IIT KHARAGPUR

NPTEL ONLINE CERTIFICATION COURSES

OPTIMIZATION IN CHEMICAL ENGINEERING

Introduction to Linear Programming

(WEEK – 8: LECTURE – 36)

Dr. Debasis Sarkar

Department of Chemical Engineering

Indian Institute of Technology Kharagpur

Introduction to Linear Programming

Week 8:

- Formulation of linear programming models
- Graphical solution
- Linear programs in standard form

Today's Topic:

- Formulation of linear programming models

Introduction to Linear Programming

Linear Programming (LP) is one of the most effective and widely used optimization techniques.

A Linear Programming model seeks to maximize or minimize a linear function, subject to a set of linear constraints.

The linear model consists of the following components:

- A set of decision variables
- An objective function which is a linear function of decision variables
- A set of linear equality or inequality constraints

Application Areas of Linear Programming

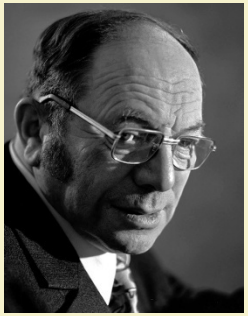
Linear programming is an extremely powerful tool for addressing a wide range of applied optimization problems. A short list of application areas is given below.

- Resource allocation
- Production scheduling
- Warehousing
- Transportation scheduling
- Facility location
- Flight crew scheduling
- Portfolio optimization
- Parameter estimation
- The Diet Problem
- Telecommunication (Call routing, Network design)
- Traveling Salesman Problem (Vehicle routing, VLSI chip design)

How do the Linear Constraints Arise in Chemical Engineering?

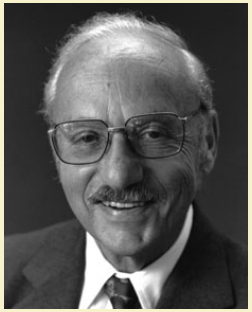
- **Production Limitation:** equipment limitations, storage limits, market constraints.
- **Raw Material Limitation.**
- **Safety Restrictions:** allowable operating ranges for temperature and pressures.
- **Physical Property Specifications:** product quality constraints when a blend property can be calculated as an average of pure component properties.
- **Mass and Energy Balances.**

A Brief History of Linear Programming

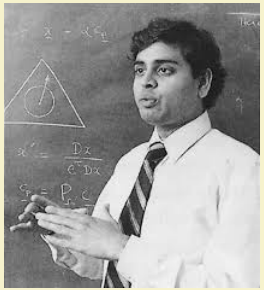


Leonid Kantorovich

Tjalling C. Koopmans



George Dantzig



Narendra Karmarkar

In 1984, Narendra Karmarkar developed "Interior Point Algorithm" for solving large scale LPP.

During World War II, Kantorovich and Koopmans made significant contributions to solution of production planning and transportation problems. They were awarded Nobel Prize in Economics in 1975.

In 1947, George Dantzig developed "Simplex Method" for solution of LPP.

Formulation of a Linear Programming Problem

There are four basic steps in the development/formulation of an LP model:

1. Identify the decision variables for the problem.

2. Determine the objective and write an expression for the objective function as a linear function of the decision variables.

3. Determine the explicit constraints, and write them as either a linear equation or a linear inequality in the decision variables.

4. Determine the implicit constraints, and write them as either a linear equation or a linear inequality in the decision variables.

Product Mix Problem: Example - 1

A fertilizer manufacturing company produces two types of fertilizer: Type A: having high phosphorus content, and Type B: having low phosphorus content.

Raw Material	Tons required per ton of fertilizer		Maximum availability per day, ton
	Type A	Type B	
Urea	2	1	3000
Potash	1	1	2400
Rock phosphate	1	0	1000
Net profit per ton	**30**	**20**	

What should be the daily production schedule to maximize the profit?

Product Mix Problem: Formulation: Step-1

Raw Material	Tons required per ton of fertilizer		Maximum availability per day
	Type A	Type B	
Urea	2	1	3000
Potash	1	1	2400
Rock phosphate	1	0	1000
Net profit per ton	30	20	

Decision variables:

Daily production of Type A: x tons, Type B: y tons

The first step in problem formulation is to identify the decision variables. These variables represent the quantifiable decisions that must be made to determine the daily production schedule.

In other words, we have to specify those quantities (variables) whose values completely determine a production schedule and the profit associated with it.

Product Mix Problem: Formulation: Step-2

Raw Material	Tons required per ton of fertilizer		Maximum availability per day
	Type A	Type B	
Urea	2	1	3000
Potash	1	1	2400
Rock phosphate	1	0	1000
Net profit per ton	30	20	

Once we have identified the decision variables, we have to specify the problem objective.

That is, we need to write an expression for the objective function as a linear function of the decision variables.

Decision Variables:

Daily production of Type A: x tons, Type B: y tons

The Objective Function:

Maximize profit, $z = 30x + 20y$

Product Mix Problem: Formulation: Step-3

Raw Material	Tons required per ton of fertilizer		Maximum availability per day
	Type A	Type B	
Urea	2	1	3000
Potash	1	1	2400
Rock phosphate	1	0	1000
Net profit per ton	30	20	

The next step in the problem formulation is to express the feasible region as the solution set of a finite collection of linear inequality and equality constraints.
-Explicit constraints
-Implicit constraints

Decision Variables:

Daily production of Type A: *x* tons, Type B: *y* tons

Explicit Constraints:

$2x + y \leq 3000$

$x + y \leq 2400$

$x \leq 1000$

The explicit constraints are those that are explicitly given in the problem statement. In the given problem, there are explicit constraints on the maximum availability of 3 Raw Materials.

Product Mix Problem: Formulation: Step-4

Raw Material	Tons required per ton of fertilizer		Maximum availability per day
	Type A	Type B	
Urea	2	1	3000
Potash	1	1	2400
Rock phosphate	1	0	1000
Net profit per ton	30	20	

Decision Variables:

Daily production of Type A: *x* tons, Type B: *y* tons

Implicit Constraints:

$x \geq 0$

$y \geq 0$

The implicit constraints are not explicitly stated in the problem statement but are present nonetheless. Typically, these constraints are associated with "natural" or "common sense" restrictions on the decision Variables. In the given problem, it is clear that one cannot produce negative amount of fertilizers. Thus x and y must be non-negative.

Product Mix Problem: Complete Formulation

Raw Material	Tons required per ton of fertilizer		Maximum availability per day
	Type A	Type B	
Urea	2	1	3000
Potash	1	1	2400
Rock phosphate	1	0	1000
Net profit per ton	30	20	

Maximize $f(x, y) = 30x + 20y$

subject to

$2x + y \leq 3000$

$x + y \leq 2400$

$x \leq 1000$

$x \geq 0$

$y \geq 0$

Decision Variables:

Daily production of Type A: x tons, Type B: y tons

Refinery Blending Problem: Example - 2

Crude Oil	Cost (INR/L)	Daily Availability (L)
A	30	6,000
B	40	10,000
C	50	12,000

An oil refinery makes two blends of fuel (Premium Petrol and Regular Petrol) by mixing three types of crude oils (A, B, and C) as shown in two Tables.

How to blend crude oils so that the refinery can maximize its profit?

Petrol Type/Daily Market Demand (L)	Selling Price (INR/L)	Composition
Regular/10,000	80	at least 30% of A, at most 50% of B, at least 30% of C
Premium/10,000	85	at most 40% of A, at least 35% of B, at most 40% of C

Refinery Blending Problem: Example - 2

Define the decision variables:

x_1 = amount (L) of Crude Oil A used in making Premium Petrol
x_2 = amount (L) of Crude oil B used in making Premium Petrol
x_3 = amount (L) of Crude Oil C used in making Premium Petrol

x_4 = amount (L) of Crude Oil A used in making Regular Petrol
x_5 = amount (L) of Crude Oil B used in making Regular Petrol
x_6 = amount (L) of Crude Oil C used in making Regular Petrol

Refinery Blending Problem: Example - 2

Define the Objective Function:

$$\text{Cost} = 30(x_1 + x_4) + 40(x_2 + x_5) + 50(x_3 + x_6)$$

$$\text{Selling Price} = 80(x_1 + x_2 + x_3) + 85(x_4 + x_5 + x_6)$$

$$\text{Profit} = \text{Selling Price} - \text{Cost}$$

$$= \left[30(x_1 + x_4) + 40(x_2 + x_5) + 50(x_3 + x_6)\right] - \left[80(x_1 + x_2 + x_3) + 85(x_4 + x_5 + x_6)\right]$$

The objective function:

$$\underset{x_1,\ldots\ldots,x_6}{\text{Maximize}}\ f = \left[30(x_1 + x_4) + 40(x_2 + x_5) + 50(x_3 + x_6)\right] - \left[80(x_1 + x_2 + x_3) + 85(x_4 + x_5 + x_6)\right]$$

Refinery Blending Problem: Example - 2

Define the Explicit Constraints:

Amount of oil available

$$x_1 + x_4 \le 6000$$
$$x_2 + x_5 \le 10000$$
$$x_3 + x_6 \le 12000$$

Daily market demand

$$x_1 + x_2 + x_3 \ge 10000$$
$$x_4 + x_5 + x_6 \ge 10000$$

Constraints on composition

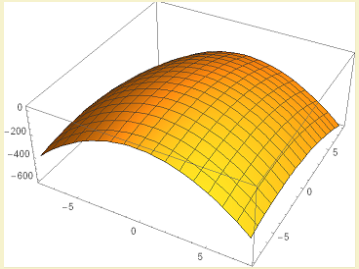
$$\frac{x_1}{x_1 + x_2 + x_3} \ge 0.3$$
$$\frac{x_2}{x_1 + x_2 + x_3} \le 0.5$$
$$\frac{x_3}{x_1 + x_2 + x_3} \ge 0.3$$
$$\frac{x_4}{x_4 + x_5 + x_6} \le 0.4$$
$$\frac{x_5}{x_4 + x_5 + x_6} \ge 0.35$$
$$\frac{x_6}{x_4 + x_5 + x_6} \le 0.4$$

Define the Implicit Constraints:

$$x_1, ..., x_6 \ge 0$$

Rewrite the 1st composition constraint: $-0.7x_1 + 0.3x_2 + 0.3x_3 \le 0$

Optimization in Chemical Engineering



Thank You

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Today's Topic:

- Graphical solution

Graphical Solution of Linear Programming Problem

Linear Programming problems in TWO variables can be very conveniently solved by Graphical Method.

1. Set up objective function and constraints in mathematical format
2. Plot the inequality constraints as equations
3. Identify the feasible solution space
4. Plot the objective function as straight line
5. Determine the optimum solution

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Graphical Solution of LPP: Feasible Region

When we plot each inequality as equation on the graph we get the feasible region.

If the inequality constraint corresponding to a constraint line is $\leq$ type, then the region below the line in the first quadrant is feasible region for that constraint.

If the inequality constraint is of $\geq$ type, then the region above the line in the first quadrant is feasible region for that constraint.

When we plot all the constraints, the points lying in common region will satisfy all the constraints simultaneously. This common region is called feasible region. We will locate the optimal solution in this feasible region.

Example: Graphical Solution of LPP

A chemical company can make two types of products: Product-A and Product-B from the same raw material using the same processing equipment.

Product	Processing time required (hour/kg)	Raw material required (kg/kg)	Profit (Rs/kg)
A	1	4	40
B	2	3	50

There are 40 hours of processing time and 120 kg of raw material available each day

How much A and B should be produced to maximize profits given processing time and materials constraints?

Problem Formulation: Graphical Solution of LPP

Resource Availability:	40 hrs of processing time per day 120 kg of raw material per day
Decision Variables:	x_1 = amount (kg) of Product-A to produce per day x_2 = amount (kg) of Product-B to produce per day
Objective Function:	Maximize $Z = 40x_1 + 50x_2$ where Z = profit per day
Resource Constraints:	$1x_1 + 2x_2 \leq 40$ hours of processing time $4x_1 + 3x_2 \leq 120$ kg of raw materials
Non-Negativity Constraints:	$x_1 \geq 0$; $x_2 \geq 0$

LPP: Feasible Solution

Maximize $Z = 40x_1 + 50x_2$

subject to:

$x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

A feasible solution does not violate any of the constraints.

Example: $x_1 = 5$ kg of A

$x_2 = 10$ kg of B

$Z = 40x_1 + 50x_2 = 700$

Processing time constraint check: $1(5) + 2(10) = 25 < 40$ hours

Raw material constraint check: $4(5) + 3(10) = 50 < 120$ kg

LPP: Infeasible Solution

Maximize $Z = 40x_1 + 50x_2$

subject to:

$x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

An infeasible solution violates at least one of the constraints:

Example:

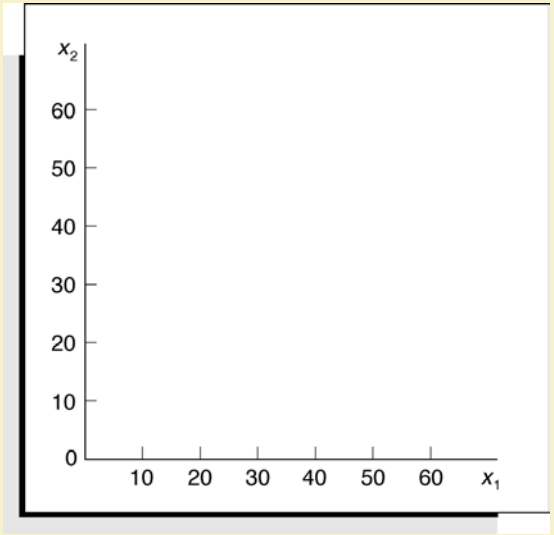
x_1 = 10 kg of A

x_2 = 20 kg of B

$Z = 40x_1 + 50x_2 = 1400$

Processing time constraint check: 1(10) + 2(20) = 50 > 40 hours

Graphical Solution of LPP: Coordinates



x_1 = amount (kg) of Product-A to produce per day

x_2 = amount (kg) of Product-B to produce per day

Maximize $Z = 40x_1 + 50x_2$

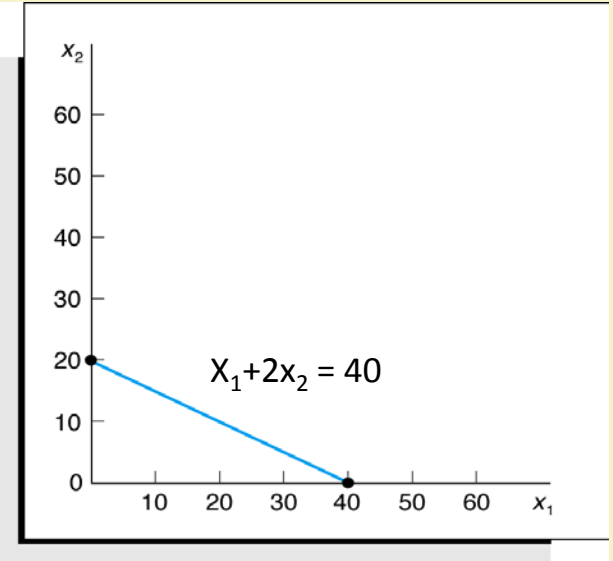
subject to:

$x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Graphical Solution of LPP: Plot the Constraints



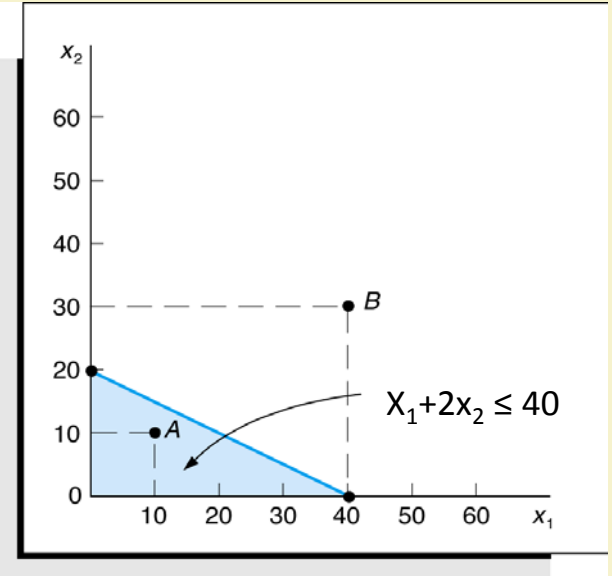
x_1 = amount (kg) of Product-A to produce per day
x_2 = amount (kg) of Product-B to produce per day

Maximize $Z = 40x_1 + 50x_2$

subject to:

$x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

Graphical Solution of LPP: Plot the Constraints



x_1 = amount (kg) of Product-A to produce per day

x_2 = amount (kg) of Product-B to produce per day

Maximize $Z = 40x_1 + 50x_2$

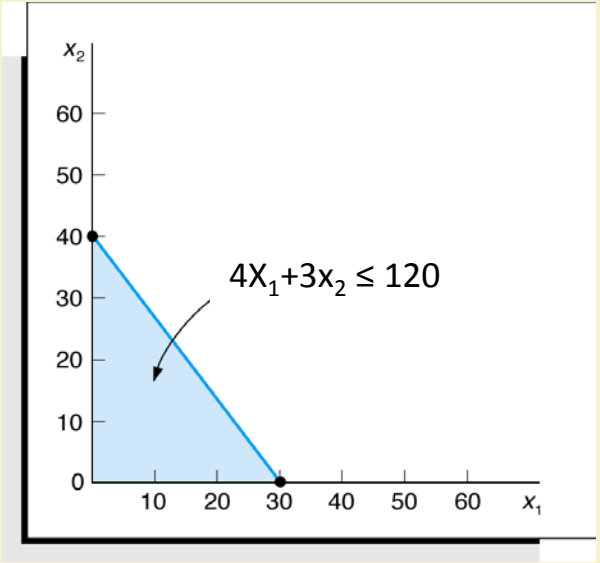
subject to:

$x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Graphical Solution of LPP: Plot the Constraints



x_1 = amount (kg) of Product-A to produce per day

x_2 = amount (kg) of Product-B to produce per day

Maximize $Z = 40x_1 + 50x_2$

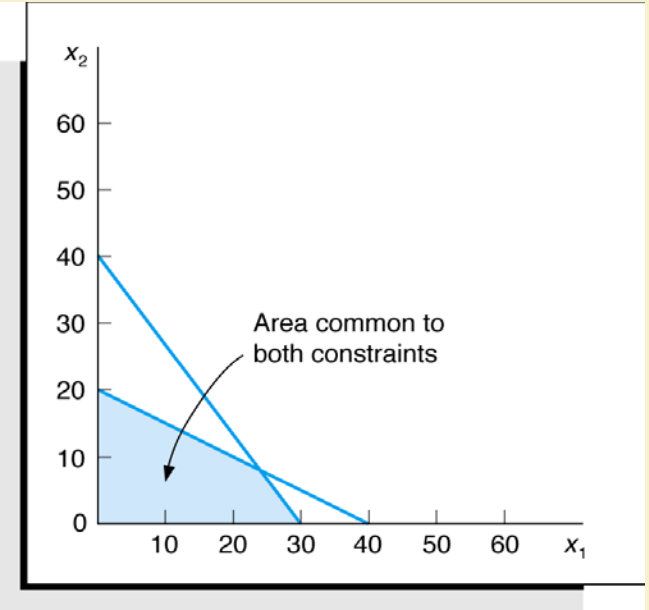
subject to:

$x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Graphical Solution of LPP: Feasible Region



x_1 = amount (kg) of Product-A to produce per day

x_2 = amount (kg) of Product-B to produce per day

Maximize $Z = 40x_1 + 50x_2$

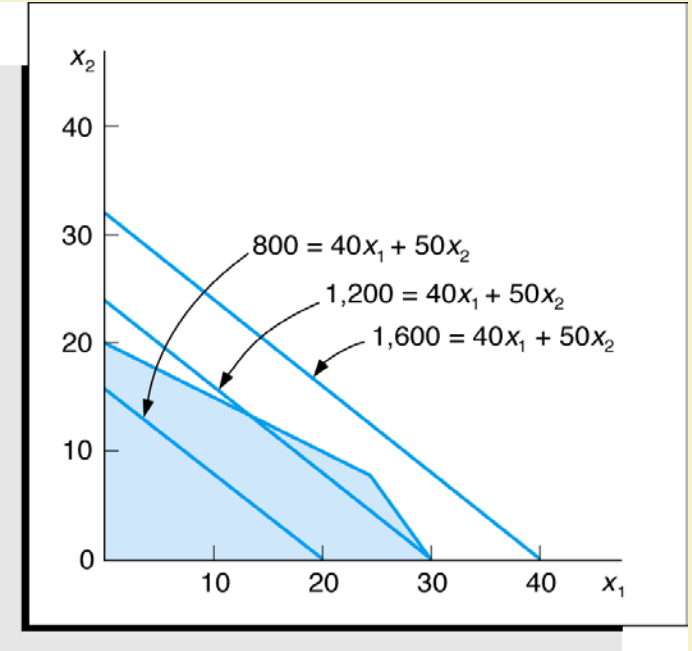
subject to:

$x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Graphical Solution of LPP: Plot Objective Function



x_1 = amount (kg) of Product-A to produce per day

x_2 = amount (kg) of Product-B to produce per day

Maximize $Z = 40x_1 + 50x_2$

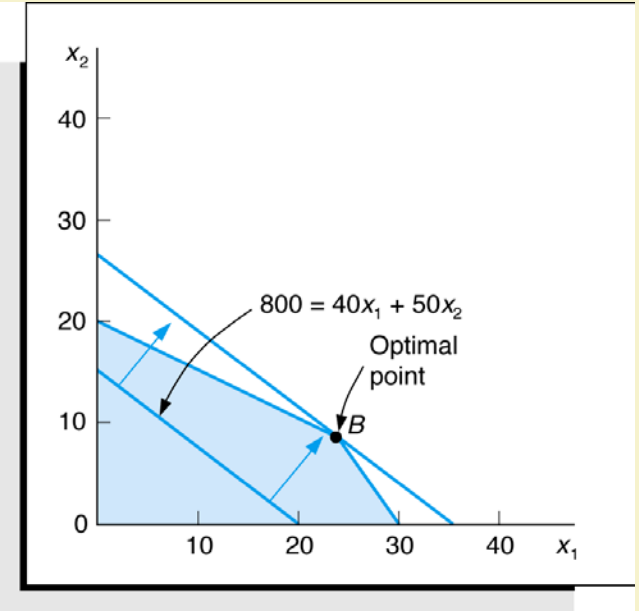
subject to:

$x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Graphical Solution of LPP: Find Optimal Solution



x_1 = amount (kg) of Product-A to produce per day

x_2 = amount (kg) of Product-B to produce per day

Maximize $Z = 40x_1 + 50x_2$

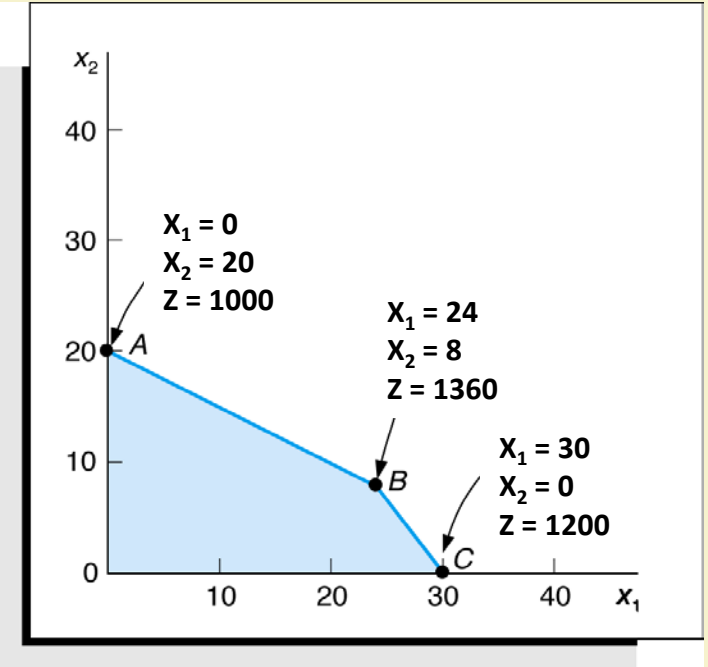
subject to:

$x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Graphical Solution of LPP: Corner Solutions



x_1 = amount (kg) of Product-A to produce per day
x_2 = amount (kg) of Product-B to produce per day

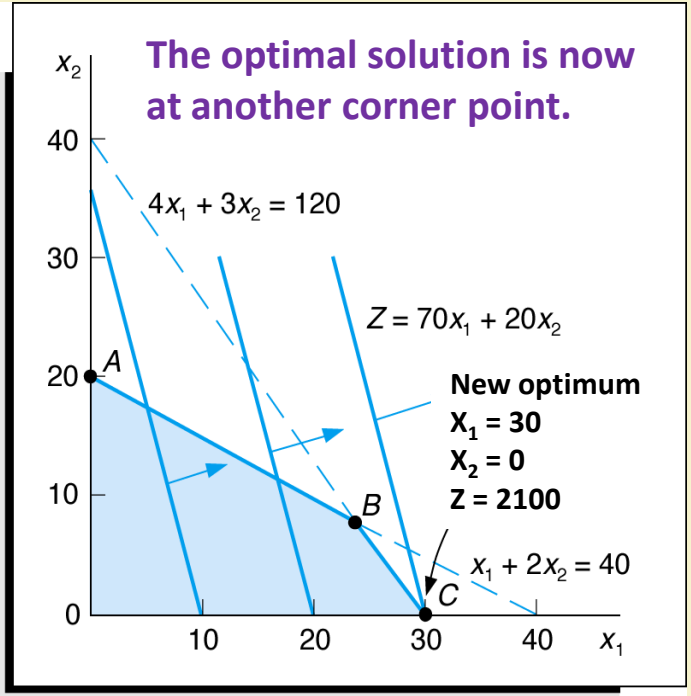
Maximize $Z = 40x_1 + 50x_2$

subject to:

$$x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$

Solution lies in one (or more) corner.
LPP is convex problem. The optimal solution is global optimal solution.

Graphical Solution of LPP: Change in Objective Function



x_1 = amount (kg) of Product-A to produce per day
x_2 = amount (kg) of Product-B to produce per day

Old Objective Function: $Z = 40x_1 + 50x_2$

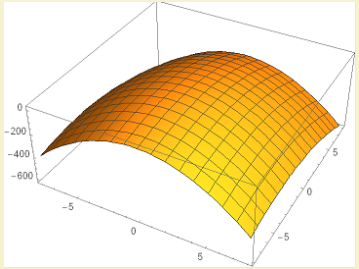
New Objective Function: $Z = 70x_1 + 20x_2$

Maximize $Z = 70x_1 + 20x_2$

subject to:

$$x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$

Optimization in Chemical Engineering



Thank You

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Today's Topic:

- Graphical solution (Cont'd)

Graphical Solution of LPP: Feasible Region: Review

When we plot each inequality as equation on the graph we get the feasible region.

If the inequality constraint corresponding to a constraint line is $\leq$ type, then the region below the line in the first quadrant is feasible region for that constraint.

If the inequality constraint is of $\geq$ type, then the region above the line in the first quadrant is feasible region for that constraint.

When we plot all the constraints, the points lying in common region will satisfy all the constraints simultaneously. This common region is called feasible region. We will locate the optimal solution in this feasible region.

Example: Graphical Solution of LPP

Two brands of fertilizer are available: N-rich and P-rich. The chemical composition and cost of each brand are shown in the Table below.

Suppose a field requires at least 16 kg of nitrogen and 24 kg of phosphate. How much of each brand of fertilizers should be purchased to minimize the total cost of the fertilizer for the filed?

Brand	Chemical composition		Cost (INR/bag)
	Nitrogen (kg/bag)	Phosphate (kg/bag)	
P-rich	2	4	600
N-rich	4	3	300

This is a cost minimization problem.

Graphical Solution of LPP: Problem Formulation

Decision Variables:

x_1 = bags of P-rich fertilizer

x_2 = bags of N-rich fertilizer

The Objective Function:

Minimize $Z = 600x_1 + 300x_2$

Where: $600x_1$ = cost of bags of P-rich fertilizer

$300x_2$ = cost of bags of N-rich fertilizer

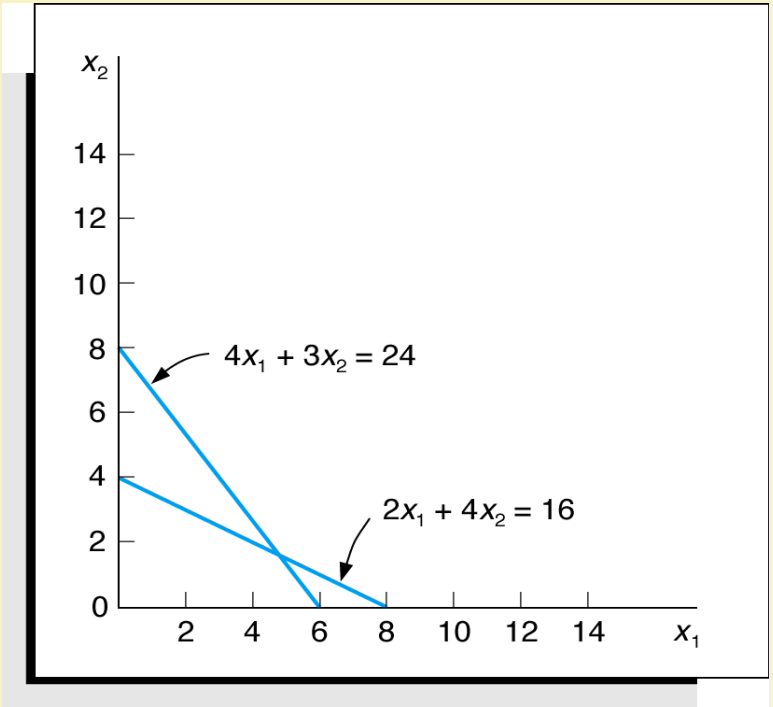
Model Constraints:

$2x_1 + 4x_2 \geq 16$ kg (nitrogen constraint)

$4x_1 + 3x_2 \geq 24$ kg (phosphate constraint)

$x_1, x_2 \geq 0$ (non-negativity constraint)

Graphical Solution of LPP: Plot the Constraints



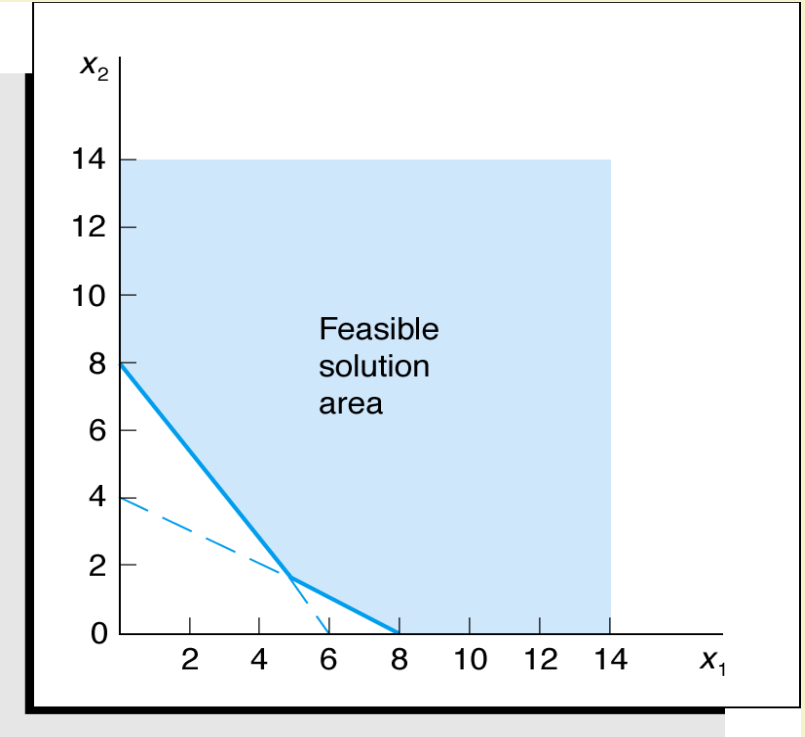
Minimize $Z = 600x_1 + 300x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_1 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$

Graphical Solution of LPP: Feasible Region



Minimize $Z = 600x_1 + 300x_2$

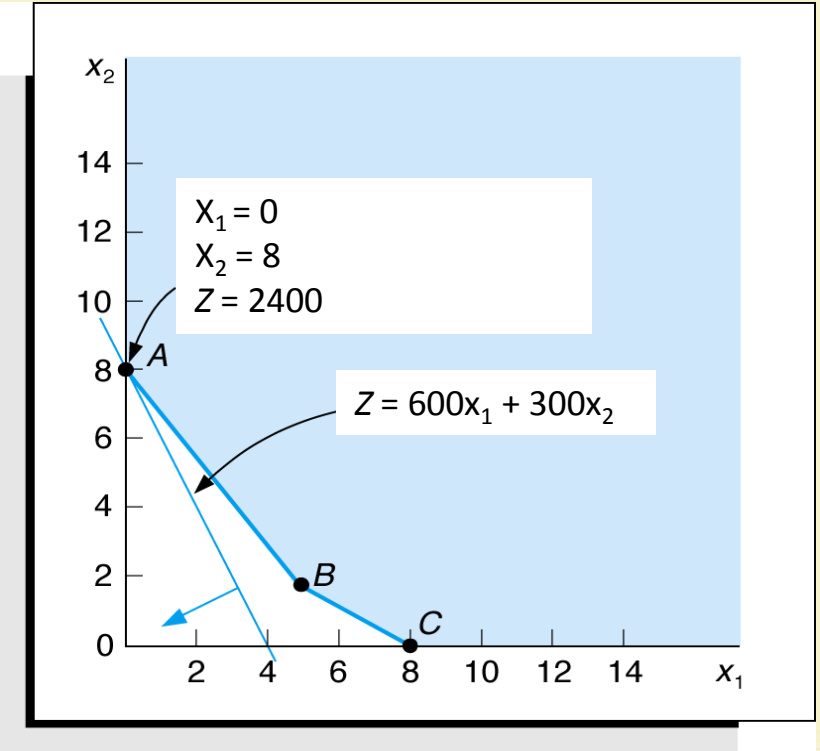
subject to: $2x_1 + 4x_2 \geq 16$

$4x_1 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$

Note: The feasible region is unbounded.

Graphical Solution of LPP: Optimal Solution



Minimize $Z = 600x_1 + 300x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_1 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$

Corner Points:

A: $x_1 = 0, x_2 = 8, Z = 2400$

B: $x_1 = 4.8, x_2 = 1.6, Z = 3360$

C: $x_1 = 8, x_2 = 0, Z = 4800$

Graphical Solution of Linear Programming Problem: Property

- **Feasible Region**: The set of points that satisfies all constraints.
- **Corner Points**: An optimal solution must lie at one or more corner points.
- **Optimal Solution**: The corner point with the best objective function value is optimal solution.

If there exists an optimal solution to a LPP, then at least one of the corner points of the feasible region will always qualify to be an optimal solution.

Graphical Solution of Linear Programming Problem: Special Cases

- The feasible region for a two-variable LP problem can be nonexistent, a single point, a line, a polygon, or an unbounded area.
- Any linear program falls in one of the following four categories:
  - LP is infeasible
  - LP has a unique optimal solution
  - LP has alternative optimal solutions
  - LP has an objective function that can be increased without bound

- A feasible region may be unbounded and yet there may be optimal solutions. This is common in minimization problems.

Special Cases: LPP: Redundant Constraints

Redundant Constraints do not affect the feasible region

Example: $x \leq 10$

$x \leq 12$

The second constraint is redundant because it is less restrictive.

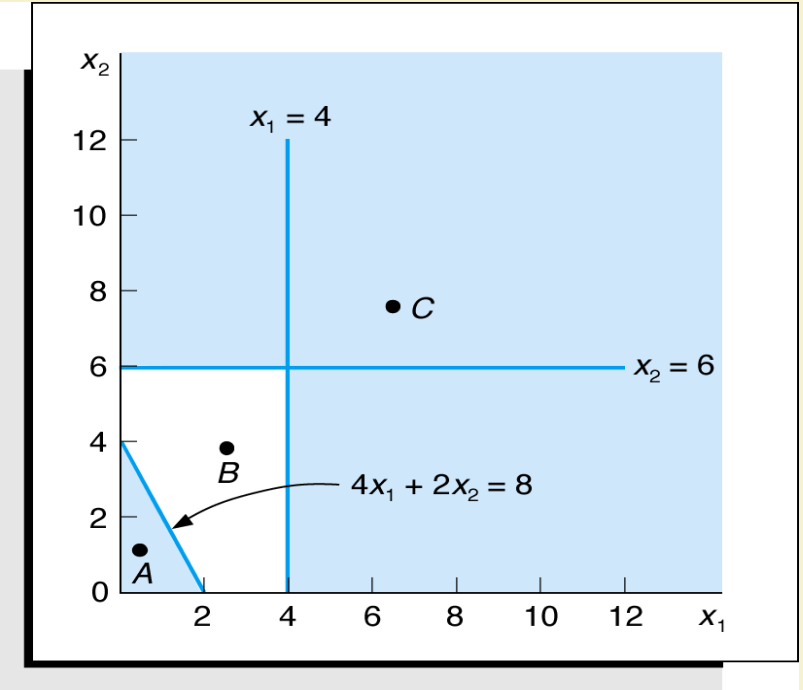
Special Cases: LPP: Infeasibility

Infeasibility – when no feasible solution exists (there is no feasible region)

Example: $x \leq 10$

$x \geq 15$

Special Cases: LPP: Infeasibility



Every possible solution violates at least one constraint. Consider points A, B, and C.

Maximize $Z = 5x_1 + 3x_2$

subject to: $4x_1 + 2x_2 \leq 8$

$x_1 \geq 4$

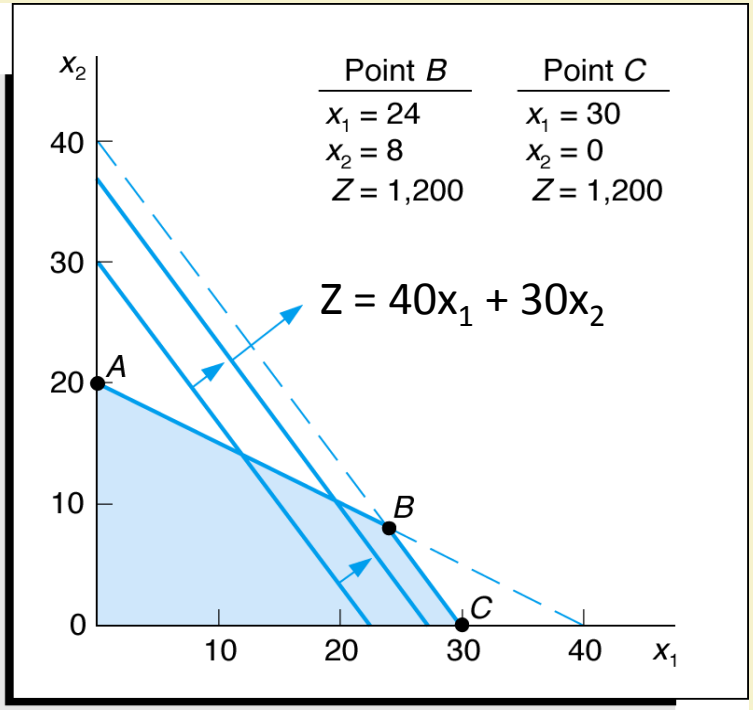
$x_2 \geq 6$

$x_1, x_2 \geq 0$

Graphical Solution of LPP: Alternate Optimal Solution

In the graphical method, if the objective function line is parallel to a constraint in the direction of optimization, there are alternate optimal solutions, with all points on this line segment being optimal.

Graphical Solution of LPP: Alternate Optimal Solution



LP will have alternate optimal solution when the objective function is parallel to a constraint line.

Maximize $Z = 40x_1 + 50x_2$
subject to: $x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

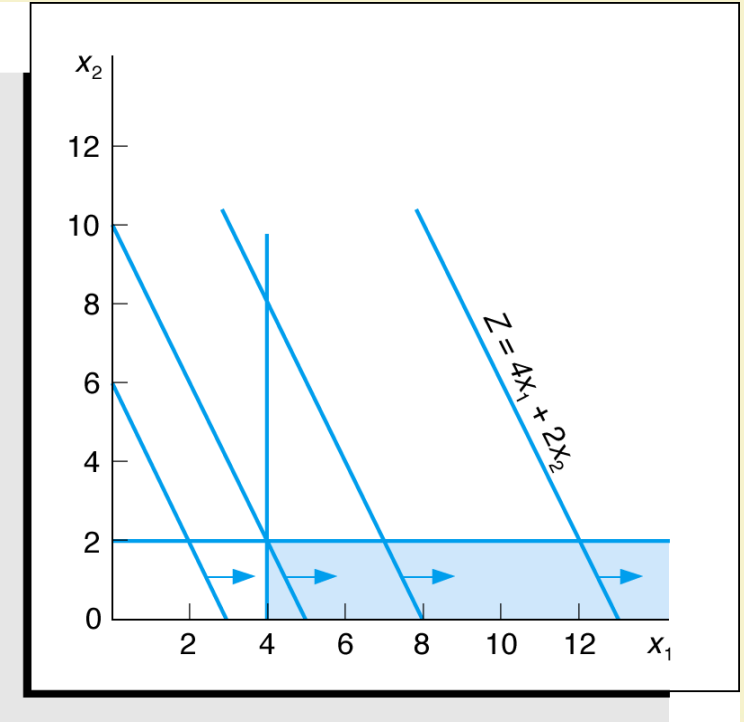
Production scheduling problem (Lecture 37)

Maximize $Z = 40x_1 + 30x_2$
subject to: $x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

Objective function is changed and it is parallel to the second constraint.

Every point on line BC is an optimal solution.

Graphical Solution of LPP: Unbounded Solution



Value of the objective function increases indefinitely.

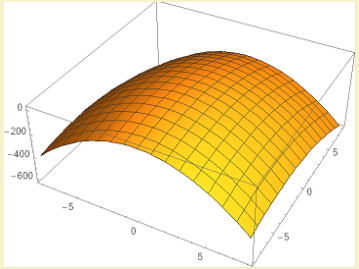
Maximize $Z = 4x_1 + 2x_2$

subject to: $x_1 \geq 4$

$x_2 \leq 2$

$x_1, x_2 \geq 0$

Optimization in Chemical Engineering



Thank You

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OPTIMIZATION IN CHEMICAL ENGINEERING

Introduction to Linear Programming

(WEEK – 8: LECTURE – 39)

Dr. Debasis Sarkar

Department of Chemical Engineering

Indian Institute of Technology Kharagpur

Introduction to Linear Programming

Week 8:

- ✓ Formulation of linear programming models
- ✓ Graphical solution
- Linear programs in standard form

Today's Topic:

- Linear programs in standard form

Linear Program in Standard Form

In general an LP could

- be either a maximization or a minimization problem
- include both equalities and inequalities in the constraints
- have negative and positive variables

But we can convert any of these forms into one equivalent *standard form*

- always a minimization/maximization problem
- all constraints are either equalities or inequalities
- only has positive variables

Linear Program in Standard Form

The standard form of an LPP with m constraints and n variables ($m \le n$) can be represented as:

Maximize (Minimize): $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n$ ← Objective Function (cost function)

Subject to:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

} Constraint Set

$$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0$$

Non-negative Variables Constraint

Non-negative Right-hand side Constants →

$$b_1 \ge 0, b_2 \ge 0, \ldots, b_m \ge 0$$

Linear Program in Standard Form: Matrix Notation

$$\text{Maximize or Minimize } Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n$$

Subject to:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$
$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$
$$b_1 \geq 0, b_2 \geq 0, \ldots, b_m \geq 0$$

In Matrix Notation:

$$\text{Maximize (Minimize): } Z = \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{Subject to : } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$
$$\boldsymbol{x} \geq 0, \quad \boldsymbol{b} \geq 0$$

A: m n matrix
x: n 1 vector
b: m 1 vector
c: n 1 vector

Linear Program in Standard Form: Matrix Notation

Maximize (Minimize): $Z = \boldsymbol{c}^T \boldsymbol{x}$

Subject to: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$\boldsymbol{x} \geq 0, \quad \boldsymbol{b} \geq 0$

where,

$$\boldsymbol{c}^T = [c_1 \; c_2 \; \ldots \; c_n]$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & & & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_m \end{bmatrix} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix}$$

LP in Standard Form: Handling Inequalities

Not all LPP come in standard form. Very often the constraints are expressed as inequalities rather than equations. In some problems, not all the design variables may be restricted to be nonnegative. Thus, the first step in solving LPP is to convert it to a problem in standard form.

Inequality constraints can be converted to equations by introducing *slack* or *surplus* variables.

$$2x_1 - 5x_2 + 4x_3 \le 8 \quad \Rightarrow 2x_1 - 5x_2 + 4x_3 + S_1 = 8$$

$$5x_1 + x_2 + 6x_3 \ge 10 \Rightarrow 5x_1 + x_2 + 6x_3 - S_2 = 10$$

S_1 = slack variable

S_2 = surplus variable

$S_1 \ge 0,\ S_2 \ge 0$

Introduce slack variable S_1 for ≤ and surplus variable S_2 for ≥ constraints. At optimal point, the values of S_1 and S_2 indicate whether the constraints are active (binding) or not.

LP in Standard Form: Handling Unrestricted Variables

Sometimes decision variables are unrestricted in sign (positive, negative or zero). In all such cases, the decision variables can be expressed as the difference between two new non-negative variables.

Consider:

$\min Z = x_1 - 2x_2$

subject to: $x_2 \geq 0$, x_1 is unrestricted

Replace x_1 as:

$x_1 = x_3 - x_4$

$\min Z = x_3 - x_4 - 2x_2$

subject to: $x_2, x_3, x_4 \geq 0$

LP in Standard Form: Example - 1

Write the following LPP in standard form. Write its as minimization LPP.

$$\begin{aligned} \max\ & x_1 + 6x_2 \\ & x_1 \le 200 \\ & x_2 \le 300 \\ & x_1 + x_2 \le 400 \\ & x_1, x_2 \ge 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \min\ & -x_1 - 6x_2 \\ & x_1 + s_1 = 200 \\ & x_2 + s_2 = 300 \\ & x_1 + x_2 + s_3 = 400 \\ & x_1, x_2, s_1, s_2, s_3 \ge 0 \end{aligned}$$

In matrix notation:

$$x^T = [x_1\ x_2\ s_1\ s_2\ s_3],\ c^T = [-1\ \ -6]$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix},\quad b = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$$

$$\min\ c^T x$$
$$\text{subject to}: Ax = b$$
$$x \ge 0,\ b \ge 0$$

LP in Standard Form: Example - 2

Write the following LPP in standard form.

$$\min Z = x_1 - 2x_2 + 3x_3$$

subject to: $x_1 + 2x_2 + x_3 \le 9$

$$2x_1 - x_2 + x_3 \ge 5$$

$$4x_1 - x_2 - 2x_3 = -6$$

$$x_1 \ge 0,\ x_2 \ge 0$$

x_3 unrestricted

LPP in standard form:

Replace x_3 as : $x_3 = x_4 - x_5$

Introduce slack variable: x_6

Introduce surplus variable: x_7

$$\min Z = x_1 - 2x_2 + 3x_4 - 3x_5$$

subject to : $x_1 + 2x_2 + x_4 - x_5 + x_6 = 9$

$$2x_1 - x_2 + x_4 - x_5 - x_7 = 5$$

$$-4x_1 + x_2 + 2x_4 - 2x_5 = 6$$

$$x_1, x_2, x_4, x_5, x_6, x_7 \ge 0$$

The LPP on Fertilizer: Revisit

Decision Variables:

x_1 = bags of P-rich fertilizer

x_2 = bags of N-rich fertilizer

The Objective Function:

Minimize $Z = 600x_1 + 300x_2$

Where: $600x_1$ = cost of bags of P-rich fertilizer

$300x_2$ = cost of bags of N-rich fertilizer

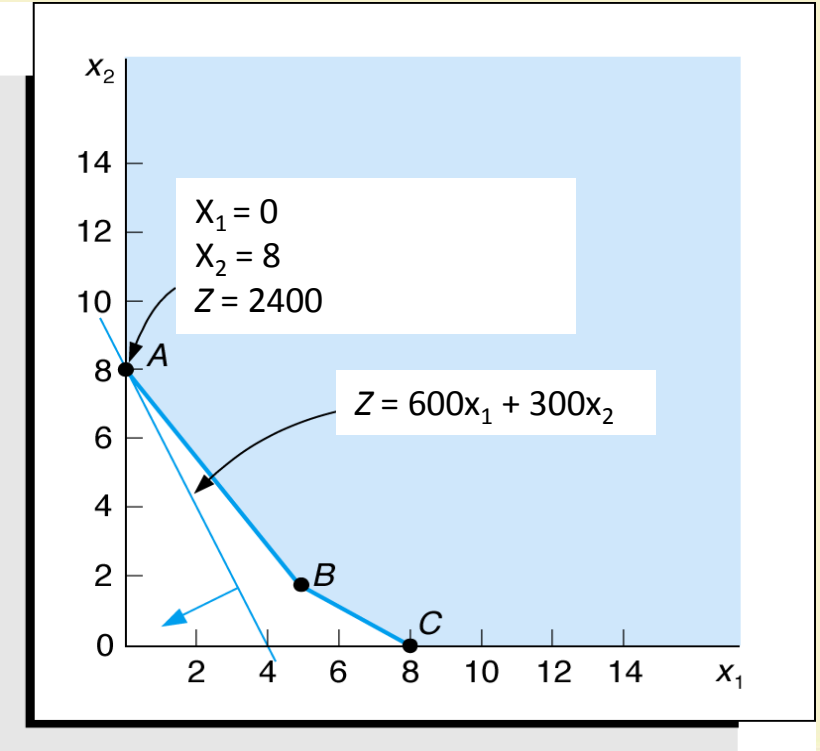
Model Constraints:

$2x_1 + 4x_2 \geq 16$ kg (nitrogen constraint)

$4x_1 + 3x_2 \geq 24$ kg (phosphate constraint)

$x_1, x_2 \geq 0$ (non-negativity constraint)

Graphical Solution of the LPP: Optimal Solution



Minimize $Z = 600x_1 + 300x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_1 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$

Corner Points:

A: $x_1 = 0, x_2 = 8, Z = 2400$

B: $x_1 = 4.8, x_2 = 1.6, Z = 3360$

C: $x_1 = 8, x_2 = 0, Z = 4800$

Transition from Graphical to Algebraic Solution

Minimize $Z = 600x_1 + 300x_2$
subject to: $2x_1 + 4x_2 \geq 16$
$4x_1 + 3x_2 \geq 24$
$x_1, x_2 \geq 0$

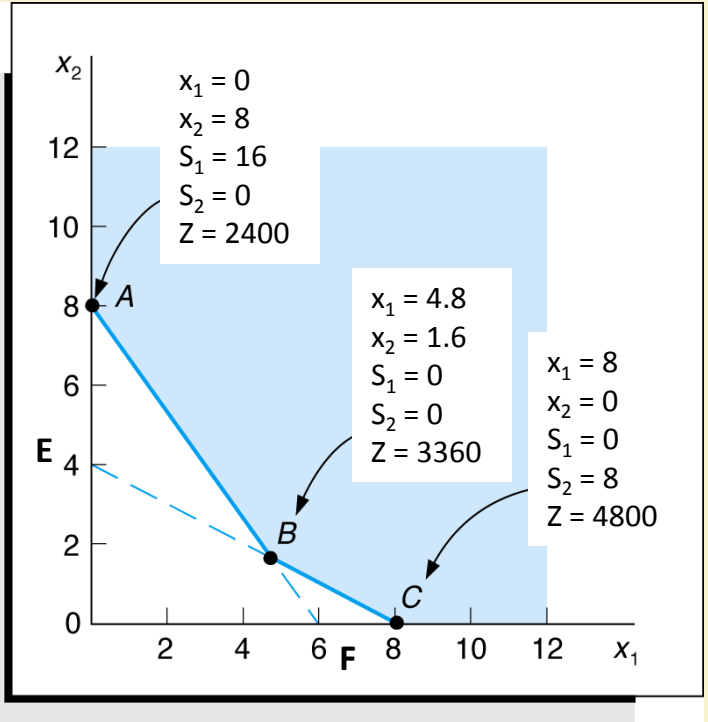
Consider algebraic solution. The solution space has 2 equations, 4 variables. Corner points can be found by setting two variables to zero and solving for the remaining two variables.

Zero variables	Solution	Feasible?
$S_1=0,\ S_2=0$	$x_1=4.8, x_2=1.6$	Yes
$x_1=0,\ x_2=0$	$S_1=-16, S_2=-24$	No
$x_1=0, S_1=0$	$x_2=4, S_2=-12$	No
$x_1=0, S_2=0$	$x_2=8, S_1=16$	Yes
$x_2=0, S_1=0$	$x_1=8, S_2=8$	Yes
$x_2=0, S_2=0$	$x_1=6, S_1=-4$	No

Slack/surplus variables do not contribute to cost function.

Minimize $Z = 600x_1 + 300x_2 + 0.S_1 + 0.S_2$
subject to: $2x_1 + 4x_2 - S_1 = 16$
$4x_1 + 3x_2 - S_2 = 24$
$x_1, x_2, S_1, S_2 \geq 0$

Transition from Graphical to Algebraic Solution

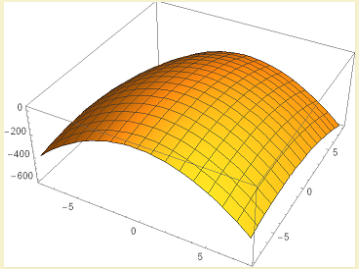


Zero variables	Solution	Corner Point	Feasible?	Objective function
$S_1=0, S_2=0$	$x_1=4.8, x_2=1.6$	B	Yes	3360
$x_1=0, x_2=0$	$S_1=-16, S_2=-24$	Origin	No	-
$x_1=0, S_1=0$	$x_2=4, S_2=-12$	E	No	-
$x_1=0, S_2=0$	**$x_2=8, S_1=16$**	**A**	**Yes**	**2400**
$x_2=0, S_1=0$	$x_1=8, S_2=8$	C	Yes	4800
$x_2=0, S_2=0$	$x_1=6, S_1=-4$	F	No	-

Optimum solution is located at point – A ($x_1=0, x_2=8$).

This equivalence leads to the Simplex method.

Optimization in Chemical Engineering



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OPTIMIZATION IN CHEMICAL ENGINEERING

Introduction to Linear Programming

(WEEK – 8: LECTURE – 40)

Dr. Debasis Sarkar

Department of Chemical Engineering

Indian Institute of Technology Kharagpur

Introduction to Linear Programming

Week 8:

- ✓ Formulation of linear programming models
- ✓ Graphical solution
- Linear programs in standard form

Today's Topic:

- Linear programs in standard form (Cont'd)
- Solution of system of linear equations

Linear Program in Standard Form: Some Definitions

Maximize: $Z = \boldsymbol{c}^T \boldsymbol{x}$

Subject to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$\boldsymbol{x} \geq 0, \quad \boldsymbol{b} \geq 0$

A: m n matrix
x: n 1 vector
b: m 1 vector
c: n 1 vector

A feasible solution is a nonnegative vector x that satisfy the constraints: Ax = b

A feasible region, denoted by S, is the set of all feasible solutions. Mathematically, we can write: $S = \{x \mid Ax = b,\ x \geq 0\}$

If the feasible set S is empty, then the linear program is infeasible.

An optimal solution is a vector x* that is feasible and its value of the objective function (cx*) is greater than any other feasible solution. Mathematically, $cx^* \geq cx \ \forall x \in S$

Linear Program in Standard Form: Some Definitions

Maximize: $Z = \boldsymbol{c}^T \boldsymbol{x}$

Subject to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$\boldsymbol{x} \geq 0, \quad \boldsymbol{b} \geq 0$

A: m n matrix
x: n 1 vector
b: m 1 vector
c: n 1 vector

The optimal value of a LP is the value of the objective function at the optimal solution. Thus, if x* is the optimal solution, then Z* = cx* is the optimal value of the LP.

If a LP has more than one optimal solution, it is said to have alternate optimal solutions. In this case there will exist more than one feasible solutions having the same optimal value Z* of the objective function.

Linear Program in Standard Form: Some Definitions

Maximize: $Z = \boldsymbol{c}^T \boldsymbol{x}$

Subject to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$\boldsymbol{x} \geq 0, \quad \boldsymbol{b} \geq 0$

A: m n matrix
x: n 1 vector
b: m 1 vector
c: n 1 vector

The optimal solution of a LP is said to be unique when there exists no other optimal solution.

When a LP does not have a finite optimum, it is said to have unbounded optimum. Then,

$$\max Z \to +\infty$$

or

$$\min Z \to -\infty$$

Solving System of Linear Equations: Square System

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \qquad (E_1)\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \qquad (E_2)\\
a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \qquad (E_3)\\
&\vdots\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \qquad (E_n)
\end{aligned}
$$

Consider the system of n equations with n variables.

The above system of equations will not be altered under the following elementary operations:

- Any equation E_r is replaced by the equation kE_r, where k is a nonzero constant
- Any equation E_r is replaced by the equation $E_r + kE_s$, where E_s is any other equation of the system

Solving System of Linear Equations: Canonical Form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \qquad (E_1)$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \qquad (E_2)$$
$$a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3 \qquad (E_3)$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \qquad (E_n)$$

Canonical form

$$1x_1 + 0x_2 + 0x_3 + \cdots + 0x_n = b_1''$$
$$0x_1 + 1x_2 + 0x_3 + \cdots + 0x_n = b_2''$$
$$0x_1 + 0x_2 + 1x_3 + \cdots + 0x_n = b_3''$$
$$\vdots$$
$$0x_1 + 0x_2 + 0x_3 + \cdots + 1x_n = b_n''$$

Operation at each step to eliminate one variable at a time, from all equations except one, is known as ***pivotal operation***. Number of *pivotal operations* are same as the number of variables in the set of equations.

Canonical form may be obtained after n pivotal operations.

Solving System of Linear Equations: Canonical Form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \qquad (E_1)$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \qquad (E_2)$$
$$a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3 \qquad (E_3)$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \qquad (E_n)$$

Canonical form

$$1x_1 + 0x_2 + 0x_3 + \cdots + 0x_n = b_1''$$
$$0x_1 + 1x_2 + 0x_3 + \cdots + 0x_n = b_2''$$
$$0x_1 + 0x_2 + 1x_3 + \cdots + 0x_n = b_3''$$
$$\vdots$$
$$0x_1 + 0x_2 + 0x_3 + \cdots + 1x_n = b_n''$$

Transformed set of equations through elementary operations is equivalent to the original set of equations. Thus, solution of transformed set of equations is the solution of original set of equations.

The solution may be easily obtained from canonical form as: $x_i = b_i''$, $i = 1, 2, \cdots, n$

Transformation to Canonical Form: General Procedure

Canonical form of the (n × n) system of equations can be obtained by performing n pivotal operations . Variable x_i (i = 1, ….., n) is eliminated from all equations except j^{th} equation for which a_{ji} is nonzero.

General procedure for one pivotal operation consists of following two steps:

1. Divide j^{th} equation by a_{ji}. Let us call it as (E'_j), i.e., $E'_j = \frac{E_j}{a_{ji}}$
2. Subtract a_{ki} times of (E'_j) equation from k^{th} equation $(k = 1, 2, \cdots j-1, j+1, \cdots, n)$ i.e., $E_k - a_{ki}E'_j$

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Solving System of Linear Equations: Non-square System

Instead of a square system, let us consider a system of m equations in n variables with $n \geq m$. This system of equations is assumed to be consistent so that it will have at least one solution

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

The solution vector X that satisfy the above equation can not be readily obtained from the equations. However, it is possible to reduce this system to an equivalent canonical system from which at least one solution can readily be deduced.

Solving System of Linear Equations

Use the Gauss-Jordan elimination procedure to solve this series of linear equations.

$$x_1 + x_2 + 3x_3 + x_4 = 7$$
$$x_1 - 2x_2 + x_3 - x_5 = 2$$

Solution:

Multiply row 1 by 2 and add to row 2.

$$x_1 + x_2 + 3x_3 + x_4 = 7$$
$$3x_1 + 7x_3 + 2x_4 - x_5 = 16$$

Divide row 2 by 3.

$$x_1 + x_2 + 3x_3 + x_4 = 7$$
$$x_1 + \frac{7}{3}x_3 + \frac{2}{3}x_4 - \frac{1}{3}x_5 = \frac{16}{3}$$

Multiply row 2 by –1 and add to row 1.

$$x_2 + \frac{2}{3}x_3 + \frac{1}{3}x_4 + \frac{1}{3}x_5 = \frac{5}{3}$$
$$x_1 + \frac{7}{3}x_3 + \frac{2}{3}x_4 - \frac{1}{3}x_5 = \frac{16}{3}$$

Solving System of Linear Equations

$$x_2 + \frac{2}{3}x_3 + \frac{1}{3}x_4 + \frac{1}{3}x_5 = \frac{5}{3}$$

$$x_1 + \frac{7}{3}x_3 + \frac{2}{3}x_4 - \frac{1}{3}x_5 = \frac{16}{3}$$

Solution:

Set $x_3 = 0$, $x_4 = 0$, $x_5 = 0$:

$$x_1 = \frac{16}{3}, x_2 = \frac{5}{3}, x_3 = 0, x_4 = 0, x_5 = 0$$

x_1 and x_2 are **basic** variables; x_3 , x_4 and x_5 are **non-basic** variables.

The solution, $x_1 = \frac{16}{3}, x_2 = \frac{5}{3}, x_3 = 0, x_4 = 0, x_5 = 0$

is referred to as a basic solution since all non-basic variables have been set to 0.

This solution is also referred to as a basic feasible solution since all basic variables are non-negative.

Every corner point of the feasible region corresponds to a Basic Feasible Solution . This is the fundamental building block for the *simplex* method.

Transformation to Canonical Form: General Case

Maximize: $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n$

Subject to
$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$
$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$
$$b_1 \geq 0, b_2 \geq 0, \ldots, b_m \geq 0$$

In general, $m < n$ which leads to infinite number of feasible solutions. Hence selection of best feasible solution which maximizes Z is not an easy problem.

To generate the solutions, use first m variables (x_1, …, x_m), reduce the system to canonical or row echelon form by Gauss-Jordon elimination.

Transformation to Canonical Form: General Case

By performing n pivotal operations for any m variables (say, x_1, x_2, …, x_m) called pivotal variables the system of equations can be reduced to canonical form as follows:

$$1x_1 + 0x_2 + \cdots\cdots + 0x_m + \bar{a}_{1,m+1}x_{m+1} + \cdots + \bar{a}_{1s}x_s + \cdots\cdots + \bar{a}_{1n}x_n = \bar{b}_1$$

$$0x_1 + 1x_2 + \cdots\cdots + 0x_m + \bar{a}_{2,m+1}x_{m+1} + \cdots + \bar{a}_{2s}x_s + \cdots\cdots + \bar{a}_{2n}x_n = \bar{b}_2$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$0x_1 + \cdots + 1x_r + \cdots + 0x_m + \bar{a}_{r,m+1}x_{m+1} + \cdots + \bar{a}_{rs}x_s + \cdots\cdots + \bar{a}_{rn}x_n = \bar{b}_r$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$0x_1 + 0x_2 + \cdots\cdots + 1x_m + \bar{a}_{m,m+1}x_{m+1} + \cdots + \bar{a}_{ms}x_s + \cdots\cdots + \bar{a}_{mn}x_n = \bar{b}_m$$

Transformation to Canonical Form: General Case

$$
\begin{aligned}
&1x_1 + 0x_2 + \cdots\cdots\cdots + 0x_m + \bar{a}_{1,m+1}x_{m+1} + \cdots + \bar{a}_{1s}x_s + \cdots\cdots + \bar{a}_{1n}x_n = \bar{b}_1 \\
&0x_1 + 1x_2 + \cdots\cdots\cdots + 0x_m + \bar{a}_{2,m+1}x_{m+1} + \cdots + \bar{a}_{2s}x_s + \cdots\cdots + \bar{a}_{2n}x_n = \bar{b}_2 \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
&0x_1 + \cdots + 1x_r + \cdots + 0x_m + \bar{a}_{r,m+1}x_{m+1} + \cdots + \bar{a}_{rs}x_s + \cdots\cdots + \bar{a}_{rn}x_n = \bar{b}_r \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
&0x_1 + 0x_2 + \cdots\cdots + 1x_m + \bar{a}_{m,m+1}x_{m+1} + \cdots + \bar{a}_{ms}x_s + \cdots\cdots + \bar{a}_{mn}x_n = \bar{b}_m
\end{aligned}
$$

Variables, x_{m+1},..., x_n of above set of equations is known as *non-pivotal variables* or independent variables.

Basic Variables, Basic Feasible Solution

One solution that can be obtained from the above set of equations is:

$$x_i = \bar{b}_i \quad \text{for} \quad i = 1, \cdots, m$$
$$x_i = 0 \quad \text{for} \quad i = (m+1), \cdots, n$$

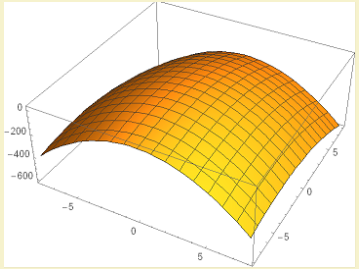
This solution is known as *basic solution*.

Pivotal variables, $x_1, x_2, \ldots, x_m$, are also known as ***basic variables***.

Non-pivotal variables, $x_{m+1}, \ldots, x_n$, are known as ***non-basic variables***.

Basic solution is also known as **basic feasible solution** because it satisfies all the constraints as well as non-negativity criterion for all the variables

Optimization in Chemical Engineering



Thank You