The Lecture Contains

1. Topics in finite elasticity: Hyperelasticity of rubber, elastomers, and biological tissues – with examples, M. F Beatty, App. Mech. Rev. Vol. 40(12), pp. 1699-1734 (1987).

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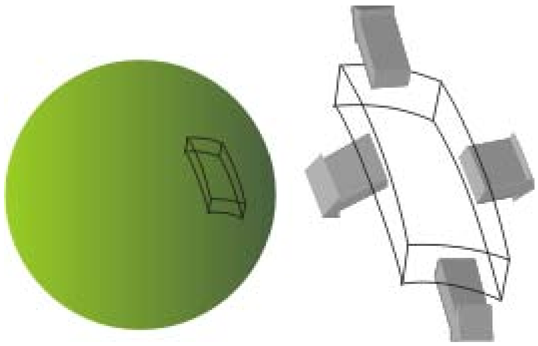
Inflation of a balloon

We model the balloon as an isotropic spherical membrane with uniform radius r_0 and thickness $(t_0 << r_0)$ and that the spherical shape is preserved due to inflation. Uniform isotropic stretch of the membrane is described by the extension ratios $\lambda_1 = \lambda_2 = \lambda = r/r_0$; and the stretch in the thickness coordinate is $\lambda_3 = t/t_0$.



Then due to incompressibility, $\lambda_3 = 1/\lambda^2$. Then the strain energy stored in the balloon is

$$\Pi = \left(4\pi r_0^2 t_0\right)\frac{E}{6}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) = \left(4\pi r_0^2 t_0\right)\frac{E}{6}\left(2\lambda^2 + \frac{1}{\lambda^4}\right) \tag{25.1}$$

Now let's say, the balloon is inflated slightly, i.e. from radius r to $r + dr$ so that, the work done against the outside atmospheric pressure is

$$p(r)4\pi r^2 dr$$

Then the net work done is

$$\int_{r_0}^{r} p(r)4\pi r^2 dr = 4\pi r_0^3 \int_{1}^{\lambda} \lambda^2 p(\lambda) d\lambda \tag{25.2}$$

This energy, which remains stored as the strain energy in the material of the balloon can be written as,

$$d\Pi = 4\pi r_0^2 t_0 \frac{4E}{6}\left(\lambda - \frac{1}{\lambda^5}\right) d\lambda = 4\pi r_0^2 t_0 \frac{4E\lambda}{6}\left(1 - \frac{1}{\lambda^6}\right) d\lambda \tag{25.3}$$

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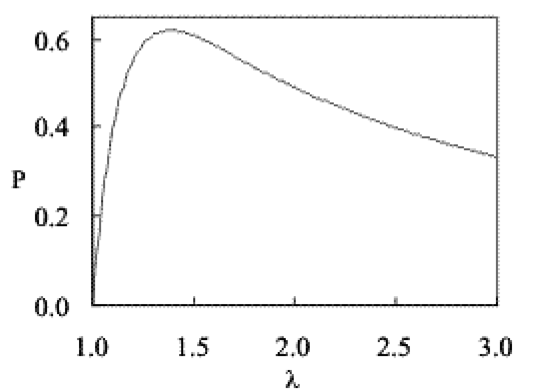
Balancing these energies, we have,

$$p(\lambda)4\pi r_0^3\lambda^2 d\lambda = 4\pi r_0^2 t_0 \frac{4E\lambda}{6}\left(1-\frac{1}{\lambda^6}\right)d\lambda$$
$$\Rightarrow p(\lambda) = \frac{2Et}{3r_0\lambda}\left(1-\frac{1}{\lambda^6}\right) \qquad (25.4)$$

This is the expression for pressure for a neo = Hookean balloon. It is easy to see that pressure required to inflate the balloon is zero at $\lambda = 1$ and also at $\lambda \to \infty$ $p(\lambda) \to 0$. Then there should be a maxima, which can be obtained by putting

$$\frac{dp(\lambda)}{d\lambda} = \frac{2Et}{3r_0}\left(-\frac{1}{\lambda^2}+\frac{7}{\lambda^8}\right) = 0 \qquad (25.5)$$

which yields $\lambda = 7^{1/6} = 1.383$, The maximum pressure is then calculated as $p(\lambda) = 0.4132\frac{Et}{r_0}$. The maximum pressure depends upon the ratio $\frac{t}{r_0}$ and also on the Young's modulus E, but the stretch at which the pressure becomes maximum remains constant and is equal to $\lambda = 1.383$. Pressure decreases to zero asymptotically. However,



in experiments we do not see that. The pressure goes through a maxima, then a minima after which it increases again. The Hookean strain energy function can not capture this complex behavior of the pressure vs extension ratio. In order to explain it we need more involved relation, one such strain energy function is proposed by Mooney and Rivlin and is known as Mooney-Rivlin equation which can be written as,

$$W = c_1\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right)+c_2\left(\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2}-3\right) = c_1(I_1-3)+c_2(I_2-3) = cJ_1+c_2J_2 \qquad (25.6)$$

Here c_1 and c_2 are constants signifying material properties such that $2(c_1+c_2)$ equates the shear modulus μ. Notice that the principal extension ratios are actually obtained as the eigen values of the matrix 8.1, so that the quantities $\lambda_1^2+\lambda_2^2+\lambda_3^2$, $\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2}$ and $\lambda_1^2\lambda_2^2\lambda_3^2$ are invariant. We can expand equation (25.6) as a power series in terms of $e_i = \lambda_i - 1$ which includes the terms for e_i^2 and e_i^3, as a result, the Mooney–Rivlin form of strain energy gives good agreement with experiment at small strains. The Neo-Hookean material is a subset of the Mooney-Rivlin material.

For the balloon inflation problem, this relation simplifies to

$$W = c_1\left(\left(2\lambda^2 + \frac{1}{\lambda^4} - 3\right) + \alpha\left(\frac{2}{\lambda^2} + \lambda^4 - 3\right)\right) \tag{25.7}$$

Then the differential strain energy:

$$d\Pi = \left(4\pi r_0^2 t_0\right) 4c_1\left(\left(\lambda - \frac{1}{\lambda^5}\right) + \alpha\left(-\frac{1}{\lambda^3} + \lambda^3\right)\right) \tag{25.8}$$

And the mechanical energy done to expand the volume:

$$dM = p(r) 4\pi r^2 dr = p(\lambda) 4\pi r_0^3 \lambda^2 d\lambda$$

Balancing these two one obtains,

$$p(\lambda) = \frac{t_0}{r_0}\frac{4c_1}{\lambda}\left(1 - \frac{1}{\lambda^6}\right)\left(1 - \alpha\lambda^2\right) \tag{25.9}$$

Then the condition for obtaining the two extremes translate to, $\frac{dp(\lambda)}{d\lambda} = 0$, or,

$$\alpha\lambda^8 + \lambda^6 + 5\alpha\lambda^2 - 7 = 0 \tag{25.10}$$

Thus neo-Hookean relation alone is not sufficient for the inflation of the balloon, we need a two parameter model like the Mooney-Rivlin equation to account for it.

The Mooney–Rivlin form agrees well with the experiment at small strains. However at modest strain beyond 10% the second term of the equation 12.61 looses importance, therefore there is a need for modifying this equation. This is done by replacing it by a logarithmic term as follows:

$$W = c_1 J_1 + c_2^{'} \ln\left(\frac{J_2 + 3}{3}\right) \tag{25.11}$$

For soft rubbers the constants c_1 and $c_2^{'}$ are found to be 0.25-0.5 MPa.

Many materials like rubber strain hardens at large strain which is not captured by the above equations. The strain hardening behaviour of rubber is accounted for further modification of equation (25.11) in which the maximum possible value of measured quantity J_1 is represented as J_m:

$$W = -c_1 J_m \ln\left(1 - \frac{J_1}{J_m}\right) + c_2^{'} \ln\left(\frac{J_2 + 3}{3}\right) \tag{25.12}$$

Calculations from 25.1 to 25.5 can now be repeated.