Module 2 :

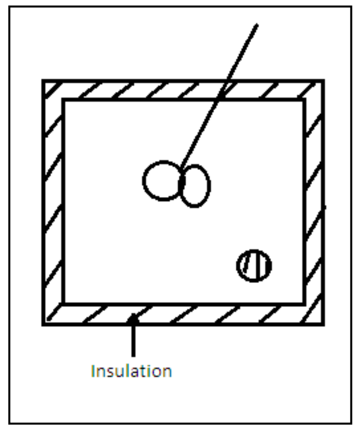
“Diffusive” heat and mass transfer

Lecture 16:

Cooling of a Sphere in contact with a Well-Stirred Fluid & Heat Conduction in Solids with Interfacial Resistance

A. Cooling of a Sphere in contact with a Well-Stirred Fluid

Consider a fluid of volume v_f, density ρ_f, heat capacity $\hat{C}_{p_f}$, and initially at temperature T_0. Let's assume that the sphere of volume v_s, density ρ_s, heat capacity $\hat{C}_{p_s}$, and initially at temperature T_1.



Assumptions

1. Well-mixed fluids
2. No external heat transfer resistance

Find the temperature of fluid as a function of time ($T_f(t)$).

Governing equations are

For sphere

$$\frac{\partial \theta_s}{\partial \tau} = \frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \theta_s}{\partial \xi}\right) \qquad (1)$$

For fluid

$$\frac{\partial \theta_f}{\partial \tau} = -\frac{3}{B}\frac{\partial \theta_s}{\partial \xi}\bigg|_{\xi=1} \qquad (2)$$

Where $B = \dfrac{\left(v_f \rho_f \hat{C}_{pf}\right)}{\left(v_s \rho_s \hat{C}_{ps}\right)}$

The initial and boundary conditions are

At $\tau = 0 \Rightarrow \theta_s = 0$
At $\xi = 1 \Rightarrow \theta_s = \theta_f$
At $\xi = 0 \Rightarrow \theta_s = \text{finite}$

Considering the dimensionless quantities

$$\theta_s = \frac{T_1 - T_s}{T_1 - T_0},\ \theta_f = \frac{T_1 - T_f}{T_1 - T_0},\ \xi = \frac{r}{R},\ \tau = \frac{\alpha_s t}{R^2}$$

It will be convenient to solve eqn. (1) using Laplace transform

Therefore

$$p\overline{\theta_s} = \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\overline{\theta_s}}{d\xi}\right) \qquad (3)$$

And eqn. (2) gives

$$p\overline{\theta_f} - 1 = -\frac{3}{B}\frac{d\overline{\theta_s}}{d\xi}\bigg|_{\xi=1} \qquad (4)$$

The solution to eqn. (3) is

$$\overline{\theta_s} = \frac{C}{\xi}\sinh\sqrt{p}\xi + \frac{D}{\xi}\cosh\sqrt{p}\xi \qquad (5)$$

The second term on right hand side is equal to zero since $\overline{\theta_s}$ is finite at $\xi = 0$.

Substituting eqn. (5) into eqn. (4), we get

$$\overline{\theta_f} = \frac{1}{p} + 3\frac{C}{Bp}\left(\sinh\sqrt{p} - \sqrt{p}\cosh\sqrt{p}\right) \qquad (6)$$

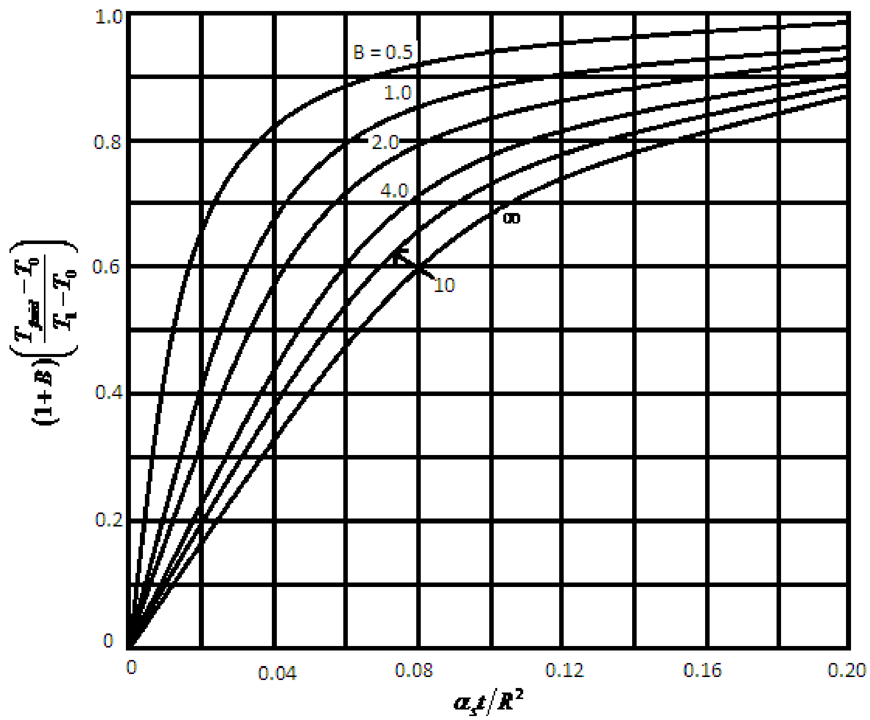
To determine C, equate $\overline{\theta_s} = \overline{\theta_f}$ at $\xi = 1$. Using eqn. (5) & (6), we get

$$\overline{\theta_f} = \frac{1}{p} + 3\left\{\frac{1-\left(\sqrt{p}\right)\tanh\sqrt{p}}{\left(3-Bp\right)\sqrt{p}\tanh\sqrt{p} - 3p}\right\} \qquad (7)$$

The inverse transform yields

$$\overline{\theta_f} = \frac{B}{1+B} + 6B\sum_{k=1}^{\infty}\frac{\exp\left(-b_k^{\,2}\tau\right)}{B^2 b_k^{\,2} + 9\left(1+B\right)} \qquad (8)$$

where b_k are the roots of $\tan b = \dfrac{3b}{3+Bb^2}$



Note that

1. At thermal equilibrium $(1+B)\left(\dfrac{T_{fluid}-T_0}{T_1-T_0}\right)=1$.

2. This plot can be used to determine α_s.

3. T_f was obtained w/o the need to solve for T_s ((ξ,t)

B. Heat Conduction with Interfacial Resistance

Consider a finite slab of thickness 2b with initial temperature T_0. Assuming, convection losses at both faces of the slab with heat transfer coefficient *h* into a medium at temperature T_∞.

The unsteady state heat conduction equation in one dimension in y-direction is given by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \qquad (9)$$

Where α is the thermal diffusivity

Defining dimensionless quantities

$$\theta = \frac{(\mathrm{T} - \mathrm{T}_\infty)}{(\mathrm{T}_0 - \mathrm{T}_\infty)},\ \eta = \frac{y}{b},\ \tau = \frac{\alpha t}{b^2},$$

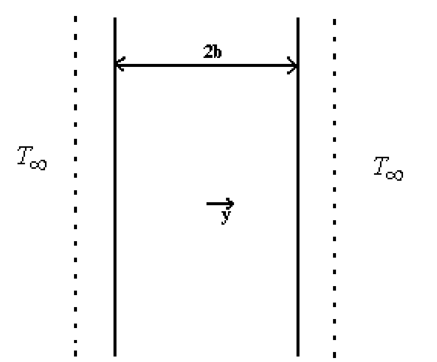


Fig.15.1. Heat transfer through slab

Biot number: It is a dimensionless number used in unsteady state heat transfer calculations. It relates the heat transfer resistance inside and at the surface of the body. Mathematically it can be defined as

$$\mathrm{Bi} = \frac{External\ heat\ transfer}{Internal\ heat\ transfer} = \frac{hb}{k}$$

If Bi>1 indicates that system is internal heat transfer controlled

If Bi<1 indicates that the system is external heat transfer controlled

Also, rate of heat transfer is given by

$$q = -k\left(\frac{\partial T}{\partial y}\right)_{y=\pm b} = h\left(T_\infty - T\right) \tag{10}$$

Using dimensionless quantities eqn. (9) becomes

$$\frac{\partial \theta}{\partial \tau} = \alpha \frac{\partial^2 \theta}{\partial \eta^2} \tag{11}$$

And from eqn. (10) and definition of Biot number

$$\frac{\partial \theta}{\partial \eta} = Bi\theta \tag{12}$$

Solution of problem given by eqn. (11) is

$$\theta = \sum_{n=1}^{\infty} \frac{2Bi\cos\beta_n\eta}{\left(\beta_n^{\,2} + Bi + Bi^2\right)\cos\beta_n} \exp\left(-\beta_n^{\,2}\tau\right) \tag{13}$$

where β_n are the positive roots of $\beta \tan\beta = Bi$.

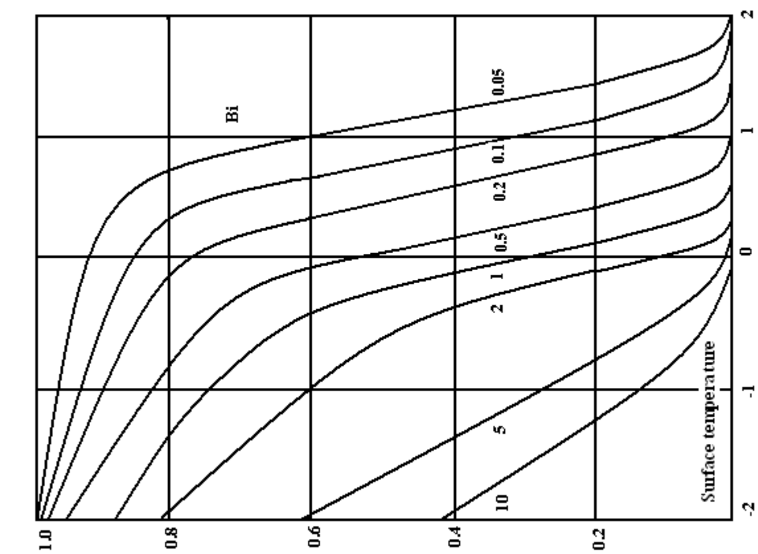


Fig.15.2. Surface temperature (θ_s) versus $\log_m \tau$

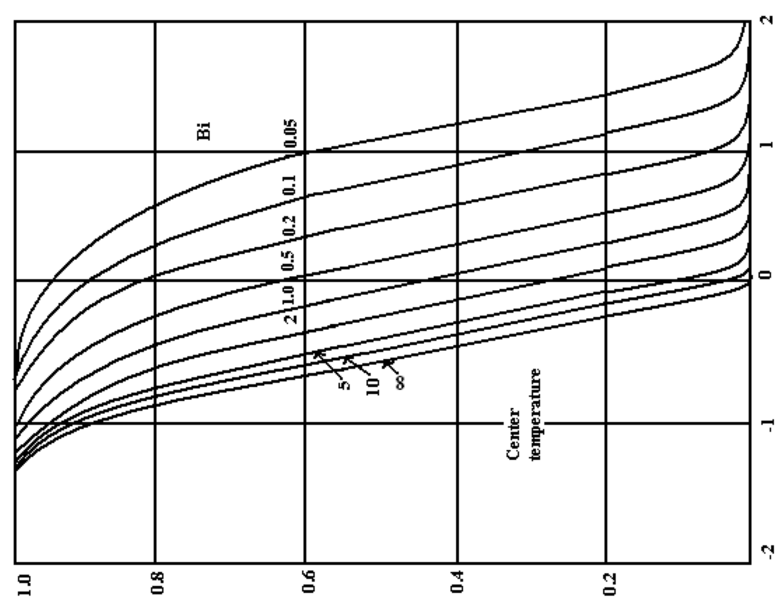


Fig.15.3. Center temperature (θ_c) versus $\log_m \tau$

Heat flux at interface

$$q_0 = -k\frac{\partial T}{\partial y}\bigg|_{y=\pm b} = \frac{k}{b}\left(T_0 - T_\infty\right)\sum_{n=1}^{\infty}\frac{2Bi^2}{\left(\beta_n^{\ 2} + Bi + Bi^2\right)}\exp\left(-\beta_n^{\ 2}\tau\right) \tag{14}$$

Rearranging, we get

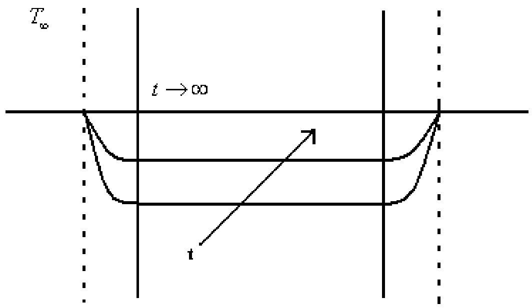
$$q^* = \frac{q_0}{h\left(T_0 - T_\infty\right)} = \sum_{n=1}^{\infty}\frac{2Bi}{\left(\beta_n^{\ 2} + Bi + Bi^2\right)}\exp\left(-\beta_n^{\ 2}\tau\right) \tag{15}$$

<u>**For finite slab with interface resistance**</u>

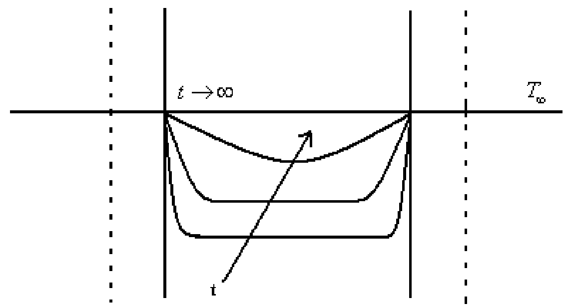
$$\theta = erf\,\frac{y}{\sqrt{4\alpha t}} + \exp\left(\frac{hb}{k} + \frac{h^2}{k^2}at\right)erfc\left(\frac{y}{\sqrt{4\alpha t}} + \frac{h}{k}\sqrt{at}\right) \tag{16}$$

Here $\frac{h^2}{k^2}at = \tau Bi^2$

$$q^* = \exp\left(Bi^2\tau\right)erfc\left(Bi\sqrt{\tau}\right) \tag{17}$$



$Bi \ll 1$



$Bi \gg 1$