Applied Time-Series Analysis

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Periodogram as PSD estimator

Learning Goals

In this lecture, we shall closely examine the following

- ▶ Periodogram

- ▶ Spectral leakage and modified periodogram

- ▶ Window functions for improving spectral leakage

- ▶ Periodogram as an estimator of p.s.d. of stochastic signals

Recap

- Periodic signals have only line spectra. Densities do not exist
- Aperiodic finite-energy signals only have energy densities. However, due to the way DFT is computed, one ends up treating the signal as periodic.
- An ad hoc density, known as the *periodogram*, for analyzing periodic deterministic signals embedded in noise is often used:

Two questions:

Q1: Does the periodogram correctly detect the frequencies in a deterministic signal?

Q2: How good is the periodogram as an estimator of PSD for stochastic signals?

Periodogram for analyzing spectra of deterministic signals

The main tool is the **periodogram.**

The periodogram obtained from N samples of a signal $x[k]$ is defined as

$$P_{xx}(f_n) = \frac{1}{N} \left| \sum_{k=0}^{N-1} x[k] e^{-j2\pi f_k n} \right|^2 = \frac{|X(f_n)|^2}{N}, \ n = 0, \cdots, N-1 \ \text{(Schuster, 1897)}$$

where $X[n] = X(f_n)$ is the nth DFT coefficient (of $x[k]$).

▶ **Issues:** (i) **spectral leakage** due to finite-length effects, (ii) **limited resolution.**
▶ **Remedies:** Use modified periodogram (to minimize leakage)

Finite-length effects: Spectral leakage

The periodogram of a sinusoidal signal $x[k] = A\sin(2\pi f_0)$ peaks at the true frequency f_0 **only if the data record contains complete cycles of the sinusoid**. If the data record contains **incomplete cycles**, the spectrum does not peak at the right frequency and also leaks to neighbouring frequencies.

Q: Why?

Finite-length effects   . . . contd.

Two viewpoints can be offered:

1. DFT implies periodic extension of the signal. When the cycles are incomplete, discontinuities occur at the borders. Alternatively,

2. Any finite-duration sequence can be treated as an infinite sequence viewed through a <u>window of finite length</u>. Mathematically, this may be written as $\tilde{x}[k] = x[k]w[k]$ where $\tilde{x}[k]$ is the length-L finite duration sequence, $x[k]$ is the infinite sequence and $w[k]$ is the **rectangular window** of length L

$$w[k] = \begin{cases} 1, & 0 \leq k \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$

Effect of finite-duration: Distortion

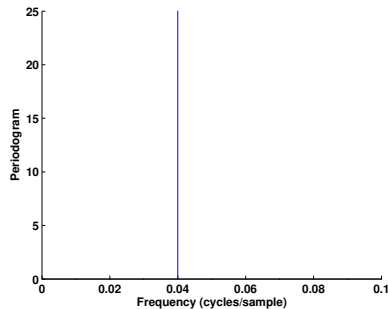▶ From the properties of FT we know that multiplication in time is equivalent to convolution in frequency domain:

$$\mathcal{F}\left\{\tilde{x}[k] = x[k]w[k]\right\} \equiv \tilde{X}(\omega) = X(\omega) \star W(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta)W(\omega - \theta) \, d\theta$$

where $X(.)$ and $W(.)$ are the Fourier Transforms of the infinite sequence and the window functions respectively
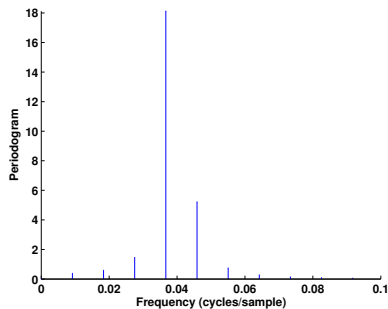
  ▶ From above, it follows that the spectral estimate we obtain is a **distorted** version of the true spectrum (convolution causes distortion)

  ▶ The distortion disappears when the window (signal) length is infinite

An Example

Suppose we have $N = 100$ and $N = 109$ observations of $x[k] = \sin(2\pi f_0 k)$, $f_0 = 0.04$ cycles/sample.

(a) $N = 100$ (integer cycles)   (b) $N = 109$ (fractional cycles)

Spectral leakage occurs primarily when the signal has not completed an integer no. of cycles in the data record

Spectral Leakage

- The peak in the spectrum should have occurred at $f = 0.04$ cycles/sample
- The corresponding bin is $1 + \frac{0.04}{\triangle f} = 5.36$ where $\triangle f = 1/109$ is he frequency resolution. Now, there is no such bin as 5.36 (bins are integers)!
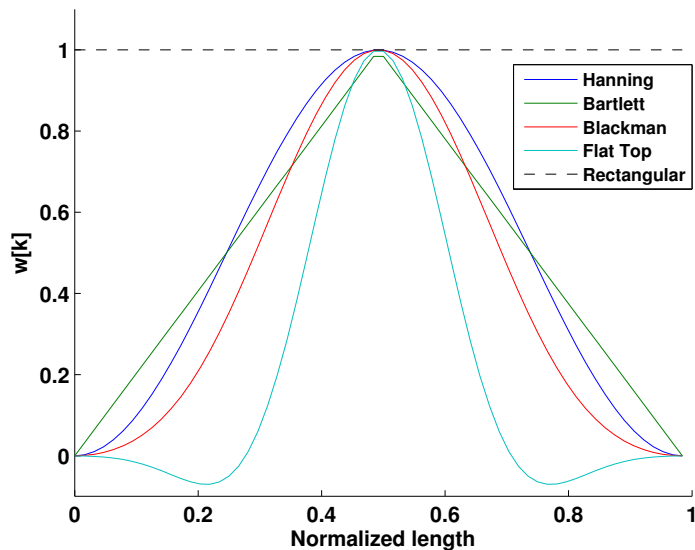 **Note:** the zero frequency corresponds to first bin
- Consequently, the power "leaks out" to surrounding bins
- The additional smearing that is observed is due to the *windowing*
- In general, therefore, spectral leakage occurs due to:
  1. Windowing (convolution of the true spectrum)
  2. Lack of exactly matching basis function in the set of N basis functions $\{e^{-j2\pi k/N}\}$
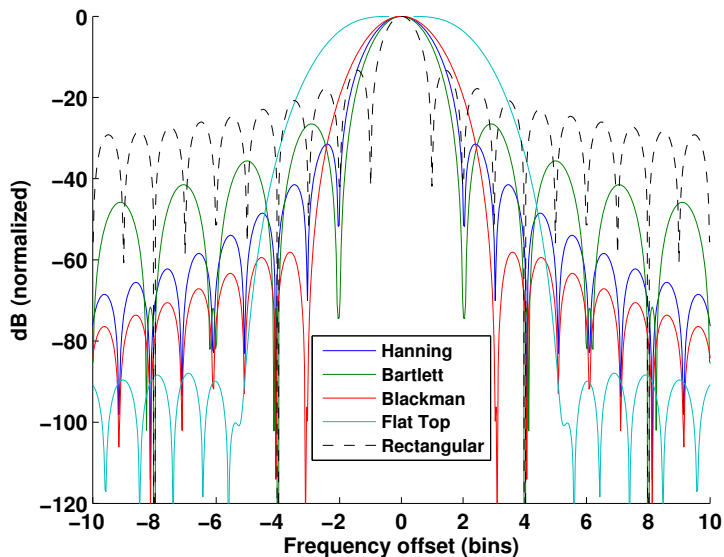
Overcoming spectral leakage

▶ Spectral leakage can be mitigated by windowing the data, thereby reducing border effects.

▶ Common windows $w[k]$ $(0 \leq n \leq N-1)$

  ▶ Rectangular: $w[k] = 1$
  ▶ Hanning: $w[k] = 0.53836 - 0.46164 \cos\left(\frac{2\pi k}{N-1}\right)$
  ▶ Hamming: $w[k] = 0.5\left(1 - \cos\left(\frac{2\pi k}{N-1}\right)\right)$
  ▶ Bartlett: $w[k] = \frac{2}{N}\left(\frac{N}{2} - \left|k - \frac{N-1}{2}\right|\right)$
  ▶ Gaussian: $w[k] = e^{-\frac{1}{2}\left(\frac{k-(N-1)/2}{\sigma(N-1)/2}\right)^2}$ where $\sigma \leq 0.5$

▶ Each window offers a different trade-off between leakage and effective $\triangle f$.

  ▶ The rectangular window has an excellent resolution but poor leakage characteristics

Window functions

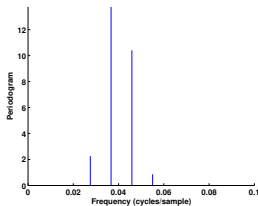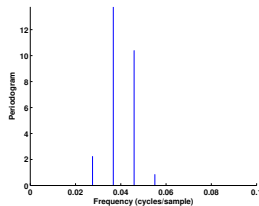# Frequency response of window function

Which window to choose?

Signal characteristics	Window features
Strong interfering components distant from the frequency of interest	High side lobe roll-off rate
Strong interfering components near the frequency of interest	Low maximum side lobe level
Two or more components very near to each other.	Very narrow main lobe
Amplitude accuracy of a component is more important than the actual frequency	Wide main lobe
Flat or broadband spectrum	Rectangular window

▶ The Hanning window offers a satisfactory trade-off between $\triangle f$ and spectral leakage.
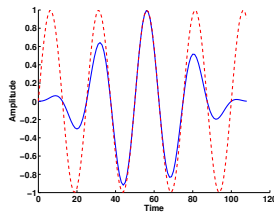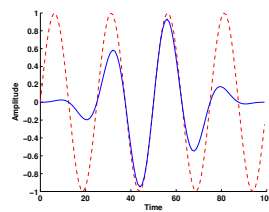
Example 1: Reduction in spectral leakage

(a) With Hanning

(b) With Blackman

(c) Hanning windowed signal

(d) Blackman windowed signal

▶ Spectral leakage has clearly diminished with the use of window functions.

▶ However, the power has been re-distributed among the two frequencies nearest to $f_0 = 0.04$

Key point

Ideally, for zero leakage, it is necessary to have a window such that $W(\omega) = \delta(\omega)$.
Unfortunately this cannot be satisfied by any window function of finite length owing
to the **duration-bandwidth principle** .

*A function that is perfectly localized in frequency is spread infinitely in time and
vice versa.*

Essentially, it is impossible to have a window that is perfectly localized in both time and
frequency! Thus, finite-length windows will always result in distortion.

ESTIMATION OF PSD FOR STOCHASTIC SIGNALS

PSD for stochastic signals: Recap

▶ A spectral density exists for all stationary signals whose ACVF is absolutely convergent. Three definitions of p.s.d. may be recalled:

$$\gamma_{vv}(\omega) = \lim_{N \to \infty} E(\gamma_{vv}^{(i,N)}(\omega)) = \lim_{N \to \infty} E\left(\frac{|V_N(\omega)|^2}{2\pi N}\right) \qquad \text{(From signal)}$$

$$\gamma_{vv}(\omega) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \sigma_{vv}[l] e^{-j\omega l} \qquad \text{(From ACVF)}$$

$$\gamma_{vv}(\omega) = |H(e^{-j\omega})|^2 \gamma_{ee}(\omega) = |H(e^{-j\omega})|^2 \frac{\sigma_e^2}{2\pi} \qquad \text{(From IR)}$$

The first two definitions lead to non-parametric estimators while the last definition produces a parametric estimator

Periodogram as an estimator of PSD for stochastic signals

▶ The estimation problem is different from that of deterministic signals because the spectral density is an average property whereas in practice, only a single realization is available

  ▶ Consequently the periodogram yiueld a very poor estimate of the p.s.d of a stochastic signal

  ▶ The primary concern is that the **periodogram is not a consistent estimator of the spectral density** for random signals.
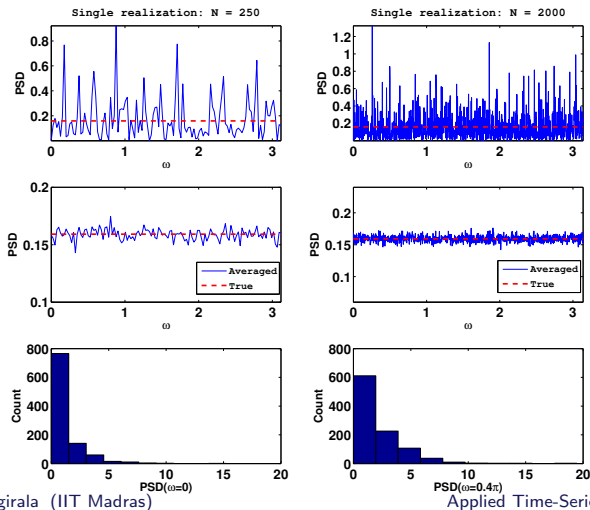
Properties of periodogram estimator

▶ For a general linear stationary process with i.i.d. driving source and rapidly decaying ACVF, the following *asymptotic* results hold:

1. $\lim_{N \to \infty} E(\mathbb{P}(\omega_n)) = \gamma(\omega_0)$ (asymptotically unbiased estimator)

2. $\lim_{N \to \infty} \mathsf{var}[P_{vv}(f)] = \gamma_{vv}^2(f)$

3. $2\dfrac{\mathbb{P}(\omega_n)}{\gamma(\omega_0)} \xrightarrow{d} \chi^2(2), \; n = 1, \cdots, \lfloor \frac{N-1}{2} \rfloor; \quad 2\dfrac{\mathbb{P}(\omega_n)}{\gamma(\omega_0)} \xrightarrow{d} 2\chi^2(1), \; n = 0, \frac{N}{2}$

4. Approximate $100(1-\alpha)\%$ C.I. for $\gamma(\omega_0)$: $\; 2\dfrac{\mathbb{P}(\omega_n)}{\chi^2_{\frac{\alpha}{2}}(2)} < \gamma(\omega_0) < 2\dfrac{\mathbb{P}(\omega_n)}{\chi^2_{1-\frac{\alpha}{2}}(2)}$

5. Estimates at two different frequencies are independent.

Example: Periodogram of WN process

It is desired to estimate the p.s.d. of a (zero-mean, unit-variance) GWN using a data set containing $N = 250$ observations of the process.

Top panel: Periodograms from a single realization with $N = 250$ and $N = 2000$ observations

Middle panel: Averaged periodograms from 1000 realizations

Bottom panel: Distribution of $\hat{\gamma}(0)/\gamma(0)$ and $2\hat{\gamma}(0.4\pi)/\gamma(0.4\pi)$ obtained from 1000 realizations.

Example   . . . contd.

Increasing N does not improve the quality of periodogram. However, averaging across realizations, as expected does bring the estimate closer to the truth.

The estimated parameters of a fitted $\Gamma(a, b)$ distribution at $\omega = 0$ and $\omega \neq 0$ are

ω_0	$\hat{a}$	$\hat{b}$	$\hat{\text{var}}(\hat{\gamma}(.))$
0	0.495	2.142	0.0571
0.4π	0.962	2.148	0.0288

in close agreement with the theoretical expectations $a(0) = 0.5, b(0) = 2, \text{var}(\hat{\gamma}(0)) = 0.0507$ and $a(0.4\pi) = 1, b(0.4\pi) = 2, \text{var}(\hat{\gamma}(0.4\pi)) = 0.0253$, respectively.

The correlation coefficient between the PSD estimates at two different frequencies $\omega_1 = 0.15\pi$ and $\omega_2 = 0.4\pi$ is found to be 0.015, a negligibly low value,

Estimation of p.s.d   . . . contd.

The lack of consistency can also be explained from three viewpoints:

i. The infinitely-long ACVF is approximated by a finite-length **estimated** ACVF and the error in ACVF estimates increases with the lag (leads to Blackman-Tukey estimators).

ii. The true p.s.d. is a **smooth** function of frequency, whereas the estimated one is *erratically* fluctuating (leads to smoothers).

iii. The true p.s.d. is an **average** property, whereas the estimated one is from a single realization (leads to Welch's average periodogram estimators).

Methods for improvement

Method	Summary	Reference
Blackman-Tukey	Fourier transformation of smoothed, truncated autocovariance function	Chatfield, 1975
Smoothed periodogram	Estimate periodogram by DFT of time series; Smooth periodogram with modified Daniell filter	Bloomfield, 2000
Welch's method	Averaged periodograms of overlapped, windowed segments of a time series	Welch, 1967
Multi-taper method (MTM)	Use orthogonal windows / tapers to get approximately independent estimates of spectrum; combine estimates	Percival and Walden, 1993
Singular spectrum analysis (SSA)	Eigenvector analysis of autocorrelation matrix to eliminate noise prior to transformation to spectral estimates	Vautard and Ghil, 1989
Maximum entropy (MEM)	Parametric method: estimate acf and solve for AR model parameters; AR model has theoretical spectrum	Kay, 1988

Summary

- Periodogram is a very good and natural estimator for deterministic signals.
- Spectral leakage is an issue that arises due to finite-length effects.
    - ▶ Remedy: Either use large sample sizes or apply tapered windows to data.
- Periodogram is an **inconsistent estimator** of the p.s.d. of a **stochastic signal.**
- Smoothed / Averaged periodogram methods induce the consistency property at the cost of losing out on the ability to resolve frequencies.

Bibliography

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