Module 3 : Method of Analyses

Lecture 14 : Methods of Characteristics [Section 14.1 : Introduction]

Objectives

In this section you will learn the following

- Introduction

METHOD OF CHARACTERISTICS

Fig. 3.5 Infinitesimal stress analysis

The equilibrium equations of an infinitesimal element are given by:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = X$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = Y$$

Considering Mohr Coulomb criteria

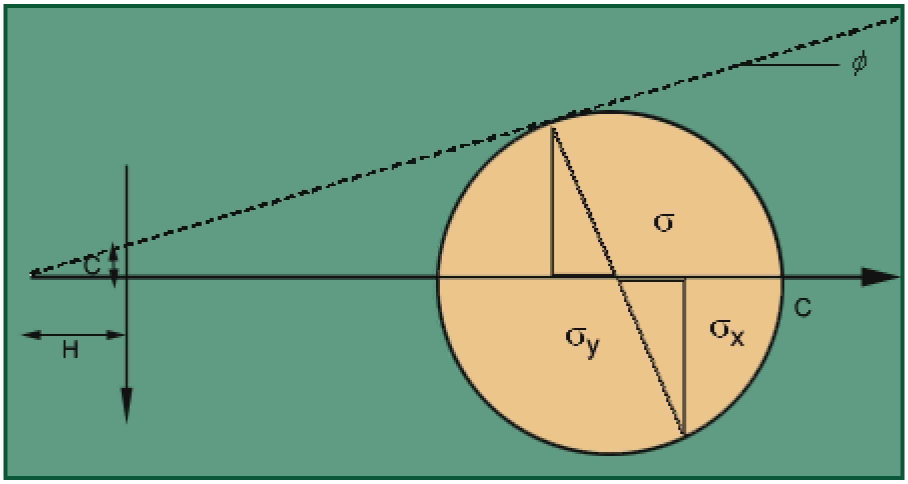


Fig. 3.6 Mohr circle of stresses

H=c cot ϕ

$$\sigma = H + \frac{\sigma_x + \sigma_y}{2}$$

$$r = \sigma \sin\ \phi$$

$$\tau_{xy} = r \sin\ 2\theta = \sigma \sin\ \phi \sin\ 2\theta \qquad \text{---------(1)}$$

$$\sigma_x = (\sigma - H) + \sigma \sin\ \phi \cos 2\theta$$

$$\sigma_y = (\sigma - H) + \sigma \sin\ \phi \cos 2\theta$$

$$\sigma_{x,y} = \sigma(1 \pm \sin\ \phi \cos 2\theta) - H$$

for c=0,H=0

$$\sigma_{x,y} = \sigma(1 \pm \sin\ \phi \cos 2\theta) \qquad \text{---------(2)}$$

equation (1) and(2) gives the stress conditions at failure for a cohesionless soil.

Recap

In this section you have learnt the following

- Introduction

Module 3 : Method of Analyses

Lecture 14 : Methods of Characteristics [Section 14.2 : Rankine earth pressure]

Objectives

In this section you will learn the following

- Rankine earth pressure
- Determination of earth pressure coefficients
- Inclination of failure plane
- Derivations of basic parameters and equations

- **Rankine earth pressure**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = X = \gamma \qquad \text{----------(3)}$$

where γ is the unit weight of the soil

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = Y = 0 \qquad \text{----------(4)}$$

$\partial \sigma_{y,} \partial \tau_{xy} = 0$ along a horizontal plane.

at a depth *x*, integrating equation (3) and (4),

$\sigma_x = \gamma x + C$

$\tau_{xy} = D$

Boundary conditions:

if there is no surcharge, C=0, D=0 at x=0.

$\tau_{xy} = \sigma \sin \phi \sin 2\theta = 0$

$\sigma \neq 0, \phi = 0, \sin 2\theta = 0$.

Hence $\theta = 0$ (active conditions) or $\theta = \dfrac{\pi}{2}$ (passive conditions)

This implies that in passive case, $\theta = \dfrac{\pi}{2}$ and in active case $\theta = 0$ .where θ is the inclination of the major principle stress with the x direction.

- **Determination of earth pressure coefficients**

$$\sigma_x = \gamma x = \sigma(1+\sin\phi\cos 2\theta)$$

$$\sigma = \frac{\gamma x}{1+\sin\phi}$$ (for active case, $\theta = 0$)

$$\sigma_y = \sigma(1-\sin\phi\cos 2\theta)$$

$$= \frac{\gamma x}{1+\sin\phi}(1+\sin\phi) \qquad \text{---------(5)}$$

$$\sigma_y = K_a \gamma x \qquad \text{---------(6)}$$

from eqn(5) and (6), coefficient of active earth pressure $K_a = \dfrac{1-\sin\phi}{1+\sin\phi}$

similarly, in the passive case , $\theta = \dfrac{\pi}{2}$

$$\sigma_x = \sigma(1-\sin\phi) \qquad \text{----------(7)}$$

$$\sigma_y = \sigma(1+\sin\phi) \qquad \text{----------(8)}$$

from eqn(7) and (8), coefficient of passive earth pressure $K_p = \dfrac{1+\sin\phi}{1-\sin\phi}$

- **Inclination of failure plane**

The failure planes at particular plane will make an angle of $\pm\mu$ with the direction of major principal stress.

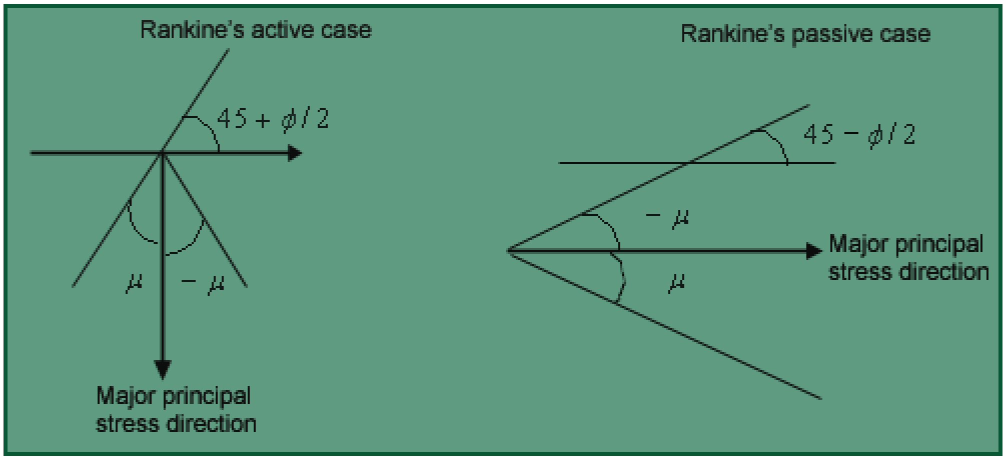


Fig .3.7 Inclination of failure planes

Inclined Ground

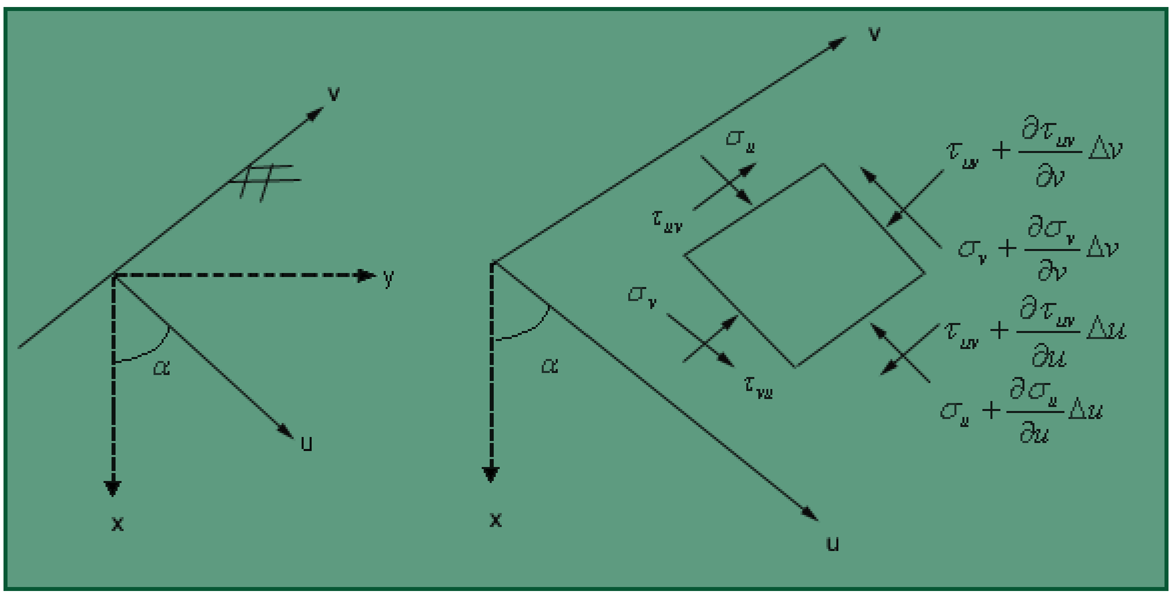


Fig.3.8 Infinitesimal stress analysis for an element for inclined ground

Considering the forces in the u and v directions,

$$\frac{\partial \sigma_u}{\partial u} + \frac{\partial \tau_{uv}}{\partial v} = \gamma \cos \alpha$$

$$\frac{\partial \sigma_v}{\partial v} + \frac{\partial \tau_{uv}}{\partial u} = -\gamma \sin \alpha$$

$$\sigma_u = \gamma u \cos \alpha \qquad \text{---------(9)}$$

$$\tau_{uv} = -\gamma u \sin \alpha \qquad \text{---------(10)}$$

dividing eqn 9 by 10 and simplifying ,

$$\frac{\gamma u \cos \alpha}{-\gamma u \sin \alpha} = \frac{1 + \sin \phi \cos 2\theta}{\sin \phi \sin 2\theta}$$

$$\sin \alpha \cos 2\theta + \sin 2\theta . \cos \alpha = \frac{\sin \alpha}{\sin \phi}$$

$$\sin(2\theta + \alpha) = \frac{-\sin \alpha}{\sin \phi}$$

thus, $\theta = \frac{1}{2}\left[\sin^{-1}\left(\frac{-\sin \alpha}{\sin \phi}\right) - \alpha\right]$

- **Derivations of basic parameters and equations**

Equations of equilibrium are,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = X \qquad \text{--------(11)}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = Y \qquad \text{--------(12)}$$

From the derivations given earlier we know,

$$\frac{\sigma_x}{\sigma_y} = \sigma(1 \pm \sin\phi \cos 2\theta) - H \qquad \text{---------(13)}$$

and $\tau_{xy} = \sigma \sin\phi \sin 2\theta$ ---------(14)

substituting the values of the equation 13, 14 in 11,

$$\frac{\partial}{\partial x}\left[\sigma(1+\sin\phi\cos 2\theta) - H\right] + \frac{\partial}{\partial y}\left[\sigma\sin\phi\sin 2\theta\right] = X$$

As c and f are not the function of x and y, i.e. H and f are not functions of x and y, then,

$$\frac{\partial \sigma}{\partial x}\left[(1+\sin\phi\cos 2\theta)\right] + \frac{\partial \theta}{\partial x}\left[\sigma(-2\sin\phi\sin 2\theta)\right] + \frac{\partial \sigma_y}{\partial y}(\sin\phi\sin 2\theta) + \frac{\partial \theta}{\partial y}(2\sigma\sin\phi\cos 2\theta) = X \qquad \text{--------15)}$$

Substituting the values of the equation 13, 14 in 12,

$$\frac{\partial}{\partial y}\left[\sigma(1-\sin\phi\cos 2\theta) - H\right] + \frac{\partial}{\partial x}\left[\sigma\sin\phi\sin 2\theta\right] = X$$

As c and f are not the function of x and y, i.e. H and f are not functions of x and y, then,

$$\frac{\partial \sigma}{\partial y}\left[(1-\sin\phi\cos 2\theta)\right] + \frac{\partial \theta}{\partial x}\left[\sigma(2\sin\phi\sin 2\theta)\right] + \frac{\partial \sigma_x}{\partial x}(\sin\phi\sin 2\theta) + \frac{\partial \theta}{\partial x}(2\sigma\sin\phi\cos 2\theta) = Y \qquad \text{--------(16)}$$

Let, $\chi = \frac{\cot\phi}{2}\ln\left(\frac{\sigma}{\sigma_0}\right)$

Where, σ_0 is the characteristic stress.

$$\therefore \ln\frac{\sigma}{\sigma_0} = 2\chi\tan\phi$$

$$\therefore \sigma = \sigma_0 e^{2\chi\tan\phi}$$

$$\therefore \frac{\partial\sigma}{\partial x} = 2\sigma\tan\phi\frac{\partial\chi}{\partial x} \qquad \text{---------(17)}$$

$$\therefore \frac{\partial\sigma}{\partial y} = 2\sigma\tan\phi\frac{\partial\chi}{\partial y} \qquad \text{---------(18)}$$

$$(1+\sin\phi\cos2\theta)\left(2\sigma\tan\phi\frac{\partial\chi}{\partial x}\right)-(2\sigma\sin\phi\sin2\theta)\frac{\partial\theta}{\partial x}+\left(2\sigma\sin\phi\sin2\theta\tan\phi\frac{\partial\chi}{\partial y}\right)+(2\sigma\sin\phi\cos2\theta)\frac{\partial\theta}{\partial y}=X$$

Substituting 17 and 18 in equation 15,

Dividing both sides by $2\sigma\tan\phi$,

$$(1+\sin\phi\cos2\theta)\left(\frac{\partial\chi}{\partial x}\right)-(\cos\phi\sin2\theta)\frac{\partial\theta}{\partial x}+\left(\sin\phi\sin2\theta\frac{\partial\chi}{\partial y}\right)+(\cos\phi\cos2\theta)\frac{\partial\theta}{\partial y}=\frac{X\cot\phi}{2\sigma} \qquad \text{--------(19)}$$

Substituting 17 and 18 in equation 16 and dividing both sides by $2\sigma\tan\phi$,

$$(1-\sin\phi\cos2\theta)\left(\frac{\partial\chi}{\partial y}\right)+(\cos\phi\sin2\theta)\frac{\partial\theta}{\partial y}+\left(\sin\phi\sin2\theta\frac{\partial\chi}{\partial x}\right)+(\cos\phi\cos2\theta)\frac{\partial\theta}{\partial x}=\frac{Y\cot\phi}{2\sigma} \qquad \text{--------(20)}$$

Multiplying eq. 19 by $\sin(\theta-\mu)$ and equation 20 by $(-\cos(\theta-\mu))$ and adding them,

Where, $\mu=\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$,

$$\therefore (1+\sin\phi\cos 2\theta)\sin(\theta-\mu)\left(\frac{\partial\chi}{\partial x}\right)-(\cos\phi\sin 2\theta)\sin(\theta-\mu)\frac{\partial\theta}{\partial x}+\left(\sin\phi\sin 2\theta\sin(\theta-\mu)\frac{\partial\chi}{\partial y}\right)$$

$$+(\cos\phi\cos 2\theta)\sin(\theta-\mu)\frac{\partial\theta}{\partial y}=\frac{X\cot\phi}{2\sigma}\sin(\theta-\mu)$$

$$\therefore -\cos(\theta-\mu)(1-\sin\phi\cos 2\theta)\left(\frac{\partial\chi}{\partial y}\right)-(\cos\phi\sin 2\theta)\cos(\theta-\mu)\frac{\partial\theta}{\partial y}-\left(\sin\phi\sin 2\theta\cos(\theta-\mu)\frac{\partial\chi}{\partial x}\right)$$

$$-(\cos\phi\cos 2\theta)\cos(\theta-\mu)\frac{\partial\theta}{\partial x}=\frac{Y\cot\phi}{2\sigma}\cos(\theta-\mu)$$

Let, $\eta=\chi-\theta$ and $\xi=\chi+\theta$,

So above two equations can be written as,

$$\frac{\partial\eta}{\partial x}+\tan(\theta-\mu)\frac{\partial\eta}{\partial y}=a \qquad \text{----------(21)}$$

$$\frac{\partial\xi}{\partial x}+\tan(\theta+\mu)\frac{\partial\xi}{\partial y}=b \qquad \text{----------(22)}$$

where,

$$\begin{matrix}a\\b\end{matrix}=\pm\frac{X\sin(\theta\pm\mu)-Y\cos(\theta\pm\mu)}{2\sigma\sin\phi\cos(\theta\mp\mu)}$$

Solve for η and ξ,

$$d\eta=\frac{\partial\eta}{\partial x}dx+\frac{\partial\eta}{\partial y}dy$$

$$d\xi=\frac{\partial\xi}{\partial x}dx+\frac{\partial\xi}{\partial y}dy$$

Substituting above two equations in 21 and 22 and solving we get the FINAL EQUATIONS,

$$\frac{\partial \eta}{\partial x} = \frac{a\,dy - \tan(\theta - \mu)\,d\eta}{dy - \tan(\theta - \mu)\,dx}$$ ------eq. I

$$\frac{\partial \xi}{\partial x} = \frac{b\,dy - \tan(\theta + \mu)\,d\xi}{dy - \tan(\theta + \mu)\,dx}$$ ------eq. II

$$\frac{\partial \eta}{\partial y} = \frac{d\eta - a\,dx}{dy - \tan(\theta - \mu)\,dx}$$ ------eq. III

$$\frac{\partial \xi}{\partial y} = \frac{d\xi - b\,dx}{dy - \tan(\theta + \mu)\,dx}$$ ------eq. IV

If in equation II we assume denominator = 0, then,

$$\frac{dy}{dx} = \tan(\theta - \mu)$$

But, if numerator ≠ 0 then, it represents undefined state of stress at each point in a medium. $\therefore \frac{\partial \eta}{\partial y} = \infty$

So, to have finite state of stress, numerator = 0, then, $\frac{\partial \eta}{\partial x} = a$

i.e. it has number of values which satisfies the condition, $\frac{\partial \eta}{\partial y} = \frac{0}{0}$

It is known as *path of characteristics*, where these partial derivatives are of the form $\frac{0}{0}$. It is called as $(\theta-\mu)$ characteristics. In x-y media of soil, infinite number of parallel lines having slope $(\theta-\mu)$ represents " $(\theta-\mu)$ characteristics". Along these lines stress condition is $\frac{\partial\eta}{\partial x}=a$.

Similarly, $(\theta+\mu)$ is another characteristics where $\frac{\partial\xi}{\partial x}=b$ is the stress condition as shown in fig. below.3.9.

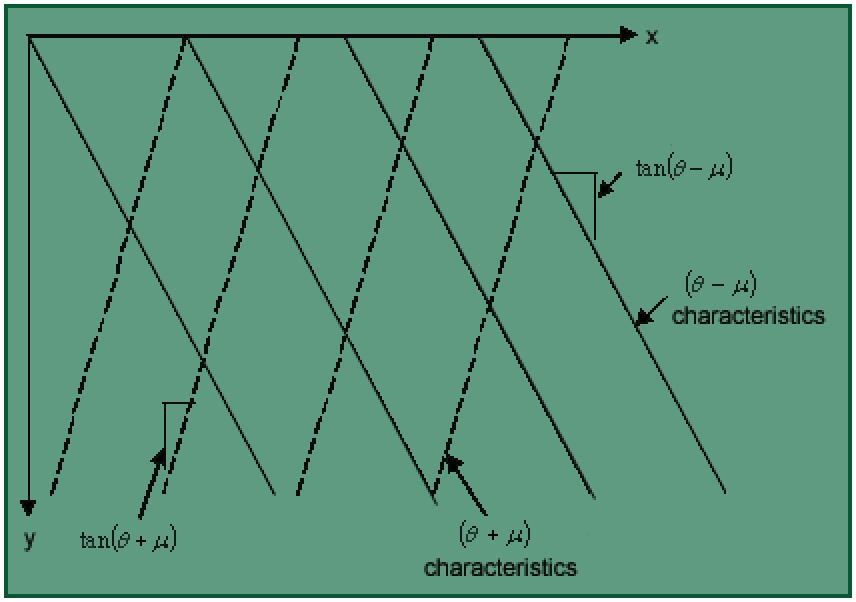
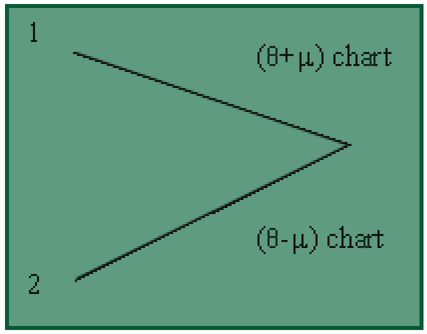


Fig. 3.9 $(\theta-\mu)$ and $(\theta+\mu)$ characteristics



At 1& 2 stress conditions are known say(x_1,y_1)(x_2,y_2) are location points and stress conditions are (ξ_1,η_1) (ξ_2,η_2)

P(x,y)=?

P(ξ, η)=?

$$\frac{y-y_1}{x-x_1}=\tan(\theta_1-\mu)$$ Considering ‘1 to P’ point.

$$\frac{y-y_2}{x-x_2}=\tan(\theta_2+\mu)$$ Considering ‘2 to P’ point.

$$a_1=\frac{\eta-\eta_1}{x-x_1}, \qquad b_2=\frac{\xi-\xi_2}{x-x_2}$$

Find x and y from above two equations and substitute in the below two equations and solve for ξ & η Value.

Recap

In this section you have learnt the following

- Rankine earth pressure
- Determination of earth pressure coefficients
- Inclination of failure plane
- Derivations of basic parameters and equations

Objectives

In this section you will learn the following

- Strip Foundation
- Two cases of failures
- Numerical example

STRIP FOUNDATIONS

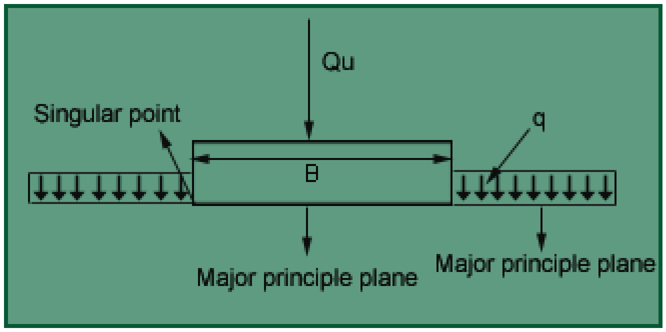


Fig 3.10 Strip foundation free body diagram

Singular pint: - where the state of stress jumps from one state of stress to the other.

At points A & C both major principal plane and minor principal plane are meeting.

$$\frac{\partial \sigma}{\partial x} = \gamma; \quad \frac{\partial \tau_{xy}}{\partial y} = 0$$

Two cases of failures,

1. $Q_u > q$ - footing goes down by keeping q constant, no yielding of surrounding soil.
2. $Q_u < q$ – surrounding soil goes down and footing goes up.

- **In Rankine passive zone**, $\theta = \pi/2$

$$\sigma_x = \gamma x + q = \sigma(1 + \sin\phi\cos 2*\frac{\pi}{2}) - H$$

$$\sigma = \frac{\gamma x + q + H}{1 - \sin\phi}$$

now, $\chi = \frac{\cot\phi}{2}\ln\frac{\sigma}{\sigma_0}$, let us assume, x=0, and find N_c & N_r values.

$$\chi = \frac{\cot\phi}{2}\ln\frac{q + H}{(1 - \sin\phi)\sigma_0}$$

$\theta = \frac{\pi}{2}$ therefore X=Y=0 and a=b=0.

$$\frac{d\eta}{dx} = 0, \qquad \frac{d\xi}{dx} = 0$$

just under the foundation,

$$\xi = \chi + \theta = \frac{\cot\phi}{2}\ln\frac{q + H}{(1 - \sin\phi)\sigma_0} + \frac{\pi}{2}$$

Now, below foundation, $\theta = 0$

Below foundation,

$$\chi = \frac{\cot\phi}{2}\ln\frac{q + H}{(1 - \sin\phi)\sigma_0}$$

- ξ **=constant**

$\xi_{RP} = \xi_{bf}$

$$\xi_{RP} = \chi_{bf} + \theta_{bf}$$

$$\chi_{bf} = \xi_{RP} = \frac{\cot\phi}{2}\ln\frac{q+H}{(1-\sin\phi)\sigma_0} + \frac{\pi}{2}$$

now, $\chi = \dfrac{\cot\phi}{2}\ln\dfrac{\sigma}{\sigma_0}$

therefore, $\sigma = \sigma_0 e^{2\chi\tan\phi}$

$$\sigma = \sigma_0 e^{\left[\ln\frac{q+H}{(1-\sin\phi)\sigma_0}+\pi\tan\phi\right]}$$

now, $\sigma_x = \sigma[1+\sin\phi\cos 2\theta] - H$

$= \sigma[1+\sin\phi] - H$

$$\sigma_x = \sigma_0 e^{\left[\ln\frac{q+H}{(1-\sin\phi)\sigma_0}+\pi\tan\phi\right]}.(1+\sin\phi) - H$$

Numerical example :

- $\sigma_0 = 1.0$ (assumed)

If q=0,C=1, ϕ=0.001

$$\sigma_x = 1.0e^{\left[\ln\frac{0+1.0\cot 0.001}{(1-\sin 0.001)1.0}+\pi\tan 0.001\right]}.(1+\sin 0.001) - 1.0\cot 0.001$$

=5.14= N_c

- σ_o =0,q=1.0,C=1.0, ϕ =0.001 Therefore, σ_x =1.0 = N_q

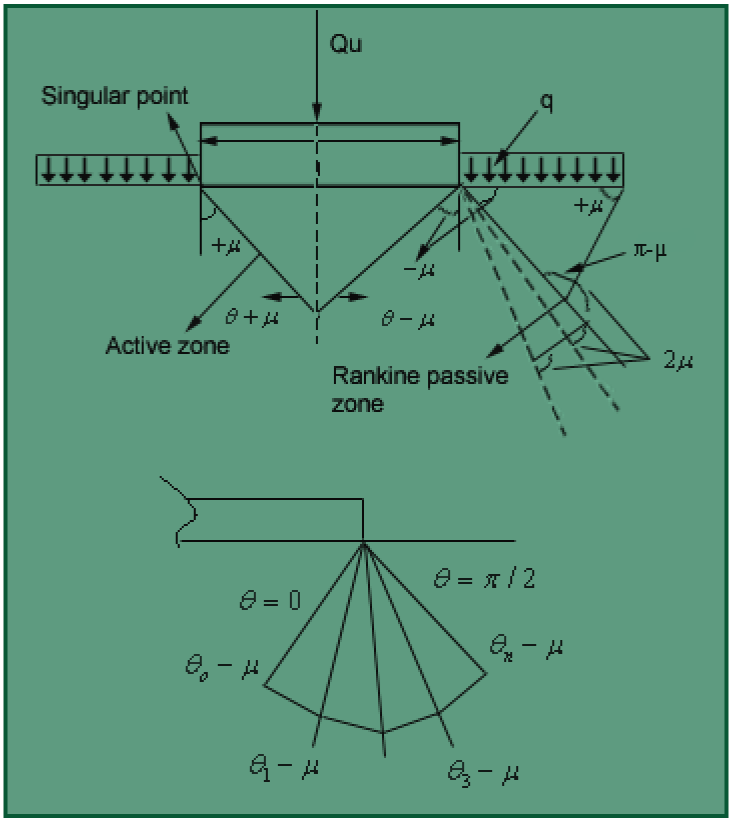


Fig 3.11 Stress path diagrams

for $\gamma=0$

ξ =constant at every point

along $(\theta-\mu),\eta$ =constant.

Draw different $(\theta-\mu)$ characteristics with different θ values at singular point.

The number of segments between ‘0' and 90 should be as maximum as possible, so as to get closed real value as possible.

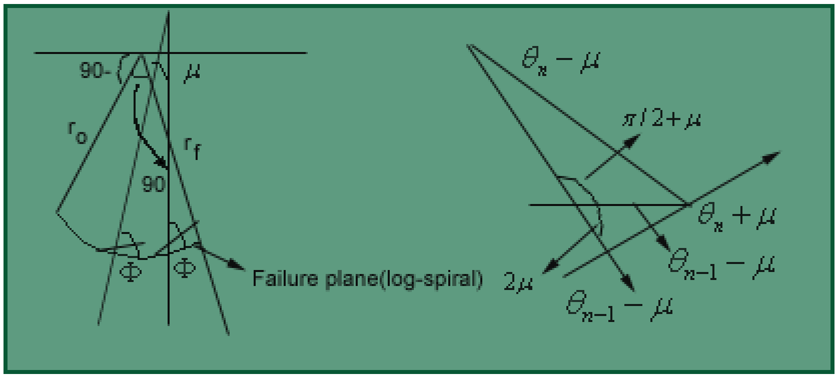


Fig 3.12 Failure plane diagram

Recap

In this section you have learnt the following

- Strip Foundation
- Two cases of failures
- Numerical example

Objectives

In this section you will learn the following

- Inclined Load --- Bearing Capacity Factors
- Rankine passive zone

Inclined Load --- Bearing Capacity Factors

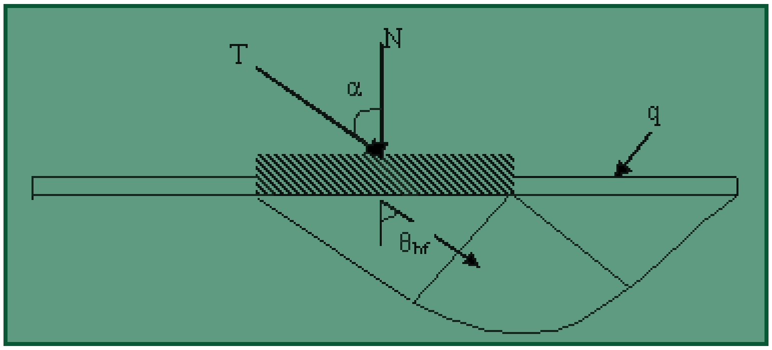


Fig.3.13 Failure surface beneath the footing

For Rough footing & inclined load there may be shear force at the base of the footing. One side Failure occurs because of inclined load. For Rough Footing, one-sided failure is considered for calculating seismic bearing capacity factors.

$$\frac{\tau_{xy}}{\sigma_x} = \tan(\alpha + \delta)$$

$$\sigma_x = \sigma\left[1 + \sin\phi.\cos 2\theta\right] - H$$

$$\tau_{xy} = \sigma.\sin\phi.\sin 2\theta$$

For $C = 0$ $\quad \tan\alpha = \dfrac{\tau_{xy}}{\sigma_x}$

$$\tan\alpha = \frac{\sin\phi.\sin 2\theta}{1 + \sin\phi.\cos 2\theta} \qquad (\because \sigma \neq 0)$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{\sin\phi.\sin 2\theta}{1 + \sin\phi.\cos 2\theta}$$

$$\sin\alpha + \sin\alpha.\sin\phi.\cos 2\theta = \cos\alpha.\sin\phi.\sin 2\theta$$

$$\sin\alpha = \sin\phi\left[\left(\cos\alpha.\sin 2\theta\right) - \sin\alpha.\cos 2\theta\right]$$

$$\sin\alpha = \sin\phi.\sin\left(2\theta - \alpha\right)$$

$$\frac{\sin\alpha}{\sin\phi} = \sin(2\theta - \alpha)$$

$$2\theta = \sin^{-1}\left(\frac{\sin\alpha}{\sin\phi}\right) + \alpha$$

$$\theta_{bf} = \frac{1}{2}\left[\sin^{-1}\left(\frac{\sin\alpha}{\sin\phi}\right) + \alpha\right]$$

(For Smooth Footing, Valid only for inclined Force & for below foundation case)

Here we get some value of θ instead of $\theta = 0$

This equation is valid only for inclined force & below foundation.

$\theta_{PR} = \pi/2$ There is no effect of inclined load on Rankine passive zone & $\tau_{xy} = 0$

$\theta = 0$; if $\alpha = 0$; Vertical load is acting.

Bearing capacity Factor

Since C = 0, γ = 0, So N_q we have to find out.

$$q_u = C.N_c + q.N_q + \frac{1}{2}\gamma B N_\gamma$$

$$q_u = C.i_c.N_c + q.i_q.N_q$$

Rankine passive zone

$$\theta = \frac{\pi}{2}, \gamma = 0, \sigma_x = q$$

$$\sigma_x = \sigma(1 + \sin\phi.\cos 2\theta)$$

$$q = \sigma(1 + \sin\phi.\cos 2.\frac{\pi}{2})$$

$$\sigma = \frac{q}{1 - \sin\phi}$$

$$\chi = \frac{\cot\phi}{2} \ln \frac{q}{(1 - \sin\phi).\sigma_0}$$

$(\theta + \mu)$ = Characteristics bend towards foundations & along ξ constant.

As ξ constant then $\xi_{bf} = \xi_{RP}$

$$(\chi + \theta)_{bf} = (\chi + \theta)_{RP}$$

Now we Know θ_{bf}, θ_{RP} & χ_{RP} then we have to find out χ_{bf}. We have to get σ_x below foundation.

$$\chi_{bf} = \chi_{RP} + \theta_{RP} - \theta_{bf}$$

$$\chi_{bf} = \frac{\cot\phi}{2} \ln \frac{q}{(1-\sin\phi)\sigma_0} + \frac{\pi}{2} - \frac{1}{2}\left[\sin^{-1}\left(\frac{\sin\alpha}{\sin\phi}\right) + \alpha\right] = \frac{\cot\phi}{2} \ln \frac{\sigma_{bf}}{\sigma_0}$$

$$\sigma_{bf} = \sigma_0 . e^{\left[\ln\frac{q}{(1-\text{Sin}\phi)\sigma_0} + \pi\tan\phi - \tan\phi\left\{\text{Sin}^{-1}\left(\frac{\text{Sin}\alpha}{\text{Sin}\phi}\right) + \phi\right\}\right]}$$

$$\sigma_{x_{bf}} = \sigma_{bf}\left[1 + \sin\phi . \cos 2\theta_{bf}\right]$$

$$\sigma_{x_{bf}} = \sigma_{bf}\left[1 + \sin\phi . \cos\left[\sin^{-1}\left(\frac{\sin\alpha}{\sin\phi}\right) + \alpha\right]\right]$$

Recap

In this section you have learnt the following

- Inclined Load --- Bearing Capacity Factors
- Rankine passive zone

Congratulations, you have finished Lecture 14. To view the next lecture select it from the left hand side menu of the page