A linear random process is a filter

The result that we just observed is well-known in the frequency-domain analysis of deterministic LTI systems. Drawing parallels, we introduce the terminology

$h[.]$: Impulse response of the random process

$H(e^{j\omega})$: **Frequency response function** (FRF) of the process

Essentially, **any linear random process acts like a filter**

Filtering perspective: Example

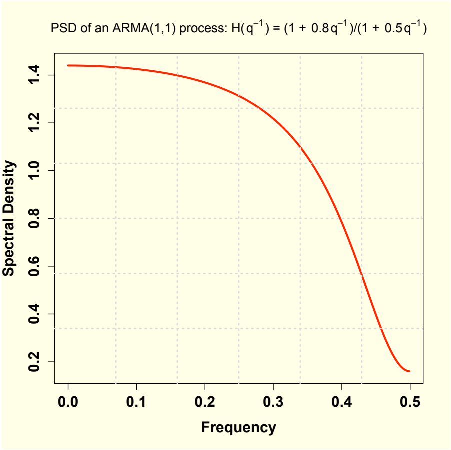
Example

An ARMA (1,1) process has the T.F.:

$$H(q^{-1}) = \frac{1 + 0.8q^{-1}}{1 + 0.5q^{-1}}$$

The p.s.d. is therefore proportional to

$$|H(e^{j\omega})|^2 = \frac{1.64 + 1.16\cos\omega}{1.25 + 0.5\cos\omega}$$



Random periodic processes

What are random periodic processes?

They are stationary processes that exhibit a periodic behaviour.

Example: Train arrival times at a station - it has a periodicity but some randomness due to minor variations in starting times or arrival times

Why do we study random periodic processes?

Random periodic processes ...contd.

Because they can model a variety of periodic phenomena, both natural and man-made, that are embedded in some stochastic processes.

How can a random process be periodic?

The periodicity of a random process is not in its amplitude, as it was for deterministic signals, but is in a mean square sense.

It turns out that this definition is also equal to requiring that the ACVF be periodic.

Harmonic Process

Definition

A discrete-time (wide-sense) *stationary* process $\{v[k]\}$ is said to be *periodic* with period N_p if

$$E((v[k+N_p]-v[k])^2)=0 \tag{3}$$

or equivalently,

1. $\sigma_{vv}[l+N_p]=\sigma_{vv}[l], \quad \forall l \in \mathbb{Z}$ (Periodic ACVF)
2. $\sigma_{vv}[N_p]=\sigma_{vv}[0]$
3. $\Pr(v[k+N_p]=v[k])=1 \;\; \forall k \in \mathbb{Z}$

Harmonic processes

A simple possible way of constructing a random, stationary periodic signal $v[k]$ is through a linear combination of sines and cosines with random coefficients (or by linearly combining sinusoids with random amplitudes and phases)

$$v[k] = \sum_{n=1}^{F} a_n \cos(2\pi f_n k) + b_n \sin(2\pi f_n k)$$

However, these coefficients cannot be arbitrarily random.

Harmonic processes

In fact, for $v[k]$ to be stationary, it is required that

a_n and b_n are independent random variables with $E(a_n) = E(b_n) = 0$ and with equal variances

With the assumption $E(a_n) = 0 = E(b_n)$, it is easy to show that:

$$\sigma_{vv}[l] = \sum_{n=1}^{F} \sigma_n^2 \cos(2\pi f_n l) \qquad \text{where} \qquad \sigma_n^2 = E(a_n^2) = E(b_n^2)$$

The ACVF of a random periodic signal is periodic with contributions from each frequency component proportional to their respective variances

DTFS for ACVF

The fact is that for this case **ACVF has a Fourier Series expansion**

Observe:

- For the deterministic case, the squared magnitude of Fourier coefficients gave the power spectrum
- For random signals, the expectation of the square of Fourier coefficients can be thought of the power spectrum of the random periodic signal

Spectral distribution function

Spectral densities can only be defined for random processes that are not periodic. However, the notion of a **spectral distribution function** can be applied to both classes of processes.

Spectral Distribution Function

The spectral distribution function of an aperiodic stationary stochastic process is defined as

$$\boxed{\Gamma(\omega) = \int_{-\pi}^{\omega} \gamma(\omega)\, d\omega} \quad \text{or} \quad \boxed{\gamma(\omega) = \frac{d\Gamma(\omega)}{d\omega}} \tag{4}$$

For periodic random processes, $\Gamma(\omega)$ is a staircase function with spikes at the frequencies.

The quantities $\Gamma(\omega)/\sigma^2$ and $\gamma(\omega)/\sigma^2$ are the **normalized spectral** distribution and density functions respectively

Unified Wiener-Khinchin theorem

Theorem (Khintchine, (1934), Wiener, (1930), and Wold, (1938))

A discrete sequence $\rho[l]$ is the auto-correlation function of a discrete-time stochastic process $v[k]$ if and only if there exists a function $F(\omega)$, such that

$$\rho[l] = \int_{-\pi}^{\pi} e^{j\omega l} dF(\omega) \qquad l \in \mathbb{Z} \tag{5}$$

where $F(\omega)$ has the properties of a (normalized) distribution function on the interval $(-\pi, \pi)$, i.e., $F(\omega)$ is right-continuous, non-decreasing, bounded on $[-\pi, \pi]$ and $F(-\pi) = 0$, $F(\pi) = 1$.

Proof is found in standard texts. See Brockwell and Davis, (1991) and Priestley, (1981).

W-K Theorem ...contd.

The function $F(.)$ in (5) is called the *normalized spectral distribution function*.

Comparing with earlier equations,

$$F(\omega) = \Gamma(\omega)/\sigma^2 \qquad \text{such that} \qquad F(\pi) = 1 \tag{6}$$

Spectral Factorization

Recall

For the causal linear time-series model

$$v[k] = \sum_{n=0}^{\infty} h[n]e[k-n] = H(q^{-1})e[k], \qquad \sum_{n=0}^{\infty} |h[n]| < \infty, \qquad e[k] \sim \mathsf{WN}(0, \sigma_e^2) \tag{7}$$

Then, we know

$$\gamma(\omega) = \frac{\sigma_e^2}{2\pi}|H(e^{-j\omega})|^2 = \frac{\sigma_e^2}{2\pi}H(e^{-j\omega})H^{\star}(e^{-j\omega}) = \frac{\sigma_e^2}{2\pi}H(e^{-j\omega})H(e^{j\omega}) \tag{8}$$

Spectral Factorization

Spectral factorization is the inverse problem, as stated below.

Given a time-series with continuous, symmetric, non-negative spectral density $\gamma(\omega)$ that is integrable over $[-\pi, \pi]$ find a factorization of the form (8). From this viewpoint, $H(e^{-j\omega})$ is known as the **spectral factor**.

Why is spectral factorization important?

A few questions

Given a time-series, building a linear model (predictor) in (7) (or even its non-causal version) amounts to factorizing the spectral density as in (8)

Q: Under what conditions is it possible to obtain the factorization (8) and when is it **unique**? Are there any restrictions on $\gamma(\omega)$ or the spectral factor $H(e^{j\omega})$?

Recall the ACVGF, also called as the spectral density:

$$\gamma(z) = \sum_{l=-\infty}^{\infty} \sigma[l] z^{-l} \tag{9}$$

Clearly, $\gamma(\omega) = \gamma(z)|_{z=e^{-j\omega}}$.

A more general problem

Spectral factorization

Find σ^2 and $H(z)$ such that the spectral density $\gamma(z)$ in (9) can be factorized as

$$\gamma(z) = \frac{\sigma^2}{2\pi} H(z^{-1}) H(z) \tag{10}$$

where

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}; \qquad H(z^{-1}) = \sum_{n=0}^{\infty} h[n] z^{n} \tag{11}$$

$H(z^{-1})$ is obtained by replacing every appearance of z in $H(z)$ with z^{-1}.

Remarks

1. The factorization in both forms (8) and (10), *is not unique*. If (σ_e^2, H) is a solution, then $(\alpha^2\sigma^2, H/\alpha),\ \ \alpha \in \mathbb{R}$ is also a solution.
   To fix the non-uniqueness issue, we require that (recall Chapter 9)

$$h[0] = 1 \qquad \Longrightarrow \qquad H(0) = 1 \tag{12}$$

2. *Spectral factors can only be identified correctly up to a phase.* If $H(z)$ is a solution, then so is $H(z)e^{-D\omega}$. Nevertheless, spectral factorization guarantees the identification of a minimum-phase filter $H(z)$.
3. Thirdly, if $H(z)$ is a solution, then $H(z^{-1})$ is an equally likely solution. This is in fact a fallout of the first issue above.

Example

Consider the ARMA process: $v[k] = \dfrac{1+3q^{-1}}{1-2q^{-1}}e[k], \quad e[k] \sim \text{WN}(0, \sigma_e^2)$. Observe that this process is neither causal nor invertible

From (1),

$$\gamma_{vv}(\omega) = \frac{\sigma_e^2}{2\pi}\frac{|1+3e^{-j\omega}|^2}{|1-2e^{-j\omega}|^2} \tag{13}$$

The spectral density can be re-written as

$$\gamma_{vv}(\omega) = \frac{\sigma_e^2}{2\pi}\frac{|1+3e^{-j\omega}|^2}{|1-2e^{-j\omega}|^2} = \frac{\sigma_e^2}{2\pi}\frac{|1+3e^{j\omega}|^2}{|1-2e^{j\omega}|^2} = \frac{9\sigma_e^2}{8\pi}\frac{|(1/3)e^{-j\omega}+1|^2}{|(-1/2)e^{-j\omega}+1|^2} \tag{14}$$

Example ...contd.

Thus, from the spectral density viewpoint, the process

$$v[k] = \frac{1 + 3q^{-1}}{1 - 2q^{-1}} e[k], \qquad e[k] \sim \mathsf{WN}(0, \sigma_e^2) \tag{15}$$

and

$$\tilde{v}[k] = \frac{1 + (1/3)q^{-1}}{1 - (1/2)q^{-1}} \tilde{e}[k] \qquad \tilde{e}[k] \sim \mathsf{WN}(0, \frac{9}{4}\sigma_e^2) \tag{16}$$

are indistinguishable.

Conditions for existence of factorization

1. A non-causal (two-sided), infinite-order, MA representation exists for all stationary processes that have continuous spectral densities. For proof, see Priestley, (1981).
2. We seek *causal* (one-sided) representations of the form (7), i.e., the IR sequence $\{h[.]\}$ is one-sided. This is guaranteed if the spectral density, satisfies the following **Paley-Wiener condition:**

$$\boxed{\int_{-\pi}^{\pi} \log \gamma(\omega)\, d\omega > -\infty} \tag{17}$$

- Satisfied by most stationary processes with continuous PSD unless $\gamma(\omega)$ is zero over a continuous interval in frequency.
- When $\gamma(\omega) \neq 0$ at almost all ω, the process is said to be **regular.**

Guaranteeing invertibility

Interestingly, the condition in (17) does not guarantee invertibility of the factor or an AR representation of the process.

An *invertible* spectral factor exists if and only if

> *The logarithm of the spectral density* $\log \gamma(z)$ *is analytic in the annulus*
> $$\beta < |z| < 1/\beta, \;\; \beta < 1$$

- Analytic $\Longrightarrow$ the function does not assume indeterminate values. This condition is a generalization of (17).
- Ensures that an AR representation of the process exists (see Priestley, (1981, Chapter 10)). Furthermore, it also leads to $h[0] = 1$ (the uniqueness issue)!

Putting together: Main result

Theorem (Spectral factorization)

Given a (discrete-time) stationary process whose spectral density is,

1. **Symmetric**: $\gamma(\omega) = \gamma(-\omega), \;\; \omega \in [-\pi, \pi]$
2. **Non-negative**: $\gamma(\omega) \geq 0$, *(cannot be zero over an interval of frequencies)*
3. **Integrable**: $0 < \displaystyle\int_{-\pi}^{\pi} \gamma(\omega)\, d\omega < \infty$ *(finite variance)*
4. **Log-Analytic**: $\log(\gamma(z))$ *possesses derivatives of all orders in the annulus* $\beta < |z| < 1/\beta, \;\; \beta < 1$

Putting together: Main result

Theorem (Spectral factorization . . . contd.)

its spectral density function is factorizable as

$$\gamma(z) = e^{c_0} H(z^{-1}) H(z) = \frac{\sigma^2}{2\pi} H(z^{-1}) H^\star(z^{-1}) \tag{18}$$

with $H(z^{-1})$ *and* $H(z)$ *as defined in* (11). *Further,*

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(\gamma(\omega))\, d\omega, \qquad h[0] = 1, \qquad |\textit{zeros}(H(z))| < 1 \;\; \textit{invertible)} \tag{19a}$$

Rational spectral densities

When the spectral density $\gamma(\omega)$ is a rational function of trigonometric polynomials,

$$\gamma(\omega) = \frac{\alpha_0 + \sum_{r=1}^{M} \alpha_r \cos(r\omega)}{\beta_0 + \sum_{s=1}^{N} \beta_s \cos(s\omega)} \tag{20}$$

the solution to the factorization simplifies considerably because all ARMA(P, M) processes possess rational spectral densities of the form above.

Example

ARMA Model from Rational Spectral Density

Suppose a random process $v[k]$ is known to possess the spectral density

$$\gamma_{vv}(\omega) = 4\frac{1.09 + 0.6\cos\omega}{1.64 - 1.16\cos\omega}$$

By visual inspection, $\gamma_{vv}(\omega)$ can be factorized as

$$\gamma_{vv}(\omega) = 4\left(\frac{1 + 0.3e^{-j\omega}}{1 - 0.8e^{-j\omega}}\right)\left(\frac{1 + 0.3e^{j\omega}}{1 - 0.8e^{j\omega}}\right)$$

Example ... contd.

There are two solutions to the filter that generate $v[k]$, one which has zeros and poles inside the unit circle and the other which has them outside the unit circle.
We choose the one that is both causal and invertible.

$$v[k] = \frac{1 + 0.3q^{-1}}{1 - 0.8q^{-1}} e[k] \qquad e[k] \sim \text{WN}(0, 8\pi) \tag{21}$$

General scenario

Theorem

If γ is a symmetric, non-negative, continuous spectral density on $[-\pi, \pi]$, then for every $\epsilon > 0$, there exists a non-negative integer M and a polynomial

$$A(z) = \prod_{i=1}^{M}(1 - \eta_i^{-1} z) = 1 + a_1 z + a_2 z^2 + \cdots + a_p z^M \qquad |\eta_j| > 1, \ \ \forall j = 1, \cdots, M \tag{22}$$

with real-valued coefficients such that

$$|K|A(e^{-j\omega})|^2 - \gamma(\omega)| < \epsilon \qquad \forall \omega \in [-\pi, \pi] \tag{23}$$

where

$$K = \frac{1}{(1 + a_1^2 + a_2^2 + \cdots + a_M^2)} \int_{-\pi}^{\pi} \gamma(\omega)\, d\omega$$

To conclude

ARMA models can be used to model most linear stationary random processes with continuous spectral densities (to be precise, those satisfying the conditions listed in the main result).

- When the true process has rational spectral density, the ARMA model provides an exact representation.
- In other cases, an approximate model with an arbitrarily small degree of error can be constructed.

Cross-spectrum and Coherence

The cross-spectral density detects linear relationship between two series as the CCVF. Extending the W-K Theorem to the bivariate case, the cross p.s.d. of two random processes $y[k]$ and $u[k]$ is

$$\gamma_{yu}(\omega) = \mathrm{DTFT}(\sigma_{yu}[l]) = \sum_{l=-\infty}^{\infty} \sigma_{yu}[l] e^{-j\omega l}$$

- It is a complex-valued quantity!
- $|\gamma_{yu}(\omega)|$ gives the strength of common power at that frequency
- $\angle\gamma_{yu}(\omega)$ (**phase**) is useful in estimating delays in the system
- Useful result: $\boxed{\gamma_{yu}(\omega) = H(e^{-j\omega})\gamma_{uu}(\omega)}$

Coherence

As with CCF, a normalized CPSD, known as **coherence function**, is used in practice:

$$\boxed{\kappa_{yu}(\omega) = \frac{\gamma_{yu}(\omega)}{\sqrt{\gamma_{yy}(\omega)\gamma_{uu}(\omega)}}}$$

Coherence

The magnitude of coherence function is **coherence**.

Coherence

Property of Coherence

A system is LTI if and only if coherence is unity at all frequencies

$$\kappa_{yu}(\omega) = 1 \ \forall \omega$$

Whenever $\kappa_{yu}(\omega) \neq 1$, same conclusions can be drawn as with correlation:

- The two series are probably non-linearly related at that frequency
- The "true" series are linearly related, but noise in the data could be masking the linear relationship.

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