CH5350: Applied Time-Series Analysis

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Spectral Representations of Random Signals

Opening remarks

We have learnt, until this point, how to represent deterministic signals (and systems) in the frequency-domain using Fourier series / transforms.

- Signal decomposition results in also energy / power decomposition (as the case maybe) by virtue of Parseval's relations.
- **Signal decomposition** is primarily useful in filtering and signal estimation, whereas **power / energy decomposition** is useful in detection of periodic components / frequency content of signals.
- Fourier analysis, as we have seen, is the key to characterizing the frequency response of LTI systems and analyzing their "filtering" nature,

Analysis of stochastic processes

Frequency-domain analysis of random signals is, however, not as straightforward, primarily because,

Fourier transforms of random signals do not exist.

- Random signals are, in general, **aperiodic**, but with infinite energy (they exist forever, by definition). They are, in fact, **power signals.**
- **Periodic** stationary random signals also exist, but, one cannot construct a Fourier series for such signals (in the usual sense) either.

Does this rule out the possibility of constructing a Fourier / spectral representation of random signals?

Fourier analysis of random signals

In the frequency-domain analysis of random signals we are primarily interested in **power / energy decomposition** rather than *signal decomposition*, because we would like to characterize the random process and **not** necessarily the realization.

Since stationary random signals (both periodic and aperiodic) are power signals, we may think of a **power spectral density** for the random process.

- However, we cannot adopt the approach used for deterministic signals.

Spectral analysis of random signals

A rigorous way of defining power spectral density (p.s.d.) of a random signal is through Wiener's **generalized harmonic analysis** (GHA). Roughly stated, the GHA is an extension of Fourier transforms / series to handle random signals, wherein the Fourier transforms / coefficients of expansion are random variables.

An alternative route takes a **semi-formal** approach to arrive at the same expression for p.s.d. as through Weiner's GHA. The most widely used route is, however, through the **Wiener-Khinchin relation.**

Three different approaches to p.s.d.

1. **Semi-formal approach:** Construct the spectral density as the ensemble average of the empirical spectral density of a finite-length realization in the limit as $N \to \infty$.

2. **Wiener-Khinchin relation:** One of the most fundamental results in spectral analysis of stochastic processes, it allows us to compute the spectral density as the Fourier transform of the ACVF. This is perhaps the most widely used approach.

3. **Wiener's GHA:** A generalization of the Fourier analysis to the class of signals which are neither periodic nor finite-energy, aperiodic signals (e.g., $\cos \sqrt{2}k$). It is theoretically sound, but also involves the use of advanced mathematical concepts, e.g., stochastic integrals.

Focus: First two approaches and the conditions for the existence of a spectral density.

Semi-formal approach

Consider a length-N sample record of a random signal. Compute the **periodogram**, i.e., the empirical p.s.d., of the finite-length realization

$$\gamma_{vv}^{(i,N)}(\omega_n) = \frac{|V_N(\omega_n)|^2}{2\pi N} = \frac{1}{2\pi N}\left|\sum_{k=0}^{N-1} v^{(i)}[k]e^{-j\omega_n k}\right|^2$$

where $V_N(\omega_n)$ is the N-point DFT of the finite length realization.

Semi-formal approach

The spectral density of the random signal is the **ensemble average (expectation)** of the density in the limiting case of $N \to \infty$

$$\boxed{\gamma_{vv}(\omega) = \lim_{N\to\infty} E(\gamma_{vv}^{(i,N)}(\omega_n)) = \lim_{N\to\infty} E\left(\frac{|V_N(\omega_n)|^2}{2\pi N}\right)}$$

The spectral density exists when the limit of average of periodogram exists.

When does the empirical definition exist?

In order to determine the conditions of existence, we begin by writing

$$|V_N(\omega_n)|^2 = V_N(\omega_n)V_N^\star(\omega_n) = \sum_{k=0}^{N-1} v[k]e^{-j\omega_n k} \sum_{m=0}^{N-1} v[m]e^{-j\omega_n m}$$

Next, take expectations and introduce a change of variable $l = k - m$ to obtain,

$$\gamma_{vv}(\omega) = \frac{1}{2\pi} \lim_{N\to\infty} \sum_{l=-(N-1)}^{N-1} f_N(l)\sigma_{vv}[l]e^{-j\omega l}, \qquad \text{where} \;\; f_N(l) = 1 - \frac{|l|}{N}$$

Conditions for existence

Now, **importantly**, assume that $\sigma_{vv}[l]$ is absolutely convergent, i.e.,

$$\sum_{l=-\infty}^{\infty} |\sigma_{vv}[l]| < \infty$$

Further, that it decays sufficiently fast, $\sum_{l=-\infty}^{\infty} |l|\sigma_{vv}[l] < \infty$.

Under these conditions, the limit converges and the p.s.d. is obtained as,

$$\boxed{\gamma_{vv}(\omega) = \sum_{l=-\infty}^{l=\infty} \sigma_{vv}[l]e^{-j\omega l}}$$

Wiener-Khinchin Theorem

Recall that the DTFT of a sequence exists only if it is absolutely convergent. Thus, the p.s.d. of a signal is defined only if its ACVF is absolutely convergent.

This leads us to the familiar **Wiener-Khinchin theorem** or the **spectral representation** theorem.

Spectral Representation / Wiener-Khinchin Theorem

W-K Theorem (Shumway and Stoffer, 2006)

Any stationary process with ACVF $\sigma_{vv}[l]$ satisfying

$$\sum_{l=-\infty}^{\infty} |\sigma_{vv}[l]| < \infty \qquad \text{(absolutely summable)}$$

has the spectral representation

$$\sigma_{vv}[l] = \int_{-\pi}^{\pi} \gamma_{vv}(\omega) e^{j\omega l}\, d\omega, \quad \text{where} \quad \gamma_{vv}(\omega) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \sigma_{vv}[l] e^{-j\omega l} \quad -\pi \leq \omega < \pi$$

$\gamma_{vv}(\omega)$ is known as the **spectral density.**

W-K Theorem: Remarks

- It is one of the milestone results in the analysis of linear random processes.
- Recall that a similar version also exists for aperiodic, finite-energy, deterministic signals. The p.s.d. is replaced by e.s.d. (energy spectral density). Thus, it provides a **unified** definition for both deterministic and stochastic signals.
- It establishes a direct connection between the second-order statistical properties in time to second-order properties in frequency domain.
- The inverse result offers an alternative way of computing the ACVF of a signal.

A more general statement of the theorem unifies both classes of random signals, the ones with absolutely convergent ACVFs and the ones with periodic ACVFs (harmonic processes).

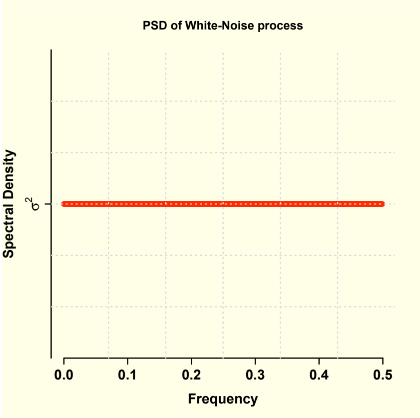
Spectral Representation of a WN process

Recall that the ACVF of WN is an impulse centered at lag $l = 0$,

The WN process is a stationary process with a constant p.s.d.

$$\gamma_{ee}(\omega) = \frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\sigma_{ee}[l]e^{-j\omega l} = \frac{\sigma_e^2}{2\pi}, \; -\pi \leq \omega \leq \pi$$

All frequencies contribute uniformly to the power of a WN process (as in white light). Hence the name.



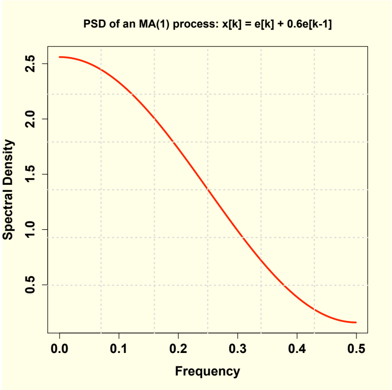
Auto-correlated processes $\equiv$ Coloured Noise

We can also examine the spectral density of AR and MA processes.

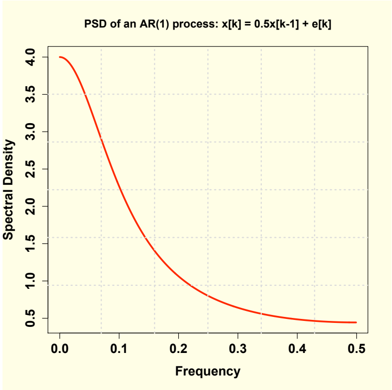
Two examples are taken up: (i) an MA(1) process and an (ii) an AR(1) process

$$\sigma_{vv}[l] = \begin{cases} 1.36 & l = 0 \\ 0.6 & |l| = 1 \\ 0 & |l| \geq 2 \end{cases} \quad \text{(MA(1))} \qquad \Bigg| \qquad \sigma_{vv}[l] = \frac{4}{3}(0.5)^{|l|} \ \forall l \quad \text{(AR(1))}$$

PSD of MA(1) and AR(1) processes



The spectral density is a function of the frequency unlike the "white" noise. Correlated processes therefore acquire the name *coloured* noise.



Obtaining p.s.d. from time-series models

The p.s.d. of a random process was computed using its ACVF and the W-K theorem. However, if a time-series model exists, the p.s.d. can be computed directly from the transfer function as:

$$\boxed{\gamma_{vv}(\omega) = |H(e^{-j\omega})|^2 \gamma_{ee}(\omega) = |H(e^{-j\omega})|^2 \frac{\sigma_e^2}{2\pi}} \tag{1}$$

$$\text{where } \; H(e^{-j\omega}) = \text{DTFT}(h[k]) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \tag{2}$$

PSD from Model

Derivation: Start with the general definition of a linear random process

$$v[k] = \sum_{m=-\infty}^{\infty} h[m]e[k-m] \qquad \Longrightarrow \sigma_{vv}[l] = \sum_{m=-\infty}^{\infty} h[m]h[l-m]\sigma_{ee}^2$$

Taking (discrete-time) Fourier Transform on both sides yields the main result.

The p.s.d. of a linear random process is $\propto$ the squared magnitude of its FRF