Module 3 : Equilibrium of rods and plates

Lecture 16 : Bending of thin plates

The Lecture Contains:

- Bending of thin plates

This lecture is adopted from the following book

1. "Theory of Elasticity, 3rd edition" by Landau and Lifshitz. Course of Theoretical Physics, vol-7

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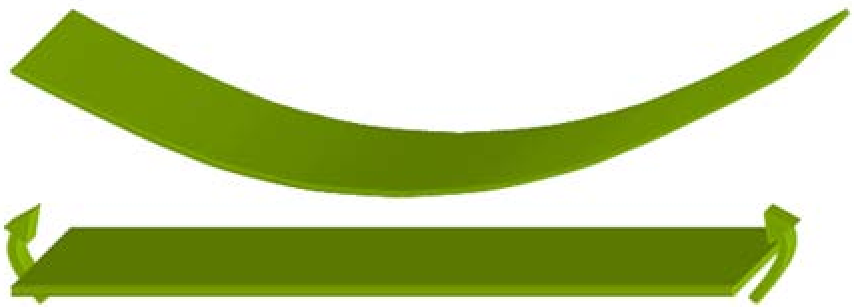
Bending of thin plates

Here we will develop the general equation for bending of thin plates whose thickness is small compared with other two dimensions. When such a plate is bent, it is stretched in the convex part and compressed in the concave side. The stretching decreases as one penetrates into the plate till becoming zero at the neutral surface where there is no stretching or compression, at the other side of the neutral surface, the deformation changes sign.



Let us consider a co-ordinate system with origin on the neutral surface and the z-axis normal to it. The xy-plane is that of the un-deformed plate. Say, ζ is the vertical displacement of a point on the neutral surface along the z coordinate. Here, we consider small bending of the plate so that $\zeta(x, y)$ is small with respect to the thickness h of the film. Other components of the displacement in the xy-plane are negligibly small relative to ζ. Thus on the neutral surface the components of displacement vector are: $u^{(0)} = v^{(0)} = 0, \quad w^{(0)} = \zeta(x, y)$.

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Assumption 1: Since the bending is small the normal vector to any point on the bent plate is along the z-axis.

Assumption 2: Since the plate is thin it bends very easily, so that a very small amount force needs to be applied on the surface of the plate, much smaller than the internal stresses produced by the stretching and compression of its parts. Hence we safely neglect the surface forces leading to $\sigma_{ik} n_k = 0$, i.e. $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$. However these quantities must be small within the plate if they are zero on the surface. Hence it can be concluded that these stress components are negligibly small with respect to the other components of the stress tensor. Thus we have,

$$\sigma_{zx} = \frac{E}{1+\sigma} e_{zx} = 0, \quad \sigma_{zy} = \frac{E}{1+\sigma} e_{zy} = 0, \quad \sigma_{zz} = \frac{E}{(1+\sigma)(1-2\sigma)}\{(1-\sigma)e_{zz} + \sigma(e_{xx} + e_{yy})\} = 0 \qquad (16.1)$$

Putting $w = \zeta(x, y)$, the components of displacement are obtained as,

$$u = -z\frac{\partial \zeta(x,y)}{\partial x}, \quad v = -z\frac{\partial \zeta(x,y)}{\partial y} \qquad (16.2)$$

and the components of the strain tensor are:

$$\begin{aligned} &e_{xx} = -z\frac{\partial^2 \zeta(x,y)}{\partial x^2}, \; e_{yy} = -z\frac{\partial^2 \zeta(x,y)}{\partial y^2}, \; e_{xy} = -z\frac{\partial^2 \zeta(x,y)}{\partial x \partial y}, \\ &e_{xz} = e_{yz} = 0, \; e_{zz} = \frac{\sigma}{1-\sigma} z\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right) \end{aligned} \qquad (16.3)$$

The free energy per unit volume is,

$$W = \frac{1}{2}\lambda e_{ii}^2 + \mu e_{ik}^2 = \frac{1}{2}\lambda(e_{xx} + e_{yy} + e_{zz})^2 + \mu\left(e_{xx}^2 + e_{yy}^2 + e_{zz}^2 + 2e_{xy}^2 + 2e_{yz}^2 + 2e_{zx}^2\right)$$

$$F = z^2\frac{E}{1+\sigma}\left\{\frac{1}{2(1-\sigma)}\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right)^2 + \left(\frac{\partial^2 \zeta}{\partial x \partial y}\right)^2 - \frac{\partial^2 \zeta}{\partial x^2}\frac{\partial^2 \zeta}{\partial y^2}\right\} \qquad (16.4)$$

Total free energy of the deformed plate of thickness h is,

$$F_{pl} = \frac{Eh^3}{24(1-\sigma^2)}\iint\left\{\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right)^2 + 2(1-\sigma)\left\{\left(\frac{\partial^2 \zeta}{\partial x \partial y}\right)^2 - \frac{\partial^2 \zeta}{\partial x^2}\frac{\partial^2 \zeta}{\partial y^2}\right\}\right\}dxdy \qquad (16.5)$$

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Let us now assume that bending occurs only in one dimension, so that ζ is a function of x only and the above expression for the free energy simplifies to,

$$F_{pl} = \frac{Eh^3}{24\left(1-\sigma^2\right)} \iint \left(\frac{\partial^2 \zeta}{\partial x^2}\right)^2 dxdy \tag{16.6}$$

If the bending of the plate is increased slightly so that ζ increases by $\delta\zeta$, then the differential amount of increase in free energy of plate is

$$\delta F_{pl} = \frac{Eh^3}{24\left(1-\sigma^2\right)} \delta \iint \left(\frac{\partial^2 \zeta}{\partial x^2}\right)^2 dxdy = \frac{Eh^3}{24\left(1-\sigma^2\right)} \iint 2\left(\frac{\partial^2 \zeta}{\partial x^2}\right) \frac{\partial^2 \delta\zeta}{\partial x^2} dxdy \tag{16.7}$$

Integration by parts of the above equation yields:

$$\begin{aligned} \delta F_{pl} &= \frac{Eh^3}{12\left(1-\sigma^2\right)} \left[\left(\frac{\partial^2 \zeta}{\partial x^2}\right) \iint \frac{\partial^2 \delta\zeta}{\partial x^2} dxdy - \iint \left(\frac{\partial^3 \zeta}{\partial x^3}\right) \frac{\partial \delta\zeta}{\partial x} dxdy\right] \\ &= \frac{Eh^3}{12\left(1-\sigma^2\right)} \left[\left(\frac{\partial^2 \zeta}{\partial x^2}\right) \frac{\partial \delta\zeta}{\partial x} \int dy - \left(\frac{\partial^3 \zeta}{\partial x^3}\right) \delta\zeta \int dy + \iint \left(\frac{\partial^4 \zeta}{\partial x^4}\right) \delta\zeta dxdy\right] \end{aligned} \tag{16.8}$$

If P be the external force acting on the plate per unit area normal to the surface, then the work done by P due to displacement ζ of the point of application of the force is $\delta U = -\int P \delta\zeta df$.

Then it can be shown after some algebraic manipulation that the minimization of the total free energy of the system which includes the free energy of the plate and the work done by the external load results in the following equilibrium equation of a bent plate:

$$D\Delta^2 \zeta - P = 0 \tag{16.9}$$

where $D = Eh^3/12\left(1-\sigma^2\right)$ is the flexural rigidity of the plate. The boundary condition for solving this equation is different depending on whether the plate is clamped or supported.

(a) For plate whose edge is clamped, the boundary conditions are $\zeta = 0$ (no vertical displacement) and $\frac{\partial \zeta}{\partial n} = 0$ (the edge remains horizontal). The reaction force F and the reaction moment M are written as

$$F = -D\frac{\partial^3 \zeta}{\partial n^3}, \; M = D\frac{\partial^2 \zeta}{\partial n^2} \tag{16.10}$$

(b) For a supported plate the moment of force becomes zero where it is supported, so that the following boundary conditions prevail: $\zeta = 0$ and $\frac{\partial^2 \zeta}{\partial n^2} = 0$.

Problem 1

Deflection of a circular plate of radius R with clamped edges, placed horizontally in a gravitational field.

$$P=\rho gh\,,\ D=\frac{Eh^3}{12\left(1-\sigma^2\right)} \tag{16.11}$$

$$\Delta=\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) \tag{16.12}$$

$$\begin{aligned}&\frac{Eh^3}{12\left(1-\sigma^2\right)}\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\zeta}{dr}\right)\right)\right)=\rho gh\\ &\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\zeta}{dr}\right)\right)\right)=64\,\frac{3\rho g\left(1-\sigma^2\right)}{16\,Eh^2}=64\,\beta\end{aligned} \tag{16.13}$$

Boundary condition: $\frac{\partial\zeta}{\partial r}=0$ at $r=0$, $\zeta=0$ and $\frac{\partial\zeta}{\partial r}=0$ at $r=R$ (16.14)

Show that the deflection of the plate can be obtained as $\zeta=\beta\left(R^2-r^2\right)^2$

Problem 2

Deflection of the circular plate of radius R with clamped edges, when a force f is applied at its center.

Governing equation: $\Delta^2\zeta=0$

Boundary condition: $\zeta=0$ and $\frac{\partial\zeta}{\partial r}=0$ at $r=R$

$$2\pi D\int_0^R r\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\zeta}{dr}\right)\right)dr=f \tag{16.15}$$

Show that the deflection of the plate can be obtained as, $\zeta=\frac{f}{8\pi D}\left[\frac{1}{2}\left(R^2-r^2\right)-r^2\ln\left(\frac{R}{r}\right)\right]$

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