Module 2 : Solid bodies in contact with and without interactions

Lecture 5 : Rigid flat punch

The Lecture Contains:

- Rigid Flat Punch
- Frictionless Punch
- No slip Boundary Condition
- Axi-symmetric loading of an Elastic half-space

This lecture is adopted from the following book:

1. "Contact Mechanics" by K.L.Johnson

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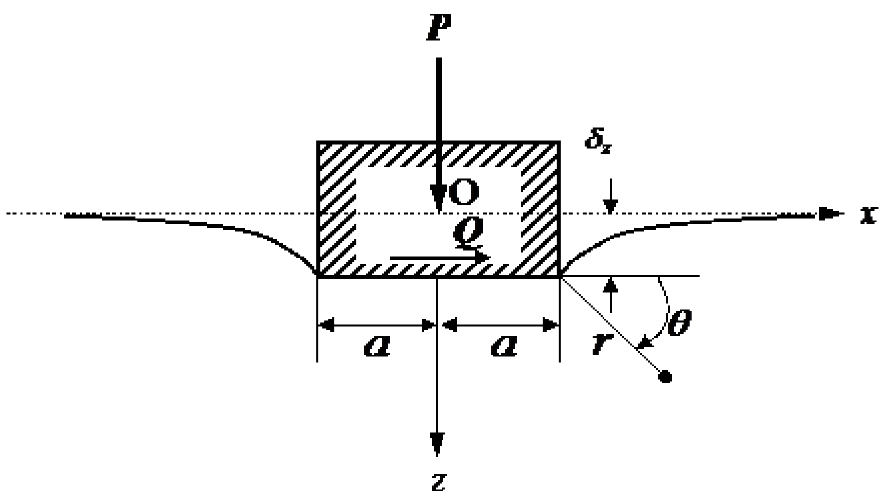
Rigid flat punch

Consider now the problem of a frictionless rigid punch which presses an elastic half space. Since the punch is rigid, the surface of the elastic solid must remain flat where it is in contact with the punch. In addition to that we assume the condition that the punch does not tilt, so that the interface remains parallel to the undeformed surface of the solid. So our first boundary condition is

$$v|_{z=0} = \text{constant} = \delta_z \qquad (5.1)$$

We can consider following situations:

(a) Surface of the punch is frictionless, so that	$q(x) = 0$	(5.2a)
(b) Friction at the interface is sufficient to prevent any slip between the punch and the surface of the solid:	$u\|_{z=0} = \delta_x$	(5.2b)
(c) Partial sip occurs to limit the tangential traction	$\|q(x)\| \le \mu p(x)$	(5.2c)



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Frictionless punch

The boundary conditions:

$$v\big|_{z=0} = \text{constant}, \quad q(x) = 0 \tag{5.3}$$

In equation 4.30 we can substitute for $g(s)$ as

$$g(s) = -\frac{\pi E}{2(1-\nu^2)}\frac{\partial v}{\partial x} = 0 \tag{5.4}$$

Which results in the following expressions for the normal component of stress

$$p(x) = \frac{C}{\pi^2(a^2 - x^2)^{1/2}} \tag{5.5}$$

The constant C is estimated by

$$P = \int_{-a}^{a} F(x)\,dx = C\int_{-a}^{a}\frac{dx}{\pi^2(a^2 - x^2)^{1/2}} = \frac{C}{\pi} \quad \text{which gives } C = P\pi \tag{5.6}$$

The pressure distribution is given as,

$$p(x) = \frac{P}{\pi(a^2 - x^2)^{1/2}} \tag{5.7}$$

Equation (5.7) shows that the pressure reaches a theoretically infinite value at the edge of the punch $x = \pm a$

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Frictionless punch (contd...)

The displacement of the surface outside the punch can be calculated from

$$v\big|_{z=0} = -\frac{2\left(1-\sigma^2\right)}{\pi E}\int_{-a}^{a} p(s)\ln|x-s|ds + \frac{(1-2\sigma)(1+\sigma)}{2E}\left\{\int_{-a}^{x} q(s)ds - \int_{x}^{a} q(s)ds\right\} + C_2$$

$$= -\frac{2\left(1-\sigma^2\right)}{\pi E}\int_{-a}^{a}\frac{P}{\pi\left(a^2-s^2\right)^{1/2}}\ln|x-s|ds + C_2 \qquad (5.8)$$

or

$$v\big|_{z=0} = \delta_z - \frac{2\left(1-\sigma^2\right)P}{\pi E}\ln\left\{\frac{x}{a} + \left(\frac{x^2}{a^2}-1\right)^{1/2}\right\}$$

The expression for the surface gradient can be obtained as:

$$\frac{\partial v\big|_{z=0}}{\partial x} = -\frac{2\left(1-\sigma^2\right)P}{\pi E}\frac{1}{\sqrt{x^2-a^2}}$$

which shows that the surface gradient at $x = \pm a$ is infinite.

We find the tangential displacements under the punch as,

$$u\big|_{z=0} = -\frac{(1-2\sigma)(1+\sigma)}{2E}\left\{\int_{-a}^{x} p(s)ds - \int_{x}^{a} p(s)ds\right\} + C_1$$

$$= -\frac{(1-2\sigma)(1+\sigma)P}{2E}\sin^{-1}\left(\frac{x}{a}\right) \qquad (5.9)$$

For a compressible material, $\sigma < 0.5$, the points on the surface moves towards the center of the punch, which can be reduced and completely prevented if there is sufficient friction on the surface.

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No slip boundary condition:

Let's say that the surface of the solid adheres completely onto the punch, so that there is no slippage of the solid, then we can have the following boundary condition:

$$u|_{z=0} = \delta_x \quad \text{and} \quad v|_{z=0} = \delta_z \tag{5.10}$$

Where δ_x and δ_z are the displacements of the punch.

We can then write following coupled equations for the displacements from equation 4.27, in which the slope of the displacements underneath the punch are zero:

$$\begin{aligned} \int_{-a}^{a} \frac{q(s)}{x-s} ds &= -\frac{\pi(1-2\sigma)}{2(1-\sigma)} p(x) \\ \int_{-a}^{a} \frac{p(s)}{x-s} ds &= \frac{\pi(1-2\sigma)}{2(1-\sigma)} q(x) \end{aligned} \tag{5.11}$$

Adding the two equations, we can have

$$p(x)+iq(x) = \frac{2(1-\sigma)}{\pi(1-2\sigma)} \int_{-a}^{a} \frac{-q(s)+ip(s)}{x-s} ds = \frac{2(1-\sigma)i}{\pi(1-2\sigma)} \int_{-a}^{a} \frac{p(s)+iq(s)}{x-s} ds \tag{5.12}$$

Equation 5.12 is of the general form

$$F(x) + \frac{\lambda}{\pi} \int_{-a}^{a} \frac{F(s)}{x-s} ds = 0, \quad \lambda = \frac{2(1-\sigma)i}{(1-2\sigma)}$$

Which has the solution,

$$F(x) = -\frac{\lambda}{\left(1+\lambda^2\right)^{1/2}} \frac{a}{\pi\left(a^2-x^2\right)} \left(\frac{a+x}{a-x}\right)^{\gamma} C, \tag{5.13}$$

Where, $e^{2\pi i\gamma} = \dfrac{i\lambda-1}{i\lambda+1}$ and $C = \displaystyle\int_{-a}^{a} F(x)dx$

So the solution is obtained as,

$$\begin{aligned} p(x)+iq(x) &= \frac{2(1-\sigma)}{(3-4\sigma)^{1/2}} \frac{P+iQ}{\pi\left(a^2-x^2\right)^{1/2}} \left(\frac{a+x}{a-x}\right)^{i\eta}, \quad \eta = \frac{1}{2\pi}\ln(3-4\sigma) \\ &= \frac{2(1-\sigma)}{(3-4\sigma)^{1/2}} \frac{P+iQ}{\pi\left(a^2-x^2\right)^{1/2}} \left[\cos\left\{\eta \ln\left(\frac{a+x}{a-x}\right)\right\} + i\sin\left\{\eta \ln\left(\frac{a+x}{a-x}\right)\right\}\right] \end{aligned} \tag{5.14}$$

Axi-symmetric loading of an elastic half-space

Concentrated normal force, point loading

The biharmonic equation can be written as

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}\right)=0 \tag{5.15}$$

The solution of the biharmonic equation has been obtained, and following are the stress and the displacements components,

$$\begin{aligned}
R_r &= \frac{P}{2\pi}\left\{(1-2\sigma)\left(\frac{1}{r^2}-\frac{z}{\rho r^2}\right)-\frac{3zr^2}{\rho^5}\right\}\\
\Theta_\theta &= -\frac{P}{2\pi}(1-2\sigma)\left(\frac{1}{r^2}-\frac{z}{\rho r^2}-\frac{z}{\rho^3}\right)\\
Z_z &= -\frac{3P}{2\pi}\frac{z^3}{\rho^5}\\
R_z &= -\frac{3P}{2\pi}\frac{rz^2}{\rho^5}
\end{aligned} \tag{5.16}$$

and

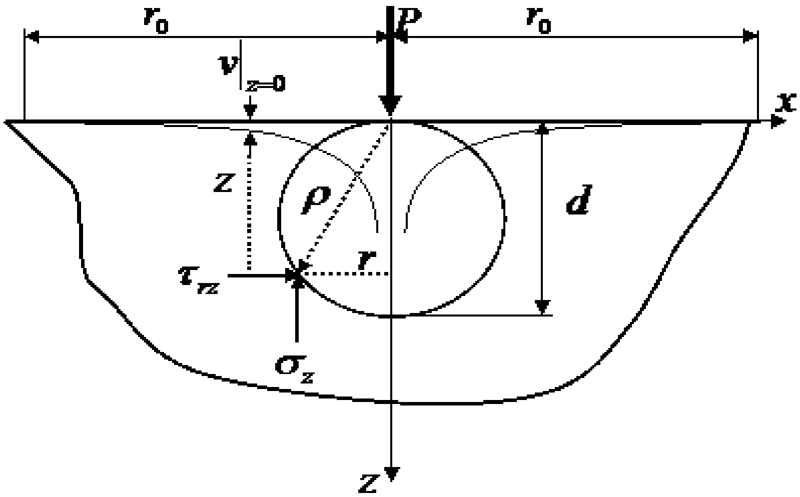
$$u=\frac{P}{4\pi\mu}\left\{\frac{rz}{\rho^3}-(1-2\sigma)\frac{\rho-z}{\rho r}\right\}$$

$$v=\frac{P}{4\pi\mu}\left\{\frac{z^2}{\rho^3}+\frac{2(1-\sigma)}{\rho}\right\}$$

At the surface, i.e. at $z=0$

$$\begin{aligned}
u &= -\frac{P}{4\pi\mu}\left\{(1-2\sigma)\frac{1}{r}\right\}\\
v &= \frac{P}{4\pi\mu}\left\{\frac{2(1-\sigma)}{r}\right\}
\end{aligned} \tag{5.17}$$

The equation shows that the profile of the deformed surface is a rectangular hyperboloid which is asymptotic to the undeformed surface at large distance from the point of application of the load.



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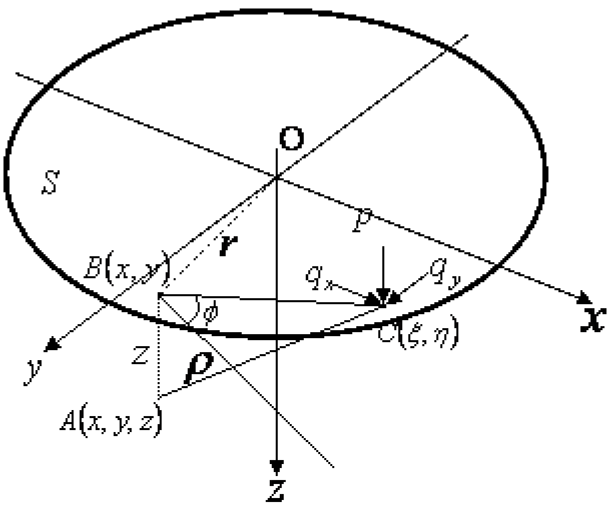
The stress and the displacement for a distributed load on an area:

We want to find out the surface depression at a location B (x, y) and stress distribution at an interior point A due to a distributed pressure $p(\xi,\eta)$ acting on the surface area S. We can change to a polar coordinate (s,ϕ) with origin at B, hence the pressure acting at an elemental area C is equivalent to a force $ps\,ds\,d\phi$.

The displacement of the surface at B due to this force can be written from equation 5.17, by putting (s,ϕ) The displacement at B, due to the distribution of load over the whole area S is given by,

$$v\big|_{z=0} = \frac{1-\sigma}{2\pi\mu}\iint_{S\phi} p(s,\phi)\,ds\,d\phi = \frac{1-\sigma^2}{\pi E}\iint_{S\phi} p(s,\phi)\,ds\,d\phi \qquad (5.18)$$



The stress components at A can be found out by integrating the stress components given in equation 5.16.

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