ACVGF

Then, one could write the general linear model in (9) as

$$v[k] = \sum_{n=-\infty}^{\infty} h[n]q^{-n}e[k] = H(q^{-1}) \tag{15}$$

where $$\boxed{H(q^{-1}) = \sum_{n=-\infty}^{\infty} h[n]q^{-n}} \tag{16}$$

is known as the **transfer function operator**

Auto-covariance generating function

ACVGF

The auto-covariance generating function is defined as

$$\boxed{g_\sigma(z) = \sum_{l=-\infty}^{\infty} \sigma_{vv}[l] z^{-l}} \tag{17}$$

where z is a variable.

ACVGF ... contd.

The key use of this ACVF generating function stems from the fact that it can be computed directly from the MA representation of the random process.

$$v[k] = H(q^{-1})e[k] \tag{18}$$

$$\Longrightarrow \boxed{g_\sigma(z) = \sigma_e^2 H(z^{-1})H(z)} \tag{19}$$

where $H(z^{-1})$ is obtained by replacing the operator q^{-1} in $H(q^{-1})$ with the variable z^{-1}

Example: ACVF of an MA(2) process

Problem: Compute the ACVF of an MA(2) process

$$v[k] = e[k] + h_1 e[k-2] + h_2 e[k-2]$$

Solution: First observe that

$$H(q^{-1}) = 1 + h_1 q^{-1} + h_2 q^{-2}$$

To compute the ACVF, construct the ACVGF by computing the product

$$\begin{aligned} g_\sigma(z) &= \sigma_e^2 H(z^{-1})H(z) = \sigma_e^2(1 + h_1 z^{-1} + h_2 z^{-2})(1 + h_1 z + h_2 z^2) \\ &= \sigma_e^2(h_2 z^{-2} + (h_1 + h_1 h_2)z^{-1} + (1 + h_1^2 + h_2^2) + (h_1 + h_1 h_2)z + h_2 z^2) \end{aligned}$$

ACVGF of an MA(2) process

Comparing with equation (17) and reading off the coefficients of z^{-l}, we obtain

$$\sigma_{vv}[l] = \begin{cases} (1 + h_1^2 + h_2^2)\sigma_e^2, & l = 0 \\ (h_1 + h_1 h_2)\sigma_e^2, & l = 1 \\ h_2 \sigma_e^2, & l = 2 \\ 0, & |l| \geq 3 \end{cases} \tag{20}$$

Thus, as expected, the ACVF of an MA(2) process vanishes at all lags $|l| > 2$ $\square$

Auto-Regressive (AR) processes: ACF

The second class of processes that we consider are the **auto-regressive (AR) processes**

For illustration, consider a first-order, i.e., AR(1) process:

$$v[k] = -d_1 v[k-1] + e[k] \tag{21}$$

where $e[k]$ is the zero-mean GWN process of variance σ_e^2 and d_1 is a finite constant.

- The current state is a linear function of the past state plus the unpredictable $e[k]$
- Assume $|d_1| < 1$ (a condition required for stationarity of $v[k]$)

ACF of an AR(1) process ...contd.

The theoretical ACF can be now obtained using the definition in (2)

Observe that $\mu_e = 0 \implies \mu_v = 0$.

$$\begin{aligned}\sigma_{vv}[l] &= E(v[k]v[k-l]) \\ &= -d_1 E(v[k-1]v[k-l]) + E(e[k]v[k-l]) \\ &= \phi_1 \sigma_{vv}[l-1] + \sigma_{ev}[l]\end{aligned}$$

where $\sigma_{ev}[l]$ is the cross-covariance function, *i.e.*, the covariance between $e[k]$ and $v[k-l]$ (see the definition of CCVF shortly)

ACF of an AR(1) process ...contd.

- By symmetry property of $\sigma_{vv}[l]$, it is sufficient to work out the derivation for $l \geq 0$. To complete the derivation, we first evaluate $\sigma_{ev}[l]$ for $l \geq 0$.
- A careful examination of (21) reveals that $v[k-l]$ contains effects of only past $e[k]$. By definition of WN, therefore, $\sigma_{ev}[l] = 0, l > 0$.

ACF of an AR(1) process ...contd.

To obtain $\sigma_{ev}[0]$, multiply both sides of (21) with $e[k]$ and take expectations on both sides to yield,

$$\begin{aligned} E(e[k]v[k]) &= -d_1 E(e[k]v[k-1]) + E(e[k]e[k]) \\ &= \sigma_e^2 \end{aligned}$$

using the same arguments as above. Thus, we have the following set of equations

$$\begin{aligned} \sigma_{vv}[0] &= -d_1\sigma_{vv}[-1] + \sigma_{ev}[0] \\ &= -d_1\sigma_{vv}[1] + \sigma_e^2 \\ \sigma_{vv}[1] &= -d_1\sigma_{vv}[0] \end{aligned}$$

ACF of an AR(1) process ...contd.

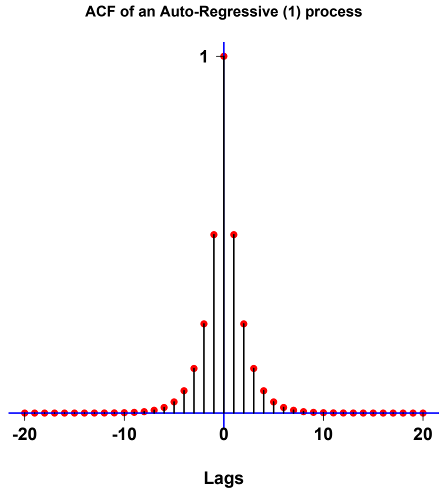
Solving equations for $\sigma_{vv}[0]$ and $\sigma_{vv}[1]$ simultaneously gives

$$\boxed{\begin{aligned} \sigma_{vv}[0] &= \frac{\sigma_e^2}{1-d_1^2} \\ \rho_{vv}[l] &= (-d_1)^{|l|} \;\; \forall \;\; |l| \geq 1 \end{aligned}} \tag{22}$$

ACF of an AR(1) process ...contd.

- Shown adjacent is the plot of the ACF of an AR(1) process with $d_1 = -0.5$
- In general wheenver $|d_1| < 1$, we have that

$$\boxed{\textbf{The ACF of an AR(1) process exhibits exponential decay}}$$



Summary

- The ACF measures linear dependencies between observations of a time-series
- For a stationary process, the ACF is a symmetric function
- The ACF coefficients at any lag determine the optimal linear model for $x[k]$ in terms of its past.
- For an MA(M) process, the ACF abruptly vanishes after lags $|l| > M$.
- For an AR(P) process, the ACF dies down only exponentially.
  - The ACF satisfies the same difference equation as the random process itself