Pressure Applied to a circular region (contd...)

This lecture is adopted from the following book

1. "Contact Mechanics" by K.L.Johnson

Now, consider the pressure distribution of the form:

$$p = p_0\left(1-\frac{t^2}{a^2}\right)^{-1/2} \tag{7.1}$$

Where,

$$t^2 = (r+s\cos\phi)^2 + (s\sin\phi)^2 = r^2+s^2+2rs\cos\phi.$$

Hence the pressure distribution can be expressed as,

$$p(s,\phi) = p_0\left(\frac{a^2-t^2}{a^2}\right)^{-1/2} = p_0 a\left(a^2-r^2-2rs\cos\phi-s^2\right)^{-1/2} \tag{7.2}$$

Similar to previous section, the displacement at the surface of the film, within the loaded region can be found out using equation, 4.49

$$v|_{z=0} = \frac{1-\sigma^2}{\pi E}\iint_{S\phi} p_0 a\left(a^2-r^2-2rs\cos\phi-s^2\right)^{-1/2} ds\,d\phi$$
$$= \frac{1-\sigma^2}{\pi E} p_0 a \int_0^{2\pi} d\phi \int_0^{s_1}\left(a^2-r^2-2rs\cos\phi-s^2\right)^{-1/2} ds \tag{7.3a}$$

Here, s_1 is the positive root of the equation $a^2-r^2-2rs\cos\phi-s^2=0$, i.e.

$$s_1 = \frac{-2r\cos\phi+\sqrt{4r^2\cos^2\phi-4\left(-a^2+r^2\right)}}{2} = -r\cos\phi+\sqrt{a^2-r^2\sin^2\phi} \tag{7.3b}$$

Note the value of the following integral,

$$\int_0^{s_1}\left(a^2-r^2-2rs\cos\phi-s^2\right)^{-1/2} ds = \sin^{-1}\left(\frac{r\cos\phi+s_1}{\sqrt{a^2-r^2\sin^2\phi}}\right) - \sin^{-1}\left(\frac{r\cos\phi}{\sqrt{a^2-r^2\sin^2\phi}}\right)$$
$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{r\cos\phi}{\sqrt{a^2-r^2}}\right) \tag{7.4}$$

And since, $\tan^{-1}\left(\frac{r\cos\phi}{\sqrt{a^2-r^2}}\right) = -\tan^{-1}\left\{\frac{r\cos(\phi+\pi)}{\sqrt{a^2-r^2}}\right\}$, it easy to show that the integration of the second term over $0<\phi<2\pi$ vanishes, so that, 4.67 yields,

$$v|_{z=0} = \frac{1-\sigma^2}{\pi E} p_0 a \int_0^{2\pi}\left\{\frac{\pi}{2} - \tan^{-1}\left(\frac{r\cos\phi}{\sqrt{a^2-r^2}}\right)\right\} d\phi = \pi\frac{1-\sigma^2}{E} p_0 a \tag{7.5}$$

which is independent of r, in other word, a pressure distribution of the type of equation 4.65 leads to uniform normal displacement of the loaded circle.

Furthermore the total pressure is obtained as,

$$P=\int_0^a \frac{2\pi t p_0}{\sqrt{1-\frac{t^2}{a^2}}}dt = 2\pi a^2 p_0 \tag{7.6}$$

Now, consider the situation for a point B which lies outside the loaded region.

The distance of an element at (s,ϕ) from the origin is

$$t^2=(r-s\cos\phi)^2+(s\sin\phi)^2=r^2+s^2-2rs\cos\phi \tag{7.7}$$

So that the pressure distribution in terms of (s,ϕ) is obtained as,

$$p(s,\phi)=p_0\left(\frac{a^2-r^2-s^2+2rs\cos\phi}{a^2}\right)^{-1/2}=p_0a\left(a^2-r^2+2rs\cos\phi-s^2\right)^{-1/2}$$

And the displacement at point B is

$$\begin{aligned} v\big|_{z=0}&=\frac{1-\sigma^2}{\pi E}\iint_{S\phi}p_0a\left(a^2-r^2-2rs\cos\phi-s^2\right)^{-1/2}ds\,d\phi \\ &=\frac{1-\sigma^2}{\pi E}p_0a\int_0^{2\pi}d\phi\int_0^{s_1}\left(a^2-r^2-2rs\cos\phi-s^2\right)^{-1/2}ds \end{aligned} \tag{7.8}$$

Here, $s_{1,2}$ are the roots of the equation $a^2-r^2+2rs\cos\phi-s^2=0$, i.e.

$$s_{1,2}=\frac{2r\cos\phi\pm\sqrt{4r^2\cos^2\phi-4\left(-a^2+r^2\right)}}{2}=r\cos\phi\pm\sqrt{a^2-r^2\sin^2\phi}$$

Then, noting the result of following integral,

$$\int_{s_1}^{s_2}\left(a^2-r^2+2rs\cos\phi-s^2\right)^{-1/2}ds=\pi \tag{7.9}$$

We have the following expression for the normal displacement of a point outside the circle of loading,

$$v\big|_{z=0}=\frac{1-\sigma^2}{\pi E}\pi p_0 a2\sin^{-1}\left(\frac{a}{r}\right)=\frac{2\left(1-\sigma^2\right)}{E}p_0a\sin^{-1}\left(\frac{a}{r}\right) \tag{7.10}$$

Previous Next

We can consider also following pressure distribution:

$$p = p_0\left(1-\frac{t^2}{a^2}\right)^{1/2} \tag{7.11}$$

From which the total force is obtained as

$$P = \int_0^a 2\pi t p_0\sqrt{1-\frac{t^2}{a^2}}dt = \frac{2\pi a^2}{3}p_0 \tag{7.12}$$

Again putting following expression for *t*

$$t^2 = (r+s\cos\phi)^2+(s\sin\phi)^2 = r^2+s^2+2rs\cos\phi$$

We have, the following expression for pressure distribution,

$$p = \frac{p_0}{a}\left(a^2-r^2-s^2-2rs\cos\phi\right)^{1/2} \tag{7.13}$$

Then the displacement at the surface of the film, within the loaded region can be found out using equation, 4.49

$$v\big|_{z=0} = \frac{1-\sigma^2}{\pi E}\frac{p_0}{a}\int_0^{2\pi}\int_0^{s_1}\left(a^2-r^2-2rs\cos\phi-s^2\right)^{1/2}ds\,d\phi \tag{7.14}$$

Where

$$s_1 = \frac{-2r\cos\phi+\sqrt{4r^2\cos^2\phi-4\left(-a^2+r^2\right)}}{2} = -r\cos\phi+\sqrt{a^2-r^2\sin^2\phi}$$ as before.

$$\int_0^{s_1}\left(a^2-r^2-2rs\cos\phi-s^2\right)^{1/2}ds = -\frac{r\cos\phi}{2}\sqrt{a^2-r^2}+\frac{1}{2}\left(a^2-r^2\sin^2\phi\right)\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{r\cos\phi}{\sqrt{a^2-r^2}}\right)\right)$$

in which integration of $-\frac{r\cos\phi}{2}\sqrt{a^2-r^2}$ and $\tan^{-1}\left(\frac{r\cos\phi}{\sqrt{a^2-r^2}}\right)$ over 0 to 2π vanishes, then the displacement of the surface within the loaded region.

$$\begin{aligned} v\big|_{z=0} &= \frac{1-\sigma^2}{\pi E}\frac{p_0}{a}\int_0^{2\pi}\int_0^{s_1}\left(a^2-r^2-2rs\cos\phi-s^2\right)^{1/2}ds\,d\phi \\ &= \frac{1-\sigma^2}{\pi E}\frac{p_0}{a}\int_0^{2\pi}d\phi\frac{\pi}{4}\left(a^2-r^2\sin^2\phi\right) = \frac{1-\sigma^2}{E}\frac{\pi p_0}{4a}\left(2a^2-r^2\right) \qquad r\le a \end{aligned} \tag{7.15}$$

We can find also the tangential displacement at B, which must be radial.

$$u\Big|_{z=0} = \int_0^{2\pi}\int_0^{s_1} \frac{(1-2\sigma)(1+\sigma)}{2\pi E}\frac{ps\,ds\,d\phi}{s}\cos\phi$$
$$= \frac{(1-2\sigma)(1+\sigma)p_0}{2\pi Ea}\int_0^{2\pi}\int_0^{s_1}\cos\phi\sqrt{a^2-r^2-2rs\cos\phi-s^2}\,ds\,d\phi \qquad (7.16)$$

Integration with respect to s is as before, so the final result is obtained as,

$$u\Big|_{z=0} = -\frac{(1-2\sigma)(1+\sigma)p_0}{2\pi Ea}$$
$$\int_0^{2\pi}\cos\phi\left[-\frac{r\cos\phi}{2}\sqrt{a^2-r^2}+\frac{1}{2}\left(a^2-r^2\sin^2\phi\right)\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{r\cos\phi}{\sqrt{a^2-r^2}}\right)\right)\right]d\phi \qquad (7.17)$$
$$= \frac{(1-2\sigma)(1+\sigma)p_0}{3E}\frac{a^2}{r}\left\{1-\left(1-\frac{r^2}{a^2}\right)^{3/2}\right\} \qquad r\le a$$

Assignment: Find out the normal and radial displacements when the point B lies outside the loaded circle.

Previous Next

Assignment:

- Find out the normal and radial displacements when the point B lies outside the loaded circle