Module 3:

“Convective” heat and mass transfer

Lecture 21:

Convective Transport: Fluid Flow to a Rotating Disk (in an infinite mass of fluid)

Consider the rotating disk in represented in cylindrical coordinates (r, θ, y) in Fig. 1 that rotates with an angular velocity of ω. v_r, v_θ, v_z are the fluid velocities along the r, θ, z, directions respectively.

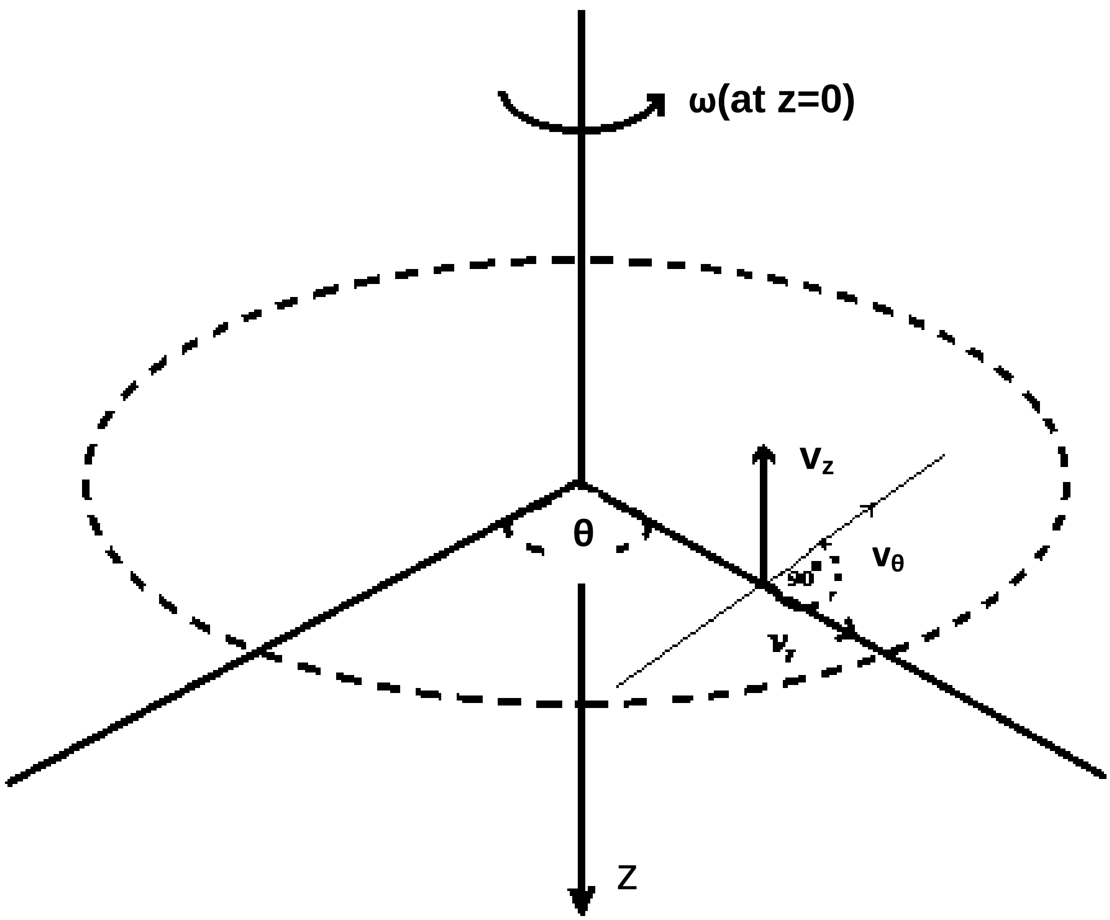

Fig. 1 Coordinates and velocity components for the rotating disk system.

Assumptions used for modelling the momentum transport in the fluid surrounding the rotating disk

1. Disk is of infinite radius (no edge effects)

2. Surrounding fluid is an incompressible, Newtonian fluid with constant μ.

3. Flow is steady state and axisymmetric

Now we write the equation of continuity for mass conservation and the Navier-Stokes equations for momentum conservation in r, r, θ, z, directions in **Cylindrical Co-ordinates.**

Continuity equation for mass conservation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right)+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0 \qquad (1)$$

Navier –Stokes equations for momentum conservation (r-component)

$$\frac{\partial \upsilon_r}{\partial t}+\upsilon_r\frac{\partial \upsilon_r}{\partial r}+\frac{\upsilon_\theta}{r}\frac{\partial \upsilon_r}{\partial \theta}-\frac{\upsilon_\theta^{2}}{r}+\upsilon_z\frac{\partial \upsilon_r}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial r}$$
$$+\nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\upsilon_r\right)\right)+\frac{1}{r^2}\frac{\partial^2 \upsilon_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial \upsilon_\theta}{\partial \theta}+\frac{\partial^2 \upsilon_r}{\partial z^2}\right]$$

(2)

Navier –Stokes equations for momentum conservation (θ-component)

$$\frac{\partial \upsilon_\theta}{\partial t}+\upsilon_r\frac{\partial \upsilon_\theta}{\partial r}+\frac{\upsilon_\theta}{r}\frac{\partial \upsilon_\theta}{\partial \theta}+\frac{\upsilon_r\upsilon_\theta}{r}+\upsilon_z\frac{\partial \upsilon_\theta}{\partial z}=-\frac{1}{\rho r}\frac{\partial P}{\partial \theta}$$
$$+\nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\upsilon_\theta\right)\right)+\frac{1}{r^2}\frac{\partial^2 \upsilon_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial \upsilon_r}{\partial \theta}+\frac{\partial^2 \upsilon_\theta}{\partial z^2}\right]$$

(3)

Navier –Stokes equations for momentum conservation (z-component)

$$\frac{\partial \upsilon_z}{\partial t}+\upsilon_r\frac{\partial \upsilon_z}{\partial r}+\frac{\upsilon_\theta}{r}\frac{\partial \upsilon_z}{\partial \theta}+\upsilon_z\frac{\partial \upsilon_z}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z} +\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \upsilon_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \upsilon_z}{\partial \theta^2}+\frac{\partial^2 \upsilon_z}{\partial z^2}\right] \tag{4}$$

In the above equation P is the dynamic pressure.

$$\underset{\substack{\text{Dynamic}\\\text{Pressure}}}{\nabla P} = \underset{\substack{\text{Thermodynaic}\\\text{Pressure}}}{\nabla p} - \underset{\substack{\text{Hydrostatic}\\\text{Pressure}}}{\rho g} \tag{5}$$

<u>Von Karman separation of variables</u>

The solution can be of the form of $\upsilon_\theta = rg(z),\ \upsilon_r = rf(z),\ \upsilon_z = h(z),\ P = p(z)$

If these expressions are substituted into the equation, above, one obtains

$$\begin{aligned}
&2f + h' = 0\\
&f^2 - g^2 + hf' = \upsilon f''\\
&2fg + hg' = \upsilon g''\\
&\rho hh' + P' = \mu h''
\end{aligned} \tag{6}$$

where prime $(')$ denotes differentiation w.r.t z

Boundary conditions are derived from the fact that

$\upsilon_\theta = r\Omega,\ \upsilon_r = 0,\ \upsilon_z = 0$ on the surface of the disk and $\upsilon_\theta = 0,\ \upsilon_r = 0$ far away from the surface

Hence $\begin{matrix} h = f = 0, g = \Omega \text{ at } z = 0 \\ f = g = 0 \text{ at } z \to \infty \end{matrix}$

Also P needs to be specified at the same point.

Let's define dimensionless variables

$$J = z\sqrt{\frac{\Omega}{\upsilon}},\ P = \mu\Omega \mathrm{P},\ \upsilon_\theta = r\Omega G,\ \upsilon_r = r\Omega F,\ \upsilon_z = \sqrt{r\Omega H}$$

$$2F + H' = 0 \qquad (7)$$

$$F^2 - G^2 + HF' = F'' \qquad (8)$$

$$2FG + HG' = G'' \qquad (9)$$

$$HH' + P' = H'' \qquad (10)$$

With $H = F = 0$, $G = 1$ at $J = 0$

$F = G = 0$ at $J \to \infty$

System (7)-(10) can be solved numerically to obtain F, H and G (See fig. 1).

Having H, P' can be obtained from (10) as

$$P = P(0) + H' - \frac{1}{2}H^2 \qquad (11)$$

For small distance, from the disk

$$H = -0.51023J^2 + \frac{1}{3}J^3 - 0.10267J^4 + ... \qquad (12)$$

For large distances from the disk

$$H = -0.88447 + 2.112\exp(-0.88447J) + ... \qquad (13)$$

These expressions are useful, since the normal velocity v_z is important for calculating the rate of heat or mass transfer to the rotating disk. The fact that $v_z \neq v_z(r)$ has important consequences for heat or mass flux uniformity along the disk surface (see fig. below)

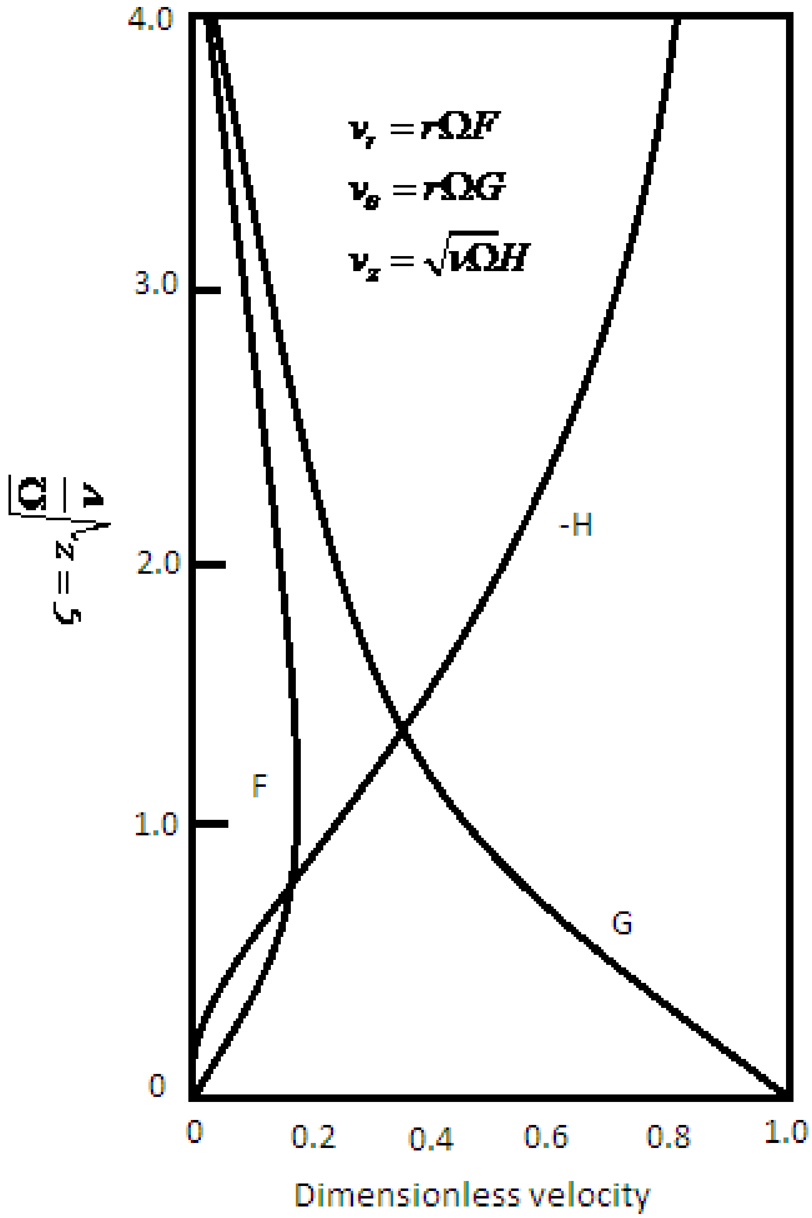

Fig.2 Velocity profile for a rotating disk.