The Lecture Contains

1. "Mechanics of Incremental Deformations" by M.A. Biot.

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We now establish the equations of equilibrium in incremental stress field.

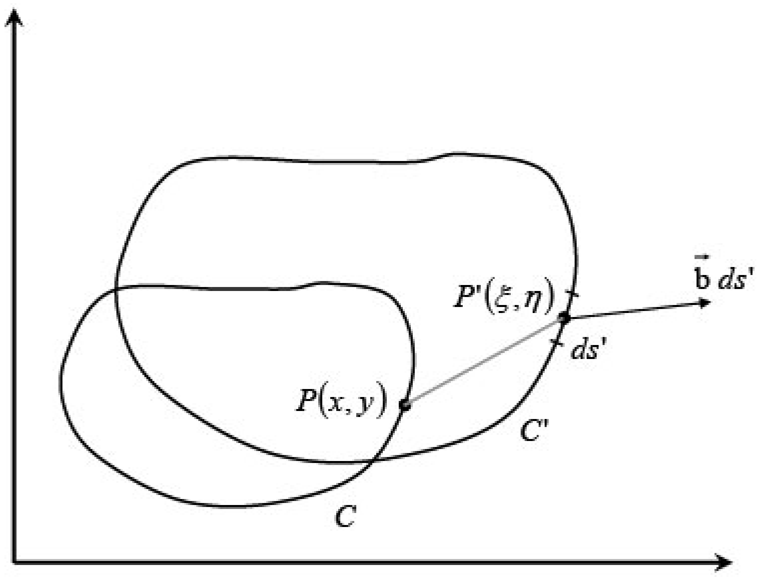

Figure 27.1

In the x, y plane we consider a body outlined by a contour *C*. The initial stress on the body are S_{11}, S_{22}, S_{12}. If X and Y are the components of body force per unit of mass and if ρ is the density of the medium before deformation, the initial stress components will satisfy the equilibrium relations,

$$\begin{aligned}\frac{\partial S_{11}}{\partial x}+\frac{\partial S_{12}}{\partial y}+\rho X(x,y)=0\\ \frac{\partial S_{12}}{\partial x}+\frac{\partial S_{22}}{\partial y}+\rho Y(x,y)=0\end{aligned} \qquad (27.1)$$

A point *P* of the material originally of coordinates x, y moves to a point *P*' of coordinates ξ, η after deformation. We denote by b_x, b_y the x and y components of the force $\vec{b}$ acting at a point P' of the boundary per unit area after deformation. This force is acting on the solid inside the contour C'.

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Module 4 : Nonlinear elasticity

Lecture 27 : Theory of incremental stresses (Contd...)

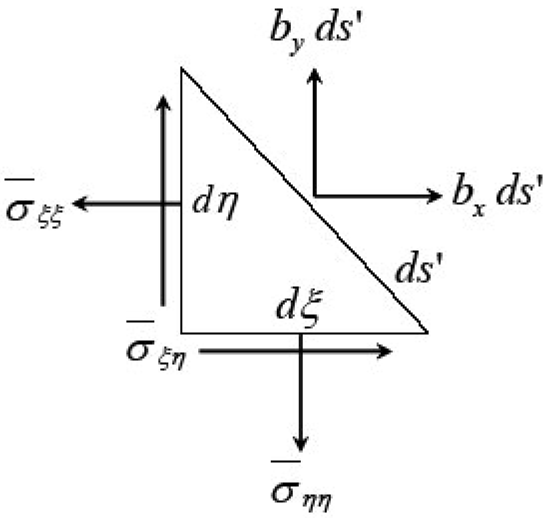

Figure 27.2

Then we can think of equilibrium of a triangular element adjacent to the boundary as shown the figure,

$$\begin{aligned} b_x ds' &= \overline{\sigma}_{\xi\xi} d\eta - \overline{\sigma}_{\xi\eta} d\xi \\ b_y ds' &= \overline{\sigma}_{\xi\eta} d\eta - \overline{\sigma}_{\eta\eta} d\xi \end{aligned} \tag{27.2}$$

Considering that other external force acting on the body are the body forces, we can write the equilibrium relation

$$\begin{aligned} \int_{C'} b_x ds' + \iint_{S'} X(\xi,\eta)\rho' dS' &= 0 \\ \int_{C'} b_y ds' + \iint_{S'} Y(\xi,\eta)\rho' dS' &= 0 \end{aligned} \tag{27.3}$$

Notice that here "s'" denotes the length of the contour and "S'" denotes the area of the surface area. Substituting the expression in equation 13.9 in 13.10 and the using following expressions for $d\xi$ and $d\eta$

$$\begin{aligned} d\xi &= \left(1+\frac{\partial u}{\partial x}\right)dx + \frac{\partial u}{\partial y}dy \\ d\eta &= \frac{\partial v}{\partial x}dx + \left(1+\frac{\partial v}{\partial y}\right)dy \end{aligned} \tag{27.4}$$

We obtain the following expression for the first integral in equation 27.3:

(27.5)

$$b_x ds' = \overline{\sigma_{\xi\xi}}\left(\frac{\partial v}{\partial x}dx + \left(1+\frac{\partial v}{\partial y}\right)dy\right) - \overline{\sigma_{\xi\eta}}\left(\left(1+\frac{\partial u}{\partial x}\right)dx + \frac{\partial u}{\partial y}dy\right)$$

$$= \left(\overline{\sigma_{\xi\xi}}\frac{\partial v}{\partial x} - \overline{\sigma_{\xi\eta}}\left(1+\frac{\partial u}{\partial x}\right)\right)dx + \left(\overline{\sigma_{\xi\xi}}\left(1+\frac{\partial v}{\partial y}\right) - \overline{\sigma_{\xi\eta}}\frac{\partial u}{\partial x}\right)dy$$

$$b_y ds' = \left(\overline{\sigma_{\xi\eta}}\frac{\partial v}{\partial x} - \overline{\sigma_{\eta\eta}}\left(1+\frac{\partial u}{\partial x}\right)\right)dx + \left(\overline{\sigma_{\xi\eta}}\left(1+\frac{\partial v}{\partial y}\right) - \overline{\sigma_{\eta\eta}}\frac{\partial u}{\partial x}\right)dy$$

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Following the law of conservation of mass, we can write $\rho' dS' = \rho dS = \rho dx dy$. We then convert the line integral to surface integral using Green's theorem to obtain the following expression:

$$\begin{aligned}
&\iint_S \left\{ \frac{\partial}{\partial x}\left(\overline{\sigma_{\xi\xi}}\left(1+\frac{\partial v}{\partial y}\right) - \overline{\sigma_{\xi\eta}}\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\overline{\sigma_{\xi\xi}}\frac{\partial v}{\partial x} - \overline{\sigma_{\xi\eta}}\left(1+\frac{\partial u}{\partial x}\right)\right) dx + X(\xi,\eta)\rho \right\} dx dy = 0 \\
&\iint_S \left\{ \frac{\partial}{\partial x}\left(\overline{\sigma_{\xi\eta}}\left(1+\frac{\partial v}{\partial y}\right) - \overline{\sigma_{\eta\eta}}\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\overline{\sigma_{\xi\eta}}\frac{\partial v}{\partial x} - \overline{\sigma_{\eta\eta}}\left(1+\frac{\partial u}{\partial x}\right)\right) dx + Y(\xi,\eta)\rho \right\} dx dy = 0
\end{aligned} \qquad (27.6)$$

Since the integrand in the above equation should be zero, we obtain the following stress equilibrium equation

$$\begin{aligned}
&\frac{\partial \overline{\sigma_{\xi\xi}}}{\partial x} + \frac{\partial \overline{\sigma_{\xi\eta}}}{\partial y} + \frac{\partial}{\partial x}\left(\overline{\sigma_{\xi\xi}}\frac{\partial v}{\partial y} - \overline{\sigma_{\xi\eta}}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\overline{\sigma_{\xi\xi}}\frac{\partial v}{\partial x} - \overline{\sigma_{\xi\eta}}\frac{\partial u}{\partial x}\right) + X(\xi,\eta)\rho = 0 \\
&\frac{\partial \overline{\sigma_{\xi\eta}}}{\partial x} + \frac{\partial \overline{\sigma_{\eta\eta}}}{\partial y} + \frac{\partial}{\partial x}\left(\overline{\sigma_{\xi\eta}}\frac{\partial v}{\partial y} - \overline{\sigma_{\eta\eta}}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\overline{\sigma_{\xi\eta}}\frac{\partial v}{\partial x} - \overline{\sigma_{\eta\eta}}\frac{\partial u}{\partial x}\right) + Y(\xi,\eta)\rho = 0
\end{aligned} \qquad (27.7)$$

When we substitute the expressions for $\overline{\sigma}_{\xi\xi}$, $\overline{\sigma}_{\eta\eta}$ and $\overline{\sigma}_{\eta\xi}$ from equation 26.7 in 27.7, we obtain the expressions which are in terms of the initial stresses and the incremental stresses:

$$\begin{aligned}
&\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - 2\frac{\partial}{\partial x}(S_{12}\omega) + \frac{\partial}{\partial y}\left[(S_{11}-S_{22})\omega\right] + \frac{\partial}{\partial x}\left(S_{11}\frac{\partial v}{\partial y} - S_{12}\frac{\partial u}{\partial y}\right) - \\
&\qquad \frac{\partial}{\partial y}\left(S_{11}\frac{\partial v}{\partial x} - S_{12}\frac{\partial u}{\partial x}\right) + \rho\left[X(\xi,\eta) - X(x,y)\right] = 0 \\
&\frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} + 2\frac{\partial}{\partial x}(S_{12}\omega) + \frac{\partial}{\partial y}\left[(S_{11}-S_{22})\omega\right] + \frac{\partial}{\partial x}\left(S_{12}\frac{\partial v}{\partial y} - S_{22}\frac{\partial u}{\partial y}\right) - \\
&\qquad \frac{\partial}{\partial y}\left(S_{12}\frac{\partial v}{\partial x} - S_{22}\frac{\partial u}{\partial x}\right) + \rho\left[Y(\xi,\eta) - Y(x,y)\right] = 0
\end{aligned} \qquad (27.8)$$

Writing the incremental body forces as,

$$\begin{aligned}
\Delta X &= X(\xi,\eta) - X(x,y) \\
\Delta Y &= Y(\xi,\eta) - Y(x,y)
\end{aligned} \qquad (27.9)$$

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