CH5350: Applied Time-Series Analysis

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Fourier Transforms for Deterministic Signals

Recap

- Correlation structure (predictability) of stationary processes is characterized by the auto-covariance function.
- A linear random process representation can be constructed only if the spectral density exists and if it satisfies the Wiener-Paley condition.
- Parametrization of the IR sequence (or the ACVF) of a linear random process leads to MA(M), AR(P) or ARMA processes.
- Trend non-stationarities are handled by applying suitable filters
- Seasonalities are detected by peaks in "spectral" plots.

Motivation

- What is meant by **spectral density?**
- Stationary processes with periodic ACVFs do not have a linear convolution form - how do we describe them?
- How do we define periodic random processes?
- Is there a method to detect periodicities embedded in noise?
- Is it possible to construct a unified spectral representation for a stationary random process?

Frequency-domain analysis

Frequency-domain characterizations of processes, also known as **spectral representations** offer a powerful framework for both a theoretical and practical analysis of random processes

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The term "spectral representation" stems from the term "spectrum" which, in signal analysis stands for a function of energy/power in the frequency domain.

What does a spectral representation mean?

Spectral representation provides a decomposition of the power / energy of the process in the frequency-domain.

In understanding this topic, we shall seek answers to several questions:

- What is the mathematical definition of spectrum?
- Is there a difference between energy and power of a signal?
- What is the utility of a spectral decomposition?
- What does spectral representation mathematically look like?
- Can any random process be given a spectral representation?
- What are the connections between time-domain and (frequency) spectral representations of a process?
- ...

Fourier transform is the main tool

The main tool for carrying out a frequency-domain analysis of signals / processes is the **Fourier transform.**

- In order to understand the various aspects of Fourier transforms and its applications to signal analysis, it is useful to first gain an understanding of how **deterministic processes** are treated in the frequency domain.
- Further, we shall categorize deterministic signals into four classes, namely, **continuous-time and discrete-time**, periodic and aperiodc signals.
- It is important to first understand quantities such as **energy**, **power** and their **densities** in the context of signal analysis.

Energy signal

Energy

The energy of a continuous-time signal $x(t)$ and a discrete-time signal $x[k]$ are, respectively, defined as,

$$\boxed{E_{xx} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \; ; \qquad\qquad E_{xx} = \sum_{-\infty}^{\infty} |x[k]|^2} \tag{1}$$

A signal with finite energy, i.e., $0 < E_{xx} < \infty$ is said to be an *energy signal*[a].

[a] The squared modulus is introduced to accommodate complex-valued signals.

Examples: exponentially decaying signals, all finite-duration bounded amplitude signals

Power signal

Power

The *average* power of a continuous-time signal $x(t)$ and a discrete-time signal $x[k]$ are, respectively, defined as,

$$\boxed{P_{xx} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \; ; \qquad P_{xx} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{k=N} |x[k]|^2} \tag{2}$$

A signal with finite power, i.e., $0 < P_{xx} < \infty$ is said to be a *power signal*.

Power signal ...contd.

Examples: periodic signals, random signals

All finite-duration (and amplitude) signals have $P_{xx} = 0$. *In general, any energy signal is not a power signal and vice versa*. However, it is possible that a signal is neither an energy nor a power signal.

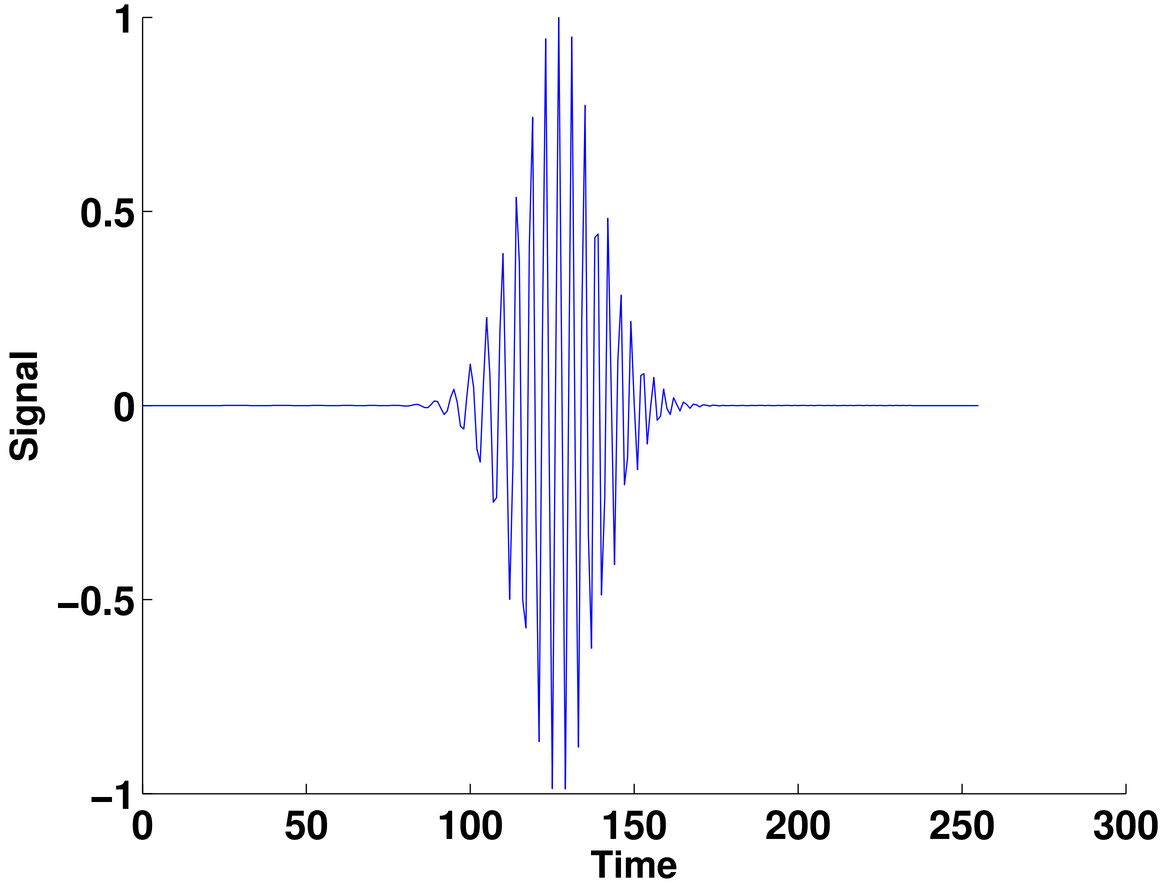
Energy Density

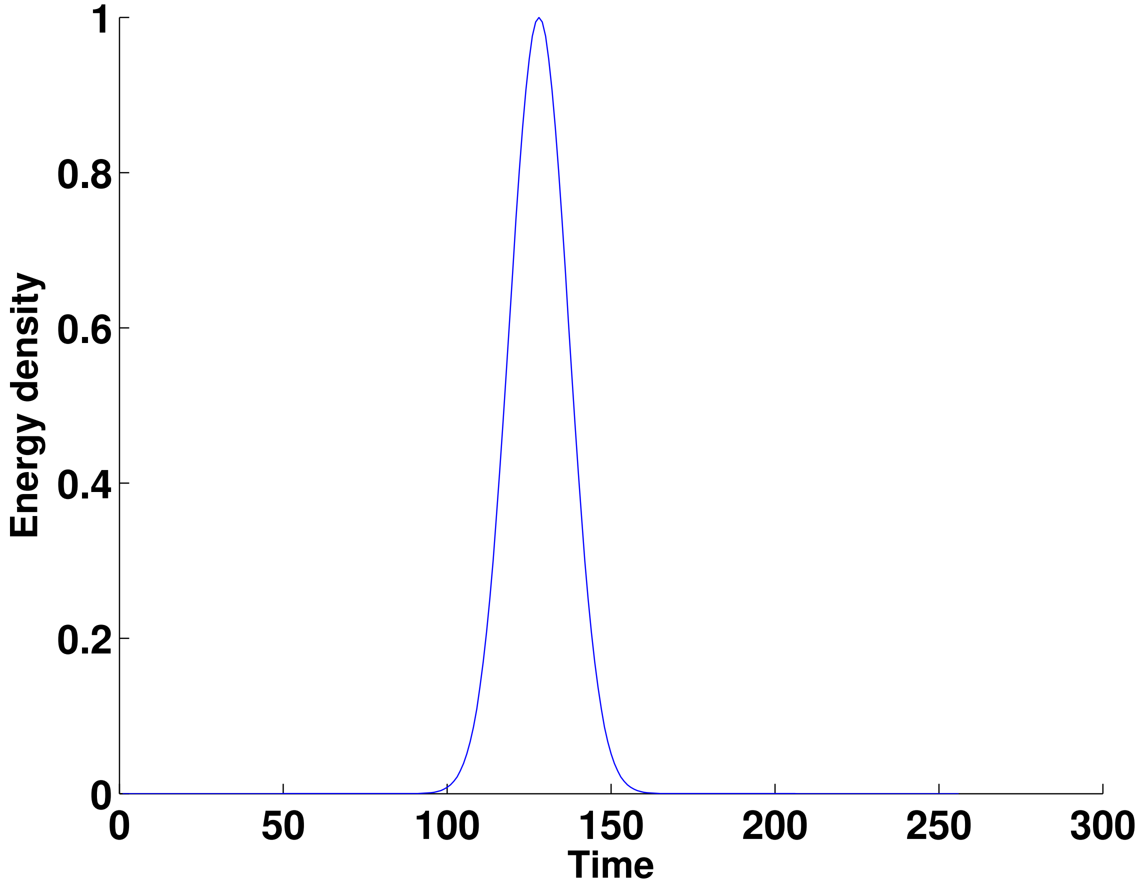
Equation (1) gives rise to the idea of an *energy density* in *time*. Drawing analogies with probability density function and densities in mechanics, the quantity

$$S_{xx}(t) = |x(t)|^2 \tag{3}$$

is termed as the energy density per unit time. It can also be thought of as an "instantaneous" power.

Energy density: Example





Power Density

Similarly, the *power density in time* can be defined as

$$\gamma_{xx}(t) = \frac{|x(t)|^2}{T} \tag{4}$$

- **For the discrete-time case, the energy and power density in time are not defined** since the time domain is not a continuum. The distribution functions exist nevertheless.

Power density ...contd.

On the other hand, we can think of energy and power densities of d.t. signals in a **transform domain**, provided that the new domain is **continuous** and that the transform is energy / power preserving.

This is the basis for defining spectral densities of c.t. and d.t. signals in the Fourier (frequency) domain.

The energy / power densities in frequency domain share a strong connection with the time-domain characteristics (properties) of the signal, specifically the **covariance functions.**

Cross-covariance function

The **cross-covariance function (CCVF) is a measure of the linear dependence** between two time-lagged (random or deterministic) signals.

- Based on the notion of **covariance**, a quantity that measures the linear dependence between two zero-lagged deterministic signals (or two random variables).
- A normalized version known as, **cross-correlation function** (CCF), is more suitable for analysis since it is invariant to the choice of units (for signals).

Caution: It is a common practice in signal processing to use the alternative terms cross-correlation and normalized cross-correlation, for CCVF and CCF, respectively.

CCVF for periodic signals

The cross-covariance function between two *zero-mean, periodic deterministic* signals $x_p[k]$ and $y_p[k]$ with a *(least) common period* N_p is defined as

$$\boxed{\sigma_{x_p y_p}[l] = \frac{1}{N_p} \sum_{k=0}^{N_p-1} x_p[k] y_p[k-l]} \tag{5}$$

CCVF for periodic signals

- The (normalized) cross-correlation function is defined as

$$\boxed{\rho_{x_p y_p}[l] = \frac{\sigma_{x_p y_p}[l]}{\sqrt{\sigma_{x_p x_p}[0]\sigma_{y_p y_p}[0]}}} \tag{6}$$

- Observe that by setting $x_p = y_p$ and $l = 0$ in (5), we obtain the **average power** of the periodic signal.

CCVF for aperiodic signals

The cross-covariance function between two *aperiodic deterministic, energy* signals $x[k]$ and $y[k]$ is defined as

$$\sigma_{xy}[l] = \sum_{k=-\infty}^{\infty} x[k]y[k-l] \tag{7}$$

CCVF for aperiodic signals

As before,

- The (normalized) cross-correlation function is defined as

$$\boxed{\rho_{xy}[l] = \frac{\sigma_{xy}[l]}{\sqrt{\sigma_{xx}[0]\sigma_{yy}[0]}}} \tag{8}$$

- Observe that by setting $x = y$ and $l = 0$ in (7), we obtain the **energy** of the aperiodic signal.

Properties and uses of CCVF

The CCVF has a few, but very useful, properties and is one of the most widely used time-domain signal analysis tools:

- The CCVF measures the linear dependence between $x[k]$ and time-shifted $y[k]$ (by l samples). This property is used in testing linear relationships between two signals.
- It is **asymmetric**, i.e., $\sigma_{xy}[l] \neq \sigma_{xy}[-l]$ (Why?).
  The asymmetric property is used in estimating time-delays between signals (by searching for peaks in the CCFs).
- The CCVF specializes to auto-covariance function (ACVF) for univariate signals, which is a widely used tool for **periodicity detection** and **echo cancellation**.

Auto-covariance functions

The ACVFs of periodic and (finite-energy) aperiodic deterministic signals are, respectively,

$$\boxed{\sigma_{x_p x_p}[l] = \frac{1}{N_p}\sum_{k=0}^{N_p-1} x_p[k]x_p[k-l]; \qquad \sigma_{xx}[l] = \sum_{k=-\infty}^{\infty} x[k]x[k-l]} \tag{9}$$

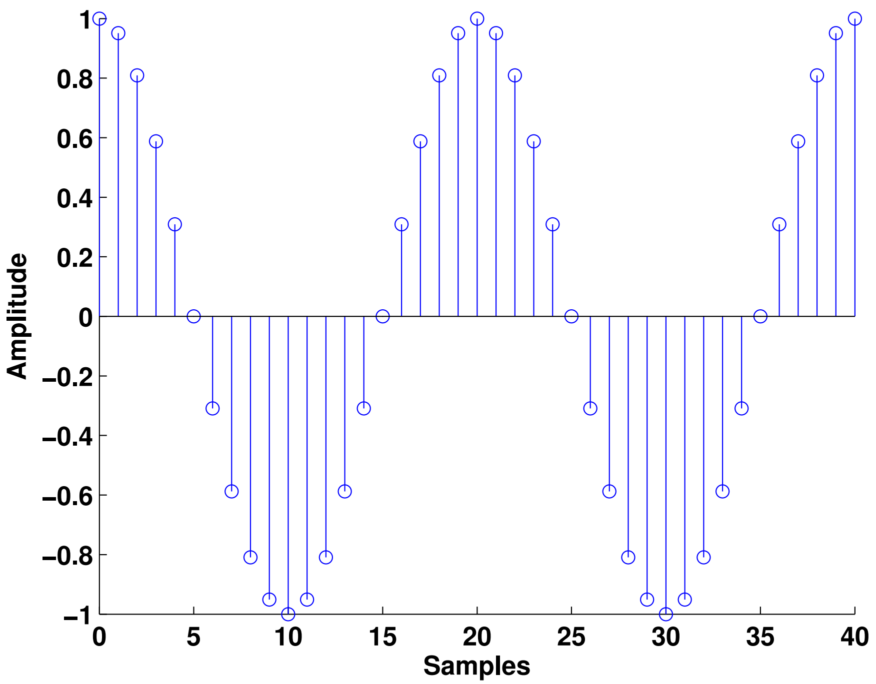
Auto-covariance functions . . . contd.

- **Unlike the CCVF**, the ACVF is a **symmetric** function.
- As before, normalized versions can be defined to obtain the respective ACFs.
- **The ACF inherits the characteristics of the signal.** For instance, **the ACVF of a periodic signal is also periodic with the same period.**

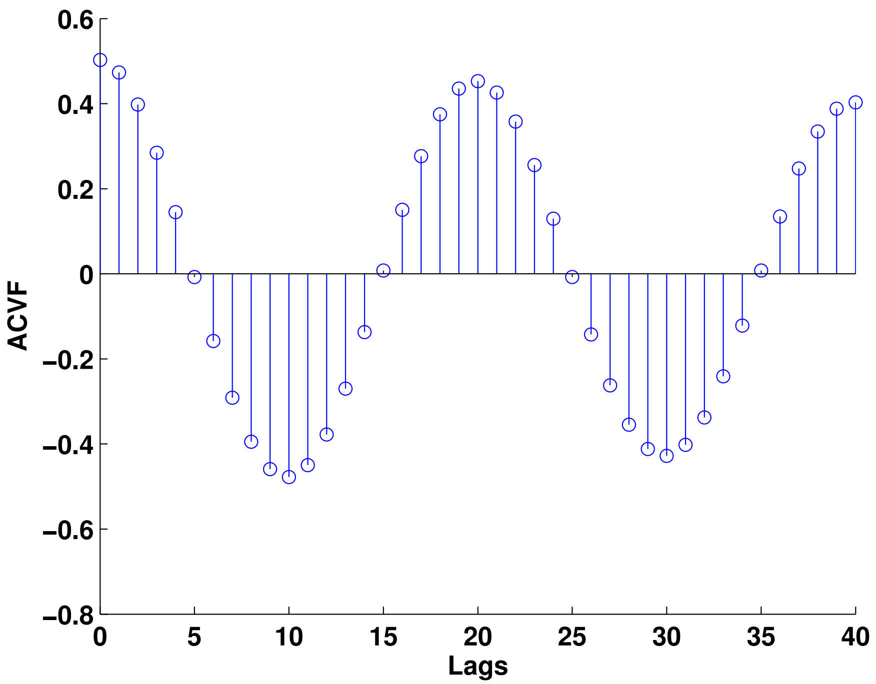
$$\boxed{\sigma_{x_p x_p}[l + N_p] = \sigma_{x_p x_p}[l]} \tag{10}$$

Example: Periodic signal

$$x_p[k] = \cos(2\pi f k) \qquad \Longrightarrow \qquad \sigma_{x_p x_p}[l] = \frac{1}{2}\cos(2\pi f l)$$



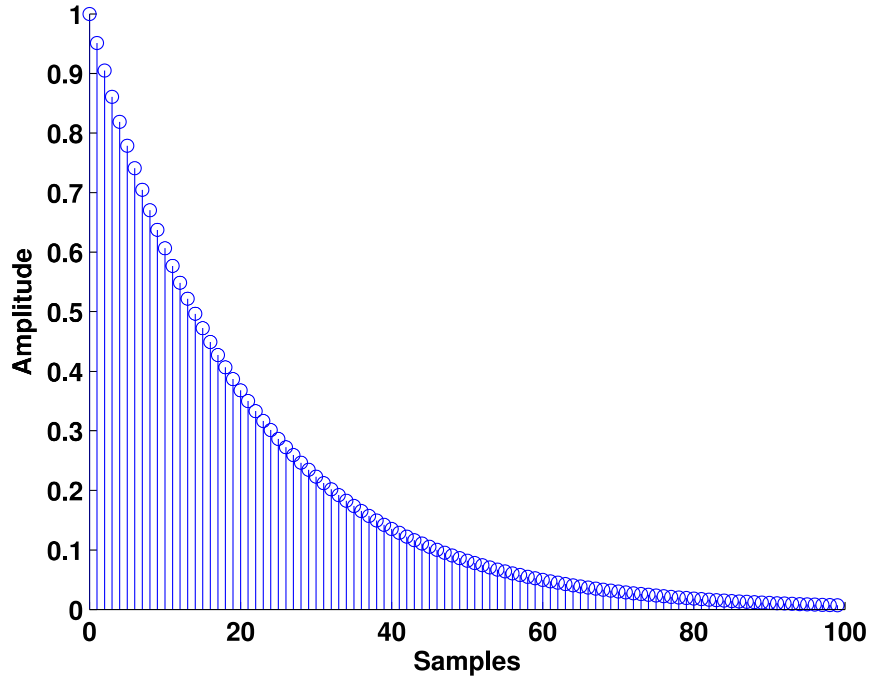
(a) Snapshot of the cosine



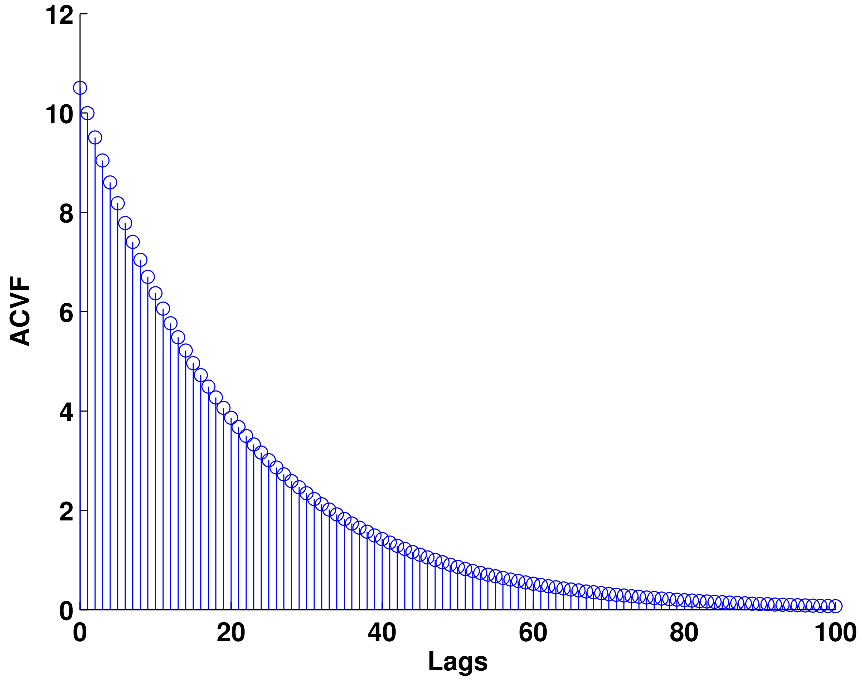
(b) ACVF of the cosine

Example: Aperiodic signal

$$x[k] = \begin{cases} e^{\alpha k}, & k \geq 0 \quad , \alpha < 0 \\ 0, & k < 0 \end{cases} \qquad \Longrightarrow \qquad \sigma_{xx}[l] = \frac{e^{\alpha l}}{1 - e^{2\alpha}}$$



(c) Exponential signal



(d) ACVF of signal

Fourier World (Series & Transforms)

Transform: Synthesis and Analysis

Every transform consists of a

i. **Synthesis equation:** Mathematical imagination of how the signal is possibly constructed from a family of *building blocks* (atoms).

ii. **Analysis equation:** Allows us to determine "which" members of the family have participated in the signal synthesis through a decomposition.

Qs: Which atoms (functions)? Is the decomposition unique, Is perfect recovery possible?

Transforms: Analysis and Filtering

The motivation of every transform is ease of analysis in the new domain.

The type of transform and approach depends on the objective:

- **Analysis:** Starting with **signal decomposition**, one proceeds to **energy / power decomposition**.
- **Filtering:** Signal is decomposed, operation(s) is / are performed in the transform domain and finally (a modified signal is) reconstructed using the synthesis equation.

A mix of both may be required in several applications.

Fixed vs. Adaptive Basis

When the building blocks are fixed a priori and independent of the signal, the transform is said to be built on **fixed basis.**

`Examples: Fourier, Wavelet transforms.`

On the other hand, when these building blocks are derived from data, the transform is said to work with **adaptive basis.**

`Examples: Wigner-Ville distributions, Principal component analysis.`

General ideas

- **Idea**: Breakdown a signal into weighted combinations of sinusoids with different frequencies:
  - Similar to expressing signals as a combination of impulses
- **Significance**
  - Weights or coefficients are in general complex
  - Magnitude of weights give the energy or power of that frequency component in the signal - Spectral Analysis
  - Angle of coefficients give how much each sinusoid (at that frequency) in the signal is aligned with respect to the basis sinusoids. Useful in time-delay or lag estimation.

Correlation perspective

Every transform of a signal can be viewed as a correlation of that signal with the analyzing function. The coefficient of transform is the "amount" of correlation or similarity of that signal with the analyzing (basis) function.

Remarks

- The breaking down of a signal is equivalent to finding the best set of sinusoids that can explain the pattern in that signal.
- In Fourier analysis, each analyzing function is a (complex) sinusoid of a certain frequency.
- The coefficient at each frequency is a measure of similarity between the signal and the sinusoid
- The complex sinusoid is chosen so as to capture shift and magnitude in a convenient manner

Continous-time Fourier Series (CTFS)

Continuous-time periodic signals: Synthesis equation

Idea: A continuous-time periodic signal with fundamental period $T_0 = 1/F_0$ is expressed as a (linear) weighted combination of (positive and negative) harmonics:

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n F_0 t}} \qquad \text{(Fourier Series)} \qquad (11)$$

Note: The summation in (11) includes both negative and positive frequencies!

Continuous-time Fourier series: Analysis equation

The coefficient c_n, is in general **a complex quantity**, and is calculated as:

$$\boxed{c_n = \frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi nF_0 t}\, dt = \frac{\langle x(t), e^{j2\pi nF_0 t}\rangle_{[0,T_p]}}{\langle e^{j2\pi nF_0 t}, e^{j2\pi nF_0 t}\rangle_{[0,T_p]}}} \tag{12}$$

- Generally useful in theoretical analysis of signals and systems

Continuous-time Fourier series

Variant	Synthesis / analysis	Parseval's relation (power decomposition) and signal requirements
Fourier Series	$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n F_0 t}$ $c_n \triangleq \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi n F_0 t}\, dt$	$P_{xx} = \frac{1}{T_p} \int_0^{T_p} \|x(t)\|^2\, dt = \sum_{n=-\infty}^{\infty} \|c_n\|^2$ $x(t)$ is periodic with fundamental period $T_p = 1/F_0$

Why do negative frequencies come in?

- The need for including negative frequencies is purely mathematical. Consider, for e.g.,

$$\sin(2\pi F_0 t) = \frac{1}{2j}(e^{j2\pi F_0 t} - e^{-j2\pi F_0 t})$$

  Observe that two exponentials, one with a positive frequency F_0 and the other with a negative frequency $-F_0$ are required to explain a sinusoid
- The corresponding coefficients are c_1 $(k = 1)$ and c_{-1} are $\frac{1}{2j}$ and $-\frac{1}{2j}$
- In general, Fourier series / transform involves expressing any signal as addition and subtraction of cosines / sines.

Power spectrum

Fourier series (and transform) is concerned with a **signal decomposition**, but gives rise to a more important result - the power (spectral) decomposition of the signal.

- A periodic signal has infinite energy, but finite power given by

$$P_{xx} = \frac{1}{T_p} \int_0^{T_p} |x(t)|^2 \; dt$$

Using the signal decomposition in a Fourier series, we can break up the average power into contributions from respective frequencies

Power spectral decomposition of CT periodic signals

The average power can be broken up as

$$\boxed{P_{xx} = \frac{1}{T_p}\int_0^{T_p} |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2} \qquad \text{(Parseval's relation)} \tag{13}$$

- $P_n = |c_n|^2$ is the contribution by the n^{th} harmonic and hence known as the power spectral density or simply *power spectrum*
- **The power in a periodic signal exists only at discrete frequencies. Hence the power spectrum is also known as line spectrum.**

Power and phase spectrum: Remarks

- Since $c_n = |c_n|e^{j\theta_n}$, c_0 represents the average component of the signal
- The power spectral density plot is independent of (or blind to) the phase
  - *Two signals having two different phases but same strengths will have identical power spectral densities*
- For a real-valued signal, $c_n^* = c_{-n}$ $\Longrightarrow$ Power spectrum of any measurement is symmetric
- As $T_p \to \infty$, $x(t)$ becomes an *aperiodic* signal and frequency spacing tends to zero.
- **Phase**: $\theta_n = \angle c_n$. A plot of θ_n vs. n shows how each frequency component is aligned w.r.t the basis functions

Example

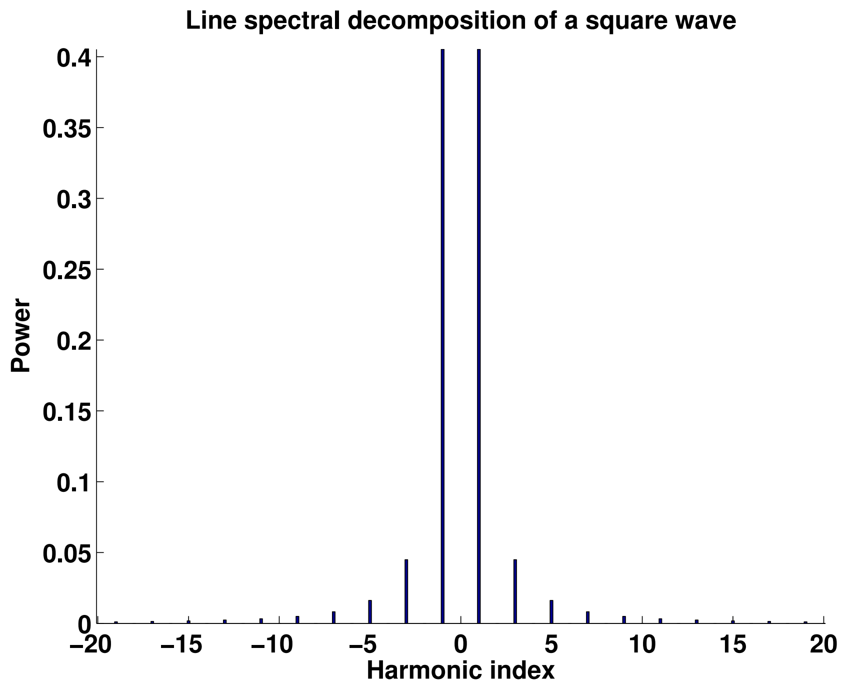
The Fourier series representation of the periodic square wave

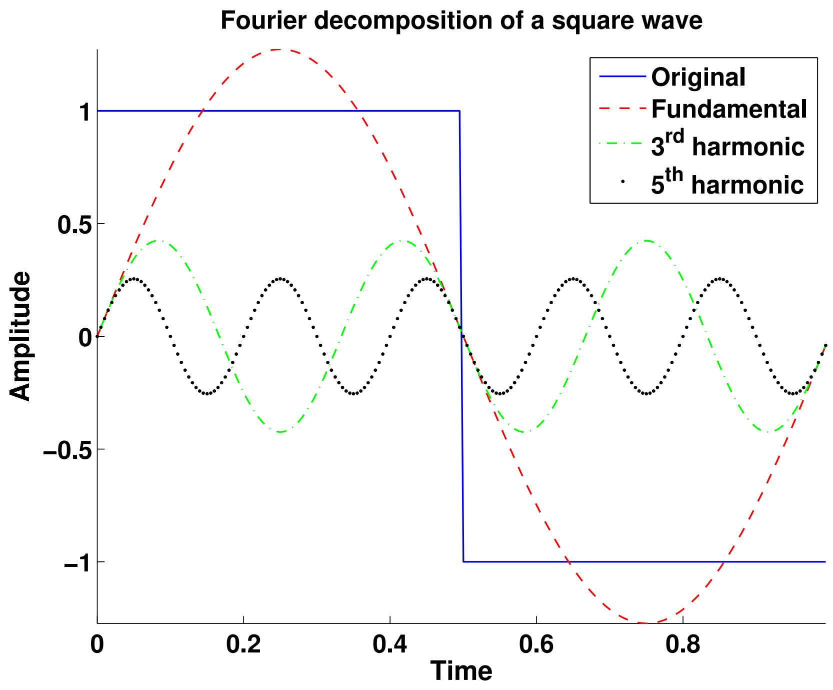
$$x(t) = \begin{cases} 1, & 0 \leq t \leq 1/2 \\ -1, & 1/2 < t \leq 1 \end{cases} \tag{14}$$

with period $T_p = 1$ is given by the coefficients

$$\begin{aligned} c_n = \frac{1}{T_p}\int_0^1 x(t)e^{-j2\pi nt}\,dt &= \int_0^{1/2} e^{-j2\pi nt}\,dt - \int_{1/2}^1 e^{-j2\pi nt}\,dt \\ &= j\sin\left(\frac{n\pi}{2}\right)\sin c\left(\frac{n\pi}{2}\right)e^{-jn\pi} \end{aligned}$$

Example contd.





- Line spectrum is plotted as a function of the harmonic index. Observe that it is symmetric and that only odd harmonics contribute to the signal.
- The signal decomposition shown above is purely mathematical.

Existence of Fourier series

- The coefficients c_n exist iff the signal $x(t)$ is absolutely convergent in $[0, T_p]$, i.e., $x(t) \in L^1(0, T_p)$.
- On the other hand, the series converges to $x(t)$ if it is continuous and of bounded variation in $[0, T_p]$. For discontinuous signals with finite extrema and finite number of discontinuities, the series converges to the average value of the left and right limits . This is termed as the *Gibbs phenomenon*
- Sufficiency conditions for any function $f(t)$ to possess a Fourier series expansion was established by Dirichlet and are popularly known as *Dirichlet conditions* (Priestley, 1981).

Existence of Fourier series

- A weaker requirement is that $x(t)$ has a finite 2-norm in the interval $(0, T_p)$. Then, the summation

$$x_M(t) = \sum_{n=-M}^{M} c_n e^{-j2\pi n F_0 t} \tag{15}$$

converges to $x(t)$ in the MS sense, i.e.,

$$\lim_{M \longrightarrow \infty} \int (x(t) - x_M(t))^2 \, dt = 0 \tag{16}$$

Continuous-time Fourier Transform (CTFT)

Opening remarks

- The signals of interest are *continuous-time, aperiodic* signals.
- Aperiodic signals can be viewed as a limiting case of periodic signals with infinite (practically very large) period, i.e., $T_p \to \infty$.
- Consequently, the spacing on **frequency axis**, $\triangle F = F_0 = 1/T_p$ now shrinks to zero, leading to a **continuum** of frequencies.
- The class of deterministic aperiodic signals under consideration are finite 1-norm and finite (2-norm) energy signals. Why?
- The line (power) spectrum is, therefore, replaced by an **energy spectral density**.

CT aperiodic signals: Synthesis equation

The aperiodic signal is imagined to be synthesized as

$$\boxed{x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi F t}\, dF \qquad \text{(Fourier Synthesis)}} \tag{17}$$

Fourier transform: Analysis equation

The "coefficient" $X(F)$ is computed using the analysis equation,

$$\boxed{X(F) \triangleq \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}\, dt \qquad \text{(Fourier Analysis)}} \qquad (18)$$

- The result $X(F)$ is known as the **Fourier transform** of $x(t)$, and has a similar interpretation as of c_n, the Fourier coefficient in Fourier series.
- As with Fourier series, the transform is useful in *theoretical* analysis of signals and systems.

Continuous-time Fourier transform

Variant	Synthesis / analysis	Parseval's relation (energy decomposition) and signal requirements
Fourier Transform	$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft}\, dF$ $X(F) \triangleq \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft}\, dt$	$E_{xx} = \int_{-\infty}^{\infty} \|x(t)\|^2\, dt = \int_{-\infty}^{\infty} \|X(F)\|^2\, dF$ $x(t)$ is aperiodic; $\int_{-\infty}^{\infty} \|x(t)\|\, dt < \infty$ or $\int_{-\infty}^{\infty} \|x(t)\|^2\, dt < \infty$ (finite energy, weaker requirement)

Conditions for existence of Fourier transform

- The Fourier transform is guaranteed to exist if the signal $x(t)$ is absolutely integrable,

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$$

- For the signal to be recovered uniquely (or the average value at points of discontinuity), $x(t)$ should have bounded variation at all points.

Conditions for existence of Fourier transform . . . contd.

- A weaker and a mathematically useful requirement is that the 2-norm of the signal be finite, i.e., $x(t) \in \mathcal{L}^2$. Almost all finite energy signals have a Fourier transform.
- The theory of generalized functions relaxes some of the above restrictions and also allows us to compute Fourier transforms of idealized functions, e.g., impulse. (Antoniou, 2006; Lighthill, 1958).

Energy spectral decomposition

The signal decomposition by Fourier transform can be shown to yield an energy decomposition in the frequency domain by virtue of Parseval's result.

$$\boxed{E_{xx} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(F)|^2 \, dF \qquad \text{(Energy decomposition)}} \tag{19}$$

Energy spectral density

- Thus, **energy is preserved by the transform**. A more general result is the preservation of inner products.
- The quantity $|X(F)|^2$ is a continuous function of the frequency and can be given the interpretation of an **energy spectral density**.
- Alternatively, $|X(F)|^2 dF$ measures the energy contributions of the frequency components within the band $(F, F + dF)$ to the total energy of the signal.

Example 1: Finite duration pulse

The Fourier transform of the **finite duration** rectangular pulse signal

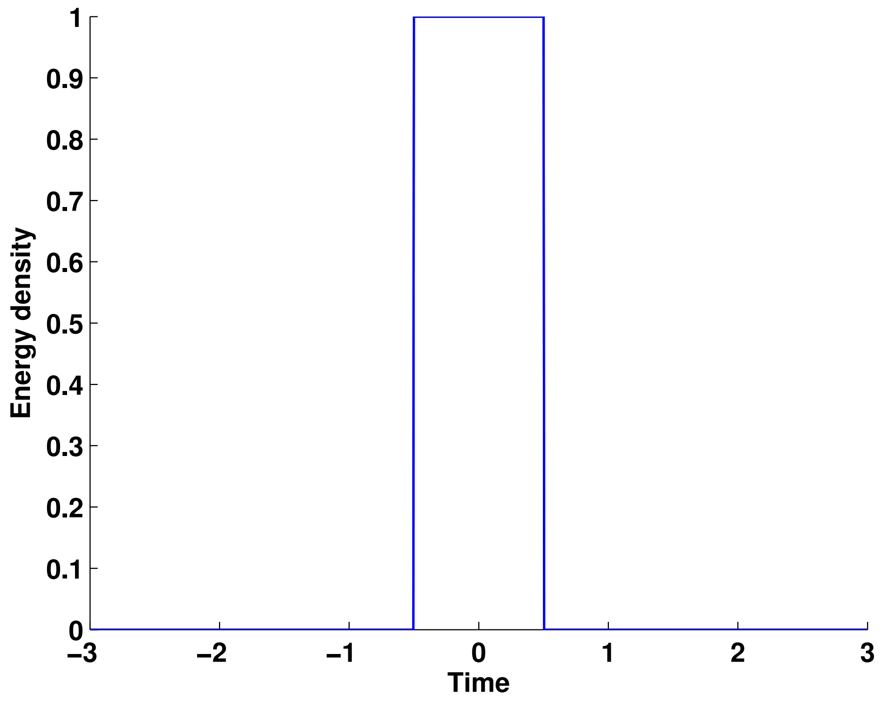
$$x(t) = A\Pi\left(\frac{t}{T}\right) = \begin{cases} A & |t| < T/2 \\ 0 & \text{otherwise} \end{cases}$$

is given by

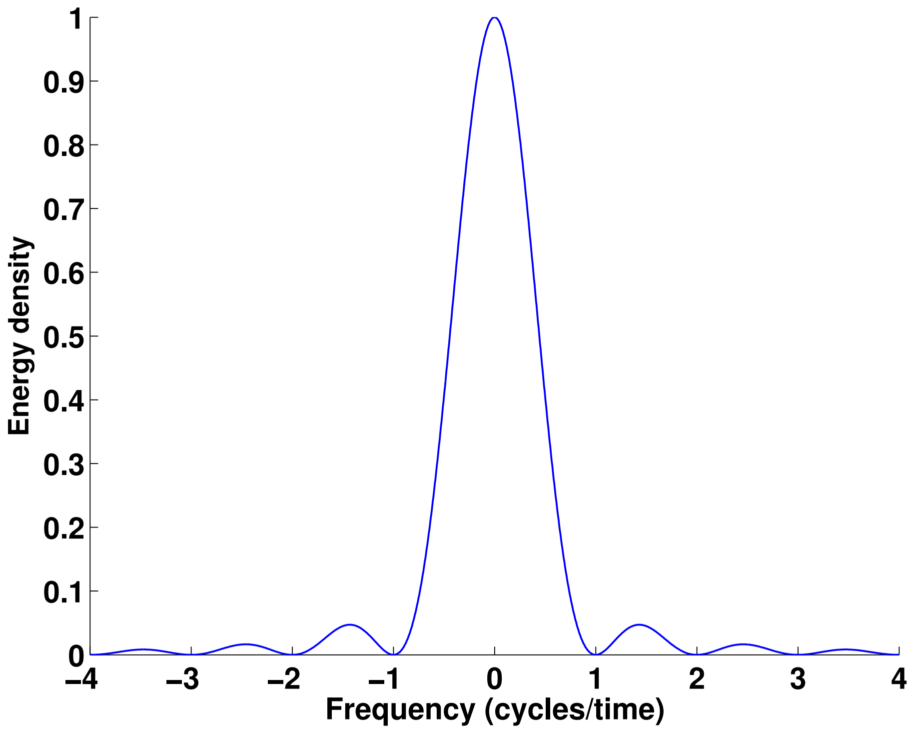
$$X(F) = \int_{-T/2}^{T/2} Ae^{-j2\pi Ft}\, dt = A\left(\left.\frac{e^{-j2\pi Ft}}{-j2\pi F}\right|_{-T/2}^{T/2}\right) = AT\frac{\sin(\pi FT)}{\pi FT} = AT\text{sinc}(\pi FT)$$

Thus, **the finite-duration pulse has an infinitely long Fourier transform.**

Example 1 contd.



(e) Energy density in time



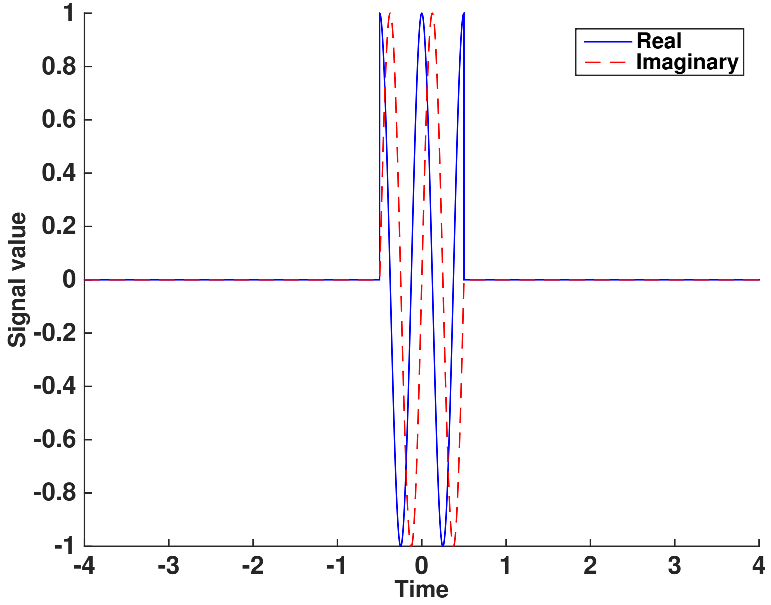
(f) Energy spectral density

Finite-duration signal has an infinitely-spread energy density in frequency.

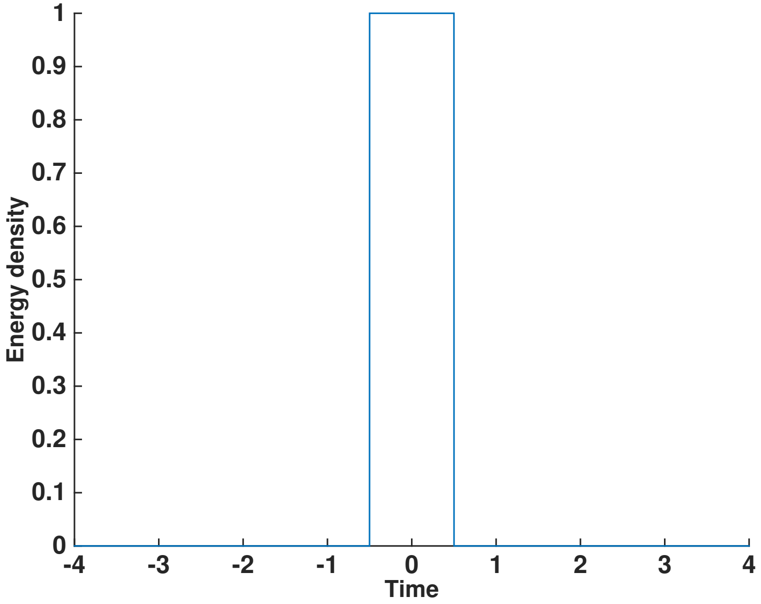
Example 2: Finite duration complex sine

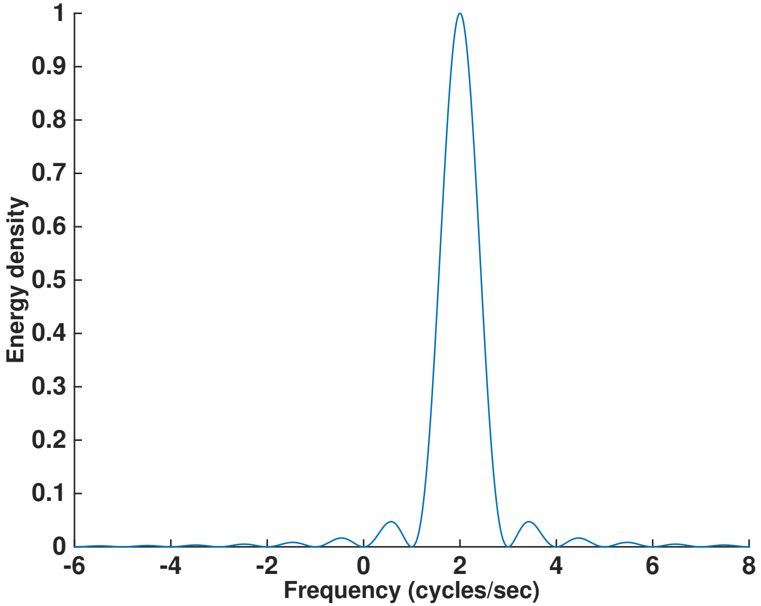
When $x(t) = \begin{cases} Ae^{j2\pi F_0 t} & |t| < T/2 \\ 0 & \text{otherwise} \end{cases}$

Therefore, $X(F) = AT\text{sinc}(\pi(F - F_0)T)$

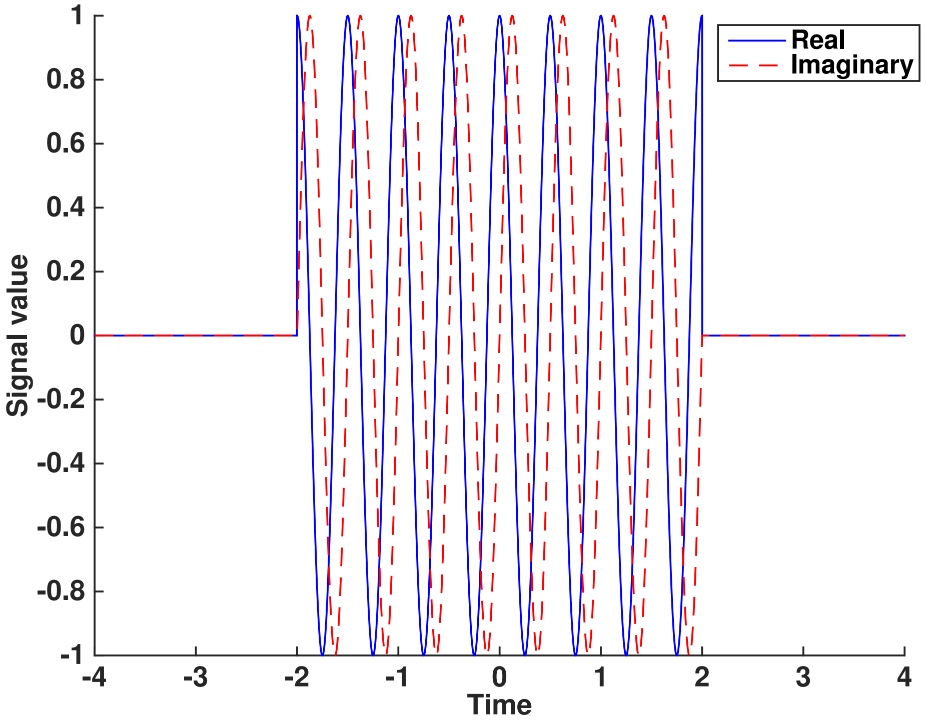


$F_0 = 2\,\text{Hz}, T = 1, A = 1$

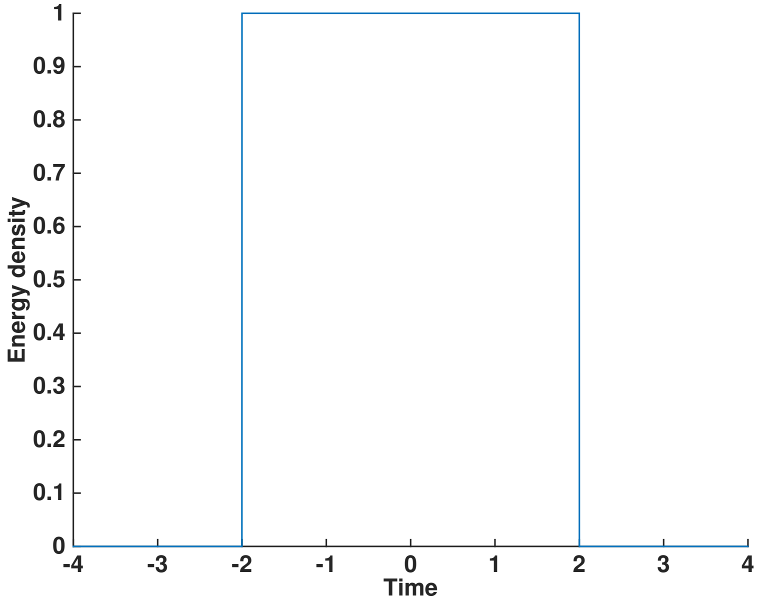


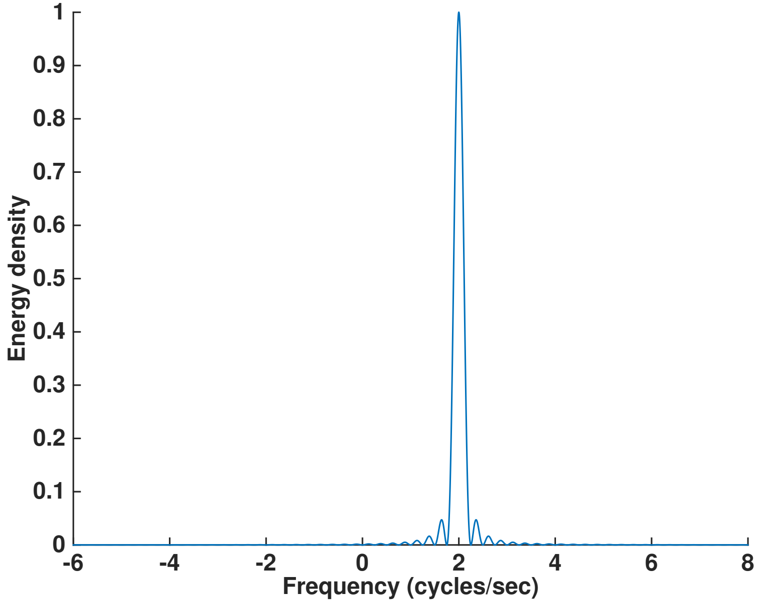


Longer finite-duration sine wave



$$F_0 = 2\,\text{Hz}, T = 4, A = 1$$





- **Longer duration** results in **narrower frequency spread** (bandwidth)

Duration and Bandwidth are tied together

All finite-duration signals have Fourier transforms that are infinitely long and vice versa. The fundamental **duration-bandwidth principle** places a lower bound on the product of the energy spreads in both domains

$$\sigma_t^2 \sigma_F^2 \geq 1/4 \tag{20}$$

where the spreads σ_t^2 and σ_F^2 are the second-order central moments of the energy densities in time and frequency, respectively (Cohen, 1994)

- The quantities σ_t and σ_F are known as the *duration* and *bandwidth*, respectively. This result has profound implications in the joint time-frequency analysis of signals.

Fourier-Stieltjes transform

The Fourier-Stieltjes transform fuses the Fourier series and transform into a single integral.

The basic idea is to re-write (17) by introducing $dX(F) = X(F)dF$ as

$$\boxed{x(t) = \int_{-\infty}^{\infty} e^{j2\pi Ft} dX(F)} \tag{21}$$

Equation (21) is known as **Fourier-Stieltjes transform.**

Fourier-Stieltjes transform

In order to accommodate periodic functions, i.e., the Fourier series, we allow $dX(F)$ to be piecewise continuous, specifically, an impulse train function so that

$$dX(F) = \begin{cases} c_n, & F = F_n,\ n \in \mathbb{Z} \\ 0, & \text{elsewhere} \end{cases}$$

It facilitates frequency-domain representations for signals (functions) that are neither periodic nor absolutely integrable, but have bounded amplitudes, e.g., random signals.