Module 3 : Equilibrium of rods and plates

Lecture 21 : Elastic effect induced by surface tension of liquid.

The Lecture Contains

The following lecture is adopted from the following journal paper:

1. Elasticity of an interfacial particle raft, D. Vella, P. Aussillous and L. Mahadevan, Europhysics Letters, Vol. 68 (2), pp. 212–218 (2004).

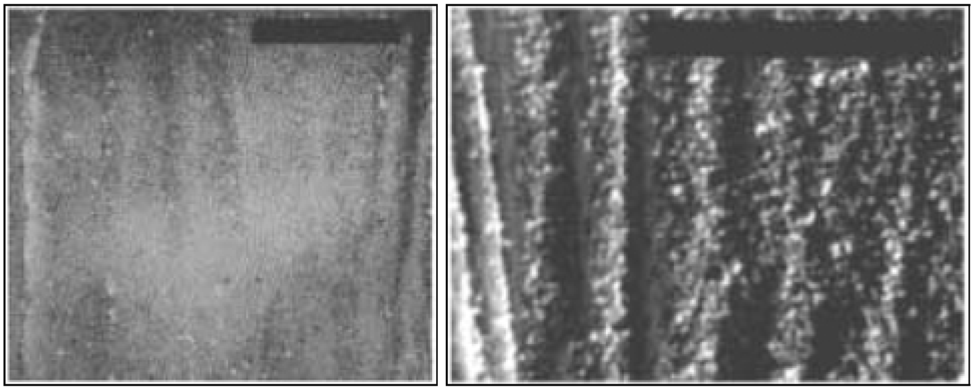
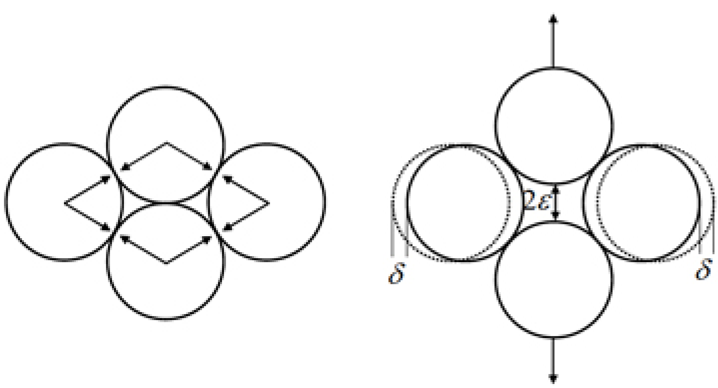


Figure 1. Optical image of a layer of hydrophobic particles spread on water. The layer is compressed which leads to the appearance of the wavy interface.

Many technological and medical applications involve particle-covered liquid interfaces, e.g. liquid drop coated with hydrophobic powder which turns the drop non-wetting with the result that the liquid marble can freely roll on a rigid surfaces or even float on water. Similarly, by encapsulating drugs within a monolayer of colloidal particles it may be possible to administer more medicines by inhalation and thus the efficacy of drug delivery.

Figure 1 shows a monolayer of hydrophobic particles sprinkled densely onto an air-water interface. Such a monolayer can be formed by particles of wide range of sizes: 2.5 μ m – 6mm. The figure shows that the layer buckles when subjected to sufficient static compressive loading, suggesting that it can support an anisotropic stress. Such a stress can only be supported by a material with a non-zero shear modulus, which is the signature of the solid. Once the compressive stress is removed, the monolayer returns rapidly to the undeformed state, ironing out the wrinkles formed by the buckling. The solid like behaviour is observed only in the presence of the liquid, because, if the liquid is evaporated, it turns to powder without any cohesion. Clearly capillary forces are responsible for the formation of the solid monolayer via the aggregation of particles at the interface and act as the restoring mechanism when the monolayer is deformed.



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For a conventional two-dimensional elastic solid, the mean stress can be written as, $\bar{\sigma} = \dfrac{\sigma_1 + \sigma_2}{2}$ and the mean strain as, $\bar{\varepsilon} = \dfrac{\varepsilon_1 + \varepsilon_2}{2}$. The stress can be related to strain as,

$$\bar{\varepsilon} = \frac{1-\nu}{E}\bar{\sigma} \qquad (21.1)$$

Where ν is the Poisson ratio of the solid. We can replace the stress by an isotropic tension averaged over the thickness of the layer, $\tau = \bar{\sigma}d$, which then yields,

$$\bar{\varepsilon} = \frac{1-\nu}{E}\bar{\sigma} \qquad (21.2)$$

We can then consider that because of infinitesimal change in the tension, the strain changes infinitesimally, so that,

$$\frac{1-\nu}{Ed} = \frac{\mathrm{d}\bar{\varepsilon}}{\mathrm{d}\tau} = \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}\tau} = \frac{1}{A_l + A_s}\frac{\mathrm{d}(A_l + A_s)}{\mathrm{d}\tau} \qquad (21.3)$$

Where, A_l and A_s are the area covered by liquid and solid particles respectively and $A = A_l + A_s$ is the total area of the system. Now, the solid particles are rigid at the individual level so that, regardless of the tension applied, the area covered by them is constant, however, the raft as a whole can be soft, because it also consists of the liquid. Equation 21.3 then simplifies to

$$\frac{1-\nu}{Ed} = \frac{1}{A_l + A_s}\frac{dA_l}{d\tau} = \frac{A_l}{A_l + A_s}\left(\frac{1}{A_l}\frac{dA_l}{d\tau}\right) \qquad (21.4)$$

Notice, that the quantity in the bracket is a measure of the surface tension of the liquid, so that $A_l \dfrac{d\tau}{dA_l} \propto \gamma$, Equation 21.4, can then be rearranged to yield an expression for the effective Young's modulus of the monolayer

$$E \propto \frac{1-\nu}{1-\phi}\frac{\gamma}{d} \qquad (21.5)$$

Where $\phi = A_s / A$ is the area fraction occupied by the solid particles at the interface. Notice that the above expression for E has the property that as $\phi \to 1$, $E \to \infty$, which is consistent with the fact that the monolayer consists of particles which are rigid.

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An estimation of ν can be obtained by considering the geometry of the packing. For example, we can assume that the particles are closed pack in hexagonal lattice, so that they interact via an attractive central force and a repulsive steric interaction. Consider a single rhombic cell in the lattice as shown in Figure 21.2. In the undeformed state, its width and height are $2\sqrt{3}R$ and $2R$ respectively where R is the particle radius. Let us say that the two central particles are displaced by a distance 2ε, so that the width of the rhombus decreases to $2\sqrt{3}R\sqrt{1-2\varepsilon/3R} \approx 2\sqrt{3}R\left(1-\varepsilon/3R\right)$. Hence the strain can be defined as $\varepsilon/3R$. At the same time, the height of the rhombus increases to $2\left(\varepsilon+r\right)$, so that the strain developed is ε/R. So that the Poisson ratio defined as the ratio of two strains is deduced as ε/R.

Analysis of the geometry yields also the numerical estimate of the Young's modulus of the layer. For close hexagonal packing of spherical particles, the area fraction of the particles can be deduced as, $\phi = \pi/2\sqrt{3}$, which gives a typical value of the Young's modulus as,

$$E \sim 2\frac{\sqrt{3}-1}{2\sqrt{3}-\pi}\frac{\gamma}{d} \approx 4.54\frac{\gamma}{d} \qquad (21.6)$$

In actual practical situations however the particles are expected to be irregular without any ordered packing which will yield some other value of the pre-factor. Nevertheless, the expression in equation 21.6 shows that the elastic property of the layer depends on the surface tension of the liquid and the diameter of the particles.

The above expressions for the elastic modulus can be verified by another experiment which involves first sprinkling on the air-water interface hydrophobic particles and then subject the two dimensional particle raft to sufficient compression. This results in elastic buckling as shown in figure 21.1, the wavelength of which can be measured by using a camera. Such measurements can be done by spreading hydrophobic particles of different particle sizes. We can then infer from this data, the value of the Young's modulus.

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We use the classical theory of elasticity in order to determine the effective mechanical properties of the two dimensional solid from the measurement of the wavelength. We assume the raft to be a thin, isotropic, homogeneous elastic sheet of thickness equal to the particle diameter *d* . Since, the wavelength is large compared to the particle size, the plane stress conditions apply. We further assume that at the onset of the buckling instability, the deflection of the raft and its vertical displacement $h(x)$measured from its equilibrium position is small. Then, balance of forces yield the following equilibrium equation for plate,

$$B\frac{\partial^4 h}{\partial x^4}+T\frac{\partial^2 h}{\partial x^2}+\rho g h=0 \qquad (21.7$$

in which $B=\dfrac{Ed^3}{12\left(1-\nu^2\right)}$ is the bending stiffness of the plate, T is the compressive force per unit length of the plate, ρ is the specific gravity of the liquid and g is the acceleration due to gravity. The above equation depicts the summation of three different of forces: the bending force of the plate occurring because of its buckling and the weight of the liquid column of height $h(x)$, both these force provide stability to the plate; the third component is the compressive force which drives the instability. Substituting $h(x)=A\sin\left(2\pi x/\lambda\right)$ into equation 21.7 we obtain,

$$BA\left(\frac{2\pi}{\lambda}\right)^4-TA\left(\frac{2\pi}{\lambda}\right)^2+\rho g A=0 \qquad (21.8)$$

Rearranging equation 21.8, we obtain an expression for the compressive load T as a function of the wavelength λ

$$T=B\left(\frac{2\pi}{\lambda}\right)^2+\rho g\left(\frac{\lambda}{2\pi}\right)^2 \qquad (21.9)$$

We then find an expression for λ at which minimum compressive load needs to be exerted on the layer for the onset of the instability, by putting $dT/d\lambda=0$:

$$\lambda=\pi\left(4Ed^3/\left(3\rho g\left(1-\nu^2\right)\right)\right)^{1/4} \qquad (21.10)$$

Rearrangement of above equation yields an expression for the Young's modulus of the layer as

$$E=\frac{3}{4\pi^4}\frac{\rho g\left(1-\nu^2\right)\lambda^4}{d^3} \qquad (21.11)$$

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