Module 2 : Solid bodies in contact with and without interactions

Lecture 10 : JKR Contact Mechanics Theory (Contd...)

The Lecture Contains:

- JKR Contact Mechanics Theory (Contd...)

This lecture is adopted from the following book

1. "Contact Mechanics" by K.L.Johnson

2. "Surface Energy and the Contact of Elastic Solid" K.L. Johnson, K. Kendall, A.D. Roberts, ***Proc. R. Soc. London, Ser. A*** 1971, ***324***, 301-313.

3. "Adhesive contact of Cylindrical lens and a Flat Sheet" by M.K. Chaudhury, T. Weaver, C.Y. Hui and E.J. Kramer, J. Appl.Phys. 1996, 80(1), 30-37

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Contact of two cylinders with their axes parallel to the y-coordinate:

Two dimensional contact of cylindrical bodies

Separation of corresponding points on the unloaded surfaces of the cylinders:

$$h = z_1 + z_2 = Ax^2 = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)x^2 \qquad (10.1)$$

For points within the contacts area after loading,

$$\nu_1\big|_{z=0} + \nu_2\big|_{z=0} = \delta - \frac{x^2}{2R} \qquad (10.2)$$

For points outside the contact area after loading,

$$\nu_1\big|_{z=0} + \nu_2\big|_{z=0} > \delta - \frac{x^2}{2R} \qquad (10.3)$$

Notice that we are concerned here with a two-dimensional problem, because here we load two parallel cylinders which result in a rectangular strip of contact area. But the 2-dimensional contact mechanics have the following mathematical problems.

Contact of two cylinders with their axes parallel to the y-coordinate:

For a change, let us assume that an elastic half space is being contacted by a distributed uniform load p which acts between $-a < x < a$ and throughout the y axis. Then the vertical displacement at a location x on the surface ($z = 0$) of the layer have the following forms within and outside the contact area respectively.

$$v\big|_{z=0} = -\frac{1-\sigma^2}{\pi E} p \left\{ (a+x)\ln\left(\frac{a+x}{a}\right)^2 + (a-x)\ln\left(\frac{a-x}{a}\right)^2 \right\} + c \qquad |x| < a \qquad (10.4a)$$

$$v\big|_{z=0} = -\frac{1-\sigma^2}{\pi E} p \left\{ (a+x)\ln\left(\frac{a+x}{a}\right)^2 - (a-x)\ln\left(\frac{a-x}{a}\right)^2 \right\} + c \qquad |x| > a \qquad (10.4b)$$

Here c is a constant, which is determined arbitrarily by considering that the displacement is zero at a distance $x = r$.

At $x < r$ the vertical displacement varies logarithmically. In other word, it is just not possible to estimate the displacement at a point without arbitrarily choosing a datum for these problems, i.e. δ can not be determined absolutely. So it is not possible to determine the pressure distribution within the contact area as was done for three dimensional problems. Therefore, for two dimensional problems, the contact between two parallel cylinders is solved in the manner discussed in the next slide..

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Contact of two cylinders with their axes parallel to the y-coordinate:

Here, we first determine the gradient of displacement, which from equation (10.2) is obtained as

$$\frac{\partial v_1|_{z=0}}{\partial x} + \frac{\partial v_2|_{z=0}}{\partial x} = -\frac{x}{R} \tag{10.5}$$

However, from equation 4.27, the gradient of displacement for a distributed normal load, in absence of shear load, can be written as,

$$\frac{\partial v_1|_{z=0}}{\partial x} + \frac{\partial v_2|_{z=0}}{\partial x} = -\frac{2}{\pi E^*}\int_{-a}^{a}\frac{p(s)}{x-s}ds \tag{10.6}$$

Where $E^* = E/(1-\nu^2)$. Matching the right hand sides of the above two equations, we have,

$$\int_{-a}^{a}\frac{p(s)}{x-s}ds = \frac{\pi E^*}{2R}x \tag{10.7}$$

Equation (10.7) has the following general form the solution of which has been worked out,

$$\int_{-a}^{a}\frac{p(s)}{x-s}ds = \frac{\pi E}{2(1-\nu^2)}(n+1)Bx^n = \frac{\pi E^*}{2}(n+1)Bx^n \tag{10.8}$$

Here the constant $B = 1/2R$ and the exponent of x is $n=1$. The integration of equation10.8 obtained as,

$$p(x) = -\frac{E^*(n+1)Ba^{n+1}}{2\pi}\frac{I_n}{\sqrt{a^2-x^2}} + \frac{P}{\pi\sqrt{a^2-x^2}} \tag{10.9}$$

Where P is the total load applied on per unit length of the contact area. The quantity I_n has the following form,

$$I_n = \pi\left\{X^{n+1} - \frac{1}{2}X^{n-1} - \frac{1}{8}X^{n-3} - \ldots\frac{1\cdot3\cdot5\cdots(n-3)}{2\cdot4\cdots n}X\right\} \text{ for } n \text{ even, } X = \frac{x}{a}$$

$$I_n = \pi\left\{X^{n+1} - \frac{1}{2}X^{n-1} - \frac{1}{8}X^{n-3} - \ldots\frac{1\cdot3\cdot5\cdots(n-3)}{2\cdot4\cdots n}X\right\} \tag{10.10}$$

Contact of two cylinders with their axes parallel to the y-coordinate:

For $n=1$, $I_1 = \pi\left\{X^2 - \frac{1}{2}\right\}$

,so that the pressure distribution is obtained as,

$$p(x) = -\frac{E^* a^2}{2\pi R}\frac{I_1}{\sqrt{a^2-x^2}} + \frac{P}{\pi\sqrt{a^2-x^2}} = \frac{1}{\pi\sqrt{a^2-x^2}}\left[P - \frac{\pi E^*}{2R}\left(x^2 - a^2/2\right)\right] \qquad (10.11)$$

The above expression for pressure distribution is not yet complete, because, the width of contact $2a$ *i*s not known. Rather the relation between P and a not yet established.

However, it is known that pressure should be positive all through the area of contact, so that,

$$p(x) > 0 \text{ , and hence } P \geq \frac{\pi E^*}{2R}\frac{a^2}{2}$$

Therefore, the total pressure has to be equal to

$$P = \frac{\pi E^* a^2}{4R} \qquad (10.12)$$

And the pressure distribution at the contact can be deduced as,

$$p(x) = \frac{2P}{\pi a^2}\sqrt{a^2-x^2} = \frac{E^*}{2R}\sqrt{a^2-x^2} \qquad (10.13)$$

The shear stresses can be obtained as,

$$\sigma_x = \sigma_y = -p(x) \qquad (10.14)$$

Equation 10.13 represents the Hertzian pressure distribution between two parallel contacting cylinders.

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Contact of two cylinders with their axes parallel to the y-coordinate:

Now we will proceed to investigate the pressure distribution when there is adhesion between the two contacting cylinders.

From equation10.11, the general form of the pressure distribution is obtained as,

$$p(x) = -\frac{\pi E^*}{2R}\frac{x^2 - a^2/2}{\pi\sqrt{a^2 - x^2}} + \frac{P}{\pi\sqrt{a^2 - x^2}}$$
$$= -\frac{1}{\pi\sqrt{a^2 - x^2}}\left[P - \frac{\pi E^*}{4R}\left(2x^2 - a^2\right)\right]$$

So that the expression for normal stress can be written as,

$$\sigma(x) = \frac{1}{\pi\sqrt{(a+x)(a-x)}}\left[P - \frac{\pi E^*}{4R}\left(2x^2 - a^2\right)\right] \qquad (10.15)$$

At the edge of the contact area, $x \to a$, the expression for stress can be approximated as,

$$\sigma(x) \approx -\frac{1}{\pi\sqrt{a-x}}\frac{1}{\sqrt{2a}}\left[P - \frac{\pi E^* a^2}{4R}\right] = \frac{K_I}{\sqrt{2\pi(a-x)}} \qquad (10.16)$$

Here K_I is called the stress intensity factor and is expressed as,

$$K_I = \frac{1}{\sqrt{\pi a}}\left[\frac{\pi E^* a^2}{4R} - P\right] \qquad (10.17)$$

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Contact of two cylinders with their axes parallel to the y-coordinate:

In the context of adhesion of two elastic bodies or one elastic body with a rigid one, the work of adhesion $W \approx G$, so that work of adhesion is expressed as,

$$W = \frac{1}{2\pi a E^*}\left[\frac{\pi E^* a^2}{4R} - P\right]^2 \tag{10.19}$$

Rearrangement of this equation and substitution of $E^* = 3K/4$, per unit length of the contact load is obtained as,

$$P = \frac{3\pi K a^2}{16R} - \sqrt{\frac{3K\pi a W}{2}} \tag{10.20}$$

Hence putting, $F = P \cdot 2L$, the measured contact load F is estimated as,

$$F = \frac{3\pi K L a^2}{8R} - \sqrt{6K\pi a W L^2} \tag{10.21}$$

Rearranging the above equation, we obtain,

$$W = \frac{\left(\frac{3\pi K L a^2}{8R} - F\right)^2}{6K\pi L^2 a} \tag{10.22}$$

The work of adhesion between two surfaces is estimated using equation 10.22. Notice that, while we deduced earlier a similar relation for adhesion between spheres using the energy approach, we obtain equation 10.22 following fracture mechanics route.