Module 3:

“Convective” heat and mass transfer

Lecture 30:

A Graetz-Nusselt Problem: Fully developed V, developing C or T

Governing equation (v_r = 0, fully developed v) is given as

$$v_z \frac{\partial C_A}{\partial z} = D_{AA}\left[\frac{1}{r}\left(r\frac{\partial C_A}{\partial r}\right)+\frac{\partial^2 C_A}{\partial z^2}\right], \quad v_z = 2\langle v_z\rangle\left(1-\frac{r^2}{R^2}\right)$$

Set $\xi = \frac{r}{R}, \; \theta = \frac{C_A - C_0}{C_b - C_0}, \; J = \frac{zD}{2\langle v_z\rangle R^2} = \frac{1}{\mathrm{Re}\,Sc}\frac{z}{R}$

where C_b is the concentration at inlet and C_0 is the concentration at the wall

$$\left(1-\xi^2\right)\frac{\partial \theta}{\partial J} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \theta}{\partial \xi}\right)+\frac{1}{Pe^2}\frac{\partial^2 \theta}{\partial J^2} \qquad \underset{\text{Axial conc. dominates axial diffusion}}{\mathrm{Pe} \gg 1}$$

With the following B.Cs

$$\theta = 1 \quad \text{at} \quad J = 0$$
$$\theta = 0 \quad \text{at} \quad \xi = 1$$
$$\frac{\partial \theta}{\partial \xi} = 0 \quad \text{at} \quad \xi = 0$$

Solve by separation of variables $\theta = f \bullet g$

Then $\frac{1}{f}\frac{\partial f}{\partial J} = \frac{1}{g}\frac{1}{\xi\left(1-\xi^2\right)}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial g}{\partial \xi}\right) = -C^2$

$f = A\exp\left(-C^2 J\right)$ and

$$\xi\frac{\partial^2 g}{\partial \xi^2}+\frac{\partial g}{\partial \xi}+C^2\xi\left(1-\xi^2\right)g = 0$$

The equation above is the Sturm-Liouville problem

B.Cs:

$$\xi = 1 \quad \text{at} \quad g = 0$$
$$\xi = 0 \quad \text{at} \quad g' = 0$$

General solution is

$$\theta = \sum_{n=1}^{\infty} P_n \exp\left(-C_n{}^2 J\right) \bullet g_n(\xi) \qquad (\text{Graetz series})$$

Normalization condition to find P_n:

$$\int_0^1 \xi\left(1-\xi^2\right) g_n(\xi)\, d\xi = \sum_{n=1}^{\infty} \int_0^1 P_n g_n(\xi)\ \xi\left(1-\xi^2\right) g_m(\xi)\, d\xi$$
$$= P_m \int_0^1 \xi\left(1-\xi^2\right)\left[g_m(\xi)\right]^2 d\xi$$

From the above equation find P_m

Total amount of material transferred to wall in a length z is

$$J = -\int_0^z D \left.\frac{\partial C_A}{\partial r}\right|_{r=R} 2\pi R dz \quad \text{and}$$

$$1-\frac{J}{\pi R^2\left(C_b-C_0\right)}\langle v_z\rangle=\sum_{n=1}^{\infty}4P_n\exp\left(-C_n{}^2J\right)\int_0^1\xi\left(1-\xi^2\right)g_n\left(\xi\right)d\xi$$
$$=\sum_{n=1}^{\infty}M_n e^{-C_n{}^2J}$$

(1)

For large n, series converges rapidly

First 10 eignvalues C_n and coefficient M_n of Graetz series

n	C_n	M_n
1	2.7043	0.8190
2	6.6790	0.09752
3	10.6733	0.0321
4	14.6710	0.01544
5	18.6698	0.008788
6	22.6691	0.0055838
7	26.66866	0.0038202
8	30.66832	0.0027564
9	34.66807	0.0020702

10	38.66788	0.0016043

Solution for short entrance length

For short J, the diffusion length is very thin. Then

$$1-\xi^2=(1-\xi)(1+\xi)\approx 2(1-\xi)$$

and curvature may be neglected (planer diffusion)

$$\frac{1}{\xi}\frac{d\theta}{d\xi}<<\frac{\partial^2\theta}{\partial\xi^2}$$

Then, the governing equation becomes

$$2(1-\xi)\frac{\partial\theta}{\partial J}=\frac{\partial^2\theta}{\partial\xi^2}$$

$\theta=1$ at $J=0$
$\theta=0$ at $\xi=1$
$\theta=1$ outside the diffusion layer

Similarity transformation

$$\eta=(1-\xi)(2/9J)^{1/3} \qquad (2)$$

PDE becomes an ODE

$$\theta'' + 3\eta^2\theta' = 0 \quad \Rightarrow \quad \theta = \frac{1}{\Gamma\left(4/3\right)}\int_0^{\eta} e^{-x^3}dx$$

In terms of the physical variables

$$\eta = y\left(\frac{4\langle v_z\rangle}{9zD_{AA}R}\right)^{1/3}$$

where y = R – r is the distance from the wall.

The total amount of material transferred to the wall up to *J* is;

$$\frac{J}{\pi R^2\left(C_b - C_0\right)\langle v_z\rangle} = \frac{\left(48\right)^{1/3}}{\Gamma\left(4/3\right)}J^{2/3} = 4.070J^{2/3} \qquad (3)$$

The mass transfer rate becomes infinite at *J* = 0 $\left(N \sim J^{-1/3}\right)$

Case IV

Plug flow $v_z = V = \text{const.}$

Developing T; const. wall T_0

$$\left.\begin{aligned} \frac{\partial T}{\partial z} &= \frac{\alpha}{V}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \\ \theta &= \frac{T_0 - T}{T_0 - T_i} \end{aligned}\right\}$$

Where T_0 is the wall temperature and T_i is the inlet temperature

$$\frac{\partial\theta}{\partial z}=\frac{\alpha}{V}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)$$

Using separation of variables technique, then we have

$$\frac{V}{\alpha}\frac{1}{f}\frac{\partial f}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right)=-C^2$$

Here $f = f_0 e^{-\frac{\alpha}{V}z}$