Module 3:

"Convective" heat and mass transfer

Lecture 28:

Application of Mises Transformation to the problem of Mass Transfer from a sphere in Creeping (Stokes) flow

Stokes flow $\mathrm{Re} = \frac{v_\infty a}{\nu} << 1$ (No momentum boundary layer)

High Sc >> 1, valid for thin mass transfer boundary layer

Also assume constant physical and transport properties (Use pseudo-steady state to find time of dissolution)

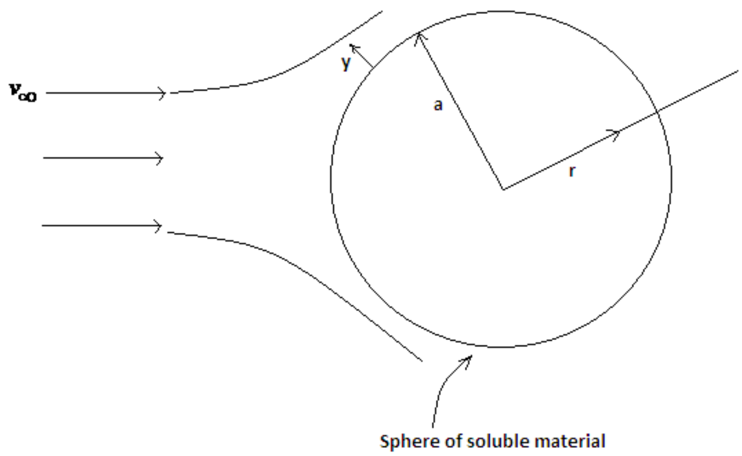

Fig. 1

Convective-diffusion equation in spherical coordinates

$$v_r \frac{\partial c}{\partial r} + \frac{v_\theta}{r} . \frac{\partial c}{\partial \theta} = D\left(\frac{\partial^2 c}{\partial r^2} + \frac{2}{r}\frac{\partial c}{\partial r} + \frac{1}{r^2 \sin\theta} . \frac{\partial}{\partial \theta}\left(\sin\theta . \frac{\partial c}{\partial \theta} \right) \right) \quad (1)$$

Assumptions: constant physical properties, no volume reactions, axis symmetry

Stream function for Stokes Flow

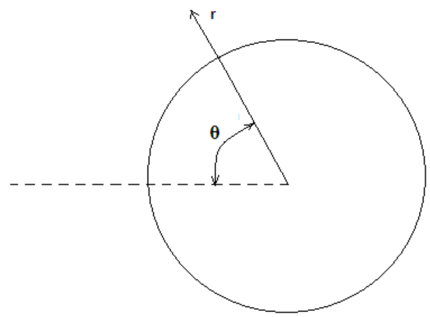

Fig. 2

$$\psi = -\frac{v_\infty}{2}\sin^2\theta\left(r^2 - \frac{3}{2}ar + \frac{1}{2}\frac{a^3}{r}\right) \qquad (2)$$

Since $r = y + a$ and $y \ll a$, eqn. (2) yields

$$\psi = -\frac{3}{4}v_\infty y^2 \sin^2\theta \qquad (3)$$

Then $v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial y} \approx -\frac{1}{a\sin\theta}\frac{\partial\psi}{\partial y} = \frac{3}{2}v_\infty\frac{y}{a}\sin\theta$ (4)

At small values of y (y << a), indicates thin boundary layer, so curvature effects can be neglected, or

$\frac{\partial^2 c}{\partial r^2} >> \frac{2}{r}\frac{\partial c}{\partial r}$ (in eqn. (1))

Using eqn. (4) into eqn. (1) in terms of variables (ψ, θ)

$$\frac{1}{a}\frac{\partial c}{\partial \theta} = Da\sin\theta \frac{\partial}{\partial \psi}\left[\left(a\sin\theta v_\theta\right)\frac{\partial c}{\partial \psi}\right]$$

Expressing v_θ in terms of ψ, one finds

$$\left(\frac{\partial c}{\partial \theta}\right)_\psi = Da^2\sin^2\theta\sqrt{3v_\infty}\,\frac{\partial}{\partial \psi}\left(\sqrt{-\psi}\,\frac{\partial c}{\partial \psi}\right) \tag{5}$$

Set $t = Da^2\sqrt{3v_\infty}\int \sin^2\theta d\theta$ eqn. (5) turns into

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial \psi}\left(\sqrt{-\psi}\,\frac{\partial c}{\partial \psi}\right) \tag{6}$$

Boundary conditions (deposition)

$c = 0$ at $\psi = 0$ (at the surface where y =0)

$c = c_0$ at $\psi \to -\infty$ (as y $\to \infty$)

$c = c_0$ at $\theta = 0,\ \psi = 0$ (at the leading edge A)

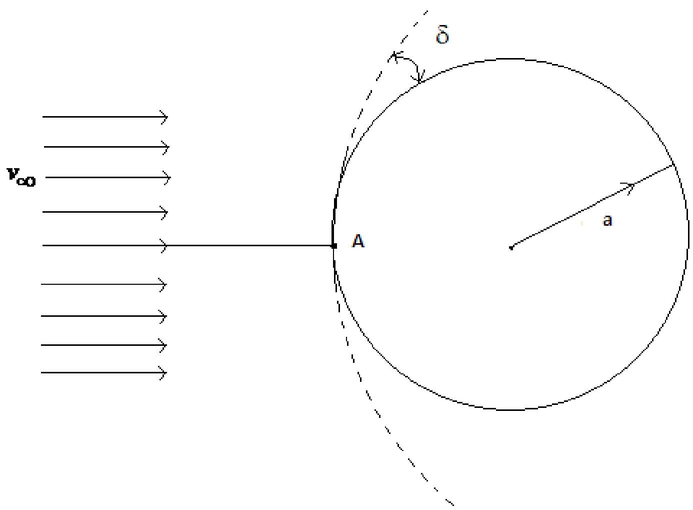

Fig. 3

Equation (6) can be solved by a combination of variables (similarity transformation)

$\eta = \dfrac{-\psi}{t^{2/3}}$, then eqn. (6) becomes

$$-\frac{2}{3}\eta\frac{dc}{d\eta} = \frac{d}{d\eta}\left(\sqrt{\eta}\,\frac{dc}{d\eta}\right)$$

Setting $z = \sqrt{\eta}$ yields

$$\frac{d^2c}{dz^2} + \frac{3}{4}z^2\frac{dc}{dz} = 0 \qquad (7)$$

Integration and use of boundary conditions yield

$$c = c_0 \frac{\int_0^z \exp\left(-\frac{4}{9}z^3\right)dz}{\int_0^\infty \exp\left(-\frac{4}{9}z^3\right)dz}$$ or

$$c = \frac{c_0}{\left(\frac{4}{9}\right)1/3\,\Gamma\left(\frac{1}{3}\right)}\int_0^z \exp\left(-\frac{4}{9}z^3\right)dz \qquad (8)$$

The surface flux is

$$N_y\big|_{y=0} = -D\frac{\partial c}{\partial y}\bigg|_{y=0} = \frac{Dc_0}{1.15}\left(\frac{3v_\infty}{4Da^2}\right)^{1/3}\frac{\sin\theta}{\left(\theta-\frac{\sin 2\theta}{2}\right)^{1/3}} \qquad (9)$$

If we consider $f(\theta) = \left(\theta - \frac{\sin 2\theta}{2}\right)^{1/3}$ then

θ	$f(\theta)$
0	1
$\pi/2$	$(2/\pi)^{1/3}$
π	0

Hence the flux is highest at the point of incidence ($\theta = 0$) decreases with θ and becomes zero at $\theta = \pi$.

The diffusion layer thickness is

$$\delta = \left(\frac{N_y\big|_{y=0}}{Dc_0} \right)^{-1} = \frac{1.15\left(\theta - \frac{\sin 2\theta}{2}\right)^{1/3}}{\sin\theta} \left(\frac{4Da^2}{3v_\infty} \right)^{1/3}$$

As θ increases δ increases and at θ = π, δ→∞

In practice δ never reaches infinity and

$$N_y\big|_{y=0,\theta=\pi} \neq 0$$

At large values of θ (close to π) analysis is breaks down because boundary layer is not thin. However, there is no significant effect of the region around θ = π on the total mass transfer rate to the particle.

$$N_t = \int N_y ds = 2\pi a^2 \int_0^\pi N_y \sin\theta \, d\theta$$
$$N_t = 7.98 c_0 D^{2/3} v_\infty^{1/3} a^{4/3} \tag{10}$$

Arranging dimensionless numbers in eqn. (10), we get

$$Sh = 0.635 \, \mathrm{Re}^{1/3} \, Sc^{1/3}$$
$$Sh = 0.635 Pe^{1/3} \tag{11}$$

Where

$$Sh = \frac{K_M a}{D} = \frac{\left(\frac{N_t}{4\pi a^2 c_0}\right) a}{D}$$

$$\mathrm{Re} = \frac{v_\infty a}{\nu}$$

$$Sc = \frac{\nu}{D}$$

Equation (11) is valid for Pe →∞. For very small Pe, diffusive transport dominates, and the governing equation becomes Laplace's equation. The concentration profile has only 'r' dependence on this case.

$$c = c_0\left(1 - \frac{a}{r}\right)$$

The diffusional flux to the surface is

$$N = \frac{Dc_0}{a} \quad \text{and Sh} = 1$$

Combination with eqn. (11) yields

$$Sh = 1 + 0.635 Pe^{1/3}, \qquad (11)$$

which reduces to the correct limits at Pe >> 1 and Pe << 1