**Course Name:** Mechanics of Soft Materials

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Module 1 : Brief Introduction

Lecture 1 : Displacement

## The Lecture Contains:



This lecture is adopted from the following book


1. "Some Basic Problems of the Mathematical Theory of Elasticity" by N.I.Muskhelishvili
2. "A Treatise on the Mathematical theory of Elasticity" by A.E.H. Love

## Introduction:

Soft solid materials are the ones which have modulus upto ~10 MPa; they are elastic or viscoelastic and they deform easily when subjected to external forces. Engineering materials, such as rubbers and thermoplastic elastomers, and soft biological tissues, such as skin, cartilage, liver and brain tissue, fall into this category. Understanding the mechanical response of these materials is important in many engineering and biomedical applications. The goal of this course is to expose the students and researchers of these diverse research interests to the principles of mechanics, its rich mathematical structure and how these tools can be useful for analyzing variety of problems related to soft deformable materials.

## Displacement:

Displacement occurs when particles in a body moves from initial state to a final state.

If the length of a line joining the two particles remains unaltered in the initial and the final state, then the displacement is called rigid body displacement.

If the displacement alters this length then the final state of the body is said to be in "strained state" and the initial state is called the "unstrained state".

Let $x, y, z$ be the location of a point occupied by a particle which in the strained state occupies a location: $x+u, y+v, z+w$, then $u, v, w$ are the projections of displacements of the particle. In simple uni-axial extension along the $x$ axis, displacement of a particle is given by $u = ex, v = 0, w = 0$ , where $e$ is the extension.

In simple shear along the $x$ axis, the planes parallel to the $x$ axis slide past each other so that particles in plane parallel to $x, y$ remain in that plane. The displacement is then given by $u = sy, v = 0, w = 0$ where $s = 2\tan\alpha, v = 0, w = 0$ .

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## Homogeneous Strain

If the components of displacements are linear function of coordinates, the strain is called homogeneous strain.

## Analysis of Strain:

Deformation in a continuous body is defined as the change in the positions of the points in the body so that their relative distances are altered.

*Say* a point at ( *x*,*y*,*z* ) in the undeformed / unstarined state of the body moves to a unique location ( $x^*, y^*, z^*$). Then the co-ordinates $x^*, y^*, z^*$ be a definite function of the coordinates $x, y, z$ of the same point before deformation:

$$x^* = f_1(x,y,z), y^* = f_2(x,y,z), z^* = f_3(x,y,z) \quad (1.1)$$

where functions $f_1, f_2, f_3$ are assumed to be continuous in the region occupied by the body. Similarly, the coordinates $x, y, z$ are also continuous functions of $x^*, y^*, z^*$.

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## Affine Transformation

The transformations of this form are called affine if the coordinates $x^*, y^*, z^*$ are linear functions of the coordinates $x, y, z$

$$\begin{aligned} x^* &= (1+a_{11})x + a_{12}y + a_{13}z + a \\ y^* &= a_{21}x + (1+a_{22})y + a_{23}z + b \\ z^* &= a_{31}x + a_{32}y + (1+a_{33})z + c \end{aligned} \quad (1.2)$$

where $a_{11}\ldots$ are constants. Above equations have nontrivial solution for $x, y, z$, so that the determinant

$$D = \begin{vmatrix} 1+a_{11} & a_{12} & a_{13} \\ a_{21} & 1+a_{22} & a_{23} \\ a_{31} & a_{32} & 1+a_{33} \end{vmatrix} \neq 0 \quad (1.3)$$

### Properties:

(a) Inverse affine transformation is also affine, because by solving 1.2 one obtains $x, y, z$ in terms of $x^*, y^*, z^*$:

$$\begin{aligned} x &= (1+b_{11})x^* + b_{12}y^* + b_{13}z^* \\ y &= b_{21}x^* + (1+b_{22})y^* + b_{23}z^* \\ z &= b_{31}x^* + b_{32}y^* + (1+b_{33})z^* \end{aligned} \quad (1.4)$$

(b) Points lying on a plane $\prod$ before transformation lies on a different plane $\prod^*$.

Say $Ax + By + cz + D = 0$ is the equation of plane $\prod$, then after the transformation they lie on a different plane $A^*x^* + B^*y^* + c^*z^* + D^* = 0$, which is the equation of plane $\prod^*$.

(c) Points lying on a straight line, which is the intersection of two different planes $\prod_1$ and $\prod_2$, will now lie again on a straight line, which is basically the intersection of two planes $\prod_1^*$ and $\prod_2^*$.
Hence, any straight segment is transformed into a straight segment and any vector to a vector.

## Affine Transformation (contd...)

Let $\vec{P} = (\xi, \psi, \zeta)$ be the vector which transforms to a vector $\vec{P^*} = (\xi^*, \psi^*, \zeta^*)$.

If $x_0, y_0, z_0$ and $x, y, z$ be the end points of vector $\vec{P} = (\xi, \psi, \zeta)$, then

$$\xi = x - x_0,\ \ \psi = y = y_0,\ \ \zeta = z - z_0 \tag{1.5}$$

Similarly

$$\xi^* = x^* - x_0^*,\ \ \psi^* = y^* - y_0^*,\ \ \zeta^* = z^* - z_0^* \tag{1.6}$$

Where $x^* = (1 + a_{11})x + a_{12}y + a_{13}z + a$ and $x_0^* = (1 + a_{11})x_0 + a_{12}y_0 + a_{13}z_0 + a$, etc.

Subtracting one obtains

$$\begin{aligned} \xi^* &= (1 + a_{11})\xi + a_{12}\psi + a_{13}\zeta \\ \psi^* &= a_{21}\xi + (1 + a_{22})\psi + a_{23}\zeta \\ \zeta^* &= a_{31}\xi + a_{32}\psi + (1 + a_{33})\zeta \end{aligned} \tag{1.7}$$

It is easy to see that $(1 + a_{11}), a_{12}, \ldots$ or in the short form $a_{ij} + \delta_{ij}$ are components of a tensor and since $\delta_{ij}$ is a tensor, then $a_{ij}$ is a tensor.

Note that the components $a_{ij}, a, b, c$ are constants.

From equation 1.7, we can conclude that two equal vectors remain equal after transformation and two parallel vectors remain parallel after transformation with the ratio of their lengths remaining unaltered due to the transformation.

Hence, it follows that two identical and identically oriented polygons remain identical and identically oriented after transformation. Since every geometric figure is some form of polygon, it follows that all parts of a body, independent of their deformations will deform in an identical manner. That is called homogeneous transformation.

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## Infinitesimal affine transformation

A transformation in equation 1.2 is called infinitesimal if $a_{ij}, a, b, c$ are small quantities so that their squares and products can be neglected.

As a result, the quantities: $x^* - x = a_{11}x + a_{12}y + a_{13}z + a$, $y^* - y = a_{21}x + a_{22}y + a_{23}z + b$ and $z^* - z = a_{31}x + a_{32}y + a_{33}z + c$, which are the differences of the coordinates of the same point before and after transformation are also infinitesimal.

It can be shown that result of two infinitesimal affine transformations is also affine.

## Pure deformation and rigid body motion

Point to be noted here is that if the quantities $a_{ij}$ are given, the transformation of the coordinates of a point will not be completely defined, because, $a, b, c$ will still remain undetermined. But these quantities do not influence the deformations because they define only the rigid translation of the body.

Let us now define the quantities $\delta\xi = \xi^* - \xi,\ \delta\psi = \psi^* - \psi,\ \delta\zeta = \zeta^* - \zeta$ which are components of vector $\delta\vec{P} = \overrightarrow{P^*} - \vec{P}$, or the displacement vector. Then

$$\begin{aligned} \delta\xi &= a_{11}\xi + a_{12}\psi + a_{13}\zeta \\ \delta\psi &= a_{21}\xi + a_{22}\psi + a_{23}\zeta \\ \delta\zeta &= a_{31}\xi + a_{32}\psi + a_{33}\zeta \end{aligned} \tag{1.8}$$

We want to know the constraints on the tensor $a_{ij}$ that will result in rigid body motion and no deformation.

Basically we want to find out the condition that the magnitude of the vector $\vec{P}$ or the square of its modulus $P^2 = \xi^2 + \psi^2 + \zeta^2$ remains unaltered due to the transformation. Restricting ourselves to infinitesimal transformation, that is neglecting higher order terms,

$$\begin{aligned} P\delta P = \xi\delta\xi + \psi\delta\psi + \zeta\delta\zeta = a_{11}\xi^2 + a_{22}\psi^2 + a_{33}\zeta^2 + \\ (a_{12}+a_{21})\xi\psi + (a_{13}+a_{31})\xi\zeta + (a_{23}+a_{32})\psi\zeta \end{aligned} \tag{1.9}$$

In order that $\delta P = \mathbf{0}$, it is imperative that

$$a_{11} = a_{22} = a_{33} = 0 \quad \text{and} \quad (a_{12}+a_{21}) = (a_{13}+a_{31}) = (a_{23}+a_{32}) = 0 \tag{1.10}$$

which implies that $a_{ij} = -a_{ji}$. Then equation 1.8 can be written as,

$$\delta\xi = q\zeta - r\psi,\ \delta\psi = r\xi - p\zeta,\ \delta\zeta = p\psi - q\xi \tag{1.11}$$

where $p = a_{32} = -a_{23},\ q = a_{13} = -a_{31},\ r = a_{21} = -a_{12}$. These quantities are infinitesimal angles of rotation about the coordinate axes and are called the components of rotation.

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## Pure deformation and rigid body motion (Contd...)

We can introduce following notations:

$$a_{11} = e_{xx},\ a_{22} = e_{yy},\ a_{33} = e_{zz},$$
$$\frac{1}{2}(a_{32}+a_{23}) = e_{yz} = e_{zy},\ \frac{1}{2}(a_{13}+a_{31}) = e_{xz} = e_{zx},\ \frac{1}{2}(a_{12}+a_{21}) = e_{xy} = e_{yx} \tag{1.12}$$

$e_{ij}$ are called the components of strain.

Furthermore we can introduce the notations:

$$p = \frac{1}{2}(a_{32}-a_{23}),\quad q = \frac{1}{2}(a_{13}-a_{31}),\quad r = \frac{1}{2}(a_{21}-a_{12}) \tag{1.13}$$

Hence we can divide the tensor $a_{ij}$ in following symmetric and anti-symmetric parts:

$$\begin{aligned} a_{32} &= e_{yz}+p,\ a_{13} = e_{zx}+q,\ a_{21} = e_{xy}+r \\ a_{23} &= e_{yz}-p,\ a_{31} = e_{zx}-q,\ a_{12} = e_{xy}-r \end{aligned} \tag{1.14}$$

Using these definitions of $a_{ij}$ in equation 1.8, we have:

$$\begin{aligned} \delta\xi &= e_{xx}\xi + e_{xy}\psi + e_{xz}\zeta + q\zeta - r\psi \\ \delta\psi &= e_{yx}\xi + e_{yy}\psi + e_{yz}\zeta + r\xi - p\zeta \\ \delta\zeta &= e_{zx}\xi + e_{zy}\psi + e_{zz}\zeta + p\psi - q\xi \end{aligned} \tag{1.15}$$

So the original affine transformation can be divided into two parts: Symmetric and Anti-Symmetric.

$$\begin{matrix} e_{xx}\xi + e_{xy}\psi + e_{xz}\zeta & \quad & q\zeta - r\psi \\ e_{yx}\xi + e_{yy}\psi + e_{yz}\zeta & \quad & r\xi - p\zeta \\ e_{zx}\xi + e_{zy}\psi + e_{zz}\zeta & \quad & p\psi - q\xi \end{matrix} \tag{1.16}$$

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## Geometric meaning of components of strain

Let us go back to equation 1.9, under new notations it has the form:

$$P\delta P = e_{xx}\xi^2 + e_{yy}\psi^2 + e_{zz}\zeta^2 + 2e_{xy}\xi\psi + 2e_{xz}\xi\zeta + 2e_{yz}\psi\zeta \tag{1.17}$$

Consider a vector $\vec{P}(\xi,0,0)$ which is parallel to the Ox axis.

For this vector: $P\delta P = e_{xx}\xi^2$ which results in $e_{xx} = \dfrac{\delta P}{P}$, i.e. $e_{xx}$ denotes the relative increase in length of the vectors.

Consider two vectors $\vec{P_1}(0,\psi_1,0)$ and $\vec{P_2}(0,0,\zeta_2)$ which are initially directed towards $y$ and $z$, after deformation turn to vectors $\vec{P_1^*}(\delta\xi_1,\psi_1+\delta\psi_1,\delta\zeta_1)$ and $\vec{P_2^*}(\delta\xi_2,\delta\psi_2,\zeta_2+\delta\zeta_2)$, the angle $\left(\dfrac{\pi}{2}-\varepsilon_{yz}\right)$ between which is written as:

$$\cos\left(\frac{\pi}{2}-\varepsilon_{yz}\right) = \frac{\delta\xi_1\delta\xi_2 + (\psi_1+\delta\psi_1)\delta\psi_2 + \delta\zeta_1(\zeta_2+\delta\zeta_2)}{\sqrt{\delta\xi_1^2 + (\psi_1+\delta\psi_1)^2 + \delta\zeta_1^2}\sqrt{\delta\xi_2^2 + \delta\psi_2^2 + (\zeta_2+\delta\zeta_2)^2}} \tag{1.18}$$

For a small $\varepsilon_{yz}$,

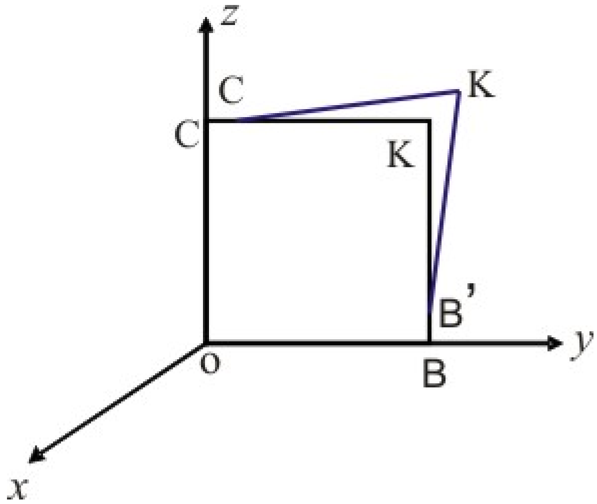
$$\cos\left(\frac{\pi}{2}-\varepsilon_{yz}\right) = \varepsilon_{yz} = \frac{\psi_1\delta\psi_2 + \delta\zeta_1\zeta_2}{\psi_1\zeta_2} = \frac{\delta\psi_2}{\zeta_2} + \frac{\delta\zeta_1}{\psi_1} \tag{1.19}$$

But for $\vec{P_1}(0,\psi_1,0)$ and $\vec{P_2}(0,0,\zeta_2)$ we have from 1.15:

$$\begin{aligned}\delta\psi_2 &= e_{yz}\zeta_2 - p\zeta_2\\ \delta\zeta_1 &= e_{zy}\psi_1 + p\psi_1\end{aligned} \tag{1.20}$$

Introducing this formula into equation 1.19, one obtains $\varepsilon_{yz} = e_{yz} + e_{zy} = \mathbf{2}e_{yz}$.

The angle $\mathbf{2}e_{yz}$ represents the decrease in angle between the two vectors originally in the directions O$y$ and O$z$ .




$$\angle BOB' = \angle COC' = e_{yz}$$

$$\varepsilon_{yz} = \angle BOB' + \angle COC' = 2e_{yz}$$

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## General Expressions of strain:

Let the new position of the point is given in terms of the deformations as $x^{*} = x+u, \quad y^{*} = y+v, \quad z^{*} = z+w$ in the strained state.

Consequently, the particle which was on a given curve in the unstarined state, now belongs to a different curve in the strained state. If $ds$ be a differential element in the original curve, then direction cosines of a tangent at any point on it are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$. Direction cosines of a tangent on an elemental arc $ds_1$ in the strained curve are $\frac{d(x+u)}{ds_1}, \frac{d(y+v)}{ds_1}, \frac{d(z+w)}{ds_1}$. Then,

$$\begin{aligned}
\frac{d(x+u)}{ds_1} &= \frac{ds}{ds_1}\left(\frac{dx}{ds} + \frac{du}{dx}\frac{dx}{ds} + \frac{du}{dy}\frac{dy}{ds} + \frac{du}{dz}\frac{dz}{ds}\right) \\
\frac{d(y+v)}{ds_1} &= \frac{ds}{ds_1}\left(\frac{dy}{ds} + \frac{dv}{dx}\frac{dx}{ds} + \frac{dv}{dy}\frac{dy}{ds} + \frac{dv}{dz}\frac{dz}{ds}\right) \\
\frac{d(x+w)}{ds_1} &= \frac{ds}{ds_1}\left(\frac{dx}{ds} + \frac{dw}{dx}\frac{dx}{ds} + \frac{dw}{dy}\frac{dy}{ds} + \frac{dw}{dz}\frac{dz}{ds}\right)
\end{aligned} \qquad (1.21)$$

However, the direction cosines are

$$\begin{aligned}
l &= \frac{dx}{ds}, & m &= \frac{dy}{ds}, & n &= \frac{dz}{ds} \\
l_1 &= \frac{d(x+u)}{ds_1}, & m_1 &= \frac{d(y+v)}{ds_1}, & n_1 &= \frac{d(z+w)}{ds_1}
\end{aligned} \qquad (1.22)$$

From the expressions of equation 1,

$$\begin{aligned}
l_1 &= \frac{ds}{ds_1}\left(l\left(1+\frac{du}{dx}\right) + m\frac{du}{dy} + n\frac{du}{dz}\right) \\
m_1 &= \frac{ds}{ds_1}\left(l\frac{dv}{dx} + m\left(1+\frac{dv}{dy}\right) + n\frac{dv}{dz}\right) \\
l_1 &= \frac{ds}{ds_1}\left(l\frac{dw}{dx} + m\frac{dw}{dy} + n\left(1+\frac{dw}{dz}\right)\right)
\end{aligned} \qquad (1.23)$$

Noting that

$$l^2 + m^2 + n^2 = 1, \qquad l_1^2 + m_1^2 + n_1^2 = 1$$

we have the following eqn,

$$\left(\frac{ds_1}{ds}\right)^2 = \left(1+2\varepsilon_{xx}\right)l^2 + \left(1+2\varepsilon_{yy}\right)m^2 + \left(1+2\varepsilon_{zz}\right)n^2 + 2\varepsilon_{yx}mn + 2\varepsilon_{zx}nl + 2\varepsilon xylm \quad (1.24)$$

where $\varepsilon_{xx}\ldots$ are the following,

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} \\
\varepsilon_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z} \\
\varepsilon_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \\
\varepsilon_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}
\end{aligned} \quad (1.25)$$

We thus obtain the general expressions for the components of strain in terns of the gradients of displacements.

The left hand side of 1.24 is a constant and the right hand side signifies an ellipsoid. It has the property that in any direction, the length of its radius is inversely proportional to $\frac{ds_1}{ds}$. Such an ellipsoid is called the reciprocal strain ellipsoid .

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## Transformation of components of strain:

Consider 1.17, here $P\delta P$ is a quadratic with respect to $\xi, \psi, \zeta$. But since $P\delta P$ has a definite meaning, it should be independent of the choice of coordinate axes or of the transformation of coordinates.

In other word, if $e_{x'x'}, \ldots, e_{x'y'}, \ldots$ are the components of strain in the new coordinate system and if $\xi', \psi', \zeta'$ are the component of the vector $\overline{P}$, then we have:

$$\begin{aligned} P\delta P &= e_{xx}\xi^2 + e_{yy}\psi^2 + e_{zz}\zeta^2 + 2e_{xy}\xi\psi + 2e_{xz}\xi\zeta + 2e_{yz}\psi\zeta \\ &= e_{x'x'}\xi'^2 + e_{y'y'}\psi'^2 + e_{z'z'}\zeta'^2 + 2e_{x'y'}\xi'\psi' + 2e_{x'z'}\xi'\zeta' + 2e_{y'z'}\psi'\zeta' \end{aligned} \tag{1.26}$$

Now let's say $l_1, m_1, n_1$ are the direction cosines of the $x'$ axis new coordinate system with respect to the old one and like wise. We can then develop following table:

$$\begin{array}{cccc} & x & y & z \\ x' & l_1 & m_1 & n_1 \\ y' & l_2 & m_2 & n_2 \\ z' & l_3 & m_3 & n_3 \end{array} \tag{1.27}$$

The coordinate of the vector $\overline{P}$ in the old system can be expressed with respect to the new one as,

$$\begin{aligned} \xi &= \xi' l_1 + \psi' m_1 + \zeta' n_1 \\ \psi &= \xi' l_2 + \psi' m_2 + \zeta' n_2 \\ \zeta &= \xi' l_3 + \psi' m_3 + \zeta' n_3 \end{aligned} \tag{1.28}$$

Substituting this into 1.26 and then matching the coefficients for $\xi'^2, \ldots, \xi'\psi', \ldots$ in the left and right hand sides,

$$\begin{aligned} e_{x'x'} &= e_{xx}l_1^2 + e_{yy}m_1^2 + e_{zz}n_1^2 + e_{yz}m_1n_1 + e_{zx}n_1l_1 + e_{xy}l_1m_1 \\ &\ldots \\ e_{y'z'} &= 2e_{xx}l_2l_3 + 2e_{yy}m_2m_3 + 2e_{zz}n_2n_3 + e_{yz}\left(m_2n_3 + m_3n_3\right) + \\ &\qquad e_{zx}\left(n_2l_3 + n_3l_2\right) + e_{xy}\left(l_2m_3 + l_3m_2\right) \\ &\ldots \end{aligned} \tag{1.29}$$

Furthermore we can rewrite 1.17 as,

$$P^2 e = e_{xx}\xi^2 + e_{yy}\psi^2 + e_{zz}\zeta^2 + 2e_{xy}\xi\psi + 2e_{xz}\xi\zeta + 2e_{yz}\psi\zeta \tag{1.30}$$

where $e = \frac{\delta P}{P}$ is the relative increase in length of vector $\vec{P} = (\xi, \psi, \zeta)$. Since $P^2 e$ does not depend on any direction but only its magnitude, hence for every direction, $P^2 e = \pm c^2$, where C is an arbitrary constant with the dimension of length. If the starting point of P lies on the origin, then its other end point lies on the surface $P^2 e = \pm c^2$, or

$$e_{xx}\xi^2 + e_{yy}\psi^2 + e_{zz}\zeta^2 + 2e_{xy}\xi\psi + 2e_{xz}\xi\zeta + 2e_{yz}\psi\zeta = c^2 \quad (1.31)$$

which is called the strain surface.

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## Transformation of components of strain (contd...)

If the axes are so chosen that they coincide with the principal axes of the surface then, 1.32 takes the form: $e_1\xi^2 + e_2\psi^2 + e_3\zeta^2 = c^2$, where $e_1$, $e_2$, $e_3$ are the $e_{xx}$, $e_{yy}$, $e_{zz}$ of the new axes.

Principal axes are the roots of the equation:

$$\begin{vmatrix} e_{xx} - e & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} - e & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} - e \end{vmatrix} = 0 \qquad (1.32)$$

$$\Rightarrow -e^3 + \theta e^2 + be + c = 0$$

where $\theta = e_{xx} + e_{yy} + e_{zz} = e_1 + e_2 + e_3$ is an invariant, since coefficients of 1.32 should be so.

Consider a right parallelepiped having sides OA= $l_1$, OB = $l_2$ and OC= $l_3$. Then its volume is $V = l_1 l_2 l_3$. After deformation the right parallelepiped remains a right parallelepiped with sides $l_1(\mathbf{1} + e_1)$, $l_2(\mathbf{1} + e_2)$, $l_3(1 + e_3)$, hence its volume is

$$V' = l_1 l_2 l_3 (1 + e_1)(1 + e_2)(1 + e_3) = V(1 + e_1)(1 + e_2)(1 + e_3) = V(1 + e_1 + e_2 + e_3) \qquad (1.33)$$

So that,

$$\frac{V' - V}{V} = e_1 + e_2 + e_3 \qquad (1.34)$$

Hence $\theta$ is the relative expansion in volume or the cubical expansion.

In a more general form, the ratio of a volume element in strained state to unstrained state is given as

$$1 + \Delta = \begin{vmatrix} 1 + \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & 1 + \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & 1 + \frac{\partial w}{\partial z} \end{vmatrix} \qquad (1.35)$$

where $\Delta$ is the cubical dilation.

## Saint-Venant's conditions of compatibility :

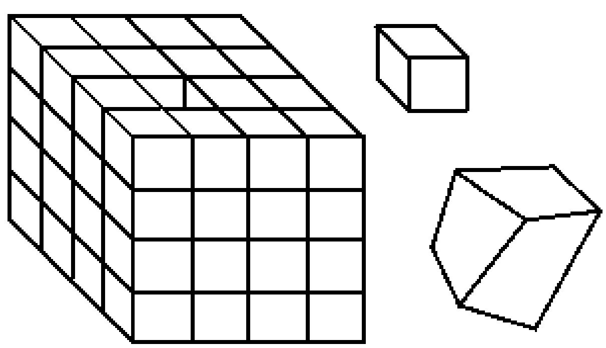
We have so far deduced the formula for components of strain to be calculated from the components of displacements which are functions of the current location $(x, y, z)$ of a point.

Let us now look at the inverse problem: say, the components of strain $e_{xx}$ ,...,$e_{yy}$,... in a body are known, then how to determine the components of displacements $u$,$v$,$w$.

This problem can not be solved in a straight forward manner, because, say somehow we are able to find out the components $u, v, w$ , but then if we add an arbitrary displacement to them, then, it does not change the strain components, because strain does not depend upon the rigid body translation.

Consider the following situation: Let's say we separate an infinitesimal cube from a body and subject it to deformations resulting in a parallelepiped. We can subject many other cubes to such deformations. Now we put them all back to the body.

Question is will the adjoining faces and lines will perfectly match without any gaps, after this exercise. It will almost be impossible to do that. Hence, it is obvious now that components of strain must satisfy certain conditions in order to result in deformations without discontinuities.



$$e_{xx} = \frac{\partial u}{\partial x},\ e_{yy} = \frac{\partial v}{\partial y},\ e_{zz} = \frac{\partial w}{\partial z},$$
$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ e_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),\ e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad (1.36)$$

$$r = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \quad q = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \text{ and } \quad p = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

where $e_{xx},\ldots,e_{xy},\ldots$ are single valued functions having continuous second order derivatives.

One has six equations and three unknown functions, which implies that the problem may not have solutions if $e_{xx},\ldots,e_{xy},\ldots$ are not subjected to additional conditions.

## Saint-Venant's conditions of compatibility (contd...):

Let $V$ be simply connected region occupied by a body. In this domain $(x,y,z)$ is the current location of a point at which $e_{xx},\ldots,e_{xy},\ldots$ are given and we need to find out the deformations.

Let $M_0(x_0,y_0,z_0)$ be any point in $V$ and $u_0,v_0,w_0$ are the components of displacement at $M_0$ and $p_0,q_0,r_0$ are the components of rotation.

Let $M_1(x,y,z)$ be any other point at $V$ at which we want to find out the components of displacements. Let $M_0M_1$ be a line that joins $M_0$ and $M_1$ and lies in $V$. Then if the partial derivatives $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$ are known, then the displacements $u_1,v_1,w_1$ can be found out as,

$$u_1 = u_0 + \int_{M_0M_1}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right)$$

where $$\frac{\partial u}{\partial x} = e_{xx}, \frac{\partial u}{\partial y} = e_{xy} - r, \frac{\partial u}{\partial z} = e_{xz} + q \tag{1.37}$$

Hence, $$u_1 = u_0 + \int_{M_0M_1}\left(e_{xx}dx + e_{xy}dy + e_{xz}dz\right) + \int_{M_0M_1}\left(qdz - rdy\right)$$

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## Saint-Venant's conditions of compatibility (contd...):

Let us focus on the second integral:

By partial integration:

$$\int_{M_0M_1}(q dz - r dy) = \int_{M_0M_1}\left(r d(y_1-y) - q d(z_1-z)\right) =$$
$$r(y_1-y)\Big|_{y_0}^{y_1} - \int_{M_0M_1}(y_1-y)dr - q(z_1-z)\Big|_{z_0}^{z_1} + \int_{M_0M_1}(z_1-z)dq \qquad (1.38)$$
$$= q_0(z_1-z_0) - r_0(y_1-y_0) - \int_{M_0M_1}\left((y_1-y)dr - (z_1-z)dq\right)$$

Now $r = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$, so that,

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial y \partial x}\right) = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial y \partial x}\right) = \frac{\partial e_{xy}}{\partial x} - \frac{\partial e_{xx}}{\partial y}$$
$$\frac{\partial r}{\partial y} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2}\right) = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial y \partial x}\right) = \frac{\partial e_{yy}}{\partial x} - \frac{\partial e_{xy}}{\partial y} \qquad (1.39a)$$
$$\frac{\partial r}{\partial z} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 u}{\partial y \partial z}\right) = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial y \partial x} - \frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 w}{\partial y \partial x}\right) = \frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{xz}}{\partial y}$$

And $q = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$, so that

$$\frac{\partial q}{\partial x} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 w}{\partial x^2}\right) = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 u}{\partial z \partial x} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 u}{\partial z \partial x}\right) = \frac{\partial e_{xx}}{\partial z} - \frac{\partial e_{xz}}{\partial x}$$
$$\frac{\partial q}{\partial y} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial z \partial y} - \frac{\partial^2 w}{\partial y \partial x}\right) = \frac{1}{2}\left(\frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 v}{\partial z \partial x} - \frac{\partial^2 w}{\partial y \partial x} - \frac{\partial^2 v}{\partial z \partial x}\right) = \frac{\partial e_{xy}}{\partial z} - \frac{\partial e_{zy}}{\partial x} \qquad (1.39b)$$
$$\frac{\partial q}{\partial z} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial x \partial z}\right) = \frac{1}{2}\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial z \partial x} - \frac{\partial^2 w}{\partial x \partial z} - \frac{\partial^2 w}{\partial x \partial z}\right) = \frac{\partial e_{xz}}{\partial z} - \frac{\partial e_{zz}}{\partial x}$$

Now substituting these expressions into

$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy + \frac{\partial r}{\partial z}dz$ and $dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy + \frac{\partial q}{\partial z}dz$, and equation 1.37 we obtain

$$u(x_1,y_1,z_1)=u_0+q_0(z_1-z_0)-r_0(y_1-y_0)+\int_{M_0M_1}(U_x dx+U_y dy+U_z dz)$$

$$v(x_1,y_1,z_1)=v_0+r_0(x_1-x_0)-p_0(z_1-z_0)+\int_{M_0M_1}(V_x dx+V_y dy+V_z dz) \quad (1.40)$$

$$w(x_1,y_1,z_1)=w_0+p_0(y_1-y_0)-q_0(x_1-x_0)+\int_{M_0M_1}(W_x dx+W_y dy+W_z dz)$$

Where,

$$U_x=e_{xx}+(y_1-y)\left(\frac{\partial e_{xx}}{\partial y}-\frac{\partial e_{xy}}{\partial x}\right)+(z_1-z)\left(\frac{\partial e_{xx}}{\partial z}-\frac{\partial e_{xz}}{\partial x}\right)$$

$$U_y=e_{xy}+(y_1-y)\left(\frac{\partial e_{xy}}{\partial y}-\frac{\partial e_{yy}}{\partial x}\right)+(z_1-z)\left(\frac{\partial e_{xy}}{\partial z}-\frac{\partial e_{yz}}{\partial x}\right) \quad (1.41)$$

$etc.$

Since $u_0,v_0,w_0$ and $p_0,q_0,r_0$ are known at $M_0(x_0,y_0,z_0)$, then, $u_1,v_1,w_1$ can be found out using 1.40.

However, it is imperative that $u_1,v_1,w_1$ be functions only of $(x_1,y_1,z_1)$ and do not depend upon the path of integration.

The necessary and sufficient condition that the integral be path independent is that,

$$\int_{M_0M_1}(U_x dx+U_y dy+U_z dz)=-\int_{M_1M_0}(U_x dx+U_y dy+U_z dz)$$

Or, by Green's theorem for integration over a closed curve,

$$\int_C(U_x dx+U_y dy+U_z dz)=\iint\left(\frac{\partial U_y}{\partial x}-\frac{\partial U_x}{\partial y}\right)dxdy+\iint\left(\frac{\partial U_z}{\partial y}-\frac{\partial U_y}{\partial z}\right)dydz$$
$$+\iint\left(\frac{\partial U_z}{\partial x}-\frac{\partial U_x}{\partial z}\right)dxdz=0$$

$$\frac{\partial U_z}{\partial y}=\frac{\partial U_y}{\partial z},\quad \frac{\partial U_x}{\partial z}=\frac{\partial U_z}{\partial x},\quad \frac{\partial U_y}{\partial x}=\frac{\partial U_x}{\partial y} \quad (1.42)$$

Analogous conditions are obtained for other equations. These equations reduce to the following six equations,

(1.43)

$$\frac{\partial^2 e_{yy}}{\partial z^2}+\frac{\partial^2 e_{zz}}{\partial y^2}=2\frac{\partial^2 e_{yz}}{\partial y\partial z},\quad \frac{\partial^2 e_{xx}}{\partial y\partial z}+\frac{\partial^2 e_{yz}}{\partial x^2}=\frac{\partial^2 e_{zx}}{\partial x\partial y}+\frac{\partial^2 e_{xy}}{\partial x\partial z}$$

$$\frac{\partial^2 e_{zz}}{\partial x^2}+\frac{\partial^2 e_{xx}}{\partial z^2}=2\frac{\partial^2 e_{zx}}{\partial z\partial x},\quad \frac{\partial^2 e_{yy}}{\partial z\partial x}+\frac{\partial^2 e_{zx}}{\partial y^2}=\frac{\partial^2 e_{xy}}{\partial y\partial z}+\frac{\partial^2 e_{yz}}{\partial y\partial x}$$

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=2\frac{\partial^2 e_{xy}}{\partial x\partial y},\quad \frac{\partial^2 e_{zz}}{\partial x\partial y}+\frac{\partial^2 e_{xy}}{\partial z^2}=\frac{\partial^2 e_{yz}}{\partial z\partial x}+\frac{\partial^2 e_{zx}}{\partial z\partial y}$$

These equations are called the conditions of compatibility of Barré de Saint-Venant.

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