The Lecture Contains

1 “Mechanics of Incremental Deformations” by M. A. Biot

Previous Next

Boundary conditions

$$\begin{aligned}&\left(S_{12}+s_{12}-S_{22}\varpi-S_{11}e_{xy}+S_{12}e_{xx}\right)\cos(n,y)+\\&\qquad\left(S_{11}+s_{11}-S_{12}\varpi+S_{11}e_{yy}-S_{12}e_{xy}\right)\cos(n,x)=f_x\\&\left(S_{12}+s_{12}+S_{11}\varpi+S_{12}e_{yy}-S_{22}e_{xy}\right)\cos(n,x)+\\&\qquad\left(S_{22}+s_{22}+S_{12}\varpi-S_{12}e_{xy}+S_{22}e_{xx}\right)\cos(n,y)=f_y\end{aligned} \quad 28.1\ (a)$$

$$\begin{aligned}&\left(s_{12}-S_{22}\varpi-S_{11}e_{xy}+S_{12}e_{xx}\right)\cos(n,y)+\\&\qquad\left(s_{11}-S_{12}\varpi+S_{11}e_{yy}-S_{12}e_{xy}\right)\cos(n,x)=\Delta f_x\\&\left(s_{12}+S_{11}\varpi+S_{12}e_{yy}-S_{22}e_{xy}\right)\cos(n,x)+\\&\qquad\left(s_{22}+S_{12}\varpi-S_{12}e_{xy}+S_{22}e_{xx}\right)\cos(n,y)=\Delta f_y\end{aligned} \quad 28.1(b)$$

$$\begin{aligned}&e_{xx}=\frac{\partial u}{\partial x}\quad e_{yz}=e_{zy}=\frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\quad \varpi_x=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\\&e_{yy}=\frac{\partial v}{\partial y}\quad e_{zx}=e_{xz}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\quad \varpi_y=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\\&e_{zz}=\frac{\partial w}{\partial z}\quad e_{xy}=e_{yx}=\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\quad \varpi_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\end{aligned} \quad (28.2)$$

Previous Next

Incompressible Elastic media:

Consider a unit cube with sides oriented along the x, y, z axes is subjected to normal stresses S_1, S_2 and S_3 on its three faces, so that the cube turns into a parallelepiped with edges having lengths λ_1, λ_2 and λ_3. These initial extension ratios represent the initial finite strain. These extension ratios must satisfy the following relation of incompressibility,

$$\lambda_1 \lambda_2 \lambda_3 = 1 \tag{28.3}$$

In other word there are two independent strain variables for extensions along the co-ordinate axes. We then consider the strain energy as a function of a set of these two extension ratios:

$$W = W(\lambda_1, \lambda_2) \tag{28.4}$$

The normal forces that act on the faces of the solid are f_1, f_2 and f_3. Then the following law of conservation of energy should hold,

$$dW = f_1 d\lambda_1 + f_2 d\lambda_2 + f_3 d\lambda_3 \tag{28.5}$$

Here, however, the three differentials are not independent but must satisfy the following relation of volume conservation,

$$\frac{d\lambda_1}{\lambda_1} + \frac{d\lambda_2}{\lambda_2} + \frac{d\lambda_3}{\lambda_3} = 0 \tag{28.6}$$

Solving for $d\lambda_3$ and substituting that in equation (28.5), we obtain

$$dW = \left(f_1 - \frac{\lambda_3}{\lambda_1} f_3\right) d\lambda_1 + \left(f_2 - \frac{\lambda_3}{\lambda_2} f_3\right) d\lambda_2 \tag{28.7}$$

Since $d\lambda_1$ and $d\lambda_2$ are independent,

$$\begin{aligned} f_1 - \frac{\lambda_3}{\lambda_1} f_3 &= \frac{\partial W}{\partial \lambda_1} \\ f_2 - \frac{\lambda_3}{\lambda_2} f_3 &= \frac{\partial W}{\partial \lambda_2} \end{aligned} \tag{28.8}$$

or

$$\begin{aligned} \lambda_1 f_1 - \lambda_3 f_3 &= \lambda_1 \frac{\partial W}{\partial \lambda_1} \\ \lambda_2 f_2 - \lambda_3 f_3 &= \lambda_2 \frac{\partial W}{\partial \lambda_2} \end{aligned} \tag{28.9}$$

The normal stresses on each face of the deformed solid are

$$S_{11} = \frac{f_1}{\lambda_2 \lambda_3} = \lambda_1 f_1$$
$$S_{22} = \frac{f_2}{\lambda_1 \lambda_3} = \lambda_2 f_2 \qquad (28.10)$$
$$S_{33} = \frac{f_3}{\lambda_1 \lambda_2} = \lambda_3 f_3$$

Hence equation 28.9 can be written as,

$$S_{11} - S_{33} = \lambda_1 \frac{\partial W}{\partial \lambda_1}$$
$$S_{22} - S_{33} = \lambda_2 \frac{\partial W}{\partial \lambda_2} \qquad (28.11)$$

The above expressions suggest that an isotropic stress produces no strain and that the superposition of an isotropic stress on existing stresses generates no incremental deformation. The incremental stress-strain relations are obtained by taking differentials of the finite stresses

$$d\left(S_{11} - S_{33}\right) = s_{11} - s_{33}$$
$$d\left(S_{22} - S_{33}\right) = s_{22} - s_{33} \qquad (28.12)$$

Differentiating the stress-strain relations in equation 28.11 we obtain,

$$d\left(S_{11} - S_{33}\right) = \left(\frac{\partial W}{\partial \lambda_1} + \lambda_1 \frac{\partial^2 W}{\partial \lambda_1^2}\right) d\lambda_1 + \lambda_1 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} d\lambda_2$$
$$d\left(S_{22} - S_{33}\right) = \lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} d\lambda_1 + \left(\frac{\partial W}{\partial \lambda_2} + \lambda_2 \frac{\partial^2 W}{\partial \lambda_2^2}\right) d\lambda_2 \qquad (28.13)$$

which yields the following expressions for the incremental strains,

$$s_{11} - s_{33} = \lambda_1 \left(\frac{\partial W}{\partial \lambda_1} + \lambda_1 \frac{\partial^2 W}{\partial \lambda_1^2}\right) \frac{d\lambda_1}{\lambda_1} + \lambda_1 \lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \frac{d\lambda_2}{\lambda_2}$$
$$s_{22} - s_{33} = \lambda_1 \lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \frac{d\lambda_1}{\lambda_1} + \lambda_2 \left(\frac{\partial W}{\partial \lambda_2} + \lambda_2 \frac{\partial^2 W}{\partial \lambda_2^2}\right) \frac{d\lambda_2}{\lambda_2} \qquad (28.14)$$

Or

$$s_{11} - s_{33} = a_1 e_{xx} + a_2 e_{yy}$$
$$s_{22} - s_{33} = a_2 e_{xx} + a_3 e_{yy} \qquad (28.15)$$

For which the coefficients a_1, a_2 and a_3 are written as,

$$a_1 = \lambda_1 \frac{\partial W}{\partial \lambda_1} + \lambda_1^2 \frac{\partial^2 W}{\partial \lambda_1^2}$$

$$a_2 = \lambda_1 \lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \qquad (28.16)$$

$$a_3 = \lambda_2 \frac{\partial W}{\partial \lambda_2} + \lambda_2^2 \frac{\partial^2 W}{\partial \lambda_2^2}$$

Previous Next

Equation 28.15 and 28.16 are the incremental stress-strain relations. They can be written in more symmetric form by using the incompressibility relation,

$$e_{xx} + e_{yy} + e_{zz} = 0 \qquad (28.17)$$

which can be used in 28.15 to yield,

$$\begin{aligned} s_{11} - s_{33} &= (a_1 - a_2)e_{xx} - a_2 e_{zz} \\ s_{22} - s_{33} &= (a_3 - a_2)e_{yy} - a_2 e_{zz} \end{aligned} \qquad (28.18)$$

We then write,

$$\begin{aligned} a_1 - a_2 &= A \\ a_3 - a_2 &= B \\ a_2 &= C \end{aligned} \qquad (28.19)$$

The stress-strain relations then have the form,

$$\begin{aligned} s_{11} - s_{22} &= Ae_{xx} - Be_{yy} \\ s_{22} - s_{33} &= Be_{yy} - Ce_{zz} \\ s_{33} - s_{11} &= Ce_{zz} - Ae_{xx} \end{aligned} \qquad (28.20)$$

Which contain three incremental elastic coefficients, A, B and C. We can write also the expressions for incremental shear stresses,

$$\begin{aligned} s_{23} &= 2Q_1 e_{yz} \\ s_{31} &= 2Q_2 e_{zx} \\ s_{12} &= 2Q_3 e_{xy} \end{aligned} \qquad (28.21)$$

The six equations 28.19 and 28.20 are the complete set of incomplete stress relations for the orthotropic incompressible medium when the principal directions of the initial stress are located in the planes of the elastic symmetry.

$$s = \frac{1}{3}(s_{11} + s_{22} + s_{33}) \qquad (28.22)$$

$$\begin{aligned} 3(s_{11} - s) &= 2Ae_{xx} - Be_{yy} - Ce_{zz} \\ 3(s_{22} - s) &= -Ae_{xx} + 2Be_{yy} - Ce_{zz} \\ 3(s_{33} - s) &= -Ae_{xx} - Be_{yy} + 2Ce_{zz} \end{aligned} \qquad (28.23)$$

Shear components can be written as

(28.24)

$$s_{23} = 2Q_1 e_{yz}$$
$$s_{31} = 2Q_2 e_{zx}$$
$$s_{12} = 2Q_3 e_{xy}$$

In which the elastic constants are obtained as (the derivation of these equations are not being presented here)

$$Q_1 = \frac{1}{2}(S_{22} - S_{33})\frac{\lambda_2^2 + \lambda_3^2}{\lambda_2^2 - \lambda_3^2}$$
$$Q_2 = \frac{1}{2}(S_{33} - S_{11})\frac{\lambda_3^2 + \lambda_1^2}{\lambda_3^2 - \lambda_1^2} \qquad (28.25)$$
$$Q_3 = \frac{1}{2}(S_{11} - S_{22})\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2}$$

In the above set of equations, in case, $\lambda_1 = \lambda_2$, the elastic constant Q_3 can be obtained as

$$Q_3 = \frac{1}{2}\frac{s_{11} - s_{22}}{e_{xx}} \qquad (28.26)$$