equation is a function of $x$ alone and the right hand side is a function of $y$ alone. Hence, for these functions to be same, they have to be a constant. Solving the second order ordinary differential equations (9.77) we obtain,

$$V_x = A\cos(kx) + B\sin(kx), \quad V_y = C\cosh(ky) + D\sinh(ky). \tag{9.78}$$

Consistent with the boundary condition (9.75), we expect the functions $V_x$ and $V_y$ to be even functions, i.e., $V_x(-x) = V_x(x)$ and $V_y(-y) = V_y(y)$. Hence,

$$B = D = 0. \tag{9.79}$$

Substituting (9.79) in the equation (9.78) and the result into equation (9.76) we obtain,

$$V = AC\cos(kx)\cosh(ky) = E\cos(kx)\cosh(ky), \tag{9.80}$$

where $E$ is some arbitrary constant. The equation (9.80) on applying the boundary condition (9.75a), i.e., $V(\pm b, y) = 0$ yields that

$$k = \frac{(2n+1)\pi}{2b}, \tag{9.81}$$

where $n$ is any integer. Since, the governing equation for $V$, (9.74) is a Laplace equation, it is a linear equation. Consequently, if two functions satisfy the given equation (9.74) then a linear combination of these functions also satisfies the Laplace equation. So the solution (9.80) can be written as,

$$V = \sum_{n=1}^{\infty} E_n \cos\left(\frac{(2n+1)\pi}{2b} x\right) \cosh\left(\frac{(2n+1)\pi}{2b} y\right), \tag{9.82}$$

where $E_n$'s are constant. Substituting (9.82) in the boundary condition (9.75b), i.e., $V(x, \pm h) = x^2 - h^2$ we obtain,

$$\sum_{n=1}^{\infty} E_n \cosh\left(\frac{(2n+1)\pi h}{2b}\right) \cos\left(\frac{(2n+1)\pi x}{2b}\right) = x^2 - b^2. \tag{9.83}$$

Now, we have to find the constants $E_n$ such that for $-b \leq x \leq b$, a linear combination of cosines approximates the quadratic function on the right hand side of equation (9.83). Using the standard techniques in fourier analysis we obtain,

$$E_n \cosh\left(\frac{(2n+1)\pi h}{2b}\right) = \int_{-b}^{b} [x^2 - b^2] \cos\left(\frac{(2n+1)\pi x}{2b}\right). \tag{9.84}$$

Integrating the above equation and rearranging, we obtain

$$E_n = \frac{32b^2(-1)^n}{(2n+1)^3 \pi^3 \cosh\left(\frac{(2n+1)\pi h}{2b}\right)}. \tag{9.85}$$

Substituting equation (9.85) in (9.82) and the resulting equation in (9.73), we obtain

$$\phi = \mu\Omega \left[ (b^2 - x^2) + \frac{32b^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n \cos\left(\frac{(2n+1)\pi x}{2b}\right) \cosh\left(\frac{(2n+1)\pi y}{2b}\right)}{(2n+1)^3 \cosh\left(\frac{(2n+1)\pi h}{2b}\right)} \right]. \tag{9.86}$$

Having found the stress function, we are now in a position to find the shear stresses and the warping function. First, we compute the stresses as,

$$\begin{aligned} \sigma_{xz} &= \frac{\partial \phi}{\partial y} \\ &= \mu\Omega \frac{32b^2}{\pi^3} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n \cos\left(\frac{(2n+1)\pi x}{2b}\right) \left[\frac{(2n+1)\pi}{2b}\right] \sinh\left(\frac{(2n+1)\pi y}{2b}\right)}{(2n+1)^3 \cosh\left(\frac{(2n+1)\pi h}{2b}\right)} \right\}, \end{aligned} \tag{9.87}$$

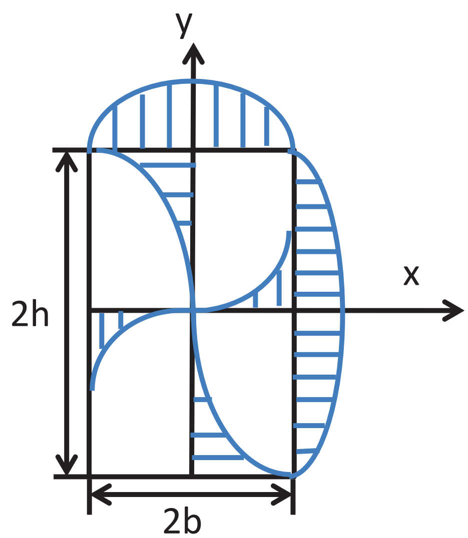



Figure 9.16: Schematic of shear stress distribution in a bar with solid rectangular cross section subjected to end torsion

$$\begin{aligned}\sigma_{yz} &= -\frac{\partial \phi}{\partial x}\\ &= \mu\Omega\left\{2x + \frac{32b^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n \sin\left(\frac{(2n+1)\pi x}{2b}\right)\left[\frac{(2n+1)\pi}{2b}\right]\cosh\left(\frac{(2n+1)\pi y}{2b}\right)}{(2n+1)^3\cosh\left(\frac{(2n+1)\pi h}{2b}\right)}\right\}.\end{aligned} \tag{9.88}$$

Figure 9.16 schematically shows the shear stress distribution in a rectangular cross section along the sides of the rectangle and along the lines $x = 0$ and $y = 0$. The shear stress is maximum on the boundary and zero at the center of the rectangle. The maximum shear stress occurs at the middle of the long side and is given by,

$$\tau_{max} = \sigma_{yz}(\pm b, 0) = \mu\Omega\left\{2b + \frac{32b}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2\cosh\left(\frac{(2n+1)\pi h}{2b}\right)}\right\}, \tag{9.89}$$

where we have assumed that $h \geq b$.

Substituting (9.87) in equation (9.45) and solving the first order ordinary

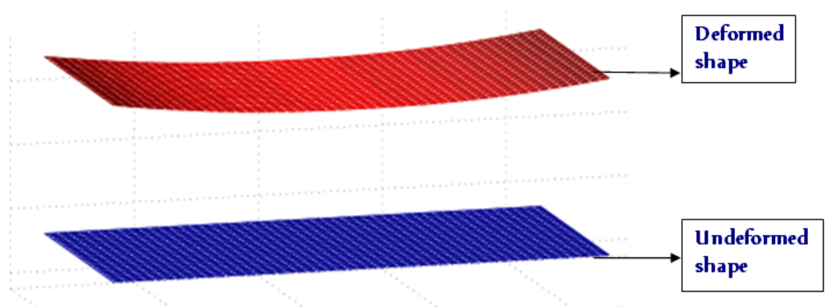



Figure 9.17: Warping deformation of a rectangular section due to end torsion

differential equation for $\psi$ we obtain

$$\psi = -\Omega\frac{32b^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n\sin\left(\frac{(2n+1)\pi x}{2b}\right)\sinh\left(\frac{(2n+1)\pi y}{2b}\right)}{(2n+1)^3\cosh\left(\frac{(2n+1)\pi h}{2b}\right)} + \Omega xy + g(y). \quad (9.90)$$

Substituting equations (9.88) and (9.90) in equation (9.46) and solving the first order ordinary differential equation for $g$ we obtain $g(y) = D_0$, a constant. Since, the bar does not move as a rigid body, we want $\psi(0,0) = 0$. Hence, $D_0 = 0$. Consequently, the warping function is,

$$\psi = -\Omega\frac{32b^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n\sin\left(\frac{(2n+1)\pi x}{2b}\right)\sinh\left(\frac{(2n+1)\pi y}{2b}\right)}{(2n+1)^3\cosh\left(\frac{(2n+1)\pi h}{2b}\right)} + \Omega xy. \quad (9.91)$$

Figure 9.17 plots the warped rectangular section.

Substituting (9.87) and (9.88) in equation (9.57) we can relate the applied torque, $T$ to the realized angle of twist per unit length, $\Omega$ as

$$T = \frac{16\mu\Omega b^3 h}{3} - \frac{1024\mu\Omega b^4}{\pi^5}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\tanh\left(\frac{(2n+1)\pi h}{2b}\right) \quad (9.92)$$

This relation is expressed as

$$T = \mu\Omega C_1(2h)(2b)^3, \quad (9.93)$$

where $C_1$ is a non-dimensional parameter which depends on $h/b$. In the same spirit, the maximum shear stress, (9.89) could be related to the applied

Table 9.1: Tabulation of parameters $C_1$ and $C_2$ as a function of $h/b$

| $h/b$ | $C_1$ | $C_2$ |
|---|---|---|
| 1 | 0.141 | 4.80 |
| 1.5 | 0.196 | 4.33 |
| 2.0 | 0.229 | 4.06 |
| 3.0 | 0.263 | 3.74 |
| 4.0 | 0.281 | 3.54 |
| 6 | 0.299 | 3.34 |
| 10 | 0.312 | 3.21 |
| $\infty$ | 0.333 | 3.00 |

torque as,

$$\tau_{max} = C_2 \frac{T}{(2h)(2b)^2}. \tag{9.94}$$

where $C_2$ is another non-dimensional parameter which depends on $h/b$. Values of $C_1$ and $C_2$ for various $h/b$ ratios are tabulated in table 9.1.

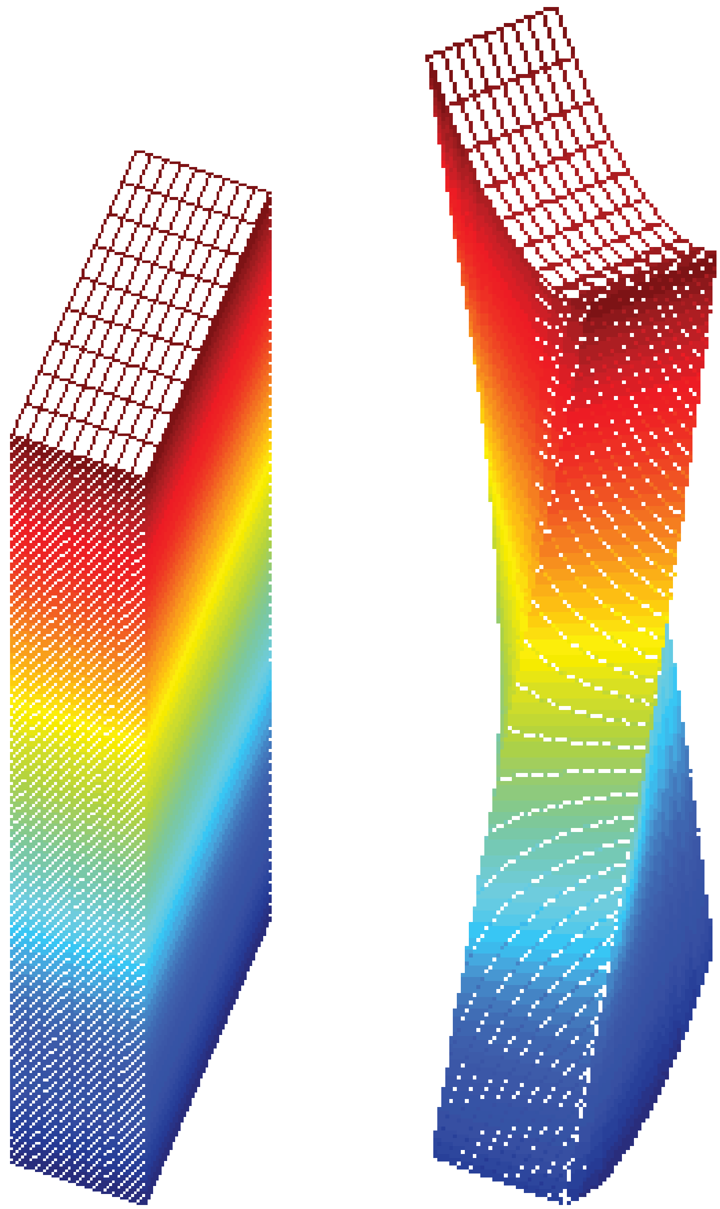
Thus, for thin rectangular sections, when $h/b$ tends to $\infty$,

$$T = \mu\Omega \frac{wt^3}{3}, \qquad \tau_{max} = \frac{3T}{wt^2}, \tag{9.95}$$

where we have assumed the section to have a thickness, $t = 2b$ and depth, $w = 2h$. It should be mentioned that irrespective of the orientation of the thin rectangular section, the equation (9.95) remains the same. This is because of the requirement that $h \geq b$. Physically, irrespective of the orientation of the section, the torsion is resisted by the spatial variation of the shear stresses through the thickness of the section as shown in figure 9.19.

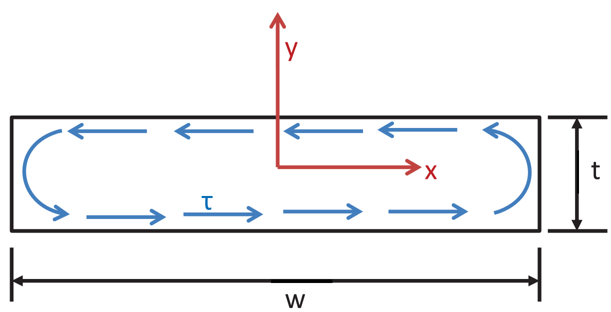
## 9.3.3 Thin rolled section

Here we study the problem of end torsion of a bar whose cross section is narrow and has small curvature such as those shown in figure 9.20. Such sections are usually made of hot rolled steel and hence are called rolled sections. Since, the shape across the thickness is similar to that of the thin rectangular section, excepting at the bent corners, it is assumed that the

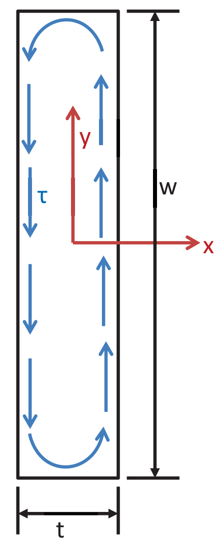


(a) Undeformed rectangular bar (b) Deformed rectangular bar

Figure 9.18: A bar with rectangular cross section subjected to end twisting moment

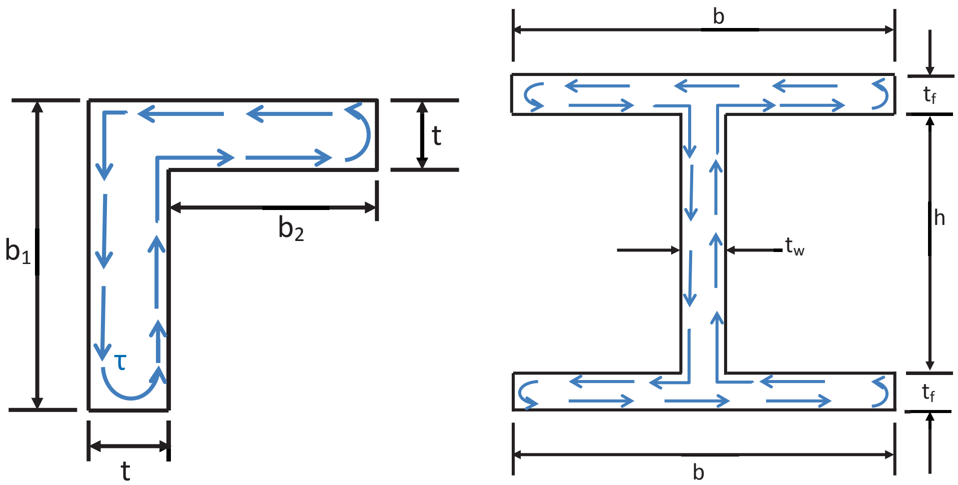



(a) Thin rectangular section oriented along x direction




(b) Thin rectangular section oriented along y direction

Figure 9.19: Schematic of shear stress distribution in a thin rectangular section in different orientations




(a) Angle section (b) I section

Figure 9.20: Schematic of shear stress distribution in a thin rolled sections

solution obtained for thin rectangular section holds. In order for the corners not to change the displacement and stress field from that obtained for a thin rectangular section, it is required that the radius of curvature of these corners be large.

Thus, for the angle section shown in figure 9.20a, the torque required to realize a given angle of twist per unit length and the maximum shear stress in the section is obtained from (9.95) as

$$T = \mu\Omega\frac{t^3(b_1 + b_2)}{3}, \qquad \tau_{max} = \frac{3T}{(b_1 + b_2)t^2}, \tag{9.96}$$

where we have treated the entire angle without the 90 degree bend, as a thin walled rectangular section. In order for the above equations to be reasonable, $(b_1 + b_2)/t > 10$ and the radius of the curvature of the bent corner large.

Similarly, for the I section shown in figure 9.20b, the torque required to realize a given angle of twist per unit length and the maximum shear stress in the section is obtained from (9.95) as

$$T = \mu\Omega\frac{2t_f^3 b_1 + t_w^3 h}{3}, \qquad \tau_{max} = \frac{3T}{2t_f^2 b_1 + t_w^2 h}, \tag{9.97}$$

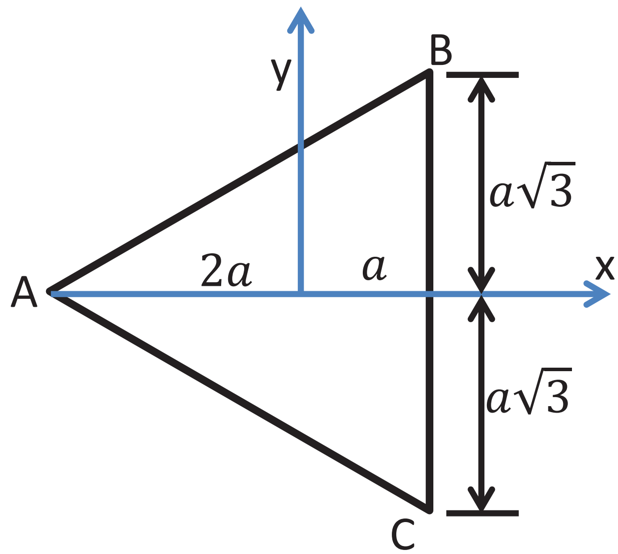



Figure 9.21: Orientation of an equilateral triangle with respect to a given coordinate basis

where the torsional stiffness due to each of the three plates - the two flange plates and the web plate - is summed algebraically. This is justified because the shear stress distribution in each of these plates is similar to that of a thin rectangular section subjected to torsion, as indicated in the figure 9.20b. However, comparison of this solution with the exact result for this geometry is beyond the scope of this notes.

### 9.3.4 Triangular cross section

Finally, we consider the torsion of a cylinder with equilateral triangle cross section, as shown in figure 9.21. The boundary of this section is defined by,

$$f(x,y) = [x - \sqrt{3}y + 2a][x + \sqrt{3}y + 2a][x - a] = 0, \tag{9.98}$$

where $a$ is a constant and we have simply used the product form of each boundary line equation. Assuming that the Prandtl stress function to be of the form,

$$\phi = K[x - \sqrt{3}y + 2a][x + \sqrt{3}y + 2a][x - a], \tag{9.99}$$

so that the boundary condition (9.50) is satisfied. It can then be verified that the potential given in equation (9.99) satisfies (9.47) if

$$K = -\frac{\mu\Omega}{6a}. \tag{9.100}$$

Substituting equation (9.99) in (9.57) and using (9.100) we obtain the torque to be

$$T = \frac{27}{5\sqrt{3}}\mu\Omega a^4 = \frac{3}{5}\mu\Omega J, \tag{9.101}$$

where the polar moment of inertia for the equilateral triangle section, $J = 3\sqrt{3}a^4$.

The shear stresses given in equation (9.44) evaluates to

$$\sigma_{xz} = \frac{\mu\Omega}{a}[x-a]y, \tag{9.102}$$

$$\sigma_{yz} = \frac{\mu\Omega}{2a}[x^2 + 2ax - y^2], \tag{9.103}$$

on using equations (9.99) and (9.100). The magnitude of the shear stress at any point is given by,

$$\tau = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2} = \frac{\mu\Omega}{a}\sqrt{(x^2+y^2)^2 + 4a^2(x+y)^2 + 4ax(x^2 - 2ay - 3y^2)}. \tag{9.104}$$

Since, for torsion the maximum shear stress occurs at the boundary of the cross section, we investigate the same at the three boundary lines. We begin with the boundary $x = a$. It is evident from (9.102) that on this boundary $\sigma_{xz} = 0$. Then, it follows from (9.103) that $\sigma_{yz}$ is maximum at $y = 0$ and this maximum value is $3a\mu\Omega/2$. It can be seen that on the other two boundaries too the maximum shear stress, $\tau_{max} = 3a\mu\Omega/2$.

Substituting (9.99) in equations (9.45) and (9.46) and solving the first order differential equations, we obtain the warping displacement as,

$$\psi = \frac{\Omega}{6a}y[3x^2 - y^2], \tag{9.105}$$

on using the condition that the origin of the coordinate system does not get displaced; a requirement to prevent the body from displacing as a rigid body.

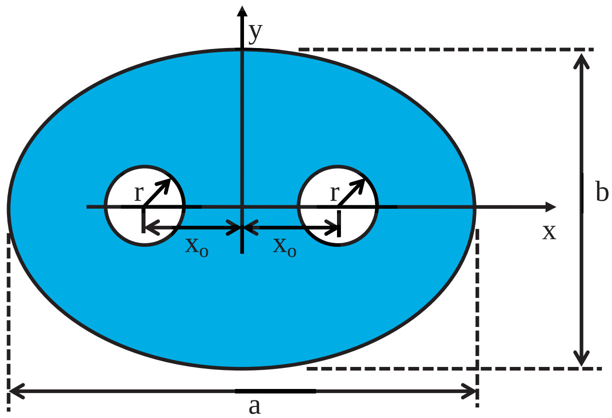



Figure 9.22: Example of a multiply connected cross section. Elliptical cross section with two circular holes.

## 9.4 Twisting of hollow section

In this section, as in the previous, we consider a bar subjected to twisting moments at its ends. The bar axis is straight and the shape of the cross section is constant along the axis. The bar is assumed to have a hollow cross section, i.e, a multiply connected domain. That is any closed curves in the cross section cannot be shrunk to a point in the domain itself without leaving the domain.

Let us assume that the boundary of the cross section is defined by a function, $f(x, y) = 0$, the set of points constituting this boundary by $\partial\mathcal{A}_o$, and the enclosed area by $A_o$. The boundary of the $i^{th}$ void in the interior of the cross section is defined by the function $g_i(x, y) = 0$, the set of points constituting this boundary of the $i^{th}$ void by $\partial\mathcal{A}_i$ and the area enclosed by the void, $A_i$. Thus, the area of the cross section, $A_{cs}$ is given by

$$A_{cs} = A_o - \sum_{i=1}^{N} A_i, \tag{9.106}$$

where we have assumed that there are $N$ voids.

Thus, for the cross section shown in figure 9.22, with the boundary of the cross section in the form of an ellipse centered about origin and oriented such that the major and minor axis coincides with the $\mathbf{e}_x$ and $\mathbf{e}_y$ directions respectively and having two circular shaped voids of radius, $r$ centered at $(\pm x_o, 0)$, the functions $f(x,y)$ and $g_i(x,y)$ would be: $f(x,y) = x^2/a^2 + y^2/b^2 - 1$, $g_1(x,y) = (x - x_o)^2 + y^2 - r^2$ and $g_2(x,y) = (x + x_o)^2 + y^2 - r^2$. Consequently, this body is assumed to occupy a region in Euclidean point space, defined by $\mathcal{B} = \{(x,y,z)|f(x,y) \leq 0, 0 \leq Z \leq L\} - \{(x,y,z)|f(x,y) \leq 0, 0 \leq Z \leq L\}$ $- \{(x,y,z)|f(x,y) \leq 0, 0 \leq Z \leq L\} - \{(x,y,z)|g_1(x,y) \leq 0, 0 \leq Z \leq L\}$ $- \{(x,y,z)|f(x,y) \leq 0, 0 \leq Z \leq L\} - \{(x,y,z)|g_2(x,y) \leq 0, 0 \leq Z \leq L\}$, where $L$ is a constant.

Since, we are interested in the case where in the bar is subjected to pure end torsional moment, the traction boundary conditions are:

$$\begin{aligned}
\mathbf{t}_{(\mathbf{e}_n)}(\partial\mathcal{A}_o, z) &= \mathbf{o}, && (9.107)\\
\mathbf{t}_{(\mathbf{e}_{m_i})}(\partial\mathcal{A}_i, z) &= \mathbf{o}, && (9.108)\\
\int_a \mathbf{t}_{(-\mathbf{e}_z)}(x,y,0)da &= \mathbf{o}, && (9.109)\\
\int_a \mathbf{t}_{(\mathbf{e}_z)}(x,y,L)da &= \mathbf{o}, && (9.110)\\
\int_a (x\mathbf{e}_x + y\mathbf{e}_y) \wedge \mathbf{t}_{(-\mathbf{e}_z)}(z,y,0)da &= -T\mathbf{e}_z, && (9.111)\\
\int_a (x\mathbf{e}_x + y\mathbf{e}_y) \wedge \mathbf{t}_{(\mathbf{e}_z)}(x,y,L)da &= T\mathbf{e}_z, && (9.112)
\end{aligned}$$

where

$$\mathbf{e}_n = \frac{[\frac{\partial f}{\partial y}\mathbf{e}_x - \frac{\partial f}{\partial x}\mathbf{e}_y]}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}, \qquad \mathbf{e}_{m_i} = \frac{[\frac{\partial g_i}{\partial y}\mathbf{e}_x - \frac{\partial g_i}{\partial x}\mathbf{e}_y]}{\sqrt{\left(\frac{\partial g_i}{\partial x}\right)^2 + \left(\frac{\partial g_i}{\partial y}\right)^2}}, \tag{9.113}$$

and $T$ is a constant. Here $\mathbf{e}_n$ is obtained such that it is in the $xy$ plane, perpendicular to $grad(f)$ and is a unit vector. Similarly, $\mathbf{e}_{m_i}$ is obtained such that it is in the $xy$ plane, perpendicular to $grad(g_i)$ and is a unit vector. Recognize that $grad(f)$ gives the tangent vector to the cross section for any point in $\partial\mathcal{A}_o$ and $grad(g_i)$ is the tangent vector to the cross section at any point in $\partial\mathcal{A}_i$. Further, since one end of this body is assumed to be fixed against twisting but free to displace axially and the other end free to displace in all directions, the displacement boundary conditions for this problem is,

$$\mathbf{u}(x,y,0) = u_z(x,y)\mathbf{e}_z, \tag{9.114}$$

where $u_z$ is any function of $x$ and $y$. Here we have assumed that the surface of the body defined by $z = 0$ is fixed against twisting but free to displace axially and the other surfaces are free to displace in all directions.

Thus, we find that the boundary value problem is similar to that in the previous section (section 9.3) except for the additional boundary condition (9.108). Hence, we assume that the stress is as given in equation (9.44), in terms of the Prandtl stress function, $\phi = \hat{\phi}(x, y)$. Following the standard procedure, as outlined in section 9.3, it can be shown that the stress function should satisfy,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2\mu\Omega, \tag{9.115}$$

where $\mu$ is the shear modulus, and $\Omega$ is the angle of twist per unit length.

For the assumed stress state, (9.44), the boundary condition (9.107) requires that $grad(\phi) \cdot grad(f) = 0$ on $\partial\mathcal{A}_o$. Thus,

$$\phi(x, y) = C_0, \quad \text{for } (x, y) \in \partial\mathcal{A}_o, \tag{9.116}$$

where $C_0$ is a constant. As before, since, the stress and displacement fields depend only in the derivative of the stress potential and not its value at a location, it suffices to find this stress function up to a constant. Therefore, we arbitrarily set $C_0 = 0$, understanding that it can take any value and that the stress and displacement field would not change because of this. Hence, equation (9.116) reduces to requiring,

$$\phi(x, y) = 0, \quad \text{for } (x, y) \in \partial\mathcal{A}_o. \tag{9.117}$$

The boundary condition (9.108) requires that $grad(\phi) \cdot grad(g_i) = 0$ on $\partial\mathcal{A}_i$. Hence,

$$\phi(x, y) = C_i, \quad \text{for } (x, y) \in \partial\mathcal{A}_i, \tag{9.118}$$

where $C_i$'s are constants. Though $\phi$ is a constant over all free surfaces, the value of this constant can differ between distinct free surfaces, i.e., $\partial\mathcal{A}_i$. Also, recognize that these constants cannot be set to zero, as we have already set $C_0 = 0$.

The boundary condition (9.109) and (9.110) for the stress state (9.44) require

$$\int_a \frac{\partial \phi}{\partial y} da = 0, \qquad \int_a \frac{\partial \phi}{\partial x} da = 0. \tag{9.119}$$

Appealing to Green's theorem for multiply connected domains, (2.278) and (2.279), the above equation (9.119) reduces to

$$\int_a \frac{\partial \phi}{\partial y} da = \oint_{\partial \mathcal{A}_o} \phi \frac{\partial f}{\partial x} ds - \sum_{i=1}^{N} \oint_{\partial \mathcal{A}_i} \phi \frac{\partial g_i}{\partial x} ds$$
$$= C_0 \oint_{\partial \mathcal{A}_o} \frac{\partial f}{\partial x} ds - \sum_{i=1}^{N} C_i \oint_{\partial \mathcal{A}_i} \frac{\partial g_i}{\partial x} ds = 0, \quad (9.120)$$

$$\int_a \frac{\partial \phi}{\partial x} da = \oint_{\partial \mathcal{A}_o} \phi \frac{\partial f}{\partial y} ds - \sum_{i=1}^{N} \oint_{\partial \mathcal{A}_i} \phi \frac{\partial g_i}{\partial y} ds$$
$$= C_0 \oint_{\partial \mathcal{A}_o} \frac{\partial f}{\partial y} ds - \sum_{i=1}^{N} C_i \oint_{\partial \mathcal{A}_i} \frac{\partial g_i}{\partial y} ds = 0, \quad (9.121)$$

where we have used (9.116) and (9.118) and the fact that $\oint_{\partial \mathcal{A}_o} grad(f) ds = 0$, $\oint_{\partial \mathcal{A}_i} grad(g_i) ds = 0$.

Finally, the boundary conditions (9.111) and (9.112) for the stress state (9.44) yields,

$$-\int_a \left[ x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \right] da = T. \quad (9.122)$$

Noting that

$$\int_a x \frac{\partial \phi}{\partial x} = \int_a \left[ \frac{\partial (x\phi)}{\partial x} - \phi \right] da = \oint_{\partial \mathcal{A}_o} x\phi \frac{\partial f}{\partial y} ds - \sum_{i=1}^{N} \oint_{\partial \mathcal{A}_i} x\phi \frac{\partial g_i}{\partial y} ds - \int_a \phi da$$
$$= -\sum_{i=1}^{N} C_i \oint_{\partial \mathcal{A}_i} x \frac{\partial g_i}{\partial y} ds - \int_a \phi da = -\sum_{i=1}^{N} C_i A_i - \int_a \phi da, \quad (9.123)$$

$$\int_a y \frac{\partial \phi}{\partial y} = \int_a \left[ \frac{\partial (y\phi)}{\partial y} - \phi \right] da = \oint_{\partial \mathcal{A}_o} y\phi \frac{\partial f}{\partial x} ds - \sum_{i=1}^{N} \oint_{\partial \mathcal{A}_i} y\phi \frac{\partial g_i}{\partial x} ds - \int_a \phi da$$
$$= -\sum_{i=1}^{N} C_i \oint_{\partial \mathcal{A}_i} y \frac{\partial g_i}{\partial x} ds - \int_a \phi da = -\sum_{i=1}^{N} C_i A_i - \int_a \phi da, \quad (9.124)$$

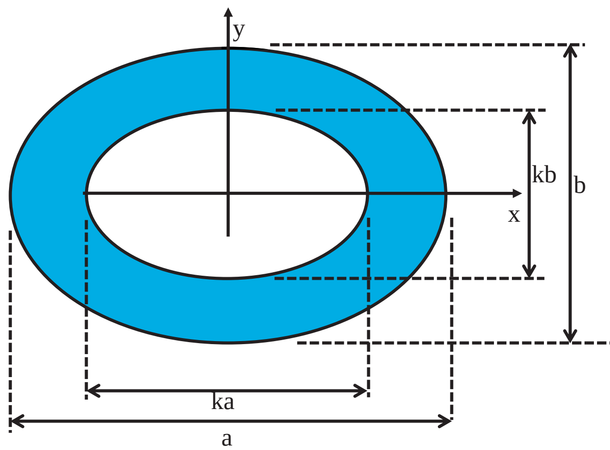



Figure 9.23: Hollow elliptical section

where we have used the Green's theorem for multiply connected domain (2.278) and (2.279) and equations (9.117) and (9.118). Recognize that from Green's theorem,

$$\oint_{\partial \mathcal{A}_i} x \frac{\partial g_i}{\partial y} ds = \int_{A_i} da = A_i, \qquad \oint_{\partial \mathcal{A}_i} y \frac{\partial g_i}{\partial x} ds = \int_{A_i} da = A_i, \tag{9.125}$$

where $A_i$ is the area enclosed by the void.

Substituting the above equations (9.123) and (9.124) in (9.122) we obtain,

$$T = 2 \int_a \phi da + 2 \sum_{i=1}^{N} C_i A_i. \tag{9.126}$$

Thus, we have to find $\phi$ such that the governing equation (9.115) has to hold along with the boundary conditions (9.117) and (9.118). Then, we use equation (9.126) to find the torsional moment required to realize a given angle of twist per unit length, $\Omega$.

## 9.4.1 Hollow elliptical section

Here we study the torsion of a bar with a hollow elliptical section as shown in figure 9.23 where the inner boundary is a scaled ellipse similar to that of the outer boundary. Since, the inner ellipse is a scaled outer ellipse, the

Prandtl stress function obtained for solid ellipse cross section in section 9.3.1 is a constant at the inner surface. Hence, the boundary condition (9.118) holds and therefore this stress function

$$\phi = \mu\Omega\frac{a^2b^2}{a^2+b^2}\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right], \tag{9.127}$$

is a solution to this boundary value problem of twisting of a hollow elliptical bar. Now, the constant $C_1$, the value of $\phi$ at the inner boundary is obtained as,

$$C_1 = \mu\Omega\frac{a^2b^2}{a^2+b^2}\left[k^2-1\right]. \tag{9.128}$$

Using (9.127) and (9.128) in (9.126), twisting moment is obtained as

$$T = \mu\Omega\pi\frac{a^3b^3}{a^2+b^2}[1-k^4]. \tag{9.129}$$

Since, the stress function is same as in the case of a solid elliptical section, the stress distribution is also identical as before. Therefore, the maximum shear stress occurs at $x = 0$, $y = \pm b$ and is given by,

$$\tau_{max} = \frac{2T}{\pi ab^2}\frac{1}{1-k^4}. \tag{9.130}$$

Before concluding this section we observe that this solution scheme can be applied to other cross sections whose inner boundary coincides with a contour line of the stress function in the corresponding solid section problem.

### 9.4.2 Thin walled tubes

In this section, we seek approximate solution to the case when, the bar, in the form of a thin walled tube is subjected to end torsion. We do not assume that the cross section of the tube to be of any particular shape. While we assume that the thickness of the cross section, $t$ is small, we do not assume that it is uniform. Let the boundary of the cross section be defined by a function, $f(x,y) = 0$, the set of points constituting this boundary be denoted by $\partial\mathcal{A}_o$, and the enclosed area by $A_o$. The boundary in the interior of the cross section is defined by the function $g_1(x,y) = 0$, the set of points constituting this boundary is denoted by $\partial\mathcal{A}_i$ and the area enclosed by, $A_i$. Then the area of the cross section is $A_{cs} = A_o - A_i$.

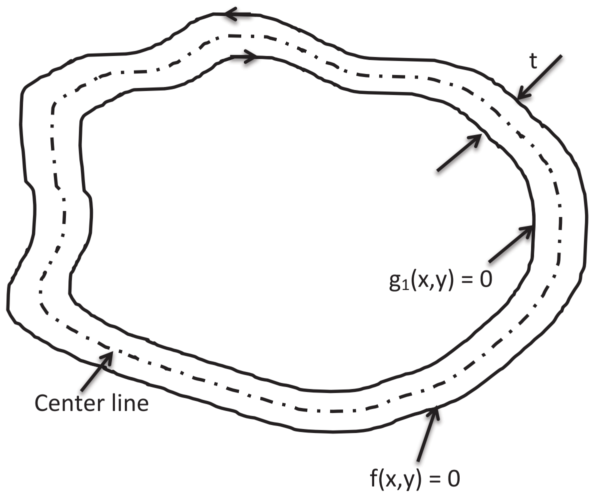



Figure 9.24: Thin walled section

In order to satisfy the requirement that the lateral surface is taction free, boundary conditions (9.107) and (9.108), the stress function has to be zero at the outer surface (9.117) and should be a constant at the inner surface (9.118). Hence, we assume that the stress function varies linearly through the thickness of the section. This assumption would be reasonable given that the thickness of the section is small. Thus,

$$\phi = \frac{C_1}{t}(r_o - r), \tag{9.131}$$

where $C_1$ is a constant, the value of $\phi$ at the inner surface, $r_o$ is the radial distance of the outer boundary from the centroid of the cross section, $r$ is the radial distance of any point in the cross section. If $r_i$ is the radial distance of the inner boundary from the centroid of the cross section, then $t = r_o - r_i$.

For the warping displacement, $\psi$ to be single valued in the thin walled tube shown in figure 9.24,

$$\oint_{\partial\mathcal{A}_i} \frac{\partial\psi}{\partial x} dx + \oint_{\partial\mathcal{A}_i} \frac{\partial\psi}{\partial y} dy = 0. \tag{9.132}$$

Substituting equations (9.45) and (9.46) in the above equation,

$$\oint_{\partial\mathcal{A}_i} \left[\frac{1}{\mu}\frac{\partial\phi}{\partial y} + \Omega y\right] dx - \oint_{\partial\mathcal{A}_i} \left[\frac{1}{\mu}\frac{\partial\phi}{\partial x} + \Omega x\right] dy = 0. \tag{9.133}$$

Using Green's theorem (2.271) we find that

$$\oint_{\partial\mathcal{A}_i} y dx - x dy = \int_a 2 dx dy = 2A_i. \tag{9.134}$$

Since, the stress function is assumed to vary linearly over the domain, $grad(\phi)$ $= -C_1/t\mathbf{e}_r$, where $\mathbf{e}_r$ is a unit vector normal to the boundary of the cross section. Consequently,

$$\oint_{\partial\mathcal{A}_i} \frac{\partial\phi}{\partial y} dx - \oint_{\partial\mathcal{A}_i} \frac{\partial\phi}{\partial x} dy = \oint_{\partial\mathcal{A}_i} grad(\phi)\cdot\mathbf{e}_r ds = -C_1 \oint_{\partial\mathcal{A}_i} \frac{ds}{t} \tag{9.135}$$

Substituting equations (9.134) and (9.135) in (9.133) we obtain

$$\Omega = \frac{C_1}{2\mu A_i} \oint_{\partial\mathcal{A}_i} \frac{ds}{t}. \tag{9.136}$$

Since, $\phi$ varies linearly through the thickness of the cross section, using trapezoidal rule for integration we find that

$$\int_a \phi da = \frac{C_1}{2} A, \tag{9.137}$$

where $A$ is the area of the cross section. Here it should be pointed out that trapezoidal rule for integration gives the exact value when the variation is linear. Now, substituting equation (9.137) in (9.126) we obtain the torque to be,

$$T = C_1 A + 2C_1 A_i = 2C_1 \left[A_i + \frac{A}{2}\right] = 2C_1 A_c, \tag{9.138}$$

where $A_c$ is the area enclosed by the centerline of the cross section. As a consequence of the section being thin walled, $A_i \approx A_c$. Therefore, from equations (9.136) and (9.138) we obtain,

$$\Omega = \frac{T}{4\mu A_c^2} \oint_{\partial\mathcal{A}_i} \frac{ds}{t}. \tag{9.139}$$

Here it should be recollected that the thickness of the section can vary along the circumference of the cross section.

Finally, we estimate the shear stress in the cross section. Since, the shear stress in the cross section is equal to the magnitude of $grad(\phi)$, we obtain

$$\tau = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2} = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2} = \frac{C_1}{t} = \frac{T}{2tA_c}. \tag{9.140}$$

Thus, $\tau t$ is constant throughout the section.

## 9.5 Summary

In this chapter, we obtained the stress and displacement fields in a straight, prismatic bar subjected to end twisting moments. For the case of thick walled closed sections and thin walled sections of any shape, we obtained an expression relating the torque with the angle of twist per unit length and the shear stress with the torque. However, for thick walled open sections, a procedure was outlined to obtain the stress and displacement fields and the same illustrated for elliptical, rectangular and triangular shaped cross sections. In all these cases, the section was assumed to be free to warp, so that no warping stresses develop. Generalization for the case when warping is restrained, is not straight forward and is beyond the scope of this lecture notes.

## 9.6 Self-Evaluation

1. Knowing that the inner diameter of an annular cylinder is $d_i = 30$ mm and that its outer diameter is $d_o = 40$ mm, determine the torque that causes a maximum shearing stress of 52 MPa in the annular cylinder.

2. Neglecting the effect of stress concentration, determine the largest torque that may be applied at A in the composite bar shown in figure 9.25. Assume that the end C is fixed against twisting and the allowable shear stress is 104 MPa in the 55 mm diameter solid steel rod BC and 55 MPa in the 40 mm diameter solid brass rod AB.

3. The composite shaft shown in figure 9.26 is to be twisted by applying a torque T at end A. Knowing that the shear modulus for steel is 77 GPa and for brass is 37 GPa, determine the largest angle through which end A may be rotated if the following allowable stresses are not to be exceeded 100 MPa in steel and 50 MPa in brass.

4. The stepped circular shaft, shown in figure 9.27, is rigidly attached to a wall at E. The diameter of the shaft AC, $d_{AC}$ is 25mm and that of the shaft CE, $d_{CE}$ is 50mm. Determine the angle of twist of the end A when the torques at B and D are applied as shown in the figure 9.27. Let $T_b = 150$ Nm and $T_d = 1000$ Nm. Assume the shear modulus $G =$ 80 GPa. Neglect stress concentration effects.

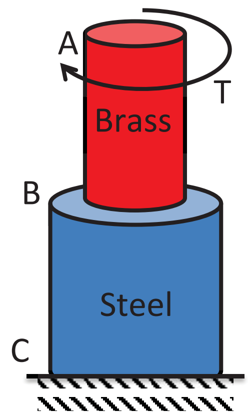



Figure 9.25: Figure for problem 2

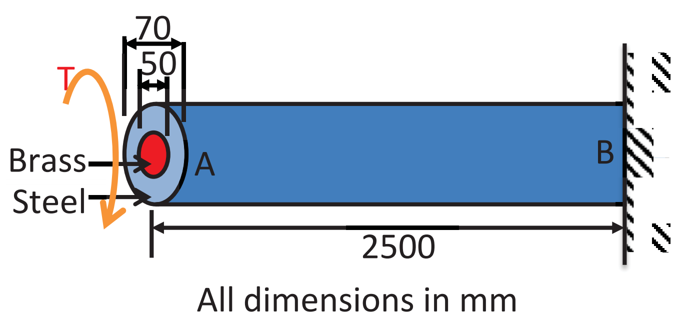



Figure 9.26: Figure for problem 3

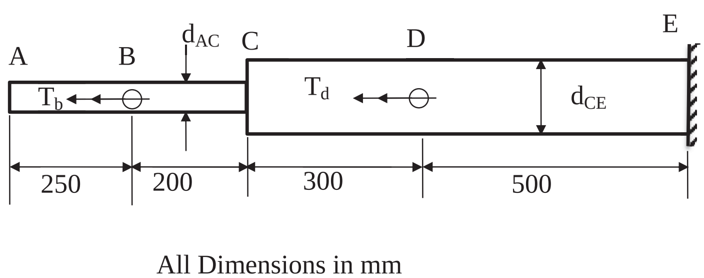



Figure 9.27: Figure for problem 4

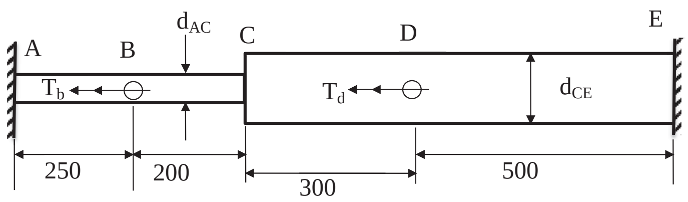



Figure 9.28: Figure for problem 5

5. (a) Determine the reactions for the circular stepped shafts shown in figure 9.28. Let the diameter of the shaft AC, $d_{AC}$ be 25mm and that of the shaft CE, $d_{CE}$ be 50mm and the torques, $T_b$ = 400 Nm and $T_d$ = 2000 Nm. (b) Plot the angle of twist diagram for the shaft along its length. The material parameters for the material with which this shaft is made are: Young's modulus, $E$ = 120 GPa and Poisson's ratio, $\nu$ = 1/3. Neglect stress concentration effects.

6. Same torque is applied to two hollow shafts of same length 0.5 m and made of same material but having cross sections in the shape of T and C, as shown in figure 9.29. Neglecting the effect of stress concentration, compare the maximum shearing stress that occurs in these sections and the angle of twist of the shaft. Assume, $t_w = t = 6$ mm, $t_f = 12$ mm, $b = 25$ mm, $h = 30$ mm. Recognizing that the area of both the cross sections is same, comment on which section is economical.

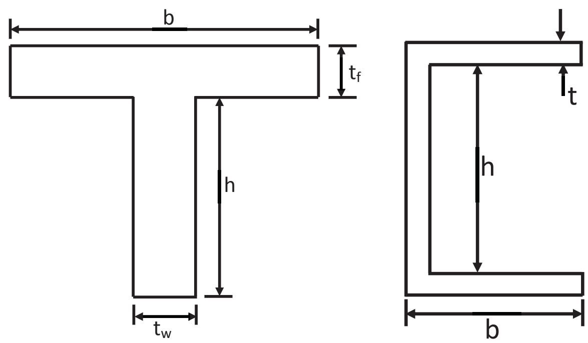
7. Verify that the admissibility of the following Prandtl stress function for a circular shaft with a keyway, as shown in figure 9.30,

$$\phi = K(b^2 - r^2)\left(1 - \frac{2a\cos(\theta)}{r}\right),$$

where $K$ is a constant. If the stress function is admissible,

   (a) Find the constant K

   (b) Compute the Cartesian components of the shear stress, $\sigma_{xz}$ and $\sigma_{yz}$




(a) T section (b) C section

Figure 9.29: Figure for problem 6

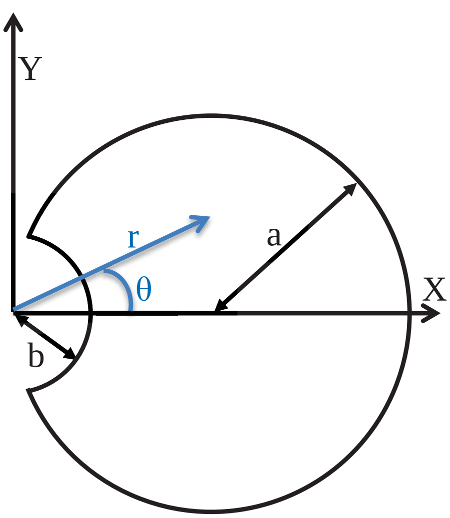



Figure 9.30: Circular shaft centered about $(a, 0)$ with a keyway. Figure for problem 7

(c) Show that the resultant shear stresses on the shaft and the keyway boundaries are given by

$$\tau_{shaft} = \mu\Omega a \left[\frac{b^2}{4a^2\cos^2(\theta)} - 1\right], \quad \tau_{keyway} = \mu\Omega[2a\cos(\theta) - b],$$

where $\Omega$ is the angle of twist per unit length and $\mu$ is the shear modulus.

(d) Determine the maximum value of the shear stress in the shaft and keyway

(e) Show that the maximum value of the keyway shear stress is approximately twice that of the shear stress in the shaft

(f) Determine the maximum shear stress for a solid shaft of circular section of radius $a$, $\tau_{max}^{solidshaft}$.

(g) Plot the $\tau_{max}^{keyway}/\tau_{max}^{solidshaft}$, where $\tau_{max}^{keyway}$ is the maximum shear stress in the keyway, for $0 \leq b/a \leq 1$ assuming same torque is applied on both the sections. Show that $\tau_{max}^{keyway}/\tau_{max}^{solidshaft} \to 2$ as $b/a \to 0$. This shows that a small notch would result in the doubling of the shear stresses in the circular section under same torsion.

(h) Show that the warping function for the shaft with a keyway is

$$\psi = -\frac{\mu\Omega}{2}\left[x^2 + y^2 - 2ax + \frac{2b^2ax}{x^2+y^2} - b^2\right],$$

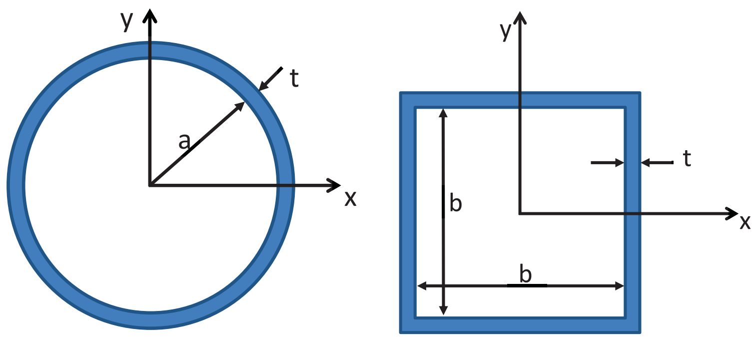
where $x$ and $y$ are the coordinates of a typical point in the cross section.

8. Two tubular thin walled sections shown in figure 9.31 have the same wall thickness $t$ and the same perimeter and hence $b = a\pi/2$. Neglecting the stress concentration, find the ratio of the shear stresses if,

(a) Both the sections are subjected to equal twisting moment

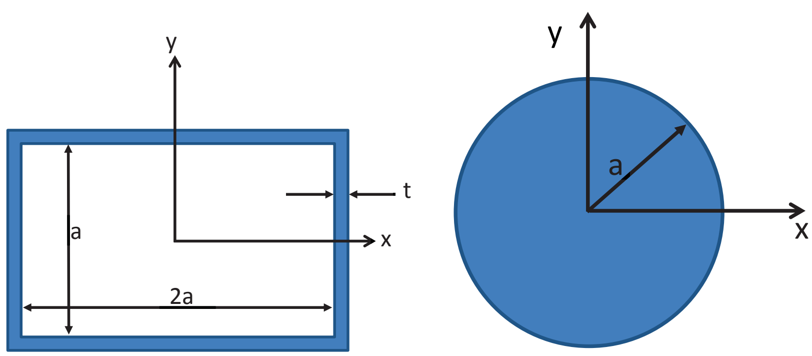
(b) Both the sections have the same angle of twist per unit length

Comment on which of these sections is economical.




(a) Annular cylinder (b) Annular square

Figure 9.31: Sections for problem 8




(a) Annular rectangular (b) Solid circular

Figure 9.32: Sections for problem 9

9. Two sections, a solid circular section and a hollow rectangular tube as shown in figure 9.32 are subjected to the same torque, $T$. If the two tubes are of same length, $L$ and made of the same material and differ only in the shape of the cross section, find the thickness of the rectangular section so that (a) the maximum shear stresses is same in both the sections (b) angle of twist per unit length is same for these two sections. Comment on which of these sections is economical.

10. During the early development of the torsion formulation, Navier assumed that there is no warping displacement for any cross section. Show that although such an assumed displacement field will satisfy all the required governing differential equations, it would not satisfy the boundary conditions and hence this is not an acceptable solution.

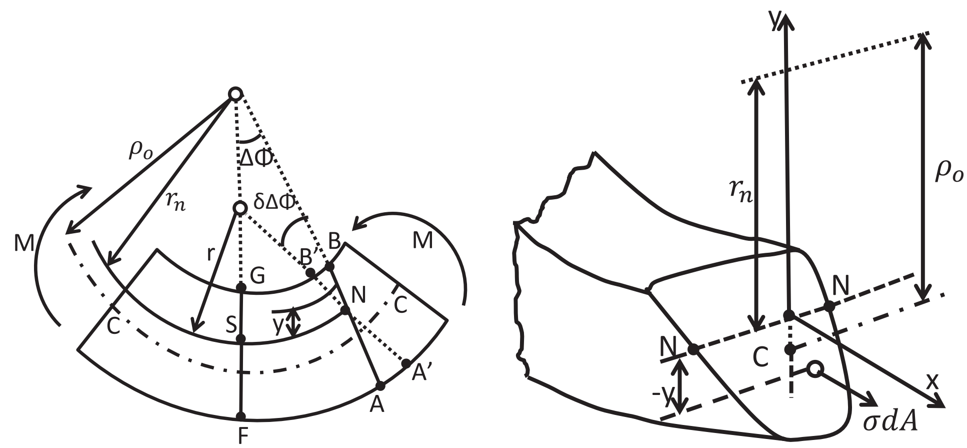
# Chapter 10

# Bending of Curved Beams

## 10.1 Overview

Till now, we have been studying members that are initially straight. In this chapter, we shall study the bending of beams which are initially curved. We do this by restricting ourselves to the case where the bending takes place in the plane of curvature. This happens when the cross section of the beam is symmetrical about the plane of its curvature and the bending moment acts in this plane. As we did for straight beams, we first obtain the solution assuming sections that are initially plane remain plane after bending. The resulting relation between the stress, moment and the deflection is called as Winkler-Bach formula. Then, using the two dimensional elasticity formulation, we obtain the stress and displacement field without assuming plane sections remain plane albeit for a particular cross section of a curved beam subjected to a pure bending moment or end load. We conclude by comparing both the solutions to find that they are in excellent agreement when the beam is shallow.

Before proceeding further, we would like to clarify what we mean by a curved beam. Beam whose axis is not straight and is curved in the elevation is said to be a curved beam. If the applied loads are along the $y$ direction and the span of the beam is along the $x$ direction, the axis of the beam should have a curvature in the $xy$ plane. On the hand, if the member is curved on the $xz$ plane with the loading still along the $y$ direction, then it is not a curved beam, as this loading will cause a bending as well as twisting of the section. Thus, a curved beam does not have a curvature in the plan. Arches




(a) Elevation view: projection in the $xy$ plane.

(b) Isotropic view.

Figure 10.1: Schematic of bending of a curved beam

are examples of curved beams.

## 10.2 Winkler-Bach formula for curved beams

As mentioned before, in this section we obtain the stress field assuming, sections that are plane before bending remain plane after bending. Consequently, a transverse section rotates about an axis called the neutral axis as shown in figure 10.1. Let us examine an infinitesimal portion of a curved beam enclosing an angle $\Delta\phi$. Due to an applied pure bending moment $M$, the section $AB$ rotates through an angle $\delta(\Delta\phi)$ about the neutral axis and occupy the position $A'B'$. $SN$ denotes the surface on which the stress is zero and is called the neutral surface. Since, the stress is zero in this neutral surface, the length of the material fibers on this plane and oriented along the axis of the beam would not have changed. However, fibers above the neutral surface and oriented along the axis of the beam would be in compression and those below the neutral surface and oriented along the axis of the beam would be in tension. Hence, for a fiber at a distance $y$ from the neutral surface, its length before the deformation would be $(r_n - y)\Delta\phi$, where $r_n$ is the radius of curvature of the neutral surface. The change in length of the same fiber after deformation due to the applied bending moment, $M$ would be $-y(\delta(\Delta\phi))$. Note that the negative sign is to indicate that the length reduces when $y$ is positive for the direction of the moment indicated in the figure 10.1. Thus, the linearized strain is given by,

$$\epsilon = -\frac{y(\delta(\Delta\phi))}{(r_n - y)(\Delta\phi)}. \tag{10.1}$$

It is assumed that the lateral dimensions of the beam are unaltered due to bending, i.e. the Poisson's effect is ignored. Hence, the quantity $y$ remains unaltered due to the deformation. Now, to estimate the quantity $\delta(\Delta\phi)/(\Delta\phi)$, we observe from figure 10.1a that

$$SN = r(\Delta\phi + \delta(\Delta\phi)), \tag{10.2}$$

where $r$ is the radius of curvature of the neutral axis after bending. However, by virtue of it being neutral surface, its length is unaltered and therefore

$$SN = r_n \Delta\phi. \tag{10.3}$$

Equating equations (10.2) and (10.3) and simplifying we obtain

$$\frac{\delta(\Delta\phi)}{\Delta\phi} = \frac{r_n}{r} - 1. \tag{10.4}$$

Substituting equation (10.4) in (10.1) we obtain,

$$\epsilon = -\frac{y}{r_n - y}\left[\frac{r_n}{r} - 1\right]. \tag{10.5}$$

Having obtained the strain, the expression for the stress becomes

$$\sigma = -E\frac{y}{r_n - y}\left[\frac{r_n}{r} - 1\right], \tag{10.6}$$

where $E$ is the Young's modulus and we have appealed to one dimensional Hooke's law to relate the strain and the stress.

Since, we assume that the section is subjected to pure bending moment and in particular no axial load, we require that

$$\int_a \sigma da = -\int_a E\frac{y}{r_n - y}\left[\frac{r_n}{r} - 1\right] da = 0, \tag{10.7}$$

where we have used (10.6). Since, $r_n$ and $r$ are constants for a given section and $r \neq r_n$, when the beam deforms, for equation (10.7) to hold,

$$\int_a \frac{Ey}{r_n - y} da = 0. \tag{10.8}$$

We have to find $r_n$ such that (10.8) holds. Observing that $y/(r_n - y) = r_n/(r_n - y) - 1$, equation (10.8) can be simplified as

$$r_n \int_a \frac{E}{r_n - y} da - \int_a E da = 0. \tag{10.9}$$

If the section is homogeneous, Young's modulus is constant over the section and therefore the above equation can be written as,

$$r_n = \frac{A}{\int_a \frac{da}{r_n - y}} \tag{10.10}$$

Assuming the bending moment at the section being studied is $M$, as shown in section 8.1, equation (8.9),

$$M = -\int_a y\sigma da, \tag{10.11}$$

where we have assumed that the origin of the coordinate system is located at the neutral axis of the section; consistent with the assumption made while obtaining the expression for the strain. Substituting equation (10.7) in (10.11) and rewriting we obtain,

$$\begin{aligned} M = \left[\frac{r_n}{r} - 1\right] \int_a E \frac{y^2}{r_n - y} da &= \left[\frac{r_n}{r} - 1\right] \int_a E \left[r_n \frac{y}{r_n - y} - y\right] da \\ &= \left[\frac{r_n}{r} - 1\right] \left[r_n \int_a E \frac{y}{r_n - y} da - \int_a E y da\right]. \end{aligned} \tag{10.12}$$

It follows from (10.8) that the first integral in the above equation is zero. Thus, equation (10.12) simplifies to

$$\left[\frac{r_n}{r} - 1\right] = -\frac{M}{\int_a E y da}. \tag{10.13}$$

Then, combining equation (10.6) and (10.13) we obtain

$$-\left[\frac{r_n}{r} - 1\right] = \frac{M}{\int_a Eyda} = \frac{\sigma}{E}\frac{r_n - y}{y}, \tag{10.14}$$

where $r_n$ is obtained by solving (10.9). If the section is homogeneous that is $E$ is constant over the section equation (10.14) simplifies to,

$$-E\left[\frac{r_n}{r} - 1\right] = \frac{M}{\int_a yda} = \sigma\frac{r_n - y}{y}, \tag{10.15}$$

where $r_n$ is obtained by solving (10.10). Note that in these equations $y$ is measured from the neutral axis of the section and the bending moment that increases the curvature (decreases the radius of curvature) is assumed to be positive.

Thus, given the moment in the section, using equation (10.14) or (10.15), we can estimate the stress ($\sigma$) distribution in the section and/or the deformed curvature ($r$) of the beam. These equations are called Winkler-Bach formula for curved beams.

Next, we illustrate a technique to find the radius of curvature of the neutral surface, $r_n$ for a homogeneous rectangular section. Consider a rectangular section shown in figure 10.2 where $\rho_o$ denotes the radius of curvature of the centroid of the cross section, $r_i$ the radius of curvature of the topmost fiber of the cross section and $r_o$ the radius of curvature of the bottommost fiber of the cross section. Let $u = r_n - y$. Now, $u$ is the location of a fiber from the center of curvature of the section,as indicated in the figure 10.2. Hence,

$$\int_a \frac{da}{r_n - y} = \int_{r_i}^{r_o} \frac{bdu}{u} = b\ln\left(\frac{r_o}{r_i}\right). \tag{10.16}$$

Consequently, the value of $r_n$, the radius of curvature of the neutral axis for a rectangular section as determined from (10.10) is

$$r_n = \frac{bh}{b\ln\left(\frac{r_o}{r_i}\right)} = \frac{h}{\ln\left(\frac{r_o}{r_i}\right)}. \tag{10.17}$$

Having obtained $r_n$, we would like to obtain the stress distribution in a curved beam with rectangular section subjected to a moment $M$. It follows from equation (10.15) that

$$\sigma = \frac{y}{r_n - y}\frac{M}{\int_a yda}. \tag{10.18}$$

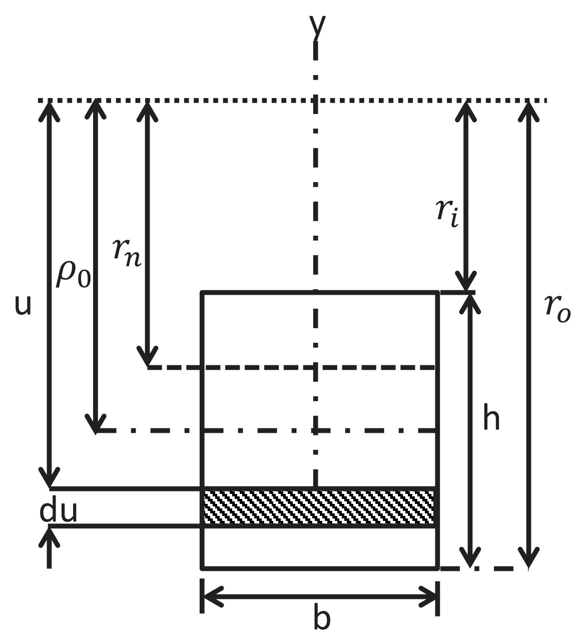



Figure 10.2: Parameters for a rectangular section to compute $r_n$

Now we need to compute $\int_a y da$ with $y$ measured from the neutral axis. Towards this,

$$\int_a y da = \int_{r_i}^{r_o} (r_n - u) b du = b \left[ r_n (r_o - r_i) - \frac{1}{2}(r_o^2 - r_i^2) \right] = bh \left[ r_n - \rho_o \right], \tag{10.19}$$

where we have used the fact that $r_o - r_i = h$ and $\rho_o = (r_o + r_i)/2$. Substituting equation (10.19) in (10.18), we obtain

$$\sigma = \frac{My}{bh(r_n - y)[r_n - (r_i + r_o)/2]}, \tag{10.20}$$

where $r_n - r_o \leq y \leq r_n - r_i$. We compare the qualitative features of this solution after obtaining the elasticity solution.

## 10.3 2D Elasticity solution for curved beams

In this section, we obtain the two dimensional elasticity solution for the curved beam subjected to pure bending and end loading. The cross section of the beam is assumed to be rectangular of width $2b$ and depth $h$. As the beam is curved, we use cylindrical polar coordinates to formulate and study this problem. The curved beam is assumed to be the annular region between two coaxial radially cut cylinders of radius $r_i$ and $r_i + h$, i.e, $\mathcal{B} = \{(r, \theta, z) | r_i \leq r \leq r_o, \alpha_1 \leq \theta \leq \alpha_2, -b \leq z \leq b\}$, where $r_i$, $r_o$, $\alpha_1$, $\alpha_2$ and $b$ are constants. Note that here $r_o = r_i + h$

### 10.3.1 Pure bending

The first example that we study, is that of pure bending of a curved beam. Here the curved beam is assumed to be subjected to end moments as shown in figure 10.3. The traction boundary conditions for this problem are

(a) The surfaces defined by $r = r_i$ and $r = r_o$ are traction free.

(b) The surfaces defined by $\theta = \pm\alpha$ though is subjected to a moment, $M$ along the $z$ direction, has no net force.

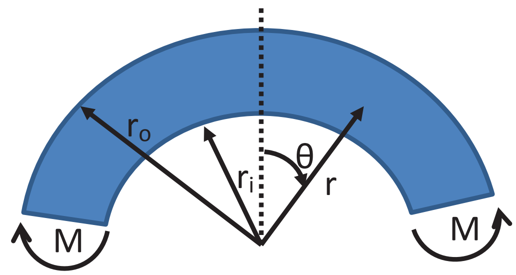



Figure 10.3: Curved beam subjected to pure bending

Translating these boundary conditions into mathematical statements:

$$\mathbf{t}_{(\mathbf{e}_r)}(r_o,\theta,z) = \mathbf{t}_{(-\mathbf{e}_r)}(r_i,\theta,z) = \mathbf{o}, \tag{10.21}$$

$$\int_a \mathbf{t}_{(\mathbf{e}_\theta)}(r,\alpha,z)drdz = \int_a \mathbf{t}_{(-\mathbf{e}_\theta)}(r,-\alpha,z)drdz = \mathbf{o}, \tag{10.22}$$

$$\int_a (r\mathbf{e}_r + z\mathbf{e}_z)\wedge \mathbf{t}_{(\mathbf{e}_\theta)}(r,\alpha,z)drdz = M\mathbf{e}_z, \tag{10.23}$$

$$\int_a (r\mathbf{e}_r + z\mathbf{e}_z)\wedge \mathbf{t}_{(-\mathbf{e}_\theta)}(r,-\alpha,z)drdz = -M\mathbf{e}_z, \tag{10.24}$$

where $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are the cylindrical polar coordinate basis.

The displacement boundary condition for this problem are

(a) The surface defined by $\theta = 0$ undergoes no tangential displacement.

(b) There is no radial displacement of the point with cylindrical polar coordinates $(r_n, 0, 0)$, where $r_n$ is the radial location on the surface $\theta = 0$ at which $\sigma_{\theta\theta}(r_n, 0, z) = 0$.

The mathematical statements of these conditions are

$$\mathbf{e}_\theta \cdot \mathbf{u}(r,0,z) = 0, \tag{10.25}$$

$$\mathbf{e}_r \cdot \mathbf{u}(r_n,0,0) = 0. \tag{10.26}$$

Assuming that the state of stress in the curved beam is plane and the cylindrical polar components of this stress are

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & 0 \\ \sigma_{r\theta} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10.27}$$

Substituting the above stress (10.27) in the traction boundary conditions (10.21) through (10.24) we obtain,

$$\sigma_{rr}(r_i, \theta, z) = \sigma_{rr}(r_o, \theta, z) = 0, \tag{10.28}$$

$$\sigma_{r\theta}(r_i, \theta, z) = \sigma_{r\theta}(r_o, \theta, z) = 0, \tag{10.29}$$

$$\int_a \sigma_{r\theta}(r, \pm\alpha, z) dr dz = 0, \tag{10.30}$$

$$\int_a \sigma_{\theta\theta}(r, \pm\alpha, z) dr dz = 0, \tag{10.31}$$

$$\int_a z\sigma_{r\theta}(r, \pm\alpha, z) dr dz = 0, \tag{10.32}$$

$$\int_a r\sigma_{\theta\theta}(r, \pm\alpha, z) dr dz = M. \tag{10.33}$$

Now, we have to find Airy's stress function, $\phi$ that would satisfy the boundary conditions (10.28) through (10.33) and the bi-harmonic equation. In this problem we expect the stresses to be such that $\boldsymbol{\sigma}(r, \theta, z) = \boldsymbol{\sigma}(r, -\theta, z)$, for any $\theta$, i.e. the stress is an even function of $\theta$. Imposing this restriction that the stress be an even function of $\theta$, on the general periodic solution to the bi-harmonic solution (7.57), results in requiring that the Airy's stress function be independent of $\theta$. Thus, Airy's stress function is,

$$\phi = a_{01} + a_{02}\ln(r) + a_{03}r^2 + a_{04}r^2\ln(r), \tag{10.34}$$

where the constants $a_{0i}$'s are to be found from the boundary conditions. The stress field corresponding to this Airy's stress function, (10.34) found using (7.53) is

$$\sigma_{rr} = 2a_{04}\ln(r) + \frac{a_{02}}{r^2} + a_{04} + 2a_{03}, \tag{10.35}$$

$$\sigma_{\theta\theta} = 2a_{04}\ln(r) - \frac{a_{02}}{r^2} + 3a_{04} + 2a_{03}, \tag{10.36}$$

$$\sigma_{r\theta} = 0. \tag{10.37}$$

It can be immediately seen that by virtue of $\sigma_{r\theta} = 0$, boundary conditions (10.29), (10.30) and (10.32) are trivially satisfied. The boundary condition (10.28) requires that

$$2a_{04}\ln(r_i) + \frac{a_{02}}{r_i^2} + a_{04} + 2a_{03} = 0, \tag{10.38}$$

$$2a_{04}\ln(r_o) + \frac{a_{02}}{r_o^2} + a_{04} + 2a_{03} = 0. \tag{10.39}$$

The boundary condition (10.31) requires that

$$r_o\left[2a_{04}\ln(r_o)+\frac{a_{02}}{r_o^2}+a_{04}+2a_{03}\right]-r_i\left[2a_{04}\ln(r_i)+\frac{a_{02}}{r_i^2}+a_{04}+2a_{03}\right]=0. \tag{10.40}$$

By virtue of the terms in the square brackets being same as those in equations (10.39) and (10.38), equation (10.40) holds if (10.38) and (10.39) are satisfied. The only remaining boundary condition (10.33) when evaluated mandates that

$$a_{04}[r_o^2\ln(r_o)-r_i^2\ln(r_i)]-a_{02}\ln\left(\frac{r_o}{r_i}\right)+(a_{04}+a_{03})(r_o^2-r_i^2)=\frac{M}{2b}. \tag{10.41}$$

Solving equations (10.38), (10.39) and (10.41) for the unknown constants $a_{0i}$'s, we obtain

$$a_{02} = \frac{M}{N}4r_i^2r_o^2\ln\left(\frac{r_o}{r_i}\right), \tag{10.42}$$

$$a_{03} = -\frac{M}{N}\left[r_o^2-r_i^2+2(r_o^2\ln(r_o)-r_i^2\ln(r_i))\right], \tag{10.43}$$

$$a_{04} = \frac{M}{N}2[r_o^2-r_i^2], \tag{10.44}$$

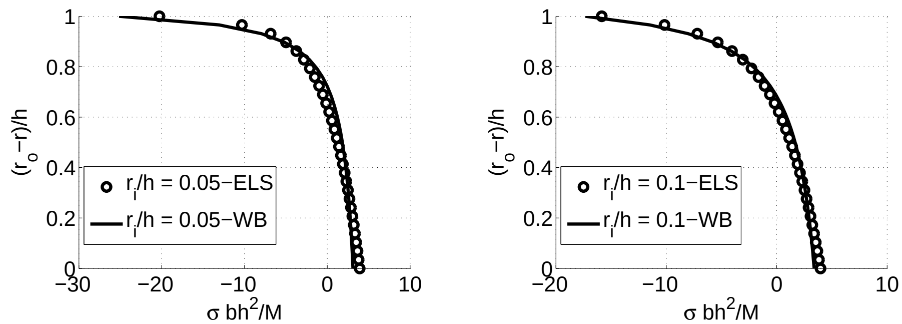
where

$$N=2b\left\{(r_o^2-r_i^2)^2-4r_i^2r_o^2\left[\ln\left(\frac{r_o}{r_i}\right)\right]^2\right\}. \tag{10.45}$$

Substituting these constants from equation (10.42) through (10.44) in the expression for the stresses (10.35) through (10.37) the stress field becomes known.

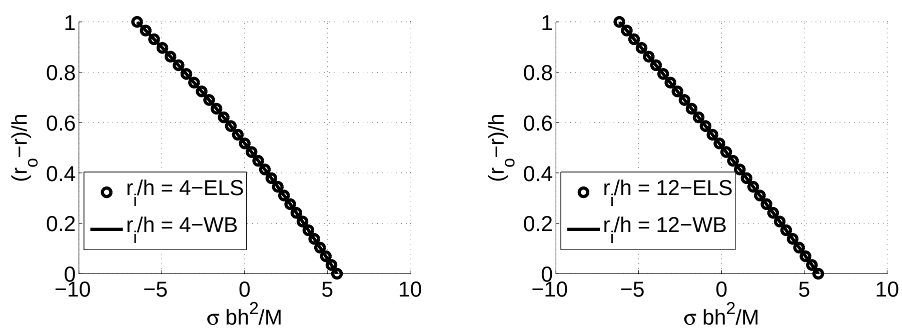
In figure 10.4 we compare the bending stress ($\sigma_{\theta\theta}$) obtained using the Winkler-Bach formula with that obtained using the two dimensional elasticity approach. We find that both these approaches though predict different expressions for the stress, evaluate to the same values as seen from figure 10.4b. However, differences between these approaches increases as $r_i/h$ value tends to zero as seen from figure 10.4a.

In figure 10.5 we study when critical curvature of the beam above which the stresses in the beam are not influenced much by the curvature. It seems that if the curvature of the innermost fiber exceeds 5 times the depth of the




(a) Beam with large initial curvature (b) Beam with moderate initial curvature

Figure 10.4: Comparison of the Winkler-Bach formula for the stress in the curved beams with the two dimensional elasticity solution for beams with different initial curvatures




(a) Beam with small initial curvature (b) Beam with negligible initial curvature

Figure 10.5: Study on the influence of initial curvature of the beam on the bending stresses developed due to a given moment

beam with rectangular cross section, one can consider the beam as straight for practical purposes.

Having obtained the stress field that satisfies the compatibility conditions, a smooth displacement field corresponding to this stress field can be determined by following the standard approach of estimating the strains for this stress field from the two dimensional constitutive relation and then integrating the resulting strains for the displacements using the strain displacement relation. On performing these calculations, we find that the cylindrical polar components of the displacement field are given by

$$u_r = \frac{1}{E}\left[2(1-\nu)(a_{04}r\ln(r) + a_{03}r) - (1+\nu)\left(\frac{a_{02}}{r} + a_{04}r\right)\right] + C_1\sin(\theta) + C_2\cos(\theta), \quad (10.46)$$

$$u_\theta = \frac{4a_{04}}{E}r\theta + C_1\cos(\theta) - C_2\sin(\theta) + C_3 r, \quad (10.47)$$

where $C_i$'s are constants to be determined from the displacement boundary conditions, $u_r$ is the radial component of the displacement and $u_\theta$ is the tangential component of the displacement.

Substituting equation (10.47) in the displacement boundary condition (10.25) we obtain,

$$C_1 + C_3 r = 0. \quad (10.48)$$

For equation (10.48) to hold,

$$C_1 = 0, \qquad C_3 = 0. \quad (10.49)$$

In order to satisfy the displacement boundary condition (10.26),

$$C_2 = -\frac{1}{E}\left[2(1-\nu)(a_{04}r_n\ln(r_n) + a_{03}r_n) - (1+\nu)\left(\frac{a_{02}}{r_n} + a_{04}r_n\right)\right]. \quad (10.50)$$

### 10.3.2 Curved cantilever beam under end load

The next example that we study, is that of end loading of a curved cantilever beam. Here a cantilever beam is assumed to be subjected to end shear force as shown in figure 10.6. Thus, the displacement boundary condition for this problem is

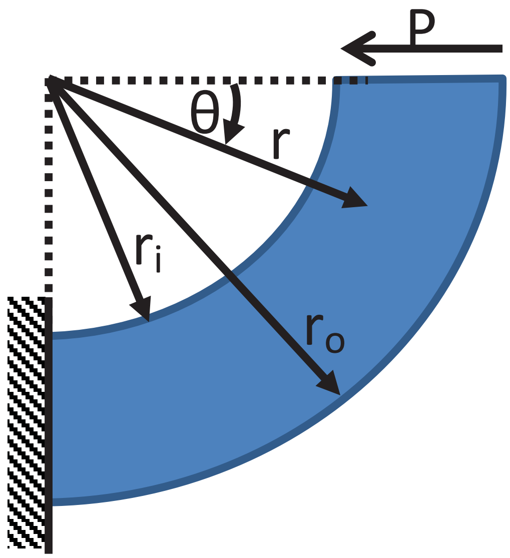



Figure 10.6: Curved cantilever beam subjected to end loading

(a) The surface defined by $\theta = \frac{\pi}{2}$ undergoes no displacement.

The mathematical statement of this condition is

$$\mathbf{u}(r, \frac{\pi}{2}, z) = \mathbf{o}, \tag{10.51}$$

The traction boundary conditions for this problem are

(a) The surfaces defined by $r = r_i$ and $r = r_o$ are traction free.

(b) The surface defined by $\theta = 0$ is subjected to a force, $-P$ along the radial direction.

(c) The surface defined by $\theta = \frac{\pi}{2}$, is subjected to a force, $P$ along tangential direction and to a moment $M$ along the $z$ direction. The value of this moment $M$ needs to be determined so that the required displacement boundary conditions are satisfied.

Translating these boundary conditions into mathematical statements:

$$\begin{aligned}
\mathbf{t}_{(\mathbf{e}_r)}(r_o, \theta, z) = \mathbf{t}_{(-\mathbf{e}_r)}(r_i, \theta, z) &= \mathbf{o}, && (10.52)\\
\int_a \mathbf{t}_{(\mathbf{e}_\theta)}(r, 0, z) dr dz &= -P\mathbf{e}_r, && (10.53)\\
\int_a \mathbf{t}_{(-\mathbf{e}_\theta)}\left(r, \frac{\pi}{2}, z\right) dr dz &= P\mathbf{e}_\theta, && (10.54)\\
\int_a (r\mathbf{e}_r + z\mathbf{e}_z) \wedge \mathbf{t}_{(\mathbf{e}_\theta)}(r, 0, z) dr dz &= \mathbf{o}, && (10.55)\\
\int_a (r\mathbf{e}_r + z\mathbf{e}_z) \wedge \mathbf{t}_{(-\mathbf{e}_\theta)}\left(r, \frac{\pi}{2}, z\right) dr dz &= -M\mathbf{e}_z, && (10.56)
\end{aligned}$$

where $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are the cylindrical polar coordinate basis.

Assuming that the state of stress in the curved beam is plane and the cylindrical polar components of this stress are

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & 0 \\ \sigma_{r\theta} & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (10.57)$$

Substituting the above stress (10.57) in the traction boundary conditions (10.52) through (10.56) we obtain,

$$\sigma_{rr}(r_i,\theta,z)=\sigma_{rr}(r_o,\theta,z)=0, \tag{10.58}$$
$$\sigma_{r\theta}(r_i,\theta,z)=\sigma_{r\theta}(r_o,\theta,z)=0, \tag{10.59}$$
$$\int_a \sigma_{r\theta}(r,0,z)drdz=-P, \tag{10.60}$$
$$\int_a \sigma_{\theta\theta}(r,0,z)drdz=0, \tag{10.61}$$
$$\int_a \sigma_{r\theta}\left(r,\frac{\pi}{2},z\right)drdz=0, \tag{10.62}$$
$$\int_a \sigma_{\theta\theta}\left(r,\frac{\pi}{2},z\right)drdz=P, \tag{10.63}$$
$$\int_a z\sigma_{r\theta}(r,0,z)drdz=0, \tag{10.64}$$
$$\int_a r\sigma_{\theta\theta}(r,0,z)drdz=0, \tag{10.65}$$
$$\int_a z\sigma_{r\theta}\left(r,\frac{\pi}{2},z\right)drdz=0, \tag{10.66}$$
$$\int_a r\sigma_{\theta\theta}\left(r,\frac{\pi}{2},z\right)drdz=M. \tag{10.67}$$

In order to satisfy the traction boundary conditions (10.58) through (10.67), we chose Airy's stress function from the general solution (7.57) such that it contains only the $\sin(\theta)$ and $\cos(2\theta)$ terms as,

$$\begin{aligned}\phi=&\left[b_{11}r+b_{12}r\ln(r)+\frac{b_{13}}{r}+b_{14}r^3\right]\sin(\theta)\\&+\left[a_{21}r^2+a_{22}r^4+\frac{a_{23}}{r^2}+a_{24}\right]\cos(2\theta).\end{aligned} \tag{10.68}$$

To proceed further, one has to follow the standard procedure and hence, we leave it as an exercise to find the stress and displacement field.

## 10.4 Summary

In this chapter we studied on how to analyze beams with initial curvature. We obtained the stress field based on the assumption that the plane section remain plane after bending. We also obtained two dimensional elasticity solution which is not based on the assumption that plane sections remain plane. However, we found that both these solutions predict the same value of stresses for practically used curved beams.

## 10.5 Self-Evaluation

1. There are two beams - a straight beam and a curved beam - each of which is subjected to a pure bending moment, M. Assuming that both the beams are homogeneous and has a square cross section with side 0.3m and are made of the same material, which is linear elastic and isotropic, compute the ratio of the maximum tensile stresses experienced in the straight and curved beams. Also, find the ratio of the maximum compressive stresses experienced in the straight and curved beams. Assume, the curved beam forms a part of a circle, with the initial radius of curvature of the center line of the bar, being $\rho_o$. Obtain the value of the ratios as a function of $\rho_o$ and determine the critical $rho_o^c$ below which the stresses would differ by more than 10 percent.

2. A curved beam with a circular center line has the T-section as shown in figure 10.7. This beam is subjected to pure bending in its plane of symmetry. Find the tensile and compressive stresses in the extreme fibers. Assume, $b_1 = 100$ mm, $b_2 = 20$ mm, the radius of curvature of the innermost fiber, $r_1 = 80$ mm, the outermost fiber, $r_2 = 180$ mm and $r_3 = 100$ mm.

3. A curved beam with a circular center line has the circular sections as shown in figure 10.8. Find the stress distribution across this section if the curved beam is subjected to a pure bending moment, $M$. Assume that the initial curvature of the center line of the beam is $\rho_o$.

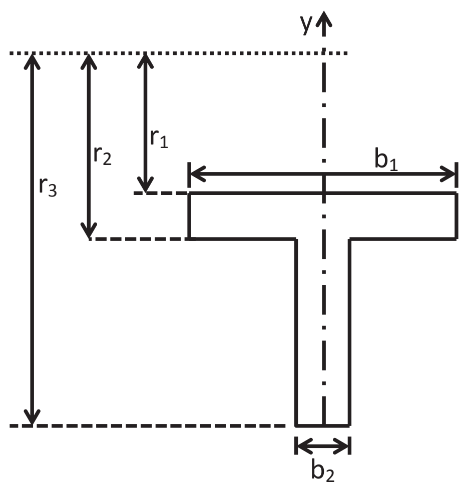



Figure 10.7: T section. Figure for problem 2

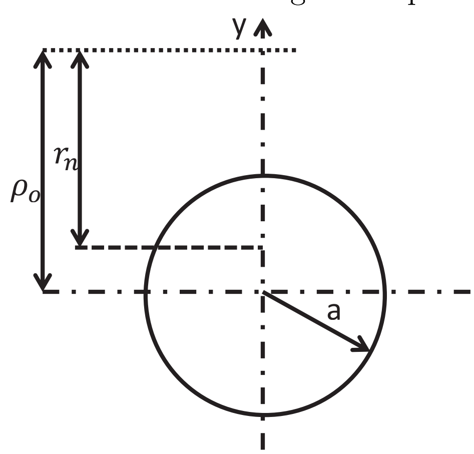



Figure 10.8: Circular section. Figure for problem 3

# Chapter 11

# Beam on Elastic Foundation

## 11.1 Overview

In some applications such as rail tracks, the member subjected to loads is supported on continuous foundations. That is the reactions due to external loading is distributed along the length of the member. Here we study on how to get the stresses and displacements in these members resting on continuous foundations. If the dimensions of this member is such that, it is longer along one of the axis, called the longitudinal axis in comparison with the dimensions along the other directions, it is called as a beam. If we assume that the reaction force offered by the continuous support is a function of the displacement that of the member, the support is called as elastic. A beam resting on an elastic support is said to be beam on elastic foundation.

In this chapter, we first formulate this problem of beam on an elastic foundation for a general loading condition. Then, we study the problem of a concentrated load at the mid point of a beam that is infinitely long. Appealing to the principle of superposition we obtain the solution to the problem of a concentrated moment at mid span and uniformly distributed load of length $L$, centered about the midpoint of the beam.

## 11.2 General formulation

In this section, we formulate the boundary value problem of beam on an elastic foundation. A beam having some cross section, resting on an elastic support is shown in figure 11.1. We assume that the reaction offered by

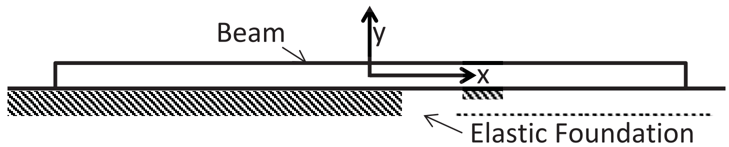



Figure 11.1: Schematic of a long beam on elastic foundation

the support at any point is directly proportional to the displacement of that point along the $y$ direction and is in a direction opposite to the displacement. Thus, if $\Delta$ is the vertical displacement of a point in the beam, $q_y$ the support reaction per unit width of the beam, then the above assumption that the reaction force is proportional to the displacement mathematically translates into requiring

$$q_y = -K_s\Delta. \tag{11.1}$$

Assuming the beam to be homogeneous, we obtained the equation (8.41) which we document here again:

$$-\frac{\sigma_{xx}}{(y-y_o)} = E\frac{d^2\Delta}{dx^2} = \frac{M_z}{I_{zz}}, \tag{11.2}$$

where $y_o$ is the $y$ coordinate of the centroid of the cross section which can be taken as 0 without loss of generality provided the origin of the coordinate system used is located at the centroid of the cross section, $E$ is the Young's modulus, $(x, y)$ is the coordinate of the point along the axis of the beam direction and the $y$ direction, $M_z$ is the $z$ component of the bending moment, $I_{zz}$ is the moment of inertia of the section about the $z$ axis.

In section 8.1, we integrated the equilibrium equations and obtained equations (8.18) and (8.25) which we record here:

$$\begin{aligned} \frac{dM_z}{dx} + V_y &= 0, && (11.3) \\ \frac{dV_y}{dx} + q_y &= 0, && (11.4) \end{aligned}$$

where $V_y$ is the shear force along the $y$ direction and $q_y$ is the transverse loading along the $y$ direction. Combining the equations (11.3) and (11.4) we obtain,

$$\frac{d^2M_z}{dx^2} = q_y. \tag{11.5}$$

Substituting equation (11.2) in equation (11.5), we obtain

$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2\Delta}{dx^2}\right) = q_y. \tag{11.6}$$

Assuming the beam to be homogeneous and prismatic, so that $EI_{zz}$ is constant through the length of the beam, and substituting equation (11.1) in equation (11.6), we obtain

$$EI_{zz}\frac{d^4\Delta}{dx^4} = -K_s\Delta. \tag{11.7}$$

Defining,

$$\beta^2 = \sqrt{\frac{K_s}{4EI_{zz}}}, \tag{11.8}$$

equation (11.7) can be written as

$$\frac{d^4\Delta}{dx^4} + 4\beta^4\Delta = 0. \tag{11.9}$$

The differential equation (11.9) has a general solution:

$$\Delta = \exp(-\beta x)[C_1\sin(\beta x) + C_2\cos(\beta x)] + \exp(\beta x)[C_3\sin(\beta x) + C_4\cos(\beta x)], \tag{11.10}$$

where $C_i$'s are constant to be determined from the boundary conditions.

Having found the deflection, the stress is estimated from (11.2) as

$$\begin{aligned}\sigma_{xx} = -E(y-y_o)&\frac{d^2\Delta}{dx^2}\\ &= -2yE\beta^2\left\{\exp(-\beta x)[C_2\sin(\beta x) - C_1\cos(\beta x)]\right.\\ &\qquad\left.+\exp(\beta x)[C_3\cos(\beta x) - C_4\sin(\beta x)]\right\},\end{aligned} \tag{11.11}$$

where we have assumed that the origin is located at the centroid of the cross section and hence have set $y_o = 0$.

## 11.3 Example 1: Point load

The first boundary value problem that we study for the beam on elastic foundation is when it is subjected to a point load at its mid span as shown

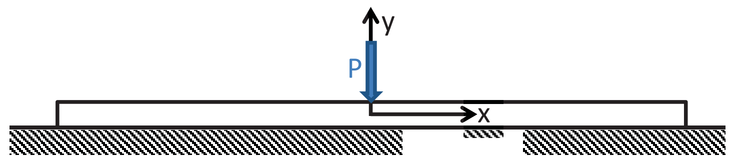



Figure 11.2: Schematic of a long beam on elastic foundation subjected to concentrated load at mid span

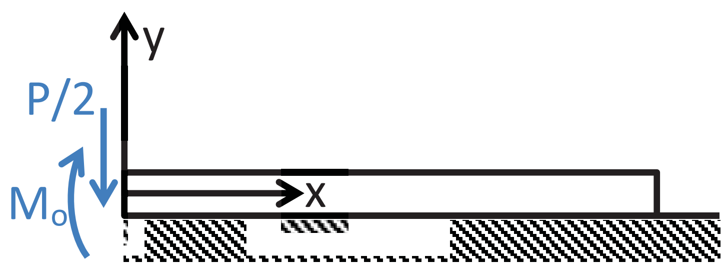



Figure 11.3: Free body diagram of half the section of long beam on elastic foundation subjected to concentrated load

in figure 11.2. The origin of the coordinate system is assumed to coincide with the point of application of the load. The beam length is assumed to be large enough compared to its lateral dimension that it can be considered to be infinitely long. (We shall quantify what length could be considered as infinitely long after we obtain the solution.)

To obtain the solution we section the beam at $x = 0$, the point of application of the concentrated load, as shown in figure 11.3. Since, there is a concentrated force acting at $x = 0$, the shear force would be discontinuous at $x = 0$ and hence the fourth derivative of the deflection, $\Delta$ does not exist. Consequently, the governing equation (11.9) is valid only in the domain $x > 0$ and $x < 0$ and not at $x = 0$. Therefore, we segment the beam at $x = 0$ and solve (11.9) on each of the segments. Then, we ensure, the differentiability of the second order derivative of deflection so that the third order derivative exist. This is required to ensure the existence of shear force at $x = 0$.

We also expect the deflection to be symmetric about $x = 0$ that is, $\Delta(x) = \Delta(-x)$ and therefore the slope of the deflection should be zero at $x = 0$, i.e.,

$$\left.\frac{d\Delta}{dx}\right|_{x=0} = 0. \tag{11.12}$$

By sectioning the beam at $x = 0$ we find the bending moment and shear force at this location. Using equation (11.2) we find that

$$\left.\frac{d^2\Delta}{dx^2}\right|_{x=0} = \frac{M(0)}{EI_{zz}} = -\frac{M_o}{EI_{zz}}, \tag{11.13}$$

where $M_o$ is the bending moment at $x = 0$, acting as shown in the figure 11.3 and the negative sign is to account for the fact that it is hogging. Since, shear force should exist, the continuity of the bending moment and $\frac{d^2\Delta}{dx^2}$ has to be ensured. Therefore the value of bending moment at both the segments of the beam should be the same.

Substituting equation (11.2) in (11.3) and assuming the beam to be homogeneous and prismatic, we obtain

$$EI_{zz}\frac{d^3\Delta}{dx^3} = -V_y. \tag{11.14}$$

Since, there is a concentrated force at $x = 0$, $V_y(0^+) = -V_y(0^-)$ and the equilibrium of an infinitesimal element centered about $x = 0$ requires that $V_y(0^+) - V_y(0^-) = P$. Hence, $V_y(0^+) = P/2$ and $V_y(0^-) = -P/2$. Thus, from equation (11.14) we obtain,

$$\left.\frac{d^3\Delta}{dx^3}\right|_{x=0^+} = -\frac{V_y(0^+)}{EI_{zz}} = -\frac{P}{2EI_{zz}}. \tag{11.15}$$

$$\left.\frac{d^3\Delta}{dx^3}\right|_{x=0^-} = -\frac{V_y(0^-)}{EI_{zz}} = \frac{P}{2EI_{zz}}. \tag{11.16}$$

Further, we require that

$$\Delta \to 0, \quad \text{as} \quad x \to \pm\infty, \tag{11.17}$$

since, we expect the effect of the load would be felt only in its vicinity.

To obtain the solution, we first focus on the right half of the beam wherein $x > 0$. Then, the requirement (11.17) implies that the constants $C_3$ and $C_4$ in the general solution (11.10) has to be zero; otherwise $\Delta \to \infty$ as $x \to$

$\infty$. Next, the condition (11.12) requires that $C_1 = C_2 = C_0$. Finally, the equation (11.15) tells us that

$$C_0 = -\frac{P}{8\beta^3 EI_{zz}} = -\frac{P\beta}{2K_s} \tag{11.18}$$

where to obtain the last equality we have made use of (11.8). Thus, in the domain, $x > 0$,

$$\Delta = -\frac{P\beta}{2K_s}\exp(-\beta x)[\cos(\beta x) + \sin(\beta x)]. \tag{11.19}$$

Now, we consider the left half of the beam, i.e., $x < 0$. The requirement (11.17) implies that the constants $C_1$ and $C_2$ in the general solution (11.10) has to be zero. Next, the condition (11.12) requires that $C_3 = -C_4 = C_5$. Finally, the equation (11.16) tells us that

$$C_5 = -\frac{P}{8\beta^3 EI_{zz}} = -\frac{P\beta}{2K_s} \tag{11.20}$$

where to obtain the last equality we have made use of (11.8). Thus, in the domain, $x < 0$,

$$\Delta = -\frac{P\beta}{2K_s}\exp(\beta x)[\cos(\beta x) - \sin(\beta x)]. \tag{11.21}$$

Thus, the distribution of the reaction from the foundation along the axis of the beam given in (11.1) evaluates to:

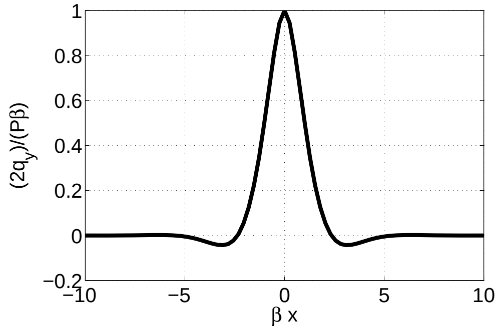
$$q_y = -K_s\Delta = \begin{cases} \frac{P\beta}{2}\exp(-\beta x)[\cos(\beta x) + \sin(\beta x)] & x > 0 \\ \frac{P\beta}{2}\exp(\beta x)[\cos(\beta x) - \sin(\beta x)] & x < 0 \end{cases}. \tag{11.22}$$

The variation of the bending moment along the axis of the beam obtained from (11.2) is:

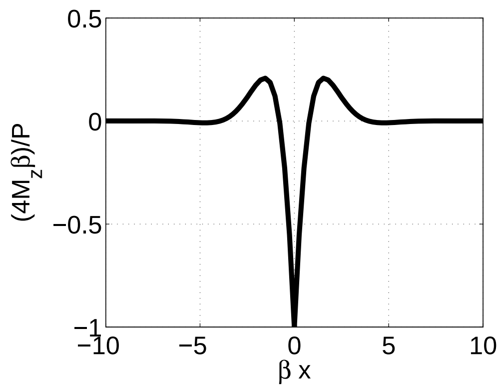
$$M_z = EI_{zz}\frac{d^2\Delta}{dx^2} = \begin{cases} -\frac{P}{4\beta}\exp(-\beta x)[\cos(\beta x) - \sin(\beta x)] & x \geq 0 \\ -\frac{P}{4\beta}\exp(\beta x)[\cos(\beta x) + \sin(\beta x)] & x \leq 0 \end{cases}. \tag{11.23}$$

The shear force variation along the axis of the beam computed using (11.14) is:

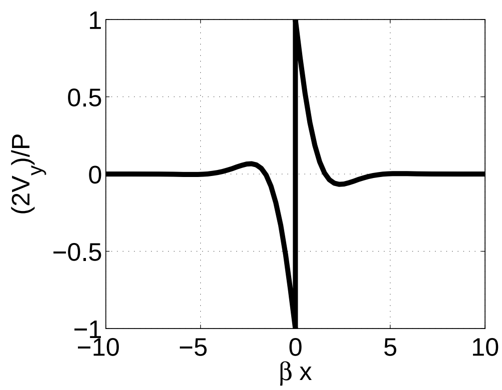
$$V_y = -EI_{zz}\frac{d^3\Delta}{dx^3} = \begin{cases} \frac{P}{2}\exp(-\beta x)\cos(\beta x) & x \geq 0 \\ -\frac{P}{2}\exp(\beta x)\cos(\beta x) & x \leq 0 \end{cases}. \tag{11.24}$$




(a) Variation of support reaction




(b) Variation of bending moment




(c) Variation of shear force

Figure 11.4: Variation of support reaction, bending moment and shear force along the axis of the beam on elastic foundation

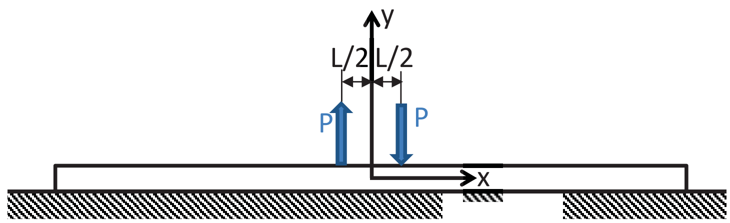



Figure 11.5: Schematic of a long beam on elastic foundation subjected to concentrated moment at mid span

In figure 11.4 we plot the variation of the support reaction, bending moment and shear force along the axis of the beam. It can be seen from the figure that though the beam is assumed to be infinitely long the reaction force, bending moment and shear forces all tend to zero for $\beta x > 5$. Hence, a beam may be considered as long if its length is greater than, $5/\beta$. It can also be seen from the figure that the maximum deflection, support reaction, bending moment and shear occurs at $z = 0$ and these values are,

$$\Delta^{max} = -\frac{P\beta}{2K_s}, \quad q_y^{max} = \frac{P\beta}{2}, \quad M_z^{max} = -\frac{P}{4\beta}, \quad V_y^{max} = \frac{P}{2}. \tag{11.25}$$

It can be seen from figure 11.4a that the support reaction changes sign. The support reaction changes sign at a point when $q_y = 0$, i.e., $\sin(\beta x) = -\cos(\beta x)$ or at $x = 3\pi/(4\beta)$. Since, the support reaction is proportional to the deflection, $\Delta$, this change in sign of the support reaction also tells us that the beam will uplift at $x = \pm 3\pi/(4\beta)$. Hence, the beams have to be adequately clamped to the foundation to prevent it from uplifting.

## 11.4 Example 2: Concentrated moment

The concentrated moment, $M_o$, is considered to be equivalent to the action of two concentrated forces, $P$, equal in magnitude but opposite in direction and separated by a distance $L$ as shown in the figure 11.5. Thus,

$$P = \frac{M_o}{L}. \tag{11.26}$$

We obtain the solution to this loading case by superposing the displacement field obtained in the above example for a single point load. Thus, it follows from equations (11.17) and (11.19) that the displacement due to the downward acting force at a distance, $L/2$ from the origin is,

$$\Delta_{L/2} = \begin{cases} -\frac{P\beta}{2K_s}\exp(-\beta(x-L/2))[\cos(\beta(x-L/2)) \\ \qquad\qquad + \sin(\beta(x-L/2))] & x \geq L/2 \\ -\frac{P\beta}{2K_s}\exp(\beta(x-L/2))[\cos(\beta(x-L/2)) \\ \qquad\qquad - \sin(\beta(x-L/2))] & x \leq L/2 \end{cases}. \tag{11.27}$$

Similarly, the displacement due to the upward acting force at a distance, $-L/2$ from the origin is

$$\Delta_{-L/2} = \begin{cases} \frac{P\beta}{2K_s}\exp(-\beta(x+L/2))[\cos(\beta(x+L/2)) \\ \qquad\qquad + \sin(\beta(x+L/2))] & x \geq -L/2 \\ \frac{P\beta}{2K_s}\exp(\beta(x+L/2))[\cos(\beta(x+L/2)) \\ \qquad\qquad - \sin(\beta(x+L/2))] & x \leq -L/2 \end{cases}. \tag{11.28}$$

Since, the displacement is small and the material obeys Hooke's law, we can superpose the solutions as discussed in section 7.5.2. Hence, the displacement under the action of both the forces, $\Delta = \Delta_{L/2} + \Delta_{-L/2}$ evaluates to,

$$\Delta = \begin{cases} \frac{M_o\beta}{2K_sL}\{\exp(-\beta(x+L/2))\,[\cos(\beta(x+L/2)) \\ \quad + \sin(\beta(x+L/2))] - \exp(-\beta(x-L/2)) \\ \quad [\cos(\beta(x-L/2)) + \sin(\beta(x-L/2))]\} & x \geq L/2 \\ \frac{M_o\beta}{2K_sL}\{\exp(-\beta(x+L/2))\,[\cos(\beta(x+L/2)) \\ \quad + \sin(\beta(x+L/2))] - \exp(\beta(x-L/2)) \\ \quad [\cos(\beta(x-L/2)) - \sin(\beta(x-L/2))]\} & -L/2 \leq x \leq L/2 \\ \frac{M_o\beta}{2K_sL}\{\exp(\beta(x+L/2))\,[\cos(\beta(x+L/2)) \\ \quad - \sin(\beta(x+L/2))] - \exp(\beta(x-L/2)) \\ \quad [\cos(\beta(x-L/2)) - \sin(\beta(x-L/2))]\} & x \leq -L/2 \end{cases}, \tag{11.29}$$

where we have used equation (11.26). When $L \to 0$ and $PL \to M_o$ the above equation (11.29) evaluates to,

$$\Delta = \begin{cases} -\frac{M_o\beta^2}{K_s}\exp(-\beta x)\sin(\beta x) & x \geq 0 \\ -\frac{M_o\beta^2}{K_s}\exp(\beta x)\sin(\beta x) & x \leq 0 \end{cases}. \tag{11.30}$$

Having found the displacement, (11.30), the variation of the bending moment along the axis of the beam obtained from (11.2) is:

$$M_z = EI_{zz}\frac{d^2\Delta}{dx^2} = \begin{cases} \frac{M_o}{2}\exp(-\beta x)\cos(\beta x) & x \geq 0 \\ -\frac{M_o}{2}\exp(\beta x)\cos(\beta x) & x \leq 0 \end{cases}, \tag{11.31}$$

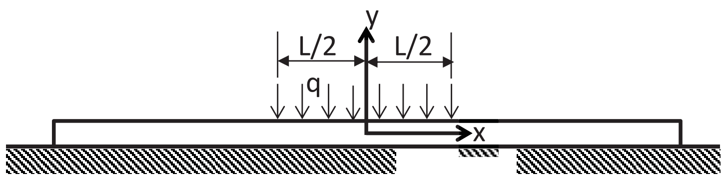



Figure 11.6: Schematic of a long beam on elastic foundation subjected to uniformly distributed load of length $L$ on either side of the mid span

and the shear force variation computed using (11.14) is:

$$V_y = -EI_{zz}\frac{d^3\Delta}{dx^3} = \begin{cases} \frac{M_o\beta}{2}\exp(-\beta x)[\cos(\beta x)+\sin(\beta x)] & x \geq 0 \\ \frac{M_o\beta}{2}\exp(\beta x)[\cos(\beta x)-\sin(\beta x)] & x \leq 0 \end{cases}. \quad (11.32)$$

## 11.5 Example 3: Uniformly distributed load

Next, we study the problem of an infinite beam on an elastic foundation subjected to a uniformly distributed load of length $L$ symmetrically on either side of the origin, as shown in figure 11.6. As before, we apply the principle of superposition to find the deflection at a point to be

$$\Delta = \begin{cases} -\frac{\beta}{2K_s}\int_{-L/2}^{L/2}\exp(-\beta(x-a))\left[\cos(\beta(x-a)) \right. \\ \qquad \left. +\sin(\beta(x-a))\right] q da & x \geq L/2 \\ -\frac{\beta}{2K_s}\left\{\int_{-L/2}^{x}\exp(-\beta(x-a))\left[\cos(\beta(x-a)) \right.\right. \\ \qquad \left. +\sin(\beta(x-a))\right] q da \\ \qquad + \int_{x}^{L/2}\exp(\beta(x-a))\left[\cos(\beta(x-a)) \right. \\ \qquad \left.\left. -\sin(\beta(x-a))\right] q da\right\} & -L/2 \leq x \leq L/2 \\ -\frac{\beta}{2K_s}\int_{-L/2}^{L/2}\exp(\beta(x-a))\left[\cos(\beta(x-a)) \right. \\ \qquad \left. -\sin(\beta(x-a))\right] q da & x \leq -L/2 \end{cases}. \quad (11.33)$$

Evaluating the integrals in equation (11.33) we obtain

$$\Delta = \begin{cases} -\frac{q}{2K_s}\exp(-\beta(x+L/2))\,[-\cos(\beta(L/2+x)) \\ \qquad +\cos(\beta(L/2-x))\exp(\beta L)] & x \geq L/2 \\ -\frac{q}{2K_s}\,\{2-\cos(\beta(L/2-x))\exp(-\beta(L/2-x)) \\ \qquad -\cos(\beta(L/2+x))\exp(-\beta(L/2+x))\} & -L/2 \leq x \leq L/2 \\ -\frac{q}{2K_s}\exp(\beta(x-L/2))\,[-\cos(\beta(L/2-x)) \\ \qquad +\cos(\beta(L/2+x))\exp(\beta L)] & x \leq -L/2 \end{cases}. \tag{11.34}$$

Having found the displacement, (11.34), the variation of the bending moment along the axis of the beam obtained from (11.2) is:

$$M_z = \begin{cases} -\frac{q}{4\beta^2}\exp(-\beta(x+L/2))\,[-\sin(\beta(x+L/2)) \\ \qquad \sin(\beta(L/2-x))\exp(\beta L)] & x \geq L/2 \\ -\frac{q}{4\beta^2}\,[\sin(\beta(L/2-x))\exp(-\beta(L/2-x)) \\ \qquad +\sin(\beta(L/2+x))\exp(-\beta(L/2+x))] & -L/2 \leq x \leq L/2 \\ -\frac{q}{4\beta^2}\exp(\beta(x-L/2))\,[\sin(\beta(L/2-x)) \\ \qquad +\sin(\beta(L/2+x))\exp(\beta L)] & x \leq -L/2 \end{cases}, \tag{11.35}$$

and the shear force variation computed using (11.14) is:

$$V_y = \begin{cases} \frac{q}{4\beta}\exp(-\beta(x+L/2))\,\{\sin(\beta(L/2+x)) \\ \qquad -\cos(\beta(L/2+x)) \\ +[\sin(\beta(L/2-x))+\cos(\beta(L/2-x))]\exp(\beta L)\} & x \geq L/2 \\ \frac{q}{4\beta}\,\{\exp(-\beta(L/2-x))\,[\cos(\beta(L/2-x)) \\ \qquad -\sin(\beta(L/2-x))] \\ \quad -\exp(-\beta(L/2+x))\,[\cos(\beta(L/2+x)) \\ \qquad -\sin(\beta(L/2+x))]\} & -L/2 \leq x \leq L/2 \\ \frac{q}{4\beta}\exp(\beta(x-L/2))\,\{\sin(\beta(L/2-x)) \\ \qquad -\cos(\beta(L/2-x)) \\ +[\sin(\beta(L/2+x))+\cos(\beta(L/2+x))]\exp(\beta L)\} & x \leq -L/2 \end{cases}. \tag{11.36}$$

## 11.6 Summary

In this chapter, we formulated and solved the problem of a concentrated load acting on a long beam on elastic foundation. Using this solution and appealing to the principal of superposition, we solved two problems. One of the problems is that of a concentrated moment on a long beam on elastic support. The other problem is that of uniformly distributed load of length $L$, on a long beam continuously supported at the bottom. These problem serve as an illustration of the use of principle of superposition.

## 11.7 Self-Evaluation

1. A steel beam of a rectangular cross section, 180 mm wide and 280 mm thick, is resting on an elastic foundation whose modulus of foundation is 6.5 N/$mm^2$. This beam is subjected to a concentrated anti-clockwise moment of 0.5 MNm at the center. Determine the maximum deflection and the maximum bending stresses in the beam. Assume Young's modulus, $E$ = 200 GPa and the Poisson's ratio, $\nu = 0.3$. Also, plot

   (a) The deflected shape of the beam

   (b) The variation in the bending moment along the axis of the beam

   (c) The variation in the shear force along the axis of the beam

   Find the length of the beam beyond which it would require clamping to prevent uplift.

2. Repeat problem - 1, for the case in which the beam is subjected to a uniformly distributed load of intensity 200 N/mm over a length 500 mm about the center, instead of a concentrated moment.

3. A four-wheel wagon runs on steel rails. The rails have a depth of 140mm; the distance from the top of a rail to its centroid is 70 mm; and its moment of inertia is $21 * 10^6 mm^4$. The Young's modulus of the rail, $E$ = 210 GPa and the Poisson's ratio, $\nu = 0.3$. The rail rests on an elastic foundation with linear spring constant, $K_s = 12 N/mm^2$. The two wheels on each side of the car are spaced 2.50 m center to center and the distance between the front and rear axle is 13 m. If each wheel load is 90 kN, determine the maximum deflection and maximum bending stress on the rail. List the assumptions made in the analysis.

4. Determine the thickness of a square foundation of side 1 m required, if it were to carry a concentrated load of 1 MN and a concentrated clockwise

moment of 0.1 MNm. Assume that the foundation rests on an elastic foundation with spring constant, $K_s = 12N/mm^2$, the Young's modulus of the foundation is 20 GPa and that the maximum permissible normal stresses on the foundation is 10 MPa.

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