Module 5

Lecture 1

Flow through unsaturated soils

The flow of water in soils is governed by the total head, as mentioned earlier, which is expressed as

$H = h_g + h_m + h_o$

where h_g is the gravitational head, h_m the matric suction head, and h_o the osmotic suction head. In the absence of gravitational head (horizontal flow) and the presence of solute in soils, matric suction head controls the flow of water in soils. Flow through soils can be either steady state or unsteady state (transient) depending on the type of soil and the boundary conditions. Steady-state flow is a time invariant flow as shown in the Fig. 5.1.

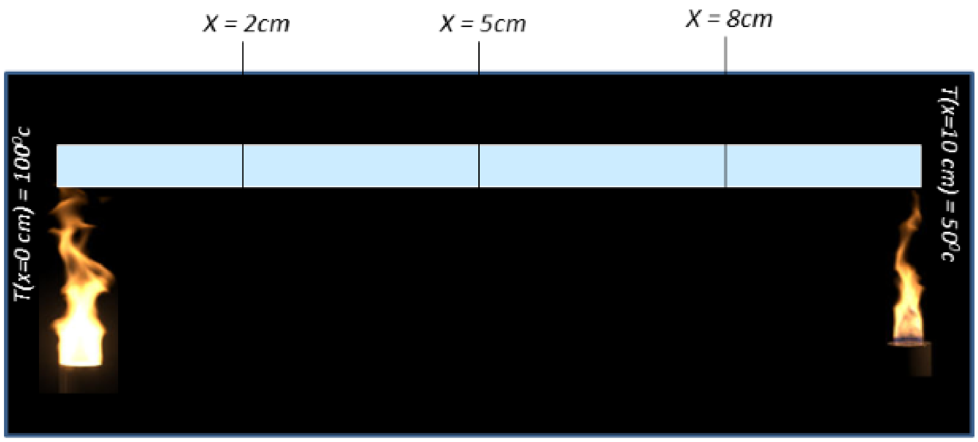

Fig. 5.1 Conceptual illustration describing the steady state heat conduction in solids

A metallic object is heated at two ends by maintaining two different, but constant temperatures. Conduction of heat in the object is governed by Fourier's law. The heat distribution in the object can be measured by sensors (thermometers) placed along the length of the object as shown in figure #. It can be observed that the heat distribution with space becomes constant and time-independent after certain time. This state is called the steady state. Similarly, the steady state and transient flows take place in saturated soils. However, can the steady state flow take place in unsaturated soils? The answer to this question is affirmative. Steady state flows can take place in unsaturated soils when a constant matric suction head values are maintained at the boundaries as shown in the Fig. 5.2a – 5.2b.

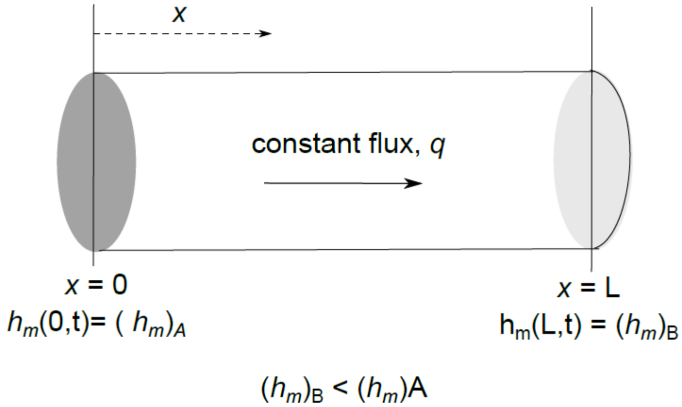

Fig. 5.2a Steady state infiltration through a horizontal soil column

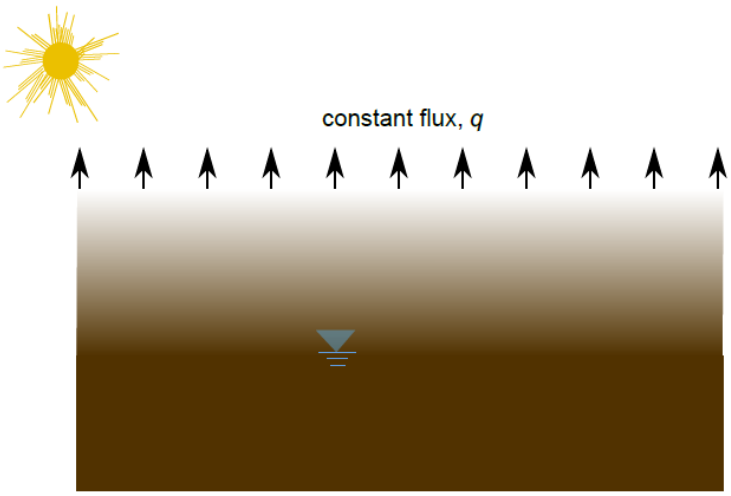

Fig. 5.2b Steady state evaporation through top layers of the soil

The steady state flow can takes place in a partially saturated soil column when the flow boundaries are time invariant as shown in Fig. 5.#a. As it was shown in Fig. 5.#b, a steady state flow can takes place in soils situated between the ground water table and top surface due to the evaporation. Similarly, it is also possible to maintain a constant water content ($\theta < \theta_s$) in unsaturated soils by applying low hydraulic gradient across the soil sample. For example, consider a pore structure in a soil as shown in the Fig. 5.3.

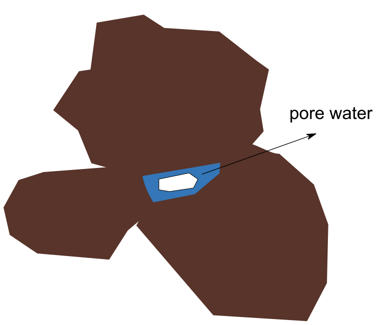

Fig. 5.3. Illustration of pore water channel in a partly saturated soil

Application of small gradients can drive the water flow through the thick water films in the pore space, but the gradients may not be sufficient to drive the air pockets out of the system (saturation). Therefore, a constant (time-independent) flows take place in unsaturated soils with water contents less that the saturation.

The head distribution in steady state is linear which can be easily verified. The Darcy's law for steady flow through saturated soils can be written as

$$q = -k_s i = -k_s \frac{dh}{dx} \tag{5.1}$$

The head distribution can be obtained by rearranging the terms and integrating along the flow length

$$q\int_0^x dx = -k_s \int_0^h dh \tag{5.2}$$

Therefore,

$$h = -\left(q/k_s \right) x \tag{5.3}$$

which describes a linear relationship between the head and the spatial distance. However, what is the head (i.e., matric suction head) distribution in unsaturated soils? It is important to qualitatively distinguish the difference in the head distribution in saturated and unsaturated soils. The head distribution in steady state unsaturated soils is evaluated and analyzed in these lectures.

Horizontal steady-state flow

Consider a one-dimensional and steady flow through a homogeneous soil column as shown in the Fig. 5.4.

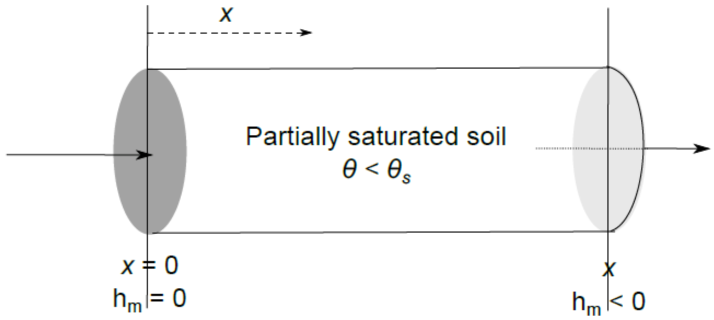

Fig. 5.4. Illustration of 1D steady-state flow through unsaturated soils

Flow through unsaturated soils is assumed to be described by modified Darcy's law which is expressed as

$$q = -k(h)\frac{dh}{dx} \tag{5.4}$$

where k is the hydraulic conductivity expressed in functional form for unsaturated soils, h is the total energy potential (total head), and x is the distance measured in the direction of flow. The total energy potential is sum of the matric suction head and the elevation head i.e., $h_m + x$ in the absence of solute.

If we assume that the Gardner's (1958) functional form for the hydraulic conductivity is valid

The Darcy's law for steady flow in unsaturated soils may be written as

$$q = -k_s \exp(\alpha h_m)\frac{dh_m}{dx} \tag{5.5}$$

where a is the fitting parameter related to air entry head (1/cm). Integrating the above expression and imposing the boundary condition results

$$q\int_0^x dx = -k_s \int_0^{h_m} \exp(\alpha h_m) dh_m \tag{5.6}$$

which gives to

$$h_m = \frac{1}{\alpha}\ln\left(1 - \frac{q\alpha x}{k_s}\right) \tag{5.7}$$

which is a non-linear distribution when a non-linear functional relationship is assumed for hydraulic conductivity.

Similarly, one can also assume a simple linear relationship (Richards, 1931) for hydraulic conductivity and derive the expression for head distribution. The hydraulic conductivity function by Richards (1931) is given as

$$k = a + bh_m \tag{5.8}$$

The discharge flux can be expressed by

$$q = -\left(a + bh_m\right)\frac{dh_m}{dx} \tag{5.9}$$

After simplification and integration,

$$q\int_0^x dx = -\int_0^{h_m}\left(a + bh_m\right)dh_m = -ah_m - \frac{bh_m^2}{2} \tag{5.10}$$

which can be expressed in the quadratic form as

$$\frac{bh_m^2}{2} + ah_m + qx = 0 \tag{5.11}$$

which is has two solutions (roots) as shown below

$$h_m = \frac{1}{b}\left(-a \pm \sqrt{a^2 - 2bqx}\right) \tag{5.12}$$

Substituting the lower boundary (x = 0, h = 0) provides the correct root which is

$$h_m = -\frac{1}{b}\left(a - \sqrt{a^2 - 2bqx}\right) \tag{5.13}$$

which is a non-linear distribution.

Therefore, the head distribution in unsaturated soils is non-linear irrespective of the functional form assumed for hydraulic conductivity.

Lecture 2

Example problem on steady state horizontal flow:

Steady water flow is taking place in a horizontal column. The time-invariant boundary conditions are given in Fig. 5.5.

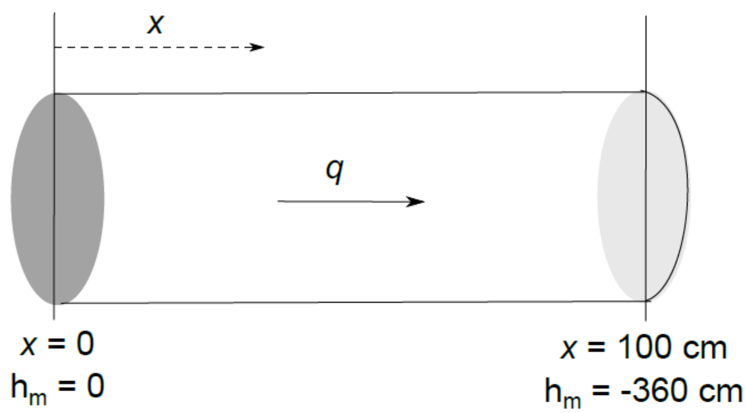

Fig. 5.5. Problem description on flow through a horizontal column

Find out the head distribution along the column length when the following functional relationships are assumed for hydraulic conductivity

(i) $k = k_s \exp(\alpha h_m)$

where k_s = 0.1 cm/day and a = 0.001/cm

(ii) $k = a + bh_m$

where a = 8 cm/day, b = 0.02/day

Solution:

(i) The suction head distribution can be expressed as (from eq. 5.7)

$$h_m = \frac{1}{\alpha}\ln\left(1 - \frac{q\alpha x}{k_s}\right)$$

where hydraulic conductivity is assumed to vary $k = k_s \exp(\alpha h_m)$.

It can be observed that the discharge flux is required to estimate the suction head distribution. It can be estimated by integrating the flux in the entire column. Therefore,

$$q\int_{0}^{100} dx = -k_s \int_{0}^{-360} \exp(0.001h_m)dh_m$$

after solving,

q = 0.302 cm/day

Therefore, $h_m = 1000\ln(1-0.00302x)$

Considering the similar approach,

(ii) The suction head distribution is $h_m = -\frac{1}{b}\left(a-\sqrt{a^2-2bqx}\right)$ (eq. 5.13) when $k = a + bh_m$ is used. The flux can be obtained by solving

$$q\int_{0}^{100} dx = -\int_{0}^{-360} (8+0.02h_m)dh_m$$

which gives, q = 15.84 cm/day

Therefore, $h_m = -50\left(8-\sqrt{64-0.6336x}\right)$

The head distribution by both the methods and the functional relationships for the conductivity are given in Fig. 5.6a-5.6b.

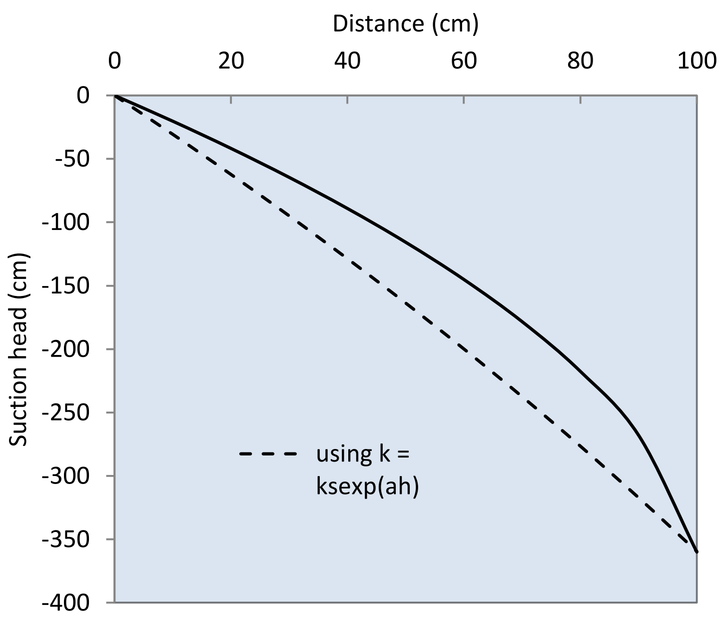

Fig. 5.6a. Suction head distribution in the horizontal column by different methods

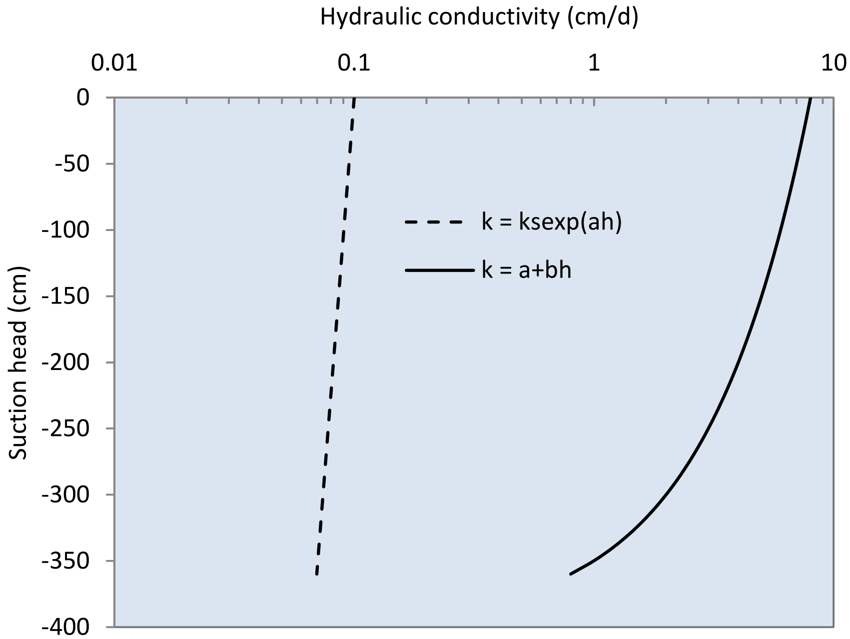

Fig. 5.6b. Assumed hydraulic conductivity functional relationships

The head distribution is completely different by both the methods due to the wide variation in the assumed conductivity relationships. The conductivities vary in several orders by the assumed relationships. The discharge fluxes got adjusted to satisfy the boundary conditions. Therefore, a correct hydraulic conductivity functional relationship is required for estimating an accurate head distribution.

Vertical steady state flow

The driving force for vertical flows in unsaturated soils is the combination of gravity and the suction head. The gravity, therefore, has a subsequent influence on the spatial distribution of the suction head. In case of one-dimensional vertical flow the governing flow equation can be written as

$$q = -k\left(\frac{dh_m}{dz} + 1\right) \qquad (5.14)$$

where z is the spatial distance. The suction head distribution can be obtained by rearranging the terms as

$$dz = -\frac{dh_m}{1 + q/k} \qquad (5.15)$$

Integration of the equation Eq. (5.10) and substitution of the boundary conditions: $h_m = 0$ at $z = 0$ and $h_m = h$ at $z = Z$ gives

$$Z = -\int_0^{h_m} \frac{1}{1+q/k(\theta)} dh_m(\theta) \tag{5.16}$$

which can be solved to obtain the profiles of the matric suction head, h_m, or total head if the steady flux q, the soil-water characteristic curve, and the hydraulic conductivity function are known. The solution can be obtained numerically if we write the above equation in discrete form as shown in the following equation

$$z = -\sum_{j=1}^{n} \frac{\Delta h_m(\theta_i)}{1+q/k(\theta_i)} \tag{5.17}$$

where n is the number of discrete data points selected from the SWCC and hydraulic conductivity function.

Rate of capillary rise

The concept of capillary rise in unsaturated soils was described in the earlier modules. The rate of capillary rise can be appreciated after going through the flow through unsaturated soils in this module. Terzaghi in his infamous work (1943) provided a simple relationship for rate of capillary rise in soils. Terzaghi assumed that Darcy's law is valid for describing the steady flow through unsaturated soils which can be expressed as

$$q = -k_s i$$

where the hydraulic conductivity of the wetting front is assumed to be described by k_s. Further, the gradient is assumed to be $\left(h_c - z \big/ z\right)$ where h_c is the capillary rise as shown in the following figure #.

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Therefore, the discharge flux becomes

$$q = -k_s\left(\frac{h_c - z}{z}\right)$$

or

$$n\frac{dz}{dt} = -k_s\left(\frac{h_c - z}{z}\right)$$

solving the equation for z results,

$$-\int\left(\frac{z}{h_c - z}\right)dz = \frac{k_s}{n}\int dt$$

which can be simplified to

$$-\int dz + h_c\int\left(\frac{dz}{h_c - z}\right) = \frac{k_s}{n}\int dt$$

Therefore,

$$-z - h_c\ln\left(h_c - z\right) = \frac{k_s t}{n} + c$$

where c is the constant of integration. The constant can be obtained by substituting the initial condition (i.e., $t = 0$ and $z = 0$) which is

$$t = \left(\frac{nh_c}{k_s t}\right)\left[\ln\left(\frac{h_c}{h_c - z}\right) - \frac{z}{h_c}\right] \qquad (5.18)$$

which is an implicit expression describing the rate of rise. However, the assumption of using k_s for unsaturated conductivity causes the predicted rate of capillary rise a higher value. If we assume Gardner's (1958) expression for describing the hydraulic conductivity function the following closed-form expression can be obtained (Lu and Likos, 2004)

$$\frac{k_s t}{n} = \sum_{j=0}^{m}\frac{\alpha^j}{j!}\left[-\sum_{s=0}^{j}\frac{h_c^s z^{j+1-s}}{j+1-s} + h_c^{j+1}\ln\frac{h_c}{h_c - z}\right] \qquad (5.19)$$

which assumes the Terzaghi's equation if the index m is set to zero.

Lecture 3

Transient flow

The flow of water in unsaturated soils may vary both spatially and temporally due to several factors. Time dependent changes in the boundary conditions (infiltration-evaporation) can significantly influence the flow mechanism. Such changes are accounted by the theoretical models by considering these changes in terms of boundary conditions for the soil domain. Other effects due to soil hydraulic characteristics are captured in the governing equation.

The governing one-dimensional, transient flow equation in soils can be expressed as

$$-\rho_w \frac{\partial q}{\partial x} = \frac{\partial(\rho_w \theta)}{\partial t} \qquad (5.20)$$

where ρ is the density of water (kg/m^3) and q is the water flux (m/s) in the x direction.

In case of flow through saturated soils, the volumetric water content is equal to the porosity, n, of the soil. After combining the Eq. (5.20) with Darcy's equation and writing n for θ gives

$$D\frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t} \qquad (5.21)$$

where h is the total head, D the hydraulic diffusivity (m^2/s), which is equal to S_s/k, and S_s the specific storage. The specific storage is defined as

$$S_s = \rho_w g(\alpha_s + n\beta_w) \qquad (5.22)$$

where α_s is the bulk compressibility of soil (m^2/N) and β_w the compressibility of pore water (m^2/N).

The flow through unsaturated soils can be described using the Darcy's law as

$$\frac{\partial}{\partial z}\left[k(h_m)\left(\frac{\partial h_m}{\partial z}+1\right)\right] = \frac{\partial \theta}{\partial t} \qquad (5.23)$$

where the additional term added to suction gradient is the gradient due to elevation. Using the chain rule,

$$\frac{\partial}{\partial z}\left[k(h_m)\left(\frac{\partial h_m}{\partial z}+1\right)\right] = \frac{\partial \theta}{\partial h_m}\frac{\partial h_m}{\partial t} \qquad (5.24)$$

where $\partial\theta/\partial h_m$ is the slope of the soil water characteristic curve, which is called specific moisture capacity, C. As SWCC is non-linear, C is expressed as a function of matric suction head. The resulting equation is called Richards' equation which is written as

$$\frac{\partial}{\partial z}\left[k\left(h_m\right)\left(\frac{\partial h_m}{\partial z}+1\right)\right]=C\left(\theta\right)\frac{\partial h_m}{\partial t} \tag{5.25}$$

The Richards' equation may often be expressed in terms of volumetric water content as shown in the following equation

$$\frac{\partial}{\partial z}\left[D\left(\theta\right)\frac{\partial \theta}{\partial z}+k\left(\theta\right)\right]=\frac{\partial \theta}{\partial t} \tag{5.26}$$

The solution to Richards' equation with appropriate boundary and initial conditions provides the spatial and temporal distribution of matric suction or moisture content in the soil. As implied, three characteristic functions are required for the solution of Eq. (5.25). The function $C\left(\theta\right)$ imposes the existence of smooth and well-defined soil-water characteristic curve. The following figures, Fig. 5.7-5.8, illustrate the nature of specific moisture capacity function for a well-defined and smooth SWCC.

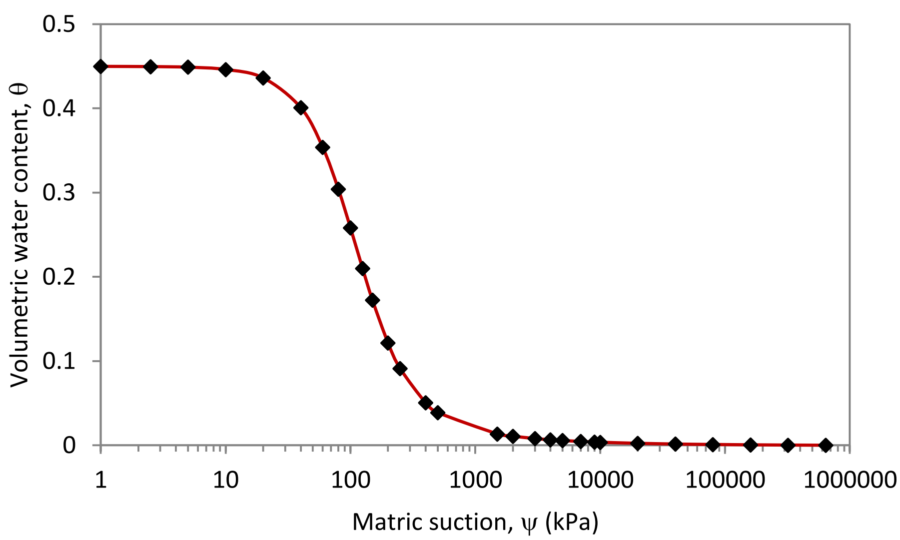

Fig. 5.7. Smooth and well-defined SWCC

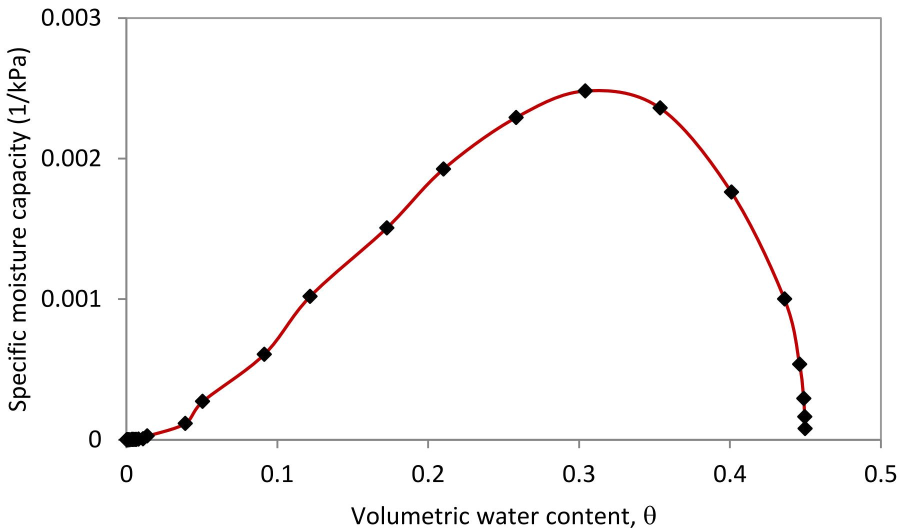

Fig. 5.8. Specific moisture capacity function for the SWCC defined in Fig. 5.2.

It is interesting to note that the magnitude of specific moisture capacity reaches a maximum value of 0.0025 cm^{-1} at volumetric water content value 0.3. This reflects the fact that for each unit change in matric suction, the change in volumetric water content is 0.0025. It is a simple exercise to compare with the specific moisture capacity functions for coarse and fine-grained soils having contrasting pore size distributions. The relatively sharp maximum will be observed in case of coarse grained soils due to their narrow pore size distribution, where the majority of the pores are drained over a narrow range of suction.

Transient horizontal flow

The transient moisture flow through an initially dry unsaturated soil column was analyzed by analytical solutions several decades ago by assuming the suction head gradient in the soil beyond the wetting front is assumed to be zero in both space and time. Thus, the water content and corresponding hydraulic conductivity of the soil beyond the wetting front can be assumed are constant in both space and time. Figure 5.9 illustrates the conceptual problem.

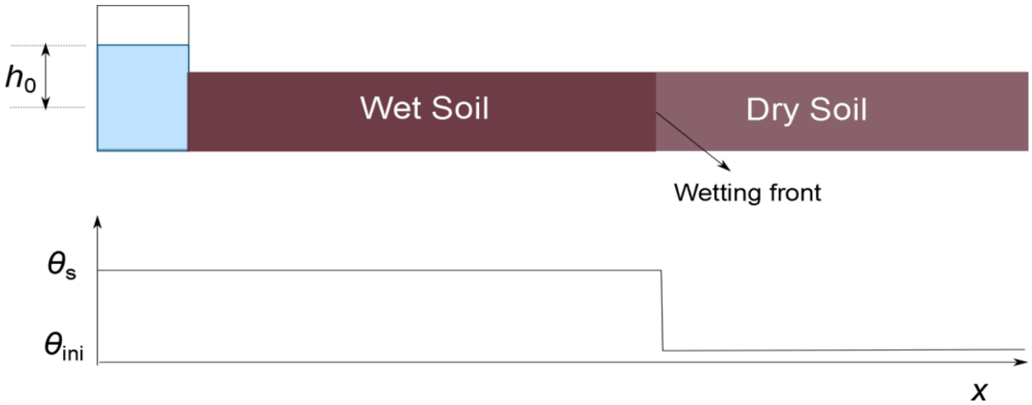

Fig. 5.9. Conceptual transient horizontal infiltration problem (After, Lu and Likos, 2004)

The infiltration rate can be predicted using the Darcy's first law which states

$$q = -k_s \frac{dh}{dx}$$

which can be written as

$$(\theta_s - \theta_{ini})\frac{dx}{dt} = -k_s\left(\frac{h_{ini} - h_0}{x}\right)$$

where h_{ini} is the suction head at the wetting front, h_0 the head ($h_0 \geq 0$) behind the wetting front, k_s the hydraulic conductivity behind the wetting front, which is often assumed to be equal to the saturated hydraulic conductivity. After integrating over the wetting front,

$$\int_0^x x\,dx = -k_s\left(\frac{h_{ini} - h_0}{x}\right)\int_0^t dt$$

After the simplification,

$$\frac{x}{\sqrt{t}} = \sqrt{2k_s\left(\frac{h_0 - h_{ini}}{\theta_s - \theta_{ini}}\right)} \qquad (5.26)$$

However, most soils do not exhibit such sharp wetting front behavior. The partial differential equation given in 5.19 has to be solved for accurate prediction of moisture distribution in soils by numerical methods.

Lecture 4

Determination of water diffusivity:

A simple approach is given in some textbooks (Refer: Lu and Likos, 2004; Rumynin, 2012*) using "Boltzmann transformation" for flow through horizontal columns. This transformation allows the partial differential equation to transform into the ordinary differential equation. The governing equation for horizontal flows can be written as

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x}\right] \tag{5.28}$$

where the boundary conditions are $\theta(x=0, t) = \theta_s$ and $\theta(x=\infty, t) = 0$. The independent variables of this equation (i.e., x, t) can be transformed into variable λ by the following substitution

$$\theta(x,t) = \theta(\lambda) \tag{5.29}$$

where $\lambda = x/\sqrt{t}$ is the Boltzmann transformation. Replacing the independent variables in eq. 5.28 with λ results the following ordinary differential equation

$$-\frac{\lambda^3}{2x^2}\frac{d\theta}{d\lambda} = \frac{1}{\sqrt{t}}\frac{d}{d\lambda}\left[\frac{D(\theta)}{\sqrt{t}}\frac{d\theta}{d\lambda}\right]$$

after simplification,

$$\frac{\lambda}{2}\frac{d\theta}{d\lambda} + \frac{d}{d\lambda}\left[D(\theta)\frac{d\theta}{d\lambda}\right] = 0 \tag{5.30}$$

where $\sqrt{t}d\lambda$ is substituted for dx and $-2x^2 d\lambda/\lambda^3$ is substituted for dt. The boundary conditions transform into

$$\theta(\lambda = 0) = \theta_s;\ \theta(\lambda \to \infty) = \theta_{ini} \tag{5.31}$$

The integration of the governing equation yields the determination of water diffusivity function using the following expression

$$D(\theta) = -\frac{1}{2}\left.\frac{d\lambda}{d\theta}\right|_{\theta}\int_{\theta_{ini}}^{\theta_s}\lambda(\theta)d\theta \tag{5.32}$$

which can be solved simple iteration methods.

Transient vertical infiltration

The following figure (Fig. 5.10) illustrates the conceptual problem of one-dimensional vertical infiltration through unsaturated soil column. Both suction and gravity play a role in the wetting front advancement in this case. The time of infiltration can be obtained a similar approach used for the horizontal case as shown below:

$$q = \left(\theta_s - \theta_{ini}\right)\frac{dz}{dt} = -k_s \frac{dh}{dz}$$

$$\left(\theta_s - \theta_{ini}\right)\frac{dz}{dt} = -k_s\left[\frac{h_{ini} - z}{z} - \frac{h_0}{z}\right] = k_s\left[1 + \frac{h_0 - h_{ini}}{z}\right]$$

Solving for infiltration rate

$$\frac{k_s}{\left(\theta_s - \theta_{ini}\right)}\int_0^t dt = \int_0^z \frac{dz}{\left[1 + \frac{h_0 - h_{ini}}{z}\right]}$$

The integration yields the following solution

$$\frac{k_s}{\left(\theta_s - \theta_{ini}\right)} t = z - \left(h_0 - h_{ini}\right)\ln\left(\frac{z + \left(h_0 - h_{ini}\right)}{\left(h_0 - h_{ini}\right)}\right) \qquad (5.33)$$

which is very similar to what Terzaghi (1943) had derived for the rate of capillary rise. It appears that this derivation had inspired Terzaghi for his derivation.

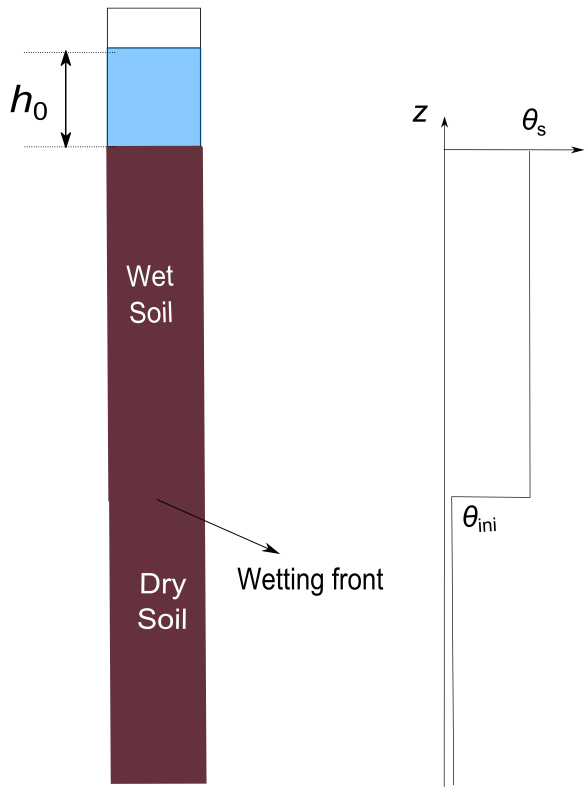

Fig. 5.10. Conceptual transient vertical infiltration problem (after, Lu and Likos, 2004)

The wetting front for downward infiltration advances nonlinearly as against the horizontal flow due to the negligible effect of gravity. Due to this reason, vertical infiltration advances faster as compared to the horizontal infiltration in coarse-grained soils.

Accurate prediction of water flow through unsaturated soils is possible with the current computational advancement. The Richards' equation can be numerically solved by considering appropriate hydraulic characteristic functions as shown below.

The Richards' equation for vertical infiltration through a homogeneous soil can be written as

$$\frac{\partial \theta}{\partial t} = -\frac{\partial}{\partial z}\left[k(\theta)\left(\frac{\partial h_m}{\partial z}+1\right)\right]$$

which can be approximated using the finite-difference numerical technique using the grid shown in Fig. 5.11 as

$$\frac{\theta_i^{n+1}-\theta_i^n}{\Delta t} = -\frac{1}{\Delta z}\left[k\left(\theta_{i+1/2}\right)\left(\frac{\partial h_m\left(\theta_{i+1/2}\right)}{\partial z}+1\right)-k\left(\theta_{i+1/2}\right)\left(\frac{\partial h_m\left(\theta_{i+1/2}\right)}{\partial z}+1\right)\right]$$

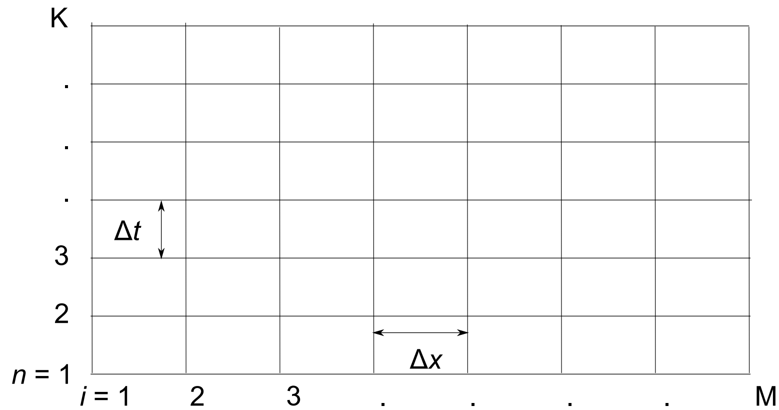

Fig. 5.11. A finite difference reference grid

$$\frac{\theta_i^{n+1}-\theta_i^{n}}{\Delta t}=-\frac{k\left(\theta_{i+1/2}\right)}{\Delta z^2}\left(h_m\left(\theta_{i+1}\right)-h_m\left(\theta_i\right)\right)+\frac{k\left(\theta_{i-1/2}\right)}{\Delta z^2}\left(h_m\left(\theta_i\right)-h_m\left(\theta_{i-1}\right)\right)+\frac{1}{\Delta z}\left[k\left(\theta_{i+1/2}\right)-k\left(\theta_{i+1/2}\right)\right]$$

which can be solved along with proper boundary and initial conditions.

Rumynin, V. G. (2012) “Subsurface Solute Transport Models and Case Histories: With Applications to Radionuclide Migration”, Springer Science & Business Media