Discrete-time Fourier Series (DTFS)

Opening remarks

The Fourier series representation for discrete-time signals has some similarities with that of continuous-time signals. Nevertheless, certain differences exist:

- Discrete-time signals are unique over the frequency range $f \in [-0.5, 0.5)$ or $]\omega \in [-\pi, \pi)$ (or any interval of this length).
- The period of ?a discrete-time signal **is expressed in samples.**

Discrete-time signals

- A discrete-time signal of fundamental period N can consist of frequency components $f = \dfrac{1}{N}, \dfrac{2}{N}, \cdots, \dfrac{(N-1)}{N}$ besides $f = 0$, the DC component
  - Therefore, the Fourier series representation of the **discrete-time periodic** signal contains only N complex exponential basis functions.

Fourier series for d.t. periodic signals

Given a periodic sequence $x[k]$ with period N, the Fourier series representation for $x[k]$ uses N harmonically related exponential functions

$$e^{j2\pi kn/N}, \quad k = 0, 1, \cdots, N-1$$

The Fourier series is expressed as

$$\boxed{x[k] = \sum_{n=0}^{N-1} c_n e^{j2\pi kn/N}} \tag{22}$$

Fourier coefficients and Parseval's relation

The Fourier coefficients $\{c_n\}$ are given by:

$$\boxed{c_n = \frac{1}{N}\sum_{k=0}^{N-1} x[k]e^{-j2\pi kn/N}} \tag{23}$$

Parseval's result for discrete-time signals provides the decomposition of power in the frequency domain,

$$\boxed{P_{xx} = \frac{1}{N}\sum_{k=0}^{N-1} |x[k]|^2 = \sum_{n=0}^{N-1} |c_n|^2} \tag{24}$$

Power (Line) Spectrum

Thus, we have the **line spectrum** in frequency domain, as in the continuous-time case.

$$\boxed{P_{xx}[n] \triangleq P_{xx}(f_n) = |c_n|^2, \qquad\qquad n = 0, 1, \cdots, N-1} \tag{25}$$

- The term $|c_n|^2$ denotes therefore the power associated with the n^{th} frequency component
- The difference between the results in the c.t. and d.t. case is only in the restriction on the number of basis functions in the expansion.

Remarks

- The Fourier coefficients $\{c_n\}$ enjoy the conjugate symmetry property

$$c_n = c^\star_{N-n-1} \qquad n \neq 0, N/2 \qquad \text{(assuming } N \text{ is even)} \tag{26}$$

- **The Fourier coefficients $\{c_n\}$ are periodic with the same period as $x[k]$**
  - *The power spectrum of a discrete-time periodic signal is also, therefore, periodic,*

$$P_{xx}[N+n] = P_{xx}[n] \tag{27}$$

- The range $0 \leq n \leq N-1$ corresponds to the fundamental frequency range $0 \leq f_n = \dfrac{n}{N} \leq 1 - \frac{1}{N}$

Example: Periodic pulse

The discrete-time Fourier representation of a periodic signal $x[k] = \{1, 1, 0, 0\}$ with period $N = 4$ is given by,

$$c_n = \frac{1}{4}\sum_{k=0}^{3} x[k]e^{-j2\pi kn/4} = \frac{1}{4}(1 + e^{-j2\pi n/4}) \qquad n = 0, 1, 2, 3$$

This gives the coefficients

$$c_0 = \frac{1}{2}; \qquad c_1 = \frac{1}{4}(1 - j); \qquad c_2 = 0; \qquad c_3 = \frac{1}{4}(1 + j)$$

Observe that $c_1 = c_3^\star$.

Power spectrum and auto-covariance function

The power spectrum of a discrete-time periodic signal and its auto-covariance function form a Fourier pair.

$$P_{xx}[n] = \frac{1}{N}\sum_{l=0}^{N-1} \sigma_{xx}[l]e^{-j2\pi ln/N} \tag{28a}$$

$$\sigma_{xx}[l] = \sum_{l=0}^{N-1} P_{xx}[n]e^{j2\pi ln/N} \tag{28b}$$

Discrete-time Fourier Series

Variant	Synthesis / analysis	Parseval's relation (power decomposition) and signal requirements
Discrete-Time Fourier Series	$x[k] = \sum_{n=0}^{N-1} c_n e^{j2\pi kn/N}$ $c_n \triangleq \frac{1}{N}\sum_{k=0}^{N-1} x[k]e^{-j2\pi kn/N}$	$P_{xx} = \frac{1}{N}\sum_{k=0}^{N-1} \lvert x[k]\rvert^2 = \sum_{n=0}^{N-1} \lvert c_n\rvert^2$ $x[k]$ is periodic with fundamental period N

Discrete-time Fourier Transform (DTFT)

Opening remarks

- The discrete-time aperiodic signal is treated in the same way as the continuous-time case, i.e., as an extension of the DTFS to the case of periodic signal as $N \to \infty$.
- Consequently, the frequency axis is a **continuum**.
- The synthesis equation is now an integral, but still restricted to $f \in [-1/2, 1/2)$ or $\omega \in [-\pi, \pi)$.

Discrete-time Fourier transform (DTFT)

The synthesis and analysis equations are given by:

$$\boxed{\begin{aligned} x[k] &= \int_{-1/2}^{1/2} X(f)e^{j2\pi fk}\,df = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega k}\,d\omega && \text{(Synthesis)} \\ X(f) &= \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk} && \text{(DTFT)} \end{aligned}}$$

(29)

(30)

DTFT

Remarks

- The DTFT is unique only in the interval $[0, 1)$ cycles/ sample or $[0, 2\pi)$ rad/sample.
- The DTFT is periodic, i.e., $X(f+1) = X(f)$ or $X(\omega + 2\pi) = X(\omega)$ (**Sampling in time introduces periodicity in frequency**)
- Further, the DTFT is also the z-transform of $x[k]$, $X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$, evaluated on the unit circle $z = e^{j\omega}$

Existence conditions

- The signal should be absolutely convergent, i.e., it should have a finite 1-norm

$$\sum_{k=-\infty}^{\infty} |x[k]| < \infty \tag{31}$$

- A weaker requirement is that the signal should have a finite 2-norm, in which case the signal is guaranteed to only converge in a sum-squared error sense.
- Essentially signals that exist forever in time, e.g., step, ramp and exponentially growing signals, do not have a Fourier transform.
- On the other hand, **all finite-length, bounded-amplitude signals always have a Fourier transform.**

Energy conservation

Energy is preserved under this transformation once again due to Parseval's relation:

$$\boxed{E_{xx} = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \int_{-1/2}^{1/2} |X(f)|^2 \, df = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega} \tag{32}$$

Energy spectral density

Consequently, the quantity

$$S_{xx}(f) = |X(f)|^2; \quad S_{xx}(\omega) = \frac{|X(\omega)|^2}{2\pi} \tag{33}$$

qualifies to be a density function, specifically as the *energy spectral density* of $x[k]$.

Given that $X(f)$ is periodic (for real-valued signals), **the spectral density of a discrete-time (real-valued) signal is also periodic** with the same period.

Example: Discrete-time impulse

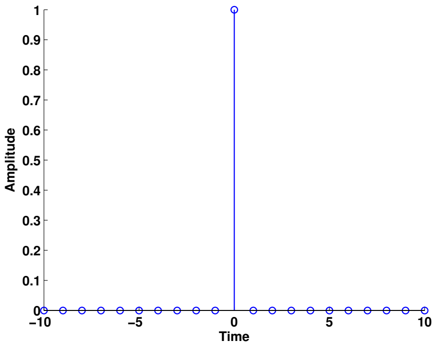
The Fourier transform of a discrete-time impulse $x[k] = \delta[n]$ (Kronecker delta) is

$$X(f) = \mathcal{F}\{\delta[n]\} = \sum_{k=-\infty}^{\infty} \delta[k] e^{-j2\pi f k} = 1 \quad \forall f \tag{34}$$

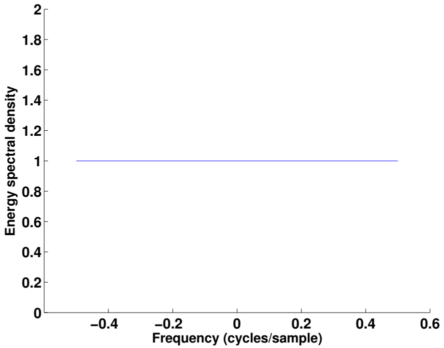
giving rise to a uniform energy spectral density

$$S_{xx}(f) = |X(f)|^2 = 1 \quad \forall f \tag{35}$$

Example: Discrete-time impulse



(g) Finite-duration pulse



(h) Energy spectral density

Example: Discrete-time finite-duration pulse

Compute the Fourier transform and the energy density spectrum of a finite-duration rectangular pulse

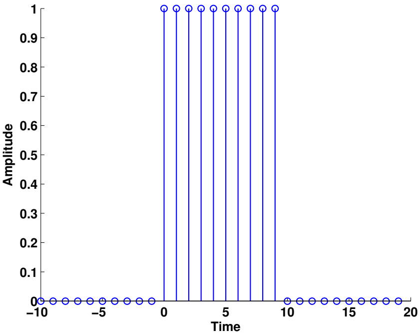
$$x[k] = \begin{cases} A, & 0 \leq k \leq L-1 \\ 0 & \text{otherwise} \end{cases}$$

Solution: The DTFT of the given signal is

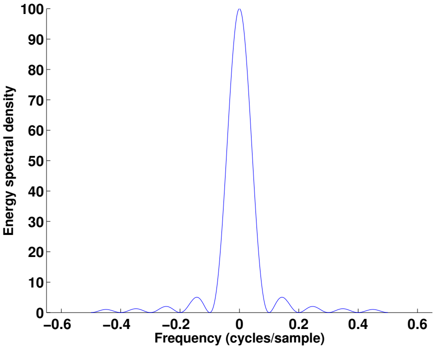
$$X(f) = \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi fk} = \sum_{k=0}^{L-1} Ae^{-j2\pi fk} = A\frac{1-e^{-j2\pi fL}}{1-e^{-j2\pi f}}$$

$$S_{xx}(f) = A^2\frac{1-\cos(2\pi fL)}{1-\cos 2\pi f}$$

Example: Discrete-time impulse contd.



(i) Finite-duration pulse



(j) Energy spectral density

Finite-length pulse and its energy spectral density for $A = 1, L = 10$.

Energy spectral density and auto-covariance function

The energy spectral density of a discrete-time aperiodic signal and its auto-covariance function form a Fourier pair.

$$S_{xx}(f) = \sum_{l=-\infty}^{\infty} \sigma_{xx}[l] e^{-j2\pi l f} \tag{36a}$$

$$\sigma_{xx}[l] = \int_{-1/2}^{1/2} S_{xx}(f) e^{j2\pi f l} \, df \tag{36b}$$

Cross-energy spectral density

In multivariable signal analysis, it is useful to define a quantity known as cross-energy spectral density,

$$\boxed{S_{x_2x_1}(f) = X_2(f)X_1^\star(f)} \tag{37}$$

The cross-spectral density measures the linear relationship between two signals in the frequency domain, whereas the auto-energy spectral density measures linear dependencies within the observations of a signal.

Cross energy spectral density ...contd.

When $x_2[k]$ and $x_1[k]$ are the output and input of a linear time-invariant system respectively, i.e.,

$$x_2[k] = G(q^{-1})x_1[k] = \sum_{n=-\infty}^{n=\infty} g[n]x_1[k-n] = g_1[k] \star x_1[k] \tag{38}$$

two important results emerge

$$\boxed{S_{x_2x_1}(f) = G_1(e^{-j2\pi f})S_{x_1x_1}(f); \qquad S_{x_2x_2}(f) = |G_1(e^{-j2\pi f})|^2 S_{x_1x_1}(f)} \tag{39}$$

Discrete-time Fourier Transform

Variant	Synthesis / analysis	Parseval's relation (energy decomposition) and signal requirements
Discrete-Time Fourier Transform	$x[k] = \int_{-1/2}^{1/2} X(f) e^{j2\pi fk}\, df$ $X(f) \triangleq \sum_{k=-\infty}^{\infty} x[k] e^{-j2\pi fk}$	$E_{xx} = \sum_{k=-\infty}^{\infty} \lvert x[k]\rvert^2 = \int_{-1/2}^{1/2} \lvert X(f)\rvert^2\, df$ $x[k]$ is aperiodic; $\sum_{k=-\infty}^{\infty} \lvert x[k]\rvert < \infty$ or $\sum_{k=-\infty}^{\infty} \lvert x[k]\rvert^2 < \infty$ (finite energy, weaker requirement)

Summary

It is useful to summarize our observations on the spectral characteristics of different classes of signals.

i. *Continuous-time* signals have *aperiodic spectra*
ii. *Discrete-time* signals have *periodic spectra*
iii. *Periodic* signals have *discrete (line) power spectra*
iv. *Aperiodic (finite energy)* signals have *continuous energy spectra*

Continuous spectra are qualified by a *spectral density function*.

Spectral Distribution Function

In all cases, one can define an energy / power **spectral distribution function**, $\Gamma(f)$.

For **periodic** signals, we have **step-like power spectral distribution** function,
For **aperiodic** signals, we have a **smooth energy spectral distribution** function,
where one could write the **spectral density** as,

$$S_{xx}(f) = d\Gamma(f)/df \qquad \text{or} \qquad \Gamma_{xx}(f) = \int_{-1/2}^{f} S_{xx}(f)\,df \tag{40}$$

.

Properties of DTFT

Linearity property

1. **Linearity:**

If $x_1[k] \xrightarrow{\mathcal{F}} X_1(\omega)$ and $x_2[k] \xrightarrow{\mathcal{F}} X_2(\omega)$ then

$$a_1 x_1[k] + a_2 x_2[k] \xrightarrow{\mathcal{F}} a_1 X_1(f) + a_2 X_2(f)$$

The Fourier transform of a sum of discrete-time (aperiodic) signals is the respective sum of transforms.

Shift property

2. **Time shifting:**

If $x_1[k] \xrightarrow{\mathcal{F}} X_1(\omega)$ then
$x_1[k-D] \xrightarrow{\mathcal{F}} e^{-j2\pi f D} X_1(f)$

- Time-shifts result in frequency-domain modulations.
- Note that the **energy spectrum of the shifted signal remains unchanged** while the phase spectrum shifts by $-\omega k$ at each frequency.

Dual:

A shift in frequency $X(f-f_0)$ corresponds to modulation in time, $e^{j2\pi f_0 k} x[k]$.

Time reversal

3. **Time reversal:**

$$\text{If } x[k] \xrightarrow{\mathcal{F}} X(\omega), \text{ then } x[-k] \xleftrightarrow{\mathcal{F}} X(-f) = X^\star(f)$$

If a signal is folded in time, then its power spectrum remains unchanged; however, the phase spectrum undergoes a sign reversal.

Dual: The dual is contained in the statement above.

Scaling property

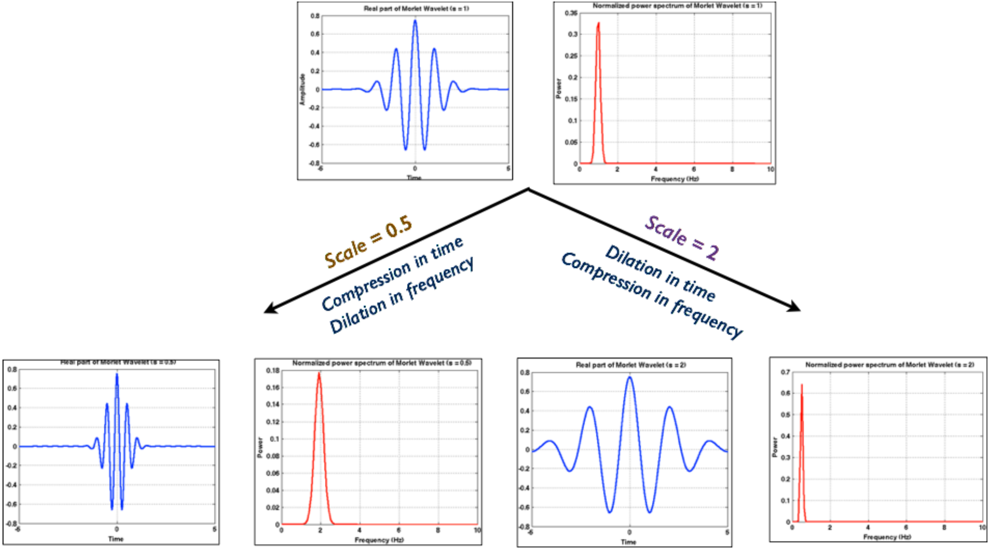
4. **Scaling:**

If $x[k] \xrightarrow{\mathcal{F}} X(\omega)$ (or $x(t) \xrightarrow{\mathcal{F}} X(F)$),
then $x\left[\dfrac{k}{s}\right] \xrightarrow{\mathcal{F}} X(sf)$ (or $x\left(\dfrac{t}{s}\right) \xrightarrow{\mathcal{F}} X(sF)$)

If $X(F)$ has a center frequency F_c, then scaling the signal $x(t)$ by a factor $\dfrac{1}{s}$ results in shifting the center frequency (of the scaled signal) to $\dfrac{F_c}{s}$

Note: For real-valued functions, it is more appropriate to refer to $|X(F)|$,

Example: Scaling a Morlet wave



Convolution

5. **Convolution Theorem:** Convolution in time-domain transforms into a product in the frequency domain.

Theorem

If $x_1[k] \xrightarrow{\mathcal{F}} X_1(\omega)$ *and* $x_2[k] \xrightarrow{\mathcal{F}} X_2(\omega)$ *and*

$$x[k] = (x_1 \star x_2)[k] = \sum_{n=-\infty}^{\infty} x_1[n]x_2[k-n]$$

$$\textit{then} \;\; X(f) \triangleq \mathcal{F}\{x[k]\} = X_1(f)X_2(f)$$

This is a highly useful result in the analysis of signals and LTI systems or linear filters.

Product

6. **Dual of convolution:** Multiplication in time corresponds to convolution in frequency domain.

$$x[k] = x_1[k]x_2[k] \xrightarrow{\mathcal{F}} \int_{-1/2}^{1/2} X_1(\lambda)X_2(f-\lambda)\,d\lambda$$

- This result is useful in studying Fourier transform of windowed or finite-length signals such as **STFT and discrete Fourier transform (DFT)**.

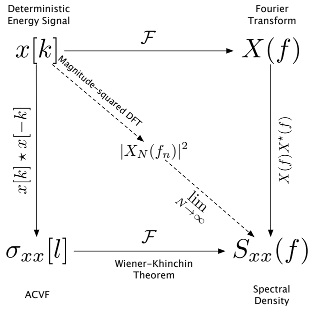
Correlation theorem

7. **Correlation Theorem** (Wiener-Khinchin theorem for deterministic signals)

Theorem

The Fourier transform of the cross-covariance function $\sigma_{x_1x_2}[l]$ is the cross-energy spectral density

$$\mathcal{F}\{\sigma_{x_1x_2}[l]\} = \sum_{l=\infty}^{\infty} \sigma_{x_1x_2}[l]e^{-j2\pi fl} = S_{x_1x_2}(f) = 2\pi S_{x_1x_2}(\omega)$$



- This result provides alternative way of computing spectral densities (esp. useful for random signals)

Discrete Fourier Transform (DFT) and Periodogram

Opening remarks

- Signals encountered in reality are not necessarily periodic.
- Computation of DTFT, i.e., the Fourier transform of discrete-time aperiodic signals, presents two difficulties in practice:
  1. **Only finite-length N measurements are available.**
  2. **DTFT can only be computed at a discrete set of frequencies.**

Computing the DTFT: Practical issues

- Can we compute the finite-length DTFT, i.e., restrict the summation to the extent observed?
- Or do we artificially extend the signal outside the observed interval? Either way what are the consequences?
- Some form of discretization of the frequency axis, *i.e.*, *sampling in frequency* is therefore inevitable.

When the DTFT is restricted to the duration of observation and evaluated on a frequency grid, we have the **Discrete Fourier Transform** (DFT)

Sampled finite-length DTFT: DFT

DFT

The discrete Fourier transform of a finite length sequence $x[k],\ k = 0, 1, \cdots, N-1$ is defined as:

$$X(f_n) = \sum_{k=0}^{N-1} x[k] e^{-j2\pi f_n k}, \tag{41}$$

The transform derives its name from the fact that it is now *discrete in both time and frequency*.

Q: What should be the grid spacing (sampling interval) in frequency?

Main result

For signal $x[k]$ of length N_l, its DTFT $X(f)$ is perfectly recoverable from its sampled version $X(f_n)$ if and only if the frequency axis is sampled uniformly at N_l points in $[-1/2, 1/2)$, i.e., iff

$$\triangle f = \frac{1}{N_l} \quad \text{or} \quad \triangle \omega = \frac{2\pi}{N_l} \tag{42}$$

See Proakis and Manolakis, (2005) for a proof.

N-point DFT

The resulting DFT is known as the N-point DFT with $N = N_l$. The associated analysis and synthesis equations are given by

$$\boxed{\begin{aligned} X[n] &\triangleq X(f_n) = \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi}{N}nk} && n = 0, 1, \cdots, N-1 \quad (43a) \\ x[k] &= \frac{1}{N} \sum_{n=0}^{N-1} X[n] e^{j\frac{2\pi}{N}kn} && k = 0, 1, \cdots, N-1 \quad (43b) \end{aligned}}$$

Unitary DFT

It is also a common practice to use a factor $1/\sqrt{N}$ on both (43a) and (43b) to achieve symmetry of expressions.

$$\boxed{\begin{aligned} X[n] &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x[k] e^{-j2\pi f_n k} & f_n &= \frac{n}{N},\ n = 0, 1, \cdots, N-1 && \text{(44a)} \\ x[k] &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X[n] e^{j2\pi f_n k} & k &= 0, 1, \cdots, N-1 && \text{(44b)} \end{aligned}}$$

The resulting transforms are known as **unitary** transforms since they are norm-preserving, i.e., $||x[k]||_2^2 = ||X[n]||_2^2$.

Reconstructing $X(f)$ from $X[n]$

The reconstruction of $X(f)$ from its N-point DFT is facilitated by the following expression (Proakis and Manolakis, 2005):

$$X(f) = \sum_{n=0}^{N-1} X\left(\frac{2\pi n}{N}\right) P\left(2\pi f - \frac{2\pi n}{N}\right) \quad N \geq N_l \tag{45}$$

$$\text{where} \quad P(f) = \frac{\sin(\pi f N)}{N \sin(\pi f)} e^{-j\pi f(N-1)}$$

- Equation (45) has very close similarities to that for a continuous-time signal $x(t)$ from its samples $x[k]$ (Proakis and Manolakis, 2005).
- Further, the condition $N \geq N_l$ is similar to the requirement for avoiding aliasing.

Consequences of sampling the frequency axis

When the DTFT is evaluated at N equidistant points in $[-\pi, \pi]$, one obtains

$$\begin{aligned} X\left(\frac{2\pi}{N}n\right) &= \sum_{k=-\infty}^{\infty} x[k]e^{-j2\pi nk/N} \qquad n = 0, 1, \cdots, N-1 \\ &= \sum_{l=-\infty}^{\infty} \sum_{k=lN}^{lN+N-1} x[k]e^{-j2\pi nk/N} \\ &= \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} x[k-lN]e^{-j2\pi nk/N} \end{aligned} \tag{46}$$

Now, define $x_p[k] = \sum_{l=-\infty}^{\infty} x[k-lN]$, with period $N_p = N$.

Equivalence between DFT and DTFS

Then (46) appears structurally very similar to that of the coefficients of a DTFS:

$$Nc_n = \sum_{k=0}^{N-1} x_p[k] e^{-j2\pi nk/N} \tag{47}$$

The N-point DFT $X[n]$ of a sequence $\mathbf{x}_N = \{x[0], x[1], \cdots, x[N-1]\}$ is equivalent to the coefficient c_n of the DTFS of the periodic extension of $\mathbf{x}_N$. Mathematically,

$$X[n] = Nc_n, \qquad c_n = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi}{N}kn} \tag{48}$$

Putting together $\cdots$

An N-point DFT *implicitly assumes the given finite-length signal to be periodic with a period equal to* N *regardless of the nature of the original signal.*

- The basis blocks are $\cos(2\pi \frac{k}{N} n)$ and $\sin(2\pi \frac{k}{N} n)$ characterized by the index n
  - The quantity n denotes the number of cycles completed by each basis block for the duration of N samples
- DFT inherits all the properties of DTFT with the convolution property replaced by *circular convolution*.

DFT: Summary

Definition

The N-point DFT and IDFT are given by

$$X[n] = \sum_{k=0}^{N-1} x[k] e^{-j2\pi kn/N}; \qquad x[k] = \frac{1}{N}\sum_{n=0}^{N-1} X[n] e^{j2\pi kn/N}$$

- Introducing $W_N = e^{j2\pi/N}$, the above relationships are also sometimes written as

$$X[n] = \sum_{k=0}^{N-1} x[k] W_N^{-kn}; \quad x[k] = \frac{1}{N}\sum_{n=0}^{N-1} X[n] W_N^{kn}$$

Points to remember

- The frequency resolution in DFT is equal to $1/N$ or $2\pi/N$. **Increasing the length artificially by padding with zeros does not provide any new information** but can only provide a better "display" of the spectrum
- DFT is calculated assuming that the given signal $x[k]$ is periodic and therefore it is a *Fourier series* expansion of $x[k]$ in reality!
- In an N-point DFT, only $N/2+1$ frequencies are unique. For example, in a 1024-point DFT, only 513 frequencies are sufficient to reconstruct the original signal.

DFT in practice: FFT

- The linear transformation relationships are useful for short calculations.
- In 1960s, Cooley and Tukey developed an efficient algorithm for fast computation of DFT which revolutionized the world of spectral analysis
  - This algorithm and its subsequent variations came to be known as the Fast Fourier Transform (FFT), which is available with almost every computational package.
- The FFT algorithm reduced the number of operations from N^2 in regular DFT to the order of $N\log(N)$
- FFT algorithms are fast when N is exactly a power of 2
  - Modern algorithms are not bounded by this requirement!

R: `fft`

Power or energy spectral density?

- Practically we encounter either finite-energy aperiodic or stochastic (or mixed) signals, which are characterized by *energy* and *power* spectral density, respectively.
- However, the practical situation is that we have a finite-length signal $\mathbf{x}^N = \{x[0], x[1], \cdots, x[N-1]\}$.
- Computing the N-point DFT amounts to treating the underlying infinitely long signal $\tilde{x}[k]$ as *periodic* with period N.

Thus, strictly speaking **we have neither densities. Instead DFT always implies a power spectrum (line spectrum) regardless of the nature of underlying signal!**

Periodogram: Heuristic power spectral density

The power spectrum $P_{xx}(f_n)$ for the finite-length signal $\mathbf{x}_N$ is obtained as

$$P_{xx}(f_n) = |c_n|^2 = \frac{|X[n]|^2}{N^2} \tag{49}$$

A *heuristic* power spectral density (power per unit cyclic frequency), known as the **periodogram**, introduced by Schuster, (1897), for the finite-length sequence is used,

$$\boxed{\mathbb{P}_{xx}(f_n) \triangleq \mathsf{PSD}(f_n) = \frac{P_{xx}(f_n)}{\triangle f} = N|c_n|^2 = \frac{|X[n]|^2}{N}} \tag{50}$$

Alternatively,

$$\boxed{\mathbb{P}_{xx}(\omega_n) = \frac{1}{2\pi N}|X[n]|^2} \tag{51}$$

Routines in R

Task	Routine	Remark
Convolution	`convolve`, `conv`	Computes product of DFTs followed by inversion (`conv` from the `signal` package)
Compute IR	`impz`	Part of the `signal` package
Compute FRF	`freqz`	Part of the `signal` package
DFT	`fft`	Implements the FFT algorithm
Periodogram	`spec.pgram`, `periodogram`	Part of the `stats` and `TSA` packages, respectively

Bibliography I

Antoniou, A. (2006). *Digital Signal Processing: Signals Systems and Filters*. USA: McGraw-Hill.

Bloomfield, P. (2000). *Fourier Analysis of Time Series: An Introduction*. 2nd edition. New York, USA: John Wiley & Sons, Inc.

Cohen, L. (1994). *Time Frequency Analysis: Theory and Applications*. Upper Saddle River, New Jersey, USA: Prentice Hall.

Hamilton, J. D. (1994). *Time Series Analysis*. Princeton, NJ, USA: Princeton University Press.

Lighthill, M. (1958). *Introduction to Fourier Analysis and Generalized Functions*. Cambridge, UK: Cambridge University Press.

Priestley, M. B. (1981). *Spectral Analysis and Time Series*. London, UK: Academic Press.

Proakis, J. and D. Manolakis (2005). *Digital Signal Processing - Principles, Algorithms and Applications*. New Jersey, USA: Prentice Hall.

Bibliography II

Schuster, A. (1897). "On lunar and solar periodicities of earthquakes". In: *Proceedings of the Royal Society* 61, pp. 455–465.

Shumway, R. and D. Stoffer (2006). *Time Series Analysis and its Applications*. New York, USA: Springer-Verlag.

Smith, S. W. (1997). *Scientist and Engineer's Guide to Digital Signal Processing*. San Diego, CA, USA: California Technical publishing.

Tangirala, A. K. (2014). *Principles of System Identification: Theory and Practice*. Boca Raton, FL, USA: CRC Press, Taylor & Francis Group.