

Module 2

Lecture 1

Fundamental Principles

Prediction of an observed phenomenon in nature is of great interest to mankind. Prediction requires a mathematical representation of a given phenomenon. The governing equations in continuum physics represent fundamental physical laws that are independent of matter. Therefore, any natural phenomenon must obey the governing equations independent of the matter. The governing equations are conservation of mass, energy, linear momentum, and angular momentum; second law of thermodynamics; and Maxwell's equations. The readers are advised to refer any standard text book on continuum mechanics for more details of the equations (Malvern, 1969). The variables in the governing equations are the state variables as they represent the state of the system. However, we require additional equations to solve the problem uniquely for satisfying the constituents (material-dependency) of the system. Such equations are called constitutive equations to establish the interdependency of several state variables from the experimental observations. For example, the proportionality equation between the stress and strain forms a fundamental constitutive relationship in understanding the mechanical behavior of solids. Further, Hooke's law and Newton's laws are the constitutive equations. Constitutive equations, therefore, explain the interdependency of different state variables (i.e., stress, strain, etc.). The proportionality constants of the constitutive relations are called the material constants that represent the fundamental constituents of the system. The dependency of material constants on the state variables is required for understanding the behavior.

Mechanical and physico-chemical behavior of soils are commonly quantified in geotechnical engineering by constitutive relationships between several state variables and interrelationships between such variables. Such identification exercise is useful for the prediction of physical phenomena or simply the engineering behavior of soils. Some of the commonly used constitutive relationships, state variables, and material constants of saturated soils are discussed here for the prediction of geotechnical behavior of soils, observed in the general soil mechanics. The variables of unsaturated soil mechanics are then identified based on the developed understanding on saturated soil mechanics.

Constitutive Relationships

Constitutive equations express the interrelationships between state variables. These equations for a given system are useful in the prediction of the physical phenomenon when qualitative understanding of the variables is available. A flow chart is provided in Fig. 2.1 which provides an in-detail explanation of a prediction process. The state variables responsible for the phenomenon need to be identified from the governing equations in the beginning. Constitutive relationships between different state variables

are established. The material constants, proportionality constants in the constitutive relationships, and their dependency on different state variables are determined. For example, the shear strength of the soils is expressed using Mohr-Coulomb criterion which is an important constitutive relationship between stress and strain (two state variables) in geotechnical engineering. The cohesion and angle of internal friction constitute the material constants. The phenomenon is then predicted and compared with the observation.

State Variables

Common stress state variables encountered in geotechnical practice are total stress, effective stress, pore water pressure, pore air pressure, and the shear stress. The variables such as strain and void ratio are the state variables used in the compressibility behavior of soils. Similarly, the degree of saturation/water content and total hydraulic head serve as the state variables in flow related problems in geotechnical practice.

The state variables can be functions of other state variables. For example, the present weather (state) can be expressed in defining several state variables such as temperature, pressure, relative humidity, and wind speed.

Material Constants

These are the proportionality constants of the constitutive relationships. Material constants may vary from one state to another state and one material to another. The angle of internal friction, cohesion, viscosity of water, hydraulic conductivity, diffusion coefficient, and swelling pressure are some of the commonly encountered material-constants in soil mechanics. The material constants are function of state variables. For example, the viscosity of water is function of a state variable, i.e. temperature. Similarly, the swelling pressure of the soil is dependent on the initial void ratio (initial stress state) of the soil sample. Therefore, the material constants vary usually from one material to another material and/or from one state to another state.

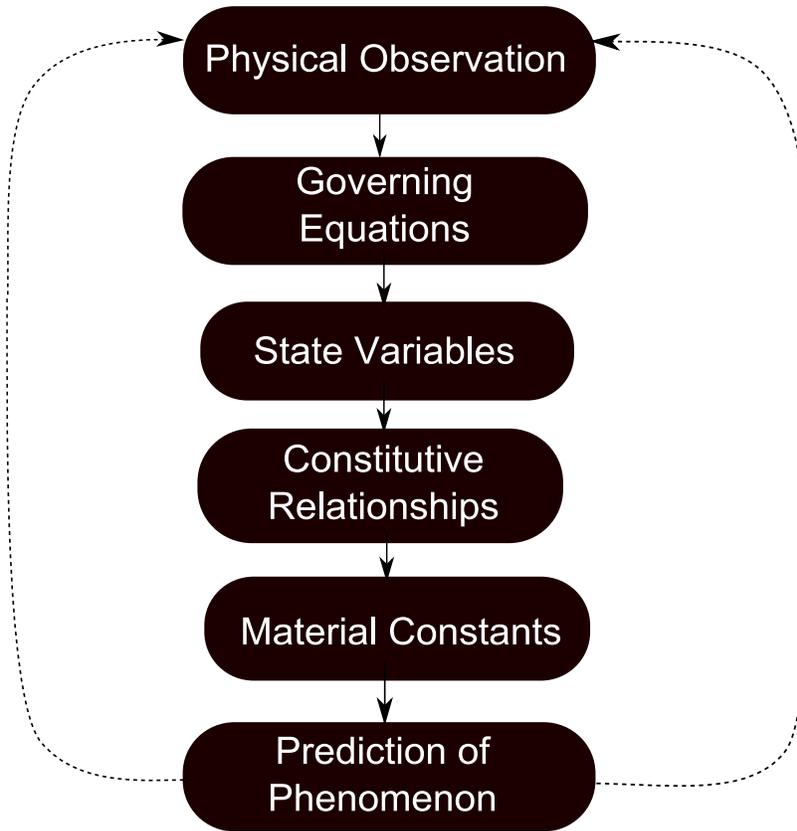


Fig. 2.1 Flow chart for the phenomenon prediction

Some of the constitutive relationships used in the soil mechanics are discussed here.

Shear strength of soils:

Existing natural and man-made slopes are stable only due to the shearing strength possessed by the composed soil matrix. The underlying laboratory and field studies in soil mechanics for stability analysis is an understanding the shearing strength of the soils and determining whether or not the shearing strength of soil sufficient to prevent the slope failure. The shear strength may be represented by a straight line at a slope of ϕ' through the origin of the Mohr diagram, called the failure envelope, as shown in Fig. 2.2.

An important constitutive relationship for quantifying the shear strength of soils is Mohr-Coulomb equation which relates the stress state variables viz. shear stress and normal stress with the material properties viz. cohesion and angle of internal friction as shown in the Fig. 2.2.

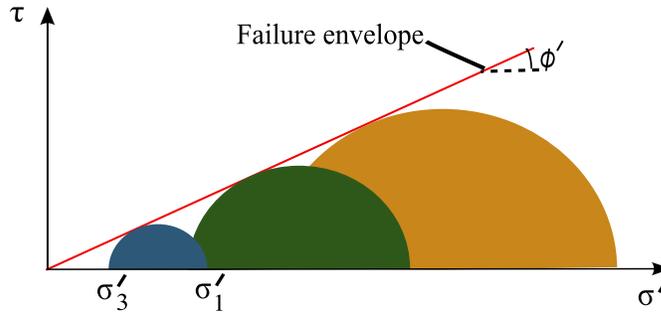


Fig. 2.2. Mohr strength diagram

The failure envelope is thought of as a property of the material and is independent of the stresses on the material. The normal stress and shear stress constitute the stress state variables and the cohesion and angle of internal friction represent the material constants. The failure envelope represents the constitutive relationship.

The representation of failure envelopes in unsaturated soils is highly complex due to the presence of the third state variable, *matric suction*, influencing the strength behavior of soils. Further, this state variable is dependent on another state variable viz. moisture content. The concept of shear strength in unsaturated soils is discussed in Chapter – 6.

Compressibility behavior of soils:

The resultant strains under the applied stresses are time-dependent in fine-grained soils. In such soils, constitutive relationships have been developed to relate the stresses (effective stresses), the strains (consolidation settlement), and the required time to achieve equilibrium under the applied stresses. The stress-strain-time relationships are important to understand the compressibility characteristics viz. coefficient of compressibility (C_c), coefficient of consolidation (C_v), etc., of a given clay soil. One-dimensional oedometer tests are conducted in the laboratory to predict the material characteristics, C_c and C_v . The relationship between consolidation settlement with time constitutes an important relationship to determine the C_v of the given clay as shown in Fig. 2.3a.

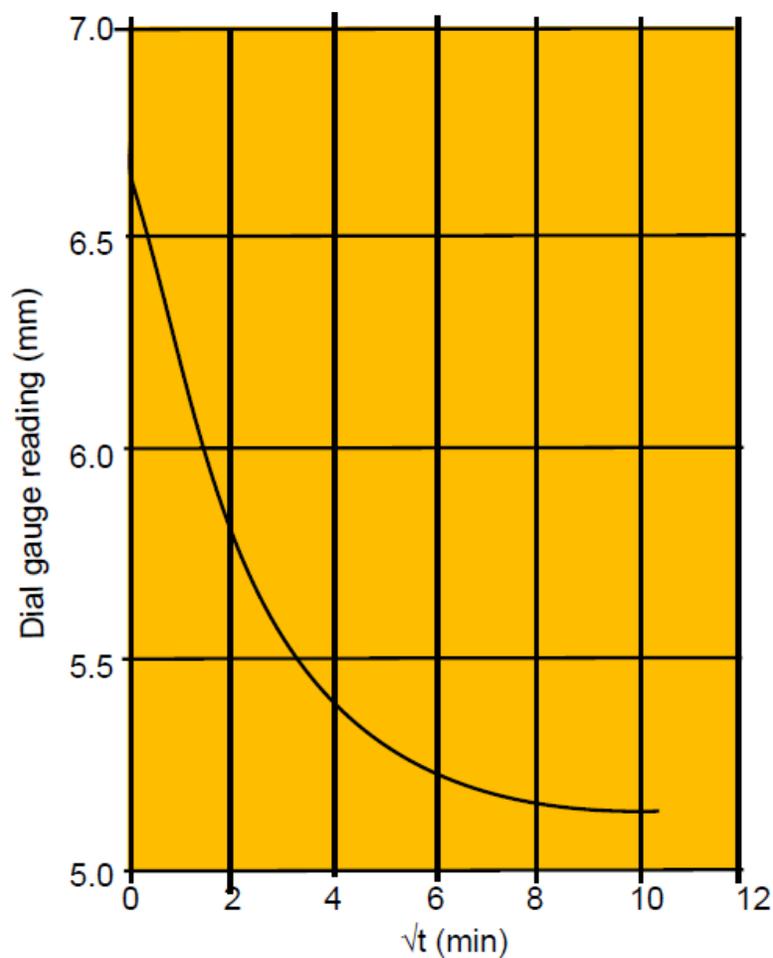


Fig. 2.3a. Settlement versus time relationship from Oedometer test

A typical constitutive relationship obtained under different applied stresses is shown in Fig. 2.3b. It shows the initial compression followed by expansion. State variable such as the preconsolidation pressure can also be obtained from this curve other than the coefficient of compression, which is a material constant.

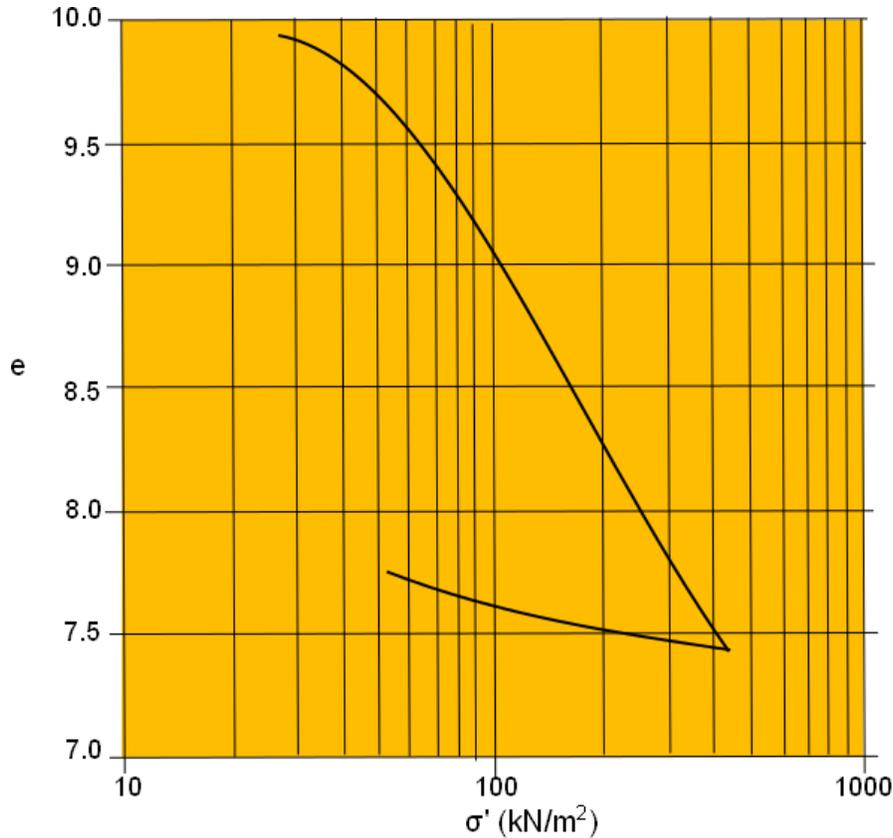


Fig. 2.3b. Consolidation and rebound obtained from the Oedometer test

As the rate of settlement (decrease in void ratio) of the clay sample under a given loading depends on the hydraulic conductivity of the soil, it constitutes a relationship. Similarly, it can be observed from Fig. 2.3b that the void ratio decreases with increase in the effective stress in the soil. Therefore, it is not difficult to demonstrate that the hydraulic conductivity decreases with increase in the effective stress or decrease in the void ratio. Fig. 2.3c shows the constitutive relationship between void ratio (state variable) and saturated hydraulic conductivity (material constant) of the sample. It appears that a simple power relationship exists between the void ratio and hydraulic conductivity.

The relationship between void ratio and hydraulic conductivity becomes highly complex in unsaturated soils as it is difficult to define the stress state and interdependency of moisture content on the conductivity. In unsaturated soils, hydraulic conductivity is a function of state variables such as *matric suction* or *water content*, which will be discussed later.

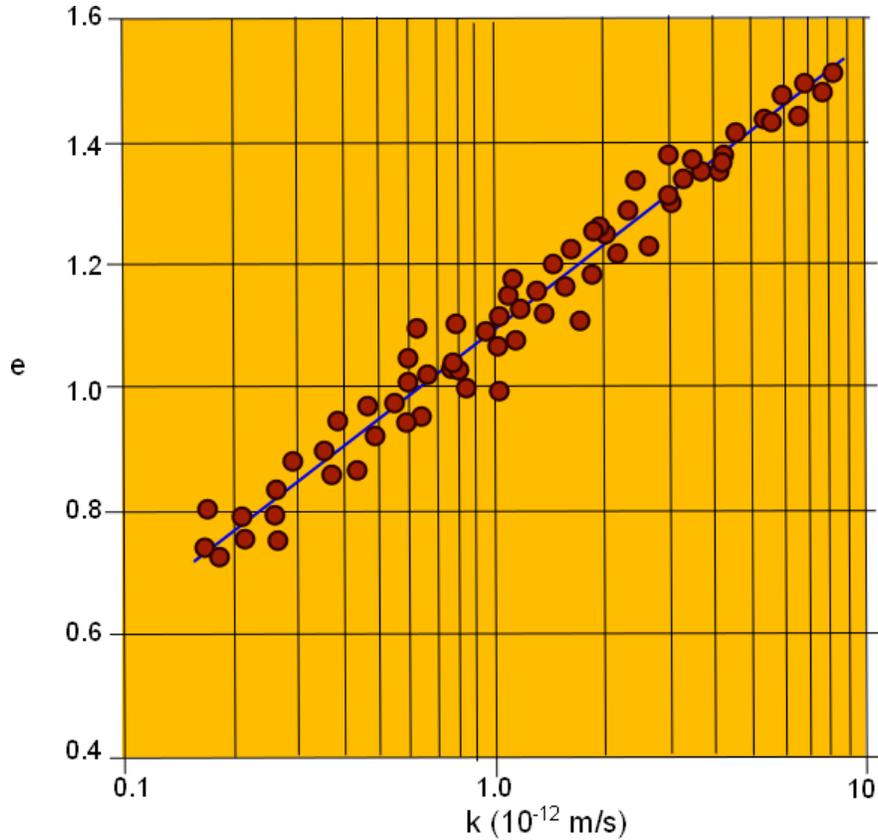


Fig. 2.3c. Hydraulic conductivity variation with void ratio obtained from Oedometer test

Lecture 2

State Variables and Material constants

Some of the state variables and material constants; their importance in the unsaturated soil mechanics are detailed here. Two important constitutive relationships, namely soil-water characteristic curve and hydraulic conductivity function are discussed in the next two chapters. The state variables viz. total head, effective stress, net normal stress, shear stress, and matric suction are commonly used in defining several geotechnical phenomena. These variables are discussed in-detail when the geotechnical aspects were discussed in the later chapters. Some of the important state variables such as density of air, density of water, and relative humidity; few important material constants such as viscosity of air and water are discussed in three chapters.

Multi-phase system

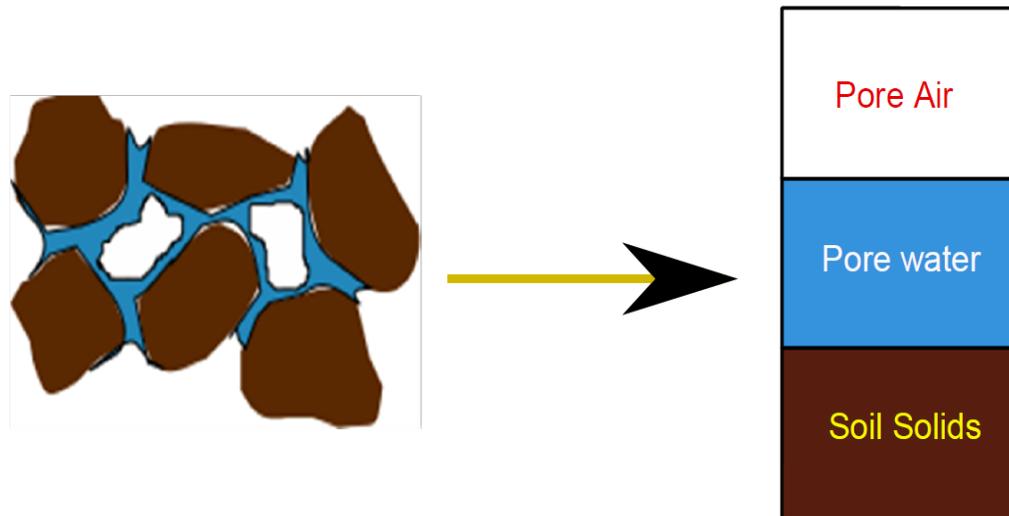


Fig. 2.4. Soil as a multi-phase system

Unsaturated soils are analyzed as a multi-phase system as depicted in Fig. 2.4. The liquid phase in the soil pore space is generally circumscribed by the pore space not occupied by air. The liquid phase could be water, oil, non-aqueous phase liquids, or combination of two or more solutions. Similarly, the air phase is circumscribed by the pore space not occupied by liquid phase. However, water can exist in air phase as vapor and air can exist in water in dissolved condition under the given atmospheric conditions. The solid phase consists of the soil solid grains and may range from fine-grained materials such as clays to coarse-grained materials such as gravels. It is important to consider the properties of the different phases existing in the soil matrix. The important state variables that are responsible for air flow and vapor flow through unsaturated soils are presented herein.

Density of Air

The pore-air plays an important role in the behavior of unsaturated soils. For example, the density of air acts as a driving force for the vapor flow in the soils. The understanding of vapor flow is important in nuclear waste repositories and other geotechnical applications involving partially saturated soils. The density of air also plays an important role in the equilibrium between pore-air and free water or pore-water and the atmospheric air. The density of air is the mass of air, M , per unit volume of the air, V_a . Air density depends on other state variables, namely, temperature and pressure of the atmosphere. Density also varies significantly with the composition of air. The natural air is comprised of dissimilar gaseous mixture, namely, oxygen, nitrogen, and other trace gases. The composition of these mixtures in the air, by volume, is: oxygen = 20.95%, nitrogen = 78.09%, and trace gases = 0.96% as illustrated in Fig.2.5.

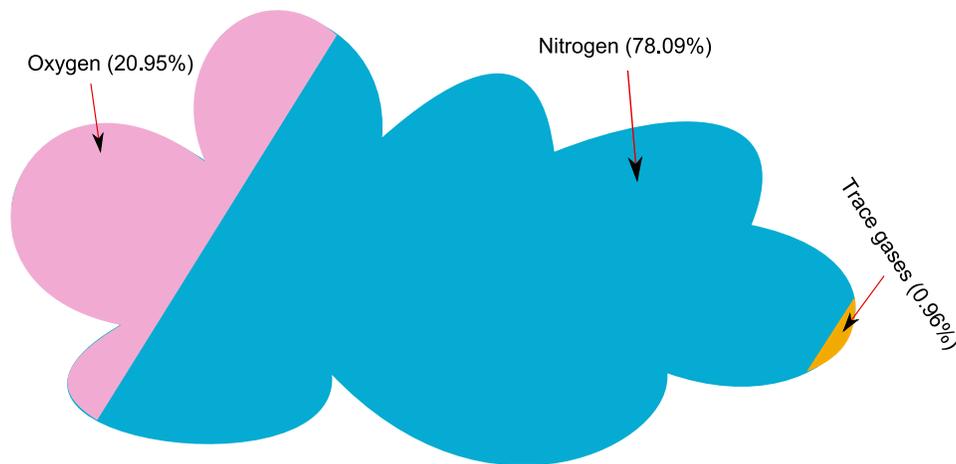


Fig. 2.5. Illustration showing the composition of air

The molecular mass of different components of air is presented in Table 1.

Table 2.1. Volume and molecular mass of air

Components in Dry Air	Volume Ratio compared to Dry Air	Molecular Mass - M (kg/kmol)	Molecular Mass in Air (kg/kmol)
Oxygen	0.2095	32.00	6.704
Nitrogen	0.7809	28.02	21.88
Carbon Dioxide	0.0003	44.01	0.013
Hydrogen	0.0000005	2.02	0
Argon	0.00933	39.94	0.373
Neon	0.000018	20.18	0
Helium	0.000005	4.00	0
Krypton	0.000001	83.8	0
Xenon	0.09×10^{-6}	131.29	0
Total Molecular Mass of Air			28.97

The density of air can be computed using the ideal gas law at a given pressure and temperature as

$$\rho_a = \frac{u_a M_a}{RT} \quad (2.1)$$

where u_a is the air pressure, M_a the molecular mass, R the ideal gas constant, and T the temperature.

The density of the air can be calculated using the air pressure (assumed to be atmospheric) as 101.325 kPa, temperature is 298 K, the universal gas constant is 8.314 J/mol-K, and assuming the molecular mass of air to be 28.97 kg/kmol

$$\rho_a = \frac{u_a M_a}{RT} = \frac{(101.325 \times 10^3 \text{ N/m}^2)(28.97 \times 10^{-3} \text{ kg/mol})}{(8.314 \text{ J/mol} \cdot \text{K})(298 \text{ K})} = 1.185 \text{ kg/m}^3$$

The air density is highly sensitive to the state variables viz. pressure and temperature. These atmospheric variables become the driving forces for the air transport from the location of higher density to the lower density. The sensitivity of these variables can be studied using the Eq. (2.2) as given by

$$\frac{\Delta \rho_a}{\rho_a} = \frac{u_a}{\Delta u_a} - \frac{\Delta T}{T} \quad (2.2)$$

An increase in the ambient pressure at a constant temperature results in the compression of air and, thus an increase in the air density. Conversely, the air density decreases with increase in the temperature while pressure is maintained constant.

Table 2.2. Dependency of air density on the temperature

Temperature, <i>T</i> in °C	Temperature, <i>T</i> in K	Density of air, ρ_a kg/m ³	Relative change in air density (%)
35	308.15	1.146	4.87
30	303.15	1.164	3.30
25	298.15	1.184	1.68
20	293.15	1.204	0.00
15	288.15	1.225	-1.74
10	283.15	1.247	-3.53
5	278.15	1.269	-5.39
0	273.15	1.292	-7.32
-5	268.15	1.316	-9.32
-10	263.15	1.341	-11.39
-15	258.15	1.367	-13.55
-20	253.15	1.394	-15.80

The sensitivity of the air density to temp and pressure is presented graphically in Fig. 2.5. It can be observed that the air density is highly sensitive to the other state variables.

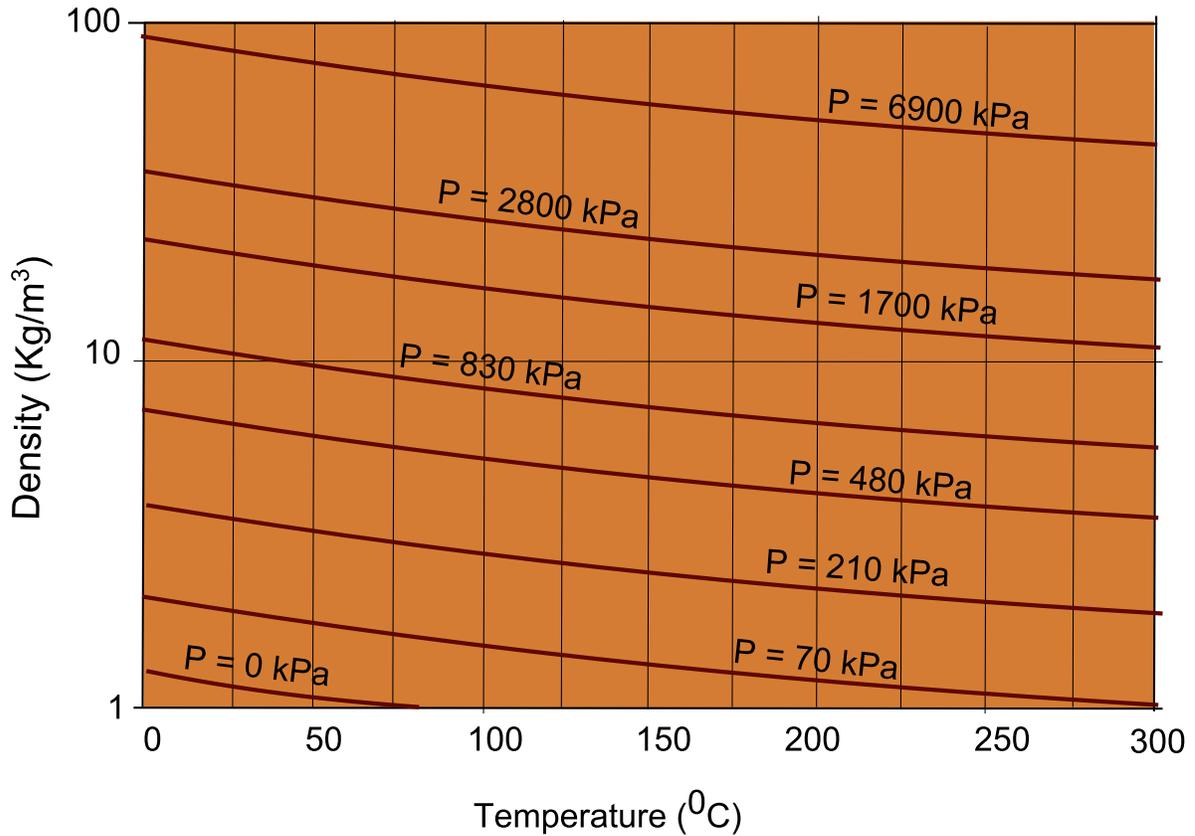


Fig. 2.5. Influence of the state variables on the density of air

The density of air decreases with the altitude as the density is directly proportional to the atmospheric air pressure as shown in Eq. (2.1). The variation in the air pressure with the altitude is shown in Fig. 2.6.

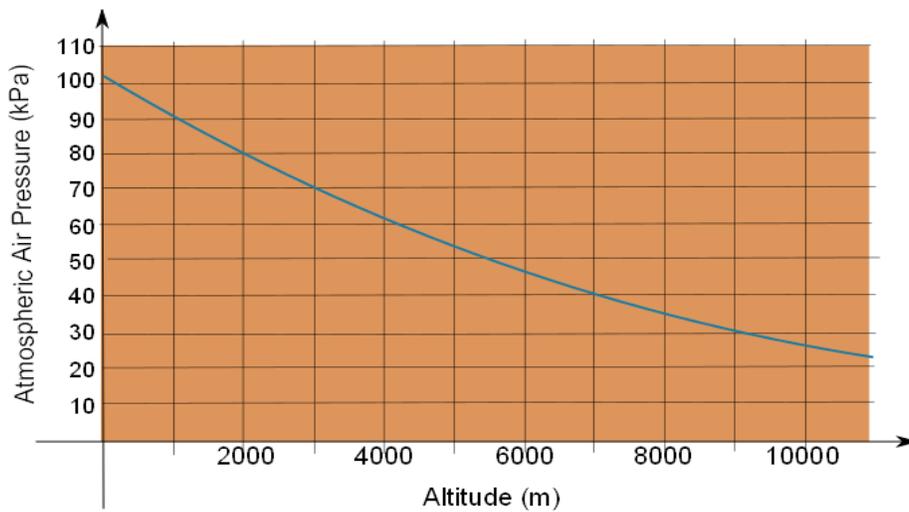


Fig. 2.6. The variation of air pressure with altitude at 15°C temperature and 0% humidity

It can be observed that the atmospheric air pressure can decrease from ~101 kPa to ~30 kPa at altitudes more than 10 KM from the mean sea level under constant temperature. The effect of change in the air pressure is explained using a simple analogy in Fig. 2.7. (http://en.wikipedia.org/wiki/Atmospheric_pressure) An empty (air-filled) plastic bottle was sealed at high altitude (~4.3 KM) and was brought to low altitudes. The bottle was observed to be deformed when it was brought to the lower altitudes (~2.7 KM). The bottle was completely crushed near the sea level (~0.3 KM) due to the increase in the atmospheric pressure compared to the air pressure inside the bottle.

The humidity also plays an influencing role on the air pressure. The effect of humidity is considered in the later part of this module which deals with the vapor.

Lecture 3

State Variables and Material Constants (Continued)

Density of Water

The density of water can directly influence the physical and mechanical behavior of unsaturated soils because many unsaturated soil parameters depend on the water density. The density of water, in-turn, is a function of temperature as shown in Fig. 2.8. Most materials (like mercury, Gold, etc.) decrease their volume consistently when the temperature decreases. Nevertheless, when water is cooled down steadily from its boiling point, the volume decreases up to 4°C and then water expands until it freezes. The freezing point temperature for water is 0°C where water undergoes phase transformation from liquid to solid. As ice starts to melt upon heating, some of the hydrogen bonds break, and water molecules can slide closer together when compared to its solid state, i.e. ice. The ice is about 10% less denser than cold water at 4°C. The volume expansion of water below 4°C temperature is called anomalous expansion of water. Such expansion causes bursting of frozen pipes and frostbites on human skin in the cold countries. Therefore, the density of water increases up to 4°C and then decreases below this temperature as shown in Fig. 2.8. The ice (< 4°C) can float on water due to these consequences. The ice displaces more water volume than its weight. Moreover, the conduction of heat is dependent on the density of the substance. A material with less dense conducts less heat than the denser material/substance. Therefore, when the lakes freeze with the formation of ice (which floats on the surface due to less density than the water) conducts less heat through it helping the deeper layers of water to maintain warm temperatures beneath the ice sheet. The aquatic life survives because of this peculiar and anomalous expansion of the water in the cold places as illustrated in Fig. 2.9. If the water had to behave similar to other substances, all water bodies including the ocean would freeze solid during the winters and only the upper layers of the ocean would thaw during the summer.

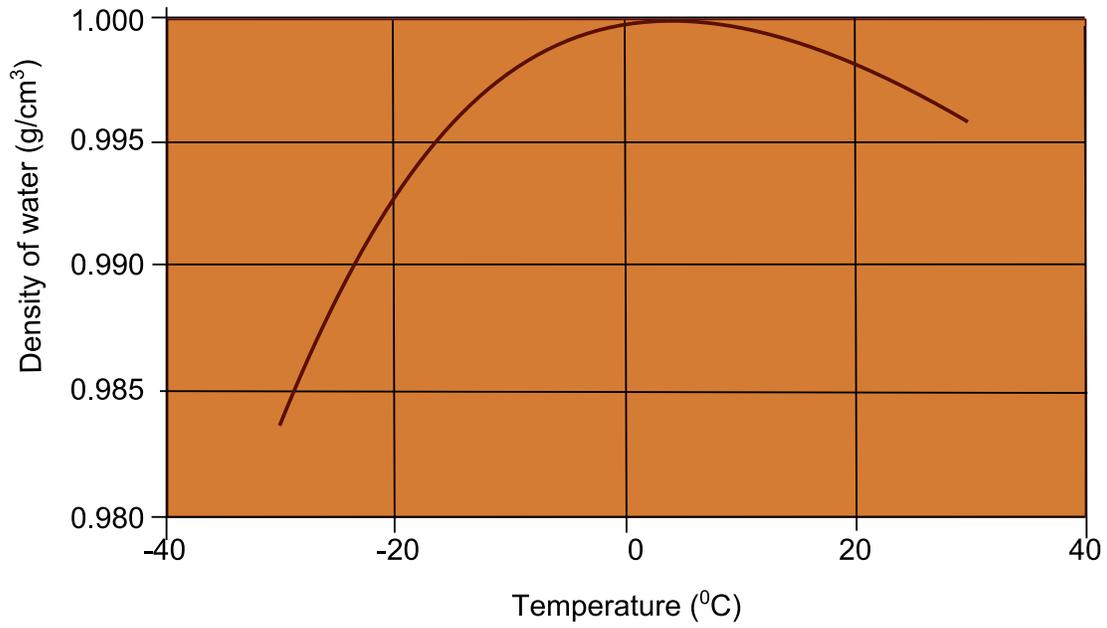


Fig. 2.8. Dependency of water density on temperature (after Lu and Lukos, 2004)

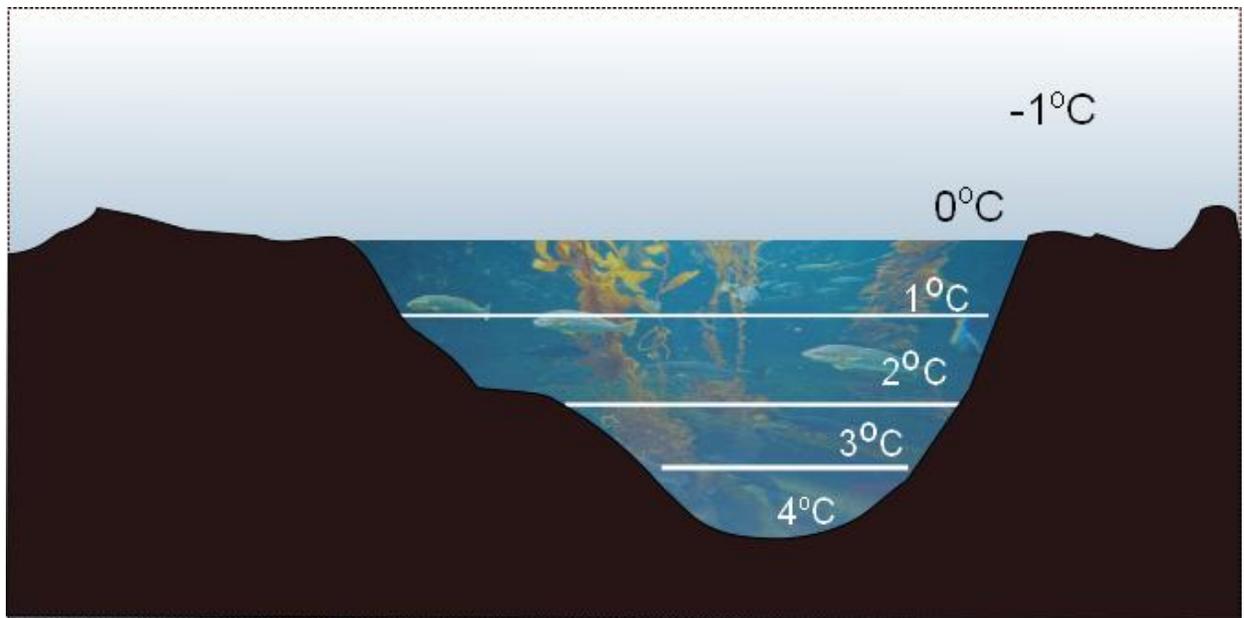


Fig. 2.9. Anomalous expansion of water and its implications on survival of aquatic creatures in the ponds

The influence of pressure on the density is not significant as the water is relatively incompressible. Other than the external state variables, the density of water also varies with the degree of saturation of the soil. As the water content approaches residual conditions, where the degree of saturation is very low, the physico-chemical factors of the clay surface play an important role on the density of water as the water presents in the form of adsorbed layer around the clay particles as shown in the Fig. 2.10.

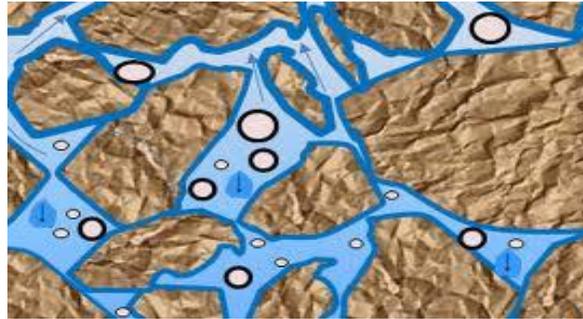


Fig. 2.10. Adsorbed and free water in the soil matrix

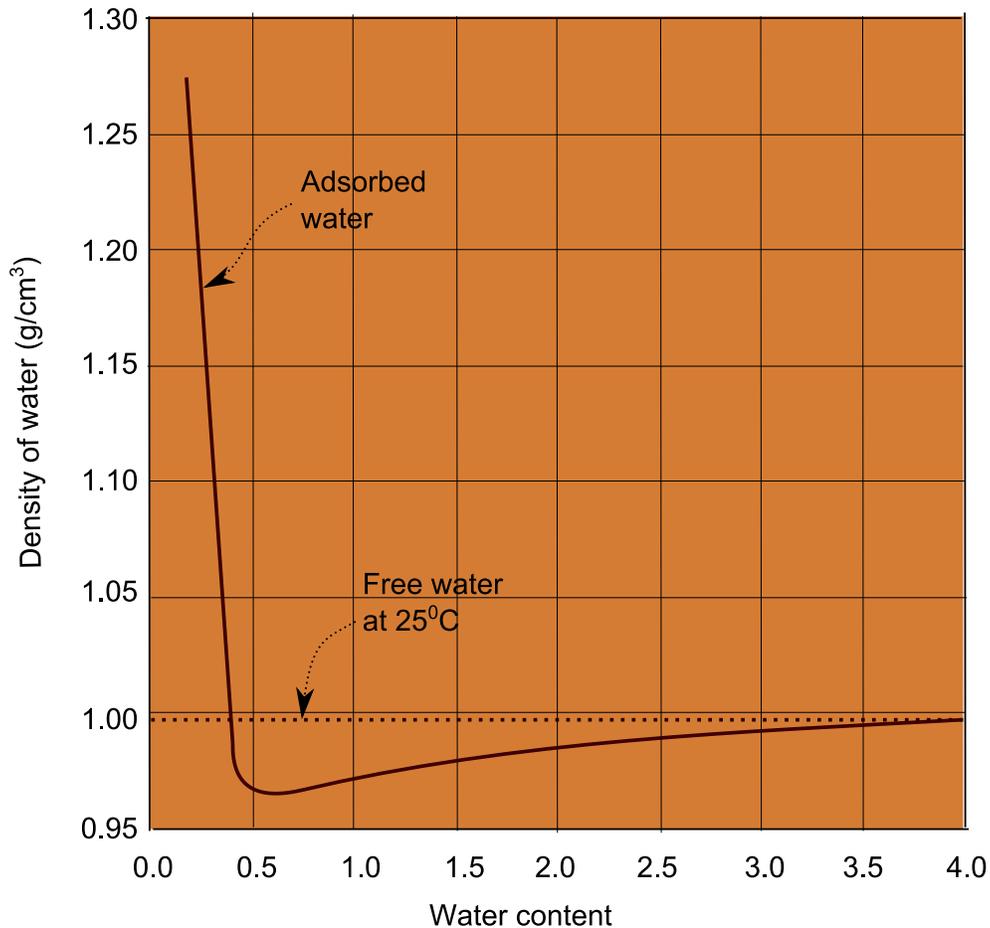


Fig. 2.11. Dependency of water density on adsorbed water content (after Lu and Lukos, 2004)

The density of water understood to be increasing with the decrease in gravimetric water content as shown in Fig. 2.11. The effect of surface characteristics of clays become more and more important for montmorillonitic clays. Such clays exhibit strong short-range interactive forces due to the presence of high specific surface area ($\sim 800 \text{ m}^2/\text{g}$) and high surface charge density. It was observed that the density of pore water in sodium montmorillonite increases from 1.0 g/cm^3 at full saturation (free water) to $\sim 1.27 \text{ g/cm}^3$ at residual condition (highly adsorbed state).

Viscosity of Air and Water

Another material constant of interest in unsaturated soil mechanics is the “Viscosity”. It is an important material constant for analyzing the flow related problems as the viscosities of soil pore air and pore water have a direct bearing on the hydraulic conductivity or, in general, the flow characteristics. Further, the rheology and the soil compressibility also directly depend on the soil pore water characteristics.

Viscosity is defined as the ability of fluid to deform under shear stresses. The resistance of the fluid is due to the friction between inter-particles. Two definitions of viscosity are used in fluid mechanics viz. dynamic viscosity (μ) and kinematic viscosity (ν) based on the conceptual applications. The dynamic or absolute viscosity ($\text{N}\cdot\text{s}/\text{m}^2$ or centiPoise) expresses resistance to shearing flows of the fluid. On the other hand, kinematic viscosity is expressed as the ratio of dynamic viscosity to the density of the fluid, which is used in analyzing Reynolds number. The dynamic viscosity of pure water (1 cP) at standard temperature is approximately 100 times higher than the pure air (0.02 cP). It is interesting to note that the dynamic viscosity of air increases with temperature contrary to the water where the viscosity decreases as shown in the Fig. 2.12. Therefore, the notion of using warm water for washing clothes can be understood. As the viscosity of the water decreases with temperature, the water will have the more able to penetrate into the smaller pores of the cloth and remove the dirt. The use of detergents would enhance the dirt removal process. The flow characteristics of water through the soils will also be improved with the low viscosity as the conductivity of the water increases. Therefore, the viscosity has a direct bearing on the flow behavior.

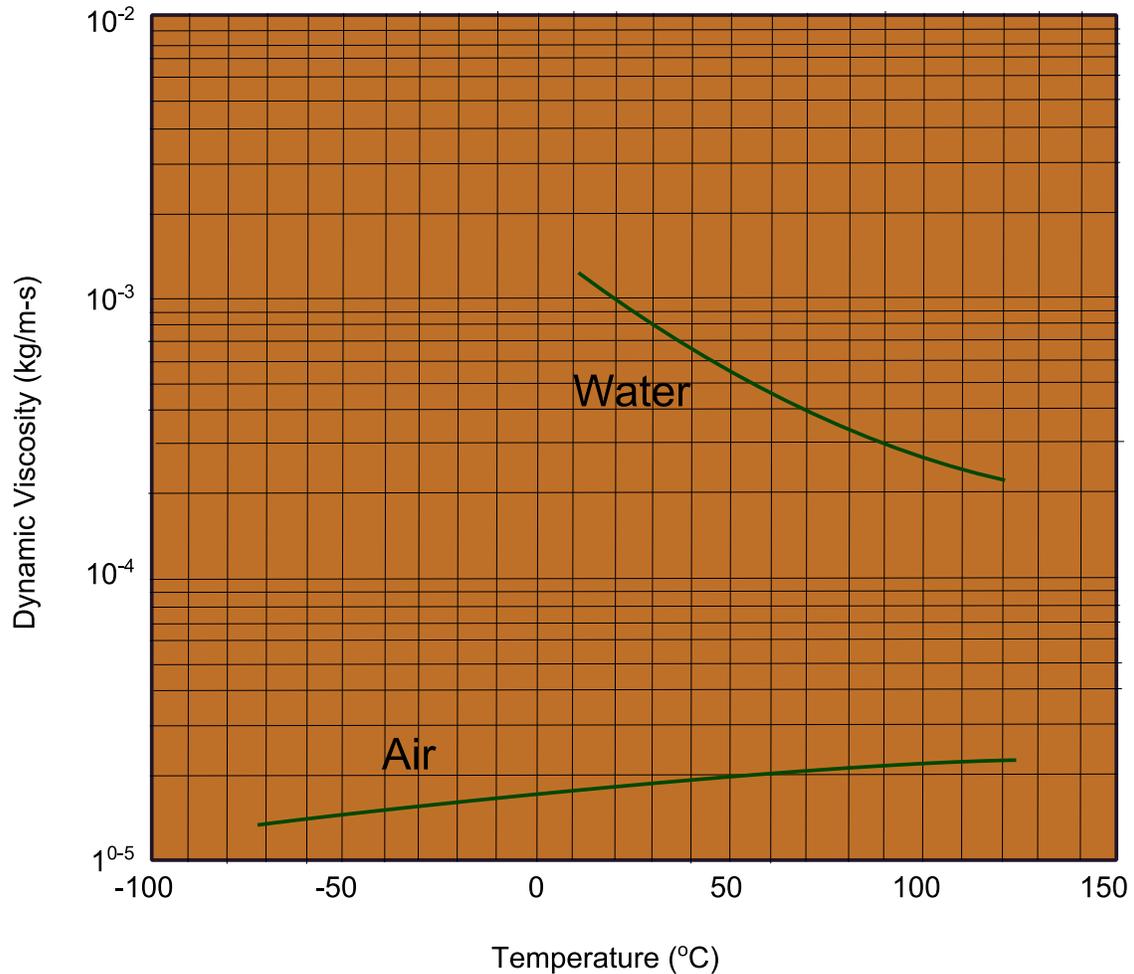


Fig. 2.12. Dependency of dynamic viscosity of water and air on temperature (after, Lu and Likos, 2004)

It was, so far, elucidated the properties of different phases in soil matrix when each phase is pure. However, each of these phases is by no means pure. The air phase may consist of water vapor and the presence of vapor depends on several atmospheric state variables. On the other hand, the air and solids may present in the water phase in the form of dissolved air and dissolved salts. Therefore, it is important to study the significance of relative presence of air, water, and solids in the other phases. Some of the important state variables are presented in the following section to explain the equilibrium between water and air phase for the co-existence.

Lecture 4

Thermodynamic equilibrium between free water and air

A qualitative understanding of an important state variable, i.e. the relative humidity, which defines equilibrium between water and air phases, requires the following assumptions to be valid (Lu and Likos, 2004).

- (i) *the composition of air, excluding the water vapor component, at a given spatial location remains essentially same over time as presented in Fig. 2.5 for the dry air.* However, in humid conditions, the atmospheric air contains a water vapor component which may significantly vary from space and time under the dynamic variables of the atmosphere. In unsaturated soils, the water vapor component is understood to be the most important component of the air which changes the overall “chemical” composition of the soil pore air. The prevailing amount of water vapor present in equilibrium with the soil is often used in the measurement of important state variable.
- (ii) *all the gas components follow the ideal gas behavior.* Therefore, the partial pressure of each air component can be obtained from the molar fraction of each component as shown below

$$\frac{u_i}{u_a} = \frac{n_i}{\sum_i n_i} \quad (2.3)$$

where the subscripts “*i*” denotes the component of air, n denotes the molar fraction, and the u_a is the atmospheric air pressure. The Eq. (2.3) can be written in terms of volume fraction as ideal gas has the constant volume, which is equal to 22.4 L/mol. Therefore, the Eq. (2.3) can be written as

$$\frac{u_i}{u_a} = \frac{v_i}{\sum_i v_i} \quad (2.4)$$

where v denotes the volume fraction of the air. The partial pressure of oxygen can, then, be calculated using the following equation if the volume fraction of oxygen in the atmospheric air and the atmospheric pressure of air are known:

$$u_{O_2} = u_a \frac{v_{O_2}}{\sum_i v_i}$$

The partial pressures of different components of air are computed and presented in Table 2.3 considering the atmospheric pressure to be 101.345 kPa.

Table 2.3. Partial pressures of the components of dry air

Components of Dry Air	Volume Ratio compared to Dry Air (from Tabel 2.2)	Partial pressure (kPa)
Oxygen	0.2095	21.23
Nitrogen	0.7809	79.14
Carbon Dioxide	0.0003	0.030
Hydrogen	0.0000005	5.07×10^{-5}
Argon	0.00933	0.946
Neon	0.000018	0.002
Helium	0.000005	5.07×10^{-4}
Krypton	0.000001	1.01×10^{-4}
Xenon	0.09×10^{-6}	9.12×10^{-6}

(iii) *all the components of air, including the water vapor, reach thermodynamic equilibrium.* It enforces that the chemical potentials among all components of all phases in a given system is same.

Let us assume equilibrium between the free water and air in a closed container at a standard temperature 20°C as shown in the first picture in Fig. 2.13.

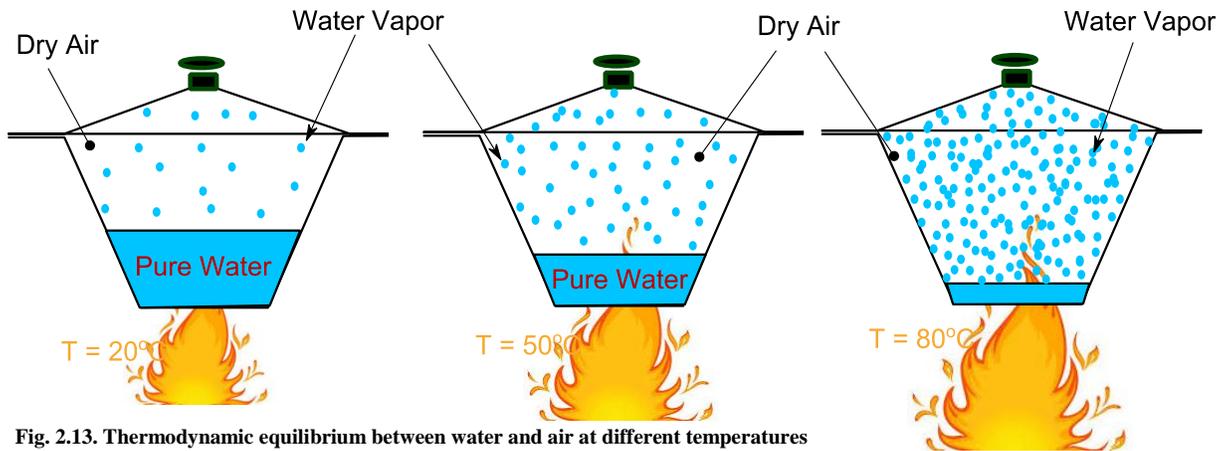


Fig. 2.13. Thermodynamic equilibrium between water and air at different temperatures

As the temperature increases, the presence of vapor in the air increases. Therefore, the air now contains water vapor as one of the components apart from other components of dry air discussed before. The water vapor and its exerted pressure increases exponentially with temperature. The vapor pressure reaches the atmospheric pressure at the boiling point temperature, 100°C. The pressure exerted by the water vapor in a unit volume of closed system at a given temperature can be calculated using the following Tetens' equation (1930: *Über einige meteorologische Begriffe*. *Zeitschrift für Geophysik*, Vol. 6:297):

$$u_{v,sat} = 0.611 \times \exp\left(17.27 \frac{t}{t+238.3}\right) \quad (2.5)$$

where t is expressed in degree Celsius. The following equation has also been used commonly to understand the dependency of saturated vapor pressure on temperature:

$$u_{v,sat} = 0.1333 \times \exp\left(20.386 - \frac{5132}{t+273}\right) \quad (2.6)$$

Another noted equation for calculating $u_{v,sat}$ is based on the Antoine's equation (1888: *Tensions des vapeurs; nouvelle relation entre les tensions et les températures*" [Vapor Pressure: a new relationship between pressure and temperature], *Comptes Rendus des Séances de l'Académie des Sciences* (in French) **107**: 681–684, 778–780, 836–837):

$$\log_{10} u_{v,sat} = K_1 - \frac{K_2}{K_3 + t} \quad (2.7)$$

Table 2.4. Coefficients of Antoine's equation (Eq. 2.7)

t_{min} (°C)	t_{max} (°C)	K_1	K_2	K_3
1	100	8.07131	1730.63	233.426
99	374	8.14019	1810.94	244.485

Table 2.5. Saturated vapor pressures and volume fractions at different temperatures

Temperature	$u_{v,sat}$ (Eq. 2.5)	u_{dry}	v_{vapor} (Eq. 2.4)	v_{dry} (= 1- v_{vapor})
20°C	2.33	99.01	0.023	0.977
50°C	12.32	89.02	0.122	0.878
80°C	47.5	53.84	0.469	0.531

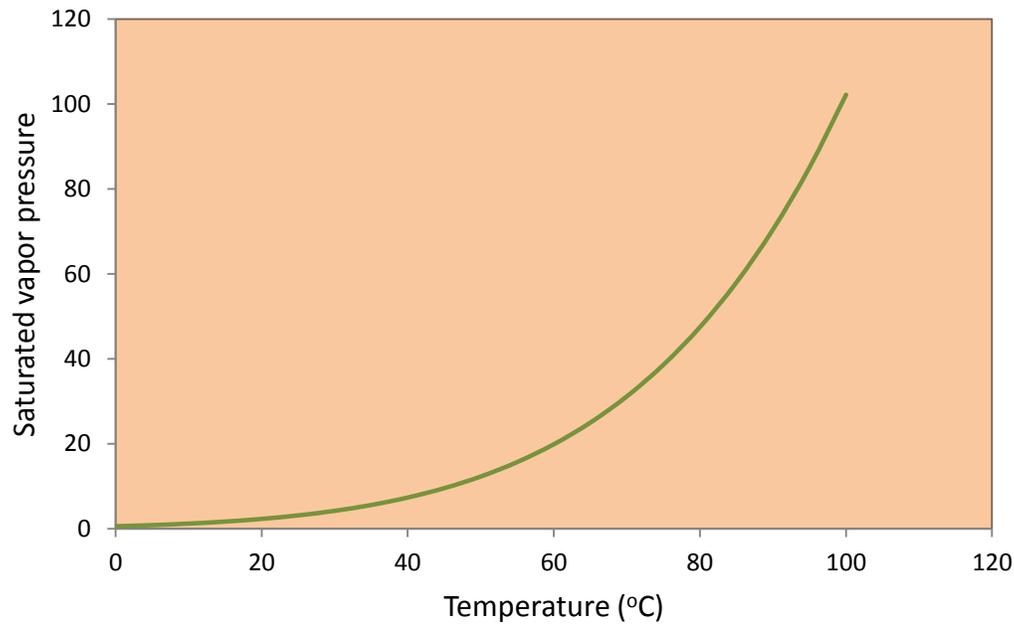


Fig. 2.14. Influence of temperature on saturated vapor pressure

Absolute and Relative Humidity

The absolute humidity expresses the amount of water vapor present in the air. Absolute humidity is a ratio of mass of water vapor to the unit volume of total air. The composition of a unit volume of dry air, at standard pressure (101.3 kPa) and temperature (298.2 K), is: Nitrogen = 78.09%, Oxygen = 20.95%, and Trace gases = 0.96%. Under such standard temperature and pressure conditions, the partial pressure of the water vapor at equilibrium between air and pure water is equal to 3.17 kPa and partial pressure of remaining components is 98.13 kPa. At this equilibrium condition, the mass exchange between water and air is not possible. If we express this equilibrium in terms of the total mass of water vapor per unit volume, the ratio is equal to 22.99 g/m³, which is called the absolute humidity. Therefore, the maximum amount of water that can be vaporized into air is 22.99 g/m³, under standard pressure and temperature conditions. Additional vaporization is only possible when either the temperature or the total air pressure is changed. The vapor pressure under the stated equilibrium condition is called saturated vapor pressure.

Relative humidity (RH) illustrates the state of thermodynamic equilibrium between air and water. RH is the ratio of partial pressure of water vapor to the saturated vapor pressure under the same equilibrium conditions. It can also be defined as the ratio of absolute humidity in equilibrium with solution to the absolute humidity in equilibrium with pure water at the same temperature. It is normally expressed in percentage. RH is an important state variable in understanding the behavior of partly saturated soils under the varying climatic conditions.

The solubility of air in water is also an important aspect which is important to understand the limitations of some of the measuring systems used for measuring the unsaturated state variables viz. multi-step outflow technique, which is beyond the scope of the present lectures.