The Lecture Contains

"Mechanics of the Cell" by David Boal, Cambridge University Press, 2002, Cambridge, UK

Network with six-fold symmetry under stress

Let us consider a network which is made of identical springs of length s_0 under zero stress. The network is then subjected to isotropic two dimensional tension τ so that the area of each triangle increases although it remains equilateral. The length of each spring increases to s_τ which can be calculated by minimizing the enthalpy of the network

$$H = E - \tau A \tag{38.1}$$

Here E is the energy of the springs and A is the area of the network. Notice, there are three springs per vertex and energy for each vertex, so that energy per vertex is

$$(3/2)k_{sp}(s - s_0)^2 \tag{38.2}$$

And the area per vertex is

$$A_v = \sqrt{3}s^2/2 \tag{38.3}$$

Hence the enthalpy per vertex is obtained by substituting in 38.1 the expressions for 37.16 and 38.3

$$H_v = (3/2)k_{sp}(s - s_0)^2 - \sqrt{3}\tau s^2/2 \tag{38.4}$$

The spring length s_τ that minimizes H_v for a given tension τ can be obtained by equating $\partial H_v/\partial s$ to zero, which yields

$$s_\tau = s_0/\left[1 - \tau/\left[\sqrt{3}k_{sp}\right]\right] \tag{38.5}$$

Substituting s_τ in equation 17.5, we obtain the minimum value for H_v,

$$H_{v,\min} = -\left(\sqrt{3}/2\right)\tau s_0^2/\left(1 - \tau/\sqrt{3}k_{sp}\right) \tag{38.6}$$

And the area per vertex is derived as,

$$A_v = \frac{\sqrt{3}}{2}\frac{s_0^2}{\left[1 - \tau/\left[\sqrt{3}k_{sp}\right]\right]^2} = \frac{A_0}{\left[1 - \tau/\left[\sqrt{3}k_{sp}\right]\right]^2} \tag{38.7}$$

The area diverges as the tension approaches a critical value $\tau_{\exp} = \sqrt{3}k_{sp}$.

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An interesting behavior of the above network under compression is not captured by the assumption of a non-equilateral triangle and that is called network collapse. For example, in the above analysis we have minimized the energy w.r.t. s, which however may not be the global minima, because, we have not really considered shapes other than equilateral triangle. Whereas, the term τA with negative τ drives the system towards smaller area, the equilateral triangle encloses largest area for a given perimeter. The relaxation of this assumption then leads to smaller area, in fact to zero area achieved by isosceles triangles with two short sides of length s_1 and one long side of length $2s_1$. The spring energy of such a network can be estimated as,

$$k_{sp}/2\left[(2s_1 - s_0)^2 + 2(s_1 - s_0)^2\right] \qquad (38.8)$$

Minimizing this energy with respect to s_1 yields, the minimum energy as $k_{sp}s_0^2/6$ which is achieved at $s_1 = 2s_0/3$. The enthalpy per vertex of the equilateral triangle increases with pressure ($\tau < 0$), until it exceeds the enthalpy of a zero area isosceles triangle $k_{sp}s_0^2/6$, at collapse tension $\tau_{coll} = -\left(\sqrt{3}/8\right)k_{sp}$, or equivalently of a spring of length $s_x/s_0 = 8/9$.

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Network with six-fold symmetry under stress (contd...)

The elastic modulii of the stressed equilateral triangle can be obtained by applying an incremental stress $d\tau$ on the isotropically stressed network and then measuring the incremental strain dA_v/A_v. In essence we can differentiate the area per vertex of the stressed triangle $A_v = \sqrt{3}s_\tau^2/2$ with respect to τ, which yields,

$$\frac{dA_v}{d\tau} = \sqrt{3}s_\tau \frac{ds_\tau}{d\tau} = \sqrt{3}s_\tau \frac{s_0}{\left(1-\frac{\tau}{\sqrt{3}k_{sp}}\right)^2}\frac{1}{\sqrt{3}k_{sp}} = \frac{s_\tau^2}{1-\frac{\tau}{\sqrt{3}k_{sp}}}\frac{1}{k_{sp}} \qquad (38.9)$$

Hence, the incremental strain in the area is given as,

$$\frac{dA_v}{A_v} = \frac{2}{\sqrt{3}s_\tau^2}\frac{s_\tau^2}{1-\frac{\tau}{\sqrt{3}k_{sp}}}\frac{1}{k_{sp}}d\tau = \frac{2}{\sqrt{3}}\frac{1}{1-\frac{\tau}{\sqrt{3}k_{sp}}}\frac{1}{k_{sp}}d\tau \qquad (38.10)$$

Rearranging, we obtain a linear relation between the incremental stress and incremental strain with a material constant which depends upon the applied stress τ on the network,

$$d\tau = \frac{\sqrt{3}}{2}\left(1-\frac{\tau}{\sqrt{3}k_{sp}}\right)k_{sp}\frac{dA_v}{A_v} \qquad (38.11)$$

Here $K_A = \frac{\sqrt{3}}{2}\left(1-\frac{\tau}{\sqrt{3}k_{sp}}\right)k_{sp}$ is the effective area compression modulus of the network. We can then obtain an expression of strain energy per vertex of this network as each chain of the equilateral triangle is extended by an incremental amount δ,

$$\Delta\Pi = 2K_A\left(\frac{\delta}{s_\tau}\right)^2 = \sqrt{3}\left(1-\frac{\tau}{\sqrt{3}k_{sp}}\right)k_{sp}\left(\frac{\delta}{s_\tau}\right)^2 \qquad (38.12)$$

Notice that K_A vanishes at $\tau = \sqrt{3}k_{sp}$ suggesting our earlier result that area per vertex grows without bound as the stress increases. This result, however, does not capture the network collapse for negative τ, i.e. for compression.

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Network with six-fold symmetry under stress (contd...)

Similarly analysis shows that the effective shear modulus of the network can be deducted as, , which shows that the effective shear modulus increases with the stress τ. The following table summarizes the above results.

Mode	$\Delta\Pi$ (microscopic)	Strain	$\Delta\Pi$ (continuum)
(a)	$\sqrt{3}k_{sp}\left(1-\tau/\left[\sqrt{3}k_{sp}\right]\right)\left(\delta/s_{I}\right)^2$	$u_{xx}=u_{yy}=\delta/s_{I}$ $u_{xy}=0$	$2K_A\left(\delta/s_{I}\right)^2$
(b)	$\left(k_{sp}/2\sqrt{3}\right)\left(1+\sqrt{3}\,\tau/k_{sp}\right)\left(\delta/s_{I}\right)^2$	$u_{xx}=u_{yy}=0$ $u_{xy}=\left(\delta/s_{I}\right)/\sqrt{3}$	$(2/3)\mu\left(\delta/s_{I}\right)^2$

The stress dependent modulii can be used to estimate the Poisson ratio σ defined as,

$$\sigma=-\frac{u_{yy}}{u_{xx}} \tag{38.13}$$

in which stress is applied along the x axis. For a three dimensional isotropic material we had shown earlier the following relations for the shear and bulk modulii:

$$\mu=\frac{E}{2(1+\sigma)}\ ,\ k=\frac{E}{3(1-2\sigma)} \tag{38.14}$$

From which the Poisson ratio can be obtained as,

$$\sigma=\frac{3K_A-2\mu}{6K_A+2\mu} \tag{38.15}$$

It can be shown that the Poisson ratio in two dimension can be obtained as,

$$\sigma=\frac{K_A-\mu}{K_A+\mu} \tag{38.16}$$

For a triangular network with zero stress with $K_A=\sqrt{3}k_{sp}/2$ and $\mu=\sqrt{3}k_{sp}/4$, the Poisson is defined as, $\sigma=\frac{1}{3}$. However, for stress dependent network, the Poisson ratio can be defined as,

$$\sigma=\frac{1-5\tau/\sqrt{3}k_{sp}}{3+\tau/\sqrt{3}k_{sp}} \tag{38.17}$$

Thus, the poisson ratio becomes negative for six-fold network under tension for $\tau/k_{sp}>\sqrt{3}/5$, implying that such a network will expand laterally when stretched in the longitudinal direction.

Net work with six-fold symmetry at non-zero temperature

So far we considered spring networks at zero temperature with all springs at identical length. In other word, the springs are in high tension relative to $k_{sp}x^2$. With increase in temperature, however, the springs increasingly fluctuate in length. We can then estimate the probability of finding its extension between x and $x+dx$ from the Boltzmann's relation,

$$\wp(x)dx = dx \cdot \frac{\exp\left(-\frac{k_{sp}x^2}{2k_BT}\right)}{\int_{-\infty}^{\infty} \exp\left(-\frac{k_{sp}x^2}{2k_BT}\right)dx} \tag{38.18}$$

The mean square value of x can be derived as,

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \wp(x)dx = \frac{k_BT}{k_{sp}} \tag{38.19}$$

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