The Lecture Contains

1. Ghatak, A. and Das, A. L., Kinking Instability of a Highly Deformable Elastic Cylinder. Physical Review Letters, 2007, Vol. 99, pp. 076101-1-076101-4.

2. "Surface Wrinkling: A Versatile Platform for Measuring Thin-Film Properties" by ***Jun Young Chung*** **,** ***Adam J. Nolte*** **,** and ***Christopher M. Stafford***. ***Adv. Mater.*** 2010, ***XX***, 1–20.

Stress induced surface instability of soft materials

We will now present few examples of surface instability which are caused by compressive stresses.

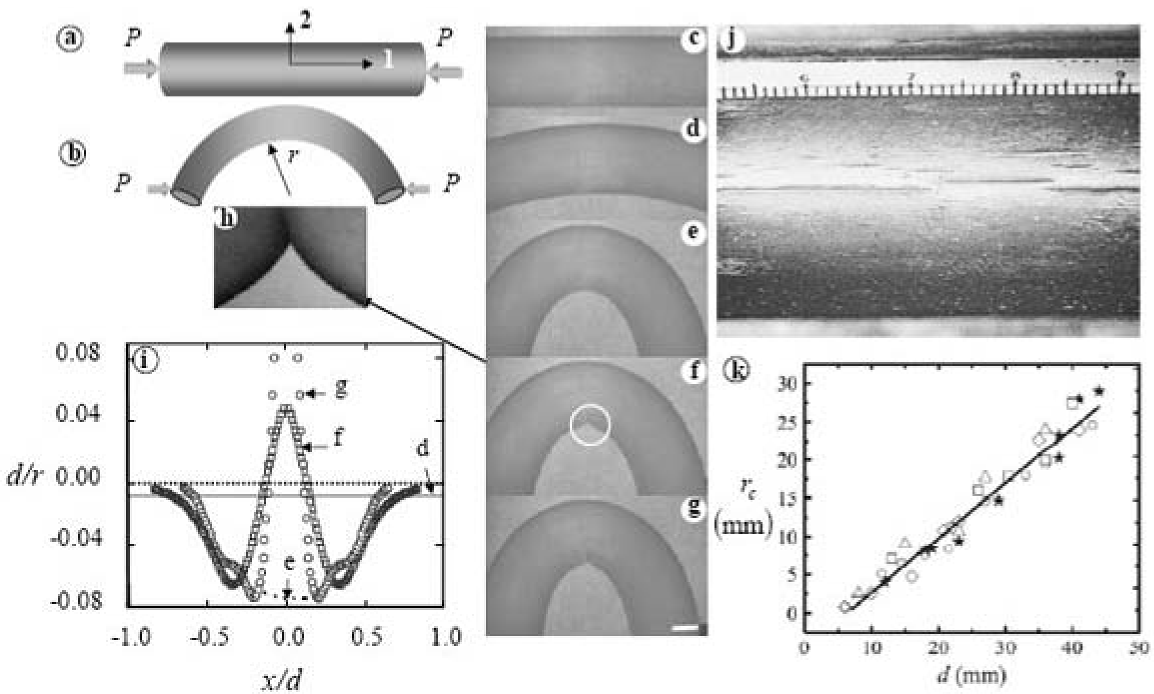

Figure 31.1

(a)-(b) Schematic of an experiment in which a straight gel cylinder of circular or rectangular cross section is subjected to compressive end loads leading to buckling. (c)–(g) A typical sequence of video micrographs show the progressive appearance of kink. Video micrograph
(h) depicts a magnified picture of the cylinder close to the kink.
(i) Dimensionless inner curvature of the cylinder plotted as a function of dimensionless length of cylinder.
(j) When a block of rubber is bent beyond a critical curvature wrinkles appear in the form of surface creases.
(k) The critical radius of curvature is plotted as a function of diameter of the gel cylinder.

Kinking Instability

Figure 1 depicts the schematic of an experiment, in which a soft, elastic, solid cylinder of poly-acrylamide gel was compressed axially between two rigid supports. The cylinder, which was essentially a straight rod initially, buckled at a critical load forming a smooth curve at both its inner and outer sides, both of which could be perfectly described by a Jacobian elliptic function corroborating with the profile of a curved elastica. This phenomenon is the classical Euler's buckling instability that we have dealt with to some detail.
However, as the cylinder was bent to an increasing extent by application of increased end load, while the outer side still followed the elliptic function, the inner side of it deviated from such a behaviour, and it deviated increasingly because of the effect of finite diameter of the cylinder. Eventually, closed to the location of maximum curvature of the cylinder, a sharp fold appeared in the form of a "kink"

with a catastrophic change in curvature. Thus, beyond a critical bending the curvature got localized within a distance ~*d* from the location of the kink; once localized, the kink acted as a hinge, so that, with further bending, the curvature did not change significantly. Video images in Figs. 1(c)–(f) represent a typical sequence leading to the appearance of the kink for a cylinder of diameter $d = 27$ mm and shear modulus $\mu = 31$ kPa. These images were grabbed at temporal resolution of 25 fps. The trace of the inner curve of each image was analysed for obtaining the radius of curvature and importantly the critical radius of curvature at which the kink appeared. The image in figure 1(j) defines the inner side of a bent rubber block for which wrinkles appear at a critical load.

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We have discussed earlier a similar problem in which an elastic slab of thickness h_0 and length L_0 is subjected to compressive axial loads. Following this compression, the slab attains dimensions *h* and *L* respectively. Under plane strain approximations, the slab has been found to buckle at a critical compressive stress P obtained from the solution of the characteristic equation:

$$4k\tanh k\gamma - \left(1+k^2\right)^2 \tanh \gamma = 0 \qquad (31.1)$$

where, $k = \sqrt{\dfrac{1-P/2\mu}{1+P/2\mu}}$ and $\gamma = \dfrac{\pi h}{L}$. For a cylinder, we replace h in the above expression by d. Solutions of equation 31.1 have been obtained for the asymptotic cases of $\gamma \to \infty$ and $\gamma \to 0$ which have yielded the corresponding critical loads at which the buckling occurs. In the context of kinking phenomenon, however, the experiments clearly show that the axial length within which the zone of compressive stress remains concentrated, is $L \sim d$, so that $\gamma = \pi$. Solution of equation 13.87 with this value of γ generates $k = 0.4$. Correspondingly the critical end load is found to be $P_c = 1.45\mu$ in terms of the shear modulus of the material of the gel. It can shown that at this value of the critical load, the curvature of the bent gel scales with the diameter as $r_c = 0.6d$ which is closed to the value observed in experiments. However a more rigorous analysis is needed in order to capture the kinking phenomenon.

Surface wrinkling

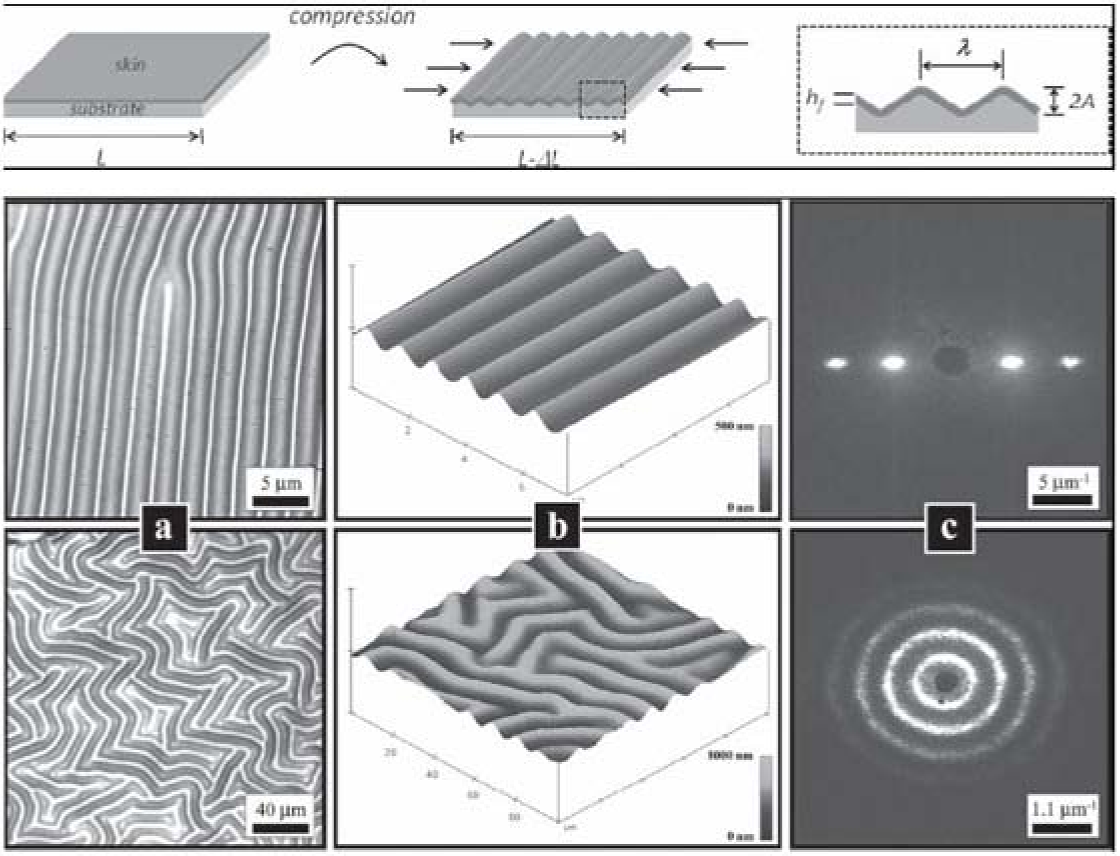

Figure 31.2. Schematic of generating wrinkles on a stiff skin adhered to a soft substrate. The laminate undergoes wrinkling upon compressive strain from an initial length (***L***) to a final length (***L*- Δ*L***). Examples of 1-dimensional and 2-dimensional wrinkling patterns and methods for measuring their wavelength.

Here we will consider buckling instability of a sandwiched system comprising of a thin laminate made of a stiffer material attached to a soft substrate. When a compressive load is applied at the ends of the sandwiched system wrinkles with well defined wavelength and amplitude begin to appear. The wavelength of the instability, λ, remains linearly dependent on the thickness h of the laminate, $\lambda \sim h$. Additionally, the amplitude of the wrinkles A increases nonlinearly with strain ε as $A \sim \varepsilon^{1/2}$.

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Surface wrinkling can be analysed by considering the classical force balance equation for bending of a stiff film on a compliant elastic substrate.

$$\overline{E_f} I \frac{d^4 z}{dx^4} + F \frac{d^2 z}{dx^2} + kz = 0 \qquad (31.2)$$

Where, $\overline{E_f} = E/(1-\nu^2)$ is the plane strain modulus and $I = wh^3/12$ is the moment of inertia of the stiff film. F is the uni-axially applied load. The constant $k = \overline{E_s} w\pi/\lambda$, is called the Winkler's modulus of an elastic half-space. The subscripts s and f denote the substrate and the film respectively. Notice the second term represents the external compressive load which causes the buckling instability, the first term represents the bending force in film which suppresses the perturbations with short wavelength, the third term defines the force due to deformation normal to the surface of the film, it suppresses large wavelength fluctuations. We assume that the perturbation in vertical deflection of the film can be represented by the following sinusoidal function

$$z(x) = A \sin \frac{2\pi x}{\lambda} \qquad (31.3)$$

Which when substituted in equation 31.2 results in following expanded form in terms of λ

$$16\overline{E_f} I \left(\frac{\pi}{\lambda}\right)^4 z - 4F\left(\frac{\pi}{\lambda}\right)^2 z + \overline{E_s} w \left(\frac{\pi}{\lambda}\right) z = 0 \qquad (31.4)$$

Solving for the applied force F, we obtain,

$$F = 4\overline{E_f} I \left(\frac{\pi}{\lambda}\right)^2 + \frac{\overline{E_s} w}{4}\left(\frac{\pi}{\lambda}\right)^{-1} \qquad (31.5)$$

The minimum force that can cause the instability can be found by putting, $\frac{\partial F}{\partial \lambda} = 0$, which yields the following relation for wavelength,

$$\lambda = 2\pi h \left(\frac{\overline{E_f}}{3\overline{E_s}}\right)^{\frac{1}{3}} \qquad (31.6)$$

An estimate of the critical stress that ensues the wrinkling phenomenon can be obtained by dividing the critical load by the cross-sectional area of the film:

$$\sigma_c = \frac{F_c}{hw} = \left(\frac{9}{64}\overline{E_f}\,\overline{E_s}^2\right)^{\frac{1}{3}} \tag{31.7}$$

Then the critical strain is written as,

$$\varepsilon_c = \frac{\sigma_c}{\overline{E_f}} = \frac{1}{4}\left(\frac{3\overline{E_s}}{\overline{E_f}}\right)^{2/3} \tag{31.8}$$

Notice that critical strain is a function only of ratio of the moduli of substrate and the film and remains independent of the thickness. The significance of the critical strain is that beyond it a wrinkling wavelength is established; with further increase in the strain, the amplitude of the wrinkling increases which accommodates the excess strain while the wavelength remains nearly constant. By assuming that wavelength of instability remains independent of the excess strain, an estimate of it, can be obtained by analysing the geometry of the wrinkles,

$$\varepsilon - \varepsilon_c = \frac{1}{\lambda}\int_0^{\lambda}\sqrt{1+\left(\frac{dz}{dx}\right)^2}\,dx - 1 \tag{31.9}$$

Notice, the quantity within the integral sign defines the contour length of the wrinkle over one wavelength. Substituting 31.3 into 31.9 and noting that $\lambda > A$, so that,

$$\sqrt{1+\left(\frac{dz}{dx}\right)^2} \approx 1 + \frac{1}{2}\left(\frac{dz}{dx}\right)^2 \tag{31.10}$$

Noting that, dz/dx is very small, we deduce,

$$\varepsilon - \varepsilon_c = \frac{\pi^2 A^2}{\lambda^2},\quad A = h\sqrt{\frac{\varepsilon - \varepsilon_c}{\varepsilon_c}} \tag{31.11}$$

Equations 31.6, 31.8 and 31.11 capture the essential features of the wrinkling instability, the wavelength, the critical strain at which the instability ensues and the growth of amplitude of the instability with the applied strain.

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