CH5350: Applied Time-Series Analysis

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Partial auto-correlation functions

Partial ACF

We learnt earlier that correlation-based measures suffer from **confounding**, *i.e.*, the common influence of a third extraneous variable can cause two variables to *appear* as correlated.

The correlation between two observations of a series (or two different series) is most likely to suffer from confounding because the intermediate samples can introduce an apparent correlation due to propagated effects. While this phenomenon holds for any random process, it becomes particularly important for auto-regressive processes.

We understand the issue of confounding in ACF by revisiting the ACF of an AR(1) process.

ACF of an AR(1) process

Recall the ACF of an AR(1) process $v[k] = -d_1 v[k-1] + e[k]$ from equation (**??**),

$$\rho_{vv}[l] = (-d_1)^{|l|}$$

The ACF suggests that $v[k]$ and $v[k-l]$ are correlated whereas the governing difference equation for the process clearly shows that only two successive samples $v[k]$ and $v[k-1]$ **directly** influence each other.

ACF of an AR(1) process

Q: What is the cause of this apparent correlation between samples separated by las $L > 1$?

A: The cause for this apparent correlation is the propagated effect. For instance, the difference equation of the process can be re-written as

$$v[k] = -d_1(-d_1 v[k-2] + e[k-1]) + e[k] = d_1^2 v[k-2] - d_1 e[k-1] + e[k]$$

Thus $v[k-2]$ appears to influence $v[k]$ **indirectly** through $v[k-1]$. The same argument can be extended to explain correlation at other lags as well.

How do we ensure ACF measures direct correlations only?

Conditioned ACF: Partial ACF

To measure the direct correlation between $v[k-l]$ and $v[k]$ we should account for the possible propagated effects of the intermediate variables $\{v[k-l+1, \cdots, v[k-1]\}$.

The procedure is illustrated for $l = 2$. The idea is to remove the presence of $v[k-1]$ in both $v[k]$ and $v[k-2]$ followed by a correlation between the respective residuals. The resulting correlation is known as **partial auto-correlation function** (PACF)

Partial ACF

Remarks:

- Partial ACF is analogous to "partial derivative" where only the effects w.r.t. a specific variable are evaluated.
- As we learnt earlier, computing partial correlation (or any other measure) is known as **conditioning** in signal processing
- The partial ACF measures **direct correlation** whereas the ACF measures **total correlation**

Procedure to compute PACF

1. Obtain the best predictor for $v[k]$ using $v[k-1]$. Denote the associated residuals by $\eta[k]$

$$\hat{v}[k|v[k-1]] = \alpha_1 v[k-1]; \qquad \eta[k] = v[k] - \alpha_1^\star v[k-1]$$

2. Obtain the best "predictor" for $v[k-2]$ using $v[k-1]$. Denote the associated residuals by $\eta[k-2]$

$$\hat{v}[k-2|v[k-1]] = \beta_1 v[k-1]; \qquad \eta[k-2] = v[k-2] - \beta_1^\star v[k-1]$$

   where $\alpha_1^\star$ and $\beta_1^\star$ are the optimal estimates of α_1 and β_1 respectively.

3. Compute $\phi_{vv}[2] = \text{corr}(\eta[k], \eta[k-2])$ to obtain the PACF of the series $v[k]$ at lag 2

Procedure to compute PACF ...contd.

The optimal estimates of α_1 and β_1 are obtained in such a way that $\eta[k]$ and $\eta[k-2]$ do not contain any (linear) effects of $v[k-1]$, *i.e.*,

$$\text{corr}(\eta[k], v[k-1]) = 0 \qquad \text{corr}(\eta[k-2], v[k-1]) = 0$$

These are also the conditions of optimality for the least squares technique. Thus, $\alpha_1^\star$ and $\beta_1^\star$ are the LS estimates.

$$\boxed{\alpha_1^\star = \rho_{vv}[1]} \qquad \boxed{\beta_1^\star = \rho_{vv}[1]}$$

General procedure

The general procedure to obtain PACF is given below.

1. Obtain the best predictors for $v[k]$ and $v[k-l]$ using $\{v[k-1], v[k-2], \cdots, v[k-l+1]\}$. Denote the associated residuals by $\eta[k]$ and $\eta[k-l]$ respectively

$$\eta[k] = v[k] - \sum_{j=1}^{l-1} \alpha_j^\star v[k-j] \qquad \eta[k-l] = v[k-l] - \sum_{j=1}^{l-1} \beta_j^\star v[k-l+j]$$

   where the * denote the optimal values (least squares) estimates.

2. Compute $\phi_{vv}[l] = \text{corr}(\eta[k], \eta[k-l])$ to obtain the PACF at lag l

Alternative procedure

The PACF coefficient at any lag p, $\phi_{vv}[p]$ can be shown to be the last coefficient of an AR(p) model fit to the series $v[k]$

1. Fit an AR(l) model at each lag l.
2. Determine the PACF at any lag l as the last coefficient of that model.

A **recursive** algorithm due to Durbin and Levinson is used in practice to compute $\phi_{vv}[p]$ using the coefficient at $l = p - 1$ and the ACF coefficients.

Remarks

- The “prediction” of $v[k-l]$ using future values is known as **backcasting**
- The PACF at lag $l = 0$ is not defined. However, to be consistent with ACF, PACF at lag $l = 0$ maybe set to unity.

PACF of an AR(1) process

Problem: Find the PACF of an AR(1) process: $v[k] = -d_1 v[k-1] + e[k]$ at lags $l = 1, 2$

Solution: The PACF coefficient at lag $l = 1$ is the ACF at lag $l = 1$ itself since there is no intermediate variable. The direct correlation at lag $l = 1$ is the same as total correlation.

Thus,

$$\phi_{vv}[1] = \rho_{vv}[1] = (-d_1)^{|l|} \tag{1}$$

PACF of an AR(1) process

To compute the PACF at lag $l = 2$, recall from the procedure

$$\begin{aligned}
\phi_{vv}[2] &= \text{corr}(v[k] - \alpha_1^\star v[k-1], v[k-2] - \beta_1^\star v[k-1]) \\
&= \frac{\text{cov}(v[k] - \rho_{vv}[1]v[k-1], v[k-2] - \rho_{vv}[1]v[k-1])}{\sqrt{\text{var}(v[k] - \rho_{vv}[1]v[k-1])\text{var}(v[k-2] - \rho_{vv}[1]v[k-1])}} \\
&= \frac{\rho[2] - \rho[1]}{1 - \rho[1]^2}
\end{aligned}$$

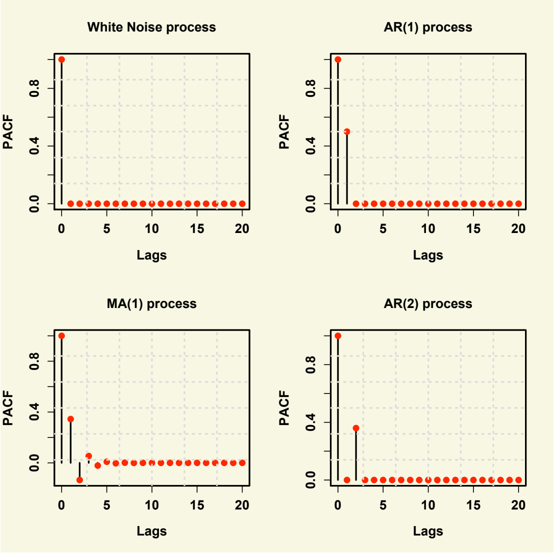
PACF of an AR(1) process ...contd.

- For an AR(1) process, $\rho[l] = (-d_1)^{|l|}$, thereby $\phi_{vv}[2] = 0$
- At a later stage, it will be shown that $\phi_{vv}[l] = 0$ for all lags $l \geq 2$ for an AR(1) process

  The PACF for an AR(1) process falls off abruptly to zero $\forall |l| > 2$.

Note: For an MA(1) process, $\phi_{vv}[1] = -c_1(1 + c_1^2)/(1 + c_1^4 - c_1^2)$

Theoretical PACF: Examples



Remarks

- The PACF of a WN process is zero at all lags (like the ACF)
- PACF of an MA(1) process dies down exponentially (somewhat analogous to the behaviour of ACF for an AR(1) process)
- The notion of PACF can be extended to handle negative lags as well. For stationary processes, PACF is symmetric like the ACF

Summary

In this chapter, we learnt / obtained

- The concepts and definitions of ACVF and ACF
- Insights into the concepts of **white noise** and its potnntial use in describing stationary random processes
- That the ACF is a measure of predictability of a given series in a linear sense
- The CCF measures the linear dependence between two shifted series and it is very useful in delay estimation and other signal processing applications

Summary ...contd.

- Partial ACF accounts for possible confounding in the ACF, particularly for auto-regressive processes
- The PACF and ACF measures are duals of each other
  - ACF decays exponentially for an AR process while the PACF falls off abruptly after an appropriate lag for the same process
  - The above behaviour is reversed for the case of an MA process