Module 3 : Equilibrium of rods and plates

Lecture 12 : Bending of a rod by couples applied at its end

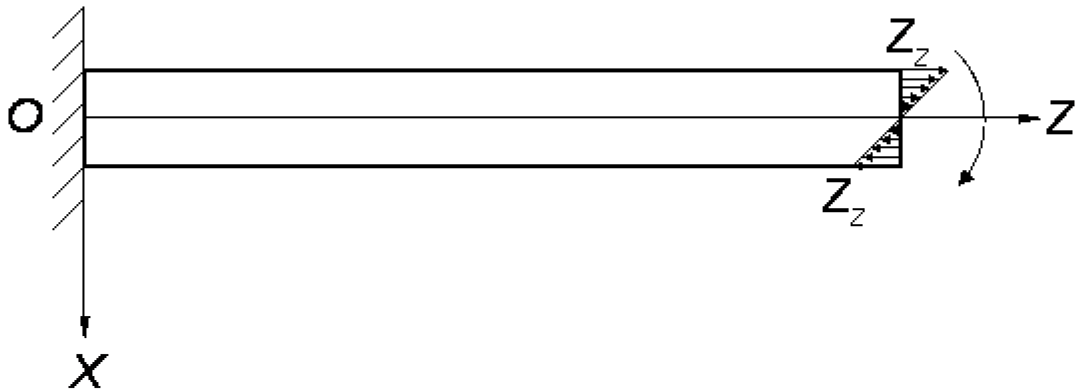
The Lecture Contains:

This lecture is adopted from the following book

1. “Theory of Elasticity, 3 rd edition” by Landau and Lifshitz. Course of Theoretical Physics, vol-7

Previous Next

Bending of a rod by couples applied at its ends:



The rod is placed horizontally and is bent by applying a couple at its end as shown in the figure. In the small bending limit, the rod is stretched in the convex side, whereas in the concave side it is under compression. The neutral surface lying in the OZ plane separates these two regions.

Here by small bending we mean that the deformation as well as the strain is small. Similar to bending of plates, external forces on the sides of the rod are small with respect to the internal stresses, so that at its sides, we have $\sigma_{ik}n_k = 0$ which ultimately results in that the all the components of the stress tensor except σ_{zz} is zero. Let there be a small element dz lying along the Z axis at a distance x from the neutral surface. On bending the length of this element becomes dz'. The elements dz and dz' lie on arcs of radii R and $R-x$ respectively. Then it can be easily shown that

$$dz' = \frac{R-x}{R}dz = \left(1-\frac{x}{R}\right)dz \qquad (12.1)$$

so that the relative expansion is $e_{zz} = (dz'-dz)/dz = -x/R$. We now find σ_{zz} as $\sigma_{zz} = -Ex/R$. The resultant traction over any cross-section is $\iint \sigma_{zz}dx\,dy$ and it is equal to zero if the z axis coincides with the line of centroid of the normal section. The component of the couple about the z axis vanishes. However, that about the y axis is nonzero and is $\iint \frac{Ex^2}{R}dx\,dy$ or $\frac{EI}{R}$ where I is the moment of inertia of the section about an axis through its centroid parallel to the y axis. Two more components of the strain tensor besides e_{zz} are non-zero: they are,

$$e_{xx} = e_{yy} = \sigma x/R \qquad (12.2)$$

Previous Next

Other components of the strain tensor are zero, so that:

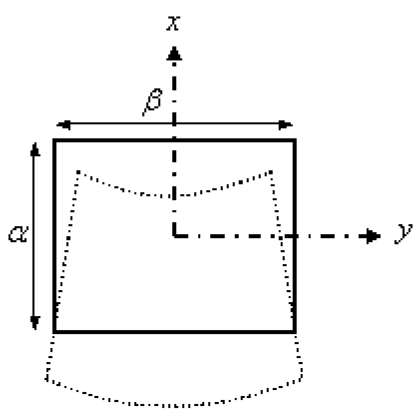
$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}=0 \qquad (12.3)$$

Integrating, the displacements can be given as

$$u=\frac{1}{2R}\left(z^2+\sigma x^2-\sigma y^2\right),\quad v=\frac{\sigma xy}{R},\quad w=-\frac{xz}{R} \qquad (12.4)$$

The formulae for the displacements show that the cross-sections remain plane but the planes are rotated so that they pass through the centre of curvature. Shape of the cross-sections is changed. For example if the cross sections initially are rectangular with boundaries $x=\pm\alpha,\quad y=\pm\beta$ in a plane $z=\gamma$, these boundaries will become the curves given respectively by,

$$x\mp\alpha-\frac{\gamma^2}{2R}-\frac{\sigma}{2R}\left(\alpha^2-y^2\right)=0,\quad y\mp\beta\mp\frac{\sigma\beta x}{R}=0 \qquad (12.5)$$



For a rod with rectangular cross-section (sides α, β) the principal moments of inertia are

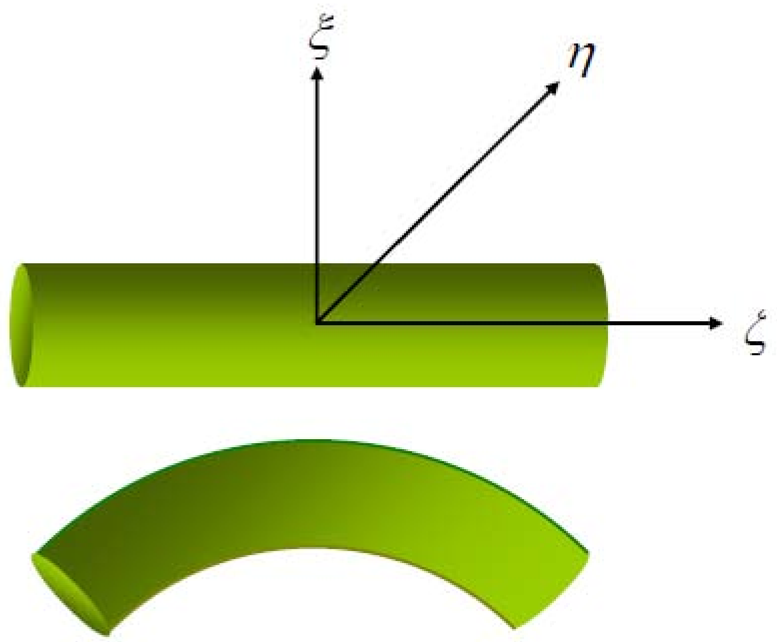
$$I_y=\frac{\alpha^3\beta}{12},\quad I_x=\frac{\alpha\beta^3}{12} \qquad (12.6)$$

For a rod with circular cross-section, with radius r, the moment of inertia is

$$I=\frac{\pi r^4}{4} \qquad (12.7)$$

The energy of a bent rod

We have so far discussed the bending deformation of only a small portion of the rod; however as we discuss the deformation whole through the rod, we will immediately see that the deformation does not remain small, rather the rod undergoes large bending deformations. Furthermore bending of the rod is also accompanied by some torsion, so that the rod undergoes both bending and twisting deformations.



To describe the deformation we divide the rod into infinitesimal elements each of which is bounded by two adjacent cross-sections. For each such element we choose a co-ordinate system ξ, η, ζ so that in the un-deformed state all these systems are parallel to each other and their ζ axes is parallel to the axis of the rod but in the deformed state the co-ordinate system for each element is rotated infinitesimally relative to each other.

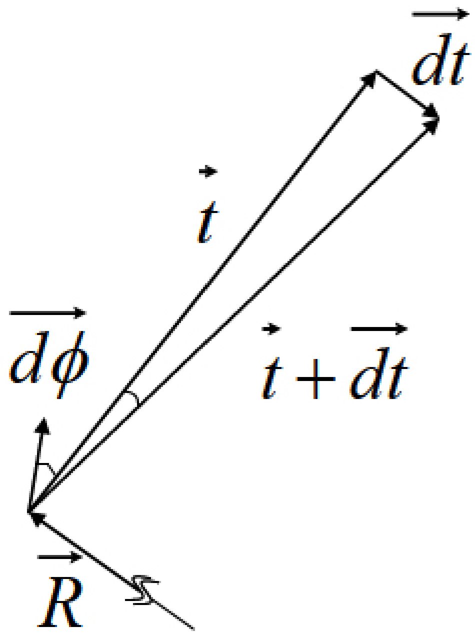
Let $d\bar{\phi}$ be the vector of the angle of relative rotation of two systems at a distance dl apart along the rod. The infinitesimal angle of rotation can be regarded as vector the components of which are the angles of rotation about the three axes of co-ordinates. We define a vector

$$\bar{\Omega} = d\bar{\phi} / dl \qquad (12.8)$$

which is the rate of rotation of the co-ordinate axes along the rod. If the deformation is pure rotation, then $\bar{\Omega}$ is the vector parallel to the axis of the rod and is magnitude is the torsion angle $\Omega_\zeta = \tau$. If the plane of bending is $\xi\zeta$ the rotation is about the η axis, i.e. $\bar{\Omega}$ parallel to the η axis.

Previous Next

Let the unit vector $\bar{t}$ is tangential to the rod. Then the derivative $d\bar{t}/dl$ is the curvature to the line: $d\bar{t}/dl = 1/R$.



The change in a vector due to infinitesimal rotation is the vector product of the rotation vector and the vector itself: $d\bar{t} = d\bar{\phi}\times\bar{t}$, or dividing by dl

$$d\bar{t}/dl = \bar{\Omega}\times\bar{t} \tag{12.9}$$

Vector multiplication by $\bar{t}$ gives

$$\bar{t}\times d\bar{t}/dl = \bar{t}\times\bar{\Omega}\times\bar{t} = \bar{\Omega}(\bar{t}\cdot\bar{t}) - \bar{t}(\bar{t}\cdot\bar{\Omega}) = \bar{\Omega} - \bar{t}(\bar{t}\cdot\bar{\Omega})$$

or

$$\bar{\Omega} = \bar{t}\times d\bar{t}/dl + \bar{t}(\bar{t}\cdot\bar{\Omega}) \tag{12.10}$$

Since the tangent vector points in the direction of the ς axis of the rod, $\bar{t}\cdot\bar{\Omega} = \Omega_\varsigma$. Say the unit vector $\bar{n}$ points to the principal normal: $\bar{n} = R\, d\bar{t}/dl$, then we have:

$$\bar{\Omega} = \bar{t}\times\bar{n}/R + \bar{t}\Omega_\varsigma \tag{12.11}$$

The unit vector $\bar{t}\times\bar{n}/R$ is a binormal vector with components Ω_ξ, Ω_η so that its magnitude is equal to the curvature $1/R$. For plane curves, binormal vector is the unit vector normal to the plane. Principal normal is the usual normal to the curve directed towards the center of curvature at that point. Since the elastic energy of the rod is a quadratic function of deformation, in this case it will be function of the components of the vector $\bar{\Omega}$. However, these terms should not contain the expressions $\Omega_\xi\Omega_\varsigma$ and $\Omega_\eta\Omega_\varsigma$ as they are dependent on the direction of the ς axis. Finally if we have ξ and η axes coinciding with the principal axes of inertia, then we should not have also the term $\Omega_\eta \Omega_\xi$ .The total energy of the rod then consists of that due to torsion about the ζ axis and bending about the ξ and η axes:

$$F_{rod} = \int \{\frac{1}{2} I_1 E \Omega_\xi^2 + \frac{1}{2} I_2 E \Omega_\eta^2 + \frac{1}{2} C \Omega_\zeta^2\} dl \qquad (12.12)$$

The moment applied about the ζ axis, $M_\zeta = dF_{rod} / d\Omega_\zeta = C\Omega_\zeta$. Other components of the moment, can similarly be obtained as,

$$M_\xi = EI_1 \Omega_\zeta, \quad M_\eta = EI_2 \Omega_\eta \qquad (12.13)$$

And the elastic energy in terms of the moments are,

$$F_{rod} = \int \{\frac{M_\xi^2}{2I_1 E} + \frac{M_\eta^2}{2I_2 E} + \frac{M_\zeta^2}{2C}\} dl \qquad (12.14)$$

Previous Next