Module 2 :

"Diffusive" heat and mass transfer

Lecture 14:

Semi-infinite Slab with time-varying surface temperature: Examples

Example 1

For $\phi(t) = C\sqrt{t}$, where C is constant. Show that

$$\theta = C\sqrt{\pi t}\ i\ erfC\left(\frac{y}{\sqrt{4\alpha t}}\right)$$

Also show that this case corresponds to the problem of semi-infinite slab with constant flux density at the wall.

Example 2

Show that the Surface Temperature is a harmonic function of time.

Assume $\phi(t) = A\cos(\omega t - \varepsilon)$ i.e. forced oscillator system

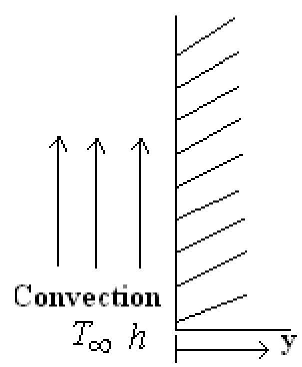

Fig. 14.1 Heat transfer in Semi-infinite solids

Using eqn. (13.8) (derived in the previous lecture), the solution to the above problem is given by

$$\theta = \frac{2A}{\sqrt{\pi}} \int_{\frac{y}{\sqrt{4\alpha t}}}^{\infty} \cos\left\{\omega\left(t - \frac{y^2}{4\alpha\mu^2}\right) - \varepsilon\right\} e^{-\mu^2} d\mu \tag{14.1}$$

$$\begin{aligned}\frac{2A}{\sqrt{\pi}} \int_0^{\infty} \cos\left\{\omega\left(t - \frac{y^2}{4\alpha\mu^2}\right) - \varepsilon\right\} e^{-\mu^2} d\mu &= \frac{2A}{\sqrt{\pi}} \int_0^{\frac{y}{\sqrt{4\alpha t}}} \cos\left\{\omega\left(t - \frac{y^2}{4\alpha\mu^2}\right) - \varepsilon\right\} e^{-\mu^2} d\mu \\ &+ \frac{2A}{\sqrt{\pi}} \int_{\frac{y}{\sqrt{4\alpha t}}}^{\infty} \cos\left\{\omega\left(t - \frac{y^2}{4\alpha\mu^2}\right) - \varepsilon\right\} e^{-\mu^2} d\mu\end{aligned} \tag{14.2}$$

The integral below is known

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \cos\left\{\omega\left(t - \frac{y^2}{4\alpha\mu^2}\right) - \varepsilon\right\} e^{-\mu^2} d\mu = \exp\left(-y\sqrt{\frac{\omega}{2\alpha}}\right) \cos\left\{\omega t - y\sqrt{\frac{\omega}{2\alpha}} - \varepsilon\right\} \tag{14.3}$$

Then from eqn. (14.1), (14.2) and (14.3), we get

$$\theta = \underbrace{A\exp\left(-y\sqrt{\frac{\omega}{2\alpha}}\right) \cos\left\{\omega t - y\sqrt{\frac{\omega}{2\alpha}} - \varepsilon\right\}}_{\text{(This term gives the periodic steady state)}} - \underbrace{\frac{2A}{\sqrt{\pi}} \int_0^{\frac{y}{\sqrt{4\alpha t}}} \cos\left\{\omega\left(t - \frac{y^2}{4\alpha\mu^2}\right) - \varepsilon\right\} e^{-\mu^2} d\mu}_{\text{(This term will "die out" at long times)}} \tag{14.4}$$

At the periodic state

$$\theta = A\exp\left(-y\sqrt{\frac{\omega}{2\alpha}}\right) \cos\left\{\omega t - y\sqrt{\frac{\omega}{2\alpha}} - \varepsilon\right\} \tag{14.5}$$

It can also be written as

$$\theta = A\,\mathrm{e}\left(-Cy\right)\cos\left\{\omega t - Cy - \varepsilon\right\}$$
$$\text{with } C=\sqrt{\frac{\omega}{2\alpha}} \qquad (14.6)$$

This is a temperature wave having wave number C and wavelength

$$\lambda = \frac{2\pi}{C} = \left(\frac{4\pi\alpha}{f}\right)^{1/2} \quad \text{where } f = \frac{1}{T} = \frac{\omega}{2\pi}, \text{ being the frequency } \left(cycles/s\right)$$

The results can be summarized as follows.

1. The amplitude of the oscillation is Ae^{-Cy} and diminishes with y. also the amplitude falls off rapidly for high ω (i.e. higher harmonics of a complex will disappear most rapidly as one moves into the solid)

   $e^{-Cy} = e^{-\left(\frac{2\pi y}{\lambda}\right)}$, shows that, at a distance of one wavelength into the solid the amplitude has been reduced by a factor of $\exp\left(-2\pi\right) = 0.0019,$ thus waves are strongly attenuated.

2. There is a phase lag of magnitude $Cy = y\sqrt{\frac{\omega}{2\alpha}}$. This lag increases with *ω*.

3. The heat flux at the surface is

   $$q = -k\left.\frac{\partial\theta}{\partial y}\right|_{y=0} = CkA\sqrt{2}\cos\left(\omega t - \varepsilon + \frac{\pi}{4}\right)$$

4. For a generic periodic function $\phi\left(t\right)$ we can use the Fourier series

   $$\phi\left(t\right) = A_0 + A_1\cos\left(\omega t - \varepsilon_1\right) + A_2\left(\cos\left(2\omega t\right) - \varepsilon_2\right) + ...$$

   and find the solution by superposition (neglecting the initial transient)

$$\theta = A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-y\sqrt{\frac{n\omega}{2\alpha}}\right)\cos\left\{n\omega t - \varepsilon_n - y\sqrt{\frac{n\omega}{2\alpha}}\right\}$$

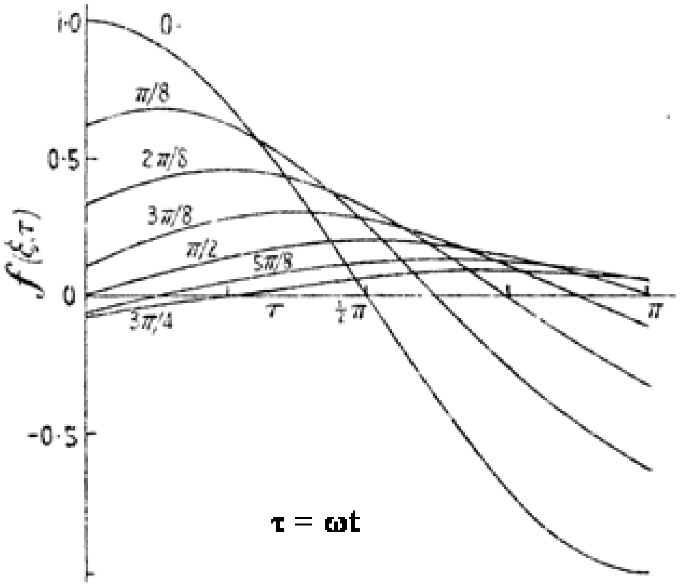

Fig. 14.3 Oscillations of temperature at various depths due to harmonic surface temperature

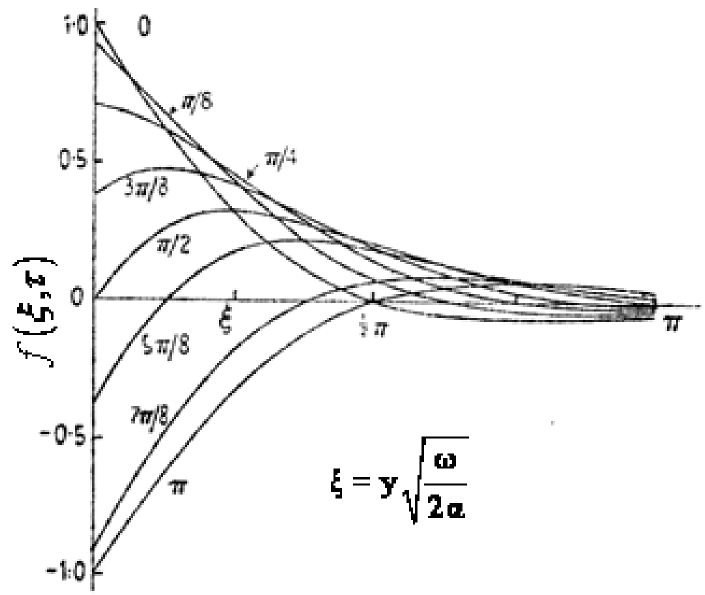

Fig. 14.4 Variation of temperature with depth when the surface temperature is harmonic

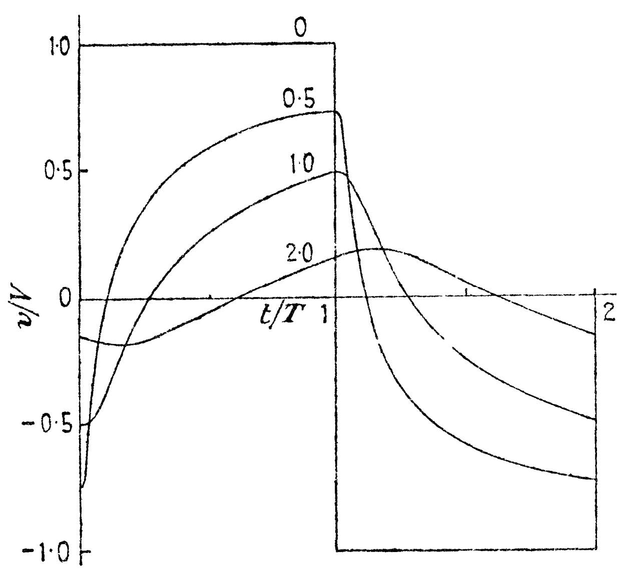

Fig.14.5 Oscillation of temperature at various depths caused by a 'square wave' surface temperature