CH5350: Applied Time-Series Analysis

Arun K. Tangirala

Department of Chemical Engineering, IIT Madras

Auto-correlation functions

Opening remarks

A prime goal of TSA is **prediction** (forecasting). Therefore, it is important to test for predictability of a series prior to any model development exercise.

Predictability is a generic term and can largely depend on what model is being considered for prediction. Most of the established theory revolves around **linear models** for predictions, which serve a large number of applications.

Recall that covariance (or correlation) is a measure of linear dependence between two random variables. **We shall now apply this concept to two observations of a series so as to test for linear dependence within that series.**

Auto-covariance function (ACVF)

The **auto-covariance function** (ACVF) is defined as the covariance between two observations of a series, $v[k_1]$ and $v[k_2]$

$$\boxed{\sigma_{vv}[k_1, k_2] = E((v[k_1] - \mu_{k_1})(v[k_2] - \mu_{k_2}))} \tag{1}$$

where μ_{k_i} is the mean of the process at k_i instant.

Note: We have switched our notation from $x[k]$ to $v[k]$ and used $v[k_i]$ to indicate both the observation as well as the associated RV.

ACVF of stationary processes

For stationary processes, recall that the mean remains invariant and the distribution is only a function of the time difference or lag, $l = k_1 - k_2$. Consequently,

ACVF of a stationary process

The auto-covariance function of a stationary process is only a function of the **lag** l between two observations,

$$\boxed{\sigma_{vv}[l] = E((v[k] - \mu_v)(v[k-l] - \mu_v))} \tag{2}$$

where $\mu_v = E(v[k])$ is the mean of the stationary process

Properties of the ACVF

- $\sigma_{vv}[l]$ measures (only) the linear dependence between $v[k]$ and $v[k-l]$.
- As with covariance, **ACVF is a symmetric measure**, *i.e.*,

$$\boxed{\sigma_{vv}[l] = \sigma_{vv}[-l]} \tag{3}$$

- It lacks directionality, *i.e.*, $\sigma_{vv}[l]$ does not provide the direction of dependence
- It is affected by confounding, *i.e.*, $\sigma_{vv}[l]$ includes the effects of other observations that can commonly influence $v[k]$ and $v[k-l]$
- The value of ACVF depends on the units in which the series is expressed.

Auto-correlation function (ACF)

In order to address the unboundedness and sensitivity to choice of units, the **auto-correlation function** (ACF) is introduced.

$$\boxed{\rho_{vv}[l] = \frac{\sigma_{vv}[l]}{\sigma_{vv}[0]}} \tag{4}$$

Remark: The ACF possesses all characteristics of correlation.

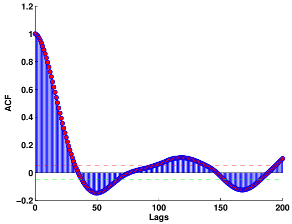
Properties of ACF

- It reaches a maximum value of 1 at lag 0 - the dependency of a sample on itself is normalized to unity.
- In fact, it is bounded like the correlation

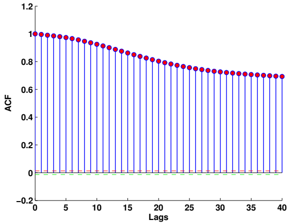
$$-1 \leq \rho_{vv}[l] \leq 1 \tag{5}$$

The equality occurs if and only when $v[k] = \alpha v]k - l]$, $\alpha \in \mathcal{R}$, *i.e.*, for a purely linear deterministic process

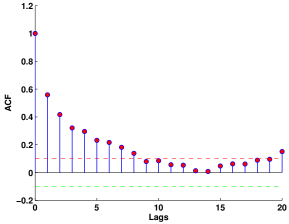
ACFs of some real series



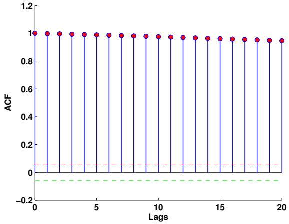
(a) Temperature



(c) ECG



(b) Wind speed



(d) Swiss SMI

$\boxed{\text{The decay rates for ACF depends on the series under analysis.}}$

By examining the ACF we obtain useful insights into the nature of correlation and the type of model that can be possibly built.

Interpreting ACF in predictions

Consider the linear forecast of a series at $k_2 = k + l$ given information only at $k_1 = k$

$$\hat{v}[k + l|v[k]] = \alpha v[k] \tag{6}$$

Then, the optimal value of α in the sense of

$$\min_{\alpha} E\left((v[k + l] - \hat{v}[k + l|v[k]])^2\right) = \min_{\alpha} E\left((v[k + l] - \alpha v[k])^2\right)$$

is

$$\boxed{\alpha^\star = \rho_{vv}[l]}$$

Thus, the ACF at any lag l is the optimal coefficient of the linear model in (6)

Discovering signatures from ACF

Methods for development / estimating time-series models implicitly or explicitly involve inverse mapping of the estimated ACF to the model parameters.

Therefore, it is important to ensure that this inverse mapping produces mathematically meaningful and correct models. For instance, can we start from any symmetric function and expect it to be the ACF of a stationary random process? The answer is NO.

Non-negative definiteness

An important property that a function to qualify as the ACF is that of the property of **non-negative definiteness**.

Definition

A sequence $\gamma[.]$ is said to be non-negative definite if it satisfies

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i \gamma[|i-j|] a_j \geq 0 \qquad \forall \;\; a_i, a_j \in \mathcal{R} \;\; , \; n > 0 \tag{7}$$

Non-negative definiteness of ACF

Theorem

The ACVF of a stationary process is non-negative definite.

Proof: Consider a process $y[k] = \sum_{i=1}^{n} a_i v[k-i+1] = \mathbf{a}^T\mathbf{v}$, where $v[.]$ is an observation of a random stationary process and $a_i \in \mathcal{R}$. Then

$$\begin{aligned}
\text{var}(y[k]) &= E\left((y[k]-\mu_y)(y[k]-\mu_y)^T\right)\\
&= E\left(\mathbf{a}^T(\mathbf{v}-\mu_{\mathbf{v}})(\mathbf{v}-\mu_{\mathbf{v}})^T\mathbf{a}\right)\\
&= E\left(\mathbf{a}^T\Gamma_n\mathbf{a}\right)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i\sigma[i-j]a_j \geq 0
\end{aligned}$$

$$\Gamma_n = \begin{bmatrix} \sigma_{vv}[0] & \cdots & \sigma_{vv}[n-1] \\ \vdots & \cdots & \vdots \\ \sigma_{vv}[1-n] & \cdots & \sigma_{vv}[0] \end{bmatrix}$$

White-noise process

One of the most important uses of ACF is in the definition of an **ideal random process**, which is the backbone of (linear) random process theory.

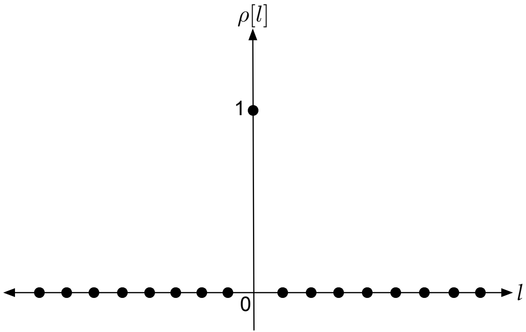
White-noise process

The white-noise process $e[k]$ is a **stationary uncorrelated** random process,

$$\rho_{ee}[l] = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases} \tag{8}$$

White-noise process ... contd.

- It is an unpredictable (in the linear sense) stationary process.
- The ACF of a white-noise process has an impulse-like shape. For any process with predictability, the ACF deviates from this shape.



White-noise process

The white-noise (WN) process is useful in two important ways:

1. It is the benchmark process for test of predictability
2. As a fictitious input (driving force) to a random process for modelling purposes

Observe that the definition of WN does not impose any conditions on the distribution - the only requirements are **stationarity** and **uncorrelated** properties

WN processes

In principle, therefore we can conceive Gaussian WN, Uniform WN and so on. The most commonly assumed one is the Gaussian WN (GWN).

Remark: A variety of random number generators can generate GWN and UWN processes. These are pseudo-random number generators in the sense that they lose their randomness when the initial condition (seed) is known.

Independence

One can extend the definition of uncorrelated process to an independent process, which demands that all higher-order moments of the joint pdf to be zero.

I.I.D. process

An identical, independent process is that process which is absolutely unpredictable (using any non-linear model).

- **Note:** A Gaussian white-noise process is an i.i.d process as well.
- In practice, it is very difficult to test for independence whereas it is quite easy to test for the absence of correlation.

ACF and time-series modelling

To quickly recap, ACF:

- Provides means of testing for predictability in a series.
- Facilitates definition of the ideal random process (white-noise process).

Taking a step further, we now move to the stage of **using ACF for determining what type (order and structure) of linear model is suitable for a given series**. To be able to do so, it is important to study the behaviour of ACF for different processes.

> Using the map between ACF and the associated random process, we would like to make an "intelligent" guess of the form of the model.

General linear random process

The general model for a linear random stationary process is given by the **convolution form**

$$v[k] = \sum_{n=-\infty}^{\infty} h[n]e[k-n], \qquad e[k] \sim \text{GWN}(0, \sigma_e^2), \sum_n |h[n]| < \infty \tag{9}$$

Typically one sets $n \geq 0$ (for causality) and $h[0] = 1$ (for uniqueness). A formal treatment of the above model appears later.

Remark: Another definition that requires $e[k]$ to be an i.i.d. process is also widely used.

Types of linear random processes

Depending on what one assumes further about the sequence of coefficients $h[n]$, (9) specializes to two types of processes:

1. **Moving Average** (MA) process
2. **Auto-regressive** (AR) process

and mixed models, i.e, auto-regressive, moving average (ARMA) models.

We shall now study the ACF signatures of the above two processes.

ACF of a Moving Average (MA) process

MA(1) process

The **MA process of first-order**, i.e., MA(1) arises when $h[1] = c_1$ and $h[n] = 0, n \neq 0, 1.$

$$v[k] = e[k] + c_1 e[k-1]$$

where $e[k] \sim \text{GWN}(0, \sigma_e^2)$ and c_1 is a finite constant.

- The current state contains the previous shock wave plus an unpredictable part $e[k]$

Theoretical ACF of an MA(1) process

The theoretical ACF is obtained using the definition in (2)

$$\begin{aligned}
\sigma_{vv}[l] &= E((v[k]-\mu_v)(v[k-l]-\mu_v s)) \\
&= E((e[k]+c_1 e[k-1])(e[k-l]+c_1 e[k-l-1])) \\
&= E(e[k]e[k-l]) + c_1 E(e[k]e[k-l-1]) \\
&\quad + c_1 E(e[k-1]e[k-l]) + c_1^2 E(e[k-1]e[k-l-1]) \\
&= \sigma_{ee}[l] + c_1\sigma_{ee}[l+1] + c_1\sigma_{ee}[l-1] + c_1^2\sigma_{ee}[l]
\end{aligned}$$

ACVF of an MA(1) process ...contd.

Using the definition of WN process in (8), the ACVF of the MA(1) process is

$$\boxed{\sigma_{vv}[l] = \begin{cases} (1+c_1^2)\sigma_{ee}^2 & l = 0 \\ c_1\sigma_{ee}^2 & l = \pm 1 \\ 0 & l = \pm 2, \pm 3, \cdots \end{cases}} \tag{10}$$

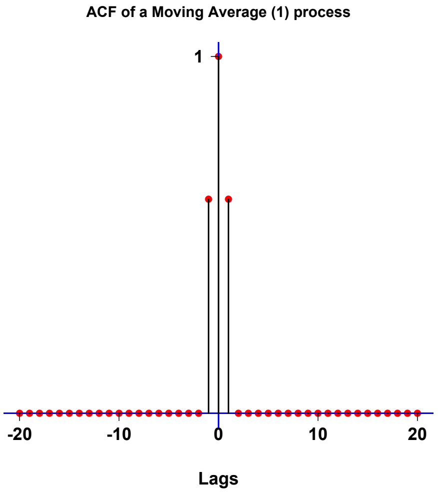
ACF of an MA(1) process ...contd.

Thus, we can write the ACF of an MA(1) process as

$$\boxed{\rho_{vv}[l] = \begin{cases} 1 & l = 0 \\ \dfrac{c_1}{(1+c_1^2)} & l = \pm 1 \\ 0 & |l| \geq 2 \end{cases}} \tag{11}$$

The ACF of an MA(1) process has a sharp cut-off after lag $l = 1$ (the order of the MA(1) process)

ACF of an MA(1) process ...contd.



- Observe that the ACF is independent of the variance of the WN process
- Since $\rho_{vv}[l] = 0, |l| \geq 2$, one cannot predict the process beyond one time-step.

General observations

- The ACF is symmetric and bounded above in magnitude by unity for all values of c_1 (verify).
- Suppose that the coefficient on $e[k]$ was c_0 instead of unity. The ACVF is then given by

$$\sigma_{vv}[l] = \begin{cases} (c_0^2 + c_1^2)\sigma_e^2, & l = 0 \\ c_1 c_0 \sigma_e^2, & |l| = 1 \\ 0, & |l| \geq 2 \end{cases} \tag{12}$$

From a modelling viewpoint, we have two equations and three unknowns (c_0, c_1 and σ_e^2). We have, therefore, an underdetermined problem. This is resolved by fixing any one of these unknowns. A reasonable and meaningful choice is to set $c_0 = 1$ (why?)

Further, recall

Not any symmetric function that is bounded above in magnitude by unity, qualifies to be the ACF of a random stationary process.

Non-negative definiteness

Non-negative definiteness of a symmetric, bounded sequence $\sigma[.]$ guarantees the existence of a stationary random process with $\sigma[.]$ as its ACVF.

Problem: For what values of c does the function $f[l] = \begin{cases} 1, & l = 0 \\ c, & l = \pm 1 \\ 0, & \text{otherwise} \end{cases}$ qualify to be the ACF of a stationary process?

Non-negative definiteness

Solution: $f[l]$ can be the ACF of a stationary process if and only if $|c| \leq 1/2$. Why? The given function resembles the ACF of an MA(1) process. Then, only when $|c| \leq 1/2$, an MA(1) process with real coefficient exists.

Alternatively, if $|c| > 1/2$, then $f[l]$ is not positive definite. See this by choosing $a = [1, -1, 1, -1, \cdots]$ in the definition of non-negative definiteness. The test sum yields $n - 2(n-1)c$, which is not positive for all $n > 0, c > 1/2$.

Alternative method for testing n.n.d.

Theorem (Bochner)

Any absolutely summable real-valued sequence $\sigma[l],\ l \in \mathcal{Z}$ is non-negative definite if and only if its Fourier transform

$$\gamma(\omega) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \sigma[l] e^{-j\omega l} \tag{13}$$

is non-negative valued at all ω, i.e., $\gamma(\omega) \geq 0, \forall\omega$.

- It is sufficient to restrict the frequency range to $[0, 2\pi)$ (why?).]