Module 1:

The equation of "continuity"

Lecture 3:

Conservation of Energy for a Pure Substance

First law of thermodynamics

The first law of thermodynamics is often called **The Law of Conservation of Energy**. According to this law, energy can be transferred from one system to another in many forms, but cannot be ***created or destroyed***. Thus the total amount of energy in the universe is constant.

Einstein suggested that the energy (E) and matter (m) are interchangeable and proved the relationship between the energy and matter with the following equation

$$E=mc^2,$$

where c is a constant (the speed of light). This relationship also suggests that the quantity of energy and matter in the universe is constant. The First Law of Thermodynamics is given by,

$$\left(\hat{U}_2-\hat{U}_1\right)+\left(K_2-K_1\right)=q-W_{sh}-P_2V_2+P_1V_1,$$

where,

h=Enthalpy= $\hat{U}+PV$

q=Heat Flow

K=Kinetic Energy= $\frac{\underline{U}^2}{2}$

P=Pressure

W = Work= $W_{sh}+P_2V_2-P_1V_1$

$\hat{U}$ = **Internal Energy**

$\underline{U}$ = **Velocity**

V=Volume

W_{sh} = **Shaft Work**

Different forms of energy

The most familiar types of energy are based on both kinetic and potential energy. However some other forms of energies are given below.

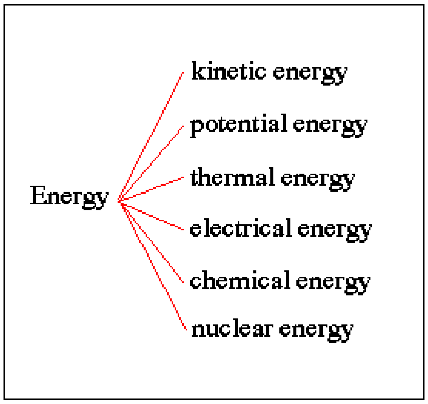

Fig. 2.1 Different forms of Energy

Internal energy

It includes the energy in all chemical bonds are associated with it. It consists of five types of energy. (i) Sensible Heat Energy (ii) Chemical Energy (iii) Nuclear Energy (iv) Latent Heat Energy (iv) Energy Interactions.

Latent Heat Energy is associated with phase of the system. Energy interactions are not stored in the system but it can be renowned in the boundary of the system. It includes the kinetic and potential energy but it is usually changes with the temperature.

Energy Equation

The energy equation mainly consists of kinetic energy and Potential energy in the form of work term against the body force, gravity. However we must include internal energy of the fluid and heat flows that result due to temperature gradients within the system. We consider a control volume fixed in space, with dimensions Δx, Δy, Δz as shown in figure below. The energy flows *in* and *out* of the control volume by conduction, convection and by diffusion due to a temperature gradient. To make the system less complicated heat energy flowing through the system due to mass diffusion can be neglected. Terms involving work being done by the fluid in the control volume against pressure forces, body forces and viscous forces are incorporated in the balance equation.

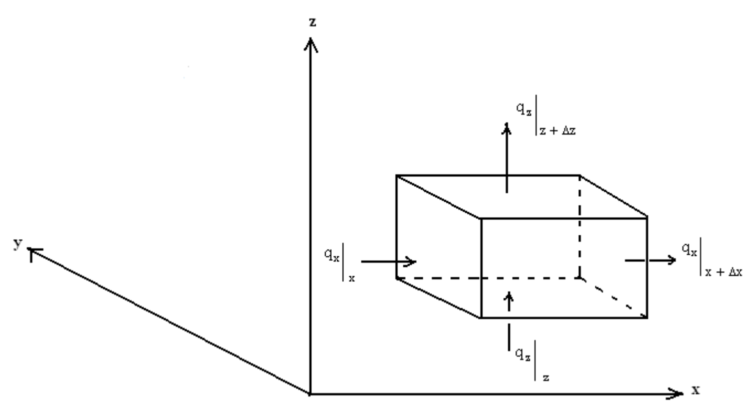

Fig.2.2 An arbitrary Control volume element

Note:

1) Potential energy is accounted for in work done by system on surroundings (see below).

2) "Other" forms of energy, e.g. electromagnetic, nuclear etc. are neglected.

So finally

Total Energy = Internal Energy + Kinetic Energy

$$\left\{\begin{array}{c}\text{Rate of accumulation of}\\ \text{Internal Energy (IE) and}\\ \text{Kinetic Energy (KE)}\\ \text{(A)}\end{array}\right\} = \left\{\begin{array}{c}\text{Rate of Internal Energy}\\ \text{(IE) and Kinetic Energy}\\ \text{(KE) in by Conduction}\\ \text{(B)}\end{array}\right\} - \left\{\begin{array}{c}\text{Rate of Internal Energy}\\ \text{(IE) and Kinetic Energy}\\ \text{(KE) out by Convection}\\ \text{(C)}\end{array}\right\} + \left\{\begin{array}{c}\text{Net rate of heat}\\ \text{addition by}\\ \text{Conduction}\\ \text{(D)}\end{array}\right\} - \left\{\begin{array}{c}\text{Net rate of work}\\ \text{done by system}\\ \text{on Surroundings}\\ \text{(E)}\end{array}\right\}$$

(2 .1)

$$(A): \Delta x \Delta y \Delta z \frac{\partial}{\partial t}\left(\rho \hat{U} + \frac{1}{2}\rho U^2\right)$$

where

ρ : Mass density

$\hat{U}$: Internal energy per unit mass

U : Magnitude of local fluid velocity

***Rate of Convection of Energy In and Out* (B)-(C):**

$$\Delta y \Delta z \left\{ U_x\left(\rho \hat{U} \frac{1}{2}\rho U^2\right)_x - U_x\left(\rho \hat{U} + \frac{1}{2}\rho U^2\right)_{x+\Delta x}\right\} + \Delta x \Delta z \left\{ U_y\left(\rho \hat{U} + \frac{1}{2}\rho U^2\right)_y - U_y\left(\rho \hat{U} + \frac{1}{2}\rho U^2\right)_{y+\Delta y}\right\} +$$
$$+ \Delta x \Delta y \left\{ U_z\left(\rho \hat{U} + \frac{1}{2}\rho U^2\right)_z - U_z\left(\rho \hat{U} + \frac{1}{2}\rho U^2\right)_{z+\Delta z}\right\}$$

***Rate of Conduction of energy In and Out* (D):**

$$\Delta y \Delta z \left\{ (q_x)_x - (q_x)_{x+\Delta x}\right\} + \Delta x \Delta z \left\{ (q_y)_y - (q_y)_{y+\Delta y}\right\} + \Delta x \Delta y \left\{ (q_z)_z - (q_z)_{z+\Delta z}\right\}$$

(E): This term consist of two parts: work against volumes forces (e.g. gravity) and work against surface forces (pressure and viscous forces)

(Work) = (Force) × (Distance in direction of force)

(Rate of doing work) = (Force) × (Velocity in the direction of force)

Thus,

$$
\begin{aligned}
(\mathrm{E}) &= \begin{pmatrix}\text{Rate of doing work}\\ \text{against gravity}\end{pmatrix} + \begin{pmatrix}\text{Rate of doing work}\\ \text{against static pressure}\end{pmatrix} + \begin{pmatrix}\text{Rate of doing work}\\ \text{against viscous forces}\end{pmatrix}\\
&= \left[-\rho\Delta x\Delta y\Delta z\left(U_x g_x + U_y g_y + U_z g_z\right)\right] +\\
&+ \left[\Delta y\Delta z\left\{(PU_x)_{x+\Delta x} - (PU_x)_x\right\} + \Delta x\Delta z\left\{(PU_y)_{y+\Delta y} - (PU_y)\right\} + \Delta x\Delta y\left\{(PU_z)_{z+\Delta z} - (PU_z)_z\right\}\right]\\
&+ \Delta y\Delta z\left\{\left(\tau_{xx}U_x + \tau_{xy}U_y + \tau_{xz}U_z\right)_{x+\Delta x} - \left(\tau_{xx}U_x + \tau_{xy}U_y + \tau_{xz}U_z\right)_x\right\} +\\
&+ \Delta x\Delta z\left\{\left(\tau_{yx}U_x + \tau_{yy}U_y + \tau_{yz}U_z\right)_{y+\Delta y} - \left(\tau_{yx}U_x + \tau_{yy}U_y + \tau_{yz}U_z\right)_y\right\} +\\
&+ \Delta x\Delta y\left\{\left(\tau_{zx}U_x + \tau_{zy}U_z + \tau_{zz}U_z\right)_{z+\Delta z} - \left(\tau_{zx}U_x + \tau_{zy}U_y + \tau_{zz}U_z\right)_z\right\}
\end{aligned}
$$

where

P is the pressure and τ is the shear stress acting on the fluid.

Substituting the above expressions for (A), (B), (C), (D) and (E) in equation 2.1, dividing by ΔxΔyΔz, and taking the limit as Δx, Δy, Δz→0, we obtain:

$$\frac{\partial}{\partial t}\left(\rho\hat{U}+\frac{1}{2}\rho U^2\right)=-\left(\frac{\partial}{\partial x}U_x\left(\rho\hat{U}+\frac{1}{2}\rho U^2\right)+\frac{\partial}{\partial y}U_y\left(\rho\hat{U}+\frac{1}{2}\rho U^2\right)+\frac{\partial}{\partial z}U_z\left(\rho\hat{U}+\frac{1}{2}\rho U^2\right)\right)-$$
$$-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)+\rho\left(U_x g_x+U_y g_y+U_z g_z\right)-\left(\frac{\partial}{\partial x}PU_x+\frac{\partial}{\partial y}PU_y+\frac{\partial}{\partial z}PU_z\right)-$$
$$-\left(\frac{\partial}{\partial x}\left(\tau_{xx}U_x+\tau_{xy}U_y+\tau_{xz}U_z\right)+\frac{\partial}{\partial y}\left(\tau_{yx}U_x+\tau_{yy}U_y+\tau_{yz}U_z\right)+\frac{\partial}{\partial z}\left(\tau_{zx}U_x+\tau_{zy}U_y+\tau_{zz}U_z\right)\right)$$

This equation may be written as:

$$\frac{\partial}{\partial t}\rho\left(\hat{U}+\frac{1}{2}U^2\right)=-\left[\nabla\bullet\rho\underline{U}\left(\hat{U}+\frac{1}{2}U^2\right)\right]-\left(\nabla\bullet\underline{q}\right)+\rho\left(\underline{U}\bullet\underline{g}\right)-\left(\nabla\bullet P\underline{U}\right)-\left(\nabla\bullet\left[\underline{\underline{\tau}}\bullet\underline{U}\right]\right) \tag{2.2}$$

(A) (B) (C) (D) (E) (F)

(A): Rate of gain of internal and kinetic energy per unit volume at the point in question

(B)- Rate of input of internal and kinetic energy per unit volume to the point by Convection

(C)-Rate of input of energy per unit volume to the point by Conduction

(D)- Rate of work done on the fluid at a point per unit volume by gravitational forces

(E)-Rate of work done on the fluid at a point per unit volume by the thermodynamic pressure

(F)-Rate of work done on the fluid at a point per unit volume by viscous forces

Let's put $\left(\hat{U}+\frac{1}{2}U^2\right)=\Omega$ in equation (2.2) then we get

$$\frac{\partial}{\partial t}(\rho\Omega)+\nabla\bullet(\rho\Omega\underline{U})=-\left(\nabla\bullet\underline{q}\right)+\rho\left(\underline{U}\bullet\underline{g}\right)-\left(\nabla\bullet P\underline{U}\right)-\left(\nabla\bullet\left[\underline{\underline{\tau}}\bullet\underline{U}\right]\right) \tag{2.3}$$

After expanding the terms on left hand side we get

$$\rho\left[\frac{\partial\Omega}{\partial t}+\underline{U}\bullet\nabla\Omega\right]+\Omega\left[\frac{\partial\rho}{\partial t}+\nabla\bullet(\rho\underline{U})\right]=-(\nabla\bullet\underline{q})+\rho(\underline{U}\bullet g)-(\nabla\bullet P\underline{U})-(\nabla\bullet[\underline{\underline{\tau}}\bullet\underline{U}]) \quad (2.4)$$

By definition, the factor in the bracket in first term on the left hand side is the Substantial derivative, and the factor in the bracket in the second term on the left hand side is zero as a consequence of the continuity equation. Therefore we can write

$$\rho\frac{D}{Dt}\left(\hat{U}+\frac{1}{2}U^2\right)=-(\nabla\bullet\underline{q})+\rho(\underline{U}\bullet\underline{g})-(\nabla\bullet P\underline{U})-(\nabla\bullet[\underline{\underline{\tau}}\bullet\underline{U}]) \quad (2.5)$$

For ***Macroscopic Mechanical Energy Balance*** (equation (1.10), described in previous section), we have

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\underline{U}^2\right)= +\rho(\underline{U}\bullet\underline{g})-P(\nabla\bullet\underline{U})-(\nabla\bullet[\underline{\underline{\tau}}\bullet\underline{U}])+(\underline{\underline{\tau}}:\nabla\underline{U})-(\nabla\bullet P\underline{U}) \quad (2.6)$$

Subtracting equation (2.6) from (2.5) yields an equation in terms of $\hat{U}$ only

$$\underbrace{\rho\frac{D\hat{U}}{Dt}}_{\begin{pmatrix}\text{Rate of gain of}\\ \text{internal energy}\\ \text{per unit volume}\end{pmatrix}} = \underbrace{-(\nabla\bullet\underline{q})}_{\begin{pmatrix}\text{Rate of internal energy}\\ \text{input by Conduction}\\ \text{per unit volume}\end{pmatrix}} - \underbrace{P(\nabla\bullet\underline{U})}_{\begin{pmatrix}\text{Rate of internal energy}\\ \text{increase per unit volume}\\ \text{by compression (reversible)}\end{pmatrix}} - \underbrace{(\underline{\underline{\tau}}:\nabla\underline{U})}_{\begin{pmatrix}\text{Rate of internal energy}\\ \text{increase per unit volume}\\ \text{by viscous dissipation}\\ \text{(irreversible)}\end{pmatrix}} \quad (2.7)$$

The last two terms represent inter conversion between mechanical and thermal energy.

If the external force $\underline{g}$ can be expressed as the gradient of a scalar function, e.g. $\underline{g}=-\nabla\hat{\Phi}$ then

$$\rho\left(\underline{U}\bullet\underline{g}\right)=-\rho\left(\underline{U}\bullet\nabla\hat{\Phi}\right)=-\rho\frac{D\hat{\Phi}}{Dt}+\rho\frac{\partial\hat{\Phi}}{\partial t} \tag{2.8}$$

The term, $\rho\dfrac{\partial\hat{\Phi}}{\partial t}=0$ for the independent $\hat{\Phi}$

<u>Equation (2.5)</u> then becomes,

$$\rho\frac{D}{Dt}\underbrace{\left(\hat{U}+\hat{\Phi}+\frac{1}{2}U^2\right)}_{Total\ energy\ E}=-\nabla\bullet\underline{q}-\left[\nabla\bullet P\underline{U}\right]-\left(\nabla\bullet\left[\underline{\underline{\tau}}\bullet\underline{U}\right]\right) \tag{2.9}$$

But changes in Internal energy due to temperature and volume is given as,

$$d\hat{U}=\left(\frac{\partial\hat{U}}{\partial\hat{V}}\right)_T d\hat{V}+\left(\frac{\partial\hat{U}}{\partial T}\right)_{\hat{V}} dT=\left[-P+T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right]d\hat{V}+\hat{C}_v\,dT \tag{2.10}$$

where $\hat{C}_v$ **is the heat capacity per unit mass of constant volume**

$$\begin{aligned}\rho\frac{D\hat{U}}{Dt}&=\left[-P+T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right]\rho\frac{D\hat{V}}{Dt}+\rho\hat{C}_v\frac{DT}{Dt}\\&=\left[-P+T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}}\right]\left(\nabla\bullet\underline{U}\right)+\rho\hat{C}_v\frac{DT}{Dt}\end{aligned} \tag{2.11}$$

Now

$$\rho\frac{D\hat{V}}{Dt}=\rho\left[\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right]=-\frac{1}{\rho}\frac{D\rho}{Dt} \tag{2.12}$$

Using continuity equation (derived in previous section)

$$\nabla \bullet (\rho \underline{U}) + \frac{\partial \rho}{\partial t} = 0$$

hence

$$\rho(\nabla \bullet \underline{U}) + \frac{D\rho}{Dt} = 0 \qquad (2.13)$$

therefore

$$-\frac{1}{\rho}\frac{D\rho}{Dt} = (\nabla \bullet \underline{U}) \qquad (2.14)$$

Substituting equation (2.14) into equation (2.12) and the result in equation (2.11) and finally into equation (2.7) we get

$$\boxed{\rho \hat{C}_v \frac{DT}{Dt} = -(\nabla \bullet \underline{q}) - T\left(\frac{\partial P}{\partial T}\right)_{\hat{V}} (\nabla \bullet \underline{U}) - (\underline{\underline{\tau}} : \nabla \underline{U})} \qquad (2.15)$$

Equation (2.15) is the equation of energy in terms of the fluid temperature *T*. In the presence of a volume source of heat generation this equation is written as:

$$\boxed{\rho \hat{C}_v \frac{DT}{Dt} = -(\nabla \bullet \underline{q}) - \mathrm{T}\left(\frac{\partial P}{\partial T}\right)_{\hat{V}} (\nabla \bullet \underline{U}) - (\underline{\underline{\tau}} : \nabla \underline{U}) + \dot{Q}} \qquad (2.16)$$

$\dot{Q}$ may arise because of, for example, **Heat generation** by the passage of electric current