Module 1 :

The equation of "continuity"

Lecture 2:

Conservation of Linear Momentum and Conservation of Mechanical energy

Definition of Fluid: Fluids and solids can be differentiated and explained qualitatively on the basis of molecular structure, but a more specific distinction is based on how they deform under the action of an *external force*. Specifically, *a fluid is defined as a substance that deforms continuously when acted on by a shearing stress of any magnitude*. This shearing stress (force/unit area) is created whenever a tangential force acts on a surface. When metals such as steel are acted on by a shearing stress, they will initially deform, but they will not continuously deform. However common fluids such as water, oil and air satisfy the definition of fluid, which is, they will flow when acted on by a shearing stress.

Newton's second law: It states that the net force on a body is equal to rate of change of its linear momentum in an inertial frame of reference. As the fluid moves form one point to another, it experiences an acceleration or deceleration. According to Newton's second law of motion, the net force acting on the fluid particle under consideration must equal its mass times its acceleration.

$$\text{F} = m\text{a} \tag{1.5}$$

Definition of an Inviscid fluid: An inviscid fluid is defined as one that has zero viscosity and can support no shear stress.

In practice, there are no inviscid fluids, since every fluid support shear stresses when it is subjected to a rate of strain displacement. For many flow situations the viscous effects are relatively small as compared with other effects, although in some circumstances they are very important. Let's assume that fluid motion is governed by pressure and gravity forces only, and examine Newton's second law as it applies to a fluid particle in the form:

Application of Newton's Second Law to a Fluid

(Net pressure force on a particle) + (net gravity force on a particle) =

(Particle mass) × (particle acceleration)

To apply Newton's law of motion to a fluid, we must define an appropriate coordinate system in which to describe the motion. In general, the motion will be three- dimensional and unsteady so that three space coordinates and time are needed to describe it.

Here if we consider all the forces mentioned above and in addition to that viscous forces, then

$$\left(\text{Time rate of change of momentum of a body}\right) = \left(\text{Sum of forces acting on the body}\right)$$

$$\frac{\partial}{\partial t}\left(\rho \underline{U}\right) = -\nabla \bullet \rho \underline{U}\underline{U} - \nabla P - \nabla \bullet \underline{\underline{\tau}} + \rho g \qquad (1.6a)$$

where g is acceleration due to gravity, P is pressure and $\underline{\underline{\tau}}$ is the second order stress tensor defined later (in equation 1.8).

Equation (1.6a) is the *Microscopic Momentum Balance* equation for the fluid.
The terms represent: The **rate of increase of momentum per unit volume of fluid.**

- term on left, time rate of change of momentum **at a point;**

- first term on right, change of momentum due to **convective velocity into or from the point**;

- second term on right, change of momentum due to **the pressure forces**;

- third term on right, change of momentum due to **the action of viscous forces**; and

- fourth term on right, change of momentum due to **the gravitational forces**

Using equation (1.4), equation (1.6a) can be written as:

$$\rho \frac{DU}{Dt} = -\nabla P - \left[\nabla \bullet \underline{\underline{\tau}} \right] + \rho g \tag{1.6b}$$

Here, $\rho \dfrac{DU}{Dt} = $ Product of mass per unit volume and acceleration $=$ The rate of change of momentum per unit volume.

Newton's Law of Viscosity

In equation (1.6), let us insert **Newton's law of viscosity**, then we obtain

$$\boxed{\tau_{yx} = -\mu \frac{d}{dy}(U_x)} \qquad \text{(for 1-D flow)} \tag{1.7}$$

Equation (1.7) relates the flux of momentum and the velocity gradient and is known as **Newton's law of Viscosity**. According to this law, the shear stress between adjacent fluid layers is proportional to the negative value of the velocity gradient between the two layers. Figure 1.4 illustrates the motion of two layers placed one above the other. The minus sign shows the momentum flows from the higher velocity to the lower velocity. The constant of proportionality, **μ** is the **viscosity** of the fluid and has the unit of Ns/m² or Poise.
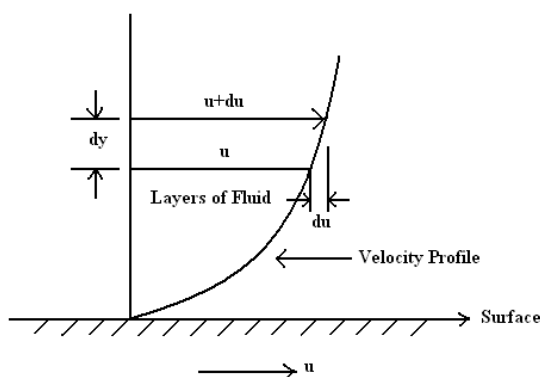

Fig.1.4. Layers of fluid flowing one above the other

Figure 1.4 shows that the two layers are separated by a distance dy and the lower layer is moving at a velocity u and the upper layer moving at a velocity u+du. When the upper layer of fluid is moving faster than the lower layer, the deformation produces the shear stress τ_{yx} proportional to velocity gradient. The fast moving layer exerts a force which tends to decelerate the faster moving layer, as if layers are rubbed against one another like the successive cards in a deck which is being spread.

Stress Tensor:

Also keep in mind that **the stress tensor is symmetric** i.e., $\underline{\underline{\tau}} = \underline{\underline{\tau}}^{t}$ (where $\underline{\underline{\tau}}^{t} = transpose\ of\ \underline{\underline{\tau}}$), as a result of the law of conservation of angular momentum (Newton's third law).

For complete definition of Newtonian fluids, the components of the stress tensor are

$$\tau_{xx} = -2\mu \frac{\partial U_x}{\partial x} + \left(\frac{2}{3}\mu - k\right)\left(\nabla \bullet \underline{U}\right) \tag{1.8a}$$

$$\tau_{yy} = -2\mu \frac{\partial U_y}{\partial y} + \left(\frac{2}{3}\mu - k\right)\left(\nabla \bullet \underline{U}\right) \tag{1.8b}$$

$$\tau_{zz} = -2\mu \frac{\partial U_z}{\partial z} + \left(\frac{2}{3}\mu - k\right)\left(\nabla \bullet \underline{U}\right) \tag{1.8c}$$

with $k = 0$, where k - Bulk Viscosity

$$\tau_{xy} = \tau_{yx} = -\mu\left(\frac{\partial Ux}{\partial y} + \frac{\partial U_y}{\partial x}\right) \tag{1.8d}$$

$$\tau_{yz} = \tau_{zy} = -\mu\left(\frac{\partial U_y}{\partial z} + \frac{\partial U_z}{\partial y}\right) \tag{1.8e}$$

$$\tau_{xz} = \tau_{zx} = -\mu\left(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}\right) \tag{1.8f}$$

NPTEL , IIT Kharagpur, Prof. Saikat Chakraborty, Department of Chemical Engineering

Navier-Stokes Equation:

We have shown before that for an **Incompressible fluid**, $\nabla \bullet \underline{U} = 0$

If the viscosity **μ** is spatially invariant, we obtain the **Navier-Stokes Equation** by inserting equation (1.8) into eqn. eqn. (1.6b). The Navier-Stokes equation for an **Incompressible Isotropic fluid** is given by

$$\rho \frac{D\underline{U}}{Dt} = -\nabla \underline{P} + \mu \nabla^2 \underline{U} + \rho \underline{g} \qquad (1.9)$$

Mechanical Energy Balance Equation:

Taking the scalar product of equation (1.6) with $\underline{U}$, we obtain:

$$\rho \frac{D}{Dt}\left(\frac{1}{2}U^2\right) = -\left(\underline{U} \bullet \nabla P\right) - \left(\underline{U} \bullet \left[\nabla \bullet \underline{\underline{\tau}}\right]\right) + \rho\left(\underline{U} \bullet g\right) \qquad (1.10)$$

Equation (1.10) can be expanded to read as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\underline{U}^2\right) = -\left(\nabla \bullet P\underline{U}\right) - P\left(-\nabla \bullet \underline{U}\right) - \left(\nabla \bullet \left[\underline{\underline{\tau}} \bullet \underline{U}\right]\right) - \left(-\underline{\underline{\tau}} : \nabla\underline{U}\right) + \rho\left(\underline{U} \bullet \underline{g}\right) \qquad (1.11)$$

Equation (1.11) is the **Microscopic Mechanical Energy Balance Equation**. It is in the Lagrangian frame of reference and describes the rate of change per unit volume of kinetic energy as one follows the fluid motion. In equation (1.11)

NPTEL , IIT Kharagpur, Prof. Saikat Chakraborty, Department of Chemical Engineering

> ➢ First term on RHS gives rate of work done per unit volume by pressure forces

> ➢ Second term on RHS represents rate of reversible conversion to internal energy per unit volume

> ➢ Third term on RHS represents rate of work done per unit volume by viscous forces

> ➢ Fourth term on RHS represents rate of irreversible conversion to internal energy per unit volume

> ➢ Fifth term on RHS represents rate of work done per unit volume by gravitational forces.

Typical Boundary Conditions on the momentum equation (given by equation (1.9)):

I.   *No-slip at the wall*: This boundary condition says that the fluid in contact with the wall will have the same velocity as the velocity of the wall. Often the walls are not moving, so the fluid velocity is zero. In drag flows, the velocity of the wall is finite and the fluid velocity is equal to the wall velocity.

Uniform approach velocity, V

Relative velocities of fluid layers
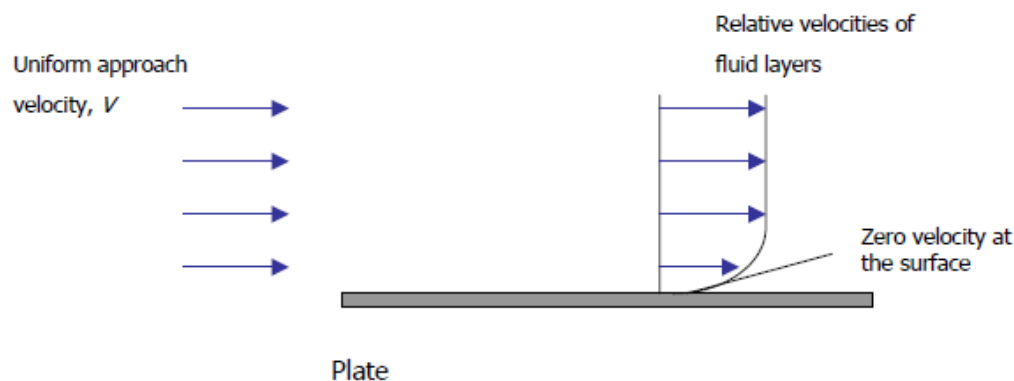
Zero velocity at the surface

Plate

Fig.1.5. *No slip condition* at the wall

A fluid flowing over stationary surface comes to complete stop at the surface because of the no-slip condition. The flow region adjacent to the wall in which viscous effects (and thus the velocity gradients) are significant is called the **boundary layer**. Mathematically, for *No slip condition* we can write,

$$\left(u\right)_{\text{at the wall}} = V_{wall}$$

II.   *Symmetry*: In some cases there is a plane of symmetry. Since the velocity field is the same on either side of the plane of symmetry, the velocity must go through a minimum or a maximum at the plane of symmetry. Thus, the boundary condition to use is that the first derivative of the velocity is zero at the plane of symmetry. Figure 1.6 below shows the symmetrical planes. Mathematically, it can be expressed as

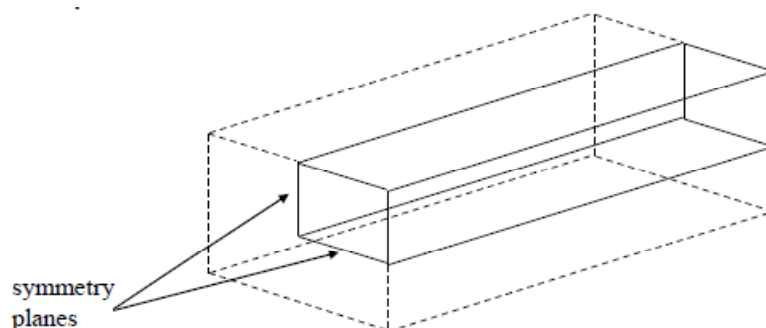$$\left(\frac{\partial u}{\partial x}\right)_{\text{at the plane of symmetry}} = 0$$

Fig.1. 6. Plane of symmetry

III. **Stress Continuity:** When a fluid forms one of the boundaries of the flow, the stress is continuous from one fluid to another. Thus for a viscous fluid in contact with an inviscid (zero or very low viscosity fluid), this means that at the fluid-fluid boundary, the stress in the viscous fluid is same as the stress in inviscid fluid. Since the inviscid fluid can support no shear stress (zero viscosity), this means that the stress is zero at the interface. The boundary condition between a fluid such as a polymer and air, for example, would be that the shear stress in the polymer at the interface would be zero.

$$\left(\tau_{jk} \right)_{\text{at the interface}} = 0$$

Alternatively, if two viscous fluids meet and form a flow boundary, this same boundary condition will require that the stress in one fluid equals the stress in the other at the interface.

$$\left(\tau_{jk} \left(\text{fluid 1} \right) \right)_{\text{at the interface}} = \left(\tau_{jk} \left(\text{fluid 2} \right) \right)_{\text{at the interface}}$$

NPTEL , IIT Kharagpur, Prof. Saikat Chakraborty, Department of Chemical Engineering

Such conditions occur at the interface and are of the result of interfacial stress. Most common example of interfacial stress is surface tension.
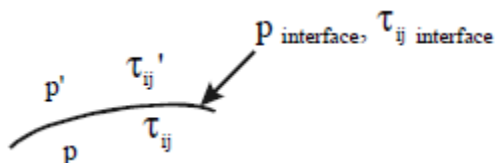

Fig.1.7. Stress continuity at the interface

At the interface of the two fluids, surface tension implies in a pressure jump across the interface.

IV.   ***Velocity Continuity***- when a fluid forms one of the boundaries of the flow as described above, the velocity is also continuous from one fluid to another.

$$\left(v_p\left(fluid\ 1\right)\right)_{at\ the\ interface} = \left(v_p\left(fluid\ 2\right)\right)_{at\ the\ interface}$$