The Lecture Contains

"Large Elastic Deformations of Isotropic Materials. I. Fundamental Concepts" by R. S. Rivlin, ***Phil. Trans. Roy. Soc. London,*** 1948, ***240***, 459-490.

Neo-Hookean elasticity:

Let the new position of the point is given in terms of the deformations as $x^{*} = x+u, \quad y^{*} = y+v, \quad z^{*} = z+w$ in the strained state. Consequently, the particle which was on a given curve in the unstarined state, now belongs to a different curve in the strained state. If ds be a differential element in the original curve, then direction cosines of a tangent at any point on it are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$. Direction cosines of a tangent on an elemental arc ds_1 in the strained curve are $\frac{d(x+u)}{ds_1}, \frac{d(y+v)}{ds_1}, \frac{d(z+w)}{ds_1}$. Then,

$$\begin{aligned}
\frac{d(x+u)}{ds_1} &= \frac{ds}{ds_1}\left(\frac{dx}{ds} + \frac{du}{dx}\frac{dx}{ds} + \frac{du}{dy}\frac{dy}{ds} + \frac{du}{dz}\frac{dz}{ds}\right) \\
\frac{d(y+v)}{ds_1} &= \frac{ds}{ds_1}\left(\frac{dy}{ds} + \frac{dv}{dx}\frac{dx}{ds} + \frac{dv}{dy}\frac{dy}{ds} + \frac{dv}{dz}\frac{dz}{ds}\right) \\
\frac{d(x+w)}{ds_1} &= \frac{ds}{ds_1}\left(\frac{dx}{ds} + \frac{dw}{dx}\frac{dx}{ds} + \frac{dw}{dy}\frac{dy}{ds} + \frac{dw}{dz}\frac{dz}{ds}\right)
\end{aligned} \tag{22.1}$$

However, the direction cosines are

$$\begin{aligned}
& l = \frac{dx}{ds}, \qquad m = \frac{dy}{ds}, \qquad n = \frac{dz}{ds} \\
& l_1 = \frac{d(x+u)}{ds_1}, \quad m_1 = \frac{d(y+v)}{ds_1}, \quad n_1 = \frac{d(z+w)}{ds_1}
\end{aligned} \tag{22.2}$$

From the expressions of equation 1,

$$\begin{aligned}
l_1 &= \frac{ds}{ds_1}\left(l\left(1+\frac{du}{dx}\right) + m\frac{du}{dy} + n\frac{du}{dz}\right) \\
m_1 &= \frac{ds}{ds_1}\left(l\frac{dv}{dx} + m\left(1+\frac{dv}{dy}\right) + n\frac{dv}{dz}\right) \\
l_1 &= \frac{ds}{ds_1}\left(l\frac{dw}{dx} + m\frac{dw}{dy} + n\left(1+\frac{dw}{dz}\right)\right)
\end{aligned} \tag{22.3}$$

Noting that

$$l^2 + m^2 + n^2 = 1, \qquad\qquad l_1^2 + m_1^2 + n_1^2 = 1$$

we have the following eqn,

$$\left(\frac{ds_1}{ds}\right)^2 = (1+2\varepsilon_{xx})l^2 + (1+2\varepsilon_{yy})m^2 + (1+2\varepsilon_{zz})n^2 + 2\varepsilon_{yz}mn + 2\varepsilon_{zx}nl + 2\varepsilon_{xy}lm \tag{22.4}$$

where ε_{xx} ... are the following,

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} \\
\varepsilon_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \\
\varepsilon_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \\
\varepsilon_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}
\end{aligned}
\tag{22.5}
$$

We thus obtain the general expressions for the components of strain in terms of the gradients of displacements.

The left hand side of 22.4 is a constant and the r.h.s. signifies an ellipsoid. It has the property that in any direction, the length of its radius is inversely proportional to $\frac{ds_1}{ds}$.

Such an ellipsoid is called the reciprocal strain ellipsoid:

$$(1+2\varepsilon_{xx})x^2+(1+2\varepsilon_{yy})y^2+(1+2\varepsilon_{zz})z^2+2\varepsilon_{yz}yz+2\varepsilon_{zx}zx+2\varepsilon_{xy}xy=1$$

Elements of length, which are parallel to the axes of the reciprocal strain ellipsoid in the un-deformed state becomes parallel to the strain ellipsoid in the deformed state. Such elements having lengths ds in the undeformed state, have lengths $\lambda_1 ds$, $\lambda_2 ds$ and $\lambda_3 ds$ respectively in the deformed state, where, λ_1-1, λ_2-1 λ_3-1 are the principal extensions and λ_1^2, λ_2^2 and λ_3^2 are the roots of the equation:

$$\begin{vmatrix} 1+2\varepsilon_{xx}-\lambda^2 & \varepsilon_{xy} & \varepsilon_{zx} \\ \varepsilon_{xy} & 1+2\varepsilon_{yy}-\lambda^2 & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{yz} & 1+2\varepsilon_{zz}-\lambda^2 \end{vmatrix}=0 \qquad (22.6)$$

An element of volume dV situated at (x,y,z) in the undeformed state has a volume τdV in the deformed state, where,

$$\tau=\begin{vmatrix} 1+u_x & u_y & u_z \\ v_x & 1+v_y & v_z \\ w_x & w_y & 1+w_z \end{vmatrix} \qquad (22.7)$$

For an incompressible material, the volume of an element in the deformed and undeformed states are equal, so that $\tau=1$. Notice that the quantities $\lambda_1^2+\lambda_2^2+\lambda_3^2$, $\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2}$ and $\lambda_1^2\lambda_2^2\lambda_3^2$ are invariant.The stress strain relationship for an incompressible material:

Hooke's law,

$$e_{xx}=\frac{1}{E}\left[(1+\sigma)t_{xx}-\sigma(t_{xx}+t_{yy}+t_{zz})\right],\text{etc.}$$

$$e_{yz}=\frac{2}{E}(1+\sigma)t_{yz},\text{etc.}$$

$$e_{xx}+e_{yy}+e_{zz}=0,\ \sigma=1/2 \qquad (22.8)$$

$$e_{xx}=\frac{3}{2E}\left[t_{xx}-p\right],\ e_{yz}=\frac{3}{E}t_{yz},\quad p=\frac{1}{3}(t_{xx}+t_{yy}+t_{zz})$$

neo-Hookean law:

$$1+2\varepsilon_{XX}=\frac{3}{E}\left[t_{XX}-p\right],\ 1+2\varepsilon_{YY}=\frac{3}{E}\left[t_{YY}-p\right]\ 1+2\varepsilon_{ZZ}=\frac{3}{E}\left[t_{ZZ}-p\right]$$

$$1+2\varepsilon_{XX}=\frac{3}{E}\left[t_{XX}-p\right] \qquad (22.9)$$

Previous Next

For a pure homogeneous deformation, suppose that $\lambda_1, \lambda_2, \lambda_3$ are the lengths in the deformed state of linear elements parallel to the axes X, Y and Z respectively which have unit lengths in the undeformed state. Then by making l, m, n successively $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, we see that

$$\lambda_1^2 = 1+2\varepsilon_{XX},\ \lambda_2^2 = 1+2\varepsilon_{YY} \text{ and } \lambda_3^2 = 1+2\varepsilon_{ZZ}$$

$$t_{XX} = \frac{E}{3}\lambda_1^2 + P,\ t_{YY} = \frac{E}{3}\lambda_2^2 + P \text{ and } t_{ZZ} = \frac{E}{3}\lambda_3^2 + P \qquad (22.10)$$

Simple extension:

Consider the simple extension along x axis of a, incompressible neo-Hookean material for whom the stress-strain relations take the form:

$$t_{XX} = \frac{E}{3}\lambda_1^2 + P,\ t_{YY} = \frac{E}{3}\lambda_2^2 + P = t_{ZZ} = \frac{E}{3}\lambda_3^2 + P = 0,$$

$$t_{YZ} = t_{ZX} = t_{XY} = 0 \qquad (22.11)$$

Since $\lambda_1\lambda_2\lambda_3 = 1$, we have $\lambda_2 = \lambda_3 = 1/\sqrt{\lambda_1}$ and $P = -\frac{E}{3}\frac{1}{\lambda_1}$ which gives,

$$t_{XX} = \frac{E}{3}\left(\lambda_1^2 - \frac{1}{\lambda_1}\right) \qquad (22.22)$$

Simple shear:

$$v = w = 0 \text{ i.e. } u_x = u_y = v_x = v_y = v_z = w_x = w_y = w_z = 0$$

$$t_{xx} = \frac{E}{3}\left(1+u_z^2\right) + P,\ t_{yy} = t_{zz} = \frac{E}{3} + P,\ \ t_{yz} = 0,\ t_{zx} = \frac{E}{3}u_z \text{ and } t_{xy} = 0 \qquad (22..13)$$

Thus shearing stresses alone can not maintain a state of simple shear in the material. If the stresses t_{yy} and t_{zz} are zero, then the stresses $t_{xx} = \frac{1}{3}Eu_z^2$; and if $t_{xx} = 0$, then $t_{yy} = t_{zz} = -\frac{E}{3}u_z^2$. Thus two possible stress systems which can maintain a simple shear:

(i) A shearing stress $\frac{1}{3}Eu_z$ in the $xz-$plane together with a normal stress $\frac{1}{3}Eu_z^2$ parallel to the $x-$axis.

(ii) A shearing stress $\frac{1}{3}Eu_z$ in the $xz-$plane together with two normal stresses of magnitude $\frac{1}{3}Eu_z^2$ parallel to the y and z axes.

The stored energy function

Consider a cubic element having unit edges is strained in such a way that in deformed state it is a cuboid having edges parallel to the axes of strain ellipsoid with lengths λ_1, λ_2 and λ_3 respectively. Then due to incompressibility,

$$\lambda_1\lambda_2\lambda_3 = 1 \tag{22.14}$$

Then the work done in straining the material quasi-statically is W which can be worked out by considering the stresses which act on the deformed element $\lambda_1 \times \lambda_2 \times \lambda_3$.

$$t_{XX} = \frac{E}{3}\lambda_1^2 + P, t_{YY} = \frac{E}{3}\lambda_2^2 + P \text{ and } t_{ZZ} = \frac{E}{3}\lambda_3^2 + P$$

$$\text{and } t_{YZ} = t_{ZX} = t_{XY} = 0 \tag{22.15}$$

The element is subjected to three mutually perpendicular forces f_1, f_2 and f_3 which are given by $f_1 = t_{XX}\lambda_2\lambda_3$, $f_2 = t_{YY}\lambda_1\lambda_3$ and $f_3 = t_{zz}\lambda_1\lambda_2$,. Here $\lambda_2\lambda_3$, $\lambda_1\lambda_3$ and $\lambda_1\lambda_2$ are the areas on which the stresses t_{XX}, t_{YY} and t_{ZZ} act. Using the expression for t_{XX}, t_{YY}, t_{ZZ} and the incompressibility relation we have,

$$f_1 = \left(\frac{E}{3}\lambda_1^2 + P\right)\frac{1}{\lambda_1} = \frac{1}{3}E\lambda_1 + \frac{P}{\lambda_1},\ f_2 = \frac{1}{3}E\lambda_2 + \frac{P}{\lambda_2} \text{ and } f_3 = \frac{1}{3}E\lambda_3 + \frac{P}{\lambda_3} \tag{22.16}$$

Work done in straining an element of volume $\lambda_1 \times \lambda_2 \times \lambda_3$ to $(\lambda_1 + \delta\lambda_1)\times(\lambda_2 + \delta\lambda_2)\times(\lambda_3 + \delta\lambda_3)$ is

$$f_1\delta\lambda_1 + f_2\delta\lambda_2 + f_3\delta\lambda_3 = \frac{1}{3}E(\lambda_1\delta\lambda_1 + \lambda_2\delta\lambda_2 + \lambda_3\delta\lambda_3) + P\left(\frac{\delta\lambda_1}{\lambda_1} + \frac{\delta\lambda_2}{\lambda_2} + \frac{\delta\lambda_3}{\lambda_3}\right) \tag{22.17}$$

Hence the work done in straining the material quasi-statically from dimension $1\times1\times1$ to $\lambda_1 \times \lambda_2 \times \lambda_3$ is

$$W = \frac{1}{3}E\left(\int_1^{\lambda_1}(\lambda_1\delta\lambda_1) + \int_1^{\lambda_2}(\lambda_2\delta\lambda_2) + \int_1^{\lambda_3}(\lambda_3\delta\lambda_3)\right) + P\left(\int_1^{\lambda_1}\frac{\delta\lambda_1}{\lambda_1} + \int_1^{\lambda_2}\frac{\delta\lambda_2}{\lambda_2} + \int_1^{\lambda_3}\frac{\delta\lambda_3}{\lambda_3}\right)$$
$$= \frac{1}{6}E(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) \tag{22.18}$$

For an ideal rubber like material the elastic modulus E is given as $3NkT$ where N is the number of segments per unit volume, k is the Boltzmann's constant and T is the absolute temperature.