Module 3:

"Convective" heat and mass transfer

Lecture 23:

Laminar Boundary Layers

Assumptions

1. Steady flow

2. Constant physical properties

Momentum Transfer

Navier-stock's equation $\underline{v} \bullet \nabla \underline{v} = -\frac{1}{\rho} \nabla \overline{P} + \nu \bullet \nabla^2 \underline{v}$ (1)

Continuity equation $\nabla \bullet \underline{v} = 0$ (2)

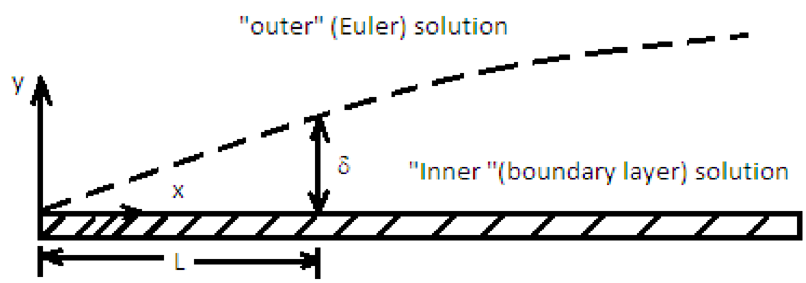


Fig. 1 Development of boundary layer on the flat plate

Limiting cases

1. Far from solid surfaces effect of viscosity may be neglected $\nu \bullet \nabla^2 \underline{v} \approx 0$. Then

$$\underline{v} \bullet \nabla \underline{v} = -\frac{1}{\rho} \nabla \overline{P}$$ Euler equation (inviscid flow)

2. Near solid surfaces velocity changes from "free stream" value far from the surface to zero on the surface; $\nu \bullet \nabla^2 \underline{v}$ is significant.

Equation for flat plate (General form)

Continuity: $\frac{\partial \upsilon_x}{\partial x}+\frac{\partial \upsilon_y}{\partial y}=0$ (3)

X-momentum: $\upsilon_x \frac{\partial \upsilon_x}{\partial x}+\upsilon_y \frac{\partial \upsilon_x}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 \upsilon_x}{\partial x^2}+\frac{\partial^2 \upsilon_x}{\partial y^2}\right)$

(4)

Y-momentum: $\upsilon_x \frac{\partial \upsilon_y}{\partial x}+\upsilon_y \frac{\partial \upsilon_y}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 \upsilon_y}{\partial x^2}+\frac{\partial^2 \upsilon_y}{\partial y^2}\right)$

(5)

One can go through an ordering argument to derive the boundary layer equations.

Flat plate boundary layer equations

$$\frac{\partial \upsilon_x}{\partial x}+\frac{\partial \upsilon_y}{\partial y}=0 \qquad (6)$$

$$v_x \frac{\partial \upsilon_x}{\partial x}+v_y \frac{\partial \upsilon_x}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left(\frac{\partial^2 \upsilon_x}{\partial y^2}\right) \qquad (7)$$

$$\frac{\partial P}{\partial y}=0 \quad \Rightarrow \quad P=P(x) \qquad (8)$$

Boundary conditions

$$U_x = U_y = 0 \quad \text{at} \quad y = 0$$
$$U_x = U_\infty(x) \quad \text{as} \quad \text{y} \to \infty$$

For a plate along the direction of the unperturbed flow (x-direction)

$$v_\infty(x) = v_\infty = \text{Const.}\,;\, P + \frac{1}{2}\rho v_\infty{}^2 = \text{Const} \Rightarrow \frac{\partial P}{\partial x} = 0\,.$$

Equation (8) also holds P = Const. in x and y.

Then equation (6) and (7) becomes

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad (9)$$

$$v_x \frac{\partial U_x}{\partial x} + v_y \frac{\partial U_x}{\partial y} = \nu\left(\frac{\partial^2 U_x}{\partial y^2}\right) \qquad (10)$$

With

$$v_x = v_y = 0 \quad \text{at} \quad y = 0$$
$$v_x \to v_\infty \quad \text{as} \quad \text{y} \to \infty$$

Use similarity variable $\eta = \dfrac{y}{\sqrt{\dfrac{\nu x}{v_\infty}}}$ to reduced PDEs into an ODE in terms of η.

$$ff'' + 2f''' = 0 \qquad (11)$$

$$f(0) = f'(0) = 0$$
$$f'(\infty) = 1$$

Stream function $\psi = \sqrt{\nu x v_\infty} f(\eta)$

Velocity components $v_x = \dfrac{\partial \psi}{\partial y}, v_y = -\dfrac{\partial \psi}{\partial x}$

Boundary Layer Thickness (Momentum)

Order of magnitude analysis of eqn. (9) yields

$$\frac{v_\infty}{L} + \frac{v_y}{\delta} = 0 \quad \Rightarrow \quad v_y \sim \frac{v_\infty \delta}{L}$$

Order of magnitude analysis of eqn. (10) yields

$$U_\infty \frac{v_\infty}{L}\left(+ \frac{v_\infty \delta}{L} \frac{v_\infty}{\delta} \right) \sim \nu \frac{U_\infty}{\delta^2} \Rightarrow \delta \sim \sqrt{\frac{\nu L}{U_\infty}} \tag{12}$$

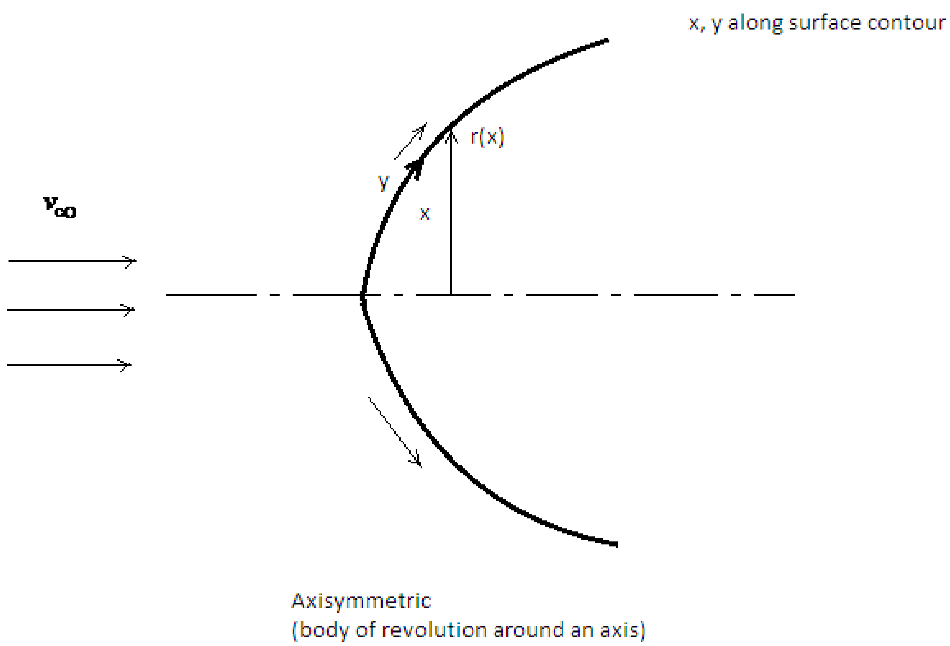
$$\frac{\delta}{L} \sim \sqrt{\frac{\nu}{L U_\infty}} \sim \frac{1}{\sqrt{\mathrm{Re}_L}} \tag{13}$$

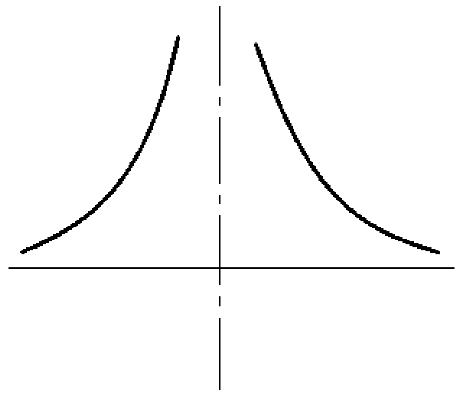
From eqn. (13), it is clear that as Re_L increases δ decreases.

One way to obtain the fluid velocity profiles

1. Assume inviscid flow and obtain solution to Euler eqns.
2. Use this solution as boundary condition to solve boundary layer equations
3. Iterate between (1) & (2), if necessary

Other shapes for which Boundary Layer theory may be applied





limiting case: Stagnation flow

Fig. 2 Different shapes for which boundary layer theory can be applied

$$v_x \frac{\partial \upsilon_x}{\partial x} + v_y \frac{\partial \upsilon_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 \upsilon_x}{\partial y^2}\right)$$

For B.L., P = P(x)

$$\frac{\partial}{\partial x}\left(r\upsilon_x\right) + \frac{\partial}{\partial y}\left(r\upsilon_y\right) = 0$$

From Bernoulli's equation (applies for inviscid flow), we can write

$$P + \frac{1}{2}\rho \upsilon_\infty{}^2\left(x\right) = \mathrm{co}\,nst$$

$$\frac{dP}{dx} = -\rho \upsilon_\infty \frac{d\upsilon_\infty}{dx}$$

Hence

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = U_\infty \frac{\partial U_\infty}{\partial x} + \nu \left(\frac{\partial^2 v_x}{\partial y^2} \right) \qquad (14)$$

Equations are identical to flat plate [eqns. (6)-(8)]

Assumption: radius of curvature >>δ

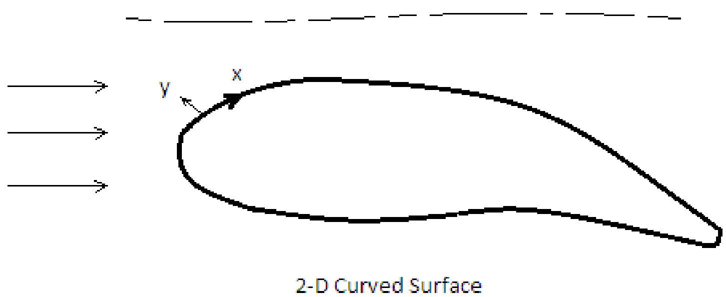


Fig. 3 Boundary layer theory applicable for curved surfaces