Module 3:

"Convective" heat and mass transfer

Lecture 26:

Heat Transfer to Boundary Layers

(Continued)

NPTEL , IIT Kharagpur, Prof. Saikat Chakraborty, Department of Chemical Engineering

The solution of the temperature field for a situation of wall temperature described by eqn. (3) of lecture 25, can be generalised to a wall temperature which varies arbitrarily along x. Such a temperature variation can be expressed as

$$T_w - T_\infty = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{i=0}^{\infty} a_i x^i \qquad (1)$$

Solutions for the individual terms a_0, a_1x etc. are superimposed to obtain the solution for B.C. given by eqn. (1).

Approximate Solutions

Use integrated B.L. energy equation. θ is approximated by a "reasonable" function of η. Integration gives δ_T vs. x. Integrated B.L. energy equation is not limited to "similar" temperature profiles.

However, this approach can be applied to flat plate with an unheated length followed by a heated section.

Also it can be applied to systems with arbitrary wall temperature distribution (by superposition Duhamel's principle)

Numerical solution (Finite element, finite difference etc.) can tackle complex geometries, complex boundary conditions, and 3-D problems.

Integrated B.L. Energy equation

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0 \tag{2}$$

$$v_x\frac{\partial\left(T-T_\infty\right)}{\partial x}+v_y\frac{\partial\left(T-T_\infty\right)}{\partial y}=\alpha\left(\frac{\partial^2\left(T-T_\infty\right)}{\partial y^2}\right) \tag{3}$$

Multiplying eqn. (2) by $(T-T_\infty)$ and adding to eqn. (3), we get

$$\frac{\partial\left(v_x\left(T-T_\infty\right)\right)}{\partial x}+\frac{\partial\left(v_y\left(T-T_\infty\right)\right)}{\partial y}=\alpha\left(\frac{\partial^2\left(T-T_\infty\right)}{\partial y^2}\right) \tag{4}$$

Integrating the above equation from 0 to δ_T

$$\frac{\partial}{\partial x}\left\{\int_0^{\delta_T} v_x\left(T-T_\infty\right)dy\right\}-\underbrace{v_x\left(T-T_\infty\right)\Big|_{\delta_T,T_\infty}}_{=0}+\underbrace{v_x\left(T-T_\infty\right)\Big|_{0,T_w}}_{=0}+\underbrace{v_y\left(T-T_\infty\right)\Big|_0^{\delta_T}}_{=0}$$

$$=\alpha\left(\underbrace{\left.\frac{\partial\left(T-T_\infty\right)}{\partial y}\right|_{\delta_T}}_{=0}-\left.\frac{\partial\left(T-T_\infty\right)}{\partial y}\right|_{0}\right)$$

(5)

$$\frac{\partial}{\partial x}\left\{\int_0^{\delta_T} v_x\left(T_\infty-T\right)dy\right\}=-\alpha\left(\left.\frac{\partial T}{\partial y}\right|_{y=0}\right) \tag{6}$$

This equation holds for both laminar and turbulent flow.

Asymptotic solutions for general plane and axis-symmetric flows

For large Pr (heavy oils) or large Sc (liquids)

Here $\delta_T << \delta \Rightarrow$ v_x can be approximated by linear function

$$\mathbf{v_x = \cancelto{=0}{v_x\big|_{y=0}} + \frac{\partial v_x}{\partial y}\bigg|_{y=0}(y-0) + \_\_\_}\ \textbf{Taylor series expansion}$$

$$v_x = B(x)\,y$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad \text{(Plane 2-D flow)}$$

$$v_y = -\frac{1}{2}\frac{dB}{dx}y^2$$

Then eqn. (2) in lecture 24 becomes

$$B(x)\,y\frac{\partial \theta}{\partial x} - \frac{1}{2}\frac{dB}{dx}y^2\frac{\partial \theta}{\partial y} = D\frac{\partial^2 \theta}{\partial y^2} \qquad (7)$$

$$\theta = 0 \quad \text{at} \quad y = 0$$
$$\theta = 1 \quad \text{at} \quad y \to \infty$$

Here $\theta = \dfrac{c - c_w}{c_\infty - c_w}$

Use combined variables $\eta = \dfrac{y}{g(x)}$

Plugging into eqn. (7)

$$\frac{d^2\theta}{d\eta^2} - \eta^2 \frac{d\theta}{d\eta}\underbrace{\left[-B\frac{g^2}{D}\frac{dg}{dx} - \frac{1}{2D}\frac{dB}{dx}g^3\right]}_{\text{Must be a constant. Set = -3}} = 0 \tag{8}$$

Then

$$\left[Bg^2\frac{dg}{dx} - \frac{g^3}{2}\frac{dB}{dx}\right] = 3D \quad \Rightarrow \quad \frac{d}{dx}\left(g^3 B^{3/2}\right) = 9B^{1/2}D$$

$$g = \frac{1}{\sqrt{B}}\left[9D\int_0^x \sqrt{B}dx\right]^{1/3}$$

and $\eta = \dfrac{y\sqrt{B}}{\left[9D\int_0^x \sqrt{B}dx\right]^{1/3}}$ (9)

Equation (9) is known as a Light hill transformation

Eqn. (8) becomes

$$\frac{d^2\theta}{d\eta^2} - 3\eta^2\frac{d\theta}{d\eta} = 0$$

Setting $P = \frac{d\theta}{d\eta} \quad \Rightarrow \quad P = P_0 e^{-\eta^3}$ and

$$\theta = \frac{1}{\Gamma\left(\frac{4}{3}\right)} \int_0^\eta e^{-x^3} dx \qquad \left(\Gamma\left(\frac{4}{3}\right) = \int_0^\infty e^{-x^3} dx = 0.89298 \right)$$

$$N_y\Big|_{y=0} = -D\frac{dc}{dy}\Big|_{y=0} = -\frac{D(c_\infty - c_w)}{\Gamma\left(\frac{4}{3}\right)} \frac{\sqrt{B}}{\left[9D\int_0^x \sqrt{B}dx\right]^{1/3}} \qquad (10)$$

For corresponding heat transfer problem

$$q_y\Big|_{y=0} = -\frac{k(T_\infty - T_w)}{\Gamma\left(\frac{4}{3}\right)} \frac{\sqrt{B}}{\left[9\alpha\int_0^x \sqrt{B}dx\right]^{1/3}} \qquad (11)$$

Gamma function

Definition $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ (for $\alpha > 0$)

Properties

$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$

$\Gamma(1) = 1$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma(k+1) = K$ for k = integer (0, 1,2,....)

Stirling formula

$$\Gamma(\alpha+1) \cong \sqrt{2\pi\alpha}\left(\frac{\alpha}{e}\right)^{\alpha} \qquad (\text{for large } \alpha)$$

with e = base of natural logarithm

Incomplete gamma function: $P(\alpha,x)=\int_0^x e^{-t}t^{\alpha-1}dt \quad (\alpha>0)$

$$Q(\alpha,x)=\int_x^\infty e^{-t}t^{\alpha-1}dt \quad (\alpha>0);\;\; \Gamma(a)=P(\alpha,x)+Q(\alpha,x)$$

(shown in Fig. 1 below)

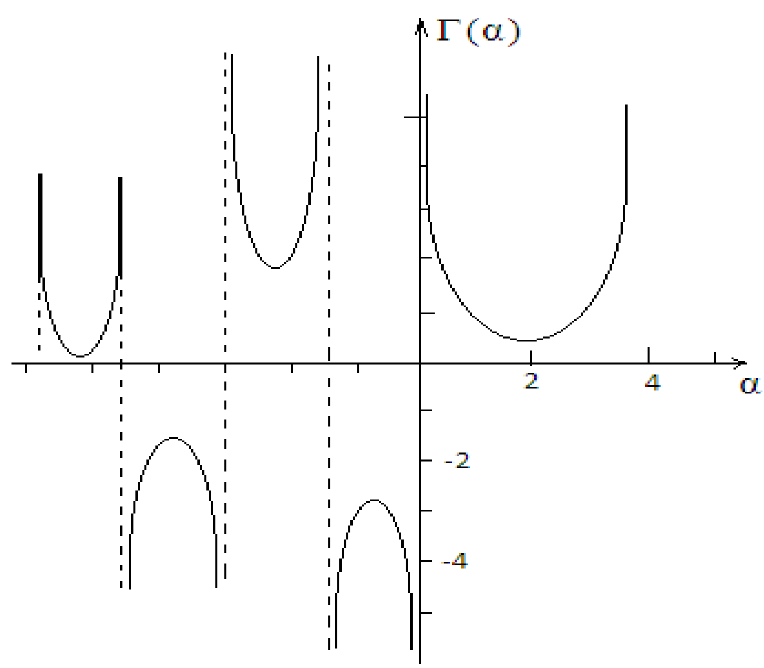

Fig. 1 Gamma functions