Module 7

Lecture 7: Statistical parameter estimation

Parameter Estimation

Methods of Parameter Estimation

1) Method of Matching Points
2) Method of Moments
3) Maximum Likelihood method

$\theta_i \rightarrow$ Population Parameter

$\hat{\theta}_i \rightarrow$ Sample Parameter

Unbiased estimation of parameter:An estimate of a parameter θ_i is said to be unbiased estimate, if $E(\hat{\theta}_i) = \theta_i$

1) Method of Matching Points

In a data set, 75 % values are less than, It is assumed to follow the distribution,

$$f(x) = \frac{e^{-x/\theta}}{\theta}; \quad x > 0, \text{ estimate the parameter } \theta.$$

$$\therefore \; P[X \le 3] = 0.75 = F(x) = \int_0^x f(x)dx = \int_0^x \frac{e^{-x/\theta}}{\theta}\,dx$$

$$= \left[\frac{e^{-x/\theta}}{\theta \times (-1/\theta)}\right]_0^x = 1 - e^{-x/\theta}$$

$$\therefore \; 1 - e^{-x/\theta} = 0.75 \quad \text{or} \quad -x/\theta = -1.3863 \quad \text{or} \quad \theta = 2.164$$

2) Method of Moments

Given a function $f(\theta_i, \ldots\ldots, \theta_j, x)$ and values $\theta_i, \ldots\ldots, \theta_j$

we need to find $x_1, x_2, \ldots\ldots\ldots x_n$

Generate number of equations by taking moments of the distribution

Take, any distribution, like $f(x) = (2\prod\theta_2^{\,2})^{-1/2} e^{\frac{-(x-\theta_1)^2}{2\theta_2^{\,2}}}$ $\quad -\alpha < x < +\alpha$

Take the 1st moment $\Rightarrow$ Mean about the origin

$$\mathbf{E(X) =} \mu \mathbf{=} \int x(fx)dx = \int_{-\alpha}^{\alpha} x(2\prod\theta_2^{\,2})^{-1/2} e^{\frac{-(x-\theta_1)^2}{2\theta_2^{\,2}}} dx$$

$$= (2\prod\theta_2^{\,2})^{-1/2} \int_{-\alpha}^{\alpha} x e^{\frac{-(x-\theta_1)^2}{2\theta_2^{\,2}}} dx$$

2) Method of Moments Contd...

substituting, y = $\left({(x-\theta_1)} \big/ {\theta_2} \right)$

x = $\theta_2 y + \theta_1$

dx = $\theta_2 dy$

E(X) = $\dfrac{1}{\sqrt{2\Pi}\,\theta_2} \displaystyle\int_{-\alpha}^{\alpha} (\theta_2 y + \theta_1) e^{\frac{-y^2}{2}} \theta_2 dx$

E(X) = $\dfrac{1}{\sqrt{2\Pi}} \left[\displaystyle\int_{-\alpha}^{\alpha} \theta_2 e^{\frac{-y^2}{2}} y dy + \theta_1 \int_{-\alpha}^{\alpha} e^{\frac{-y^2}{2}} dy \right]$

[As odd multiplier, h(-y) = -h(y)

$\Downarrow$

E(X) = $\dfrac{1}{\sqrt{2\Pi}} \left[0 \quad + \theta_1 \displaystyle\int_{-\alpha}^{\alpha} e^{\frac{-y^2}{2}} dy \right]$

= 0 + $\dfrac{\theta_1}{\sqrt{2\Pi}} \sqrt{2\Pi}$ **=** θ_1

$\therefore \theta_1 = E(x) = \mu$

2) Method of Moments Contd…

Second moment about the mean,

$$\mathbf{E}\left[(\mathbf{X}-\mu)^2\right] = \sigma^2 = \int_{-\alpha}^{\alpha} (x-\mu)^2 f(x)dx$$

$$= \int_{-\alpha}^{\alpha} (x-\mu)^2 (2\prod \theta_2{}^2)^{-1/2} e^{\frac{-(x-\theta_1)^2}{2\theta_2{}^2}} dx$$

Substituting, $\theta_1 = \mu$

and $\mathbf{y} = \dfrac{x-\mu}{\sqrt{2}\theta_2}$ **and we will get,**

$$\sigma^2 = \theta_2$$

3) Maximum Likelihood method

Sample,

$$\begin{Bmatrix} x_1 \\ \vdots \\ \vdots \\ x_i \end{Bmatrix} = \begin{Bmatrix} \theta_1 \\ \vdots \\ \vdots \\ \theta_i \end{Bmatrix}$$ We have the following, $f(x_1;\theta)\ f(x_2;\theta)\ f(x_3;\theta)$

$f(x_i;\theta) \rightarrow pdf$ evaluated at x = x_i

- Product of $f(x_1;\theta)\ \times\ \ldots\ldots\ldots \times\ f(x_3;\theta)$ is "likelihood" $\approx$ L
- If $\mathrm{L}(x\ ;\theta_i) > f(x\ ;\theta_i)$, then θ is the estimate preferred,

which maxmizes the likelihood function.

$*$ Because ln(L) is an increasing function of L, it reaches maximum value at the same pt., as ln(L) does, $\frac{\partial \ln(L)}{\partial \theta} = 0$

$\Rightarrow$ (When there is no other method feasible, this method is best one)

$f(x) = \beta e^{-\beta x}; x > 0 \quad \beta$ is a parameter

$\{x_1, x_2, \ldots\ldots\ldots, x_n\} \rightarrow$ Sample available;

$\therefore \mathrm{L} = f(x_1, \beta) \times f(x_2, \beta) \times f(x_3, \beta) \times \ldots\ldots\ldots \times f(x_n, \beta)$

$= \beta e^{-\beta x_1} \times \beta e^{-\beta x_2} \times \ldots\ldots\ldots \times \beta e^{-\beta x_n}$

$= \beta^n e^{-\beta(x_1 + x_2 + \ldots\ldots + x_n)} = \beta^n e^{-\beta \sum_{i=1}^{n} x_i}$ (formulation of likelihood function)

Now set L maximum

$$\therefore \ln(L) = n\ln\beta - \beta e^{\sum_{i=1}^{n} x_i}$$

$$\therefore \frac{\partial \ln(L)}{\partial \beta} = 0 \quad \text{as we want to set the value of } \beta$$

$$\therefore \frac{\partial \ln(L)}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i = 0$$

$$\therefore \beta = \frac{n}{\sum_{i=1}^{n} x_i} = \left(\frac{1}{\overline{x}}\right)$$

$\overline{x}$ = Arithmetic average

Max. likelihood estimate

3) Maximum Likelihood method **Contd…**

$$f(x) = \frac{1}{\sqrt{2\prod}\sigma} \exp\left[-\tfrac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

[Take, σ as parameter not S.D. and μ also] *or* $\mu = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$

$$\therefore \text{L} = f(x_1,\mu,\sigma) \bullet f(x_2,\mu,\sigma) \ldots\ldots\ldots f(x_n,\mu,\sigma) = \left[\frac{1}{\left(\sqrt{2\prod}\sigma\right)^n}\right] \exp\left[-\tfrac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$

or $\ln(L) = -\frac{n}{2}\ln(2\prod\sigma^2) - -\tfrac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$

or $\frac{\partial \ln(L)}{\partial \mu} = 0 = \frac{\partial \ln(L)}{\partial \sigma}$ set μ & σ ,Now

$$\frac{\partial \ln(L)}{\partial \mu} = 0 = \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma^2}\right) \quad \therefore \sum (X_i - \mu) = 0$$

3) Maximum Likelihood method **Contd…**

$$f(x) = \frac{1}{\sqrt{2\prod}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

[Take, σ as parameter not S.D. and μ also]

$$\therefore \mathbf{L} = f(x_1,\mu,\sigma) \bullet f(x_2,\mu,\sigma) \ldots\ldots\ldots f(x_n,\mu,\sigma) = \left[\frac{1}{\left(\sqrt{2\prod}\sigma\right)^n}\right] \exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$

$$or \quad \ln(L) = -\frac{n}{2}\ln(2\prod\sigma^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$$

$$or \quad \frac{\partial \ln(L)}{\partial \mu} = 0 = \frac{\partial \ln(L)}{\partial \sigma}$$ **set μ & σ ,Now**

$$\frac{\partial \ln(L)}{\partial \mu} = 0 = \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma^2}\right)$$

$$\therefore \sum (X_i - \mu) = 0$$

or $$\mu = \frac{\sum_{i=1}^{n} x_i}{n} = \overline{x}$$

and $$\frac{\partial \ln(L)}{\partial \mu} = 0 = -\frac{n}{2}\left(\frac{4\prod\sigma}{2\prod\sigma^2}\right) - \frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2(-2)}{\sigma^3}$$

or $$-\frac{n}{\sigma} + \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^3} = 0$$

or $$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n}$$

[But it is not the best method. It depends upon situation]

Highlights in the Module

- In hydrology, most of the phenomena are random in nature.

  E.g. rainfall-runoff model

  - Random variables involved in a hydrological process may be dependent or independent.
  - The 'random variables' X & Y are 'stochastically independent' if and only if their 'joint density' is equal to the product of 'marginal density functions'.

- Joint density function : Simultaneous occurrence
- Marginal density function : Distribution of one variable irrespective of the value of the other variables
- Conditioned distribution: Distribution of one variable conditioned on the other variable.

Highlights in the Module Contd…

❖ **Measures of Central Tendency:**

- Mean
- Arithmetic average (for sample)
- Mode
- Median

❖ **Measures of Spread or Dispersion:**

- Range [(xmax-xmin)]
- Relative Range [=(range/mean)]
- Variance
- Standard deviation,
- Coefficient of variation

Highlights in the Module **Contd…**

- **Measures of Symmetry:**
  - Coefficient of skewness,
  - Kurtosis
- **Correlation coefficient** shows the degree of linear association between two random variables
- **Commonly used distributions in hydrology :**
  - Normal distribution
  - Uniform distribution
  - Exponential distribution
  - Gamma distribution
  - Log-normal distribution
  - Extreme value distribution

Highlights in the Module

Contd…

❖ **Methods of statistical parameter estimation**

1) Method of Matching Points

2) Method of Moments

3) Maximum Likelihood method