Module
3

Limit State of Collapse - Flexure (Theories and Examples)

Lesson
5

Determination of Neutral Axis Depth and Computation of Moment of Resistance

Instructional Objectives:

At the end of this lesson, the student should be able to:

* determine the depth of the neutral axis for a given cross-section with known value of A_{st} and grades of steel and concrete,

* write and derive the expressions of $x_{u,max}$, $p_{t,lim}$, $M_{u,lim}/bd^2$ and state the influences of grades of steel and concrete on them separately,

* derive the corresponding expression of M_u when $x_u < x_{u,max}$, $x_u = x_{u,max}$ and $x_u > x_{u,max}$,

* finally take a decision if $x_u > x_{u,max}$.

3.5.1 Introduction

After learning the basic assumptions, the three equations of equilibrium and the computations of the total compressive and tensile forces in Lesson 4, it is now required to determine the depth of neutral axis (NA) and then to estimate the moment of resistance of the beams. These two are determined using the two equations of equilibrium (Eqs. 3.1 and 3.3). It has been explained that the depth of neutral axis has important role to estimate the moment of resistance. Accordingly, three different cases are illustrated in this lesson.

3.5.2 Computation of the Depth of Neutral Axis x_u

From Eqs. 3.1, 9 and 14, we have

$$0.87\ f_y\ A_{st}\ =\ 0.36\ b\ x_u\ f_{ck}$$
(3.15)

or $$x_u\ =\ \frac{0.87\ f_y\ A_{st}}{0.36\ b\ f_{ck}}$$
(3.16)

We can also write:

$$\frac{x_u}{d}\ =\ \frac{0.87\ f_y\ A_{st}}{0.36\ b\,d\ f_{ck}}$$
(3.17)

Substituting the expression of $\frac{x_u}{d}$ from Eq. 3.17 into Eq. 3.13 of Lesson 4, we have

$$\text{lever arm} = d\left\{1 - 0.42\left(\frac{0.87\,f_y\,A_{st}}{0.36\,f_{ck}\,b\,d}\right)\right\}$$

$$= d\left(1 - 1.015\frac{A_{st}\,f_y}{f_{ck}\,b\,d}\right)$$

(3.18)

Ignoring multiplying factor 1.015 in Eq. 3.18, we have

$$\text{lever arm} = d\left(1 - \frac{A_{st}\,f_y}{f_{ck}\,b\,d}\right)$$

(3.19)

3.5.3 Limiting Value of x_u (= $x_{u,\,max}$)

Should there be a limiting or maximum value of x_u?

Equation 3.17 reveals that $\frac{x_u}{d}$ increases with the increase of percentage of steel reinforcement $\frac{A_{st}}{b\,d}$ for fixed values of f_y and f_{ck}. Thus, the depth of the neutral axis x_u will tend to reach the depth of the tensile steel. But, that should not be allowed. However, let us first find out that value of x_u which will satisfy assumptions (ii) and (vi) of sec. 3.4.2 and designate that by $x_{u,\,max}$ for the present, till we confirm that x_u should have a limiting value.
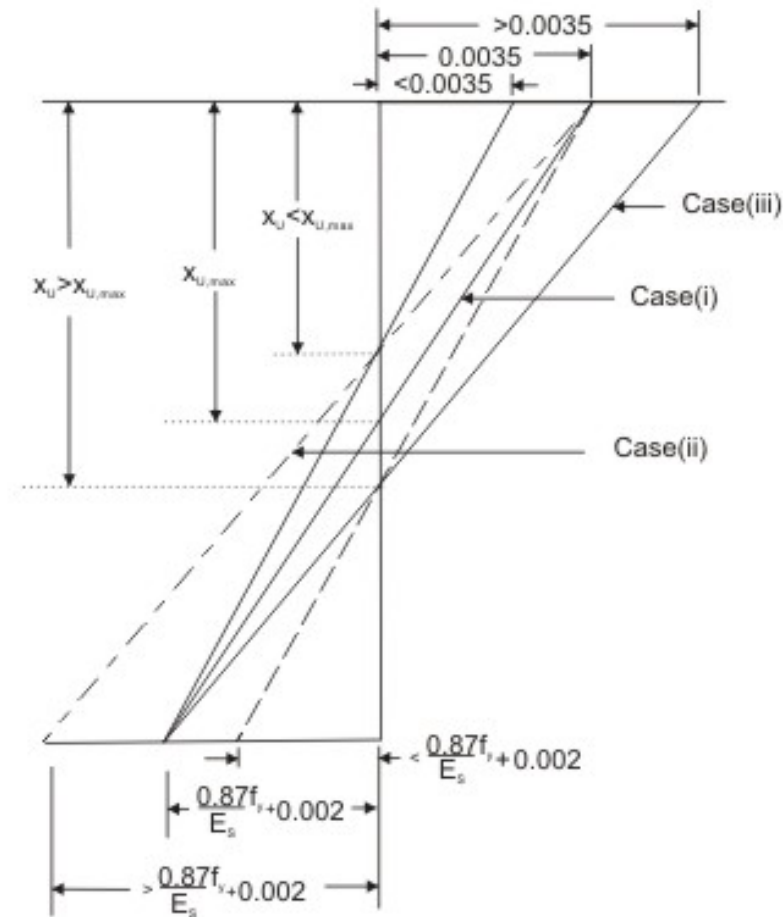
Fig.3.5.1: Strain of steel for three cases $x_u <=> x_{u,max}$

Figure 3.5.1 presents the strain diagrams for the three cases: (i) when $x_u = x_{u, max}$; (ii) when x_u is less than $x_{u, max}$ and (iii) when x_u is greater than $x_{u, max}$. The following discussion for the three cases has the reference to Fig. 3.5.1.

(i) When $x_u = x_{u, max}$

The compressive strain at the top concrete fibre $= 0.0035$ and the tensile strain at the level of steel $= \left(\dfrac{0.87\, f_y}{E_s} + 0.002 \right)$. Thus, it satisfies the assumptions (ii) and (vi) of sec. 3.4.2.

(ii) When x_u is less than $x_{u, max}$

There are two possibilities here:

(a) If the compressive strain at the top fibre = 0.0035, the tensile strain is more than $\left(\dfrac{0.87\,f_y}{E_s} + 0.002\right)$. Thus, this possibility satisfies the two assumptions (ii) and (vi) of sec. 3.4.2.

(b) When the steel tensile strain is $\left(\dfrac{0.87\,f_y}{E_s} + 0.002\right)$, the compressive concrete strain is less than 0.0035. Here also, both the assumptions (ii) and (vi) of sec. 3.4.2 are satisfied.

(iii) When x_u is more than $x_{u,\,max}$

There are two possibilities here:

(a) When the top compressive strain reaches 0.0035, the tensile steel strain is less than $\left(\dfrac{0.87\,f_y}{E_s} + 0.002\right)$. It violets the assumption (vi) though assumption (ii) of sec. 3.4.2 is satisfied.

(b) When the steel tensile strain is $\left(\dfrac{0.87\,f_y}{E_s} + 0.002\right)$, the compressive strain of concrete exceeds 0.0035. Thus, this possibility violets assumption (ii) though assumption (vi) is satisfied.

The above discussion clearly indicates that the depth of x_u should not become more than $x_{u,\,max}$. Therefore, the depth of the neutral axis has a limiting or maximum value = $x_{u,\,max}$. Accordingly, if the A_{st} provided yields $x_u > x_{u,\,max}$, the section has to be redesigned.

Since x_u depends on the area of steel, we can calculate $A_{st,\,lim}$ from Eq. 3.17.

From Eq. 3.17 (using $x_u = x_{u,\,max}$ and $A_{st} = A_{st,\,lim}$), we have

$$\frac{x_{u,\max}}{d} = \frac{0.87\,f_y \;\; A_{st,\lim}}{0.36 \;\; b\,d\,f_{ck}}$$

or $\qquad \dfrac{A_{st,\lim}}{b\,d}\,(100) \;=\; p_{t,lim} \;=\; \left(\dfrac{0.36\,f_{ck}\;\;x_{u,\max}}{0.87\,f_y\;\;d}\right)(100)$ (3.20)

In the above equation $\dfrac{x_{u,\max}}{d}$ can be obtained from the strain diagram of Fig. 3.5.1 as follows:

$$\frac{x_{u,max}}{d} = \frac{0.0035}{\dfrac{0.87\,f_y}{E_s} + 0.0055}$$

(3.21)

3.5.4 Values of $\frac{x_{u,max}}{d}$ and $p_{t,\,lim}$

Equation 3.20 shows that the values of $p_{t,\,lim}$ depend on both the grades of steel and concrete, while Eq. 3.21 reveals that $\frac{x_{u,max}}{d}$ depends on the grade of steel alone and not on the grade of concrete at all. The respective values of $p_{t,\,lim}$ for the three grades of steel and the three grades of concrete are presented in Table 3.1. Similarly, the respective values of $\frac{x_{u,max}}{d}$ for three grades of steel are presented in Table 3.2.

Table 3.1   Values of $p_{t,\,lim}$

f_{ck} in N/mm^2	$f_y = 250$ N/mm^2	$f_y = 415$ N/mm^2	$f_y = 500$ N/mm^2
20	1.76	0.96	0.76
25	2.20	1.19	0.94
30	2.64	1.43	1.13

Table 3.2   Values of $\frac{x_{u,max}}{d}$

f_y in N/mm^2	250	415	500
$\frac{x_{u,max}}{d}$	0.531 = 0.53 (say)	0.479 = 0.48 (say)	0.456 = 0.46 (say)

A careful study of Tables 3.1 and 3.2 reveals the following:

(i) The $p_{t,\,lim}$ increases with lowering the grade of steel for a particular grade of concrete. The $p_{t,\,lim}$, however, increases with increasing the grade of concrete for a specific grade of steel.

(ii) The maximum depth of the neutral axis $x_{u,max}$ increases with lowering the grade of steel. That is more area of the section will be utilized in taking the compression with lower grade of steel.

3.5.5 Computation of M_u

Equation 3.3 of Lesson 4 explains that M_u can be obtained by multiplying the tensile force T or the compressive force C with the lever arm. The expressions of C, lever arm and T are given in Eqs. 3.9, 3.13 (also 3.19) and

3.14 respectively of Lesson 4. Section 3.5.3 discusses that there are three possible cases depending on the location of x_u. The corresponding expressions of M_u are given below for the three cases:

(i) When $x_u < x_{u, max}$

Figure 3.5.1 shows that when concrete reaches 0.0035, steel has started flowing showing ductility (Strain $> \frac{0.87 f_y}{E_s} + 0.002$). So, the computation of M_u is to be done using the tensile force of steel in this case. Therefore, using Eqs. 3.13 and 3.14 of Lesson 4, we have

$M_u = T\,(\text{lever arm}) = 0.87\ f_y\, A_{st}\,(d - 0.42\,x_u)$
(3.22)

Substituting the expressions of T and lever arm from Eqs. 3.14 of Lesson 4 and 3.19 respectively we get,

$$M_u = 0.87\, f_y\, A_{st}\, d\left(1 - \frac{A_{st}\, f_y}{f_{ck}\, b\, d}\right)$$

(3.23)

(ii) When $x_u = x_{u, max}$

From Fig. 3.5.1, it is seen that steel just reaches the value of $\frac{0.87 f_y}{E_s} + 0.002$ and concrete also reaches its maximum value. The strain of steel can further increase but the reaching of limiting strain of concrete should be taken into consideration to determine the limiting M_u as $M_{u,\,lim}$ here. So, we have

$M_{u,\,lim} = C\,(\text{lever arm})$

Substituting the expressions of C and lever arm from Eqs. 3.9 of Lesson 4 and 3.19 respectively, we have

$$M_{u,\mathrm{lim}} = 0.36\, \frac{x_{u,\mathrm{max}}}{d}\left(1 - 0.42\, \frac{x_{u,\mathrm{max}}}{d}\right) b\, d^2\, f_{ck}$$

(3.24)

(iii) When $x_u > x_{u, max}$

In this case, it is seen from Fig. 3.5.1 that when concrete reaches the strain of 0.0035, tensile strain of steel is much less than ($\frac{0.87 f_y}{E_s} + 0.002$) and any further increase of strain of steel will mean failure of concrete, which is to be

avoided. On the other hand, when steel reaches $\frac{0.87\,f_y}{E_s} + 0.002$, the strain of concrete far exceeds 0.0035. Hence, it is not possible. Therefore, such design is avoided and the section should be redesigned.

However, in case of any existing reinforced concrete beam where $x_u > x_{u,\,max}$, the moment of resistance M_u for such existing beam is calculated by restricting x_u to $x_{u,\,max}$ only and the corresponding M_u will be as per the case when $x_u = x_{u,\,max}$.

3.5.6   Computation of Limiting Moment of Resistance Factor

Equation 3.24 shows that a particular rectangular beam of given dimensions of b and d has a limiting capacity of $M_{u,\,lim}$ for a specified grade of concrete. The limiting moment of resistance factor $R_{,lim}$ (= $M_{u,lim}/bd^2$) can be established from Eq. 3.24 as follows:

$$R_{,\lim} \;=\; \frac{M_{u,\lim}}{b\,d^2} \;=\; 0.36\,\frac{x_{u,\max}}{d}\left(1 - 0.42\,\frac{x_{u,\max}}{d}\right) f_{ck}$$

(3.25)

It is seen that the limiting moment of resistance factor $R_{,lim}$ depends on $\frac{x_{u,\max}}{d}$ and f_{ck}. Since $\frac{x_{u,\max}}{d}$ depends on the grade of steel f_y, we can say that $R_{,lim}$ depends on f_{ck} and f_y. Table 3.3 furnishes the values of $R_{,lim}$ for three grades of concrete and three grades of steel.

Table 3.3   Limiting values of $R_{,\lim} = \dfrac{M_{u,\lim}}{b\,d^2}$ factors (in N/mm^2)

f_{ck} in N/mm^2	$f_y = 250$ N/mm^2	$f_y = 415$ N/mm^2	$f_y = 500$ N/mm^2
20	2.98	2.76	2.66
25	3.73	3.45	3.33
30	4.47	4.14	3.99

A study of Table 3.3 reveals that the limiting moment of resistance factor $R_{,lim}$ increases with higher grade of concrete for a particular grade of steel. It is also seen that this factor increases with lowering the grade of steel for a particular grade of concrete. The increase of this factor due to higher grade of concrete is understandable. However, such increase of the factor with lowering the grade of steel is explained below:

Lowering the grade of steel increases the $\frac{x_{u,\max}}{d}$ (vide Table 3.2) and this enhanced $\frac{x_{u,\max}}{d}$ increases M_u as seen from Eq. 3.24. However, one may argue that Eq. 3.24 has two terms: $\frac{x_{u,\max}}{d}$ and $\left(1 - 0.42 \frac{x_{u,\max}}{d}\right)$.

With the increase of $\frac{x_{u,\max}}{d}$, $\left(1 - 0.42 \frac{x_{u,\max}}{d}\right)$ is decreasing. Then how do we confirm that the product is increasing with the increase of $\frac{x_{u,\max}}{d}$? Actual computation will reveal the fact. Otherwise, it can be further explained from Table 3.1 that as the grade of steel is lowered for a particular grade of concrete, the $p_{t,\ lim}$ gets increased. Therefore, amount of steel needed to have $M_{u,\ lim}$ with lower grade of steel is higher. Thus, higher amount of steel and higher values of $\frac{x_{u,\max}}{d}$ show higher $\frac{M_{u,\lim}}{b\,d^2}$ factor with the lowering of grade of steel for a particular grade of concrete (see Table 3.3).

3.5.7 Practice Questions and Problems with Answers

Q.1:   Which equation is needed to determine the depth of the neutral axis?

A.1:   Eq. 3.16 or 3.17.

Q.2:   How to find the lever arm?

A.2:   Eq. 3.13 of Lesson 4 or Eq. 3.19.

Q.3:   Should there be limiting or maximum value of u_p? If so, why? What is the equation to find the maximum value of x_u? How to find $A_{st,\ lim}$ for such case?

A.3:   Sec. 3.5.3 is the complete answer.

Q.4:   State the effects of grades of concrete and steel separately on $p_{t,\ lim}$ and $\frac{x_{u,\max}}{d}$.

A.4:   (i) $p_{t,\ lim}$ (see Eq. 3.20 and Table 3.1)

(a) $p_{t,\ lim}$ increases with lowering the grade of steel for a particular grade of concrete

(b) $p_{t, lim}$ increases with increasing the grade of concrete for a particular grade of steel

(ii) $\frac{x_{u,\max}}{d}$ (see Eq. 3.21 and Table 3.2)

(a) $\frac{x_{u,\max}}{d}$ increases with lowering the grade of steel

(b) $\frac{x_{u,\max}}{d}$ is independent on the grade of concrete.

Q.5: Write the corresponding expression of M_u when (i) $x_u < x_{u, max}$; and (ii) $x_u = x_{u, max}$

A.5: (i) Eq. 3.23, when $x_u < x_{u, max}$

(ii) Eq. 3.24, when $x_u = x_{u, max}$

Q.6: What is to be done if $x_u > x_{u, max}$ and why ?

A.6: When $x_u > x_{u, max}$, the section has to be redesigned as this does not ensure ductile failure of the beam.

Q.7: Write the expression of limiting moment of resistance factor $R_{,lim} = \frac{M_{u,lim}}{b\,d^2}$. What is its unit?

A.7: $R_{,\lim} = \frac{M_{u,\lim}}{b\,d^2} = 0.36 \frac{x_{u,\max}}{d} \left(1 - 0.42 \frac{x_{u,\max}}{d} \right) f_{ck}$ (Eq. 3.25). Its unit is N/mm^2.

Q.8: State separately the effects of grades of concrete and steel on the limiting moment of resistance factor $R_{,lim} = \frac{M_{u,\lim}}{b\,d^2}$.

A.8: (i) $R_{,lim}$ increases with increasing the grade of concrete for a particular grade of steel (see Table 3.3).

(ii) $R_{,lim}$ increases with lowering the grade of steel for a particular grade of concrete (see Table 3.3).

3.5.8 References

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3.5.9 Test 5 with Solutions

Maximum Marks = 50,   Maximum Time = 30 minutes

Answer all questions.

TQ.1:   Tick the correct answer: (4 x 5 = 20)

(i) The depth of the neutral axis is calculated from the known area of steel and it should be

      (a) less than 0.5 times the full depth of the beam

      (b) more than 0.5 times the effective depth of the beam

      (c) less than or equal to limiting value of the neutral axis depth

      (d) less than 0.43 times the effective depth of the beam

A.TQ.1: (i): (c)

(ii) For a particular grade of concrete and with lowering the grade of steel, the $p_{t,lim}$

      (a) increases

      (b) decreases

      (c) sometimes increases and sometimes decreases

      (d) remains constant

A.TQ.1: (ii): (a)

(iii) For a particular grade of steel and with increasing the grade of concrete, the $p_{t,lim}$

      (a) decreases

      (b) increases

      (c) remains constant

      (d) sometimes increases and sometimes decreases

A.TQ.1: (iii): (b)

(iv) Which of the statements is correct?

      (a) $x_{u,max}/d$ is independent of grades of concrete and steel

      (b) $x_{u,max}/d$ is independent of grade of steel but changes with grade of steel

(c) $x_{u,max}/d$ changes with the grade of concrete and steel

(d) $x_{u,max}/d$ is independent of the grade of concrete and changes with the grade of steel

A.TQ.1: (iv): (d)

TQ.2: Derive the expression of determining the depth of neutral axis and lever arm of a singly reinforced rectangular beam with known quantity of tension steel.   (10)

A.TQ.2: Section 3.5.2 is the complete answer.

TQ.3: Establish the expressions of the moment of resistance of a singly reinforced rectangular beam when (i) $x_u < x_{u,max}$, (ii) $x_u = x_{u,max}$ and (iii) $x_u > x_{u,max}$.

(8+7=15)

A.TQ.3: Section 3.5.5 is the complete answer.

TQ.4: Derive the expression of limiting moment of resistance factor and explain how it is influenced by the grades of concrete and steel.
            (5)

A.TQ.4: Section 3.5.6 is the full answer.

3.5.10 Summary of this Lesson

Understanding the various assumptions of the design, stress-strain diagrams of concrete and steel, computations of the total compressive and tensile forces and the equations of equilibrium in Lesson 4, this lesson illustrates the applications of the two equations of equilibrium. Accordingly, the depth of neutral axis and the moment of resistance of the beam can be computed with the expressions derived in this lesson. Different cases that arise due to different values of the depth of neutral axis are discussed to select the particular expression of the moment of resistance. Further, the expressions of determining the limiting values of percentage of steel, depth of neutral axis, moment of resistance and moment of resistance factor are established. The influences of grades of concrete and steel on them are also illustrated.