The Lecture Contains

1. "Mechanics of Incremental Deformations" by M. A. Biot

Previous Next

The slide modulus

Let us consider a rectangle block which is subjected to initial stresses S_{11} and S_{22} along two principal directions 1 and 2 respectively.

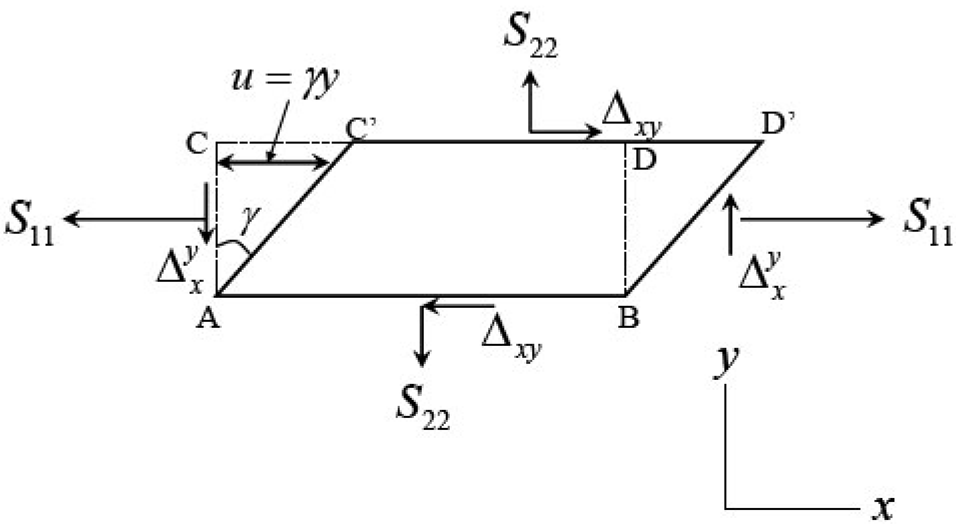

Figure 29.1

Over and above these initial stresses, let us say that it is subjected to shear displacement parallel to the x direction. Then the displacement components are,

$$u = \gamma y \quad v = w = 0 \tag{29.1}$$

This is an in plane deformation in the xy plane. The rectangle ABDC in this plane is deformed into the parallelogram ABD'C'. In order to produce this deformation, we have applied a tangential force on the CD which can be evaluated from the relation of boundary condition as in equations 13.17; substituting $S_{12} = 0$, we obtain

$$\begin{aligned} \Delta f_x &= \left(s_{11} + S_{11} e_{yy}\right)\cos(n, x) + \left(s_{12} - S_{22}\varpi - S_{11} e_{xy}\right)\cos(n, y) \\ \Delta f_y &= \left(s_{12} + S_{11}\varpi - S_{22} e_{xy}\right)\cos(n, x) + \left(s_{22} + S_{22} e_{yy}\right)\cos(n, y) \end{aligned} \tag{29.2}$$

These expressions represent the incremental forces acting on a face of the material when the normal directions are defined by the direction cosines, $\cos(n, x)$ and $\cos(n, y)$. For the shear displacement we may write,

$$\begin{aligned} e_{xy} &= -\varpi = \frac{1}{2}\gamma \\ e_{xx} &= e_{yy} = e_{zz} = 0 \\ e_{yz} &= e_{zx} = 0 \end{aligned} \tag{29.3}$$

From the stress-strain relations of 13.39 and 13.40 we may derive,

$$s_{11} = s_{22} = 0$$
$$s_{12} = Q_3 \gamma \quad (29.4)$$

Hence the expression in 29.2 yields,

$$\Delta f_x = 0.\cos(n,x) + \left(Q_3 + \frac{1}{2}S_{22} - \frac{1}{2}S_{11}\right)\gamma\cos(n,y) = \Delta_{xy}\cos(n,y)$$
$$\Delta f_y = \left(Q_3 - \frac{1}{2}S_{11} - \frac{1}{2}S_{22}\right)\gamma\cos(n,x) + 0.\cos(n,y) = \Delta_x^y\cos(n,x) \quad (29.5)$$

$\Delta_{xy} = L_{12}\gamma$ represents the tangential stress applied to the face CD in order to produce the tangential stress γ in which the quantity $L_{12} = Q_3 + \frac{1}{2}S_{22} - \frac{1}{2}S_{11}$ is called the measurable slide modulus. Δ_x^y represents the vertical stress which is applied to the sides AC and BD. This analysis may be repeated for other co-ordinate planes.

Previous Next

Hookean Elastic Material

Let us now consider a Hookean rubbery elastic material, for which the strain energy density function can be written as,

$$W(\lambda_1,\lambda_2)=\frac{\mu_0}{2}\left(\lambda_1^2+\lambda_2^2+\frac{1}{\lambda_1^2\lambda_2}-3\right) \tag{29.6}$$

Then the initial stress relations as in 28.11 are obtained as,

$$\begin{aligned} S_{22}-S_{33}&=\mu_0\left(\lambda_2^2-\lambda_3^2\right)\\ S_{33}-S_{11}&=\mu_0\left(\lambda_3^2-\lambda_1^2\right)\\ S_{11}-S_{22}&=\mu_0\left(\lambda_1^2-\lambda_2^2\right)\end{aligned} \tag{29.7}$$

Similarly the relations for incremental stresses are obtained as,

$$\begin{aligned} s_{22}-s_{33}&=2\mu_0\left(\lambda_2^2e_{yy}-\lambda_3^2e_{zz}\right)\\ s_{33}-s_{11}&=2\mu_0\left(\lambda_3^2e_{zz}-\lambda_1^2e_{xx}\right)\\ s_{11}-s_{22}&=2\mu_0\left(\lambda_1^2e_{xx}-\lambda_2^2e_{yy}\right)\end{aligned} \tag{29.8a}$$

Hence,

$$\begin{aligned} A&=2\mu_0\lambda_1^2\\ B&=2\mu_0\lambda_2^2\\ C&=2\mu_0\lambda_3^2\end{aligned} \tag{29.8b}$$

$$\begin{aligned} 3(s_{11}-s)&=2\mu_0\left(2\lambda_1^2e_{xx}-\lambda_2^2e_{yy}-\lambda_3^2e_{zz}\right)\\ 3(s_{22}-s)&=2\mu_0\left(-\lambda_1^2e_{xx}+2\lambda_2^2e_{yy}-\lambda_3^2e_{zz}\right)\\ 3(s_{33}-s)&=2\mu_0\left(-\lambda_1^2e_{xx}-\lambda_2^2e_{yy}+2\lambda_3^2e_{zz}\right)\end{aligned} \tag{29.8c}$$

Finally we can derive the elastic constants,

$$\begin{aligned} Q_1&=\frac{1}{2}\mu_0\left(\lambda_2^2+\lambda_3^2\right)\\ Q_2&=\frac{1}{2}\mu_0\left(\lambda_3^2+\lambda_1^2\right)\\ Q_3&=\frac{1}{2}\mu_0\left(\lambda_1^2+\lambda_2^2\right)\end{aligned} \tag{29.8d}$$

If the incremental stresses are applied in plane strain, the equations 29.8 get modified to the following:

(29.9)

$$s_{22} = Be_{yy}$$
$$s_{11} = Ae_{xx}$$
$$s_{12} = 2Q_3 e_{xy}$$
$$s = \frac{1}{2}\left(s_{11} + s_{22}\right) = \frac{1}{2}\left(Ae_{xx} + Be_{yy}\right)$$

Which finally results in

$$s_{11} - s = 2\mu e_{xx}$$
$$s_{22} - s = 2\mu e_{yy}$$
$$s_{12} = 2\mu e_{xy} \qquad (29.10)$$
$$\mu = \frac{1}{2}\mu_0\left(\lambda_1^2 + \lambda_2^2\right)$$

Surface instability of a homogeneous half-space

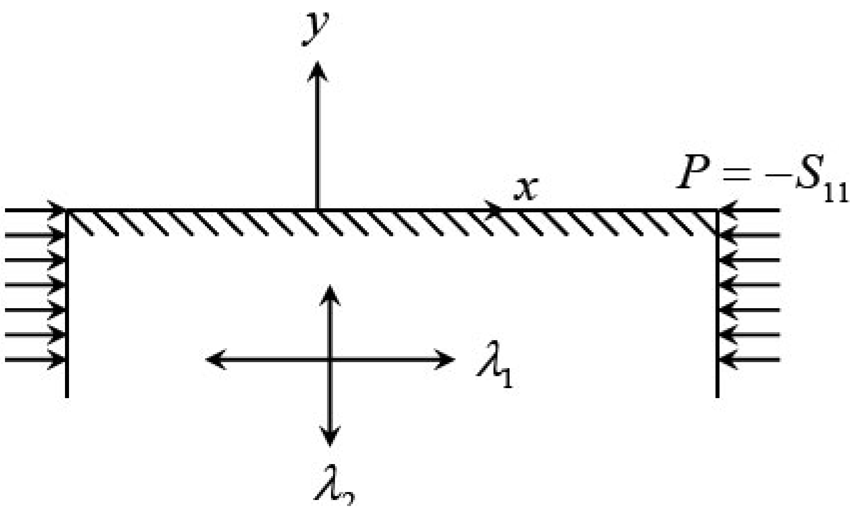

Figure 29.2

Consider a rubber block, an elastic half space, is subjected to a state of initial stress such that it is under compression in x direction but under zero stress along the y direction:

$$\begin{aligned} P &= -S_{11} \\ S_{22} &= 0 \end{aligned} \tag{29.11}$$

Since the material is incompressible, the extension ratios satisfy the following relation:

$$\lambda_1 \lambda_2 \lambda_3 = 1 \tag{29.12}$$

Here λ_1 and λ_2 are extension ratios along x and y directions respectively. And λ_3 is the extension ratio along the direction normal to the xy plane. Then the stress-strain relations in equation 29.7 yield:

$$\begin{aligned} -S_{33} &= \mu_0\left(\lambda_2^2 - \lambda_3^2\right) \\ S_{33} + P &= \mu_0\left(\lambda_3^2 - \lambda_1^2\right) \\ -P &= \mu_0\left(\lambda_1^2 - \lambda_2^2\right) \end{aligned} \tag{29.13}$$

Where, μ_0 is the shear modulus of the rubber block in unstressed state. Under this initial stress condition the block undergoes incremental deformation in the xy plane. Then the incremental stresses and strain can be represented by equations 29.10 . In addition, we have the incompressibility condition:

$$e_{xx} + e_{yy} = 0 \tag{29.14}$$

The equilibrium relations for incremental stresses can be written as,

$$\frac{\partial s_{11}}{\partial x}+\frac{\partial s_{12}}{\partial y}-P\frac{\partial \varpi}{\partial y}=0$$
$$\frac{\partial s_{12}}{\partial x}+\frac{\partial s_{22}}{\partial y}-P\frac{\partial \varpi}{\partial x}=0 \qquad (29.15)$$

Notice that equation 29.14 can be satisfied by putting the following substitutions:

$$u=-\frac{\partial \phi}{\partial y}, \qquad v=\frac{\partial \phi}{\partial x} \qquad (29.16)$$

Where $\phi(x,y)$ is a stress function which is deduced by solving the stress equilibrium relations subjected to relevant boundary conditions. We can write the stress components s_{ij} in equation 29.10 in terms of that presented in 29.15, which yield,

$$\frac{\partial s}{\partial x}-\left(\mu+\frac{P}{2}\right)\frac{\partial}{\partial y}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)=0$$
$$\frac{\partial s}{\partial y}+\left(\mu-\frac{P}{2}\right)\frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)=0 \qquad (29.17)$$

Previous Next

Further eliminating the mean incremental stress s in equation 29.16, we obtain the following equation in terms of

$$\left(\mu-\frac{P}{2}\right)\frac{\partial^4\phi}{\partial x^4}+2\mu\frac{\partial^4\phi}{\partial x^2\partial y^2}+\left(\mu+\frac{P}{2}\right)\frac{\partial^4\phi}{\partial y^4}=0 \qquad (29.18)$$

After algebraic manipulation of equation 29.18, we obtain

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left[\left(\mu-\frac{P}{2}\right)\frac{\partial^2\phi}{\partial x^2}+\left(\mu+\frac{P}{2}\right)\frac{\partial^2\phi}{\partial y^2}\right]=0 \qquad (29.19)$$

Notice that the perturbation solutions can be considered sinusoidal along the x direction which vanishes at infinite depth $y=-\infty$. Such a solution can be represented by the following functional forms:

$$\begin{aligned}\phi&=\frac{1}{l^2}\left(c_1e^{ly}+c_2e^{-kly}\right)\sin(lx)\\ s&=c_2Pke^{kly}\cos(lx)\end{aligned} \qquad (29.20)$$

In which, $2\pi/l$ defines the wavelength of the instability. The other parameters can be related as

$$\begin{aligned}k&=\sqrt{\frac{1-\zeta}{1+\zeta}}=\frac{\lambda_1}{\lambda_2}\\ \zeta&=\frac{P}{2\mu}=\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2+\lambda_1^2}\end{aligned} \qquad (29.21)$$

Since, $0<\zeta<1$, k is real and satisfies the inequality $0<k<1$.

We will now consider the boundary conditions at the surface of the block, which from equation 28.1(b) can be written as

$$\begin{aligned}\Delta f_x&=s_{12}+Pe_{xy}=(2Q+P)e_{xy}\\ \Delta f_y&=s_{22}=s+2\mu e_{yy}\end{aligned} \qquad (29.22)$$

The condition that no incremental force is applied at the surface demands that at $y=0$,

$$\Delta f_x=\Delta f_y=0 \qquad (29.23)$$

Substituting the solutions ϕ and s in equation 29.22 and eliminating for constants we obtain the characteristic equation,

$$(1+\zeta)^2k-1=0 \qquad (29.24)$$

Substituting the expression for k from 29.21 into 29.24 yields

(29.25)

$$\zeta^3 + 2\zeta^2 - 2 = 0$$

Which has one real root, $\zeta = 0.839$ at which the surface turns unstable. From equation 29.21 , the value of k at which instability occurs is obtained as $k = 0.295$. Notice that the characteristic equation is independent of l, which implies that surface can become unstable for all wavelengths.

We can further calculate the initial extension ratios at which the surface becomes unstable, by putting the following,

$$\lambda_3 = 1$$
$$\lambda_2 = \frac{1}{\lambda_1} \qquad (29.26)$$

Which yields the extension ratio corresponding to instability as $\lambda_1 = 0.544$ and the compressive stress as $P = 3.08\mu_0$.

Previous Next

Previous Next