Module 3:

“Convective” heat and mass transfer

Lecture 31:

Dispersion

One dimensional "Diffusive" Transport (without flow): Governing Equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial Z^2}$$

Dispersion Coefficient (can be molecular diffusivity)

Boundary and Initial Conditions:

B.C. $C = 0$ as $Z \to \pm\infty$ $\quad t \geq 0$

Total quantity of substance released

I.C. $C = \frac{M}{A}\delta(Z)$: Pulse at $t = 0$

area delta function

$$\int_{-\infty}^{+\infty} CA\,dz = \int_{-\infty}^{+\alpha} \frac{M}{A} A\delta(Z)\,dZ = M$$

<u>Solve the above Partial Differential Equation by Fourier Transform</u>

$$\frac{\partial \bar{C}}{\partial t} = -Ds^2\bar{C} \quad \text{and} \quad \bar{C} = \frac{M}{A} \text{ at t=0}$$

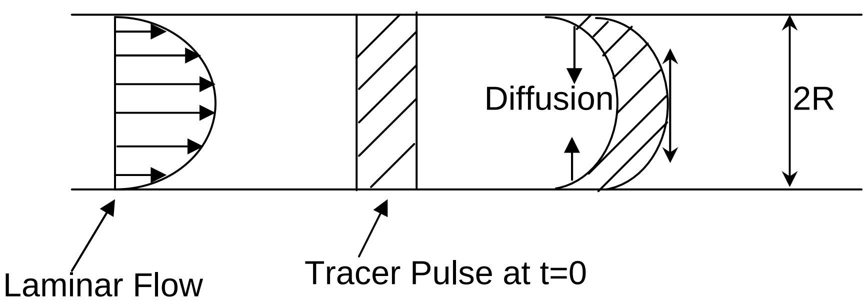
The solution is $\bar{C} = Be^{-s^2Dt} = \frac{M}{A}e^{-s^2Dt}$

From tables find inverse

$$\boxed{C = \frac{M}{A}\frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{X^2}{4Dt}\right)} \qquad (3.5.1)$$

Generalize to three-dimensional system

Dispersion in Laminar Flow inside a tube (Taylor Dispersion)



Taylor Dispersion in a circular tube of radius R with laminar flow

Assumptions:

1. Constant physical properties
2. Dilute tracer
3. Fully developed velocity

Governing equation

$$2\langle U\rangle\left\{1-\frac{r^2}{R^2}\right\}$$

$$\frac{\partial C}{\partial t}+U_0(r)\frac{\partial C}{\partial X}=D\left(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r}+\frac{\partial^2 C}{\partial X^2}\right) \qquad (3.5.2)$$

Boundary and Initial Conditions

B.C. $X=0$ and $t=0$ $\quad \Rightarrow C=\frac{M}{\pi R^2}\delta(Z)$

$t>0,\quad r=R \qquad \Rightarrow \frac{\partial C}{\partial r}=0$, impermeable wall

$t>0,\quad r=0 \qquad \Rightarrow \frac{\partial C}{\partial r}=0$, symmetry

$t\geq 0\ ,\quad X\rightarrow\pm\infty \quad \Rightarrow C=0$

It is convenient to work in a coordinate system that moves with the average flow velocity $\langle U\rangle$ Therefore, define

$$U(r)=U_0(r)-\langle U\rangle=\langle U\rangle\left(1-2\frac{r^2}{R^2}\right)$$

Then equation (3.5.2) becomes (neglecting axial "diffusion" i.e., $\frac{\partial^2 C}{\partial X^2}=0$)

$$\frac{\partial C}{\partial t}+\langle U\rangle\left(1-\frac{2r^2}{R^2}\right)\frac{\partial C}{\partial X}=D\left(\frac{1}{r}\frac{\partial C}{\partial r}+\frac{\partial^2 C}{\partial r^2}\right) \qquad (3.5.3)$$

Where x and t refer to the new coordinate system

Note: Assume that $\frac{\partial C}{\partial X}=\text{Constant}$ and $\frac{\partial C}{\partial t}=0$

Then equation (3.5.3) yields

$$C = C_0 + \frac{R^2 U}{4D}\frac{\partial C_0}{\partial X}\left(\eta^2 - \frac{1}{2}\eta^4\right) \qquad (3.5.4)$$

With $C_0 = C\left(r = 0\right)$, *and* $\eta = \frac{r}{R}$

The mean concentration C_m over the tube cross section is defined by

$$C_m = 2\int_0^1 C\,\eta\, d\eta \qquad (3.5.5)$$

Using C_m, equation (3.5.4) can be modified to

$$C = C_m + \frac{R^2 U}{4D}\frac{\partial C_m}{\partial X}\left(-\frac{1}{3} + \eta^2 - \frac{1}{2}\eta^4\right) \qquad (3.5.6)$$

We also have $\frac{\partial C}{\partial t} = 0$ (as before) in the case $\frac{\partial C_m}{\partial X}$ = Constant

Equation (3.5.6) is a solution of (3.5.3) when $\frac{\partial C_m}{\partial X}$ = Constant

The rate at which material is transported across a section of the tube is

$$Q = 2\pi R^2 \int_0^1 CU\left(\eta\right)\eta\, d\eta$$

Inserting the values of C from (3.5.6) and υ one obtains

$$Q = -\pi R^2\left(\frac{R^2 U^2}{48D}\right)\frac{\partial C_m}{\partial X} \qquad (3.5.7)$$

Equation (3.5.7) shows that the combined effect of **longitudinal convection and radial molecular diffusion** is to give rise to transport across planes which move with the average flow velocity equivalent to transport across a stationary plane with diffusivity

$$\boxed{D_{eff} = \frac{R^2 U^2}{48D}} \qquad (3.5.8)$$

<u>If it is assumed that equation (3.5.7) applies even when</u> $\frac{\partial C_m}{\partial X} \neq$ <u>Constant and one can describe the dispersion process (relative to axes moving with Speed U) by</u>

$$\frac{\partial C_m}{\partial t} = D_{eff} \frac{\partial^2 C_m}{\partial X^2} \qquad (3.5.9)$$

If a certain amount M of a substance is introduced into the system at t=0 and X=0 , and the substance occupies a small section of the tube compared to R, then

$$C_m = \frac{M}{\pi R^2} \delta(X) \quad at \quad t=0$$

The solution to this problem may given as equation (3.5.1)

$$\boxed{C_M = \frac{M}{\pi R^2} \frac{1}{\sqrt{4\pi D_{eff} t}} \exp\left(-\frac{(X - Ut)^2}{4D_{eff} t}\right)} \qquad (3.5.10)$$

Note : In equation (3.5.10) the co-ordinate has been transformed has been transformed back to a stationary plane.