The Lecture Contains:

This lecture is adopted from the following book:

1. "Contact Mechanics" by K.L.Johnson

Herttzian Mechanics

The profile of each surface in the vicinity of the origin may be expressed in the form

$$z_1 = A_1 x^2 + B_1 y^2 + C_1 xy \tag{8.1}$$

where, we neglect the higher order terms. Furthermore, we can choose the orientation of the axes such that terms like xy vanish. Then equation 5.1 may be written as:

$$z_1 = -\left(\frac{1}{2R_1'} x_1^2 + \frac{1}{2R_1'} y_1^2\right) \tag{8.2}$$

Where R_1' and R_2' are the principal radii of curvature. Note that they are the minimum and maximum value of the radii of curvature of the surface at the point. If a cross-sectional plane of symmetry exists one principal of curvature lies in that plane.

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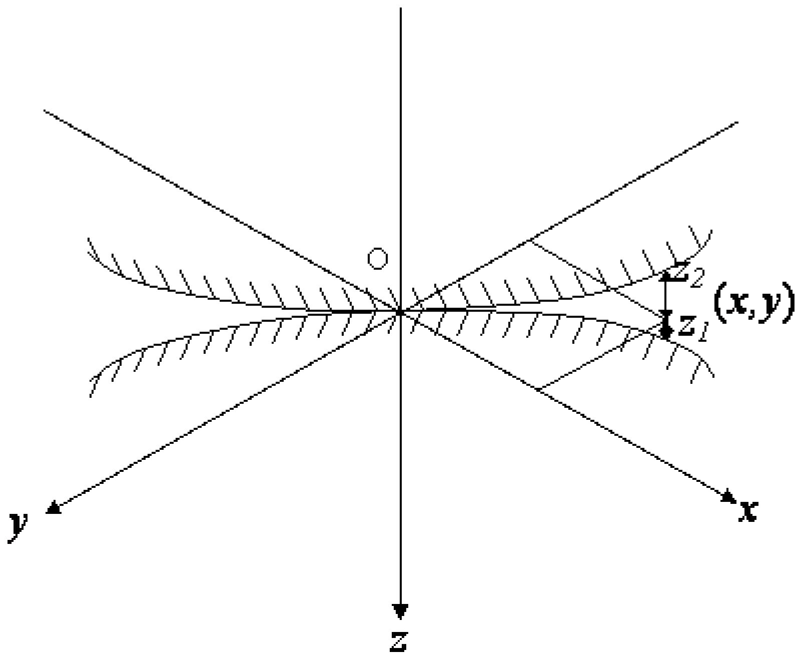
Herttzian Mechanics (contd...)

A similar expression can be written for the second surface:

$$z_2 = -\left(\frac{1}{2R_2^{'}}x_2^2 + \frac{1}{2R_2^{''}}y_2^2\right) \tag{8.3}$$



The separation between the two surfaces is given as $h = z_1 - z_2$.

We can now transpose equation 8.1 to a common set of axes x and y,

$$h = Ax^2 + By^2 + Cxy \tag{8.4}$$

In which after suitable choice of axes, C vanishes, leaving

$$h = Ax^2 + By^2 = \frac{1}{2R^{'}}x^2 + \frac{1}{2R^{''}}y^2 \tag{8.5}$$

Where A, B are constants and R', R'' are principal relative radius of curvature.

If the axes of the principal curvature of each surface, i.e. the x_1 and x_2 are inclined to each other by an angle α , then it can be shown that

$$(A+B) = \frac{1}{2}\left(\frac{1}{R'} + \frac{1}{R''}\right) = \frac{1}{2}\left(\frac{1}{R_1^{'}} + \frac{1}{R_1^{''}} + \frac{1}{R_2^{'}} + \frac{1}{R_2^{''}}\right) \tag{8.6a}$$

and

$$|B - A| = \frac{1}{2}\left\{\left(\frac{1}{R_1^{'}} - \frac{1}{R_1^{''}}\right)^2 + \left(\frac{1}{R_2^{'}} - \frac{1}{R_2^{''}}\right)^2 + 2\left(\frac{1}{R_1^{'}} - \frac{1}{R_1^{''}}\right)\left(\frac{1}{R_2^{'}} - \frac{1}{R_2^{''}}\right)\cos 2\alpha\right\}^{1/2} \tag{8.6b}$$

Herttzian Mechanics (contd...)

We further introduce an equivalent radius defined by

$$R_e = \left(R'R''\right)^{1/2} = \frac{1}{2}\left(AB\right)^{-1/2} \tag{5.7}$$

So far in our discussions we have considered only convex surface, which has positive radius. But it is applicable also for concave or saddle surface, with a negative sign to the radius of curvature.

It is evident from equation 8.5 that contours of constant gap h between the two surfaces is an ellipse with, the length of axes are in the ratio $\left(B / A\right)^{1/2} = \left(R' / R''\right)^{1/2}$.

Let's say two cylindrical lenses each with radius $R_1' = R_2' = R$, $R_1'' = R_2'' = \infty$ and $\alpha = 45°$ then $\left(A + B\right) = \frac{1}{R}$ and $B - A = \frac{1}{\sqrt{2}}\frac{1}{R}$, i.e. $A = \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{2R}$, $B = \left(1 + \frac{1}{\sqrt{2}}\right)\frac{1}{2R}$.

Thus, the relative radii of curvatures are, $R' = \frac{1}{2A} = 3.42R$ and $R'' = \frac{1}{2B} = 0.585R$.

The equivalent radius $R_e = \left(R'R''\right)^{1/2} = \frac{1}{\sqrt{2}}R$ and $\left(\frac{R'}{R''}\right)^{1/2} = \left(\frac{B}{A}\right)^{1/2} = 2.41$.

This is the ratio of major to minor axes of the contours of constant separation as discussed earlier.

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Let's say two solids are in point contact initially, which on application of load P now contacts through an area. Before deformation the separation between two corresponding points $S_1(x, y, z_1)$ and $S_2(x, y, z_2)$ is given by equation 8.5.

Due to the symmetry of this expression about O, the contact region spans equal distance on its either side. During compression two distant points T_1 and T_2 move towards each other along the z axis by distances δ_1 and δ_2 respectively.

Furthermore, due to the contact pressure the surfaces of the two bodies are displaced by distances $v_1|_{z=0}$ and $v_2|_{z=0}$ respectively. Then considering that the two points S_1 and S_2 coincide at the surface, we should have,

$$v_1|_{z=0} + v_2|_{z=0} + h = \delta_1 + \delta_2 \tag{8.8}$$

Writing $\delta_1 + \delta_2 = \delta$ and making use of 8.5,

$$v_1|_{z=0} + v_2|_{z=0} = \delta - Ax^2 - By^2 \tag{8.9}$$

If S_1 and S_2 lie outside the contact area, so that they do not touch each other,

$$v_1|_{z=0} + v_2|_{z=0} > \delta - Ax^2 - By^2 \tag{8.10}$$

Putting $\delta_1 = v_1(0)|_{z=0}$ and $\delta_2 = v_2(0)|_{z=0}$, equation 5.8 can be written in dimensionless form:

$$\frac{v_1(0)|_{z=0}}{a} - \frac{v_1(x)|_{z=0}}{a} + \frac{v_2(0)|_{z=0}}{a} - \frac{v_2(x)|_{z=0}}{a} = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{x^2}{a} \tag{5.11}$$

Putting $x = a$ and writing $v_1(0)|_{z=0} - v_1(a)|_{z=0} = d$, the deformation within the contact area becomes,

$$\frac{d_1}{a} + \frac{d_2}{a} = \frac{a}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{5.12}$$

Given that the deformation is small, i.e. $d << a$, the state of strain in each solid is characterized by the ratio d/a . Now the magnitude of the strain will be proportional to the contact pressure divided by the elastic modulus. If p_m is the average pressure acting mutually on each solid, equation 5.11 becomes,

$$\frac{p_m}{E_1} + \frac{p_m}{E_2} \propto a\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \text{ i.e. } p_m \propto \frac{a\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}{\frac{1}{E_1} + \frac{1}{E_2}} \tag{8.13}$$

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Let's say two cylinders are in contact, so that the load per unit of the axial length is $P = 2ap_m$, which results in,

$$a \propto \left(\frac{P\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}{\frac{1}{E_1} + \frac{1}{E_2}} \right)^{1/2} \text{ and } p_m \propto \left(\frac{P\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}{\frac{1}{E_1} + \frac{1}{E_2}} \right)^{1/2} \qquad (8.14)$$

i.e. both contact pressure and the contact width both increase with the square root of the applied load.

Let's say we are now dealing with solids of revolution so that the contact area is a circle of radius, then $P = p_m \pi a^2$,

$$a \propto \left(\frac{P\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}{\frac{1}{E_1} + \frac{1}{E_2}} \right)^{1/3} \text{ and } p_m \propto \left(\frac{P\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^2}{\left(\frac{1}{E_1} + \frac{1}{E_2}\right)^2} \right)^{1/3} \qquad (8.15)$$

In this case, the radius of contact circle and contact pressure increases with the cube root of the load.

While in last paragraph we deduced the relation of contact width to applied load by dimensional analysis, now we will do that in a more rigorous manner by solving the elasticity equations.

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Solids of Revolution

We first consider solids of revolution ($R_1^{'} = R_1^{''} = R_1; R_2^{'} = R_2^{''} = R_2$) for which the contact area is circular with radius a . The boundary condition for displacement within the contact area is given by

$$v_1\big|_{z=0} + v_2\big|_{z=0} = \delta - \frac{r^2}{2R} \tag{8.16}$$

where $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ is the relative curvature. A distribution of pressure which gives the displacement of the form 8.16 is $p = p_0\left(1 - \frac{r^2}{a^2}\right)^{1/2}$ for which the normal displacement has been obtained as

$$v_1\big|_{z=0} = \frac{1-\sigma_1^2}{E_1}\frac{\pi p_0}{4a}\left(2a^2 - r^2\right) \qquad r << a \tag{8.17}$$

Since pressure acting on either body is equal, we add the expressions for displacements on the body 1 and 2

$$v_1\big|_{z=0} = \frac{1-\sigma_1^2}{E_1}\frac{\pi p_0}{4a}\left(2a^2 - r^2\right) \quad \text{and} \quad v_2\big|_{z=0} = \frac{1-\sigma_2^2}{E_2}\frac{\pi p_0}{4a}\left(2a^2 - r^2\right) \tag{8.18}$$

Furthermore we define an equivalent elastic modulus $\frac{1}{E^*} = \frac{1-\sigma_1^2}{E_1} + \frac{1-\sigma_2^2}{E_2}$ and add the normal displacements, to deduce an expression very similar to 8.16.

$$v_1\big|_{z=0} + v_2\big|_{z=0} = \frac{1}{E^*}\frac{\pi p_0}{4a}\left(2a^2 - r^2\right) = \delta - \frac{r^2}{2R} \tag{8.19}$$

The expressions for δ and a is now readily obtained as,

$$\delta = \frac{\pi p_0 a}{2E^*}, \qquad a = \frac{\pi p_0 R}{2E^*} \tag{8.20}$$

Since in most practical situations we have the data of total load compressing the solids, we obtain it as in equation 5.21:

$$P = \int_0^a p(r) 2\pi r dr = \int_0^a 2\pi r p_0 \left(1 - \frac{r^2}{a^2}\right)^{1/2} dr = \frac{2}{3} p_0 \pi a^2 \tag{8.21}$$

Then we can obtain the expressions for a , δ and p_0:

$$a = \left(\frac{3PR}{4E^*}\right)^{1/3} \qquad \delta = \frac{1}{2E^*}\frac{3P}{2}\frac{1}{a} = \left(\frac{9P^2}{16RE^{*2}}\right)^{1/3} \qquad p_0 = \left(\frac{6PE^{*2}}{\pi^3 R^2}\right)^{1/3} \tag{8.22}$$