# Module 2 :

# "Diffusive" heat and mass transfer

# Lecture 13:

# Semi-infinite Slab with time-varying surface temperature: Theory

Let us consider a semi-infinite slab (bounded only by one face), initially at a temperature $T_i$.

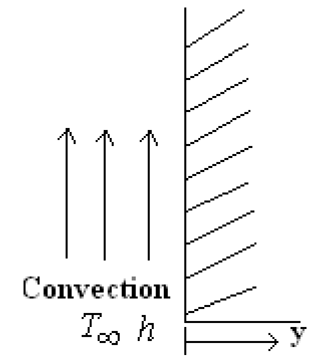

Fig. 13.1 Heat transfer in Semi-infinite solids

Let

$$\theta = T(y, y) - Ti$$

with $T_i$ = initial temperature = Const.

The one-dimensional unsteady-state heat conduction equation is given by

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial y^2} \qquad (13.1)$$

with I.C. and B.Cs,

$\theta(y, 0) = 0$

$\theta(0, t) = \phi(t)$ and

$\theta(\infty, t) = 0$

Equation (13.1) can be solved by Laplace transform; however, let's use another approach.

## Duhamel's Superposition Principle

If $T = F(x, y, z, t)$ represents the temperature at (x, y, z) at a time t in a solid in which the initial temperature is zero, while its surface is kept at temperature unity [or, in the case of convection from the surface, while convection takes place into a medium at temperature unity], then the solution of the problem when the surface is kept at temperature $\phi(t)$[or, in the case of convection, while takes place into a medium at temperature $\phi(t)$], is given by $T = \int_0^t \phi(\lambda)\frac{\partial}{\partial t}F(x, y, z, t-\lambda)\,d\lambda$ (13.2)

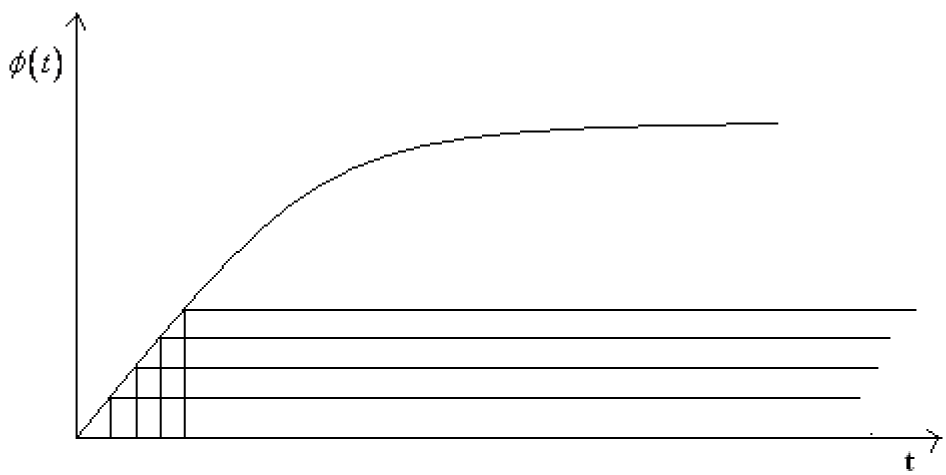


**Fig.13.2 A series of step functions**

Returning to the problem of semi-infinite slab with time-varying surface temperature,

We have seen that for the case of $\theta = 1$ at y=0 , the solution is eqn.(11.14)

$$\theta = 1-\operatorname{erf}\left(\frac{y}{\sqrt{4\ \alpha\ t}}\right) = 1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{y}{\sqrt{4\alpha t}}} e^{-\xi^2}\,d\xi$$

$$=\frac{2}{\sqrt{\pi}}\int_{\frac{y}{\sqrt{4\alpha t}}}^{\infty} e^{-\xi^2}\,d\xi \qquad (13.3)$$

Then the solution to the problem defined above with

$\theta|_{y=0} = \phi(t)$ is given as

$$\left.\begin{aligned} &T = \int_0^t \phi(\lambda)\frac{\partial}{\partial t}\theta(y,t-\lambda)d\lambda \\ &\text{with} \\ &\theta(y,t-\lambda) = \frac{2}{\sqrt{\pi}}\int_{\frac{y}{\sqrt{4\alpha(t-\lambda)}}}^{\infty} e^{-\xi^2}\,d\xi \end{aligned}\right\} \qquad (13.4)$$

In this case

$$\frac{\partial}{\partial t}F(y,t-\lambda) = -\frac{2}{\sqrt{\pi}}\exp\left(-\frac{y^2}{4\alpha(t-\lambda)}\right)\frac{\partial}{\partial t}\left[\frac{y}{\sqrt{4\alpha(t-\lambda)}}\right] \qquad (13.5)$$

$$\left(\text{Using Leibnitz rule and eqn.}(13.4)\right)$$

## <u>Leibnitz's rule</u>

Leibnitz's integral rule is useful in differentiation of a definite integral whose limits are the functions of the differential variable, which is given as

$$\frac{\partial}{\partial t}\int_{a(t)}^{b(t)} f(x,t)\,dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}dx + f\left(b(t),t\right)\frac{\partial b(t)}{\partial t} - f\left(a(t),t\right)\frac{\partial a(t)}{\partial t}.$$

It is also known as differentiation under the integral sign.

Simplifying eqn. $(13.5)$, we get

$$= \frac{y}{\sqrt{4\pi\alpha(t-\lambda)^3}}\exp\left(-\frac{y^2}{4\alpha(t-\lambda)}\right) \tag{13.6}$$

Therefore, the solution to our problem is

$$\theta = \frac{y}{\sqrt{4\alpha\pi}}\int_0^t \phi(\lambda)\frac{\exp\left(-\frac{y^2}{4\alpha(t-\lambda)}\right)}{(t-\lambda)^{3/2}}d\lambda \tag{13.7}$$

Setting $\frac{y}{\sqrt{4\alpha(t-\lambda)}} = \mu$ we have $(t-\lambda) = \frac{y^2}{4\alpha\mu^2}$

Therefore eqn. (13.7) becomes

$$\theta = \frac{2}{\sqrt{\pi}}\int_{\frac{y}{\sqrt{4\alpha t}}}^{\infty} \phi\left(t - \frac{y^2}{4\alpha\mu^2}\right)e^{-\mu^2}d\mu \tag{13.8}$$

**Note**

$$erfC(x) = 1 - erf\,(x)$$

$$i\ erfC(x) = i^1 erfC(x) = \int_x^\infty erfC(\xi)\,d\xi$$

$$i^n erfC(x) = \int_x^\infty i^{n-1} erfC(\xi)\,d\xi \qquad n = 2,3,4......$$

## Supplement to Duhamel's Superposition Principle

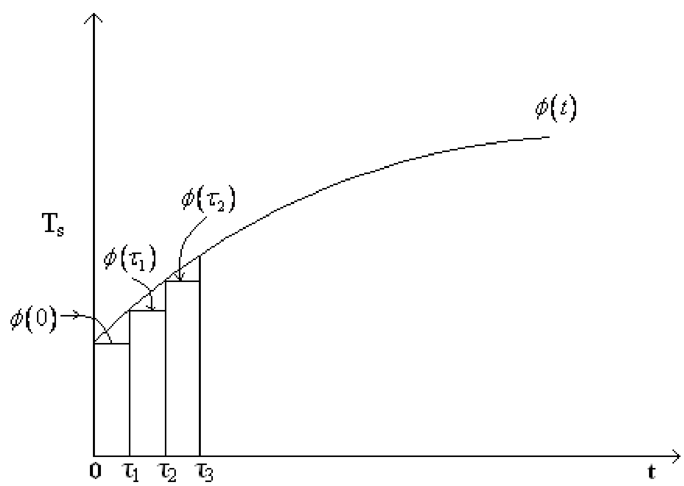


**Fig 13.3 Variation of Surface temperature according to** $\phi(t)$

One-dimensional unsteady-state conduction in a medium with constant thermal diffusivity and without any heat generation in x-direction is described by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{13.9}$$

Since governing equation is linear, individual solutions can be superimposed. Then

$$\begin{aligned} T = \phi(0)F(x_1\ t) + \left(\phi(\tau_1) - \phi(0)\right)F(x_1\ t-\tau_1) \\ + \left(\phi(\tau_2) - \phi(\tau_1)\right)F(x_1\ t-\tau_2) + ....... \\ + \left(\phi(\tau_n) - \phi(\tau_{n-1})\right)F(x_1\ t-\tau_n) + ...... \end{aligned} \tag{13.10}$$

Denote

$$\begin{aligned} \phi(\tau_k) - \phi(\tau_k - 1) = \Delta\phi_k \\ \tau_k - \tau_{k-1} = \Delta\tau_k \end{aligned}$$

Then equation (11.13) becomes

$$T(x_1\ t) = \phi(0)F(x_1\ t) + \sum_{k=1}^{\infty} F(x_1\ t-\tau_k)\left(\frac{\Delta\phi}{\Delta\tau}\right)\Delta\tau_k \tag{13.11}$$

In the limit of $\Delta\tau_k \to 0$, eqn.(13.11) can be written as

$$T(x_1\ t) = \phi(0)F(x_1\ t) + \int_0^t F(x_1\ t-\tau)\phi(\tau)\, d\tau \tag{13.12}$$

On integration by parts

$$T(x_1\ t) = \phi(0)F(x_1\ t) + \Big[F\big(x_1\ t-\tau\big)\phi\big(\tau\big)\Big]_0^t - \int_0^t \phi(\tau)\frac{\partial}{\partial\tau}\left\{F(x_1\ t-\tau)\right\}d\tau \tag{13.13}$$

Since $\dfrac{\partial}{\partial\tau}\left(F(x_1\ t-\tau)\right) = -\dfrac{\partial}{\partial t}\left(F(x_1\ t-\tau)\right)$ we have

$$T\left(x_1\ t\right)=F\left(x_1\ 0\right)\phi\left(t\right)+\int_0^t \phi\left(\tau\right)\frac{\partial}{\partial t}\left(F\left(x_1\ t-\tau\right)\right)d\tau \tag{13.14}$$

If $F\left(x_1\ 0\right)=0$

$$T\left(x_1\ t\right)=\int_0^t \phi\left(\lambda\right)\frac{\partial}{\partial t}F\left(x_1\ t-\lambda\right)\, d\lambda \tag{13.15}$$

Equation (13.15) is again in the same form as discussed above in Duhamel's Superposition Principle. This can be solved in a similar way by evaluating $\frac{\partial}{\partial t}F\left(x_1\ t-\lambda\right)$ with the help of Leibnitz rule to get the final solution.