Module 5:

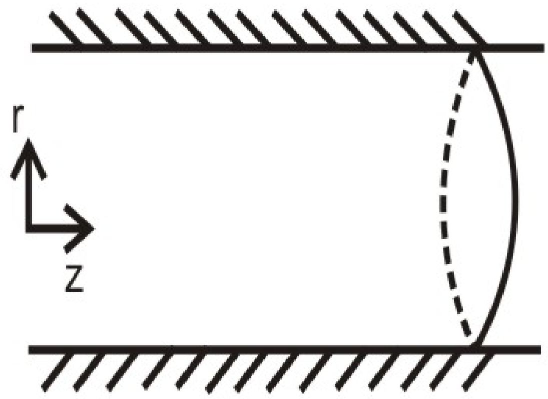
Mass transfer in turbulant flows

Lecture 40:

Turbulent Flow in a Pipe

Two methods for applying the concept of the Universal Velocity Profile (UVP) to pipe flow.

a) Assume that UVP applies even though τ should be zero at the centerline.



$$< \bar{u}_z > = \frac{2}{R^2} \int_0^R \bar{u}_z \, r \, dr$$

$$u^+ = \frac{\bar{u}_z}{u_*}, \; y^+ = \frac{y u_*}{\nu} = \frac{(R-r) u_*}{\nu} \quad with \;\; u_* = \sqrt{\frac{\tau_0}{\rho}}$$

Then

$$R_e = \frac{2R < \bar{u}_z >}{\nu} = 4 \int_0^{y^+_{\max}} u^+(y^+)[1 - \frac{y^+}{y^+_{\max}}] dy^+ \quad (1)$$

With $y^+_{\max} = y^+_{r=0} = \dfrac{R\sqrt{\tau_0/\rho}}{\nu}$

Further, since:

$$\tau_0 = f\,\frac{1}{2}\rho < \bar{u}_z >^2 \Rightarrow f = \frac{2\tau_0}{\rho < \bar{u}_z >^2},$$

We get

$$y^+_{\max} = \mathrm{Re}\sqrt{f/8} \qquad (2)$$

Universal Velocity Profile (UVP)

$$u^+ \approx y^+ \qquad y^+ < 20$$
$$u^+ \approx 2.5 \ln y^+ + 5.5 \qquad y^+ > 20$$

$$u^+ = 2.5 \ln y^+ + 5.5$$
$$\Rightarrow \frac{du^+}{dy^+} = \frac{2.5}{y^+}$$
$$\Rightarrow \quad \frac{du^+}{dy^+}\Big|_{r=0} = \frac{2.5}{y^+_{\max}}$$
$$\rightarrow 0 \quad only\, if \quad y^+_{\max} \rightarrow 0, \;\; i.e. \;\; Re \rightarrow \infty$$

The velocity profile shows a peak at the center of the pipe and the velocity derivative does not go to zero there, as it shows

Given Re, use UVP and eq. (1) to find y_{max}^{+} and find f from eq. (2).

$y_{\max}^{+}$	116	315	815	2366	6349	1.6×10^{4}	5.2×10^{4}	1.58
Re	3000	10^{4}	3.10^{4}	10^{5}	3.10^{5}	10^{6}	3.10^{6}	10^{7}
$10^{3} f$	12.01	7.96	5.91	4.48	3.58	2.88	2.40	1.99
$10^{3} f_{\text{exp.}}$	11	7.6	5.8	4.5	3.5	2.8	2.4	2.00

Now, for pipe flow (for both turbulent and laminar flow)

$$\overline{\tau}_{rz} = \tau_0 \frac{r}{R}$$

$$\overline{\tau}_{rz} = -(\mu + \mu^{(t)}) \frac{\partial \overline{u}_z}{\partial r} = \frac{(\mu + \mu^{(t)})}{\mu} \tau_0 \frac{du^+}{dy^+} \Rightarrow$$

$$(1 + \frac{\mu^{(t)}}{\mu}) \frac{du^+}{dy^+} = \frac{r}{R} = 1 - \frac{y^+}{y_{\max}^{+}}$$

See Figure 1 below.

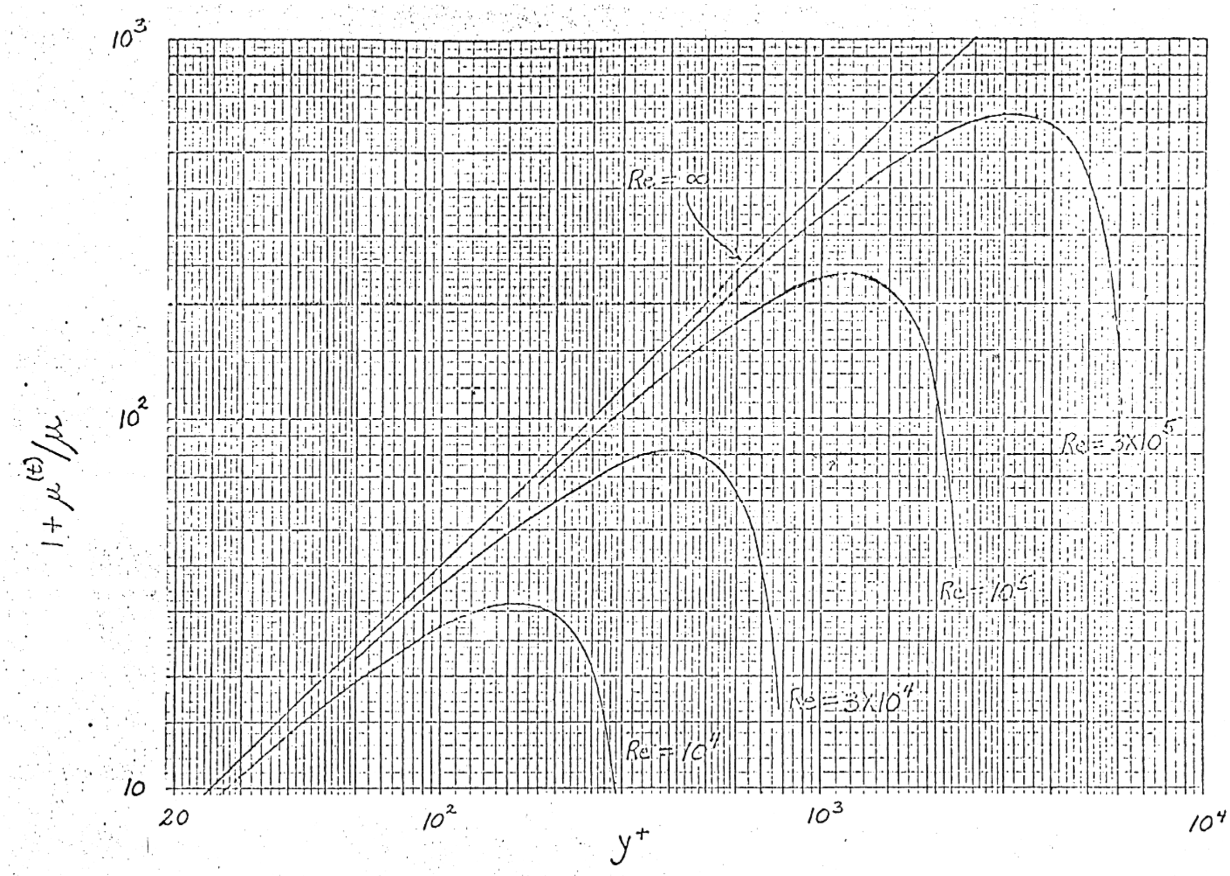


Figure 1. Total viscosity for pipe flow if the universal velocity profile applies all the way to the center of the pipe.

Therefore, method a) yields $\mu^{(t)} = -\mu$ at the centerline which is unreasonable.

b) Assume that the total viscosity is given by Fig. 2 (below) and utilize the actual shear stress distribution

$$\left(1+\frac{\mu^{(t)}}{\mu}\right)\frac{du^+}{dy^+} = 1-\frac{y^+}{y^+_{\max}} \Rightarrow u^+ = \int_0^{y^+} \frac{1-y^+/y^+_{\max}}{1+\frac{\mu^{(t)}}{\mu}}\,dy$$

At various values of $y^+_{\max}$ evaluate integral (1) (Method a)) to find *Re* and then find *f* again.

The integration using u^+ just presented should be carried out numerically.

$y^+_{\max}$	125	333	851	2455	6561	1.95×10^4	5.33×10^4	1.62
Re	3×10^3	10^4	3×10^4	10^5	3×10^5	10^6	3×10^6	10^7
$10^3 f$	13.89	8.85	6.44	4.82	3.83	3.05	2.53	2.10

The gross results are not much different, but $(1+\mu^{(t)}/\mu)$ does not go to zero at *r=0*. This does not mean that the total viscosity at the center of the pipe is correct, but it seems that has a more reasonable value than zero.

See Figures 2 and 3 below.

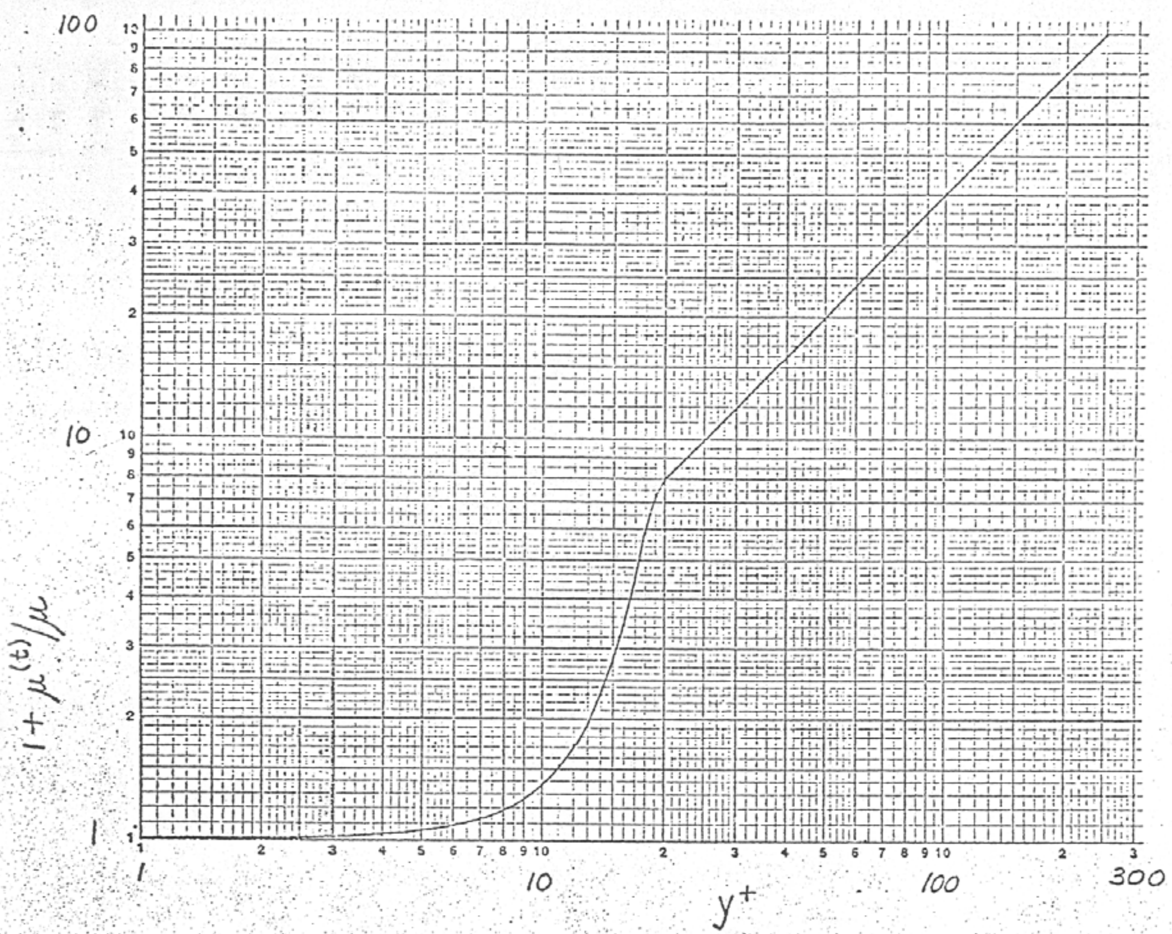


Figure 2. Representation of the eddy viscosity as a "universal" function of the distance from the wall.

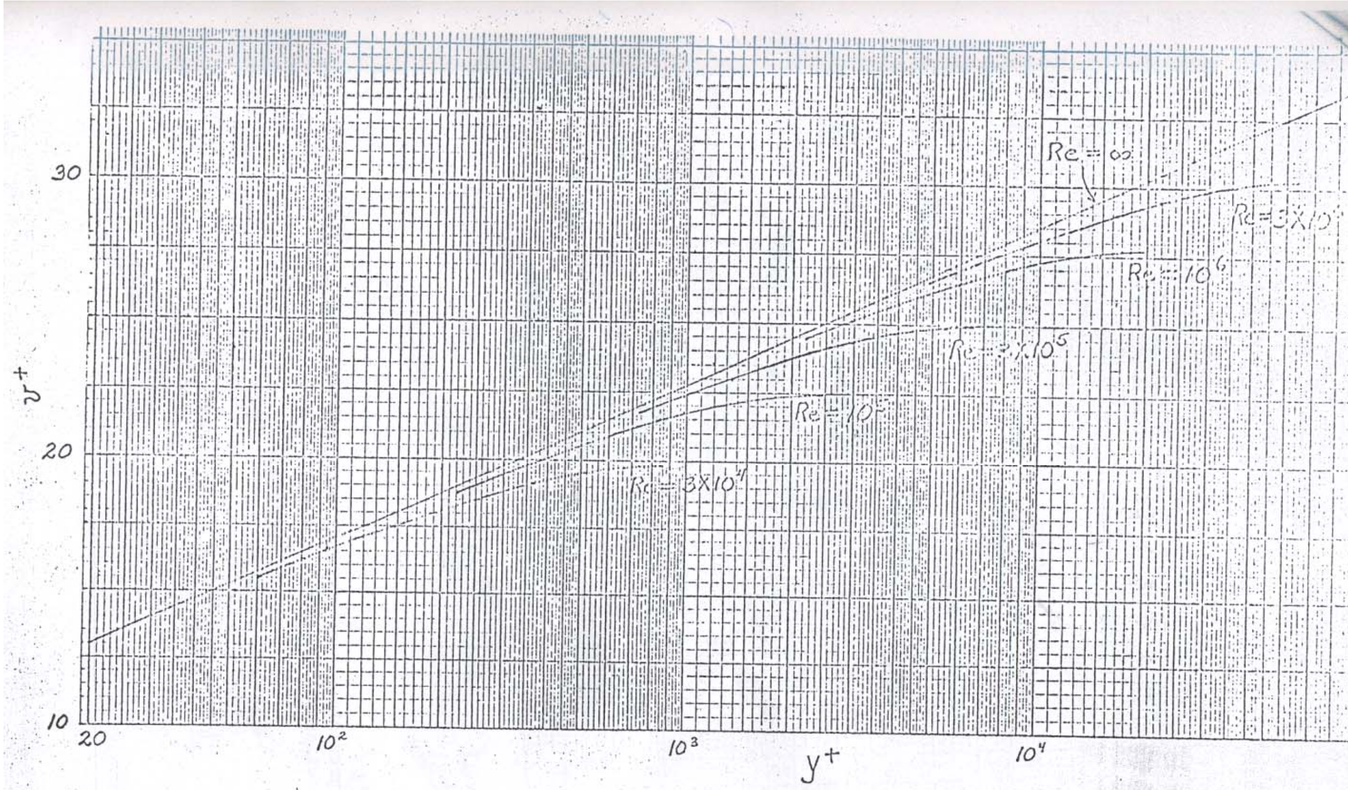


Figure 2. Velocity profiles for pipe flow for several values of the Reynolds number, calculated with the assumption that $\mu^{(t)}/\mu$ is a universal function of y^+.