Module 3:

“Convective” heat and mass transfer

Lecture 24:

Heat Transfer to Boundary Layers

Because we assumed constant physical properties, the problem of momentum transfer and heat (or mass) transfer are decoupled, i.e. we solve for fluid velocity profile first, then use these velocity profiles to determine temperature profiles, and then find heat transfer rate.

Energy balance

For constant density fluid, w/o energy production $\left(\dot{Q}=0\right)$

$$\rho C_p \frac{DT}{\partial t} = K\nabla^2 T \qquad (1)$$

This eqn. Neglects viscous dissipation and assumes constant K.

For steady state 2-D flow, using Cartesian coordinates

$$\underbrace{v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y}}_{\text{Convective}} = \alpha \underbrace{\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)}_{\text{Conductive}}$$

Now

$\alpha\left(\frac{\partial^2 T}{\partial x^2}\right) << v_x \frac{\partial T}{\partial x}$ i.e. axial convection dominates axial diffusion

$$Pe = \frac{U_\infty L}{\alpha} >> 1$$

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial y^2}\right) \qquad (2)$$

B.C.s $\quad T = T_w$ at $y = 0$

$\quad T = T_\infty$ at $y \to \infty$

Define $\theta = \frac{T - T_w}{T_\infty - T_w}$

Then eqn. (2) becomes

$$v_x \frac{\partial \theta}{\partial x} + v_y \frac{\partial \theta}{\partial y} = \alpha\left(\frac{\partial^2 \theta}{\partial y^2}\right) \qquad (3)$$

With $\quad \theta = 0$ at $y = 0$

$\quad \theta = 1$ at $y \to \infty$

Prandtl number $\qquad \Pr = \frac{\mu C_p}{k} = \frac{\nu}{\alpha}$

For Pr = 1 Eqn. (3) is identical to $v_x \frac{\partial U_x}{\partial x} + v_y \frac{\partial U_x}{\partial y} = \nu \left(\frac{\partial^2 U_x}{\partial y^2} \right)$ with θ corresponding to $\frac{v_x}{v_\infty}$.

BCs are also identical. Hence the solution is identical (Refer lecture 23).

Since $\frac{v_x}{v_\infty} = f'(\eta) \quad \Rightarrow \quad \theta = f'(\eta)$

For most gases 0.7 < Pr < 1.0

<u>For Pr≠1</u>

Using quantities $\eta = y\sqrt{\frac{v_\infty}{\nu \bullet x}}, \quad f = \frac{\psi}{\sqrt{\nu \bullet x \bullet v_\infty}}, \quad v_x = v_\infty f'$

Inserting into eqn. (3), we get

$$\Pr f \frac{d\theta}{d\eta} + 2\frac{d^2\theta}{d\eta^2} = 0 \qquad (4)$$

Integrating eqn. (4) we get

$$\theta = \frac{\int_0^\eta \left[\exp\left(-\frac{1}{2}\Pr \int_0^\eta f d\eta \right) \right] d\eta}{\int_0^\infty \left[\exp\left(-\frac{1}{2}\Pr \int_0^\eta f d\eta \right) \right] d\eta} \qquad (5)$$

Use temperature distribution to obtain local heat transfer coefficient

$$-k\left.\frac{\partial T}{\partial y}\right|_{y=0,x=x} = h_x\left(T_w - T_\infty\right)$$

$$-k\left(T_\infty - T_w\right)\left.\frac{d\theta}{d\eta}\right|_{\eta=0}\sqrt{\frac{v_\infty}{\nu x}} = h_x\left(T_w - T_\infty\right)$$

$$h_x = k\sqrt{\frac{v_\infty}{\nu x}}\left.\frac{d\theta}{d\eta}\right|_{\eta=0}$$

$$Nu_x = \frac{hx}{k} = \sqrt{\frac{v\infty x}{\nu}}\left.\frac{d\theta}{d\eta}\right|_{\eta=0} = \sqrt{\mathrm{Re}_x}\left.\frac{d\theta}{d\eta}\right|_{\eta=0} \qquad \text{or}$$

$$\underset{\substack{\text{dimensionless}\\ \text{heat transfer coeff.}}}{Nu_x} = \sqrt{\mathrm{Re}_x}\,\frac{1}{\int_0^\infty\left[\exp\left(-\frac{1}{2}\mathrm{Pr}\int_0^\eta f\,d\eta\right)\right]d\eta} \qquad (6)$$

The term on right hand side under integral sign can be found out numerically.

It turns out that over a wide range of Pr (Pr>0.6)

$$Nu_x = 0.332\,\mathrm{Re}_x^{1/2}\,\mathrm{Pr}^{1/3} \qquad (7)$$

$$\left(2\ \%\ \text{error compared to eqn. } (6)\right)$$

Thickness of Thermal Boundary Layer

From eqn. (2)

$$v_\infty \frac{T}{L} + \underbrace{v_\infty \frac{\delta T}{L}}_{\text{Neglegible quantity}} \approx \alpha \frac{T}{\delta_T^{\,2}}$$

$$\delta_T \sim \sqrt{\frac{\alpha L}{v_\infty}} \Rightarrow \frac{\delta_T}{L} \sim \frac{1}{\sqrt{\Pr}} \bullet \frac{1}{\sqrt{\mathrm{Re}_L}} \qquad (8)$$

Combining equation (8) with equation $\frac{\delta}{L} \sim \sqrt{\frac{\nu}{L v_\infty}} \sim \frac{1}{\sqrt{\mathrm{Re}_L}}$, we get

$$\frac{\delta_T}{\delta} \sim \frac{1}{\sqrt{\Pr}}. \qquad (9)$$

It turns out that this equation applies for small *Pr*.

For large Pr

$$\frac{\delta_T}{\delta} \sim \frac{1}{\Pr^{1/3}} \qquad (10)$$

For mass transfer replace *Pr* with $Sc = \frac{\nu}{D}$

Hence as $Sc \to 0 \quad \Rightarrow \quad \frac{\delta_c}{\delta} \sim \frac{1}{\sqrt{Sc}}$ (11)

Dividing eqn. (10) by eqn. (11), we get

$$\frac{\delta_T}{\delta_c} = \frac{1}{Le^{1/2}} \qquad (12)$$