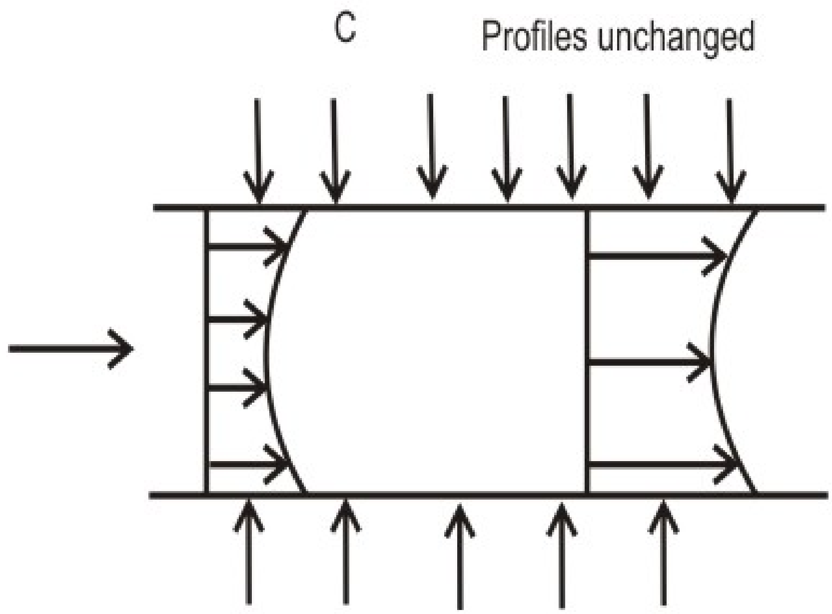
Module 5:

Mass transfer in turbulant flows

Lecture 41:

Mass Transfer in Turbulent Pipe Flow



Assumptions

(i) Fully developed Concentration profile

(ii) Constant properties

(iii) Constant wall flux (J_0)

Experiments like this provide information about variation of eddy diffusivity near the wall.

Steady-state Governing equation for the case of no axial conduction/diffusion

$$\overline{u}_z \frac{\partial \overline{c}_i}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(\underline{r}\overline{J}_{ir})$$

$$\overline{J}_{ir} = \overline{J}_{ir}^{(l)} + \overline{J}_{ir}^{(t)} = -(D + D^{(t)})\frac{\partial \overline{c}_i}{\partial r}$$

Observed similarity in the governing equation with that in laminar flow

For fully developed C profile $\frac{\partial \overline{c}_i}{\partial r} = \text{constant}$. (Remember discussion of heat transfer in laminar flow with constant q_{wall})

Integration yields : $-r\,\overline{J}_{ir} = \frac{\partial \overline{c}_i}{\partial z}\int_0^r r\overline{u}_z\,dr \qquad (3)$

But

$$-2\pi R J_0 \Delta z = \quad \Delta\overline{c}_i \quad \int_0^R 2\pi r\ \overline{u}_z \quad d\underline{r}$$

$$\frac{moles}{\mathfrak{I}} \quad \frac{moles}{cm^3} \quad Val.\ flux;\ cm^3/s$$

Or, $-RJ_0 = \frac{\partial \overline{c}_i}{\partial z}\int_0^R r\overline{u}_z\,d\underline{r} \qquad (4)$

Dividing (3) by (4), we get

$$\frac{\bar{J}_{ir}}{J_0} = \frac{R}{r}\,\frac{\int_0^r r\,\bar{u}_z\,dr}{\int_0^R r\,\bar{u}_z\,dr} = \frac{1}{1-\dfrac{y^+}{y^+_{\max}}}\;\frac{\int_{y^+}^{y^+_{\max}} u^+\left(1-\dfrac{y^+}{y^+_{\max}}\right)dy}{\int_0^{y^+_{\max}} u^+\left(1-\dfrac{y^+}{y^+_{\max}}\right)dy}$$

Where we have used $r = R - y$ and

$$\frac{\int_0^r}{\int_0^R} = \frac{\int_r^0}{\int_R^0} = \frac{\int_{y^+}^{y^+_{\max}}}{\int_0^{y^+_{\max}}}$$

Or,

$$\frac{\bar{J}_{ir}}{J_0} = \frac{4/\mathrm{Re}}{1-\dfrac{y^+}{y^+_{\max}}}\int_{y^+}^{y^+_{\max}} u^+\left(1-\dfrac{y^+}{y^+_{\max}}\right)dy^+ \quad (5)$$

Where the denominator has been replaced by the *Re* as obtained by eqn. (1) in the previous lecture and reproduced below:

$$R_e = \frac{2R < \overline{u}_z >}{\nu} = 4 \int_0^{y^+_{\max}} u^+(y^+)[1 - \frac{y^+}{y^+_{\max}}] dy^+ \quad (1)$$

We now introduce a dimensionless concentration given by

$$\theta^+ = \frac{u_*}{J_0}\left[\overline{c}_i - \overline{c}_{i|y=0}\right]$$

(compare to laminar flow with constant q_{wall}). But

$$\overline{J}_{ir} = -\left[D + D^{(t)}\right]\frac{\partial \overline{c}_i}{\partial r} \Rightarrow$$

$$\frac{\overline{J}_{ir}}{J_0} = \frac{D + D^{(t)}}{\nu}\,\frac{d\theta^+}{dy^+} \Rightarrow$$

$$\theta^+ = \int_0^{y^+} \frac{\nu}{D + D^{(t)}}\,\frac{\overline{J}_{ir}}{J_0}\,dy^+$$

Use $\frac{\overline{J}_{ir}}{J_0}$ as found in eq. (5) to obtain θ^+ distribution

$$Nu = \frac{2R\,J_0}{D[c_{aver} - \overline{c}_i\,|_{y=0}},$$

where

$$c_{aver=}\frac{\int_0^R \overline{u}_z\,\overline{c}_i\,r\,dr}{\int_0^R \overline{u}_z\,r\,dr}$$

Thus:

$$[c_{avg} - \overline{c}_{i|y=0}] = \frac{4J_0}{u_* \operatorname{Re}} \int_0^{y^+_{\max}} u^+ \theta^+ \left(1 - \frac{y^+}{y^+_{\max}}\right) dy^+,$$

$$St = \frac{Nu}{\operatorname{Re} Sc} = \frac{y^+_{\max}}{2\int_0^{y^+_{\max}} u^+\,\theta^+ \left(1 - \frac{y^+}{y^+_{\max}}\right) dy^+}.$$

a) If $\theta^+ \approx const = \theta^+_{\max}$

$$St = \frac{2y^+_{\max}}{\operatorname{Re}\,\Theta^+_{\max}} = \sqrt{\frac{f}{2}}\;\frac{1}{\Theta^+_{\max}},$$

where

$$\Theta^+_{\max} = \int_0^{y^+_{\max}} \frac{\nu}{D + D^{(t)}} \left(\frac{\overline{J}_{ir}}{J_0}\right) dy^+$$

Let $\frac{\bar{J}_{ir}}{J_0} \cong 1$ Since $\frac{\partial c}{\partial r} \neq 0$ only near the periphery:

$$\theta^+_{\max} = \int_0^{y^+_{\max}} \frac{\nu}{D + D^{(t)}} dy^+$$

Here we can't say $D^{(t)} < D$ near the wall,

although $\mu^{(t)} << \mu$ for $y^+ < 5$

Express Universal Velocity Profile (UVP):

$$u^+ = y^+ - A_1 {y^+}^4 + A_2 {y^+}^5 \qquad y^+ \leq 20$$
$$u^+ = 2.5 \ln y^+ + 5.5 \qquad y^+ > 20$$

With
$$A_1 = 1.0972 \times 10^{-4}$$
$$A_2 = 3.295 \times 10^{-6}$$

By differentiation of UVP we get $v^{(t)}$ as $f(y^+)$ or

$$D^{(t)} = f(y^+) \text{ if } D^{(t)} = v^{(t)}$$

Hence,

$$\frac{D^{(t)}}{v} = \frac{4A_1 y^{+^3} - 5A_2 y^{+^4}}{1 - 4A_1 y^{+^3} - 5A_2 y^{+^4}} \qquad y \le 20$$

$$\frac{D^{(t)}}{v} = \frac{y^+}{2.5} - 1 \qquad y > 20$$

Diffusion layer close to the wall => $y^+ << 1$ =>

$$\frac{D^t}{v} \cong 4A_1 y^{+^3}$$

$$\theta^+_{\max} = \int_0^{y^+_{\max}} \frac{v}{D + D^{(t)}} dy^+ \approx \int_0^{y^+_{\max}} \frac{dy^+}{\frac{1}{Sc} + 4A_1 y^{+^3}}$$

$$= \frac{Sc^{2/3}}{(4A_1)^{1/3}} \int_0^{y^+_{\max}} \frac{dy^+}{1 + y^{+^3}} \qquad \text{(using binomial theorem)}$$

$$= \frac{Sc^{2/3}}{(4A_1)^{1/3}} \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{9}{3}\right) \qquad \text{(if you set } y^+_{\max} \to \infty)$$

Finally $St = \frac{0.0429}{1.2092} \; Sc^{-2/3}\sqrt{f/2}$ as $Sc \rightarrow \infty$.

Chiltan-Colburn analogy:

$$St = \frac{f}{2}\, Sc^{-2/3}$$

(compares with the derived result up to $\mathrm{Re} \approx 10^5$)