Module 2 :

“Diffusive” heat and mass transfer

Lecture 15:

Heating of a finite slab

Consider heat conduction in y- direction in a finite slab of thickness 2b, initially at a uniform temperature of T_0.

The governing equation is given as

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial y^2} \qquad (1)$$

The initial and boundary conditions are

At $t=0 \Rightarrow T=T_0$ for all y

At $y=\pm\, b \Rightarrow T=T_1 \;\; (t>0)$

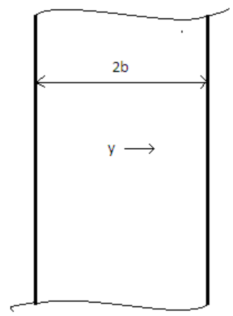


Fig. 1 Finite slab of thickness 2b

Here we assumed that symmetry will hold for all t

Solve $\dfrac{\partial \theta}{\partial \tau}=\dfrac{\partial^2 \theta}{\partial \eta^2}$ (2)

Considering dimensionless parameters

$$\theta=\frac{T_1-T}{T_1-T_0},\;\; \eta=\frac{y}{b},\;\; \tau=\frac{\alpha t}{b^2}\left(\text{non-dimnesional time}\right)$$

α: thermal diffusivity

So the new I.C. and B.Cs are

At $\tau=0 \Rightarrow \theta=1$

At $\eta=\pm\, 1 \Rightarrow \theta=0$

<u>Solution by separation of variables</u>

Let's say, $\theta(\eta,\tau) = f(\eta)g(\tau)$ (3)

Substituting eqn. (3) in (2), we get

$$\frac{1}{g}\frac{dg}{d\tau} = \frac{1}{f}\frac{d^2 f}{d\eta^2} = -C^2 \qquad (4)$$

$$\frac{dg}{d\tau} = -C^2 g$$

$$\therefore\ g = A\exp\left(-C^2\tau\right) \qquad (5)$$

$$\frac{d^2 f}{d\eta^2} = -C^2 f$$

$$\therefore\ f = B\sin C\eta + D\cos C\eta \qquad (6)$$

where C,A,B,D are constants

θ, hence f must be symmetric, therefore B = 0

From B.C $\eta = \pm 1 \Rightarrow \theta = 0$

$D\cos C = 0,$ and $D \neq 0$

Therefore

$$C = \left(n + \frac{1}{2}\right)\pi \qquad \text{where n = 0, } \pm 1, \pm 2, ...$$

The general solution is

$$\theta = \sum \theta_n = \sum_{n=-\infty}^{+\infty} A_n D_n \exp\left[-\left(n + \frac{1}{2}\right)^2 \pi^2 \tau\right] \bullet \cos\left(n + \frac{1}{2}\right)\pi\eta \qquad (7)$$

$$\text{or} \quad \theta = \sum_{n=-\infty}^{+\infty} E_n \exp\left[-\left(n + \frac{1}{2}\right)^2 \pi^2 \tau\right] \bullet \cos\left(n + \frac{1}{2}\right)\pi\eta \qquad (8)$$

with $E_n = A_n D_n + A_{-(n+1)} D_{-(n+1)}$

Determine E_n by using I.C.

$$1 = \sum_{n=0}^{\infty} D_n \cos\left(n + \frac{1}{2}\right)\pi\eta \qquad (9)$$

Multiplying both sides of eqn. (9) by $\cos\left(m + \frac{1}{2}\right)\pi\eta d\eta$ and integrate from -1 to +1, we get

$$\int_{-1}^{1} \cos\left(m + \frac{1}{2}\right)\pi\eta d\eta = \sum_{n=0}^{\infty} Dn \int_{-1}^{+1} \cos\left(m + \frac{1}{2}\right)\pi\eta \bullet \cos\left(n + \frac{1}{2}\right)\pi\eta d\eta$$

$$(10)$$

Therefore,

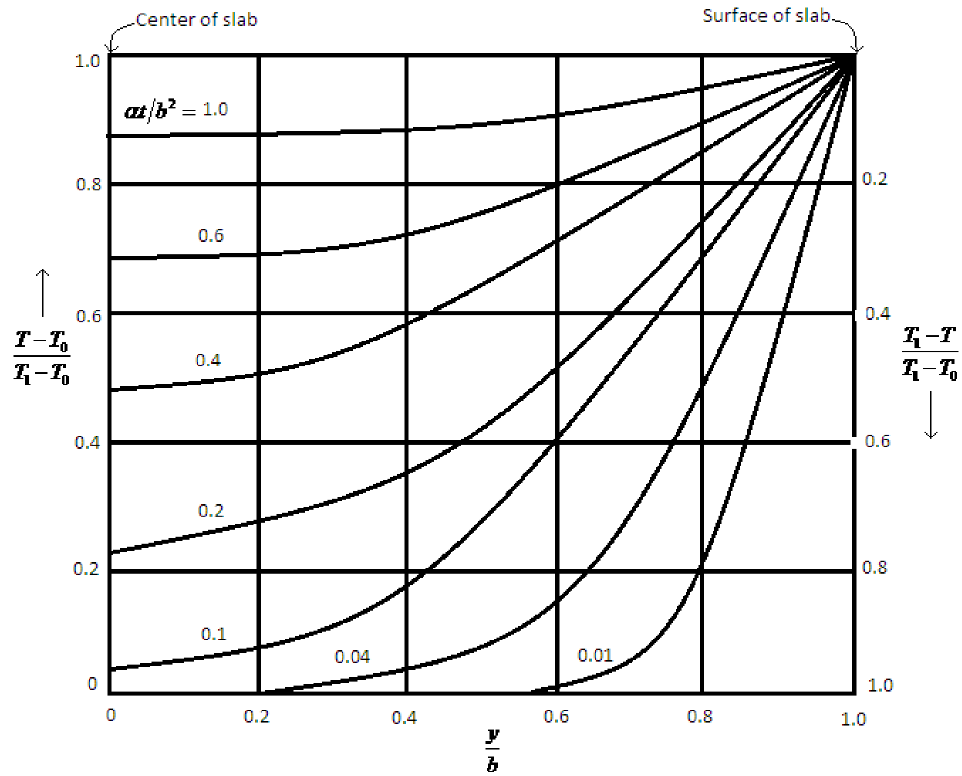
$$D_m = \frac{2\left(-1\right)^m}{\left(m+\frac{1}{2}\right)\pi} \tag{11}$$

The final solution is

$$\frac{T_1 - T}{T_1 - T_0} = 2\sum_{n=0}^{\infty} \frac{\left(-1\right)^n}{\left(n+\frac{1}{2}\right)\pi} \exp\left[-\left(n+\frac{1}{2}\right)^2 \pi^2 \frac{\alpha t}{b^2}\right] \cos\left(n+\frac{1}{2}\right)\frac{\pi y}{b}$$

(12)

The infinite series, in eqn. (12) converges rapidly at long times. For short times convergence is slow. But then the solution to the semi-infinite slab (eqn., $\theta = 1 - erf\left(\frac{y}{\sqrt{4\alpha t}}\right)$) may be used.



The solution can be extended to a three dimensions for a rectangular block of dimensions a, b and c. It is the product of three solutions of the form of eqn. (12)

$$\frac{T_1 - T(x,y,z)}{T_1 - T_0} = 8\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{m+n+p}}{\left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\left(p+\frac{1}{2}\right)\pi^3}$$

$$\times\exp\left[-\left(\frac{\left(m+\frac{1}{2}\right)^2}{a^2}+\frac{\left(n+\frac{1}{2}\right)^2}{b^2}+\frac{\left(p+\frac{1}{2}\right)^2}{c^2}\right)\pi^2\alpha t\right]$$

$$\times\cos\left(m+\frac{1}{2}\right)\frac{\pi x}{a}\cos\left(n+\frac{1}{2}\right)\frac{\pi y}{b}\cos\left(p+\frac{1}{2}\right)\frac{\pi z}{c}$$

(13)

A similar result applies for cylinders of finite length.