Module 3 : Equilibrium of rods and plates

Lecture 14 : Bending of a rod under concentrated load

The Lecture Contains:

- Bending of a rod under concentrated load
- Bending of a rod under concentrated vertical load
- Euler buckling instability

This lecture is adopted from the following book

1. "Theory of Elasticity, 3 rd edition" by Landau and Lifshitz. Course of Theoretical Physics, vol-7

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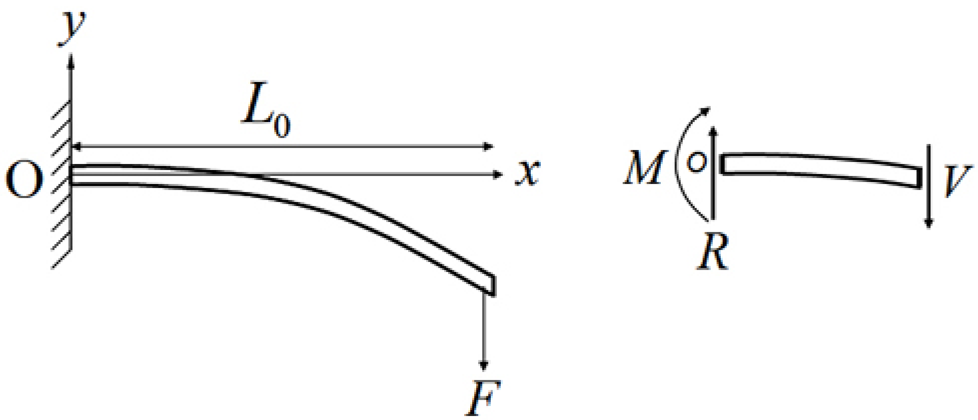
Bending of a rod under concentrated load



At the free end of the rod it is not acted upon by any reaction force. Say at any cross-section at a length L from wall, the reaction force is V, then from force balance,

$$V = R = F \Rightarrow \frac{dV}{dL} = 0 \qquad (14.1)$$

$\overline{V} = -F\overline{e_y}$ and $-\overline{K} = -F\overline{e_y}$

From equation 13.1, $K_y = F$. Then from , $\overline{M} = EI\bar{t} \times \frac{d\bar{t}}{dL} + \bar{t}C\Omega_\varsigma$ we have the following:

For a circular rod $\Omega_\varsigma = 0$, so that $\frac{d\overline{M}}{dL} = EI\bar{t} \times \frac{d^2\bar{t}}{dL^2} = EI\frac{d\bar{r}}{dL} \times \frac{d^3\bar{r}}{dL^3}$.

But from equation 13.3, $\frac{d\overline{M}}{dL} = \overline{V} \times \bar{t}$, so that

$$EI\frac{d\bar{r}}{dL} \times \frac{d^3\bar{r}}{dL^3} = \overline{V} \times \bar{t} = \overline{V} \times \frac{d\bar{r}}{dL} \qquad (14.2)$$

Now from the following expression of the tangent to any point on the rod and its derivatives,

$$\bar{t} = \frac{d\bar{r}}{dL} = \sin\theta\overline{e_x} - \cos\theta\overline{e_y}$$

$$\frac{d^3\bar{r}}{dL^3} = \left(\cos\theta\overline{e_x} + \sin\theta\overline{e_y}\right)\frac{d^2\theta}{dL^2} + \left(\sin\theta\overline{e_x} - \cos\theta\overline{e_y}\right)\left(\frac{d\theta}{dL}\right)^2$$

$$\frac{d\bar{r}}{dL} \times \frac{d^3\bar{r}}{dL^3} = \left(\sin\theta\overline{e_x} - \cos\theta\overline{e_y}\right) \times \left(\left(\cos\theta\overline{e_x} + \sin\theta\overline{e_y}\right)\frac{d^2\theta}{dL^2} + \left(\sin\theta\overline{e_x} - \cos\theta\overline{e_y}\right)\left(\frac{d\theta}{dL}\right)^2\right) \qquad (14.3a)$$

$$= \frac{d^2\theta}{dL^2}\overline{e_x} \times \overline{e_y}$$

$$\overline{V} \times \frac{d\overline{r}}{dL} = -F\overline{e_y} \times \left(\sin\theta\overline{e_x} - \cos\theta\overline{e_y}\right) = F\sin\theta\overline{e_x} \times \overline{e_y} \tag{14.3b}$$

From equation 14.3a and b, we derive,

$$EI\frac{d^2\theta}{dL^2} = F\sin\theta \tag{14.4}$$

Putting $L = L_0\eta$ and $\kappa = \frac{EI}{FL_0^2}$, we finally obtain,

$$\frac{d^2\theta}{d\eta^2} = \frac{1}{\kappa}\sin\theta \tag{14.5}$$

Equation 8.36 is solved with the boundary conditions:

at $\eta = 0$, $\theta = \pi/2$ and at $\eta = 1$, $\theta = \theta_0$ and $d\theta/d\eta = 0$ (14.6)

Integrating equation 14.5 and using b.c. 14.6,

$$\frac{d\theta}{d\eta} = \pm\sqrt{\frac{2}{\kappa}}\sqrt{\cos\theta_0 - \cos\theta} \tag{14.7}$$

Integrating the above expression we obtain the following integral, known as the elliptic integral

$$\sqrt{\frac{2}{\kappa}} = \int_{\theta_0}^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta_0 - \cos\theta}} \tag{14.8}$$

and, $\theta(\eta)$ is obtained in terms of elliptic functions.

The shape of the rod is finally obtained in terms of dimensionless lengths as: $\eta_x = \int\sin\theta\, d\eta$, $\eta_y = \int\cos\theta\, d\eta$

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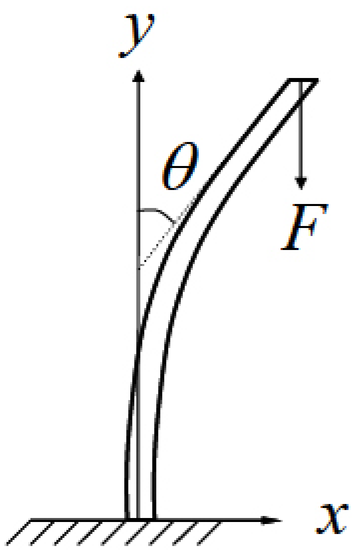
Bending of a rod under concentrated vertical load



At any cross section, the internal stress is constant and is equal to $\overline{F} = -F\overline{e_y}$. Similar to earlier examples, the expression of the tangent to the rod at any location and its derivatives can be written in terms of the angle θ,

$$\bar{t} = \frac{d\bar{r}}{dL} = \sin\theta\overline{e_x} + \cos\theta\overline{e_y}$$

$$\frac{d^3\bar{r}}{dL^3} = \left(\cos\theta\overline{e_x} - \sin\theta\overline{e_y}\right)\frac{d^2\theta}{dL^2} - \left(\sin\theta\overline{e_x} + \cos\theta\overline{e_y}\right)\left(\frac{d\theta}{dL}\right)^2$$

$$\frac{d\bar{r}}{dL}\times\frac{d^3\bar{r}}{dL^3} = \left(\sin\theta\overline{e_x} + \cos\theta\overline{e_y}\right)\times\left(\left(\cos\theta\overline{e_x} - \sin\theta\overline{e_y}\right)\frac{d^2\theta}{dL^2} - \left(\sin\theta\overline{e_x} + \cos\theta\overline{e_y}\right)\left(\frac{d\theta}{dL}\right)^2\right) \qquad (14.9a,b)$$

$$= -\frac{d^2\theta}{dL^2}\overline{e_x}\times\overline{e_y}$$

$$\overline{F}\times\frac{d\bar{r}}{dL} = -F\overline{e_y}\times\left(\sin\theta\overline{e_x} + \cos\theta\overline{e_y}\right) = F\sin\theta$$

From equation (14.9)a and b,

$$-EI\frac{d^2\theta}{dL^2} = F\sin\theta \text{ or } EI\frac{d^2\theta}{dL^2} + F\sin\theta = 0 \qquad (14.10)$$

Boundary conditions are:

$$\text{At } L = 0: \quad \theta = 0, \quad L = L_0 : \theta = \theta_0 \text{ and } \frac{d\theta}{dL} = 0 \qquad (14.11)$$

Solution of equation 14.10:

Multiplying both left and write hand side of equation 14.10

$$2\frac{d^2\theta}{dL^2}\frac{d\theta}{dL}+\frac{2F}{EI}\frac{d\theta}{dL}\sin\,\theta=0 \quad \Rightarrow \frac{d}{dL}\left[\frac{d\theta}{dL}\right]^2-\frac{2F}{EI}\frac{d}{dL}(\cos\theta)=0$$

Integrating,

$$\left[\frac{d\theta}{dL}\right]^2-\frac{2F}{EI}\cos\theta+c=0 \tag{14.12}$$

Using boundary condition $L=L_0:\theta=\theta_0$ and $\frac{d\theta}{dL}=0$, we have,

$$-\frac{2F}{EI}\cos\theta_0+c=0 \quad \Rightarrow c=\frac{2F}{EI}\cos\theta_0$$

From equation 14.12 we have,

$$\left[\frac{d\theta}{dL}\right]^2=\frac{2F}{EI}(\cos\theta-\cos\theta_0)$$
$$\Rightarrow L=\sqrt{\frac{EI}{2F}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{(\cos\theta-\cos\theta_0)}} \tag{14.13}$$

Then, for small θ, we have

$$L=\sqrt{\frac{EI}{2F}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{2\sin^2\frac{\theta_0}{2}-2\sin^2\frac{\theta}{2}}} \qquad \Rightarrow L=\sqrt{\frac{EI}{F}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\theta_0^2-\theta^2}}$$
$$\Rightarrow L=\sqrt{\frac{EI}{F}}\sin^{-1}\left(\frac{\theta}{\theta_0}\right)\Bigg|_0^{\theta_0}=\sqrt{\frac{EI}{F}}\frac{\pi}{2} \tag{14.14}$$

The above expression yields a critical force for buckling of the rod under vertical load

$$F_{cr}>\frac{\pi^2 EI}{4L^2} \tag{14.15}$$

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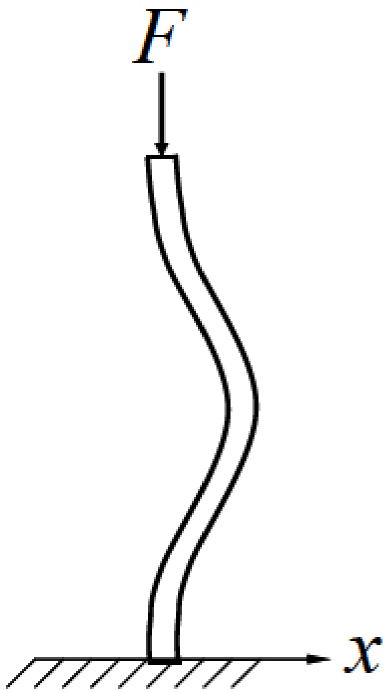
Euler buckling instability

Bending of a rod with both ends constrained to be in the same straight line:



The force balance equation for the above situation can be written as,

$$\left[\frac{d\theta}{dL}\right]^2 - \frac{2F}{EI}\cos\theta + c = 0 \qquad (14.16)$$

We can integrate this equation using boundary condition that at $L = \frac{L_0}{2}$: $\theta = \frac{\pi}{2}$ and $\frac{d\theta}{dL} = 0$, which yields $c = 0$, leading to

$$\left[\frac{d\theta}{dL}\right]^2 = \frac{2F}{EI}\cos\theta \qquad (14.17)$$

Notice that equation 14.16 is the analog of oscillation of a suspended pendulum for which the equation of motion can be written as

$$m\frac{d^2x}{dt^2} + (mg)\sin\theta = 0$$

The critical load for buckling for the above problem can be obtained as,

$$F_{cr} > \frac{4\pi^2 EI}{L^2} \qquad (14.18)$$