Lecture 17 : Longitudinal deformation of plates

The Lecture Contains:

- Longitudinal deformation of plates
- Large deflection of plates

This lecture is adopted from the following book

1. “Theory of Elasticity, 3rd edition” by Landau and Lifshitz. Course of Theoretical Physics, vol-7

Previous Next

Longitudinal deformation of plates

Here we consider longitudinal deformations of plate subjected to either forces applied at the edges or by body forces applied on its plane. These form a special class of problems in which the plate does not undergo bending. Since the plate is sufficiently thin the deformations may be regarded as uniform over its thickness, the strain is then only a function of the x, y coordinates. The boundary conditions on both the surfaces of the plate are $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. In fact these conditions should hold even when surface forces are applied on the plate, because these forces are still smaller than the internal longitudinal stresses. Hence we have,

$$e_{xz} = e_{yz} = 0,\ e_{zz} = -\frac{\sigma}{1-\sigma}\left(e_{xx} + e_{yy}\right) \tag{17.1}$$

Substituting these expressions into those of the non-zero components of the strain tensor, we have,

$$\sigma_{xx} = \frac{E}{1-\sigma^2}(e_{xx} + \sigma e_{yy}),\quad \sigma_{yy} = \frac{E}{1-\sigma^2}(e_{yy} + \sigma e_{xx}),\quad \sigma_{xy} = \frac{E}{(1+\sigma)}e_{xy} \tag{17.2}$$

If P_x and P_y are the components of the external body forces applied per unit area of the plate, the general equations of equilibrium are,

$$\begin{aligned} h\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y}\right) + P_x = 0 \\ h\left(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right) + P_y = 0 \end{aligned} \tag{17.3}$$

which when combined with equation 10.17 to result in the following equilibrium relation in terms of the deformations in the plate.

$$\begin{aligned} Eh\left(\frac{1}{1-\sigma^2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2(1+\sigma)}\frac{\partial^2 u}{\partial y^2} + \frac{1}{2(1-\sigma)}\frac{\partial^2 v}{\partial x \partial y}\right) + P_x = 0 \\ Eh\left(\frac{1}{1-\sigma^2}\frac{\partial^2 v}{\partial y^2} + \frac{1}{2(1+\sigma)}\frac{\partial^2 v}{\partial x^2} + \frac{1}{2(1-\sigma)}\frac{\partial^2 u}{\partial x \partial y}\right) + P_y = 0 \end{aligned} \tag{17.4}$$

Problem 3

Determine the deformation of a cylinder rotating uniformly about its axis.

$$\frac{\partial R_r}{\partial r} + \rho\Omega^2 r = 0 \quad \Rightarrow (\lambda + 2\mu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (ru)}{\partial r}\right) + \rho\Omega^2 r = 0 \tag{17.5}$$

Boundary condition: $\sigma_{rr} = 0$ for $r = R$

Ans: $u = \dfrac{\rho\Omega^2(1+\sigma)(1-2\sigma)}{8E(1-\sigma)} r\left[(3-2\sigma)R^2 - r^2\right]$

Previous Next

Large deflection of plates

The theory described in the above two sections are applicable only for small deflections in which $\zeta < h$. Here we will derive relations for large deflection of the plate, so that $\zeta > h$. It should be mentioned here that ζ must still be small with respect to the lateral dimension of the plate, $\zeta << l$. While for small bending of the plate, ζ goes from positive to negative, so there is a neutral surface inside the plate, for large deflection, however, bending is also accompanied by stretching, so that there is no such existence of a neutral surface within the plate. Hence, the total energy of the plate in this case includes the pure bending energy and also the energy due to the stretching. In the last two sections we have already considered the two situations of pure bending and pure stretching of the plate; here we will use these results for deriving the equilibrium relations for large deflection of a bent plate. For us, the plate is very thin, a two dimensional surface, and its struction across its thickness is not important. Let u, v be the in plane displacements along the x and y directions respectively and ζ is the transverse displacement due to its bending. Then an element of length $dl = \sqrt{\left(dx^2 + dy^2\right)}$ is transformed into an element of length dl' whose square is given by $dl'^2 = (dx + du)^2 + (dy + dv)^2 + d\zeta^2$. Substituting in this expression, $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$, $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$ and $d\zeta = \frac{\partial \zeta}{\partial x}dx + \frac{\partial \zeta}{\partial y}dy$, yields following expression for dl'^2 :

$$dl'^2 = dl^2 + 2\left(\begin{array}{l} \frac{\partial u}{\partial x}(dx)^2 + \frac{\partial v}{\partial y}(dy)^2 + \left(\frac{\partial \zeta}{\partial x}\right)^2 (dx)^2 + \left(\frac{\partial \zeta}{\partial y}\right)^2 (dy)^2 + \\ \frac{\partial u}{\partial y}(dx)(dy) + \frac{\partial v}{\partial x}(dx)(dy) + \frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y}(dx)(dy) \end{array}\right) \quad (17.6)$$

$$= dl^2 + 2\left(\frac{\partial u}{\partial x} + \left(\frac{\partial \zeta}{\partial x}\right)^2\right)(dx)^2 + 2\left(\frac{\partial v}{\partial y} + \left(\frac{\partial \zeta}{\partial y}\right)^2\right)(dy)^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y}\right)(dx)(dy)$$

$$= dl^2 + 2e_{xx}(dx)^2 + 2e_{yy}(dy)^2 + 2e_{xy}(dx)(dy) \quad (17.7)$$

Here we have neglected other quadratic terms in derivatives of displacements u and v. The same can not be done for ζ as there is no corresponding first order term. These expressions for the components of strain tensor can now be used in equation (10.17) for finding the expressions for the components of stresses. The corresponding expression for stretching energy is estimated from the following integration $F_{stretching} = \int \frac{1}{2} h e_{\alpha\beta} \sigma_{\alpha\beta} df$ ($\alpha, \beta = x, y$) over the area of the plate while the bending energy ($F_{bending}$) is obtained from the equation (10.5).

The total free energy of the plate undergoing large deformation is obtained as, $F_{pl} = F_{stretching} + F_{bending}$

Previous Next

The condition of minimum energy requires that $\delta F_{pl} + \delta U = 0$, where U is the potential energy of the external forces and $\delta U = -\int P\delta\zeta df$. After some algebraic manipulation, the details of which will not be discussed here, following equations are obtained for finding the displacements u, v and ζ :

$$D\Delta^2\zeta - h\frac{\partial}{\partial x}\left(X_x\frac{\partial\zeta}{\partial x} + X_y\frac{\partial\zeta}{\partial y}\right) - h\frac{\partial}{\partial y}\left(Y_x\frac{\partial\zeta}{\partial x} + Y_y\frac{\partial\zeta}{\partial y}\right) = P \tag{17.8}$$

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0$$
$$\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} = 0 \tag{17.9}$$

We can introduce the stress function

$$X_x = \frac{\partial^2\chi}{\partial y^2}, \quad Y_y = \frac{\partial^2\chi}{\partial x^2}, \quad X_y = -\frac{\partial^2\chi}{\partial x\partial y}$$

and simplify equation (10.13a) as,

$$D\Delta^2\zeta - h\left(\frac{\partial^2\chi}{\partial y^2}\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\chi}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2} - 2\frac{\partial^2\chi}{\partial x\partial y}\frac{\partial^2\zeta}{\partial x\partial y}\right) = P \tag{17.10}$$

Let us write the strain components in terms of stresses:

$$e_{xx} = \frac{X_x - \sigma Y_y}{E} \qquad e_{yy} = \frac{Y_y - \sigma X_x}{E} \qquad e_{xy} = \frac{(1+\sigma)X_y}{E}$$

In which we can substitute the expressions for the strains from equation 17.6

$$\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial\zeta}{\partial y}\right)^2 = \frac{1}{E}\left(\frac{\partial^2\chi}{\partial x^2} - \sigma\frac{\partial^2\chi}{\partial y^2}\right)$$
$$\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial\zeta}{\partial x}\right)^2 = \frac{1}{E}\left(\frac{\partial^2\chi}{\partial y^2} - \sigma\frac{\partial^2\chi}{\partial x^2}\right) \tag{17.11}$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial\zeta}{\partial x}\frac{\partial\zeta}{\partial y} = -\frac{2(1+\sigma)}{E}\frac{\partial^2\chi}{\partial x\partial y}$$

Deriving the three equations of 10.26 by $\partial^2/\partial y^2$, $\partial^2/\partial x^2$ and $-\partial^2/\partial x\partial y$ respectively, we obtain,

$$\frac{\partial^2}{\partial y^2}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial\zeta}{\partial x}\right)^2\right] = \frac{1}{E}\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\chi}{\partial y^2} - \sigma\frac{\partial^2\chi}{\partial x^2}\right)$$

(17.12)

$$\frac{\partial^2}{\partial x^2}\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial \zeta}{\partial y}\right)^2\right]=\frac{1}{E}\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \chi}{\partial x^2}-\sigma\frac{\partial^2 \chi}{\partial y^2}\right)$$

$$-\frac{\partial^2}{\partial x \partial y}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial \zeta}{\partial x}\frac{\partial \zeta}{\partial y}\right]=\frac{2(1+\sigma)}{E}\frac{\partial^2}{\partial x \partial y}\left[\frac{\partial^2 \chi}{\partial x \partial y}\right]$$

By adding these three, we obtain,

$$\Delta^2 \chi + E\left(\frac{\partial^2 \zeta}{\partial x^2}\frac{\partial^2 \zeta}{\partial y^2}-\left(\frac{\partial^2 \zeta}{\partial x \partial y}\right)^2\right)=0 \qquad (17.13)$$

Equations 17.10 and 17.13 represent the complete system of equations for solving problems of large deflections of thin plate. These nonlinear set of equations are called Föppl-von Karman equations. These are not amenable for solving analytically except in some one dimensional cases, so that one needs to use numerical or semi-analytical techniques to handle them.

Previous Next