Lecture 24 : Simultaneous Stretching and torsion of a solid neo-Hookean cylinder

The Lecture Contains

lecture is adopted from the following journal paper:

1. Solenoids and plectonemes in stretched and twisted elastomeric filament, A. Ghatak and L. Mahadevan, Phys. Rev. Lett., Vol. 95, pp. 057801 (2005).

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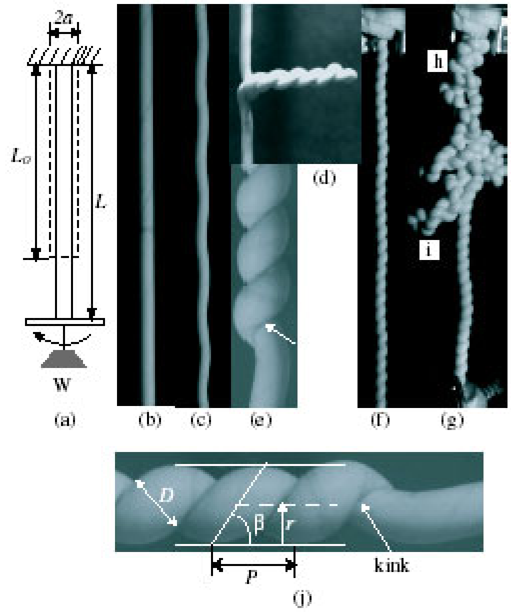
Simultaneous Stretching and torsion of a solid neo-Hookean cylinder:



In this lecture we will consider the behavior of a naturally-straight, highly extensible, elastic, cylindrical filaments and rod which is subjected to large extensional and torsional strains. The rod initially extends but finally leads to the formation of solenoidal structures as shown by the optical micrograph (e).

Let us say that the cylinder of length L_0 and radius a is stretched to extension ratio λ by applying a dead weight W and is twisted to torsional density ψ_S by applying a torque M. Since $\lambda_3 = \lambda$, $\lambda_1 = \lambda_2 = 1/\sqrt{\lambda}$ The extensional energy of the cylinder is obtained by multiplying the strain energy density function by the volume of the rod:

$$\Pi_S = \frac{\pi a^2 E}{6} L_0\left(\lambda^2 + \frac{2}{\lambda} - 3\right) \tag{24.1}$$

Similarly, the torsional energy of the cylinder is obtained as:

$$\Pi_T = \frac{C}{2} L_0 \lambda \psi^2 \tag{24.2}$$

where, the torsional rigidity is defined as $C = \dfrac{\pi a^4 E}{6}$.

Hence the total elastic energy of the rod is obtained by summing these two energies,

$$\Pi = \frac{\pi a^2 E}{6} L_0\left(\lambda^2 + \frac{2}{\lambda} - 3\right) + \frac{\pi a^4 E}{12} L_0 \lambda \psi^2 \tag{24.3}$$

The total energy of the system is obtained by adding to it the mechanical energy spent by an extensional load: $\Pi_W = -fL_0(\lambda - 1)$ and that done by the torque: $\Pi_M = -ML_0\lambda\psi$. Then the total energy is written as:

$$\Pi = \frac{\pi a^2 E}{6} L_0\left(\lambda^2 + \frac{2}{\lambda} - 3\right) + \frac{\pi a^4 E}{12} L_0\lambda\psi^2 - fL_0(\lambda - 1) - ML_0\lambda\psi \tag{24.4}$$

Minimizing Π with respect to ψ and λ we can obtain the expressions for W and M.
For example, the torque is obtained as

$$M = \frac{\pi a^4 E}{6}\psi = C\psi \tag{24.5}$$

And the expression for load experienced by the rod is obtained as

$$f = \frac{\pi a^2 E}{3}\left(\lambda - \frac{1}{\lambda^2} + \frac{1}{4}(\psi a)^2\right) \tag{24.6}$$

Notice that this load is somewhat larger than that would actually be required to extend the rod to an extension ratio of λ without application of any twist. Therefore, in a load controlled experiment, the actual extension of the rod in presence of torsion is somewhat larger and is expressed as

$$\lambda_0 - \frac{1}{\lambda_0^2} = \lambda - \frac{1}{\lambda^2} + \frac{1}{4}(\psi a)^2 \tag{24.7}$$

In other word, the dead weight required to generate extension ratio λ will be

$$W = \frac{\pi a^2 E}{3}\left(\lambda - \frac{1}{\lambda^2} - \frac{1}{4}(\psi a)^2\right) \tag{24.8}$$

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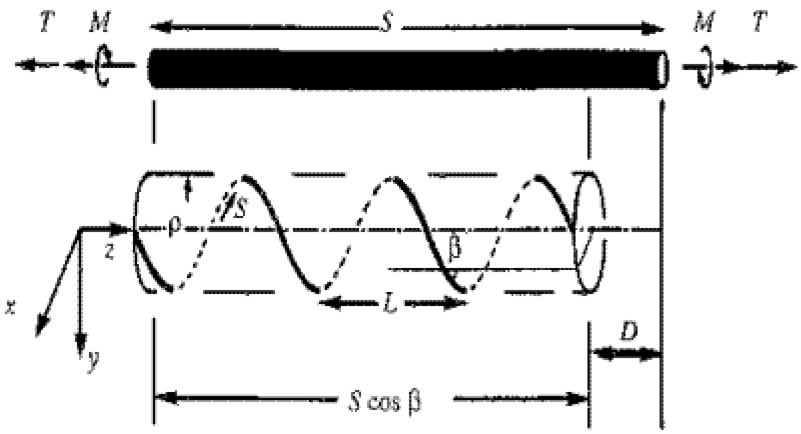

Let us now consider that the cylindrical rod of length L_0 under the action of both extension and torsion does not remain straight but undergoes spontaneous transformation to solenoidal helices characterized by the helix radius r and helix angle β. Question arises at which threshold twist density this transformation occurs?

The threshold condition can be obtained by carrying out the minimization of the total energy of the system. Let us consider that the contour length of the undeformed rod that goes to form the helix is L_S within which the extension ratio is λ_S. The twist density $\overline{\psi}_S$ of the rod in the solenoidal portion comprises of two components: that due to actual internal torsion of the rod and that due to kinematic torsion associated with bending of the rod: $\overline{\psi}_S = \psi_S + \dfrac{\sin\beta\cos\beta}{r}$. In addition to torsion, the solenoid is also characterized by the total curvature given as $\kappa = \dfrac{\sin^2\beta}{r}$.

Combining the extensional energy of the rod in its straight and solenoidal portion, the total extensional energy is written as:

$$\Pi_S = \frac{C}{a^2}\left[L_S\left(\lambda_S^2 + \frac{2}{\lambda_S} - 3\right) + (L_0 - L_S)\left(\lambda^2 + \frac{2}{\lambda} - 3\right)\right] \tag{24.9}$$

Similarly, the total torsional energy is obtained as,

$$\Pi_T = \frac{C}{2}\left[L_S\lambda_S\overline{\psi}_S^2 + (L_0 - L_S)\lambda\psi^2\right] \qquad (24.10$$

The bending energy of the deformed rod is written as $\left(B\kappa^2/2\right)\lambda_S L_S$, where B, bending rigidity is defined as EI. Here I is the moment of inertia of the deformed rod, writes as, $\pi a^4/4\lambda_S^2$. Then the bending energy of the rod is obtained as

$$\Pi_B = \left(\frac{\pi a^4 E L_S}{8\lambda_S}\right)\frac{\sin^4\beta}{r^2} = \frac{BL_S}{2\lambda_S}\frac{\sin^4\beta}{r^2} \tag{24.11}$$

The energy spent by the dead weight W in extending is obtained by multiplying the dead weight with the net vertical distance traveled,

$$\Pi_W = -W\left[L_S\left(\lambda_S \cos\beta - 1\right) + \left(L_0 - L_S\right)\left(\lambda - 1\right)\right] \tag{24.12}$$

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Notice that for the solenoidal portion of the rod, the vertical distance traveled by the dead weight is $L_S(\lambda_S \cos\beta - 1)$ and is not $L_S(\lambda_S - 1)$. For the straight portion, however, the distance traveled by the dead weight remains $(L_0 - L_S)(\lambda - 1)$.
Similarly, the energy spent by the external torque M in twisting the rod is obtained as,

$$\Pi_M = -M\left[L_S\lambda_S(\psi_S + \sin\beta/r) + (L_0 - L_S)\lambda\psi\right] \tag{24.13}$$

Notice that the total angle turned by the end moment is given as: $\psi_S + \sin\beta/r$

Then the total energy of the system and that of the external load and torque are written as

$$\Pi(\lambda_S, \psi_S, \beta, r) = \Pi_S + \Pi_T + \Pi_B + \Pi_W + \Pi_M \tag{24.14}$$

Minimizing the total energy with respect to ψ_S yields

$$M = C\psi = C\overline{\psi}_S \tag{24.15}$$

Minimizing the total energy with respect to r yields

$$\begin{aligned} &-\frac{C}{2}L_S\lambda_S 2\overline{\psi}_s \frac{\sin\beta\cos\beta}{r^2} - \frac{BL_S}{2\lambda_S}\frac{2\sin^4\beta}{r^3} + ML_S\lambda_S\frac{\sin\beta}{r^2} = 0 \\ &\Rightarrow M = C\overline{\psi}_s\cos\beta + \frac{B\sin^3\beta}{\lambda_S^2 r} \end{aligned} \tag{24.16}$$

Similarly minimizing the total energy with respect to β yields

$$\begin{aligned} &\frac{C}{2}L_S\lambda_S 2\overline{\psi}_S\frac{\cos 2\beta}{r} + \frac{BL_S}{2\lambda_S}\frac{4\sin^3\beta\cos\beta}{r^2} + WL_S\lambda_S\sin\beta - ML_S\lambda_S\frac{\cos\beta}{r} = 0 \\ &\Rightarrow C\overline{\psi}_S\frac{\cos 2\beta}{r} + \frac{B}{2\lambda_S^2}\frac{4\sin^3\beta\cos\beta}{r^2} + W\sin\beta - M\frac{\cos\beta}{r} = 0 \\ &\Rightarrow C\overline{\psi}_S\frac{\cos 2\beta}{r} + \frac{B}{\lambda_S^2}\frac{2\sin^3\beta\cos\beta}{r^2} + W\sin\beta - \frac{C\overline{\psi}_S\cos^2\beta}{r} - \frac{B\sin^3\beta\cos\beta}{\lambda_S^2 r^2} = 0 \\ &\Rightarrow W = \frac{C\overline{\psi}_S\sin\beta}{r} - \frac{B\sin^2\beta\cos\beta}{(\lambda_S r)^2} \end{aligned} \tag{24.17}$$

Substitution of expression for M in equation 24.14 into 24.15 and solution of equations 24.15 and 24.16, for r and $\overline{\psi}_S$ yields

$$r = \sqrt{\frac{B}{W}}\frac{\sin\beta}{\lambda_S}, \quad \psi_S = \sqrt{\frac{W}{B}}\left(\frac{3(1+\cos\beta)}{2\lambda_S} - \lambda_S\cos\beta\right) \tag{24.18}$$

Equation 24.15 to 24.18 present us six equations for eight unknowns: W, M, β, r, Ψ, ψ_S, λ

and λ_s. The ideas here is to fix the values for, say W and β, and calculate the values for six other unknowns.

The threshold values of different parameters at which the straight rod would turn to a helix is obtained by considering the stability of the above solutions. The solutions are stable when the eigen values of the Hessian matrix written as $\left.\frac{\partial^2 \Pi}{\partial \alpha_i \partial \alpha_j}\right|_{a_k, k \neq i, j}$ with $\alpha_i = \beta,\ \psi_s$ and r remain positive.