The Lecture Contains

“Mechanics of the Cell” by David Boal, Cambridge University Press, 2002, Cambridge, UK

We have derived earlier the following general relation of strain tensor u_{ij} in terms of the rate of change of displacement vector $\vec{u}$ with position vector $\vec{x}$

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_k \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right] \qquad (37.1)$$

The subscripts i, j and k represent the axes in Cartesian coordinates. In two dimension there are four components of e_{ij} and in three dimension there are nine components. It was shown earlier that e_{ij} is symmetric with respect to the indices i and j. We derived also that for small deformation we can neglect the last term in equation 37.1 yielding

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \qquad (37.2)$$

Using Hooke's law, the stress components was related to the strains as;

$$\sigma_{ij} = \sum_{k,l} C_{ijkl} e_{kl} \qquad (37.3)$$

In which C_{ijkl} are called the material constants, elastic modulii. The corresponding expression for strain energy density is a quadratic function of strain tensor components and hence can be written as,

$$\Delta\Pi = \frac{1}{2}\sum_{i,j,k,l} C_{ijkl} e_{ij} e_{kl} \qquad (37.4)$$

Symmetry considerations greatly reduce the number of independent constants from $3^4 = 81$ for three dimension and $2^4 = 16$ for two dimension to much smaller numbers, minimum number of constants being required for isotropic systems for which all directions are equivalent. For example, since e_{ij} is symmetric for exchange between i and j, C_{ijkl} is symmetric for pair exchange between i and j, k and l, so that,

$$C_{ijkl} = C_{jikl} = C_{jilk} \qquad (37.5)$$

Further since product $e_{ij}e_{kl}$ are symmetric for interchange of indices ij and kl, so that

$$C_{ijkl} = C_{klij} \qquad (37.6)$$

These two symmetry conditions decreases the number of constants to 21 for three dimension and 6 for two dimensions. For 2D, these constants are,

C_{xxxx},

C_{yyyy} ,

$$C_{xxyy} = C_{yyxx} \, , \tag{37.7}$$

$$C_{xyxy} = C_{xyyx} = C_{yxyx} = C_{yxxy} \, ,$$

$$C_{xxxy} = C_{xxyx} = C_{xyxx} = C_{yxxx} \text{ and}$$

$$C_{yyyx} = C_{yyxy} = C_{yxyy} = C_{xyyy}$$

Symmetry in the material can further reduce the number of constants. We will now discuss two dimensional elastic networks with four fold and six fold symmetry to demonstrate this fact.

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Six fold networks in 2D

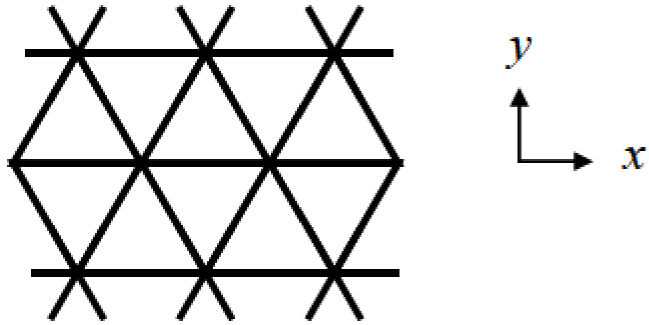

Figure 37.1 shows a two dimensional network which has six fold rotational symmetry through the vertices. We analyze this problem by changing the coordinate system from Cartesian coordinates x and y to complex coordinates $\xi = x + iy$ and $\eta = x - iy$. Then the free energy $\Delta\Pi$ contains terms $C_{\xi\xi\eta\eta} e_{\xi\xi} e_{\eta\eta}$. Now a rotation about the origin of the x, y coordinate by an angle θ changes the coordinates from (x, y) to $(x\cos\theta - y\sin\theta)$ and $(x\sin\theta + y\cos\theta)$, in other word from (ξ, η) to $\xi \rightarrow \xi\exp(i\theta)$ $\eta \rightarrow \eta\exp(-i\theta)$. Since six fold symmetry implies that the modulii remain unchanged because of rotation of the axis through $\theta = \dfrac{\pi}{3}$ i.e. $\xi \rightarrow \xi\exp\left(\dfrac{i\pi}{3}\right)$ and $\eta \rightarrow \eta\exp\left(-\dfrac{i\pi}{3}\right)$. Only non-zero components of C_{ijkl} that remains unchanged by this transformation are those which contains equal number of times ξ and η because $\exp\left(\dfrac{i\pi}{3}\right)\exp\left(-\dfrac{i\pi}{3}\right) = 1$. Only two components of C_{ijkl} satisfy this symmetry $C_{\xi\xi\eta\eta}$ and $C_{\xi\eta\xi\eta}$. The change in free energy density can be then written as

$$\Delta\Pi = \frac{1}{2}\left(4C_{\xi\eta\xi\eta} e_{\xi\eta} e_{\xi\eta} + 2C_{\xi\xi\eta\eta} e_{\xi\xi} e_{\eta\eta}\right) \tag{37.8}$$

Which contains results from four combinations involving $C_{\xi\eta\xi\eta}$ and two combinations involving $C_{\xi\xi\eta\eta}$. We can replace the strain components $e_{\xi\eta}$ by those in the Cartesian components in which we use components of tensor transform as products of the corresponding coordinates. Since,

$$\begin{aligned} \xi^2 &= (x+iy)^2 = x^2 - y^2 + 2ixy \text{ ,} \\ \eta^2 &= (x-iy)^2 = x^2 - y^2 - 2ixy \text{ and} \\ \xi\eta &= (x+iy)(x-iy) = x^2 + y^2 \end{aligned} \tag{37.9}$$

we can write,

$$e_{\xi\xi} = e_{xx} - e_{yy} + 2ie_{xy} \text{ ,}$$

$$e_{\eta\eta} = e_{xx} - e_{yy} - 2ie_{xy}$$
$$e_{\xi\eta} = e_{xx} + e_{yy} \qquad (37.10)$$

From equation 37.8, the expression for energy density can be written as,

$$\Delta\Pi = 2C_{\xi\eta\xi\eta}\left(e_{xx} + e_{yy}\right)^2 + C_{\xi\xi\eta\eta}\left\{\left(e_{xx} - e_{yy}\right)^4 + 4e_{xy}^2\right\} \qquad (37.11)$$

We can relate the C_{ijkl} to modulii to more common forms of modulii, e.g. area compression modulus K_A and shear modulus μ:

$$K_A = 4C_{\xi\eta\xi\eta} \qquad \mu = 2C_{\xi\xi\eta\eta} \qquad (37.12)$$

Hence, equation 37.11 changes to

$$\Delta\Pi = \left(K_A/2\right)\left(u_{xx} + u_{yy}\right)^2 + \mu\left\{\left(u_{xx} - u_{yy}\right)^4 + 4u_{xy}^2\right\} \qquad (37.13)$$

Equation 37.13 is very similar to that for isotropic deformation implying that in two dimension, both isotropic materials and six fold symmetry are represented by two different elastic moduli.

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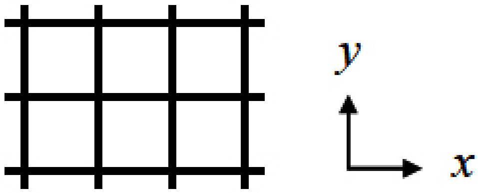
Four fold symmetry

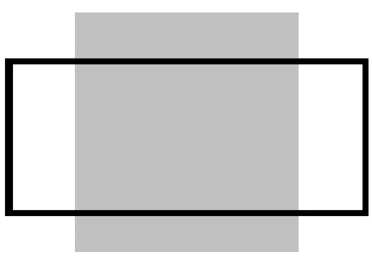


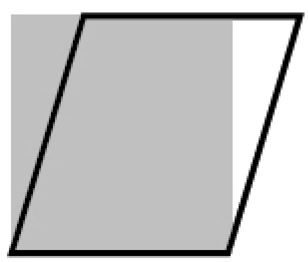
In this case the number of independent is reduced from 6 by the condition that the relation remains independent of the co-ordinate inversions: $x \to -x$ and $y \to -y$. Since, components of a tensor transform as products of the corresponding co-ordinates, the component of C_{ijkl} with odd number of x and y should change sign. Hence these components should vanish because stress-strain should not change sign due to single inversion. Secondly, upon clock-wise rotation by an angle of $\pi/2$ about an axis with four-fold symmetry yields $x \to -y$ and $y \to x$, implying $C_{xxxx} = C_{yyyy}$. Thus these symmetry considerations result in three different modulii: C_{xxxx}, C_{xxyy} and C_{xyxy}. Combining these modulii three different constants are defined:

$$\begin{aligned} K_A &= (C_{xxxx} + C_{xxyy})/2 \\ \mu_p &= (C_{xxxx} - C_{xxyy})/2 \quad \text{(pure shear)} \\ \mu_s &= C_{xyxy} \quad \text{(simple shear)} \end{aligned} \tag{37.14}$$

The expression for energy density can be written as:

$$\Delta\Pi = \frac{K_A}{2}(e_{xx} + e_{yy})^2 + \frac{\mu_p}{2}(e_{xx} - e_{yy})^2 + 2\mu_s e_{xy}^2 \tag{37.15}$$

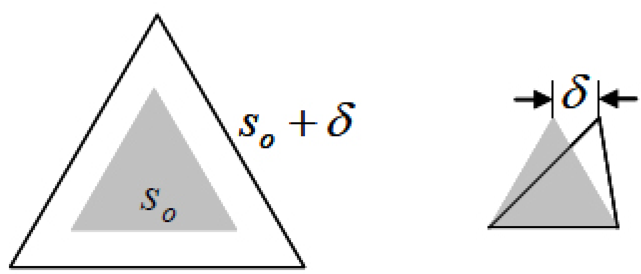




Network of Springs

Let us assume that our system behaves as a harmonic spring and can therefore be represented as a spring network with appropriate symmetry. We can then define a microscopic quantity spring constant k_{sp} for each spring which can finally yield the elastic modulii of the network. Let us consider the systems with six-fold symmetry as presented in figure below. Let the network be stretched slightly from its equilibrium position, so that the initial and final length of each springs are s_0 and s respectively, then the potential energy of each spring can be written as

$$V_{sp} = k_{sp}(s - s_0)^2/2 \qquad (37.16)$$



We can then estimate the potential energy density of the network. Consider the above triangular unit, the number of vertices is 1 while the number of springs is 3 which results in potential energy for each vertex as $\Delta U = 3\Delta V_{sp} = k_{sp}\,\delta^2/2$. The network area per vertex is $A_v = 2\times\sqrt{3}\,s_0^2/4 = \sqrt{3}\,s_0^2/2$, so that the potential energy density per unit area of the network deformed to a small extent from the equilibrium configuration is defined as

$$\Delta\Pi = \Delta U/A_v = \sqrt{3}k_{sp}(\delta/s_0)^2 \qquad (37.17)$$

Notice that the deformations are uniform along *x* and *y* directions respectively, so that we can write the normal components of the strain tensors in terms of s_0 and δ:

$$u_{xx} = u_{yy} = \delta/s_0 \qquad (37.18)$$

The displacement along *y* is independent of position along *x*, so that the shear components can be written as $u_{xy} = 0$. Use of these expressions for the strain tensor in the energy expression of equation 32.13 yield,

$$\Delta\Pi = 2K_A(\delta/s_0)^2 \qquad (37.19)$$

Comparing equation 37.19 and 37.17 we obtain the area compression modulus in terms of the spring constant of individual springs,

$$K_A = \sqrt{3}k_{sp}/2 \qquad (37.20)$$

A similar analysis yields also the expressions for shear modulus. Suppose the network is sheared so that the vertex of a triangle at a particular layer is displaced to the right, say by a distance δ resulting in increase in its left arm by a distance $\delta/2$ and shortening of the length of its right arm by a distance $\delta/2$. Since the bottom arm remains undeformed the total potential energy is written as

$$\Delta U = k_{sp}\,\delta^2/4 \qquad (37.21)$$

The energy density per unit area is then obtained by dividing the above quantity by the area of the network:

$$\Delta\Pi = \Delta U/A_v = k_{sp}\left(\delta/s_0\right)^2/2\sqrt{3} \qquad (37.22)$$

Since we are considering pure shear, the normal components of the strain tensor are zero, i.e. $u_{xx} = u_{yy} = 0$. The shear components can be estimated as $u_{xy} = \delta/2/\sqrt{3}s_0/2 = \delta/\sqrt{3}s_0$. Using these expressions in equation 37.13, we obtain,

$$\Delta\Pi = \left(2\mu/3\right)\left(\delta/s_0\right)^2 \qquad (37.23)$$

which when compared with equation 32.22, yields,

$$\mu = \sqrt{3}k_{sp}/4 \qquad (37.23)$$

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