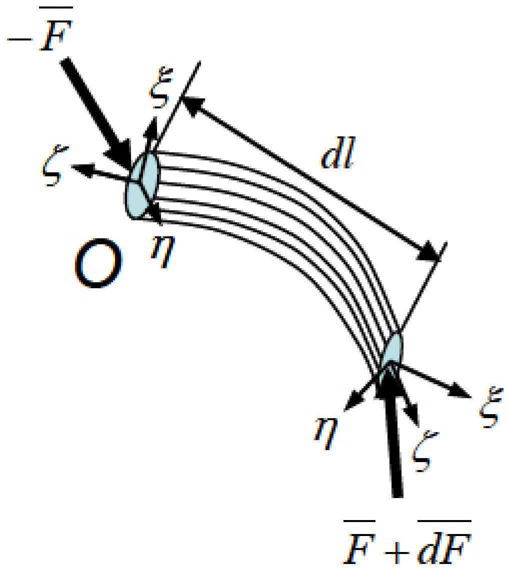
The Lecture Contains:

This lecture is adopted from the following book

1. “Theory of Elasticity, 3 rd edition” by Landau and Lifshitz. Course of Theoretical Physics, vol-7

The equations of equilibrium of rods



Consider an infinitesimal element of length dl bounded by two adjacent cross-sections of the rod. Let $\overline{F}$ be the resultant internal stress on a cross section; then the force acting on this cross-section of the rod is $-\overline{F}$ and that acting on the other end is $\overline{F}+\overline{dF}$. If $\overline{K}$ is the external force acting per unit length of the rod, then, the total force acting on the element of length dl is $\overline{K}dl$. Since the rod is in equilibrium under the action of these two forces, we have,

$$\overline{dF}+\overline{K}dl=0 \quad \text{or} \quad \overline{dF}/dl=-\overline{K} \tag{13.1}$$

Similarly, the moment of the internal stresses are $-\overline{M}$ and $\overline{M}+\overline{dM}$ respectively. And moment of the internal stresses about point O' is $-d\bar{l}\times-\overline{F}$. Summing up the total moments is obtained as:

$$d\,\overline{M}+d\bar{l}\times\overline{F}=0 \tag{13.2}$$

Dividing by dl and noting that $d\,\bar{l}\,/\,dl=\bar{t}$: the unit vector tangential to the rod, we have

$$d\,\overline{M}\,/\,dl=\overline{F}\times\bar{t} \tag{13.3}$$

If $\overline{F}$ is a concentrated force applied only at its free end, then $\overline{F}$ = constant. Furthermore, putting $\bar{t}=d\,\bar{r}\,/\,dl$ and by integrating, we have $\overline{M}=\overline{F}\times\bar{r}+$ constant

Similarly, we can differentiate equation 13.3 with respect to l to obtain

$$\frac{d^2\overline{M}}{dl^2}=\frac{d\overline{F}}{dl}\times\bar{t}+\overline{F}\times\frac{d\bar{t}}{dl}=-\overline{K}\times\bar{t}+\overline{F}\times\frac{d\bar{t}}{dl} \tag{13.4}$$

Bending of a rod with circular cross-section:

When a rod of arbitrary cross-section is bent, it undergoes also twisting although no external twisting moment may have been applied at the end. However for a circular rod no torsion results from bending. We can show this result from our earlier derivations. Consider the following derivative:

$$\frac{d}{dl}\left(\overline{M}\cdot\bar{t}\right)=\frac{d\overline{M}}{dl}\cdot\bar{t}+\overline{M}\cdot\frac{d\bar{t}}{dl} \tag{13.5}$$

Now $\overline{M}\cdot\bar{t}=C\overline{\Omega}\cdot\bar{t}=C\Omega_\varsigma$ and $\frac{d\overline{M}}{dl}=\overline{F}\times\bar{t}$. Here C is called the torsional rigidity.

Substituting these expressions we have,

$$C\frac{d\Omega_\varsigma}{dl}=\overline{M}\cdot\frac{d\bar{t}}{dl} \tag{13.6}$$

Noting that for a rod with circular cross-section, $I_1=I_2=I$, $\overline{M}$ can be written as

$$\overline{M}=EI\bar{t}\times\frac{d\bar{t}}{dl}+\bar{t}C\Omega_\varsigma \tag{13.7}$$

which when substituted in eqn. (13.6) yields,

$\frac{d\Omega_\varsigma}{dl}=0$ or $\Omega_\varsigma=$ constant, i.e. torsion angle is constant along the length of the rod. Hence if no twisting moment is applied at the end of the rod, $\Omega_\varsigma=0$. Hence pure bending of a rod with circular cross-section, we can write,

$$\overline{M}=EI\bar{t}\times\frac{d\bar{t}}{dl}=EI\frac{d\bar{r}}{dl}\times\frac{d^2\bar{r}}{dl^2} \tag{13.8}$$

Substitution of eqn. (13.8) into eqn. (11.17) yields,

$$EI\frac{d\bar{r}}{dl}\times\frac{d^3\bar{r}}{dl^3}=\overline{F}\times\frac{d\bar{r}}{dl} \tag{13.9}$$

For simple bending of the rod about the z axis, the above expressions can be simplified by writing the tangent in terms of angle θ it makes with the y axis:

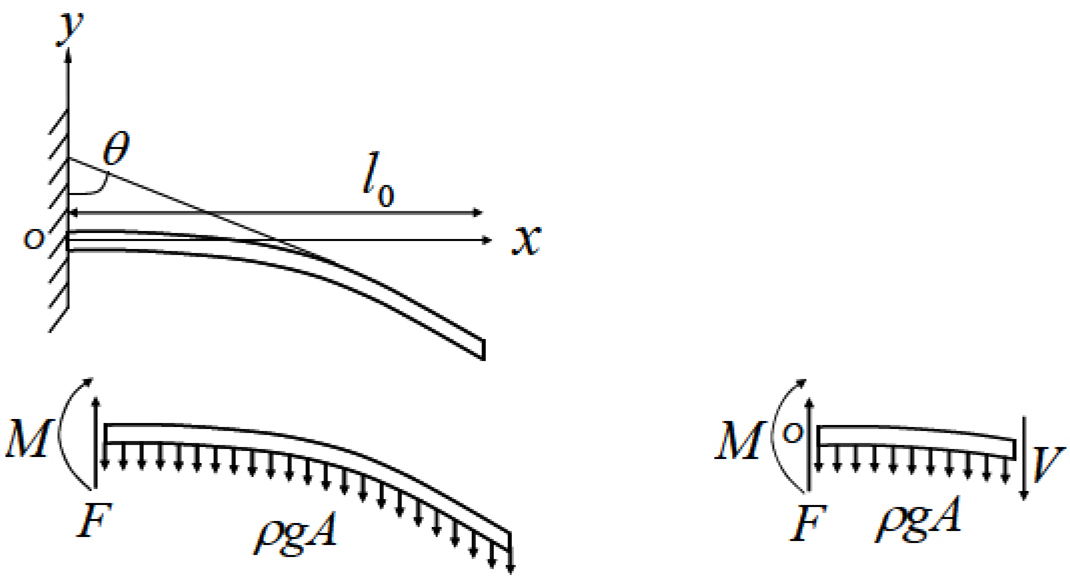
$$\begin{aligned}\bar{t}&=t_x\overline{e_x}+t_y\overline{e_y}=\sin\theta\,\overline{e_x}-\cos\theta\,\overline{e_y}\\ \frac{d\bar{t}}{dl}&=\left(\cos\theta\,\overline{e_x}+\sin\theta\,\overline{e_y}\right)\frac{d\theta}{dl}\end{aligned} \tag{13.10}$$

Similarly, the expression for the torque and its derivative can be written as,

$$\overline{M} = EI\,\bar{t} \times \frac{d\bar{t}}{dl} = EI\left(-\cos^2\theta\,\overline{e_y} \times \overline{e_x} + \sin^2\theta\,\overline{e_x} \times \overline{e_y}\right)\frac{d\theta}{dl} = EI\frac{d\theta}{dl}\overline{e_x} \times \overline{e_y}$$

$$\frac{d^2\overline{M}}{dl^2} = EI\frac{d^3\theta}{dl^3}\overline{e_x} \times \overline{e_y} \qquad (13.11)$$

Bending of a rod under distributed load



Here we will consider large bending of a rod which remains strongly attached to a rigid wall. Let us say that the rod of length L_0 and area of cross-section A bends under the action of a uniform load, e.g. gravity. If the density of the rod is ρ, then assuming that the rod does not undergo any extension, the reaction force at the rigid wall can be written as:

$$F = \rho g A L_0 \tag{13.12}$$

At the free end of the rod it is not acted upon by any reaction force. Say at any cross-section at a length L from wall, the reaction force is V, then from force balance,

$$V + \rho g A L = \rho g A L_0 \quad => V = \rho g A (L_0 - L) \quad => \frac{dV}{dL} = -\rho g A \tag{13.13}$$

Integrating the vectorial form of above expression,

$$\overline{V} = -\rho g A (L_0 - L)\overline{e_y} \quad \text{and} \quad -\overline{K} = \rho g A \overline{e_y} \tag{13.14}$$

From equation 13.14, $K_y = -\rho g A$

For a circular rod $\Omega_\varsigma = 0$, we have,

$$\frac{d\overline{M}}{dL} = \overline{V} \times \overline{t} \text{ , so that, } EI \frac{d\overline{r}}{dL} \times \frac{d^3\overline{r}}{dL^3} = \overline{V} \times \overline{t} = \overline{V} \times \frac{d\overline{r}}{dL} \tag{13.15}$$

Now

$$\bar{t} = \frac{d\bar{r}}{dL} = \sin\theta\overline{e_x} - \cos\theta\overline{e_y}$$

$$\frac{d^3\bar{r}}{dL^3} = \left(\cos\theta\overline{e_x} + \sin\theta\overline{e_y}\right)\frac{d^2\theta}{dL^2} + \left(\sin\theta\overline{e_x} - \cos\theta\overline{e_y}\right)\left(\frac{d\theta}{dL}\right)^2$$

$$\frac{d\bar{r}}{dL}\times\frac{d^3\bar{r}}{dL^3} = \left(\sin\theta\overline{e_x} - \cos\theta\overline{e_y}\right)\times\left(\left(\cos\theta\overline{e_x} + \sin\theta\overline{e_y}\right)\frac{d^2\theta}{dL^2} + \left(\sin\theta\overline{e_x} - \cos\theta\overline{e_y}\right)\left(\frac{d\theta}{dL}\right)^2\right) \quad (13.16a)$$

$$= \frac{d^2\theta}{dL^2}\overline{e_x}\times\overline{e_y}$$

$$\overline{V}\times\frac{d\bar{r}}{dL} = -\rho g A(L_0 - L)\overline{e_y}\times\left(\sin\theta\overline{e_x} - \cos\theta\overline{e_y}\right) = \rho g A(L_0 - L)\sin\theta\overline{e_x}\times\overline{e_y} \quad (13.16a)$$

Thus from equation 13.16a and b we can write the following force balance equation for bending of the rod

$$\frac{d^2\theta}{dL^2} = \frac{\rho g A}{EI}(L_0 - L)\sin\theta \quad (13.17)$$

Putting $L = L_0\eta$ and $\kappa = \frac{EI}{\rho g A L_0^3}$, we have $\frac{d^2\theta}{d\eta^2} = \frac{1}{\kappa}(1-\eta)\sin\theta$

This equation is solved with the following boundary conditions:

$$\eta = 0,\ \theta = \pi/2 \text{ and } \eta = 1,\ \frac{d\theta}{dl} = 0 \quad (13.18)$$

Solution of equation 13.17 along with the b.c. 13.18 yields the following graph,