Module 2 : Solid bodies in contact with and without interactions

Lecture 11 : Compression of an elastic film sandwiched between two rigid parallel plates

The Lecture Contains:

This lecture is adopted from the following book

1. A. N. Gent, “Compression of Rubber Blocks”, Rubber Chemistry and Technology, 67, 549-558.

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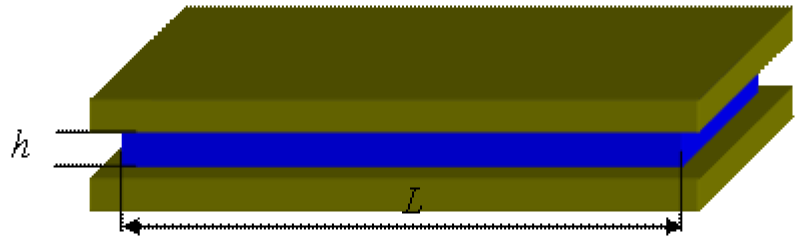
Compression of an elastic film sandwiched between two rigid parallel plates:



The stress equilibrium relations for a compressible elastic film of thickness h and width ϖ sandwiched between two rigid substrates is given in plane strain approximation by,

$$(1-2\sigma)\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}+2(1-\sigma)\frac{\partial^2 u}{\partial x^2}=0$$
$$(1-2\sigma)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial z}+2(1-\sigma)\frac{\partial^2 w}{\partial z^2}=0 \qquad (11.1)$$

Where σ is the Poisson ratio and u and w are the displacements in the film along the x and z directions respectively. We can write equations 11.1 in terms of the following dimensionless quantities:

$$U=u/L,\quad X=x/L,\quad W=w/h,\quad Z=z/h$$

Here h is the film thickness and L is a characteristic spatial length scale, such that $h<<L$. Equation 11.1 then transforms into

$$(1-2\sigma)\frac{L^2}{h^2}\frac{\partial^2 U}{\partial Z^2}+\frac{\partial^2 W}{\partial X\partial Z}+2(1-\sigma)\frac{\partial^2 U}{\partial X^2}=0 \qquad (11.2a)$$

$$(1-2\sigma)\frac{h^2}{L^2}\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 U}{\partial X\partial Z}+2(1-\sigma)\frac{\partial^2 W}{\partial Z^2}=0 \qquad (11.2b)$$

Notice that the first term in the right hand side of equation 11.2b is smaller than the rest of the terms, so that we can neglect it, yielding,

$$\frac{\partial^2 U}{\partial X\partial Z}+2(1-\sigma)\frac{\partial^2 W}{\partial Z^2}=0 \qquad (11.3)$$

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Integration of this equation then results in,

$$\frac{1}{2(1-\sigma)}\frac{\partial U}{\partial X}+\frac{\partial W}{\partial Z}+A(X)=0 \qquad \Rightarrow \frac{\partial W}{\partial Z}=-\frac{1}{2(1-\sigma)}\frac{\partial U}{\partial X}-A(X) \tag{11.4a}$$

Or,

$$\frac{\partial^2 W}{\partial Z\partial X}=-\frac{1}{2(1-\sigma)}\frac{\partial^2 U}{\partial X^2}-\frac{\partial A(X)}{\partial X} \tag{11.4b}$$

Substitution of equation 7.4b in 7.2a then results in,

$$\begin{aligned}
&\frac{(1-2\sigma)}{\varepsilon^2}\frac{\partial^2 U}{\partial Z^2}-\frac{1}{2(1-\sigma)}\frac{\partial^2 U}{\partial X^2}+2(1-\sigma)\frac{\partial^2 U}{\partial X^2}-\frac{\partial A(X)}{\partial X}=0\\
&\Rightarrow \frac{(1-2\sigma)}{\varepsilon^2}\frac{\partial^2 U}{\partial Z^2}+\left(-\frac{1}{2(1-\sigma)}+2(1-\sigma)\right)\frac{\partial^2 U}{\partial X^2}-\frac{\partial A(X)}{\partial X}=0\\
&\Rightarrow \frac{(1-2\sigma)}{\varepsilon^2}\frac{\partial^2 U}{\partial Z^2}+\left(\frac{4(1-\sigma)^2-1}{2(1-\sigma)}\right)\frac{\partial^2 U}{\partial X^2}-\frac{\partial A(X)}{\partial X}=0\\
&\Rightarrow (1-2\sigma)\frac{\partial^2 U}{\partial Z^2}+\varepsilon^2\left(\frac{(1-2\sigma)(3-2\sigma)}{2(1-\sigma)}\right)\frac{\partial^2 U}{\partial X^2}-\varepsilon^2\frac{\partial A(X)}{\partial X}=0
\end{aligned} \tag{11.5}$$

Again notice that the middle term of the right hand side of the last equation is smaller than the other terms, which result in following simplification,

$$(1-2\sigma)\frac{\partial^2 U}{\partial Z^2}-\varepsilon^2\frac{\partial A(X)}{\partial X}=0 \tag{11.6}$$

Integration of equation 7.6, with respect to Z yields the following expression,

$$U=\frac{\varepsilon^2}{(1-2\sigma)}\frac{\partial A(X)}{\partial X}\frac{Z^2}{2}+B(X)Z+C(X) \tag{11.7}$$

We can use the following boundary conditions,

$U=0$ at $Z=0$, $\Rightarrow C(X)=0$
$U=0$ at $Z=1$

Resulting in following simplifications,

$$0=\frac{\varepsilon^2}{(1-2\sigma)}\frac{\partial A(X)}{\partial X}\frac{1}{2}+B(X)$$

$$B(X)=-\frac{\varepsilon^2}{(1-2\sigma)}\frac{\partial A(X)}{\partial X}\frac{1}{2}$$

We then have the following expression for dimensionless displacement U,

$$U = \frac{\varepsilon^2}{(1-2\sigma)}\frac{\partial A(X)}{\partial X}\frac{(Z^2-Z)}{2} \tag{11.8}$$

We then obtain the derivative of U with respect to X

$$\frac{\partial U}{\partial X} = \frac{\varepsilon^2}{(1-2\sigma)}\frac{\partial^2 A(X)}{\partial X^2}\frac{(Z^2-Z)}{2}$$

Which we use in equation 11.4a, to produce,

$$\frac{\partial W}{\partial Z} = -\frac{1}{4(1-\sigma)}\frac{\varepsilon^2}{(1-2\sigma)}\frac{\partial^2 A(X)}{\partial X^2}(Z^2-Z) - A(X) \tag{11.9}$$

Integration of 11.9 and the use of boundary condition at $W=0$ at $Z=0$ yields,

$$W = -\frac{1}{4(1-\sigma)}\frac{\varepsilon^2}{(1-2\sigma)}\frac{\partial^2 A(X)}{\partial X^2}\left(\frac{Z^3}{3}-\frac{Z^2}{2}\right) - A(X)Z \tag{11.10}$$

Furthermore, the boundary condition $W=\Delta$ at $Z=1$ yields,

$$\Delta = \frac{1}{24(1-\sigma)}\frac{\varepsilon^2}{(1-2\sigma)}\frac{\partial^2 A(X)}{\partial X^2} - A(X) \tag{11.11}$$

Equation 7.11 after re-arrangement generates,

$$\frac{\partial^2 A(X)}{\partial X^2} - \frac{24(1-\sigma)(1-2\sigma)}{\varepsilon^2}A(X) = \frac{24(1-\sigma)(1-2\sigma)}{\varepsilon^2}\Delta$$

$$\frac{\partial^2 A(X)}{\partial X^2} - c_1 A(X) = c_2 \tag{11.12}$$

Solution for equation is obtained as,

$$A(X) = -\frac{c_2}{c_1} + p_1 e^{\sqrt{c_1}X} + p_2 e^{-\sqrt{c_1}X} \Rightarrow A(X) = -\Delta + p_1 e^{\sqrt{c_1}X} + p_2 e^{-\sqrt{c_1}X} \tag{11.13}$$

Differentiation of $A(X)$ with respect to X yields

$$\frac{\partial A(X)}{\partial X} = \sqrt{c_1}\left(p_1 e^{\sqrt{c_1}X} - p_2 e^{-\sqrt{c_1}X}\right)$$
$$\frac{\partial^2 A(X)}{\partial X^2} = c_1\left(p_1 e^{\sqrt{c_1}X} + p_2 e^{-\sqrt{c_1}X}\right)$$

Finally, W is obtained as,

$$\begin{aligned} W &= -\frac{1}{4(1-\sigma)}\frac{\varepsilon^2}{(1-2\sigma)}c_1\left(p_1 e^{\sqrt{c_1}X} + p_2 e^{-\sqrt{c_1}X}\right)\left(\frac{Z^3}{3} - \frac{Z^2}{2}\right) - \left(-\Delta + p_1 e^{\sqrt{c_1}X} + p_2 e^{-\sqrt{c_1}X}\right)Z \\ &= \Delta Z - \left(p_1 e^{\sqrt{c_1}X} + p_2 e^{-\sqrt{c_1}X}\right)\left(Z + \frac{1}{4(1-\sigma)}\frac{\varepsilon^2}{(1-2\sigma)}c_1\left(\frac{Z^3}{3} - \frac{Z^2}{2}\right)\right) \\ &= \Delta Z - \left(p_1 e^{\sqrt{c_1}X} + p_2 e^{-\sqrt{c_1}X}\right)\left(2Z^3 - 3Z^2 + Z\right) \end{aligned} \tag{11.14}$$

And U as,

$$U = \frac{\varepsilon^2}{2(1-2\sigma)}\sqrt{c_1}\left(p_1 e^{\sqrt{c_1}X} - p_2 e^{-\sqrt{c_1}X}\right)\left(Z^2 - Z\right) \tag{11.15}$$

$U = 0$ at $X = 0$ which gives, $p_1 = p_2$, which finally gives,

$$U = \frac{\varepsilon^2}{(1-2\sigma)}\sqrt{c_1}\,p_1 \sinh\left(\sqrt{c_1}X\right)\left(Z^2 - Z\right) \tag{11.16}$$

$$W = \Delta Z - 2p_1 \cosh\left(\sqrt{c_1}X\right)\left(2Z^3 - 3Z^2 + Z\right) \tag{11.17}$$

Now, we differentiate U and W with respect to X and Z

$$\begin{aligned} \frac{\partial U}{\partial X} &= \frac{\varepsilon^2 c_1}{(1-2\sigma)}p_1 \cosh\left(\sqrt{c_1}X\right)\left(Z^2 - Z\right) = 24(1-\sigma)p_1 \cosh\left(\sqrt{c_1}X\right)\left(Z^2 - Z\right) \\ \frac{\partial W}{\partial Z} &= \Delta - 2p_1 \cosh\left(\sqrt{c_1}X\right)\left(6Z^2 - 6Z + 1\right) \end{aligned} \tag{11.18}$$

And add them up,

$$\begin{aligned} \frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z} &= \Delta + 2p_1 \cosh\left(\sqrt{c_1}X\right)\left(12(1-\sigma)\left(Z^2 - Z\right) - \left(6Z^2 - 6Z + 1\right)\right) \\ &= \Delta + 2p_1 \cosh\left(\sqrt{c_1}X\right)\left(-1 + 6\left(Z^2 - Z\right)(1-2\sigma)\right) \end{aligned} \tag{11.19}$$

Finally the expression for the dimensionless hydrostatic pressure $P = -\overline{P}/K$ (K is the bulk modulus) is obtained as,

$$P = \left(\frac{\partial U}{\partial X} + \frac{\partial W}{\partial Z}\right) = \left(\Delta + 2p_1 \cosh\left(\sqrt{c_1}X\right)\left(-1 + 6\left(Z^2 - Z\right)(1-2\sigma)\right)\right) \qquad (11.20)$$
$$= \left(\Delta + 2p_1 \cosh\left(\sqrt{c_1}X\right)\left(-1 + 6\left(Z^2 - Z\right)(1-2\sigma)\right)\right)$$

Thus the expression for pressure is a function of Z. However, let's say we are considering a nearly incompressible material, so that at $X = a = \varpi/2L$, $P = 0$,

Then,

$$P = 0 = \left(\Delta - 2p_1 \cosh\left(\sqrt{c_1}a\right)\right) \Rightarrow p_1 = +\frac{\Delta}{2\cosh\left(\sqrt{c_1}a\right)}$$

By substituting the above expression in 7.20, we obtain

$$P = \Delta\left(1 - \frac{\cosh\left(\sqrt{c_1}X\right)}{\cosh\left(\sqrt{c_1}a\right)}\right) \qquad (11.21)$$

Now we integrate equation 7.21 to obtain the dimensionless compressive force,

$$F = 2\int_0^a P\,dX = 2\Delta\int_0^a \left(1 - \frac{\cosh\left(\sqrt{c_1}X\right)}{\cosh\left(\sqrt{c_1}a\right)}\right)dX = 2a\Delta\left(1 - \frac{\tanh\left(\sqrt{c_1}a\right)}{\sqrt{c_1}a}\right) \qquad (11.22)$$

Expanding $\tanh(x)$ in series, we have

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{5} - \ldots$$

So that, equation 7.22 turns to

$$F = 2a\Delta\left(1 - \left(1 - \frac{c_1 a^2}{3} + \frac{2c_1^2 a^4}{5} - \ldots\right)\right) = 2a\Delta\left(\frac{c_1 a^2}{3} - \frac{2c_1^2 a^4}{5} + \ldots\right) \qquad (11.23)$$

Considering only the first term in the above equation and noting that the dimensionless load is given as, $F = f/KL$, we obtain an expression for force on as,

$$f = 2aKL\Delta\frac{c_1 a^2}{3} = 2aKL\Delta\frac{24(1-\sigma)(1-2\sigma)}{h^2}\frac{(aL)^2}{3}$$
$$= 2\Delta\frac{2\mu(1+\sigma)}{3(1-2\sigma)}\frac{\varpi}{2}\frac{24(1-\sigma)(1-2\sigma)}{h^2}\frac{\varpi^2}{12} = 2\Delta\frac{2\mu(1+\sigma)}{3}\frac{\varpi}{1}\frac{(1-\sigma)}{h^2}\frac{\varpi^2}{1} \qquad (11.24)$$
$$\frac{f}{w\Delta\mu} = \frac{4\left(1-\sigma^2\right)}{3}\frac{\varpi^2}{h^2}$$

Hence for an incompressible material, $\sigma = 1/2$, so that, $\dfrac{f}{w\Delta} = \dfrac{\mu\varpi^2}{h^2}$

At $z = 1$,

$$Z_z = -p + 2\mu\frac{\partial w}{\partial z}$$

$$p = -\frac{\xi K}{h}\left(1 - \frac{\cosh\left(\sqrt{c_1}x/L\right)}{\cosh\left(\sqrt{c_1}\varpi/2L\right)}\right)$$

$$\frac{\partial w}{\partial z} = \frac{\xi}{h} - 2\frac{\xi}{2h\cosh\left(\sqrt{c_1}\varpi/2L\right)}\cosh\left(\sqrt{c_1}\frac{x}{L}\right)$$

$$-p + 2\mu\frac{\partial w}{\partial z} = \frac{\xi K}{h}\left(1 - \frac{\cosh\left(\sqrt{c_1}x/L\right)}{\cosh\left(\sqrt{c_1}\varpi/2L\right)}\right) + \frac{2\mu\xi}{h} - \frac{2\mu\xi}{h}\frac{\cosh\left(\sqrt{c_1}x/L\right)}{\cosh\left(\sqrt{c_1}\varpi/2L\right)}$$

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Let us solve the same equation, for an incompressible elastic material.

$$\frac{\partial p}{\partial x}=\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)$$
$$\frac{\partial p}{\partial z}=\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{11.25}$$

Simplifying using long-scale approximation,

$$\frac{\partial p}{\partial x}=\mu\left(\frac{\partial^2 u}{\partial z^2}\right)$$
$$\frac{\partial p}{\partial z}=0 \tag{11.26}$$

Integration of equation 7.26,

$$u=\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{z^2}{2}+A(x)z$$

Using the boundary condition, $u=0$, $w=0$ at $z=0$

$$u=\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{z^2}{2}+A(x)z \qquad w=-\frac{z^3}{6\mu}\frac{\partial^2 p}{\partial x^2}-\frac{z^2}{2}\frac{\partial A}{\partial x} \tag{11.27}$$

$u=0$ at $z=h$ which further gives,

$$A(x)=-\frac{h}{2\mu}\frac{\partial p}{\partial x}$$
$$w=-\frac{z^3}{6\mu}\frac{\partial^2 p}{\partial x^2}-\frac{z^2}{2}\frac{\partial A}{\partial x}=-\frac{z^2}{2\mu}\frac{\partial^2 p}{\partial x^2}\left(\frac{z}{3}-\frac{h}{2}\right) \tag{11.28}$$

Furthermore, $w=\xi$ at , $z=h$ so that

$$\xi=\frac{h^3}{12\mu}\frac{\partial^2 p}{\partial x^2} \tag{11.29}$$

Integration of equation 7.29 gives,

$$p=\frac{12\mu}{h^3}\xi\frac{x^2}{2}+Cx+D \tag{11.30}$$

Which is simplified using the boundary condition:

$$\frac{\partial p}{\partial x}=0 \text{ at } x=0 \text{ and } p=0 \text{ at } x=w/2$$

$$p = \frac{12\mu}{h^3}\xi\frac{x^2}{2} + D \qquad D = -\frac{12\mu}{h^3}\xi\frac{(w/2)^2}{2}$$

Finally, pressure is obtained as,

$$p = \frac{6\mu}{h^3}\xi\left(x^2 - \frac{w^2}{4}\right) \tag{11.31}$$

Integrating with respect to x in $-w/2 < x < w/2$

$$f = 2\int_0^{w/2} p\,dx = \frac{12\mu\xi}{h^3}\int_0^{w/2}\left(x^2 - \frac{w^2}{4}\right)dx = \frac{12\mu\xi}{h^3}\left(\frac{1}{3}\frac{w^3}{8} - \frac{w^3}{8}\right) = \frac{\mu\xi w^3}{h^3} \tag{11.32}$$

Putting $\Delta = \xi/h$, we have

$$\frac{f}{\Delta w} = \frac{\mu w^2}{h^2} \tag{11.33}$$

which is same as equation 7.24.

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It can be shown that for a circular punch, the pressure distribution can be calculated as,

$$p = -\frac{\xi K}{h}\left(1 - \frac{J_0\left(\sqrt{c_1}\,r/L\right)}{J_0\left(\sqrt{c_1}\,R/2L\right)}\right) \tag{11.34}$$

Where $J_0(x)$ is the zeroth-order Bessel function.

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