The Lecture Contains:

This lecture is adopted from the following book:

1. "Contact Mechanics" by K.L.Johnson

2. "Surface Energy and the Contact of Elastic Solid" K.L. Johnson, K. Kendall, A.D. Roberts, ***Proc. R. Soc. London, Ser. A*** 1971, ***324***, 301-313.

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JKR Contact Mechanics Theory

It is a common experience that one needs to apply force to pull apart two bodies placed in intimate contact with each other signifying that adhesive forces act between the two bodies. While the origin of the adhesive forces is not a subject of this course, here we will worry about how to estimate the effect of these forces in contact deformation.

Let's consider the deformation of two smooth elastic spheres of radius R_1 and R_2 pressed together under load P_0. If E_1 and E_2 are the elastic modulii of the two spheres, then according to the Hertzian mechanics, the expressions for contact radius a, displacement δ and contact pressure p_0 are obtained as,

$$a_0 = \left(\frac{3P_0 R}{4E^*}\right)^{1/3} \qquad \delta = \frac{1}{2E^*}\frac{3P_0}{2}\frac{1}{a_0} = \left(\frac{9P_0^2}{16RE^{*2}}\right)^{1/3} \qquad p_0 = \left(\frac{6P_0 E^{*2}}{\pi^3 R^2}\right)^{1/3} \tag{9.1}$$

Where, $\frac{1}{E^*} = \frac{1-\sigma_1^2}{E_1} + \frac{1-\sigma_2^2}{E_2}$ and $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

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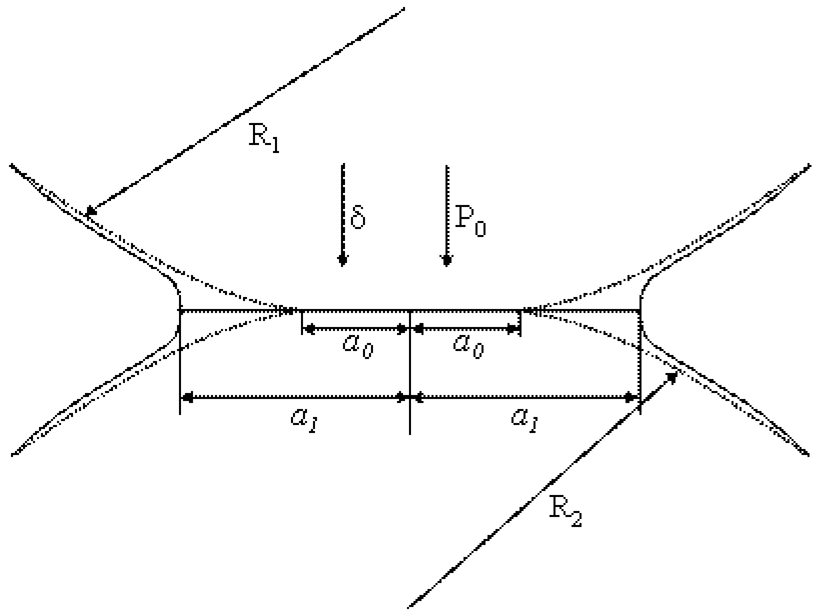
However the data from many experiments with rubber spheres in contact with glass does not quite agree with the Hertzian theory as a finite contact radius is observed even as the load approaches to zero. These observations suggest that attractive surface forces operate between solids and although they are not all significant at high loads, they become increasingly important as the loads are reduced to zero.



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How does the pressure distribution then alter from that in the Hertzian contact between surfaces without any interaction. The pressure distribution in that case was found to be

$$p(r) = p_0\sqrt{1-\frac{r^2}{a^2}}$$

which resulted in total displacement at the interface of the form

$$v_1\big|_{z=0} + v_2\big|_{z=0} = \delta - \frac{r^2}{2R}$$

It was observed that pressure distribution could have a component of the form $\boxed{p(r) = \frac{p_0^{'}}{\sqrt{1-\frac{r^2}{a^2}}}}$

which could have resulted in a constant displacement.

But it was discarded! Because, if $p_0^{'}$ were positive, it would result in interference of surfaces outside the contact area.
On the other hand, if it were negative, infinite tension could not be sustained outside the contact area.

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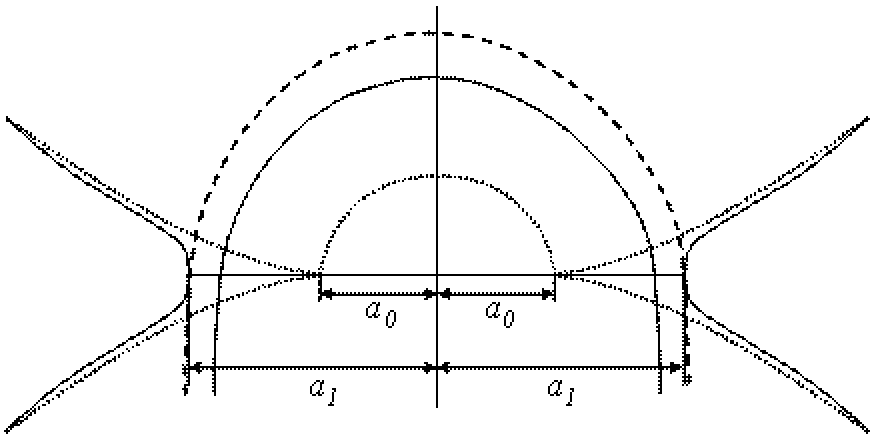
However this is not a problem, when the surfaces attract each other, so that the contact pressure is compressive not through the whole area of contact but only towards ther center and towards the periphery it is tensile:

$$p(r) = p_0\sqrt{1-\frac{r^2}{a^2}} + \frac{p_0^{'}}{\sqrt{1-\frac{r^2}{a^2}}}$$



This pressure distribution results in how the deformed profiles of the spheres contact each other, whereas in Hertzian mechanics, the interfaces meet tangentially, in this situation they meet perpendicularly with adhesive forces.

As a result, the actual contact radius is not equal to a_0 and a_1. Here we will find out how to obtain this actual contact radius.

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Here we'll find out how to obtain this actual contact radius

Consider the situation when no surface forces act, the contact radius is given as $a_0^3 = \frac{3P_0R}{4E^*}$. The movement δ of the applied load is given by $\delta = \frac{1}{2E^*}\frac{3P_0}{2}\frac{1}{a_0} = \frac{a_0^2}{R}$

When attractive forces act between the surfaces the contact radius in equilibrium will be $a_1 > a_0$. Although, applied load remains at P_0 , an apparent Hertz load P_1 corresponding to the contact radius a_1 , defined as:

$$a_1^3 = \frac{3P_1R}{4E^*} \tag{9.2}$$

Then the energy required to load the system of spheres to a contact radius a_1 with a load P_1 , in the absence of surface forces, can be calculated as:

Say, an intermediate load is P , and the corresponding contact radius is a , then $a = \left(\frac{3PR}{4E^*}\right)^{\frac{1}{3}}$ and the movement of the applied load is,

$$\delta = \frac{1}{R}\left(\frac{3PR}{4E^*}\right)^{\frac{2}{3}} = \frac{P^{\frac{2}{3}}}{K^{\frac{2}{3}}R^{\frac{1}{3}}}, \quad K = \frac{4E^*}{3} \tag{9.3}$$

The differential movement and the energy is

$$d\delta = \frac{2}{3}\frac{P^{-\frac{1}{3}}}{K^{\frac{2}{3}}R^{\frac{1}{3}}}dP \quad \Rightarrow \quad P\cdot d\delta = \frac{2}{3}\frac{P^{\frac{2}{3}}}{K^{\frac{2}{3}}R^{\frac{1}{3}}}dP \tag{9.4}$$

Hence the total energy stored in the spheres is

$$U_1 = \int_0^{P_1}\frac{2}{3}\frac{P^{\frac{2}{3}}}{K^{\frac{2}{3}}R^{\frac{1}{3}}}dP = \frac{2}{5}\frac{P_1^{\frac{5}{3}}}{K^{\frac{2}{3}}R^{\frac{1}{3}}} \tag{9.5}$$

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Now say by keeping the contact radius constant at a_1, the load is reduced to p_0. The situation is very similar to the loading on the contact area of the form $p(r)=\dfrac{p_0}{\sqrt{1-\dfrac{r^2}{a^2}}}$ in which the displacement is constant with respect r and is obtained as (in 4.69 and 4.70)

$$\delta=\frac{P}{2a_1E^*}=\frac{2P}{3Ka_1} \tag{9.6}$$

Then the energy that gets stored in the spheres, in decreasing the pressure from P_1 to P_0 is

$$U_2=\int_{P_1}^{P_0}\frac{2}{3}\frac{P}{Ka_1}dP=\frac{1}{3Ka_1}\left(P_0^2-P_1^2\right)=\frac{1}{3K^{\frac{2}{3}}R^{\frac{1}{3}}}\left(\frac{P_0^2-P_1^2}{P_1^{\frac{1}{3}}}\right) \tag{9.7}$$

The total energy stored in the spheres,

$$U_E=U_1+U_2=\frac{2}{5}\frac{P_1^{\frac{5}{3}}}{K^{\frac{2}{3}}R^{\frac{1}{3}}}-\frac{1}{3}\frac{\left(P_1^2-P_0^2\right)}{K^{\frac{2}{3}}R^{\frac{1}{3}}P_1^{\frac{1}{3}}} \tag{9.8}$$

The mechanical potential energy of the system is given by,

$$U_M=-P_0\delta_2=-P_0\left(\delta_1-\frac{2}{3}\frac{P_1-P_0}{Ka_1}\right)=-P_0\left(\frac{P_1^{\frac{2}{3}}}{K^{\frac{2}{3}}R^{\frac{1}{3}}}-\frac{2}{3}\left(\frac{K}{RP_1}\right)^{\frac{1}{3}}\frac{P_1-P_0}{K}\right) \tag{9.9}$$

$$=-\frac{P_0}{K^{\frac{2}{3}}R^{\frac{1}{3}}}\left(\frac{1}{3}P_1^{\frac{2}{3}}+\frac{2}{3}P_0P_1^{-\frac{1}{3}}\right) \tag{9.10}$$

The surface energy lost due to adhesion of the two surface is given by

$$U_s=-\pi a_1^2W_A=-W_A\pi\left(\frac{RP_1}{K}\right)^{\frac{2}{3}} \tag{9.11}$$

Then the total energy is given by

$$U_T=U_E+U_M+U_s$$

$$=\frac{1}{K^{\frac{2}{3}}R^{\frac{1}{3}}}\left(\frac{1}{15}P_1^{\frac{5}{3}}+\frac{1}{3}P_0^2P_1^{-\frac{1}{3}}\right)-\frac{1}{K^{\frac{2}{3}}R^{\frac{1}{3}}}\left(\frac{P_0P_1^{\frac{2}{3}}}{3}+\frac{2}{3}P_0^2P_1^{-\frac{1}{3}}\right)-W_A\pi\frac{R^{\frac{2}{3}}P_1^{\frac{2}{3}}}{K^{\frac{2}{3}}} \tag{9.12}$$

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The equilibrium is attained at the minimum energy state of the system, which happens when,

$$\frac{\partial U_T}{\partial a} = 0 \Rightarrow \frac{\partial U_T}{\partial P_1} = 0$$

$$\Rightarrow \frac{1}{K^{\frac{2}{3}} R^{\frac{1}{3}}} \left(\frac{1}{9} P_1^{\frac{2}{3}} - \frac{1}{9} P_0^2 P_1^{-\frac{4}{3}} \right) - \frac{1}{K^{\frac{2}{3}} R^{\frac{1}{3}}} \left(\frac{2}{9} P_0 P_1^{-\frac{1}{3}} - \frac{2}{9} P_0^2 P_1^{-\frac{4}{3}} \right) - \frac{2}{3} W_A \pi \frac{R^{\frac{2}{3}} P_1^{-\frac{1}{3}}}{K^{\frac{2}{3}}} = 0$$

$$\Rightarrow \frac{P^{-\frac{4}{3}}}{9 K^{\frac{2}{3}} R^{\frac{1}{3}}} \left(P_1^2 - P_0^2 - 2P_0 P_1 + 2P_0^2 - 6W_A \pi R P_1 \right) = 0 \tag{9.13}$$

$$\Rightarrow P_1^2 - 2P_1 \left(P_0 + 3W_A \pi R \right) + P_0^2 = 0$$

Solving we obtain an expression for P_1 :

$$P_1^2 - 2P_1 \left(P_0 + 3W_A \pi R \right) + P_0^2 = 0$$

$$P_1 = \left(P_0 + 3W_A \pi R \right) \pm \sqrt{\left(P_0 + 3W_A \pi R \right)^2 - P_0^2} \tag{9.14}$$

Stable solution is obtained for

$$P_1 = \left(P_0 + 3W_A \pi R \right) + \sqrt{6W_A \pi P_0 R + \left(3W_A \pi R \right)^2} \tag{9.15}$$

Hence the apparent Hertz load is bigger than the actual load due to the work of adhesion W_A and the radius of contact of the two spheres is given as,

$$a_1^3 = \frac{R}{K} \left(\left(P_0 + 3W_A \pi R \right) + \sqrt{6W_A \pi P_0 R + \left(3W_A \pi R \right)^2} \right) \tag{9.16}$$

From which we recover the Hertzian equation $a_1^3 = \frac{RP_0}{K}$ when $W_A = 0$.

At zero load the contact radius is finite and is given by,

$$a_1^3 = \frac{R}{K} \left(6W_A \pi R \right) \tag{9.17}$$

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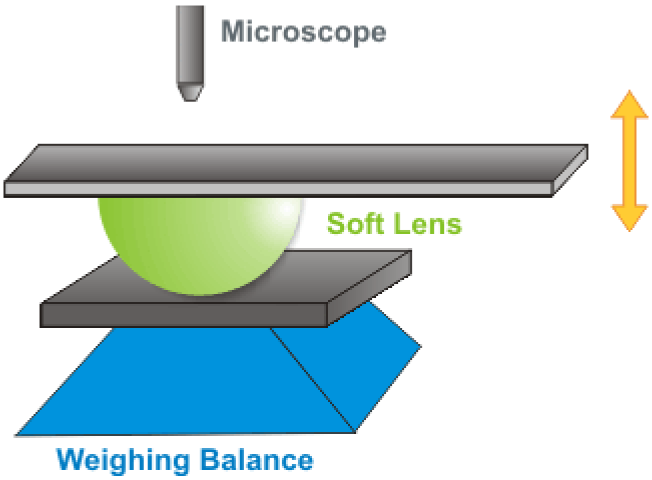
A real solution to equation 9.15 exist when

$$6W_A\pi P_0 R + (3W_A\pi R)^2 \geq 0 \quad \Rightarrow \quad P_0 \geq -\frac{3}{2}W_A\pi R \tag{9.18}$$

In other word, the separation of the spheres occur when

$$P_0 = -\frac{3}{2}W_A\pi R \tag{9.19}$$



We can rewrite equation 9.14 using 9.2 as,

$$W_A = \frac{\left(\frac{a_1^3 K}{R} - P_0\right)^2}{6\pi a_1^3 K} \tag{9.20}$$

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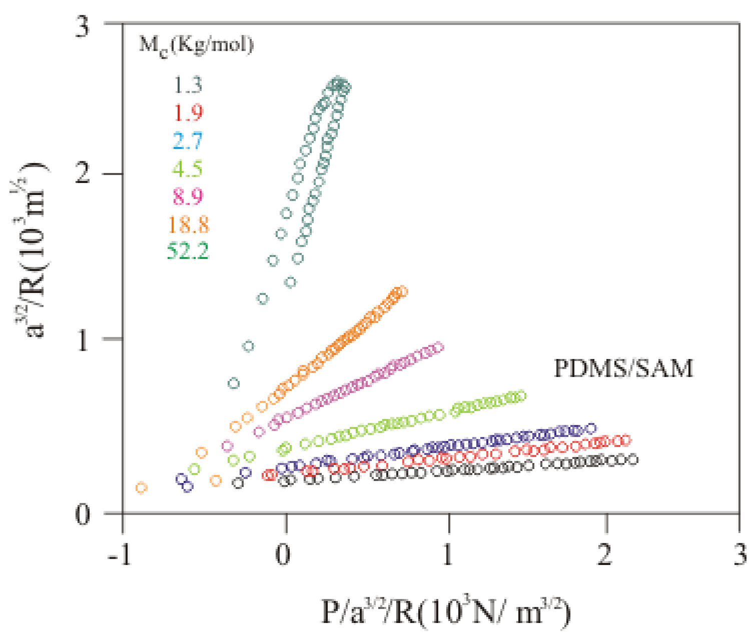
Let's say a hemispherical lens radius R made of an incompressible elastic material of modulus E is pressed against a rigid substrate. Then

$$K = \frac{4E^*}{3} = \frac{4}{3}\frac{E}{1-1/4} = \frac{16E}{9}$$

And further rearranging 9.20, one obtains

$$\frac{a_1^{\frac{3}{2}}}{R} = \frac{9}{16E}\frac{P_0}{a_1^{\frac{3}{2}}} + \frac{3}{4}\left(\frac{6W_A\pi}{E}\right)^{1/2} \qquad (9.21)$$



Thus, plotting $a_1^{3/2}/R$ vs. $P/a_1^{3/2}$ we can obtain both the elastic modulus and the work of adhesion of the material of the lens.

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