Module 4 : Nonlinear elasticity

Lecture 26 : Theory of Incremental stresses

The Lecture Contains

1. "Mechanics of Incremental Deformations" by M.A. Biot.

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Two dimensional stress at a point in a plane is defined by three components σ_{xx}, σ_{yy} and σ_{xy} referred to orthogonal axes x and y. The stress components represent the forces in the xy plane acting per unit area on the sides of an infinitesimal element of size dx, dy cut out on the slab. The tangential component σ_{xy} is same on both the sides dx and dy, because the torque of the resulting stresses on the element must be zero, this feature is known as the symmetric property as discussed in Lecture 2.

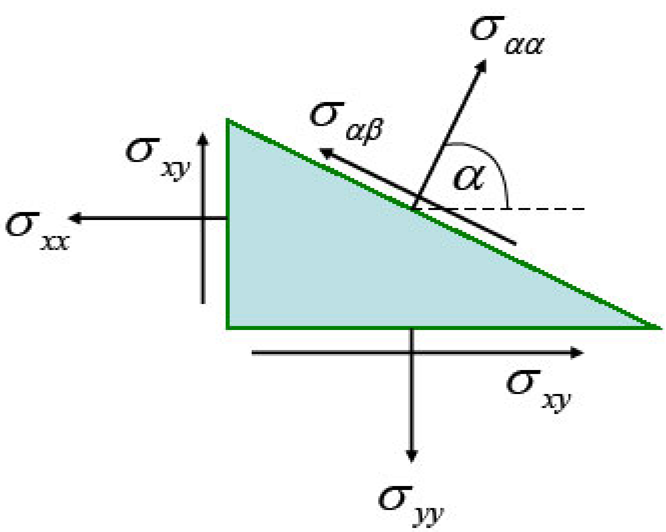

Figure 26.1

If we cut a small right-angled triangle OAB, then the normal and tangential tractions acting on the side AB are

$$\begin{aligned} \sigma_{\alpha\alpha} &= \sigma_{xx} Cos^2\alpha + \sigma_{yy} Sin^2\alpha + \sigma_{xy} Sin 2\alpha \\ \sigma_{\alpha\beta} &= \frac{1}{2}\left(\sigma_{yy} - \sigma_{xx}\right) Sin 2\alpha + \sigma_{xy} Cos 2\alpha \end{aligned} \qquad (26.1)$$

The angle α measures the inclination of the normal to AB with the x direction. The relation 26.1 generate the stress components w.r.t. axes 1 and 2 which are rotated by an angle α from the original directions $x$and y.

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Consider an initial stress field

$$\begin{matrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{matrix} \tag{26.2}$$

If the plane continuum is deformed, any point *P* is displaced to a point *P*' of co-ordinates x, h and the stress at that point *P*' acquires a new value defined by the components

$$\begin{aligned} \overline{\sigma_{\xi\xi}} &= S_{11} + \overline{s_{\xi\xi}} \\ \overline{\sigma_{\eta\eta}} &= S_{22} + \overline{s_{\eta\eta}} \\ \overline{\sigma_{\xi\eta}} &= S_{12} + \overline{s_{\xi\eta}} \end{aligned} \tag{26.3}$$

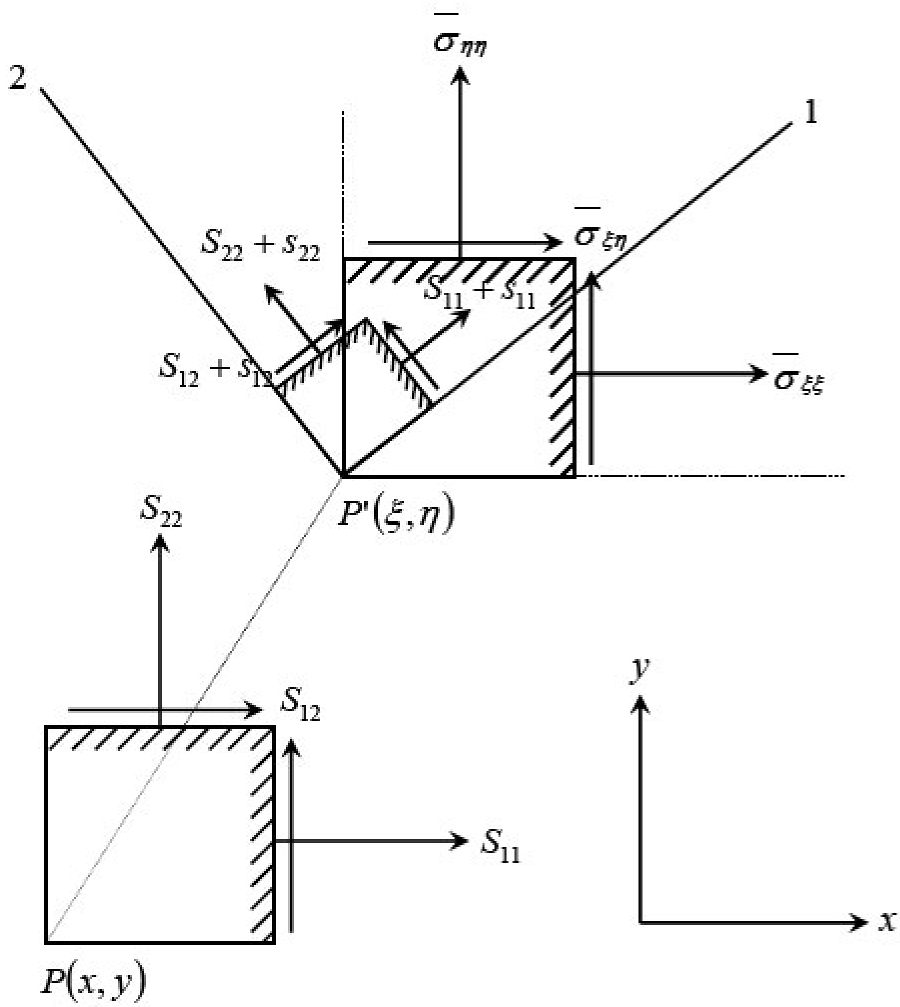

Figure 26.2

These components are referred to the fixed directions x, y. The components $\overline{s_{\xi\xi}}, \overline{s_{\eta\eta}}, \overline{s_{\xi\eta}}$ represent the increment of the total stress at the displaced point P' of co-ordinates ξ, η after deformation. Now let's introduce the idea that the incremental components are not only due to the strain but also to the fact that the initial stress field has been rotated by a certain angle when moving from P to P'.

In other words, if there is only translation followed by a solid body rotation there will be incremental stress components $\overline{s_{\xi\xi}}, \overline{s_{\eta\eta}}, \overline{s_{\xi\eta}}$ due to this rotation, this is of geometric origin. In addition, if the material undergoes a strain, there will be an incremental stress of purely physical origin. Hence it is needed to separate the geometry from the physics in expressing the incremental stress components. This can be done, if we in stead of referring them to the original directions x, y, we refer to a new direction ξ, η. These new directions are rotated with respect to the original direction by angle θ which is equal to the local rotation of the material. Its approximate value to the first order is,

$$\theta = \varpi = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (26.4)$$

The stress components referred to the rotated axes are,

$$\begin{aligned} \sigma_{11} &= S_{11} + s_{11} \\ \sigma_{22} &= S_{22} + s_{22} \\ \sigma_{12} &= S_{12} + s_{12} \end{aligned} \quad (26.5)$$

The quantities s_{11}, s_{22}, s_{12} are the increments of stresses referred to axes which rotate with the medium. It is possible to express the stresses $\overline{\sigma_{\xi\xi}}, \overline{\sigma_{\eta\eta}}, \overline{\sigma_{\xi\eta}}$ in terms of the stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}$ by using the transformation formulas presented earlier,

$$\begin{aligned} \overline{\sigma_{\xi\xi}} &= \sigma_{11} Cos^2 \varpi + \sigma_{22} Sin^2 \varpi - \sigma_{12} Sin 2\varpi \\ \overline{\sigma_{\eta\eta}} &= \sigma_{11} Sin^2 \varpi + \sigma_{22} Cos^2 \varpi + \sigma_{12} Sin 2\varpi \\ \overline{\sigma_{\xi\eta}} &= \frac{1}{2}(\sigma_{11} - \sigma_{22}) Sin 2\varpi + \sigma_{12} Cos 2\varpi \end{aligned} \quad (26.6)$$

For small angle of rotation, to the first order approximation ($\cos\varpi = \cos 2\varpi \cong 1$, $\sin\varpi = \frac{1}{2}\sin 2\varpi \cong \varpi$), we have:

$$\begin{aligned} &\overline{\sigma_{\xi\xi}} = S_{11} + \overline{s_{\xi\xi}} = \sigma_{11} + \sigma_{22} Sin^2 \varpi - \sigma_{12} 2\varpi = S_{11} + s_{11} - (S_{12} + s_{12}) 2\varpi \\ &\Rightarrow \overline{s_{\xi\xi}} = s_{11} - (S_{12} + s_{12}) 2\varpi = s_{11} - 2S_{12}\varpi \end{aligned} \quad (26.7a)$$

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