Module 2 :

"Diffusive" heat and mass transfer

Lecture 18:

Melting and Solidification (An example of Moving Boundary problem)

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Melting and solidification find application in the geophysical sciences, industrial forming operations such as casting and laser drilling; latent heat energy storage system; and pharmaceutical processing.

Melting and solidification processes can be classified as

1. One-region
2. Two –region
3. Multiple-region

The classification depends on the properties of phase change material (PCM) involved and the initial conditions. For a single component PCM, melting or solidification occurs at a single temperature. For example, pure water melts at a uniform temperature of $0\ ^oC$, while pure n-Octadecane melts at $28\ ^oC$.

For the melting (or solidification) process, if the initial temperature, T_i of the PCM, equals the melting point, T_m, the temperature in the solid phase remains uniformly equal to the melting point throughout the process. In this case only the temperature distribution in the liquid (solid) phase needs to be determined. Thus, the temperature of only one-phase is unknown and the problem is called a *one-region problem*.

For the melting process, if the initial temperature of PCM, T_i, is below the melting point of the PCM, T_m, (or above, for solidification), the temperature distribution of both the liquid and the solid phase must be determined; which is called as a *two-region problem*.

(a) Melting ($T_0 > T_m$) (b) Solidification ($T_0 < T_m$)
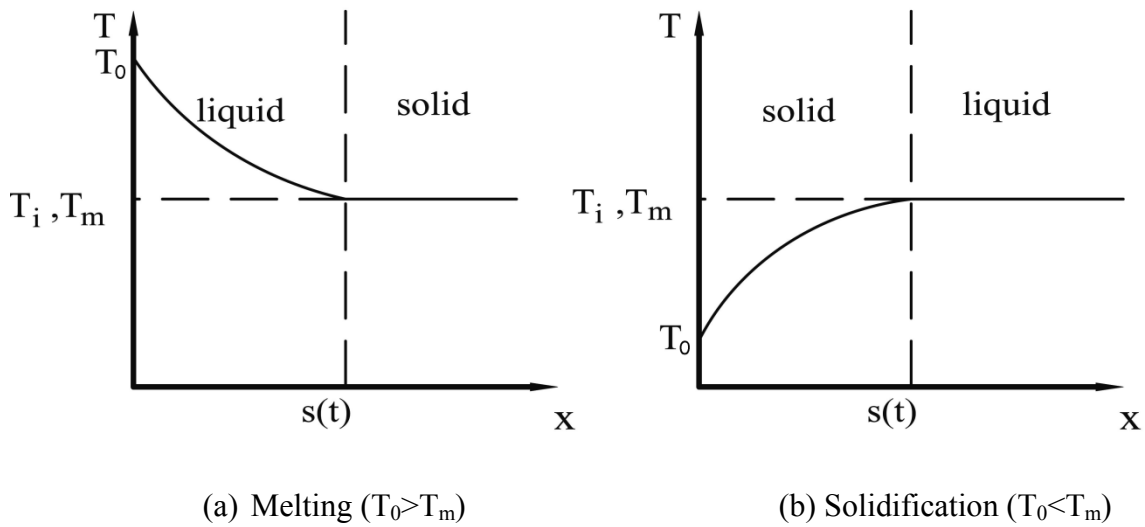
Fig.1. One-region melting and solidification

(a) Melting ($T_0 > T_m > T_i$) (b) Solidification ($T_0 < T_m < T_i$)
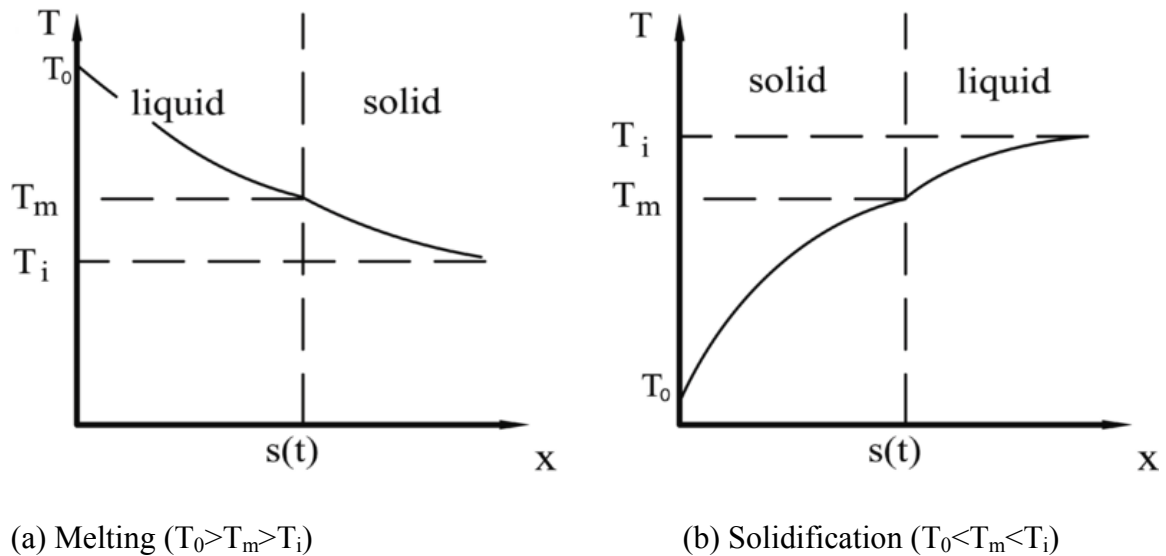
Fig.2. Two-region melting and solidification

Example

Let's assume that the x>0 is filled with liquid at temperature T_2. At time t=0 the surface temperature is lowered to T_1 (below melting point T_p) and is maintained constant

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thereafter. The solid-liquid interface starts moving. At some later time t, the region

x<X(t) is occupied by liquid with property k_1, α_1, ρ_1, $\hat{C}_{p1}$. The region x> X (t) is

occupied by liquid with property k_2, α_2, ρ_2, $\hat{C}_{p2}$.
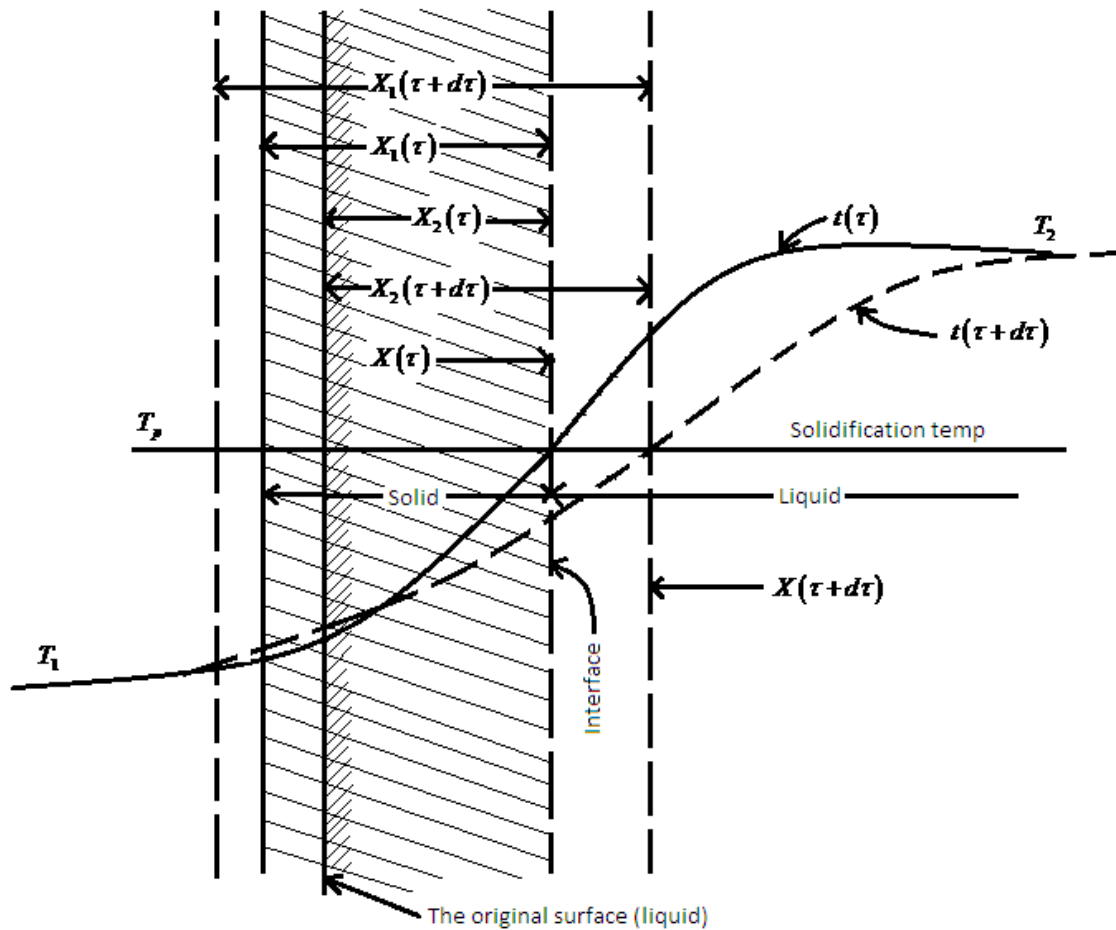
The governing equations are

$$\frac{\partial^2 t_1}{\partial x_1^2} - \frac{1}{\alpha_1}\frac{\partial t_1}{\partial t} = 0$$

(17.1)

with $t_1 = T_1$ at $x_1 = 0$

and

$$\frac{\partial^2 t_2}{\partial x_2^2} - \frac{1}{\alpha_2}\frac{\partial t_2}{\partial t} = 0 \tag{17.2}$$

with $t_2 = T_2$ at $x_2 \to \infty$

When going from liquid to solid state there is a change in density, hence we can write

$$\frac{X_1}{X_2} = \frac{\rho_2}{\rho_1} = \beta \tag{17.3}$$

At the interface $t_1 = t_2 = T_p$ at $x_1 = X_1(t)$ or $x_2 = X_2(t)$

If Q_L is the latent heat of fusion of the material, an energy balance at the interface gives

$$k_1\left(\frac{\partial t_1}{\partial x_1}\right)_{x_1=X_1} - k_2\left(\frac{\partial t_2}{\partial x_2}\right)_{x_2=X_2} = Q_L\rho_1\frac{dX_1}{dt} = Q_L\rho_2\frac{dX_2}{dt} \tag{17.4}$$

Assuming solution of the form

$$t_1 - T_p = (T_1 - T_p) + A\,erf\left(\frac{x_1}{\sqrt{4\alpha_1 t}}\right) \tag{17.5}$$

$$t_2 - T_p = (T_2 - T_p) + B\,erfc\left(\frac{x_2}{\sqrt{4\alpha_2 t}}\right) \tag{17.6}$$

which satisfy the conditions at $x_1=0$ and $x_2 \to \infty$

Requiring $t_1 = t_2 = T_p$ at the interface yields

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$$T_p - T_1 = A\ erf\left(\frac{X_1}{\sqrt{4\alpha_1 t}}\right) \tag{17.7}$$

$$T_p - T_2 = B\ erfc\left(\frac{X_2}{\sqrt{4\alpha_2 t}}\right) \tag{17.8}$$

Since these equations must hold for all values of X_1 and X_2, these must be proportional to $\sqrt{t}$.

Thus we can write

$$\begin{aligned}X_1 &= K\beta\sqrt{t} \\ X_2 &= K\sqrt{t}\end{aligned} \tag{17.9}$$

where K is a constant to be determined

Using eqn.(17.5), (17.6) and (17.9) into eqn. (17.4), we get

$$\frac{Ak_1}{\sqrt{\pi\alpha_1}}\exp\left(-K^2\beta^2\big/4\alpha_1\right) + \frac{Bk_2}{\sqrt{\pi\alpha_2}}\exp\left(-K^2\big/4\alpha_2\right) = \frac{Q_L\rho_1 K\beta}{2} \tag{17.10}$$

Also using eqn. (17.7) and (17.8) into eqn.(17.10), we get

$$\frac{\left(T_p - T_1\right)k_1\exp\left(-K^2\beta^2\big/4\alpha_1\right)}{\sqrt{\pi\alpha_1}\,erf\left(K\beta\big/2\sqrt{\alpha_1}\right)} - \frac{\left(T_2 - T_1\right)k_2\exp\left(-K^2\big/4\alpha_2\right)}{\sqrt{\pi\alpha_2}\,erfc\left(K\big/2\sqrt{\alpha_2}\right)} = \frac{Q_L\rho_1 K\beta}{2} \tag{17.11}$$

Eqn. (17.11) can be solved numerically for the determination of K.

After determining K value constants A and B can be found from eqns. (17.7-17.9).

The temperature distribution thus can be given as

Solid: $\dfrac{t_1 - T_p}{T_1 - T_p} = 1 - \dfrac{erf\left(\dfrac{x}{\sqrt{4\alpha_1 t}}\right)}{erf\left(\dfrac{K\beta}{\sqrt{4\alpha_1}}\right)}$

(17.12)

Liquid: $\dfrac{t_2 - T_p}{T_2 - T_p} = 1 - \dfrac{erfc\left(\dfrac{x}{\sqrt{4\alpha_2 t}}\right)}{erfc\left(\dfrac{K}{\sqrt{4\alpha_2}}\right)}$

(17.13)

For the water-ice system $\beta = 1.09$ and

$$K \cong \sqrt{\dfrac{2\left(T_p - T_1\right)k_1}{Q_L \rho_1 \beta^2}}$$

(17.14)

Since $\beta > 1$ (i.e. density of ice is less than that of water) a slower freezing time is calculated if the change in density is considered (i.e. if the assumption $\beta = 1$ is not made)

Special Case

When the liquid is initially at the melting point i.e. $T_q = T_p$, eqn. (17.11) simplifies to

$$\frac{\left(T_p - T_1\right)k_1 \exp\left(-K^2\beta^2\big/4\alpha_1\right)}{\sqrt{\pi\alpha_1}\,erf\left(K\beta\big/2\sqrt{\alpha_1}\right)} = \frac{Q_L\rho_1 K\beta}{2}$$