Module 3 : Equilibrium of rods and plates

Lecture 18 : Wrinkling of a thin sheet under uni-axial flexible strain

The Lecture Contains:

- Wrinkling of a thin sheet under uni-axial tensile strain

This lecture is adopted from the following book

1. Geometry and Physics of Wrinkling, E. Cerda and L. Mahadevan, Physical Review Letters, Vol. 90(7), 074302-1 -- 074302-4 (2003).

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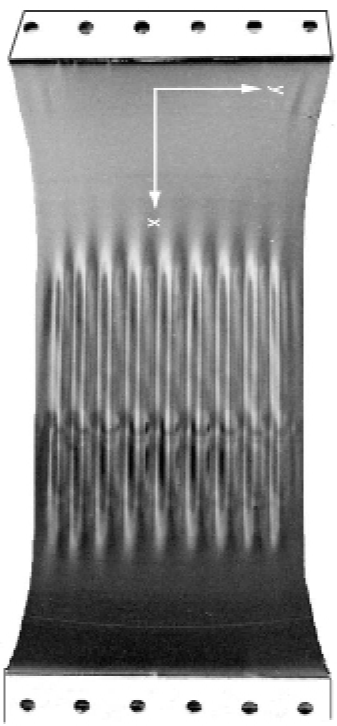
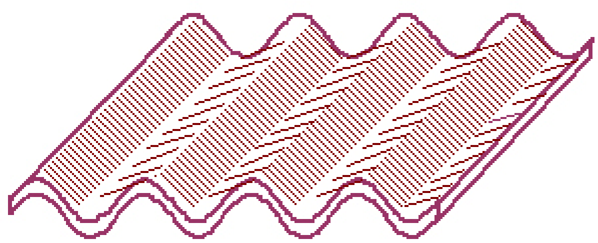
Wrinkling of a thin sheet under uni-axial tensile strain

A thin isotropic elastic sheet of thickness t, width W and length L ($t << W << L$) made of a material with Young's modulus E and Poisson's ratio ν remains clamped at two sides along y while it is subjected to longitudinal stretching strain γ along x by applying a tensile stress $T(x)$. So long as, the strain is less than a critical strain, $\gamma < \gamma_c$, the sheet stretches uniformly, however, at $\gamma > \gamma_c$, the stretching results in appearance of wrinkles along y. The wrinkles appear as the clamped boundaries prevent the sheet from contracting locally at their vicinity by creating a bi-axial state of stress. The bi-axial stress is tensile near the clamped boundary and compressive slightly away from it. When $\gamma \sim \gamma_c$, the sheet buckles to accommodate the in-plane strain incompatibility generated via the Poisson effect.





Bending energy due to short wavelength wrinkles in the y direction,

$$U_B = \frac{1}{2}\int_A D\left(\frac{\partial^2 \zeta}{\partial y^2}\right)^2 dA \tag{18.1}$$

And the strain due to extension along x direction,

$$e_{xx} = \frac{\left[\left((\partial x)^2 + (\partial \zeta)^2\right)^{\frac{1}{2}} - \partial x\right]}{\partial x} = \frac{1}{2}\left(\frac{\partial \zeta}{\partial x}\right)^2 \tag{18.2}$$

Hence, the stretching energy is written as,

$$U_S = \frac{1}{2}\int_A T(x)\left(\frac{\partial \zeta}{\partial x}\right)^2 dA \qquad (18.3)$$

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In addition to this, there is a fictitious transverse force which causes the wrinkles to appear. This unknown transverse force is $b(x)$ per unit length along y direction which causes extension $\frac{1}{2}\left(\frac{\partial\zeta}{\partial y}\right)^2-\frac{\Delta}{W}$, where $\Delta(x)$ is the imposed compressive transverse displacement. Then the work done by this force is

$$U_M=-\int_A b(x)\left[\frac{1}{2}\left(\frac{\partial\zeta}{\partial y}\right)^2-\frac{\Delta}{W}\right]dA \tag{18.4}$$

Hence, total energy is written as

$$U=U_B+U_S+U_M \tag{18.5}$$

Minimization of the total energy with respect to out of place displacement ζ yields Euler-Lagrange equation

$$D\frac{\partial^4\zeta}{\partial y^4}-T(x)\frac{\partial^2\zeta}{\partial x^2}+b(x)\frac{\partial^2\zeta}{\partial y^2}=0 \tag{18.6}$$

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For the stretched sheet, $T(x) \sim Et\gamma =$ constant, while $\Delta(x) \sim \nu\gamma W =$ constant far from the clamped boundaries, so that $b(x) =$ constant. At the clamped boundaries the boundary conditions are written as,

$$\zeta(0, y) = \zeta(L, y) = 0 \tag{18.7}$$

Substituting in equation (18.7), a periodic solution of the form $\zeta(x, y) = \sum_n e^{ik_n y} X_n(x)$, where $k_n = 2\pi n/W$ and n is the number of wrinkles, yields a Sturm-Liouville-like problem,

$$\frac{\partial^2 X_n}{\partial x^2} + \frac{bk_n^2 - Dk_n^4}{T} X_n = 0, \quad X_n(0) = X_n(L) = 0 \tag{18.8}$$

Writing, $\omega_n^2 = \frac{bk_n^2 - Dk_n^4}{T}$, equation 18.8 simplifies to $\frac{\partial^2 X_n}{\partial x^2} + \omega_n^2 X_n = 0$. We can use solutions of the form for $X_n(x) = A_n \sin \omega_n x, \quad \omega_n = m\pi/L$, Since the solution with least bending energy corresponds to $m = 1$, so that $\frac{bk_n^2 - Dk_n^4}{T} = \frac{\pi^2}{L^2}$, rearranging, we have $b = \frac{\pi^2 T}{L^2 k_n^2} + Dk_n^2$. Then the solution of $\zeta(x, y)$ is written as, $\zeta = A_n \cos(k_n y + \phi_n) \sin\left(\frac{\pi x}{L}\right)$. Plugging this expression in 18.4, yields $\frac{A_n^2 k_n^2 W}{8} \approx \Delta$, relating the wave number and amplitude, so that we can write total energy as $U = U_B + U_S + U_M$. Minimizing U, i.e. from $\partial U/\partial k_n = 0$ we obtain an expression for the wavelength:

$$\lambda = 2\sqrt{\pi}\left(\frac{D}{T}\right)^{1/4} L^{1/2} \tag{18.9}$$

Substituting this result in the condition of inextensibility along y,

$$\int_0^L \left[\frac{1}{2}\left(\frac{\partial \zeta}{\partial y}\right)^2 - \frac{\Delta}{W}\right] dy = 0 \tag{18.10}$$

We obtain an expression for the amplitude

$$A = \frac{\sqrt{2}}{\pi}\left(\frac{\Delta}{W}\right)^{1/2} \lambda \tag{18.11}$$

Putting the expressions for flexural rigidity of the sheet, $D = \frac{Et^3}{12(1-\nu^2)}$, tensile stress, $T(x) \sim Et\gamma$ and transverse displacement, and substituting these expressions in equation 18.9, we

$$\Delta(x) = \nu\gamma W$$

obtain λ in terms of measurable quantities as,

$$\lambda = \frac{(2\pi Lt)^{1/2}}{\left[3\gamma\left(1-\nu^2\right)\right]^{1/4}} \quad \text{and} \quad A = (\nu Lt)^{1/2}\left[\frac{16\gamma}{3\pi^2\left(1-\nu^2\right)}\right]^{1/4} \qquad (18.12)$$

The above expressions have been verified from experiments.

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