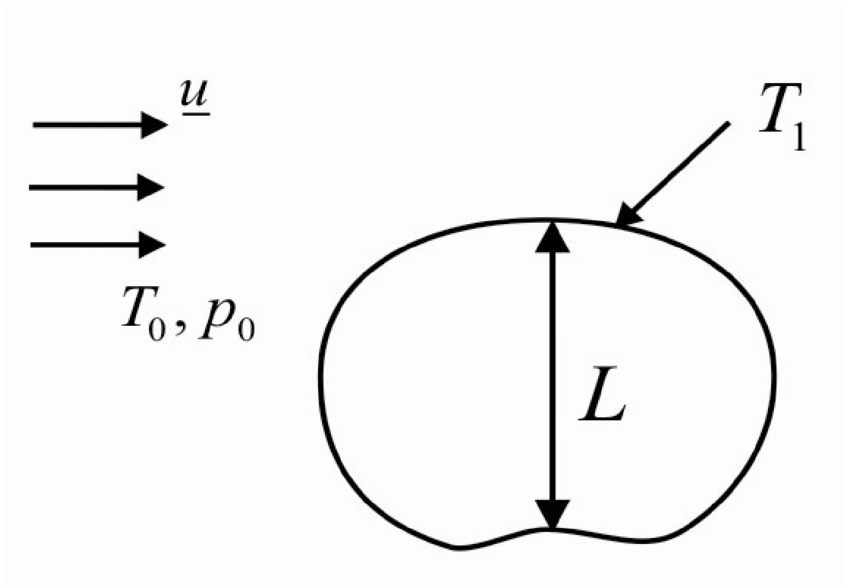
Module 4:

Multicomponent Transport

Lecture 37:

Dimensional Analysis



Equation of Continuity: $$\nabla.\underline{u} = \begin{cases} 0 & \text{(incompressible)} \\ -\dfrac{1}{\rho}\dfrac{D\rho}{DT} & \text{(compressible)} \end{cases}$$

Navler–Stokes Equation: (Constant physical properties, Newtonian fluid, Boussinesq approximation)

$$\rho\frac{D\underline{u}}{Dt} = \mu\nabla^2\underline{u} + \begin{cases} -\nabla p + \rho\underline{g} & \text{(forced convection)} \\ -\rho\beta\underline{g}\left(T - T_0\right) & \text{(free convection)} \end{cases}$$

Energy Balance Equation:

$$\rho C_p \frac{DT}{Dt} = k\nabla^2 T + \mu\Phi_u + \beta T\frac{Dp}{Dt}$$

Note: $\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$:coefficient of volume expansion

For ideal gases $\Rightarrow \beta = \frac{1}{T}$

Define dimensionless variables (for forced convection)

$$\underline{x}^* = \underline{x}/L,\ \underline{U}^* = \underline{u}/u,\ T^* = \frac{T - T_0}{T_1 - T_0},\ t^* = \frac{tu}{L}$$

$$P^* = \frac{p - p_0}{\rho u^2}$$

Dimensionless Conservation Equations (Incompressible fluid, Forced Convection case):

$$\nabla^* . \underline{U}^* = 0$$

$$\frac{D\underline{U}^*}{Dt^*} = -\nabla^* P^* + \frac{1}{Fr}\frac{\underline{g}}{g} + \frac{1}{\mathrm{Re}}\nabla^{*2}\underline{U}^*$$

$$\frac{DT^*}{Dt^*}=\frac{1}{\mathrm{Re}\,\mathrm{Pr}}\nabla^{*2}T^*+\frac{Br}{\mathrm{Re}\,\mathrm{Pr}}\Phi_U^*+\frac{Br}{\mathrm{Pr}}\left[\beta T_0+\beta\Delta T\ T^*\right]\frac{DP^*}{Dt^*}$$

Dimensionless Numbers:

$$\mathrm{Re}=\text{Reynolds}\,\#=\frac{uL}{\nu}=\frac{\rho u^2/L}{\mu u/L^2}=\frac{\text{Inertial forces}}{\text{Viscous forces}}$$

$$Fr=Froude\,\#=\frac{u^2}{L_g}=\frac{\rho u^2/L}{\rho g}=\frac{\text{Inertial forces}}{\text{Gravity forces}}$$

$$\mathrm{Re}\,\mathrm{Pr}=\frac{\rho C_P uL}{k}=Peclet(heat)=Pe_h=\frac{\ell C_P u\left(T_0-T_1\right)/L}{k\left(T_0-T_1\right)/L^2}=\frac{\text{heat transport by convection}}{\text{heat transport by conduction}}$$

$$Br=Brinkman\,\#=\frac{\mu u^2}{k\left(T_0-T_1\right)}=\frac{\mu\left(u/L\right)^2}{k\left(T_0-T_1\right)/L^2}=\frac{\text{heat production by viscous dissipation}}{\text{heat transport by conduction}}$$

(important for turbo-machinery)

$$Ec=Eckert\,\#=\frac{Br}{\mathrm{Pr}}=\frac{\rho u^2}{\rho C_P\left(T_0-T_1\right)}=\frac{\text{Kinetic energy}}{\text{Internal energy}}$$

(important for hypersonic flows)

For free convection there is no readily available reference velocity. So it is more convenient to define

$$\underline{U}^{**} = \frac{\underline{u}Le}{\mu} = \text{dimensionless velocity}$$

$$t^{**} = \frac{t\mu}{\rho L^2} = \text{dimensionless time}$$

The conservation equation then become

$$\nabla^*_{.}\underline{U}^{**} = 0$$

$$\frac{D\underline{U}^{**}}{Dt^{**}} = \nabla^{*2}\underline{U}^{**} - T^* Gr\frac{\underline{g}}{g}$$

$$\frac{DT^*}{Dt^{**}} = \frac{1}{\text{Pr}}\nabla^{*2}T^* + \text{neglible terms}$$

Thus, for free convection only Gr and Pr appear

The $\quad \text{Grashof \# Gr} = \dfrac{g\rho^2\beta\left(T_0 - T_1\right)L^3}{\mu^2}$

$$\rightarrow \frac{Gr}{\text{Re}^2} = \frac{\beta g\left(T_0 - T_1\right)L}{u^2} = \frac{\rho\beta g\left(T_0 - T_1\right)}{\rho u^2 / L} = \frac{Buoyancy\ forces}{Inertial\ forces}$$

The above number decides if natural convection is important as compared to forced convection.

Temperature Distribution: The temperature distribution will be a function of numerous parameters (say, for forced convection)

$$T^* = F\left(t^*, \underline{X}^*; \mathrm{Re}, \mathrm{Pr}, Fr, \underbrace{Br, \beta T_0, \beta \Delta T}_{\text{Usually not important}}\right)$$

Not important if there is no free surface flow

So, for a wide range of probleness, $T^* = F\left(t^*, \underline{X}^*, \mathrm{Re}, \mathrm{Pr}\right)$

$$\text{Nusselt number } = \frac{hL}{k} = \frac{Q}{A\Delta T}\frac{L}{k},$$

Where h is the average heat transfer coefficient

$$Q = \int_s -k\frac{\partial T}{\partial \underline{x}}.\underline{n}\,dA = -kL\left(T_1 - T_0\right)\int_{s^*}\frac{\partial T^*}{\partial \underline{x}^*}.\underline{u}\,dA^*$$

$$= f\left(geometry, t^*, \mathrm{Re}, \mathrm{Pr}\right)$$

Analogies between Heat and Mass Transfer: Recall correlations for boundary layer heat transfer and also for heat transfer inside tubes.

$$\mathrm{Pr} \quad \rightarrow \quad Sc = \frac{\nu}{D}$$

$$\Rightarrow For\, mass\, transfer \qquad Pe_h \quad \rightarrow \quad Pe_m = \frac{uL}{D}$$

$$Nu \quad \rightarrow \quad Sh = \frac{k_m L}{D}$$

$$Gr \rightarrow Gr_{AB} = \frac{\rho^2 \Im g\left(X_{A1} - X_{A0}\right)L^3}{\mu^2},$$

$$\text{where } \Im = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial x_A}\right)_{p,T}.$$