Module 4 : Nonlinear elasticity

Lecture 23 : Homogeneuos Strain

The Lecture Contains

"Large Elastic Deformations of Isotropic Materials. I. Fundamental Concepts" by R.S. Rivlin, ***Phil. Trans. Roy. Soc. London,*** 1948, ***240***, 459-490.

"Large Elastic Deformations of Isotropic Materials. III. Some Uniqueness Theorems for Pure, Homogeneous Deformation" by R. S. Rivlin, ***Phil. Trans. Roy. Soc. London,*** 1948, ***240***, 491-508.

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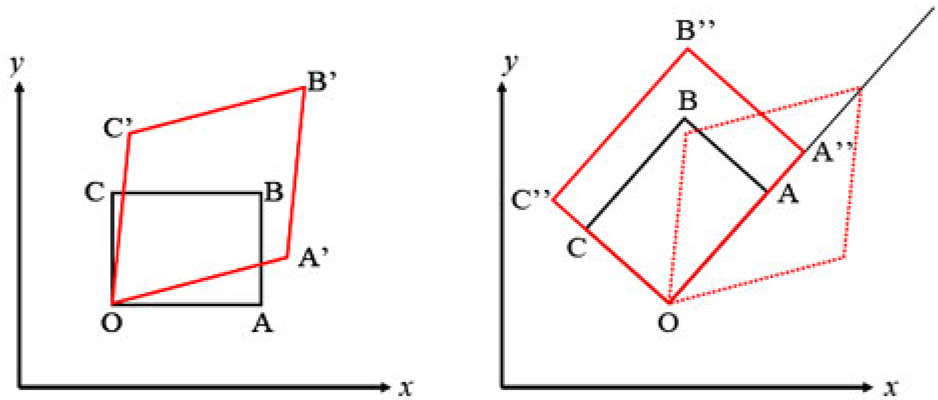
Homogeneous strain:



A homogeneous strain can be considered to consist of a rotation in which the axes of reciprocal strain ellipsoid are brought to the position of strain ellipsoid followed by pure homogeneous strain. Note that no work is done on the element due to rotation. The neo-Hookean relations for pure Homogeneous strains are:

$$1+2\varepsilon_{XX} = \frac{3}{E}\left[t_{XX} - p\right] \quad 1+2\varepsilon_{YY} = \frac{3}{E}\left[t_{YY} - p\right] \quad 1+2\varepsilon_{ZZ} = \frac{3}{E}\left[t_{ZZ} - p\right]$$
$$\lambda_1^2 = \frac{3}{E}\left[t_{XX} - p\right] \qquad \lambda_2^2 = \frac{3}{E}\left[t_{YY} - p\right] \qquad \lambda_3^2 = \frac{3}{E}\left[t_{ZZ} - p\right] \tag{23.1}$$

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Other stress components are

$$t_{ZX} = t_{XY} = t_{YZ} = 0$$
$$e_{ZX} = e_{XY} = e_{YZ} = 0$$

Since, pure homogeneous strains are the same as those for inhomogeneous strains, the stress strains will apply to general deformation of an elastic body. They can be transformed to any fixed, rectangular, Cartesian co-ordinate system (x, y, z). Suppose the direction cosines of the axes (X, Y, Z) of strain ellipsoid referred to the co-ordinate system (x, y, z) are given by

	X	Y	Z
x	l'_1	m'_1	n'_1
y	l'_2	m'_2	n'_2
z	l'_3	m'_3	n'_3

Then the components of stress referred to axes (x, y, z) given in terms of those referred to (X, Y, Z) are,

$$\begin{aligned}
t_{xx} &= l_1'^2 t_{XX} + l_2'^2 t_{YY} + l_3'^2 t_{ZZ} \\
t_{yy} &= m_1'^2 t_{XX} + m_2'^2 t_{YY} + m_3'^2 t_{ZZ} \\
t_{zz} &= n_1'^2 t_{XX} + n_2'^2 t_{YY} + n_3'^2 t_{ZZ} \\
t_{yz} &= m_1' n_1' t_{XX} + m_2' n_2' t_{YY} + m_3' n_3' t_{ZZ} \\
t_{zx} &= n_1' l_1' t_{XX} + n_2' l_2' t_{YY} + n_3' l_3' t_{ZZ} \\
t_{xy} &= l_1' m_1' t_{XX} + l_2' m_2' t_{YY} + l_3' m_3' t_{ZZ}
\end{aligned} \qquad (23.2)$$

Finally we obtain the following relation:

(23.3)

$$t_{xx}=\frac{E}{3}\left[(1+u_x)^2+u_y^2+u_z^2\right]+p$$

$$t_{yy}=\frac{E}{3}\left[u_x^2+(1+u_y)^2+u_z^2\right]+p$$

$$t_{zz}=\frac{E}{3}\left[u_x^2+u_y^2+(1+u_z)^2\right]+p$$

$$t_{yz}=\frac{E}{3}\left[v_xw_x+(1+v_y)w_y+v_z(1+w_z)\right]+p$$

$$t_{zx}=\frac{E}{3}\left[w_x(1+u_x)+w_yu_y+(1+w_z)u_z\right]+p$$

$$t_{xy}=\frac{E}{3}\left[(1+u_x)v_x+u_y(1+v_y)+u_zv_z\right]+p$$

Now consider the following situation in which a cube is extended along the three axes so that we have the following extension ratios:

$$t_{XX} - P = \frac{E}{3}\lambda_1^2, \; t_{YY} - P = \frac{E}{3}\lambda_2^2 \text{ and } t_{ZZ} - P = \frac{E}{3}\lambda_3^2 \tag{23.4}$$

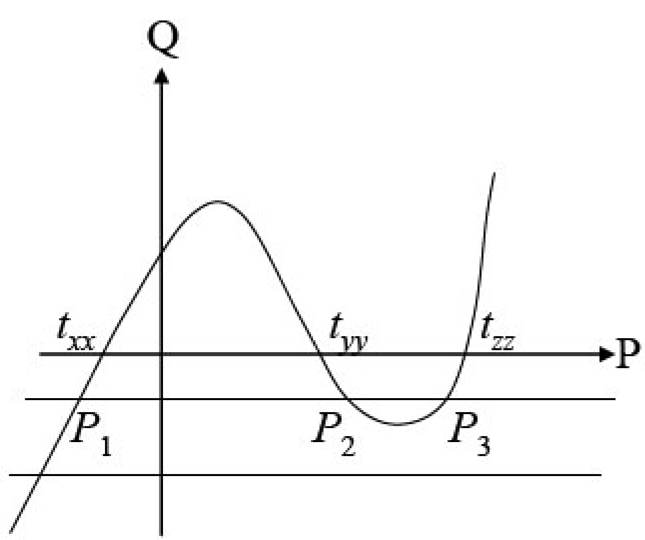
$$(P - t_{XX})(P - t_{YY})(P - t_{ZZ}) = -\frac{E^3}{27}\lambda_1^2\lambda_2^2\lambda_3^2 = -\frac{E^3}{27}$$

What can we say about the uniqueness of the pressure P, or, can it be uniquely defined? Let us say, for simplicity that $t_{ZZ} > t_{YY} > t_{XX}$. The we can think about the following two equations, at their intersections lie the solution of P.

$$Q = (P - t_{XX})(P - t_{YY})(P - t_{ZZ})$$

$$Q = -\frac{E^3}{27}$$

Needless to say that P will intersect $Q = 0$ at t_{XX}, t_{YY} and t_{ZZ}. Furthermore, when, $P \to \infty$, $Q \to \infty$ and when $P \to -\infty$, $Q \to -\infty$. Hence, the P curve look like the one plotted in figure.



Since E is positive, the $Q = -E^3/27$ line will lie beneath the $Q = 0$ line, so that curve intersect $Q = -E^3/27$ either at one location or at three locations P_1, P_2 and P_3. The figure shows that one of the solutions, for example P_1 should be less than t_{XX}, whereas, other two solutions P_2 and P_3 should be between t_{YY} and t_{ZZ}. Then from equation (23.4)

$$\frac{E}{3}\lambda_1^2 = t_{XX} - P, \; \frac{E}{3}\lambda_2^2 = t_{YY} - P \text{ and } \frac{E}{3}\lambda_3^2 = t_{ZZ} - P$$

λ_1 is real and λ_2 and λ_3 are complex numbers. Therefore, it is obvious that there is only solution possible for P.

Stability of Equilibrium

Let us consider a cube of an incompressible, neo-Hookean material with unit edges in the unstrained state. It is strained to dimensions $\lambda_1 \times \lambda_2 \times \lambda_3$ under the action of three forces f_1, f_2 and f_3 so that the strain energy stored in it can be written as,

$$W = \frac{E}{6}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) \tag{23.5}$$

and the work done by these forces is

$$f_1(\lambda_1 - 1) + f_2(\lambda_2 - 1) + f_3(\lambda_3 - 1) \tag{23.6}$$

The net energy can be expressed as the summation of the above two:

$$\Phi = \frac{E}{6}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) - \left(f_1(\lambda_1 - 1) + f_2(\lambda_2 - 1) + f_3(\lambda_3 - 1)\right) \tag{23.7}$$

The equilibrium of forces for all possible values of λ_1, λ_2 and λ_3 can be obtained by finding the minima of the above total energy, such that the following conditions are satisfied:

$$\delta\Phi = 0 \text{ and } \delta^2\Phi > 0 \tag{23.8}$$

Writing $\lambda_1\lambda_2\lambda_3 = 1$ we can obtain the allowable variations $\delta\lambda_1$, $\delta\lambda_2$ and $\delta\lambda_3$ of λ_1, λ_2 and λ_3 respectively, which are to satisfy the following condition:

$$\frac{\delta\lambda_1}{\lambda_1} + \frac{\delta\lambda_2}{\lambda_2} + \frac{\delta\lambda_3}{\lambda_3} = 0 \tag{23.9}$$

Then Φ can be rewritten as a function of λ_1 and λ_2:

$$\Phi = \frac{E}{6}\left(\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} - 3\right) - \left[f_1(\lambda_1 - 1) + f_2(\lambda_2 - 1) + f_3\left(\frac{1}{\lambda_1\lambda_2} - 1\right)\right] \tag{23.10}$$

Then the quantities $\delta\Phi$ and $\delta^2\Phi$ can be written as,

$$\begin{aligned} \delta\Phi &= \frac{\partial\Phi}{\partial\lambda_1}\delta\lambda_1 + \frac{\partial\Phi}{\partial\lambda_2}\delta\lambda_2 \\ \delta^2\Phi &= \frac{\partial^2\Phi}{\partial\lambda_1^2}(\delta\lambda_1)^2 + 2\frac{\partial^2\Phi}{\partial\lambda_1\partial\lambda_2}\delta\lambda_1\delta\lambda_2 + \frac{\partial^2\Phi}{\partial\lambda_2^2}(\delta\lambda_2)^2 \end{aligned} \tag{23.11}$$

Let us write $\left(\sqrt{\frac{\partial^2\Phi}{\partial\lambda_1^2}}\delta\lambda_1+\sqrt{\frac{\partial^2\Phi}{\partial\lambda_2^2}}\delta\lambda_2\right)^2>0$ which implies,

$$\left(\frac{\partial^2\Phi}{\partial\lambda_1^2}\right)(\delta\lambda_1)^2+\left(\frac{\partial^2\Phi}{\partial\lambda_2^2}\right)(\delta\lambda_2)^2+2\sqrt{\frac{\partial^2\Phi}{\partial\lambda_1^2}}\sqrt{\frac{\partial^2\Phi}{\partial\lambda_2^2}}\delta\lambda_1\delta\lambda_2>0 \qquad (23.12)$$

Then adding $2\frac{\partial^2\Phi}{\partial\lambda_1\partial\lambda_2}\delta\lambda_1\delta\lambda_2$ to both sides of equation (23.12), we obtain the following:

$$\begin{aligned}&\left(\frac{\partial^2\Phi}{\partial\lambda_1^2}\right)(\delta\lambda_1)^2+\left(\frac{\partial^2\Phi}{\partial\lambda_2^2}\right)(\delta\lambda_2)^2+2\frac{\partial^2\Phi}{\partial\lambda_1\partial\lambda_2}\delta\lambda_1\delta\lambda_2+\\&\qquad 2\sqrt{\frac{\partial^2\Phi}{\partial\lambda_1^2}}\sqrt{\frac{\partial^2\Phi}{\partial\lambda_2^2}}\delta\lambda_1\delta\lambda_2>2\frac{\partial^2\Phi}{\partial\lambda_1\partial\lambda_2}\delta\lambda_1\delta\lambda_2\end{aligned} \qquad (23.13)$$

Notice that the conditions

$$\sqrt{\left(\frac{\partial^2\Phi}{\partial\lambda_1^2}\right)}\sqrt{\left(\frac{\partial^2\Phi}{\partial\lambda_2^2}\right)}>\frac{\partial^2\Phi}{\partial\lambda_1\partial\lambda_2}\ ,\ \frac{\partial^2\Phi}{\partial\lambda_1^2}>0 \ \text{ and } \ \frac{\partial^2\Phi}{\partial\lambda_2^2}>0 \qquad (23.14)$$

ensure that, $\left(\frac{\partial^2\Phi}{\partial\lambda_1^2}\right)(\delta\lambda_1)^2+\left(\frac{\partial^2\Phi}{\partial\lambda_2^2}\right)(\delta\lambda_2)^2+2\frac{\partial^2\Phi}{\partial\lambda_1\partial\lambda_2}\delta\lambda_1\delta\lambda_2>0$ or $\delta^2\Phi>0$.

We can then obtain the following expressions for $\frac{\partial^2\Phi}{\partial\lambda_1^2}$, $\frac{\partial^2\Phi}{\partial\lambda_2^2}$ and $\frac{\partial^2\Phi}{\partial\lambda_1\partial\lambda_2}$,

$$\begin{aligned}\frac{\partial^2\Phi}{\partial\lambda_1^2}&=\frac{E}{6}\left(2+\frac{6}{\lambda_1^4\lambda_2^2}\right)-f_3\left(\frac{2}{\lambda_2\lambda_1^3}\right)=\frac{E}{3}\left(1+\frac{3\lambda_3^2}{\lambda_1^2}\right)-f_3\left(\frac{2\lambda_3}{\lambda_1^2}\right)\\ 2\frac{\partial^2\Phi}{\partial\lambda_2^2}&=\frac{E}{6}\left(2+\frac{6}{\lambda_2^4\lambda_1^2}\right)-f_3\left(\frac{2}{\lambda_1\lambda_2^3}\right)=\frac{E}{3}\left(1+\frac{3\lambda_3^2}{\lambda_2^2}\right)-f_3\left(\frac{2\lambda_3}{\lambda_2^2}\right)\\ \frac{\partial^2\Phi}{\partial\lambda_1\partial\lambda_2}&=\frac{E}{6}\left(\frac{4}{\lambda_2^3\lambda_1^3}\right)-f_3\left(\frac{1}{\lambda_1^2\lambda_2^2}\right)=\left(\frac{2}{3}E\lambda_3-f_3\right)\lambda_3^2\end{aligned} \qquad (23.15)$$

The conditions as in equation (23.14) are expanded by using the expressions in (23.15), which yield the following results:

$$\left[\frac{E}{3}\left(1+\frac{3\lambda_3^2}{\lambda_1^2}\right)-f_3\left(\frac{2\lambda_3}{\lambda_1^2}\right)\right]\left[\frac{E}{3}\left(1+\frac{3\lambda_3^2}{\lambda_2^2}\right)-f_3\left(\frac{2\lambda_3}{\lambda_2^2}\right)\right]>\left(\frac{2}{3}E\lambda_3-f_3\right)^2\lambda_3^4 \qquad (23.16)$$

$$\frac{E}{3}\left(1+\frac{3\lambda_3^2}{\lambda_1^2}\right)-f_3\left(\frac{2\lambda_3}{\lambda_1^2}\right)>0 \text{ and } \frac{E}{3}\left(1+\frac{3\lambda_3^2}{\lambda_2^2}\right)-f_3\left(\frac{2\lambda_3}{\lambda_2^2}\right)>0 \qquad (23.17)$$

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We can use the relation for f_3 in equations (23.16) and (23.17), $f_3\lambda_3 = \frac{E}{3}\lambda_3^2 + P$, which yields for (23.16)

$$\left[\frac{E}{3}\left(1+\frac{3\lambda_3^2}{\lambda_1^2}\right)-\frac{2}{\lambda_1^2}\left(\frac{E}{3}\lambda_3^2+P\right)\right]\left[\frac{E}{3}\left(1+\frac{3\lambda_3^2}{\lambda_2^2}\right)-\frac{2}{\lambda_2^2}\left(\frac{E}{3}\lambda_3^2+P\right)\right]>\left(\frac{1}{3}E\lambda_3^2-P\right)^2\lambda_3^2$$

$$\Rightarrow\left[\frac{E}{3}\left(\lambda_1^2+\lambda_3^2\right)-2P\right]\left[\frac{E}{3}\left(\lambda_2^2+\lambda_3^2\right)-2P\right]>\left(\frac{1}{3}E\lambda_3^2-P\right)^2\lambda_1^2\lambda_2^2\lambda_3^2$$

$$\Rightarrow\frac{E^2}{9}\left(\frac{1}{\lambda_3^2}+\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}\right)+\left(\frac{E}{3}\lambda_3^2\right)^2+4P^2-2P\frac{E}{3}\left(\lambda_1^2+\lambda_3^2+\lambda_2^2+\lambda_3^2\right)>$$

$$\left(\frac{1}{3}E\lambda_3^2\right)^2+P^2-2P\frac{1}{3}E\lambda_3^2 \qquad (23.18)$$

$$\Rightarrow\frac{E^2}{9}\left(\frac{1}{\lambda_3^2}+\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}\right)+3P^2-2P\frac{E}{3}\left(\lambda_1^2+\lambda_3^2+\lambda_2^2\right)>0$$

$$\Rightarrow\left(\frac{E}{3}\lambda_1^2-P\right)\left(\frac{E}{3}\lambda_2^2-P\right)+\left(\frac{E}{3}\lambda_2^2-P\right)\left(\frac{E}{3}\lambda_3^2-P\right)+\left(\frac{E}{3}\lambda_3^2-P\right)\left(\frac{E}{3}\lambda_1^2-P\right)>0$$

Similarly, for equation (23.17), we obtain,

$$\frac{E}{3}\left(1+\frac{3\lambda_3^2}{\lambda_1^2}\right)-\frac{2}{\lambda_1^2}\left(\frac{E}{3}\lambda_3^2+P\right)>0 \text{ and } \frac{E}{3}\left(1+\frac{3\lambda_3^2}{\lambda_2^2}\right)-\frac{2}{\lambda_2^2}\left(\frac{E}{3}\lambda_3^2+P\right)>0$$

$$\frac{E}{3}\left(\lambda_1^2+\lambda_3^2\right)>2P \text{ and } \frac{E}{3}\left(\lambda_2^2+\lambda_3^2\right)>2P \qquad (23.19)$$

These results then suggest that if $P<\frac{E}{3}\lambda_1^2$, $P<\frac{E}{3}\lambda_2^2$ and $P<\frac{E}{3}\lambda_3^2$ are satisfied then all three necessary and sufficient conditions as in equations (23.18) and (23.19) will be satisfied and the corresponding equilibrium state is stable. If P is negative, the equilibrium state is uniquely determined.