Module 2 :

"Diffusive" heat and mass transfer

Lecture 11:

Heat Conduction in semi-infinite Slab

with Constant wall Temperature

Semi-infinite Solid

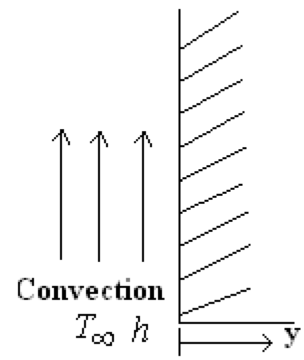

Fig. 11.1 Heat transfer in Semi-infinite solids

Semi-infinite solids can be visualized as very thick walls with one side exposed to some fluid. The other side, since the wall is very thick remains unaffected by the fluid temperature. The condition applicable can be expressed as

At $T(\infty, t) = T_0$, where T_0 is the initial wall temperature

The condition at the exposed side of the wall is called the boundary condition

Unsteady-state Heat Transfer

Consider a heat conduction in y-direction in a semi-infinite slab (bounded only by one face) initially at a temperature T_0, whose face suddenly at time equal to zero is raised to and maintained at T_1. Assuming constant thermal diffusivity and with no heat generation, a differential equation in one space dimension and time is given by

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial y^2}, \text{ where } \alpha \text{ is the thermal diffusivity} \tag{11.1}$$

with the following I.C and B.Cs

$$\left.\begin{array}{lll} \text{At } t \le 0 & \theta = 0 & \text{for all y} \\ \text{At } y = 0 & \theta = 1 & \text{for all } t > 0 \\ \text{At } y \to \infty & \theta = 0 & \text{for all } t > 0 \end{array}\right\} \qquad (11.2)$$

An example is heating of a semi-infinite slab, initially at temperature T_0. The slab surface temperature is raised suddenly to T_1 and maintained at that level for all t>0. For this situation, we have considered dimensionless parameter, $\theta = \frac{(T - T_O)}{(T_1 - T_O)}$ in equation (11.1), and $\alpha = \frac{k}{\rho C_p}$ is the thermal diffusivity.

Solution: One way to solve this problem is by Combination of Variables

Assume $\theta = y^a t^b$ (11.3)

where a and b are constants

Differentiating equation (11.3) w.r.t ''t' we get

$$\frac{\partial \theta}{\partial t} = y^a\, b\, t^{(b-1)} \qquad (11.4)$$

Also on differentiating w.r.t 'y' we get

$$\frac{\partial^2 \theta}{\partial y^2} = t^b\, a\,(a-1)\, y^{(a-2)} \qquad (11.5)$$

Substituting equation (11.4) and (11.5) into (11.1) we get

$$y^a\, b\, t^{(b-1)} = \alpha\, t^b\, a\,(a-1)\, y^{(a-2)} \qquad (11.6)$$

After some algebraic manipulations, we get

$$\frac{y^2}{\alpha\, t} = \frac{a(a-1)}{b} = \text{constant}$$

Hence the new variable can be any of the form $\left[C\frac{y^2}{\alpha\,\mathrm{t}}\right]^{\mathrm{d}}$

where C and d are constants

Define $\eta = \frac{y}{\sqrt{4\alpha\,\mathrm{t}}}$ (11.7)

(Here we choose $C = \frac{1}{4}$ and $\mathrm{d} = \frac{1}{2}$)

Then $\theta = \theta(y,t) = \theta(\eta)$

Differentiating θ (eqn. (11.7)) w.r.t.'t' we get

$$\frac{\partial\theta}{\partial \mathrm{t}} = -\frac{y}{2\mathrm{t}\sqrt{4\alpha\,\mathrm{t}}}\frac{\mathrm{d}\theta}{\mathrm{d}\eta} \tag{11.8}$$

Also differentiating θ w.r.t. 'y' we get

$$\frac{\partial^2\theta}{\partial \mathrm{y}^2} = -\frac{1}{2\alpha\,\mathrm{t}}\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2} \tag{11.10}$$

Substituting eqn. (11.8) and (11.10) into eqn. (11.1) then leads to a second-order ordinary differential equation of the form

$$\frac{\partial^2\theta}{\partial\eta^2}+2\eta\frac{d\theta}{d\eta}=0 \tag{11.11}$$

On solving equation (11.11) we get

$$\theta = B + A\int_0^{\eta} e^{-\eta^2}\,d\eta \tag{11.12}$$

Note

For the method of combination of variables to work, the boundary conditions must be expressible in terms of η only. This method is usually applied for media of infinite or semi-infinite extend.

Boundary Conditions

At $\eta = 0 \quad \Rightarrow \quad \theta = 1$ and

At $\eta \to \infty \quad \Rightarrow \quad \theta = 0$

Applying above B.Cs. to equation (11.12) we get

$$\theta = 1-\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}\,d\eta \tag{11.13}$$

Note

In mathematics, the **error function** also called as **Gauss error function** is a special function (non-elementary) of sigmoid shape which generally occurs in problems related to probability, statistics, materials science and partial differential equations. It can be defined as

$$erf\left(x\right)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

The complimentary error function denoted by erfc, is defined in terms of the error function:

$$\begin{aligned} erfc\left(x\right) &= 1-erf\left(x\right) \\ &= \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt \end{aligned}$$

Some useful results of error functions are given below.

$$erf\left(0\right)=0, \quad erf\left(\infty\right)=1, \quad erf\left(-x\right)=-erf\left(x\right)$$

So utilizing the results given for error function, we can write

$$\theta = 1-\mathrm{erf}\left(\frac{y}{\sqrt{4\alpha\,\mathrm{t}}}\right) \quad \text{or} \quad \theta = 1-\mathrm{erf}\left(\eta\right) \tag{11.14}$$

Fig.(11.3) shows the variation of the surface temperature along the length of the slab with time.

Diffusion length

Diffusion length can be defined as a measure of how far the temperature (or concentration) has propagated in the y-direction by diffusion in time t. It can be given as

$\delta \equiv 4\sqrt{\alpha\, t}$ at this value, the value of η is

$$\eta = \frac{y}{\sqrt{4\alpha\, t}} = 2$$

Therefore from equation (11.13) we can write

$$\theta = 1 - \mathrm{erf}(2) = 1 - 0.995322 = 0.004678$$

This implies that, at a distance $l > \delta$, the field variables has changed by < 0.5%.

Solution (11.14) is a good approximation even for finite length slabs if the diffusion length $\delta \ll L$.

The heat flux can be obtained as

$$(q)_{y=0} = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{k}{\sqrt{\pi\alpha t}}\left(T_1 - T_o\right) \tag{11.15}$$

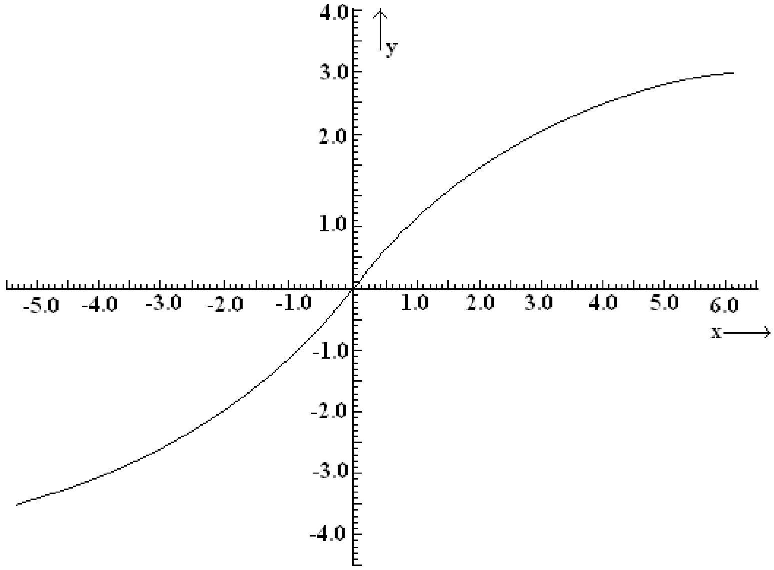

Fig.11.2 Error function

From the above solution to the problem on semi-infinite slab we can conclude that the diffusion length (or penetration depth) varies as square root of time ($t^{1/2}$) and the wall flux varies as inverse of as square root of time ($t^{-1/2}$).

Solution by the method of Laplace Transform

Definition

Let *f(z)* be any function, then the Laplace transform of *f(z)* can be given by

$$£\{f(z)\} = \overline{f}(p) = \int_0^\infty e^{-pt} f(t)dt \qquad (11.16)$$

Some of the standard Laplace results are shown below:

if $f\left(\mathrm{t}\right)=1$, then after taking Laplace transform, $\overline{f}\left(\mathrm{p}\right)=\frac{1}{\mathrm{p}}$

In this way

$$f\left(\mathrm{t}\right)=\mathrm{e}^{\mathrm{at}} \Rightarrow \overline{f}\left(\mathrm{p}\right)=\frac{1}{\left(\mathrm{p}-\mathrm{a}\right)}$$

$$f\left(\mathrm{t}\right)=\sin\left(\mathrm{wt}\right) \Rightarrow \overline{f}\left(\mathrm{p}\right)=\frac{\mathrm{w}}{\left(\mathrm{p}^2+\mathrm{w}^2\right)}$$

Problem

Consider a diffusion of component A in the y-direction in a semi-infinite (bounded only by one face) slab initially at a uniform concentration, C_0, whose face suddenly at time equal to zero is raised to and maintained at C.

The governing differential equation is given by

$$\frac{\partial C}{\partial t}=D\frac{\partial^2 C}{\partial y^2} \quad \text{, } D \text{ is the binary diffusivity} \tag{11.17}$$

with following I.C. and B.Cs.

$$\left.\begin{array}{lll} C=C_O & \text{at} \;\; \mathrm{y}=0, & \mathrm{t}>0 \\ C=0 & \text{at} \;\; \mathrm{y}>0, & \mathrm{t}=0 \\ C\rightarrow 0 & \text{at} \;\; \mathrm{y}\rightarrow\infty, & \mathrm{t}\geq 0 \end{array}\right\} \tag{11.18}$$

The Laplace transform of equation (11.17) yields

$$D\frac{\partial^2 \overline{C}}{\partial y^2} = \text{p}\overline{\text{C}} - \overline{\text{C}}(y,0) \tag{11.19}$$

Using I.C. given in equation (11.18), equation (11.19) becomes

$$D\frac{\partial^2 \overline{C}}{\partial y^2} = \text{p}\overline{\text{C}} \tag{11.20}$$

Applying B.Cs, so

$$\overline{C} = \frac{C_O}{\text{p}} \quad \text{at} \quad \text{y} = 0 \quad \text{and;}$$

$$\overline{C} = 0 \quad \text{as} \quad \text{y} \rightarrow \infty$$

Therefore the solution for equation (11.20) is

$$\overline{C} = \frac{C_O}{\text{p}} e^{-\sqrt{p/D}\,y} \tag{11.21}$$

Taking the inverse Laplace transform of equation (11.21) and from the table of Laplace transform

$$C = C_o\left[1 - erf\left(\frac{y}{\sqrt{4D\,\text{t}}}\right)\right] \tag{11.22}$$

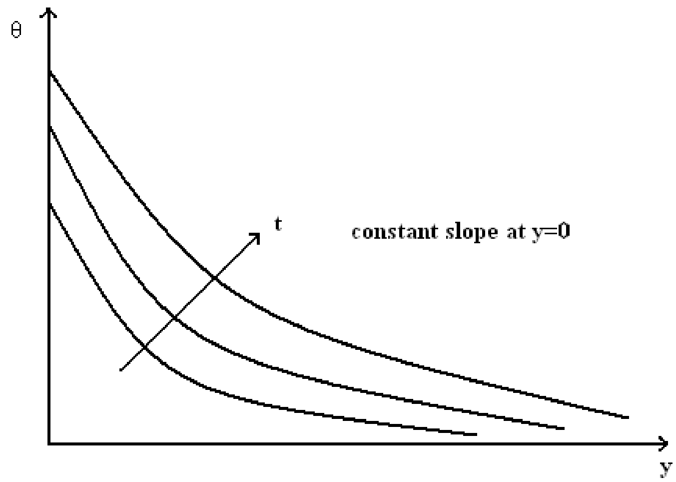

Fig.11.3 Temperature variation along the length at different time period