Module 7 : Design of Machine Foundations

Lecture 31 : Basics of soil dynamics [Section 31.1: Introduction]

Objectives

In this section you will learn the following

- Dynamic loads
- Degrees of freedom

Introduction

Dynamic loads

The foundations are subjected to static or dynamic load or combination of both. The static loads are imposed on the foundations slowly and gradually in such a way as to avoid any vibration of the foundation soil system.

The dynamic loads are time variable load, e.g. Earthquake, impact, blast loads etc. Displacement and stresses are time dependent. The inertia forces are part of the loading system.

But all the time varying activities are not dynamic such as filling of reservoir or construction activities etc. the dynamic loads lead to vibration of the soil and foundation system.

In this, Work done = Potential energy + Kinetic energy,

Displacement is connected to the stiffness of the structure and acceleration is connected to the inertia of the structure.

D'Alambert's principle:

A system may be set in state of dynamic equilibrium by adding to the external forces a fictitious force which is commonly known Inertia force. The resulting displacements are associated with accelerations which produce inertial forces resisting the acceleration.

Degrees of freedom

This can be defined as number of independent coordinates required to completely specify the response of vibrating system.

For a single particle, there are 3 $DOF's$: F_x, F_y, F_z (Forces in x,y,z directions).

For a rigid body, there are 6 $DOF's$: F_x, F_y, F_z (Forces in x,y,z directions), M_x, M_y, M_z (Moments in x,y,z directions).

DOF is not intrinsic property, if $DOF's$ are less than infinity then it is a discrete system and if $DOF's$ are tending to infinity then it is a continuous system.

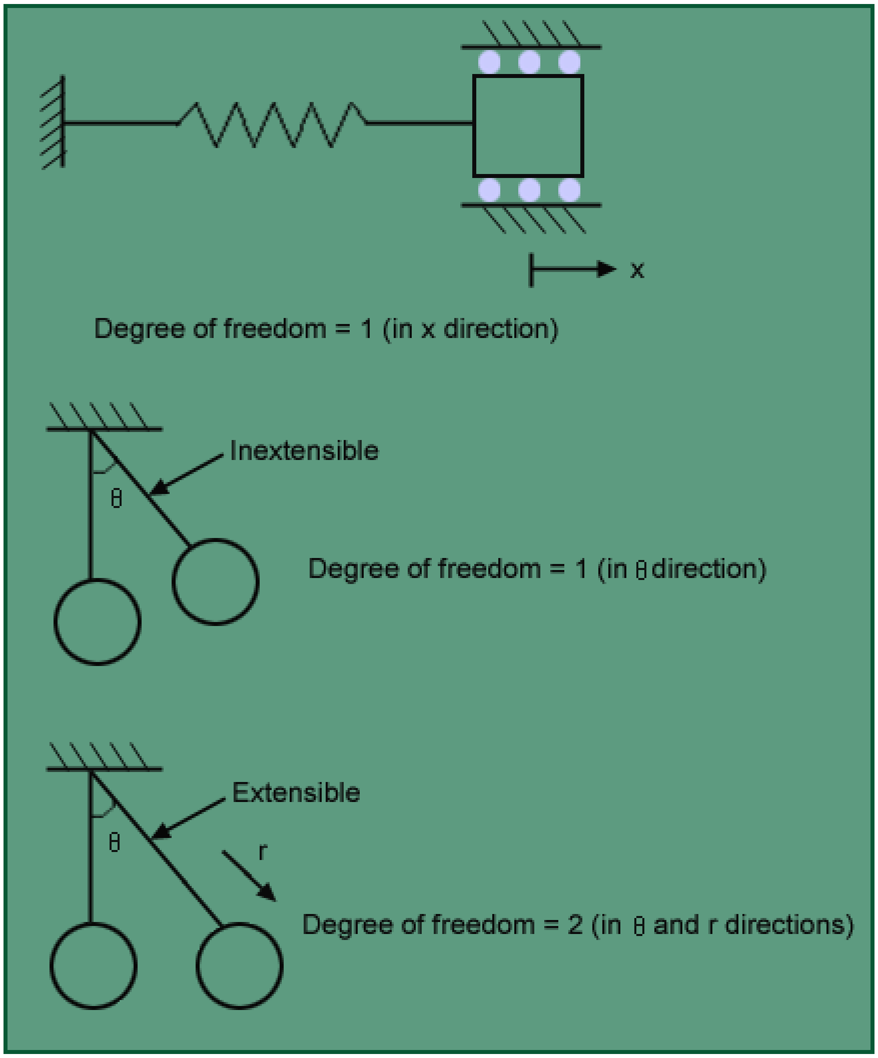

Fig. 8.1 Different degrees of freedom

Recap

In this section you have learnt the following.

- Dynamic loads
- Degrees of freedom

Objectives

In this section you will learn the following

- Undamped free vibration
- Undamped forced vibration
- Damped free vibration
- Damped forced vibration
- Undamped free vibration system

- **Undamped free vibration :**

Free vibrations means the structure is disturbed from equilibrium and then vibrates without any applied forces. The damping forces cause the dissipation of the motion. It is undamped i.e. the coefficient of damping "c"=0. This system is also called as a Single Degree Of Freedom system.(SDOF).

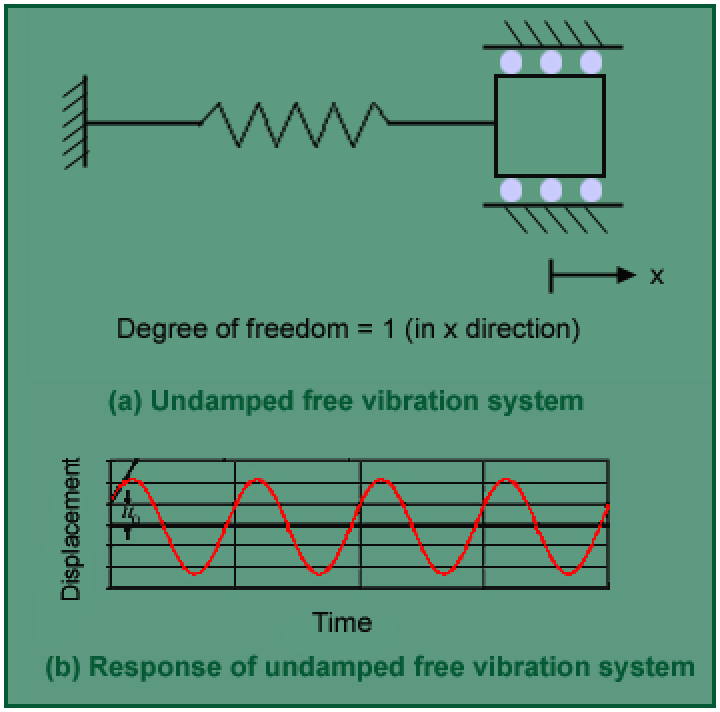

Fig. 8.2 Undamped free vibration

- **Undamped forced vibration :**

  In this external force f(t) is applied and coefficient of damping "c" = 0.

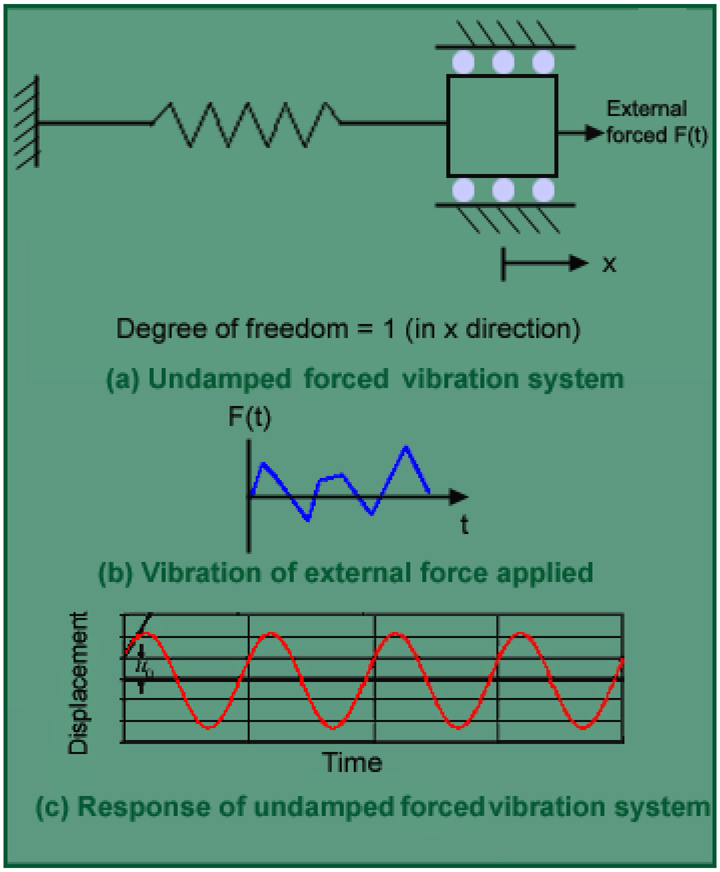

Fig. 8.3 Undamped forced vibration

- **Damped free vibration :**

  In this no external force is applied i.e. f(t) = 0 and damping is present i.e. coefficient of damping "c" $\neq$ 0.

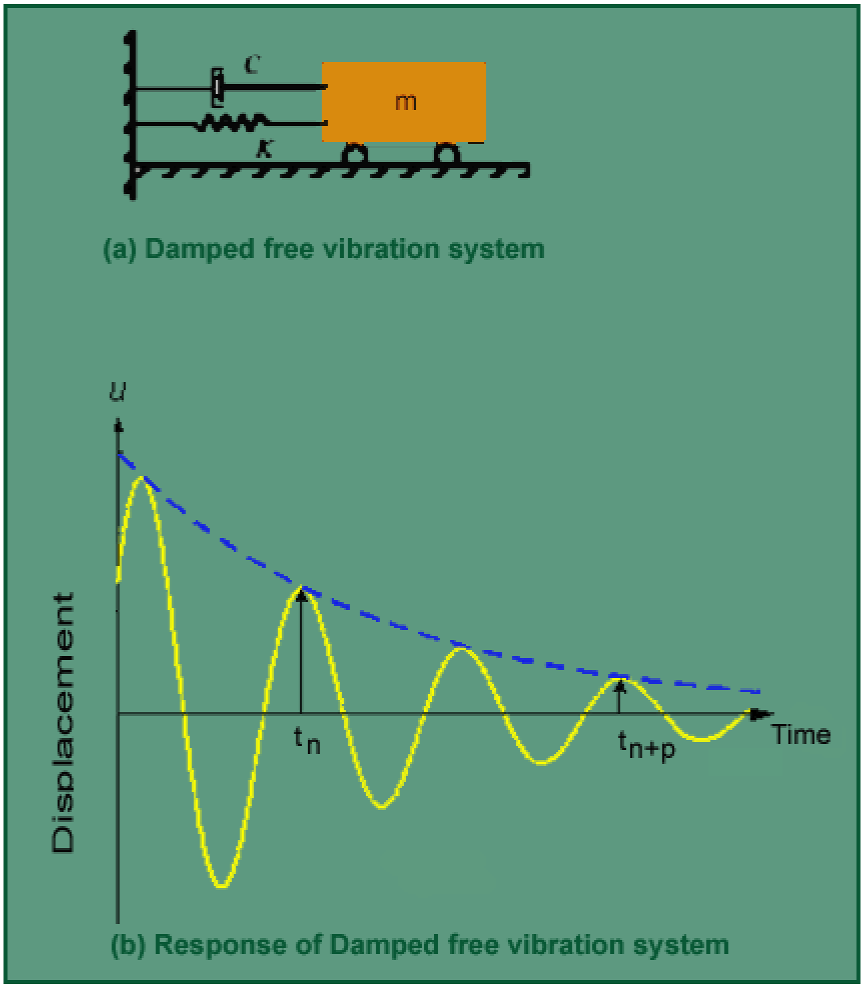

Fig. 8.4 Damped free vibration

- **Damped forced vibration :**

  In this external force is applied i.e. f(t) $\neq$ 0 and also damping is present i.e. coefficient of damping "c" $\neq$ 0.

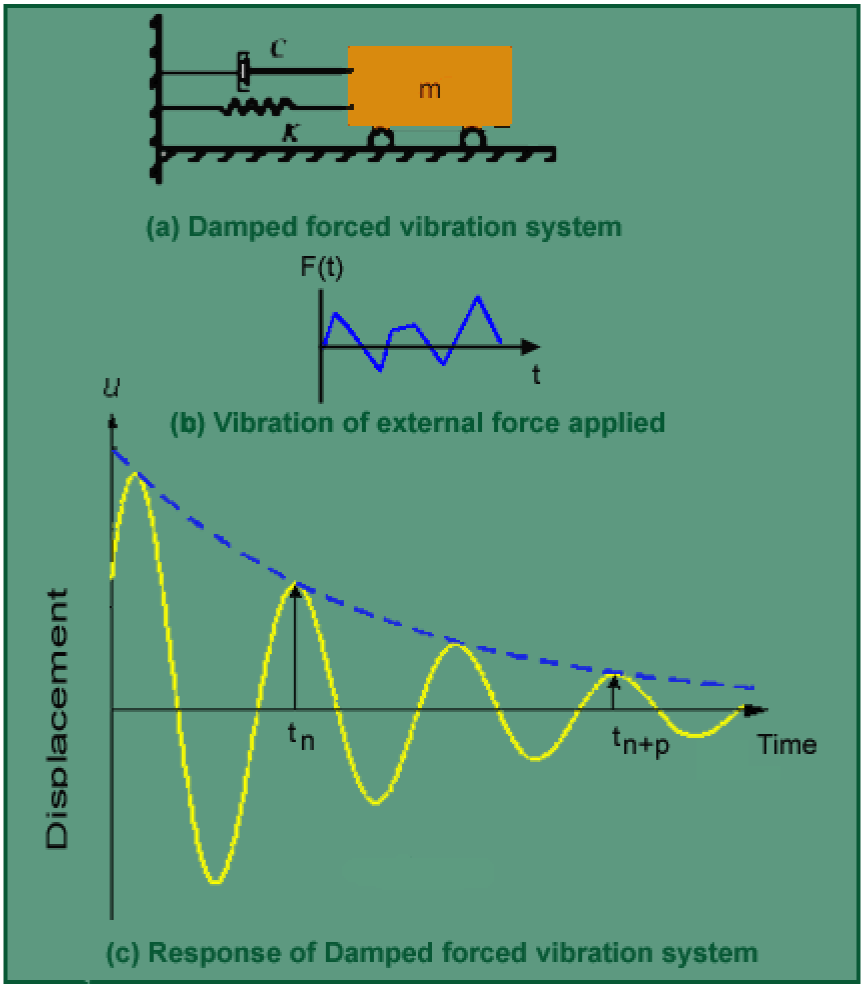

Fig. 8.5 Damped forced vibration

- **Undamped free vibration system**

The governing equation is,

$$m\ddot{x} + kx = 0 \qquad \text{---------- (1)}$$

where,
m is the mass of the system,
k is the stiffness of the system,
x is the displacement of the system and ,
$\ddot{x}$ is the acceleration.

Solving as a initial value problem, the boundary conditions are,

he initial displacement and velocity is given as,

$$x(0) = x_0 \ and \ \dot{x}(o) = \dot{x}_0$$

Where,

x_0 is the displacement at time (t) =0 $\dot{x}_0$ and is the velocity at time (t)=0.

Substituting in equation (1), x(t) = e^{st} ,we get,

$ms^2 e^{st}$ + k e^{st} = 0

If e^{st} = 0, i. e. trivial solution.
So, nontrivial solution is,
$ms^2 + k = 0$.

$$\therefore s = \pm i\sqrt{\frac{k}{m}} = \pm i\omega$$

where,

ω is the natural frequency of the system (rad/sec).

General solution is,

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t} \quad \text{---------(2)}$$

Applying boundary conditions,

$$x(0) = x_0 \text{ and } \dot{x}(o) = \dot{x}_0$$

Substituting in equation (2),

$$x(o) = x_0 = A + B$$

$$\dot{x}(o) = \dot{x}_0 = i\omega(A - B)$$

As x_0 and $\dot{x}_0$ are real values ,

$$\text{Imaginary}\,(x_0) = 0 = \text{Imaginary}\,(A + B)$$

$$\text{Real}\,(\dot{x}_0) = 0 = \text{Real}\,(A - B)$$

Where, A and B are complex conjugates.

$B = \alpha - i\beta \;\; A = \alpha + i\beta$

$\therefore x(t) = (\alpha + i\beta)(\cos \omega t + i \sin \omega t) + (\alpha - i\beta)(\cos \omega t - i \sin \omega t)$

$\therefore x(t) = (\alpha + i\beta) e^{iwt} + (\alpha - i\beta) e^{-iwt}$

$\therefore x(t) = 2\alpha \cos \omega t - 2\beta \sin \omega t$

$\therefore x(t) = x_0 \cos \omega t + \frac{\dot{x}_0}{w} \sin \omega t$

Alternatively,

x(t) = a cos ωt + b sin ωt ---------(3)

where, s = $\pm$ i ω.

It can be represented as shown in fig.

x_0 = R cos θ and,

$\frac{\dot{x}_0}{\omega}$ = R sin θ

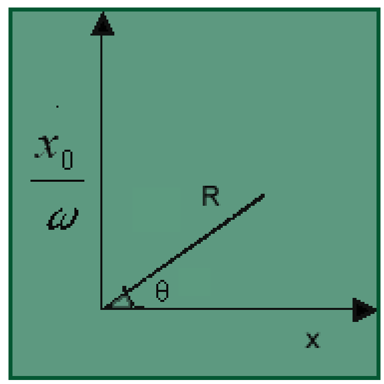

Fig. 8.6 Representation of solution

substituting in equation (3),
$x(t) = R \cos\theta \cos \omega t + R \sin\theta \sin \omega t$

$= R \cos(\omega t - \theta)$

$\tan\theta = \frac{\dot{x}_0 \omega}{x_0}$ where, $R = \sqrt{x_o^2 + \frac{\dot{x}_0^2}{\omega^2}}$ and

Fig. 8.7 Displacement (x(t)) vs time (t)

$\therefore \omega = \sqrt{\frac{k}{m}}$

Which is independent of the initial conditions, it is called as Natural frequency of the system.

$T = \frac{2\pi}{\omega}$ is the natural period of the system.

Recap

In this section you have learnt the following.

- Undamped free vibration
- Undamped forced vibration
- Damped free vibration
- Damped forced vibration
- Undamped free vibration system

Objectives

In this section you will learn the following

- Damped free vibration system

$$\delta = \ln \frac{x_0}{x_1} = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

It is measure of decay of successive max amplitude of viscously damped vibrations & expressed by the eq & in which X_0 & X_1 are two peaks amplitude.

This property is useful in practice because we can find out damping ratio of any material.

$$\ln \frac{x_n}{x_{n+m}} = \frac{2\pi . m\xi}{\sqrt{1-\xi^2}}$$

Where X_n = Initial Displacement

X_{n+m} = Displacement after n+m Cycle

Recap

In this section you have learnt the following.

- Damped free vibration system

Damped free vibration system

Refer fig.

The governing equation is,

$$m\ddot{x} + c\dot{x} + kx = 0$$

where,
m is the mass of the system,
k is the stiffness of the system,
x is the displacement of the system

$\dot{x}$, is the velocity and, $\ddot{x}$ is the acceleration.

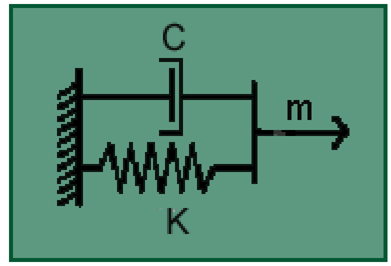

Fig. 8.8 Damped free vibration system

Solving as a initial value problem, the boundary conditions are,
The initial displacement and velocity is given as,

$$x(0) = x_0 \ and \ \dot{x}(o) = \dot{x}_0$$

Where,

x_0 is the displacement at time (t) =0 and $\dot{x}_0$ is the velocity at time (t)=0.

$$\therefore \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

Substituting in the above equation, $\omega_n = \sqrt{\frac{k}{m}}$, i.e. undamped natural frequency.

$$\therefore \ddot{x} + \frac{c}{m}\dot{x} + \omega_n^2 x = 0$$

The solution is in the form of x(t) = e^{st}

$$\therefore \left(s^2 + \frac{c}{m}s + \omega_n^2\right)e^{st} = 0$$

If e^{st} = 0, i. e. trivial solution.

So, nontrivial solution is,

$$\therefore \left(s^2 + \frac{c}{m}s + \omega_n^2\right) = 0$$

$$\therefore s_{1,2} = \frac{-c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \omega_n^2}$$

Materials	Damping ratio (ξ) values
Concrete	0.05 to 0.07
Steel	0.01 to 0.02
Aluminum	0.001 to 0.01
Soil	0.1 to 0.2

There are three possibilities,

Case 1) $s_1 = s_2$

$$\therefore \frac{c^2}{4m^2} = \omega_n^2$$

$$\therefore \frac{c}{2m} = \omega_n$$

So,

- Critical damping constant c_c = (2m ω_n),
- Damping ratio (ξ) = (c/ c_c). For different materials ξ values are given in table 8.1.
- Damped natural frequency (ω_{nd}) = $\omega_n\sqrt{1-\xi^2}$, where ω_n is the undamped natural frequency of the system.

Table 8.1 Different values of Damping ratio (ξ)

Note: For soil ξ values are high i.e. soil is a good damper.

Case 2) $s_1 \neq s_2$ and root are real (i.e. ξ >1)

$$s_{1,2} = \frac{-\xi c_c}{2m} \pm \frac{1}{2}\sqrt{\frac{\xi c_c^{\,2}}{m^2} - 4\omega_n^2} \quad s_{1,2} = \frac{-c}{2m} \pm \frac{1}{2}\sqrt{\frac{c^2}{m^2} - 4\omega_n^2}$$

$$s_{1,2} = -\xi\,\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$s_{1,2} = -\xi\,\omega_n \pm \omega_d{}'$$

So, final solution is,

$$\therefore x(t) = e^{-\xi\omega_n t}\left(x_0 \cosh \omega_d{}' t + \frac{\dot{x}_0 + \xi\omega_n x_0}{\omega_d{}'} \sinh \omega_d{}' t\right.$$

Case 3) $s_1 \neq s_2$ and root are complex (i.e. ξ <1)

$$\therefore \frac{c^2}{4m^2} - \omega_n^2 = 0$$

$$s_{1,2} = \frac{-c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \omega_n^2}$$

$$s_{1,2} = -\xi\,\omega_n \pm \omega_n\sqrt{1 - \xi^2}$$

$s_{1,2} = -\xi\,\omega_n \pm \omega_d{}'$ The different responses are given in fig.

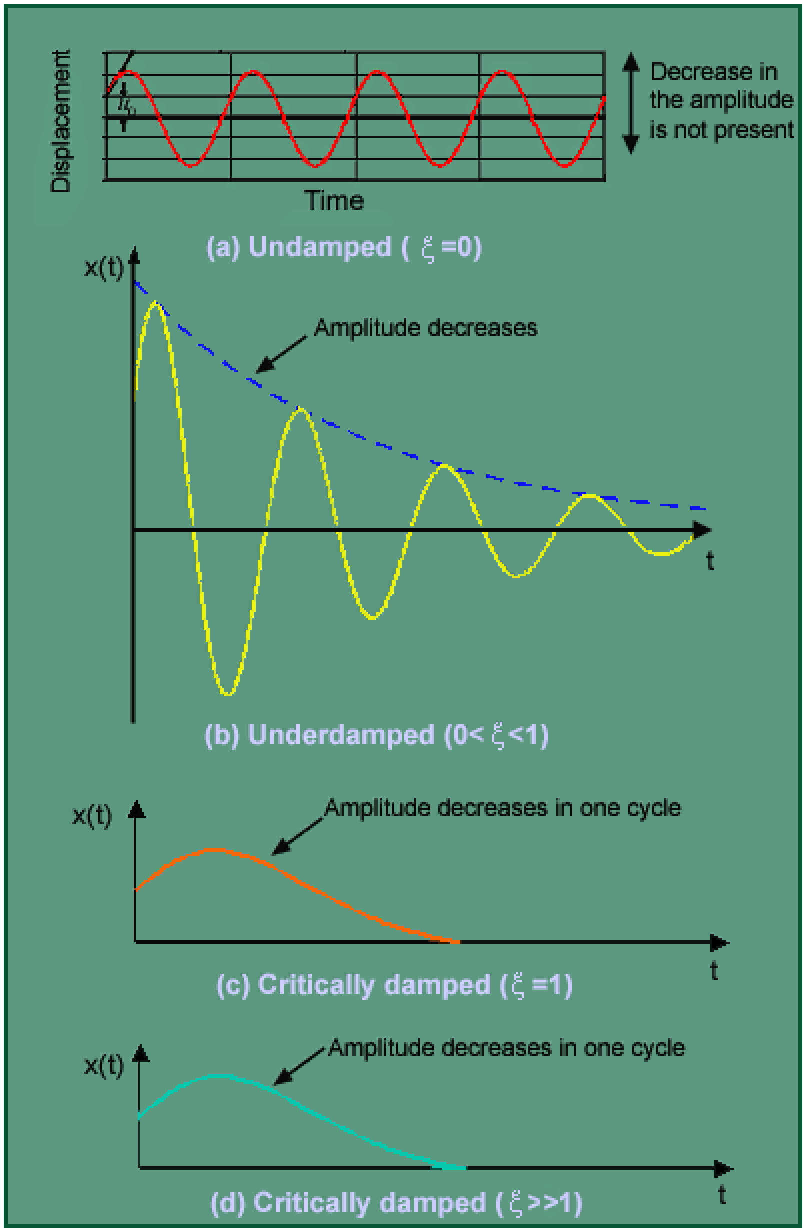

Fig. 8.9 Different responses

i Undamped: where, $\xi = 0$, practically not present.

ii Under damped: It is a oscillatory damped motion and ξ is between o to 1. All practical vibration examples are mostly under damped.

iii Over damped: Here, $\xi > 1$. It is decay without any oscillation. E.g. Door damper, music system cassette box, galvanometer and ammeter (needle should not move from one side to another side, it should move in only one direction).

iv Critically damped: Here, $\xi = 1$. It is the smallest value of the damping necessary to prevent the oscillation when system is under going damped free vibration. It is very difficult to achieve it practically.

- **Damped Force Vibrations Under Harmonic Excitations:**

$$m\ddot{x} + C\dot{x} + Kx = P\cos\lambda t \qquad \frac{c}{m} = \text{natural frequency}$$

$$\ddot{x} + \frac{C}{m}\dot{x} + w_n^2 x = \frac{P}{m}\cos\lambda t \qquad c_c = 2.m.w_n$$

$$\ddot{x} + 2\xi w_n \dot{x} + w_n^2 x = \frac{P}{m}\cos\lambda t \qquad \xi = \frac{c}{c_c} = \frac{c}{2.m.w_n}$$

Let $\xi < 1.0$ Under Damped Conditions, Practical Case

Generalized Solution

X(t) = CF + PI

X(t) = $e^{-\xi.w_n.t}\left[A\cos w_{nd}.t + B\sin w_{nd}.t\right]$ = CF

Here w_{nd} = damped natural frequency

$$x(t) = \frac{(P/m).Cos(\lambda.t-\theta)}{\sqrt{(w_n^2-\lambda^2)^2+(2\xi w_n \lambda)^2}}$$

$$\theta = \tan^{-1}\left[\frac{2\xi w_n \lambda}{w_n^2-\lambda^2}\right]$$

$$\lim_{t\to\infty} = x(t)_{PI}$$

$$x(t)_{PI} = \frac{\left(\frac{P}{mw_n^2}\right).cos(\lambda.t-\theta)}{\sqrt{1-\frac{\lambda^2}{w^2}+\left(2\xi\frac{\lambda}{w_n}\right)^2}} = \text{Dynamic Displacement}$$

$$\frac{P}{mw_n^2} = \frac{P.m}{mK} = \frac{P}{K} = X_{st} = \text{Static Displacement}$$

Frequency Ratio

$$r = \frac{\lambda}{w_n}$$

$$x_s(t) = \frac{x_{st} \cdot \cos(\lambda . t - \theta)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$\text{Amplitude of} \quad \frac{x_s(t)}{X_{st}} = \text{D.M.F.}$$

$$\text{D.M.F.} = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

Because of Harmonic Excitation in Static, It gives displacement in that direction.

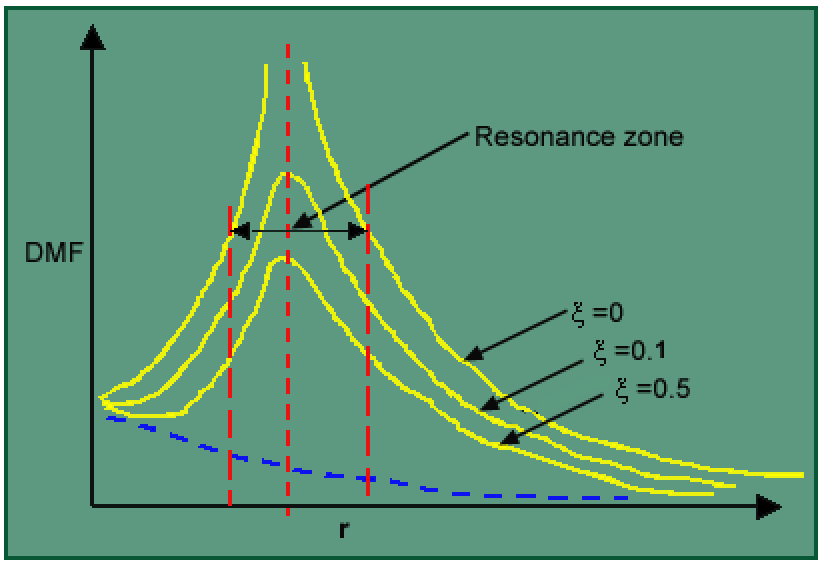

Fig 8.10 Graph between DMF Vs Frequency Ratio

Rotating Mass

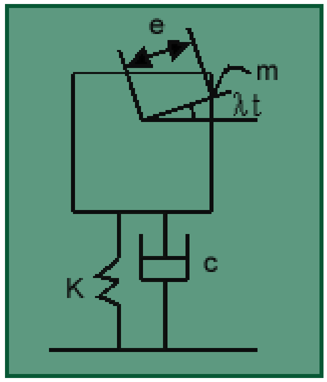

fig 8.11 model representing rotating mass

$$m\ddot{x} + C\dot{x} + K\dot{x} = me\lambda^2.(\sin\lambda t) \qquad me\lambda^2 = Q.\sin\lambda t$$

$$x(t) = \frac{\frac{Q}{K}\sin(\lambda t - \theta)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$\theta = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right)$$

$$x_s(t) = X.\sin(\lambda.t - \theta)$$

$$X = \frac{Q/K}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$X = \frac{\left(\frac{me\lambda^2}{Mw^2}\right)}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$\frac{X_m}{me} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \qquad r = \frac{\lambda}{w_n}$$

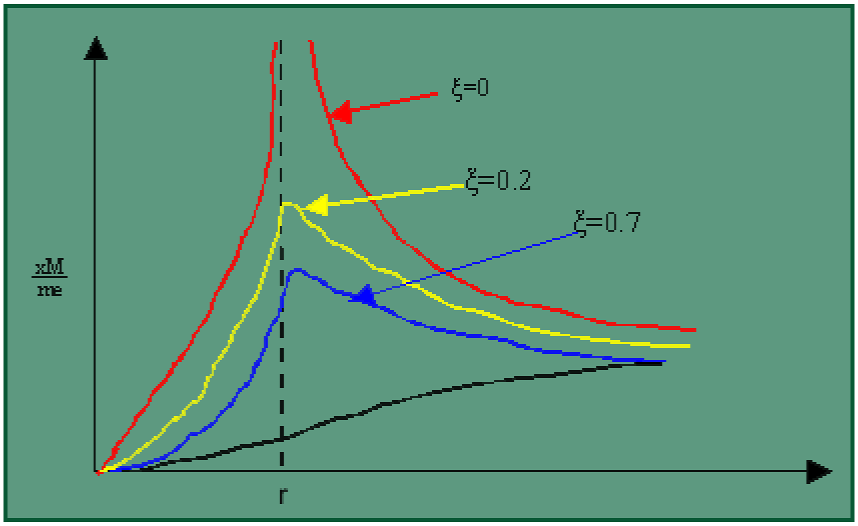

Fig 8.12 Graph between DMF Vs Frequency Ratio

Logarithmic Decrement

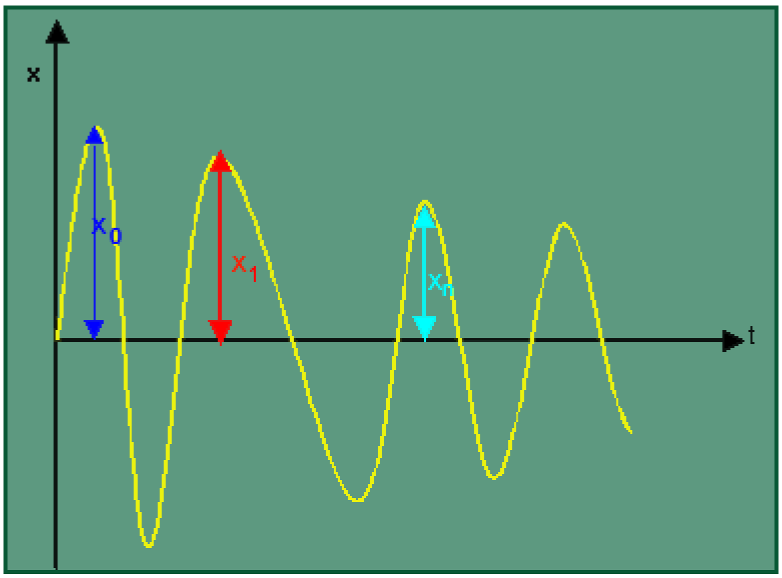

Fig. 8.13 Graph showing the decrement of amplitude with time