Module 1 : Brief Introduction

Lecture 2 : Stress

The Lecture Contains:

This lecture is adopted from the following book

1. "Some Basic Problems of the Mathematical Theory of Elasticity" by N.I.Muskhelishvili

2. "A Treatise on the Mathematical theory of Elasticity" by A.E.H. Love

3. "Advanced Engineering Mathematics" by Erwin Kreyszig

Stress

For complete specification of the stress at any point in a body, we need to know the force per unit area across any plane passing through the point and the direction of the force.

For complete specification of the state of stress in a body, we need to know the stress at every point in the body.

Here we are interested to develop the theory of equilibrium and motion of a body when subjected to body and surface forces.

Surface Traction

Let's consider a given plane S passing through a point O within a body.

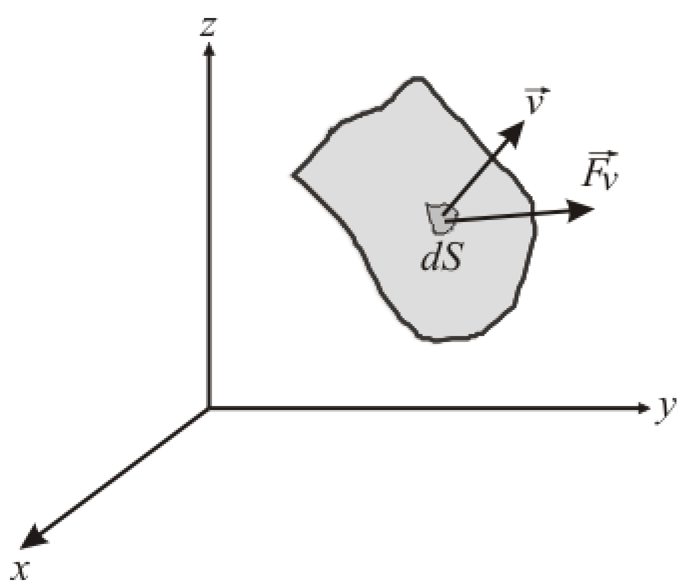

Figure 2.1

Let's say ds is a small area of the plane which is normal to direction ν. Then if F is a force which acts per unit area on the plane along the direction ν, then it has projections X_ν, Y_ν, Z_ν along the three axes of coordinates, so that

$$F = X_\nu \cos(x,\nu) + Y_\nu \cos(y,\nu) + Z_\nu \cos(z,\nu) \tag{2.1}$$

and the portion of the body which is on the other side of the plane acts along ν the following surface tractions on the element of area dS,

$$(X_\nu dS,\ Y_\nu dS,\ Z_\nu dS) \tag{2.2}$$

Body Forces

In addition to the surface tractions, other forces act on each part of a body. These are called the body forces like the gravity force, intermolecular force. The magnitude of such forces varies proportionally with the mass of the particle on which they act and their direction depends on the position of the particles in the field of force.

If (X, Y, Z) are the components of the intensity field at any point and *m* is the mass of a particle at that point, then (mX, mY, mZ) are the forces of the field that act on the particle forming part of the body.

Equation of Motion

Body forces applied on any portion of the body is statically equivalent to a single force applied at one point together with a couple .
The components parallel to the axes of the single force are,

$$\iiint \rho X\, dxdydz, \quad \iiint \rho Y\, dxdydz, \quad \iiint \rho Z\, dxdydz \tag{2.3}$$

ρ is the density of the body at the point (x, y, z) and the integration is taken throughout the volume of the body.

Similarly the tractions on the elements of area of the surface of the portion of the body are equivalent to a resultant force and a couple . The components of the force are,

$$\iint X_\nu dS, \quad \iint Y_\nu dS, \quad \iint Z_\nu dS \tag{2.4}$$

where integration is taken over the surface of the portion.

The center of mass of the portion moves like a particle under the action of these two types of forces. If (f_x, f_y, f_z) are the acceleration of the particle which is located at (x, y, z) at time *t* , then the equation of motion of the particle are

$$\begin{aligned}
\iiint \rho f_x dx\, dy\, dz &= \iiint \rho X\, dx\, dy\, dz + \iint X_\nu dS \\
\iiint \rho f_y dx\, dy\, dz &= \iiint \rho Y\, dx\, dy\, dz + \iint Y_\nu dS \\
\iiint \rho f_z dx\, dy\, dz &= \iiint \rho Z\, dx\, dy\, dz + \iint Z_\nu dS
\end{aligned} \tag{2.5}$$

The change of moment of momentum of the portion of the body are

$$\begin{aligned}
\iiint \rho(yf_z - zf_y) dx\, dy\, dz &= \iiint \rho(yZ - zY) dx\, dy\, dz + \iint (yZ_\nu - zY_\nu) dS \\
\iiint \rho(zf_x - xf_z) dx\, dy\, dz &= \iiint \rho(zX - xZ) dx\, dy\, dz + \iint (zX_\nu - xZ_\nu) dS \\
\iiint \rho(xf_y - yf_x) dx\, dy\, dz &= \iiint \rho(xY - yX) dx\, dy\, dz + \iint (xY_\nu - yX_\nu) dS
\end{aligned} \tag{2.6}$$

Simplifications

At equilibrium the accelerations (f_x, f_y, f_z) are zero.

Secondly, the volume of integration is very small in all its dimensions and l^3 be the volume, then both sides of eqn 2.5 in the limit $l \rightarrow 0$ results in

$$\iint X_\nu dS = 0, \quad \iint Y_\nu dS = 0 \quad \text{and} \quad \iint Z_\nu dS = 0 \qquad (2.7)$$

Similarly, eqn (2.6) simplifies to

$$\iint (yZ_\nu - zY_\nu)dS = 0, \iint (zX_\nu - xZ_\nu)dS = 0 \text{ and } \iint (xY_\nu - yX_\nu)dS = 0 \qquad (2.8)$$

Let's now see how these components of forces will change with change in orientation of the planes. In order to do that, let's consider the orthogonal rectilinear system

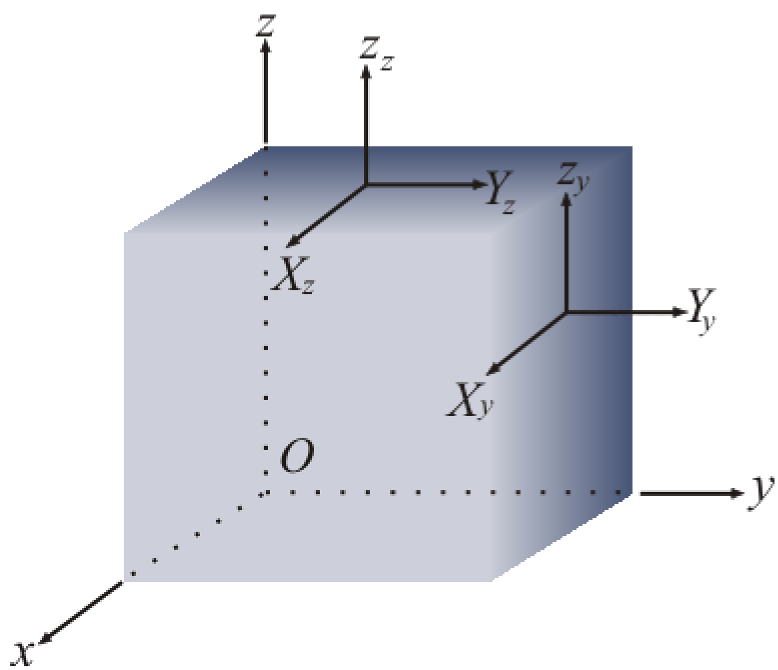

Figure 2.2

Z_z, Y_z, X_z are the components of the stresses that act on the plane normal to the z axis. While Z_z is the normal stress, Y_z, X_z are the tangential or the shear stresses. Similarly X_x, Y_x, Z_x and X_y, Y_y, Z_y are the components of stresses acting on planes normal to the x and y axes respectively.

Question is how these stresses are related to the components of stress on the plane normal to $\vec{\nu}$?

Previous Next

Simplifications (Contd...)

Consider the equilibrium of a tetrahedral portion of the body which has three edges along the three axes of the co-ordinate and the vertex at which they meet at the origin O. Say ν is the direction normal to the plane ABC drawn away from the interior of the plane. Then the direction cosines of this plane are $\cos(x,\nu)$, $\cos(y,\nu)$ and $\cos(z,\nu)$. Let's say that the plane is at a distance h from origin O with $h \to 0$. If Δ is the area of this plane then the areas of the remaining faces are $\Delta\cos(x,\nu)$, $\Delta\cos(y,\nu)$ and $\Delta\cos(z,\nu)$ respectively.

When the tetrahedron is small, the traction across the face ν be $X_\nu\Delta, Y_\nu\Delta, Z_\nu\Delta$ and those on the remaining faces to be $-X_x\Delta\cos(x,\nu),\ldots,-Y_y\Delta\cos(y,\nu),\ldots$ and $Z_z\Delta\cos(z,\nu),\ldots$.

$$\begin{aligned}
X_\nu &= X_x\cos(x,\nu)+X_y\cos(y,\nu)+X_z\cos(z,\nu)\\
Y_\nu &= Y_x\cos(x,\nu)+Y_y\cos(y,\nu)+Y_z\cos(z,\nu)\\
Z_\nu &= Z_x\cos(x,\nu)+Z_y\cos(y,\nu)+Z_z\cos(z,\nu)
\end{aligned} \tag{2.9}$$

by this equation we express traction across any plan through O in terms of the traction across planes parallel to the coordinate planes.

Now, consider a very small cube with edges parallel to the coordinate axes. The value $\iint(yZ_\nu - zY_\nu)dS$ for the cube can be taken as $l\Delta(Z_y - Y_z)$, where l is the length of any edge. Hence from the results of the last paragraph, we have,

$Z_y = Y_z, X_z = Z_x$ and $Y_x = X_y$. Thus, the number of components that must be specified in order that the stress at a point may be specified is six, three normal components and three tangential tractions.

Previous Next

The stress equations of motion and equilibrium:

Consider equation 2.5 and 2.7, the stress equilibrium relation can be obtained as

$$\iiint \rho f_x \, dx\, dy\, dz = \iiint \rho X \, dx\, dy\, dz + \iint \left\{ X_x \cos(x,\nu) + X_y \cos(y,\nu) + X_z \cos(z,\nu) \right\} dS$$

Using Gauss's transformation, the surface integral can be transformed into a volume integral as

$$\iiint \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \rho X - \rho f_x \right) dx\, dy\, dz = 0$$

The integration can be taken through any volume within the body and it can not be satisfied, if the integrand is not zero at every point, so that

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \rho X - \rho f_x = 0 \qquad (2.10)$$

If the body is held in equilibrium, then the equations of equilibrium are

$$\begin{aligned} \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \rho X &= 0 \\ \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + \rho Y &= 0 \\ \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + \rho Z &= 0 \end{aligned} \qquad (2.11)$$

If, the body has not yet reached equilibrium, then the acceleration at any point (X, Y, Z)

$$\begin{aligned} \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \rho X &= \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + \rho Y &= \rho \frac{\partial^2 \upsilon}{\partial t^2} \\ \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + \rho Z &= \rho \frac{\partial^2 w}{\partial t^2} \end{aligned} \qquad (2.12)$$

Moment of external forces:

Let's write the resultant moment about the OX axis of the body forces and the surface stresses acting on a surface S containing the volume V:

$$\iiint_V (yZ - zY)dV + \iint_S (yZ_\nu - zY_\nu)dS = 0 \tag{2.13}$$

Using the relations for Z_ν and Y_ν from 2.7, we have,

$$\iint_S (yZ_\nu - zY_\nu)dS = \iint_S \begin{pmatrix} y\left(Z_z\cos(\nu,z) + Z_y\cos(\nu,y) + Z_x\cos(\nu,x)\right) \\ -z\left(Y_z\cos(\nu,z) + Y_y\cos(\nu,y) + Y_x\cos(\nu,x)\right) \end{pmatrix} dS$$
$$= \iint_S \left((yZ_z - zY_z)\cos(\nu,z) + (yZ_y - zY_y)\cos(\nu,y) + (yZ_x - zY_x)\cos(\nu,x)\right)dS$$

Now, using of Green's Transformation, we obtain,

Gauss's Transformation

$$\iint_S \{\xi\cos(x,\nu) + \psi\cos(y,\nu) + \zeta\cos(z,\nu)\}dS = \iiint_V \left(\frac{\partial\xi}{\partial x} + \frac{\partial\psi}{\partial y} + \frac{\partial\zeta}{\partial z}\right)dV$$

Proof of the Green's transformation can be found in Kreyszig

$$\iint_S (yZ_\nu - zY_\nu)dS = \iiint_V \left(y\left(\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}\right) - z\left(\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}\right) + Z_y - Y_z \right)dV$$
$$\Rightarrow \iiint_V \left(y\left(Z + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}\right) - z\left(Y + \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}\right) + Z_y - Y_z \right)dV = 0 \tag{2.14}$$
$$\Rightarrow \iiint_V (Z_y - Y_z)dV = 0$$

In order that the last integral(s) is(are) zero in any arbitrary region ν, we should have,

$$Z_y = Y_z, \quad X_z = Z_x, \quad Y_x = X_y \tag{2.15}$$

Which implies that

the stress tensor $\begin{bmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{bmatrix}$ is symmetric.

Previous Next

Transformation of stress components:

Actually above discussions can be generalized in the form of following proposition.

Let there be two planes passing through the one and the same point: the projection of stress, acting on the first plane on the normal to the second plane is equal to the projection of stress, acting on the second plane on the normal to the first plane .

It should be possible to express the components of the stress at any point with respect to a given system of axes in terms of those referred to another orthogonal system of axes.

Let $\overline{\nu'}$ be the normal to plane 1. Its direction cosines are α', β', γ'. Then F_ν be the stress vector acting on plane 1, its components are,

$$\begin{aligned} X_{\nu'} &= X_x\alpha' + X_y\beta' + X_z\gamma' \\ Y_{\nu'} &= Y_x\alpha' + Y_y\beta' + Y_z\gamma' \\ Z_{\nu'} &= Z_x\alpha' + Z_y\beta' + Z_z\gamma' \end{aligned} \qquad (2.16)$$

Let $\overline{\nu''}$ be the normal to plane 2. Its direction cosines are $\alpha'', \beta'', \gamma''$. Then the projections of components of F_ν on normal $\overline{\nu''}$ are:

$$\begin{aligned} \overline{(F_{\nu'})_{\nu''}} &= X_{\nu'}\alpha'' + Y_{\nu'}\beta'' + Z_{\nu'}\gamma'' \\ &= (X_x\alpha' + X_y\beta' + X_z\gamma')\alpha'' + (Y_x\alpha' + Y_y\beta' + Y_z\gamma')\beta'' + (Z_x\alpha' + Z_y\beta' + Z_z\gamma')\gamma'' \\ &= X_x\alpha'\alpha'' + Y_y\beta'\beta'' + Z_z\gamma'\gamma'' + X_y(\beta'\alpha'' + \alpha'\beta'') + Y_z(\beta''\gamma' + \beta'\gamma'') + Z_x(\alpha'\gamma'' + \alpha''\gamma') \end{aligned} \qquad (2.17)$$

The above expression is quite symmetric with respect to α', β', γ' and $\alpha'', \beta'', \gamma''$ so that the two planes can be interchanged without altering the expression, but that proves the proposition.

Previous Next

Transformation of stress components (contd...)

Let's now consider transition of stress components from one coordinate system x, y, z to a new coordinate system x', y', z'. Let the direction cosines of the new system w.r.t the old one is

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

Let's say, $X'_{x'}, Y'_{y'}, Z'_{z'}, X'_{y'}, Y'_{z'}, Z'_{x'}$ represent the stress components in the new system, while, $X_x, Y_y, Z_z, X_y, Y_z, Z_x$ represent those in the old system. Then following previous discussions,

$X'_{x'} = \left(\overline{F_x}\right)_{x'}$, so that $\alpha' = \alpha'' = l_1$, $\beta' = \beta'' = m_1$ and $\gamma' = \gamma'' = n_1$

Then from 2.17,

$$X'_{x'} = l_1^2 X_x + m_1^2 Y_y + n_1^2 Z_z + 2m_1 l_1 X_y + 2m_1 n_1 Y_z + 2n_1 l_1 Z_x \qquad (2.18a)$$

Similarly, we obtain other equations,

$$X'_{y'} = l_1 l_2 X_x + m_1 m_2 Y_y + n_1 n_2 Z_z + (l_2 m_1 + m_2 l_1) X_y + (n_2 m_1 + n_1 m_2) Y_z + (n_2 l_1 + n_1 l_2) Z_x \qquad (2.18b)$$

If we add the three equations 2.18a and use the following relations,

$$\begin{aligned} l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2 = 1 \\ l_1 m_1 + l_2 m_2 + l_3 m_3 = m_1 n_1 + m_2 n_2 + m_3 n_3 = n_1 l_1 + n_2 l_2 + n_3 l_3 = 0 \end{aligned} \qquad (2.19)$$

we obtain the following:

$$X'_{x'} + Y'_{y'} + Z'_{z'} = X_x + Y_y + Z_z$$

That is, the expression $\Theta = X_x + Y_y + Z_z$ is invariant with respect to transformation of coordinates, i.e. the sum of normal stress components acting on three mutually perpendicular planes do not depend upon the orientation of the planes.

Consider again 2.17, if l, m, n are the direction cosines of the normal ν, then the equation yields following expression for the normal stress,

$$N = l^2 X_x + m^2 Y_y + n^2 Z_z + 2mlX_y + 2mnY_z + 2nlZ_x \qquad (2.20)$$

Now consider a vector $\overline{P}$ which is of length P and has end points as the origin of the coordinate system and the point ξ, ψ, ζ; then using,

$$l = \frac{\xi}{P}, m = \frac{\psi}{P}, n = \frac{\zeta}{P}$$

equation 2.20 yields,

$$NP^2 = \xi^2 X_x + \psi^2 Y_y + \zeta^2 Z_z + 2\xi\psi X_y + 2\psi\zeta Y_z + 2\zeta\xi Z_x \qquad (2.21)$$

in which N has a physical meaning, so can not depend on the particular choice of axes and the quantity P^2 does not depend on the choice of coordinates. Consequently, the quadratic $NP^2 = 2\Omega(\xi, \psi, \zeta)$ is an invariant of transformation of coordinates. Then, let us put $NP^2 = \pm c^2$, where c^2 is an arbitrary constant and it has the dimension of a force. It has the same direction as that of N i.e. for a tensile stress it is positive and for a compressive stress it is negative.

Previous Next

Transformation of stress components (contd...)

Let one end of the vector $\vec{P} = \overline{OH}$ be at the origin O of the coordinate system, then the other end $H(\xi, \psi, \zeta)$ of the vector $\vec{P}$ will lie on the surface $2\Omega(\xi, \psi, \zeta) = \pm c^2$, i.e.

$$\xi^2 X_x + \psi^2 Y_y + \zeta^2 Z_z + 2\xi\psi X_y + 2\psi\zeta Y_z + 2\zeta\xi Z_x = \pm c^2 \qquad (2.22)$$

Surface 2.22 is a quadratic with the center at the origin; it is called a stress surface.

A direction with the property that only a normal stress acts on the plane normal to it is called a principal direction of stress or a principal axis of stress. Corresponding stress is called the principal stress.

If I select the coordinate axes along the three principal axes of stress, then its equation is known to have the form:

$$\xi^2 N_1 + \psi^2 N_2 + \zeta^2 N_3 = \pm c^2$$

N_1, N_2, N_3 are the values of the quantities X_x, Y_y, Z_z for the new coordinate axes. These values are obtained by solving the following equation:

$$\begin{vmatrix} X_x - N & X_y & X_z \\ Y_x & Y_y - N & Y_z \\ Z_x & Z_y & Z_z - N \end{vmatrix} = -N^3 + \Theta N^2 + AN + B = 0 \qquad (2.23)$$

where $\Theta = X_x + Y_y + Z_z$, $A = Y_z^2 + Z_x^2 + X_y^2 - Y_y Z_z - Z_z X_x - X_x Y_y$

and B = $B = X_x Y_y Z_z + 2Y_z Z_x X_y - Y_y Z_x^2 - Z_z X_y^2 - X_x Y_z^2$

Since N_1, N_2, N_3 do not depend on the choice of the coordinate system, the coefficients Θ, A, B do not depend on it, i.e. these quantities are invariant w.r.t. the transformation of orthogonal rectilinear system of axes.

If the quadric in equation 2.23 are referred to as principal axes, then the tangential tractions across the coordinate planes become zero. There are three such systems of orthogonal planes which are called principal plane of stress and the normal tractions are called the principal stresses. Hence in order to completely specify the stresses at any point in a body we need to know the principle stresses and their directions.