Module 2 : Solid bodies in contact with and without interactions

Lecture 4 : Loading on an elastic half space

The Lecture Contains:

This lecture is adopted from the following book

1. "Contact Mechanics " by K.L. Johnson

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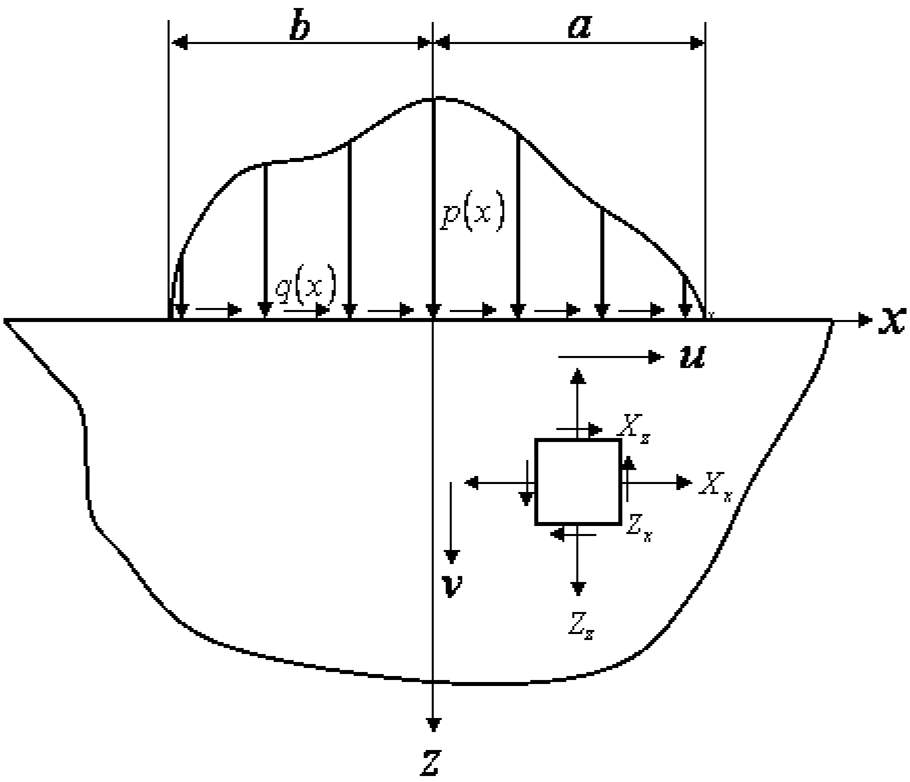
Loading on an elastic half space:

Line loading

Here we will discuss the stresses and deformations on an elastic half space loaded one dimensionally over a strip.



Consider normal and shear loading of the elastic half space under plane strain conditions. Stress equilibrium relation under plane strain approximation:

$$\begin{aligned}\frac{\partial X_x}{\partial x}+\frac{\partial X_z}{\partial z}&=0\\ \frac{\partial Z_x}{\partial x}+\frac{\partial Z_z}{\partial z}&=0\end{aligned} \qquad (4.1)$$

where the normal and shear stress components are related to the shear stresses as:

$$\begin{aligned}X_x&=\frac{E\sigma}{(1+\sigma)(1-2\sigma)}\Delta+\frac{E}{(1+\sigma)}e_{xx}=\frac{E}{(1+\sigma)(1-2\sigma)}\left((1-\sigma)e_{xx}+\sigma e_{zz}\right)\\ Z_z&=\frac{E\sigma}{(1+\sigma)(1-2\sigma)}\Delta+\frac{E}{(1+\sigma)}e_{zz}=\frac{E}{(1+\sigma)(1-2\sigma)}\left((1-\sigma)e_{zz}+\sigma e_{xx}\right)\\ Z_x&=X_z=\frac{E}{(1+\sigma)}e_{zx}, \quad \Delta=e_{xx}+e_{zz}\end{aligned} \qquad (4.2)$$

Conversely, the strain components in terms of stresses are written as:

$$e_{xx} = \frac{1}{E}\left(\left(1-\sigma^2\right)X_x - \sigma(1+\sigma)Z_z\right)$$
$$e_{zz} = \frac{1}{E}\left(\left(1-\sigma^2\right)Z_z - \sigma(1+\sigma)X_x\right) \quad (4.3)$$
$$e_{xz} = \frac{2(1+\sigma)}{E}Z_x$$

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Boundary conditions:

at $z = 0, x < -b, x > a, \; Z_z = X_z = 0$

at $z = 0, -b < x < a, \qquad Z_z = -p(x), X_z = q(x)$ (4.4)

Similarly, in cylindrical co-ordinate system the strain components can be written as,

$$e_{rr} = \frac{1}{E}\left(\left(1-\sigma^2\right)R_r - \sigma(1+\sigma)\Theta_\theta\right) \qquad R_r = \frac{E}{(1+\sigma)(1-2\sigma)}\left((1-\sigma)e_{rr} + \sigma e_{\theta\theta}\right)$$

$$e_{\theta\theta} = \frac{1}{E}\left(\left(1-\sigma^2\right)\Theta_\theta - \sigma(1+\sigma)R_r\right) \qquad \Theta_\theta = \frac{E}{(1+\sigma)(1-2\sigma)}\left((1-\sigma)e_{\theta\theta} + \sigma e_{rr}\right)$$

$$e_{r\theta} = \frac{2(1+\sigma)}{E}\Theta_r \qquad \Theta_r = \frac{E}{2(1+\sigma)}e_{r\theta}$$

Compatibility relation

$$\frac{\partial^2 e_{xx}}{\partial z^2}+\frac{\partial^2 e_{zz}}{\partial x^2}=2\frac{\partial^2 e_{zx}}{\partial z\partial x} \tag{4.4}$$

Let's say, we consider a function $\phi(x,z)$, such that,

$$X_x=\frac{\partial^2\phi}{\partial z^2},\quad Z_z=\frac{\partial^2\phi}{\partial x^2},\quad X_z=-\frac{\partial^2\phi}{\partial x\partial z} \tag{4.6}$$

$$\begin{gathered}\frac{\partial^2}{\partial z^2}\left((1-\sigma)\frac{\partial^2\phi}{\partial z^2}-\sigma\frac{\partial^2\phi}{\partial x^2}\right)+\frac{\partial^2}{\partial x^2}\left((1-\sigma)\frac{\partial^2\phi}{\partial x^2}-\sigma\frac{\partial^2\phi}{\partial z^2}\right)=2\frac{\partial^2}{\partial z\partial x}\left(-\frac{\partial^2\phi}{\partial x\partial z}\right)\\ \frac{\partial^4\phi}{\partial z^4}+\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial z^2}=0\ \Rightarrow\ \left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial x^2}\right)=0\end{gathered} \tag{4.7}$$

In some situation it is convenient to use the cylindrical polar coordinates (r,θ,z), hence we write equation 3.7 as

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)=0 \tag{4.8}$$

And the normal stresses along (r,θ,z) as,

$$\begin{aligned}R_r&=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} & & e_{rr}=\frac{\partial u}{\partial r}\\ \Theta_\theta&=\frac{\partial^2\phi}{\partial r^2} & \text{or}\quad & e_{\theta\theta}=\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\\ R_\theta&=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) & & e_{r\theta}=\frac{1}{r}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\end{aligned} \tag{4.9}$$

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Concentrated normal force:

Let us consider the case of a concentrated normal force of intensity p per unit length distributed along the y axis, acting in a direction normal to the surface. The solution of such a problem was given by the stress function,

$$\phi(r,\theta) = Ar\theta \sin\theta \tag{4.10}$$

Using this stress function, the stress components are found out as,

$$R_r = 2A\frac{\cos\theta}{r}, \quad \Theta_\theta = R_\theta = 0 \tag{4.11}$$

The stress distribution is radial directed towards the point of application of the force at O.

At the surface, $\theta = \pm\pi/2$, the force is zero, except at the origin itself.

At $r \rightarrow \infty$ the stress approaches zero.

Thus it satisfies all the boundary conditions.

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Concentrated normal force (contd...)

Then we can integrate the vertical component of the force over a semi-circle of radius r , and equate it to the applied force p. Hence we have,

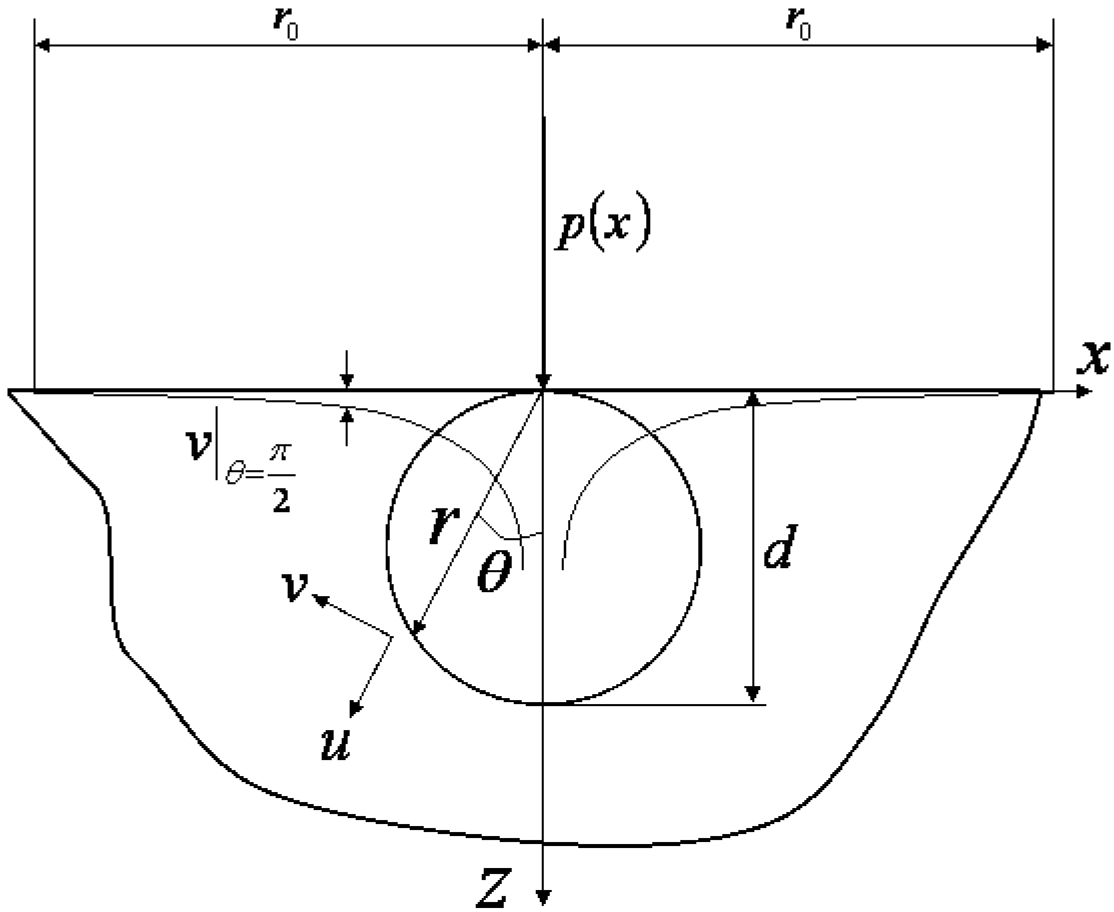
$$-P = \int_{-\pi/2}^{\pi/2} R_r \cos\theta \; r d\vartheta = \int_{0}^{\pi/2} 4A\cos^2\theta \, d\theta = A\pi \tag{4.12}$$

from which we can easily obtain the constant A. T

The radial stress component R_r is obtained as,

$$R_r = -\frac{2P}{\pi}\frac{\cos\theta}{r} \tag{4.13}$$

Note that if we introduce a variable $d = r/\cos\theta$, then the radial stress component has a constant magnitude $R_r = -\frac{2P}{\pi d}$ over a circle of diameter d.



And the strain components are

$$
\begin{aligned}
&\frac{\partial u}{\partial r} = e_{rr} = -\frac{\left(1-\sigma^2\right)}{E}\frac{2P}{\pi}\frac{\cos\theta}{r} \\
&\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = e_{\theta\theta} = \frac{\sigma(1+\sigma)}{E}\frac{2P}{\pi}\frac{\cos\theta}{r} \\
&\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = 0
\end{aligned}
\tag{4.14}
$$

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Concentrated normal force (contd...)

Integrating the strain components, one can obtain the following solutions for the displacements.

$$\begin{aligned} u &= -\frac{\left(1-\sigma^2\right)}{\pi E}2P\cos\theta\ln r - \frac{(1-2\sigma)(1+\sigma)}{\pi E}P\theta\sin\theta + C_1\sin\theta + C_2\cos\theta \\ v &= \frac{\left(1-\sigma^2\right)}{\pi E}2P\sin\theta\ln r + \frac{\sigma(1+\sigma)}{\pi E}2P\sin\theta - \frac{(1-2\sigma)(1+\sigma)}{\pi E}P\theta\cos\theta + \\ &\qquad \frac{(1-2\sigma)(1+\sigma)}{\pi E}P\sin\theta + C_1\cos\theta - C_2\sin\theta + C_3 r \end{aligned} \tag{4.15}$$

If the points along the z axis displaces only along the z axis, then from the relation of v, $C_1 = C_3 = 0$.

At the surface, where $\theta = \pm\frac{\pi}{2}$,

$$\begin{aligned} u\big|_{\theta=\frac{\pi}{2}} &= -\frac{(1-2\sigma)(1+\sigma)P}{2E} \\ v\big|_{\frac{\pi}{2}} &= \frac{\left(1-\sigma^2\right)}{\pi E}2P\ln r + \frac{(1+\sigma)}{\pi E}P - C_2 = \frac{\left(1-\sigma^2\right)}{\pi E}2P\ln r + C \end{aligned} \tag{4.16a}$$

and

$$\begin{aligned} u\big|_{\theta=-\frac{\pi}{2}} &= -\frac{(1-2\sigma)(1+\sigma)P}{2E} \\ v\big|_{-\frac{\pi}{2}} &= -\frac{\left(1-\sigma^2\right)}{\pi E}2P\ln r - \frac{(1+\sigma)}{\pi E}P + C_2 = -\frac{\left(1-\sigma^2\right)}{\pi E}2P\ln r - C \end{aligned} \tag{4.16b}$$

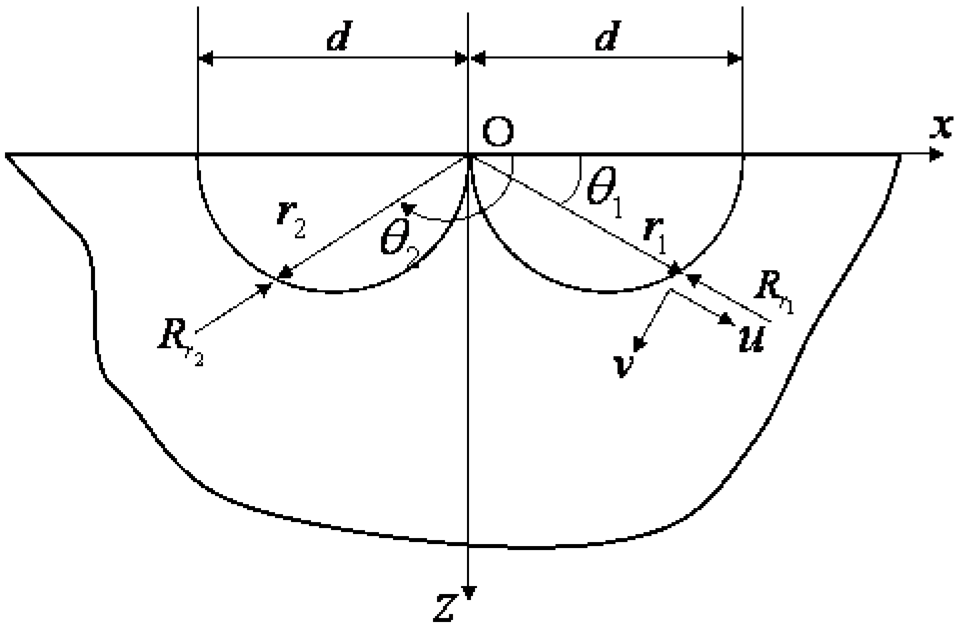
Where C is a constant which can be obtained by choosing a point on the surface at a distance r_0, where normal displacements is zero. Then we have.

$$v\big|_{\frac{\pi}{2}} = -v\big|_{-\frac{\pi}{2}} = -\frac{\left(1-\sigma^2\right)}{\pi E}2P\ln\left(\frac{r_0}{r}\right) \tag{4.17}$$

Transforming the expressions for radial stress distributions into rectangular coordinates, we have

$$\begin{aligned} X_x &= R_r\sin^2\theta = -\frac{2P}{\pi}\frac{x^2 z}{\left(x^2+z^2\right)^2}, \quad Z_z = R_r\cos^2\theta = -\frac{2P}{\pi}\frac{z^3}{\left(x^2+z^2\right)^2} \\ X_z &= R_r\sin\theta\cos\theta = -\frac{2P}{\pi}\frac{xz^2}{\left(x^2+z^2\right)^2} \end{aligned} \tag{4.18}$$

Concentrated tangential force



A concentrated force Q per unit length of the y-axis acts tangential to the surface and produces a stress field which is rotated through 90^o . If we measure θ from the line of action of the force, i.e. the Ox direction, the expression for stresses are same as for the normal force,

$$R_r = -\frac{2Q}{\pi}\frac{\cos\theta}{r}, \Theta_\theta = R_\theta = 0 \qquad (4.19)$$

Ahead of the force, in the quadrant of positive x, R_r is compressive and in the quadrant of negative x, it is tensile.

The expressions for stress components in the $x-z$ coordinates may be obtained as,

$$X_x = -\frac{2Q}{\pi}\frac{x^3}{\left(x^2+z^2\right)^2}, \quad Z_z = -\frac{2Q}{\pi}\frac{xz^2}{\left(x^2+z^2\right)^2}$$
$$X_z = -\frac{2Q}{\pi}\frac{x^2 z}{\left(x^2+z^2\right)^2} \qquad (4.20)$$

The expressions for distributions of displacements remain same, except that p is replaced by Q.

$$u = -\frac{\left(1-\sigma^2\right)}{\pi E}2Q\cos\theta\ln r - \frac{(1-2\sigma)(1+\sigma)}{\pi E}Q\theta\sin\theta + C_1\sin\theta + C_2\cos\theta$$
$$v = \frac{\left(1-\sigma^2\right)}{\pi E}2Q\sin\theta\ln r + \frac{\sigma(1+\sigma)}{\pi E}2Q\sin\theta - \frac{(1-2\sigma)(1+\sigma)}{\pi E}Q\theta\cos\theta + \frac{(1-2\sigma)(1+\sigma)}{\pi E}Q\sin\theta + C_1\cos\theta - C_2\sin\theta + C_3 r \qquad (4.21)$$

The surface displacements turn out to be,

$$-u\big|_{\theta=\pi} = u\big|_{\theta=0} = -\frac{\left(1-\sigma^2\right)}{\pi E} 2Q \ln r + C$$
$$v\big|_{\theta=\pi} = v\big|_{\theta=0} = \frac{(1-2\sigma)(1+\sigma)}{\pi E} Q \qquad (4.22)$$

Surface ahead of the application of force is depressed by an amount proportional to Q

Distributed Normal and Tangential tractions:

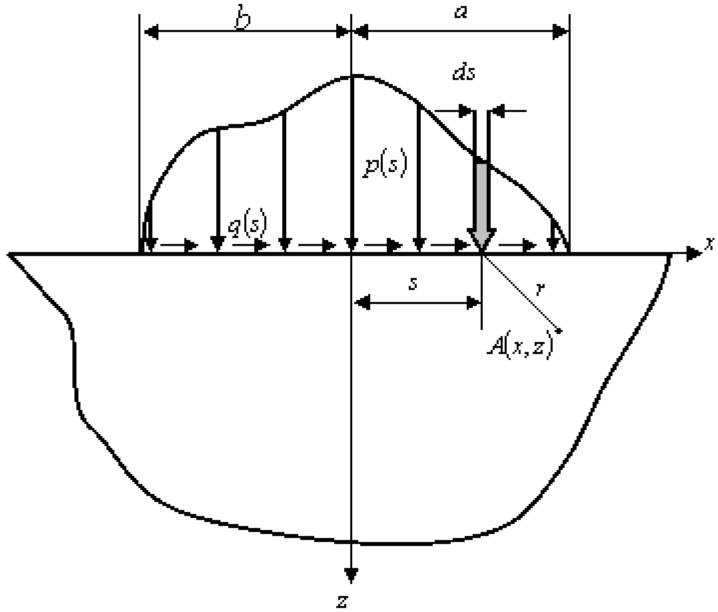
We want to find out the stress components at any point A in the body and the displacement field due to stress components $p(x)$ and $q(x)$ at the boundary.

We will consider that the tractions acting on the body at a distance s from the origin a as concentrated stress. Then we will consider the stress components at the location $A(x,z)$ due to this concentrated stress.

$$
\begin{aligned}
X_x(s) &= -\frac{2p(s)}{\pi}\frac{(x-s)^2 z}{\left((x-s)^2+z^2\right)^2}-\frac{2q(s)}{\pi}\frac{(x-s)^3}{\left((x-s)^2+z^2\right)^2}\\
Z_z(s) &= -\frac{2p(s)}{\pi}\frac{z^3}{\left((x-s)^2+z^2\right)^2}-\frac{2q(s)}{\pi}\frac{z^2(x-s)}{\left((x-s)^2+z^2\right)^2}\\
X_z(s) &= -\frac{2p(s)}{\pi}\frac{(x-s)z^2}{\left((x-s)^2+z^2\right)^2}-\frac{2q(s)}{\pi}\frac{z(x-s)^2}{\left((x-s)^2+z^2\right)^2}
\end{aligned}
\qquad (4.23)
$$

Now integrating these stress components over $-b < s < a$, we obtain the net stresses due to distributed loads $p(s)$ and $q(s)$. Thus we obtain,

$$
\begin{aligned}
X_x &= -\frac{2z}{\pi}\int_{-b}^{a}\frac{p(s)(x-s)^2\,ds}{\left\{(x-s)^2+z^2\right\}^2}-\frac{2}{\pi}\int_{-b}^{a}\frac{q(s)(x-s)^2\,ds}{\left\{(x-s)^2+z^2\right\}^2}\\
Z_z &= -\frac{2z^3}{\pi}\int_{-b}^{a}\frac{p(s)\,ds}{\left\{(x-s)^2+z^2\right\}^2}-\frac{2z^2}{\pi}\int_{-b}^{a}\frac{q(s)(x-s)\,ds}{\left\{(x-s)^2+z^2\right\}^2}\\
X_z &= -\frac{2z^2}{\pi}\int\frac{p(s)(x-s)\,ds}{\left\{(x-s)^2+z^2\right\}^2}-\frac{2z}{\pi}\int_{-b}^{a}\frac{q(s)(x-s)^2\,ds}{\left\{(x-s)^2+z^2\right\}^2}
\end{aligned}
\qquad (4.24)
$$



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Distributed Normal and Tangential tractions (contd...)

Elastic displacements at the surface are deduced in the same way, by integrating over distance

$$\begin{aligned} u|_{z=0} &= -\frac{(1-2\sigma)(1+\sigma)}{2E}\left\{\int_{-b}^{x} p(s)ds - \int_{x}^{a} p(s)ds\right\} - \frac{2(1-\sigma^2)}{\pi E}\int_{-b}^{a} q(s)\ln|x-s|ds + C_1 \\ v|_{z=0} &= -\frac{2(1-\sigma^2)}{\pi E}\int_{-b}^{a} p(s)\ln|x-s|ds + \frac{(1-2\sigma)(1+\sigma)}{2E}\left\{\int_{-b}^{x} q(s)ds - \int_{x}^{a} q(s)ds\right\} + C_2 \end{aligned} \quad (4.25)$$

Sometimes more useful information obtained by differentiating the surface displacements w.r.t. x

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)dx = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}dx + f(b(z),z)\frac{\partial b}{\partial z} - f(a(z),z)\frac{\partial a}{\partial z}$$

$$\begin{aligned} \frac{\partial u|_{z=0}}{\partial x} &= -\frac{(1-2\sigma)(1+\sigma)}{E}p(x) - \frac{2(1-\sigma^2)}{\pi E}\int_{-b}^{a}\frac{q(s)}{x-s}ds \\ \frac{\partial v|_{z=0}}{\partial x} &= \frac{(1-2\sigma)(1+\sigma)}{E}q(x) - \frac{2(1-\sigma^2)}{\pi E}\int_{-b}^{a}\frac{p(s)}{x-s}ds \end{aligned} \quad (4.26)$$

$\frac{\partial u|_{z=0}}{\partial x}$ is the tangential component of strain and $\frac{\partial v|_{z=0}}{\partial x}$ is the slope of deformation of the surface.

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Displacements specified in the loaded region:

So far in our discussion, we have specified the surface tractions as the boundary condition. But in most problems it is the displacements or a combination of displacement and traction that is specified.

We consider these problems by actually solving equation 4.26 which we rewrite as,

$$\begin{aligned}\int_{-b}^{a} \frac{q(s)}{x-s}ds &= -\frac{\pi E}{2(1-\sigma^2)}\frac{\partial u|_{z=0}}{\partial x} - \frac{\pi(1-2\sigma)}{2(1-\sigma)}p(x) \\ \int_{-b}^{a} \frac{p(s)}{x-s}ds &= -\frac{\pi E}{2(1-\sigma^2)}\frac{\partial v|_{z=0}}{\partial x} + \frac{\pi(1-2\sigma)}{2(1-\sigma)}q(x)\end{aligned} \quad (4.27)$$

Thus we have a set of coupled integral equations for the unknown displacements $p(x)$ and $q(x)$.

However there is a singularity at $x=s$, which has been taken care of by different authors. We will not worry about their derivations, instead we will present only the end results.

Let's say the given boundary conditions are $\frac{\partial v|_{z=0}}{\partial x}$ and $q(x)$ or $\frac{\partial u|_{z=0}}{\partial x}$ and $p(x)$. Then it is easy to see that above equations get decoupled, taking general form

$$\int_{-b}^{a} \frac{F(s)}{x-s}ds = g(x) \quad (4.28)$$

where $g(x)$ is a known function. T

he unknown component of traction $F(x)$ has a general solution of the form

$$F(x) = \frac{1}{\pi^2\left((x+b)(a-x)\right)^{1/2}}\int_{-b}^{a}\frac{\{(s+b)(a-s)\}^{1/2} g(s)ds}{x-s} + \frac{C}{\pi^2\left((x+b)(a-x)\right)^{1/2}} \quad (4.29)$$

If the origin is taken at the center of the loaded region then,

$$F(x) = \frac{1}{\pi^2\left(a^2-x^2\right)^{1/2}}\int_{-a}^{a}\frac{\left(a^2-s^2\right)^{1/2} g(s)ds}{x-s} + \frac{C}{\pi^2\left(a^2-x^2\right)^{1/2}} \quad (4.30)$$

The constant C is determined by the total load, normal or tangential, from the relationship:

$$C = \pi\int_{-a}^{a} F(x)dx \quad (4.31)$$