The Lecture Contains:

This lecture is adopted from the following book

1. “Theory of Elasticity, 3 rd edition” by Landau and Lifshitz. Course of Theoretical Physics, vol-7

2. “The use of soap films in solving torsion problems” by G. I. Taylor and A. A. Griffith, Proceedings of the institute of Mechanical Engineers (1917), pp. 755-789.

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Torsion of rods

You can deform a thin rod in a number of ways, e.g. you can bend it, twist it and you can both bend and twist it. In all these cases, even though the relative displacement of neighboring points may be small, i.e. the strain may be small but the overall deformation may still be large. We first consider here the problem of torsion of a long thin rod.

In torsional deformation, while the rod remains straight, each transverse element in it rotates through some angle relative to the one below it. Consequently, after the rod is twisted through some angle, any generator line parallel to the axis no linger remains straight but turns helical. The complete solution of the displacement should satisfy the following stress equilibrium relations

$$
\begin{aligned}
&\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}=0\\
&\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}=0\\
&\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}=0
\end{aligned}
\qquad (15.1)
$$

and the boundary conditions are:

$$
\left.\begin{aligned}
&X_x\cos(n,x)+X_y\cos(n,y)=0\\
&Y_x\cos(n,x)+Y_y\cos(n,y)=0\\
&Z_x\cos(n,x)+Z_y\cos(n,y)=0
\end{aligned}\right\}
$$

on the side surface

and X_z, Y_z, Z_z equal to end loads at $z=0, \quad z=l$.

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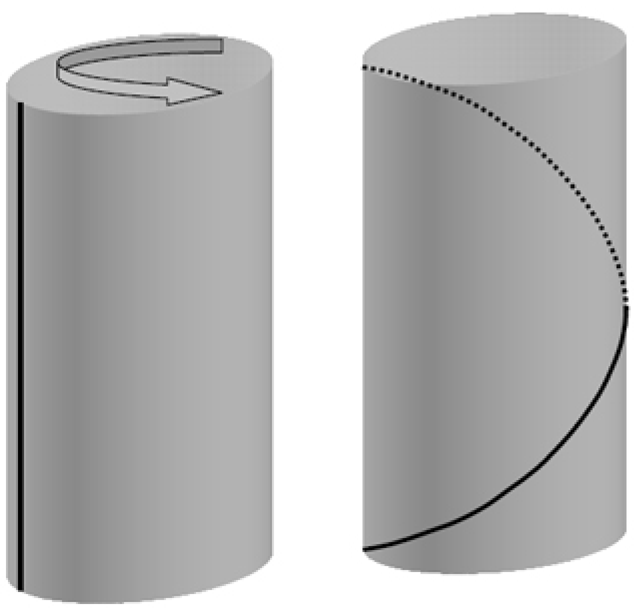
Let's consider a straight rod with arbitrary cross section whose co-ordinate system is as follows: while the z axis remains along the axis of the rod, its origin is somewhere inside and the xy plane can be taken as the undisplaced cross section.. We define the torsion angle τ as the angle through which the rod is twisted per unit length, so that two neighboring elements dz distance apart turns through an angle $d\phi$: $\tau = d\phi / dz$. τ is rather small so that when this distance is of the order of the transverse dimension R of the rod, it satisfies $\tau R << 1$. Let's estimate the displacements $\bar{u}$ in a small portion of the rod near the origin of the rod. When the radius vector $\bar{r}$ turns through an angle $\delta\phi$, the displacement of its end is given by

$$\delta\bar{r} = \delta\bar{\phi} \times \bar{r} \qquad (15.2)$$

where $\delta\bar{\phi}$ is the vector whose magnitude is the angle of rotation and direction is that of the axis of rotation i.e. the z axis. For points with coordinates z relative to the xy plane the angle of rotation is τz, hence, the components u, v and w of the displacement vector along the x, y and z directions respectively are,

$$u = -\tau z y, \quad v = \tau z x, \quad w = 0 \qquad (15.3)$$

It is easy to see that the components of the stress tensor calculated from these displacements satisfy the stress equilibrium relations and the boundary conditions.



In general points undergo displacement along the z axis also. Let's assume that for small τ, this displacement is,

$$w = \tau\psi(x, y) \qquad (15.4)$$

Here $\psi(x, y)$ is a single valued function, called the torsion function, which signifies that any cross section of the rod does not remain planer bur become curved. Furthermore, the origin of the co-ordinate system which was fixed at a particular point in the xy plane can now move along the z co-ordinate. Then the components of the strain tensor are as follows:

$$e_{xx} = e_{yy} = e_{xy} = e_{zz} = 0$$

$$e_{xz} = \frac{1}{2}\tau\left(\frac{\partial\psi}{\partial x} - y\right), \qquad e_{yz} = \frac{1}{2}\tau\left(\frac{\partial\psi}{\partial y} + x\right) \tag{15.5}$$

it should be noted that $e_{xx} + e_{yy} + e_{zz} = 0$, signifying that torsion does not result in change in volume, it is a pure shear deformation. Components of stress tensor are,

$$X_x = Y_y = X_y = Z_z = 0$$

$$X_z = 2\mu e_{xz} = \mu\tau\left(\frac{\partial\psi}{\partial x} - y\right), \qquad Y_z = 2\mu e_{yz} = \mu\tau\left(\frac{\partial\psi}{\partial y} + x\right) \tag{15.6}$$

Then the general equation of equilibrium reduces to,

$$\frac{\partial X_z}{\partial x} + \frac{\partial X_y}{\partial y} = 0 \tag{15.7}$$

in which substitution of the expressions in (15.5) results in the two dimensional Laplacian:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

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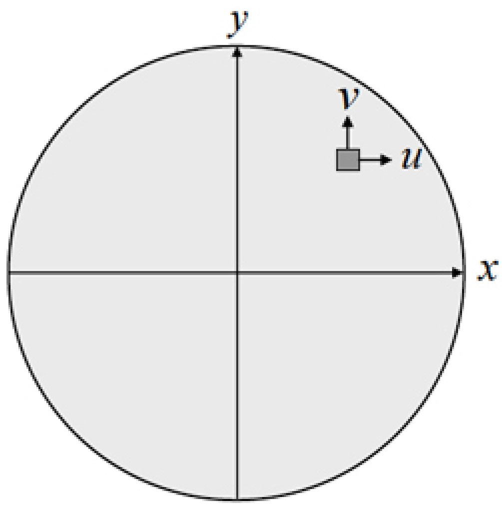
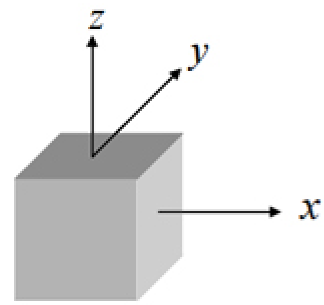
In other word ψ is a harmonic function of the two variables x, y. The boundary condition $Z_x \cos(n,x) + Z_y \cos(n,y) = 0$ at the curved surface L leads to

$$\left(\frac{\partial \psi}{\partial x} - y\right)\cos(n,x) + \left(\frac{\partial \psi}{\partial y} + y\right)\cos(n,y) = 0 \qquad (15.8)$$





Noting that $\frac{\partial \psi}{\partial x}\cos(n,x) + \frac{\partial \psi}{\partial y}\cos(n,y) = \frac{\partial \psi}{\partial n}$, the b.c. is obtained as

$\frac{\partial \psi}{\partial n} = y\cos(n,x) - x\cos(n,y)$ on L. The problem of finding ψ is one of the fundamental problems of the potential theory called "the Neumann problem".

It can be shown that the resultant of tangential stresses at any cross section is zero:

$$\iint_S X_z dx\, dy = 0, \quad \iint_S Y_z dx\, dy = 0 \qquad (15.9)$$

The resultant of moment of external stresses applied to one end is given by

$$M = \iint_S (xY_z - yX_z) dx\, dy = \mu\tau \iint_L \left(x^2 + y^2 + x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x}\right) dxdy \qquad (15.10)$$

$$D = \mu \iint_L \left(x^2 + y^2 + x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x}\right) dxdy$$

or, $M = \tau D$ where,

the proportionality constant is

called the torsional rigidity of the rod. It is a product of the shear modulus which is a material property of the rod and an integral which depends only upon the shape of the cross section of the rod. Torsional rigidity is always a positive quantity which can be proved from calculating the potential energy stored in the twisted bar. It can also be proved by applying the Green's function.

Let us consider the case of a rod with circular cross section. If the origin is placed at the centre, one has $y\cos(n,x) - x\cos(n,y) = 0$ on the boundary. Therefore,

$$\frac{d\psi}{dn} = 0 \tag{15.11}$$

on the entire boundary. Hence $\psi =$ constant which may be taken as 0. In other word in this case the cross sections of the rod remain plane, and the displacements are given as,

$u = -\tau z y, \quad v = \tau z x, \quad w = 0, \quad X_z = -\mu\tau y, \quad Y_z = \mu\tau x$. The torsional rigidity is given by,

$$D = \mu\iint_{\Sigma}\left(x^2 + y^2\right)dx\,dy = \mu I = \frac{\mu\pi R^4}{2} \tag{15.12}$$

For an annular ring one has, $D = \dfrac{\mu\pi\left(R_2^4 - R_1^4\right)}{2}$ where R_1 and R_2 are inner and outer radius respectively.

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Torsional energy

Energy density:

$$F=\frac{1}{2}\sigma_{ik}u_{ik}=Z_x u_z+Z_y v_z=\frac{1}{2\mu}\left(Z_x^2+Z_y^2\right)=\frac{1}{2\mu}(\mu\tau)^2\left(\left(\frac{\partial\psi}{\partial x}-y\right)^2+\left(\frac{\partial\psi}{\partial y}+x\right)^2\right)$$
$$=\frac{\mu\tau^2}{2}\left(\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2-2\frac{\partial\psi}{\partial x}y+2\frac{\partial\psi}{\partial y}x+x^2+y^2\right) \qquad (15.13)$$

Total energy =

$$\int_V F dV=\int_V\frac{\mu\tau^2}{2}\left(\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2-2\frac{\partial\psi}{\partial x}y+2\frac{\partial\psi}{\partial y}x+x^2+y^2\right)dV$$
$$=\frac{\mu L}{2}\iint\tau^2\left(\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2-\frac{\partial\psi}{\partial x}y+\frac{\partial\psi}{\partial y}x\right]+x^2+y^2-\frac{\partial\psi}{\partial x}y+\frac{\partial\psi}{\partial y}x\right)dxdy \qquad (15.14)$$
$$=\frac{\mu L}{2}\iint\tau^2\left(x^2+y^2-\frac{\partial\psi}{\partial x}y+\frac{\partial\psi}{\partial y}x\right)dxdy=\frac{D\tau^2L}{2}$$

Here we present a propose method by which the function can be estimated for rods of any arbitrary cross-section. In essence we define a new function $\kappa(x,y)$ such that it satisfies the following equations:

$$X_z=2\mu\tau\frac{\partial\kappa}{\partial y},\ Y_z=-2\mu\tau\frac{\partial\kappa}{\partial x} \qquad (15.15)$$

Then from the definition of equation 15.6, we obtain,

$$\frac{\partial\psi}{\partial x}-y=2\frac{\partial\kappa}{\partial y},\quad \frac{\partial\psi}{\partial y}+x=-2\frac{\partial\kappa}{\partial x} \qquad (15.16)$$

So that,

$$\frac{\partial\psi}{\partial x}=y+2\frac{\partial\kappa}{\partial y},\ \frac{\partial\psi}{\partial y}=-x-2\frac{\partial\kappa}{\partial x} \qquad (15.17)$$

We can then equate the expression for $\frac{\partial^2\psi}{\partial x\partial y}$ from the above equation, which results in the following relation:

$$\frac{\partial\psi}{\partial x}=y+2\frac{\partial\kappa}{\partial y},\ \frac{\partial\psi}{\partial y}=-x-2\frac{\partial\kappa}{\partial x} \qquad (15.17)$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = 1 + 2\frac{\partial^2 \kappa}{\partial y^2} = -1 - 2\frac{\partial^2 \kappa}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 \kappa}{\partial y^2} + \frac{\partial^2 \kappa}{\partial x^2} + 1 = 0 \qquad (15.18)$$

Equation 9.18 is interesting as it is akin to the soap film equation written as,

$$\frac{\partial^2 \zeta}{\partial y^2} + \frac{\partial^2 \zeta}{\partial x^2} + \frac{P}{2\gamma} = 0 \qquad (15.19)$$

Here we consider that a hole of an arbitrary shape is cut into a solid plate and a soap film is allowed to form by dipping the plate inside a soap solution and then withdrawing it out. The soap film does not remain flat but assumes a curved shape because of pressure difference P between the two sides of the soap film. We can then obtain the displacement ζ of the film at various location (x, y) which satisfies equation 9.19. Notice that we can recover equation 9.18 by simply substituting,

$$\zeta = \frac{P}{2\gamma}\kappa \qquad (15.20)$$

This result then suggests a simple way to obtain the function $\psi(x, y)$: we can form a hole of the shape of the cross-section of the rod in a rigid plate and then obtain apply different amount of differential pressure P across the film and obtain the displacement ζ. We obtain the function $\kappa(x, y)$ from this data by using equation 9.20, following which we can obtain $\psi(x, y)$ from the set of relations in 15.17