Module 2 :

"Diffusive" heat and mass transfer

Lecture 12:

Heat Conduction in Semi-infinite Slab with Constant Flux density at the wall

Consider heat conduction in y-direction in a semi-infinite slab (bounded only by one face) initially at a temperature T_i, whose face suddenly at time equal to zero is raised to and maintained at T_1. Assuming constant thermal diffusivity and with no heat generation, a differential equation in one space dimension and time is given by

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{12.1}$$

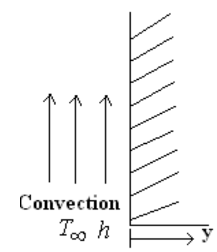

Fig. 12.1 Heat transfer in Semi- infinite solids

The initial condition (I.C) and boundary conditions (B.C) are

I.C. $T(y,0) = T_i = \text{Constant}\ (y > 0)$

The condition of constant flux density, q_0, can be achieved in a system where the object is exposed to a uniform source of energy. Such conditions can be observed in a fired heater, nuclear reactor, or solar collector, where the tube is exposed to a uniform source of radiation.

B.C.1. $-k\left(\frac{\partial T}{\partial y}\right)_{y=0} = q_0 = \text{Constant}\ \ (t > 0)$

B.C.2. $\lim_{y\to\infty} T(y,t) = T_i\ \ (t > 0)$

Setting $\theta = T - T_i$, so equation (12.1) becomes

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial y^2} \tag{12.2}$$

Accordingly I.C. and B.Cs will be

I.C. $\theta(y,0) = 0$

B.C.1. $-k\left(\frac{\partial \theta}{\partial y}\right)_{y=0} = q_0$

B.C.2. $\theta \to 0 \quad \text{as} \quad y \to \infty$

Using Laplace transform to solve equation (12.2), we get

$$p\,\bar{\theta}(y,p) - \bar{\theta}(y,0) = \alpha\,\bar{\theta}_{yy}(y,p) \tag{12.3}$$

Using I.C. in equation (12.3), we get

$$p\bar{\theta} = \alpha \frac{\partial^2 \bar{\theta}}{\partial y^2} \tag{12.4}$$

The solution to equation (12.4) is

$$\bar{\theta} = C_1 \exp\left(y\sqrt{\frac{p}{\alpha}}\right) + C_2 \exp\left(-y\sqrt{\frac{p}{\alpha}}\right) \tag{12.5}$$

Applying B.C.2 in eqn. (12.5), we get

$$C_1 = 0$$

From B.C.1 and differentiation of eqn. (12.5) w.r.t y gives

$$-kC_2\left(-\sqrt{\frac{p}{\alpha}}\right) = \frac{q_0}{p}$$

Evaluating C_2 and substituting in equation (12.5), we get

$$\bar{\theta} = \frac{q_0 \sqrt{a}}{K_p \sqrt{p}} \exp\left(-y\sqrt{\frac{p}{\alpha}}\right) \tag{12.6}$$

From tables (refer H.S. Mickley, T.S. Sherwood, and C.E. Reed, Applied Mathematics in Chemical Engineering., McGraw-Hill, 1979)

$$£^{-1}\left\{\frac{1}{\sqrt{p}} \exp\left(-y\sqrt{\frac{p}{a}}\right)\right\} = \frac{1}{\sqrt{\pi\, t}} \exp\left(-\frac{y^2}{4\alpha\, t}\right) = F(y) \tag{12.7}$$

Division of the transform by 'p' corresponds to integration between 0 and t.

$$£^{-1}\left\{\frac{1}{p} f(p)\right\} = \int_0^t F(z)\, dz \tag{12.8}$$

Thus using equation (12.6), (12.7) and (12.8) we can write

$$\theta = £^{-1}\left\{\frac{q_0 \sqrt{\alpha}}{k} \frac{1}{p\sqrt{p}} \exp\left(-y\sqrt{\frac{p}{\alpha}}\right)\right\} = \frac{q_0 \sqrt{\alpha}}{k} \int_0^t \frac{1}{\sqrt{\pi\, z}} \exp\left(-\frac{y^2}{4\alpha\, z}\right) dz \tag{12.9}$$

Set $\quad x = \dfrac{y}{\sqrt{4\alpha\, z}}$, equation (12.9) then becomes

$$\theta = \frac{q_0 y}{k\sqrt{\pi}} \int_{\frac{y}{\sqrt{4\alpha\, t}}}^{\infty} \frac{1}{x^2} \exp\left(-x^2\right) dx \tag{12.10}$$

On integration by parts, eqn. (12.11) gives

$$\theta(y,t)=\frac{q_0}{k\sqrt{\pi}}\left[2\sqrt{\alpha\, t}\exp\left(-\frac{y^2}{4\,\alpha\, t}\right)-2y\int_{\frac{y}{\sqrt{4\alpha t}}}^{\infty}e^{-y^2}dy\right] \tag{12.11}$$

or

$$\theta(y,t)=\frac{q_0}{k}\left[2\sqrt{\frac{\alpha\, t}{\pi}}\exp\left(-\frac{y^2}{4\,\alpha\, t}\right)-y\,\mathrm{erfc}\left(\frac{y}{\sqrt{4\,\alpha\, t}}\right)\right] \tag{12.12}$$

From equation (12.12) we can conclude that surface temperature increases as $\sqrt{t}$.

$$\theta(0,t)=\frac{2\,q_0\,\sqrt{\alpha}}{k\sqrt{\pi}}\sqrt{t} \tag{12.13}$$

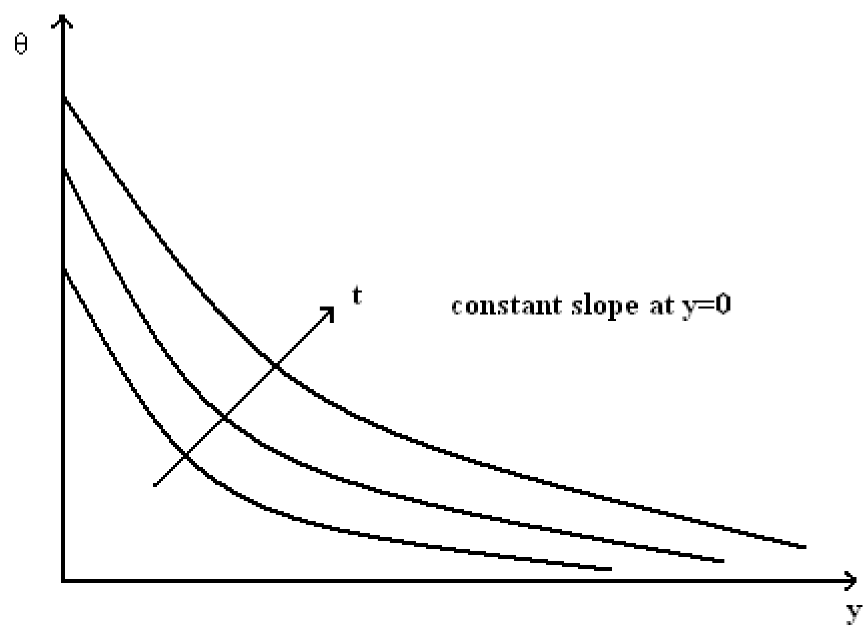

Fig. 12.1 Temperature varies with length at different time period