Module 3: "Thin Film Hydrodynamics"

Lecture 11: ""

The Lecture Contains:

Micro and Nano Scale Hydrodynamics with and without Free Surfaces

- Order of Magnitude Analysis

Micro and Nano Scale Hydrodynamics with and without Free Surfaces

When studying free surface flows, the dynamics of interfaces comes into play. So the equations of motion are important here. These equations are mentioned below.

Equation of continuity

$$\nabla.\vec{v} = 0 \qquad (3.1)$$

Navier-Stokes equation

$$\rho\frac{D\vec{v}}{Dt} = \rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\ \nabla)\vec{v}\right] = -\nabla_{\rho} - \nabla_{\emptyset} + \mu^{\nabla^2}\vec{v} \qquad (3.2)$$

There are four unknowns – pressure field and three components of the velocity vector, and four equations. Therefore, theoretically, this system of equations can be solved. But due to their complexity, these equations have not been analytically solved yet.

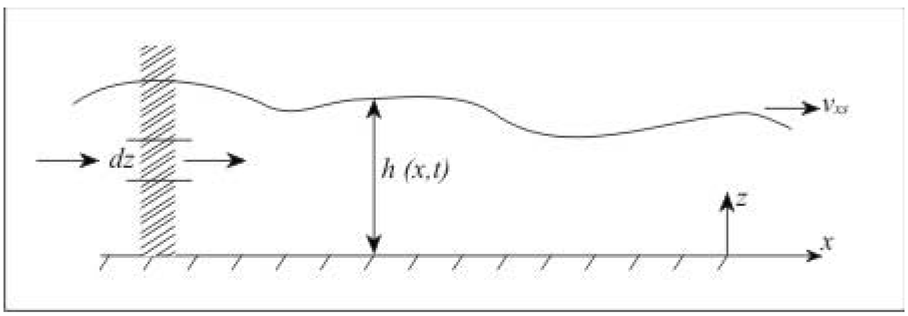

Figure 3.1. Free surface flow

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Consider a free fluid surface of variable height $h(x,t)$ (Fig. 3.1)

λ is the length scale along the x-direction. Since $\lambda \gg h, h_x \ll 1$.

$\frac{Dh}{Dt}$ is the rate of change in the film thickness recorded at a point.

$$\frac{Dh}{Dt} = v_{zs} = \frac{\partial h}{\partial t} + v_{xs}\frac{\partial h}{\partial x}$$

$$\Longrightarrow \frac{\partial h}{\partial t} + u_s \frac{\partial h}{\partial x} = w_s \qquad (3.3)$$

where, u and w are the velocity components along $x-\text{direction}$ and $z-\text{direction}$, respectively.

The above-mentioned relation is called the 'kinematic condition'

Alternatively, the kinematic condition can also be derived as follows. Consider a pill box (Fig. 3.1). Mass balance on the pill-box leads to

$$\frac{a(hdx)}{\partial t} = \left[\int_0^h udz\right]_x - \left[\int_0^h udz\right]_{x+dx}$$

$$\Longrightarrow \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^{h(x)} udz = 0$$

which can be simplified using the Leibnitz rule as

$$\frac{\partial h}{\partial t} + \int_0^{h(x)} \frac{\partial u}{\partial x} dz + u(h)\frac{\partial h}{\partial x} - u(0)\frac{\partial (0)}{\partial x} = 0$$

the last term on the LHS vanishes because of the no slip condition at the solid boundary.

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From equation of continuity, we have

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\Longrightarrow \frac{\partial u}{\partial x} = -\frac{\partial w}{\partial z}$$

therefore,

$$\frac{\partial h}{\partial t} - \int_0^{h(x)} \frac{\partial w}{\partial z} dz + u(h)\frac{\partial h}{\partial x} = 0$$

$$\Longrightarrow \frac{\partial h}{\partial t} - w(h) + w(0) + u_s\frac{\partial h}{\partial x} = 0$$

In case of an impermeable solid boundary, as in the present case,

$$w(0) = 0$$

which leads to p_g.

$$\frac{\partial h}{\partial t} - w_s + u_s\frac{\partial h}{\partial x} = 0$$

which is same as Eq (39).

Order of magnitude analysis

In this case,

$$\lambda \gg \mathrm{h}$$

where, λ is the length scale and h is the mean film thickness.

Equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$z \sim \mathrm{O(h)\,and\;x} \sim \mathrm{O}(\lambda).$$

If we assume, $z \sim \mathrm{O}(1)$

then

$$x \sim (\lambda/h) - (1/\varepsilon)$$

where

$$\varepsilon = h/\lambda \ll 1$$

Now if

$$\mathrm{u} \sim \mathrm{O}(1)$$

then

$$\mathrm{w} \sim \mathrm{O}(\varepsilon)$$

Kinematic condition

$$\frac{\partial h}{\partial t} + u_s \frac{\partial h}{\partial x} = w_s$$

On LHS, the first term is $O(1/t)$, and the second is $O(\varepsilon)$

Therefore, for consistency, t should be of $O(1/\varepsilon)$

Navier-Stokes equation (***x***-component)

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} - \frac{\partial \emptyset}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

or

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial(\rho + \phi)}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

This means that the unsteady term is as important as the inertial terms.

And,

$$(\rho + \phi) \sim O(1/\varepsilon)$$

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Navier-Stokes equation (*z*-component)

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial(\rho+\phi)}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

After considering the orders of all the terms and taking the limit $\varepsilon \rightarrow 0$ we get

$$-\frac{\partial(p+\phi)}{\partial x}+u\frac{\partial^2 u}{\partial z^2}=0$$

$$-\frac{\partial(p+\phi)}{\partial z}=0$$

This equation shows that the pressure distribution is a hydrostatic pressure distribution with respect to z-direction.

It further means that the flow is largely in the x-direction.

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Example 1. Squeezing of fluid between two flat surfaces (Fig. 3.2).

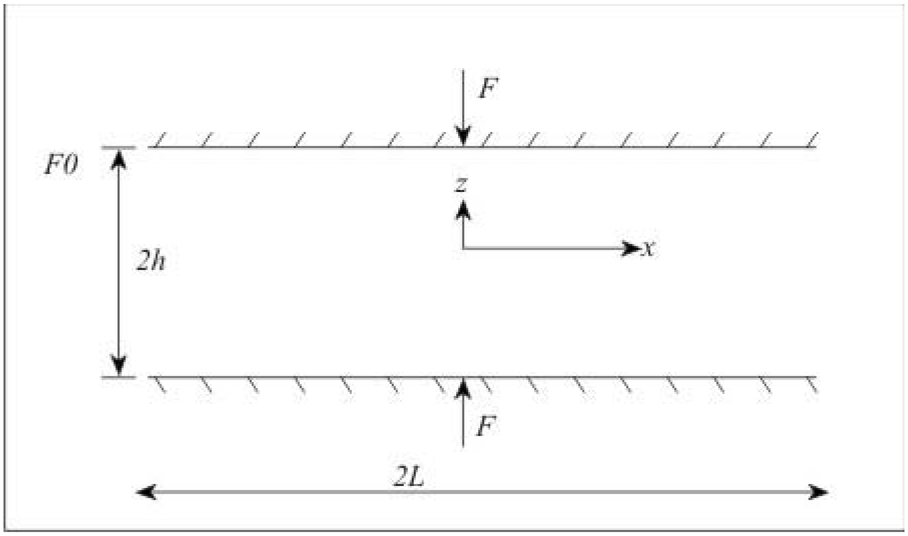

Figure 3.2: Flow between two flat plates

We want to obtain expressions for the velocity with which the two surfaces approach each other and the time it takes to do so. Here the distance 2h between the two plates is not a function of x and is only a function of time.

$$P = p + \phi$$

$$\mu \frac{\partial^2 u}{\partial z^2} = \frac{\partial P}{\partial x} = P_x$$

$$\Rightarrow u \frac{\partial u}{\partial z} = P_x z + c_1$$

$$\frac{\partial u}{\partial z} = 0 \; at \; z = 0$$

$$\Rightarrow \mu u = P_x \frac{z^2}{2} + c_2$$

$$\mathrm{At}\ z = h, u = 0$$

$$\Rightarrow \mu u = \frac{P_x}{2}(z^2 - h^2)$$

Using

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

we get,

$$\mu \frac{\partial w}{\partial z} = -\left[\frac{P_x}{2}(z^2 - h^2)\right]_x$$

$$\Longrightarrow \mu w = -\left[\frac{p_x}{2}\left(\frac{z^3}{3} - h^2 z\right)\right]_x + c_3$$

$$\Longrightarrow \mu w = -\frac{P_{xx}z^3}{6} + \frac{1}{2}(h^2 P_x)x^z$$

$$At\ z = 0, w = 0$$

$h(t)$ is to be obtained

$$\frac{\partial h}{\partial t} + u_s \frac{\partial h}{\partial x} = w_s$$

u_s vanishes due to the no slip condition and $\frac{\partial h}{\partial x}$ vanishes because h is independent of x.

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Therefore, we get,

$$\frac{\partial h}{\partial t} = \frac{h^3}{3_\mu}\frac{\partial^2 P}{\partial x^2}$$

P is still an unknown.

$$F = 2lp_a$$

$$\mu u = \frac{P_{xx}}{2}(z^2 - h^2)$$

$$\mu u = \frac{P_{xx}}{2}\frac{z^3}{3} + \frac{1}{2}(h^2 P_x)x^z$$

$$\frac{\partial h}{\partial t} + u_s\frac{\partial h}{\partial x} = w_s$$

$$h \neq h(x)$$

$$\frac{\partial h}{\partial t} = \frac{h^3}{3_\mu}\frac{\partial^2 P}{\partial x^2} = -V$$

Note that V is a positive quantity.

$$\frac{\partial^2 P}{\partial x^2} = \frac{-3\mu V}{h^3}$$

Integrate once to get

$$\frac{\partial P}{\partial x} = -\frac{-3\mu V}{h^3}x + c_1$$

Symmetry of P about mid-plane i.e. $x = 0$ implies that $c_1 = 0$

$$P = P + \Phi = \frac{-3\mu V}{2h^3}x^2 + c_2$$

At $x = L, P = p_0$, surrounding fluid pressure.

$$\Longrightarrow P = \frac{-3\mu V}{2h^3}(x^2 - L^2) + p_0$$

This P is what causes the symmertic drainage.

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Obtaining P in Terms of Velocity of Thinning

A quasi-steady analysis leads to

$$F = 2L(P_0 + P_a)$$

$$F_f = 2\int_0^L p\,dx$$

F_f is the force applied by the fluid on the plate.

$$F_f = -2\int_0^L x\frac{dP}{dx} + 2_{P0}L$$

Equating the two pressures gives

$$2LP_a = 2\int_0^L \frac{3\mu V}{h^3} x^2 dx$$

$$\Longrightarrow F_a = \frac{2\mu V}{h^3} L^3$$

$$V = \frac{F_a h^3}{2\mu L^3} = \frac{P_a h^3}{2} = -\frac{dh}{dt}$$

$$t = t_0, h = h_0$$

Integrate, to get

$$\Longrightarrow \frac{1}{h^2} - \frac{1}{h_0^2} = \frac{2P_a t}{\mu L^2}$$

$$\Longrightarrow h^2 = \frac{1}{h_0^{-2} + \frac{2P_a t}{\mu L^2}}$$

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Analysis of the solution

1. $V \propto \frac{1}{L^2}$. This means that if the length is large, drainage would slow down. This is because of the viscous resistance.
2. As $h \longrightarrow 0, t \longrightarrow \infty$. This may lead us to believe that the two surfaces will not come together. But actually this isn't true because when the surfaces are close together, Van-Der Waals forces will come into play and they will affect the system.

$$P = P_a + \Phi$$

When the film thickness reduces to molecular scale, the intermolecular interactions become significant.

$$\Phi^{\mathrm{Lw}} = \frac{\mathrm{A}}{\mathrm{h}^3}$$

$$\therefore P = P_a + \frac{A}{h^3} \qquad (3.6)$$

Substituting this in the differential equation

$$\frac{\left(p_a + \frac{A}{h^3}\right) h^3}{\mu L^3} = -\frac{dh}{dt}$$

Now we can observe that as *h* becomes very small, velocity becomes constant. This implies that coalescence would occur in a finite time.

If the exponent of *h* is smaller it a leads to deceleration as the surfaces approach each other and contact occurs in infinite time. On the other hand, if the exponent of h is larger it results in acceleration and the two surfaces crash into each other.

Thus we can see that Lipshitz-van der Waals interactions lead to macroscopically verifiable interactions.

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Example. Liquid coating on a solid substrate (Fig. 3.3).

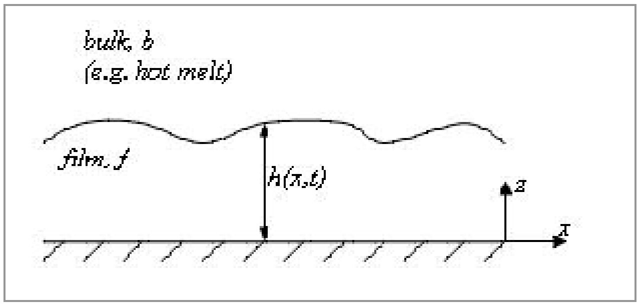

Fig. 3.4 :Liquid film on a solid surface

$$\frac{\partial P}{\partial z} = 0 \Longrightarrow \frac{\partial(p+\Phi)}{\partial z} = 0$$

$$p + \Phi = P(h) + \Phi(h)$$

Navier-Stokes equation reduces to-

$$\mu\frac{\partial^2\mu}{\partial z} = \frac{\partial P}{\partial x}P_x$$

$$\Longrightarrow u\frac{\partial u}{\partial z} = P_X z + c_1$$

From the no-slip condition, at $z = 0, u = 0$

We take another assumption – that the bounding fluid is less viscous than the film. This would mean that the bulk fluid does not offer any drag or shear force on the film.

$$\Longrightarrow \mu\frac{\partial u}{\partial z} = 0 \; at \; z = h$$

$$\Longrightarrow c_1 = P_x h$$

$$\Longrightarrow \mu u = P_x\frac{z^2}{2} - P_x hz + c_2$$

$c_2 = 0$ from the no-slip condition.

Using the continuity equation,

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$$

$$\mu\frac{\partial u}{\partial x}=P_{xx}\frac{z^2}{2}-(P_xh)x^z$$

We will use this equation to find the normal component of the velocity.

$$\Longrightarrow w=-\frac{1}{\mu}[P_{xx}\frac{z^3}{2}-(P_xh)_x\frac{z^2}{2}]+c_3$$

Substrate (a solid) is impermeable, therefore

$$w=0\ at\ z=0$$

$$\Longrightarrow c_3=0$$

$$\frac{\partial h}{\partial t}-\frac{1}{\mu}(P_xh^2)h_x-\frac{1}{3\mu}P_{xx}h^3=0 \qquad (3.7)$$

$$\frac{\partial h}{\partial t}-\frac{1}{3\mu}(P_xh^3)_x=0$$ Conservative form of the equation (3.8)

This is also the governing equation for the film thickness.

$$h=h(x,t)$$

Conservative form is more suitable for solving this problem numerically.

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Why should the surface deform?

1. Because a heavier fluid is on top of a lighter fluid,
2. Because of spreading of a fluid on a non-wettable surface.

A point to be noted is that these are not equilibrium situations.

$$P = P(h) + \Phi(h) = P_b - \gamma h_{xx} + \Phi(h)$$

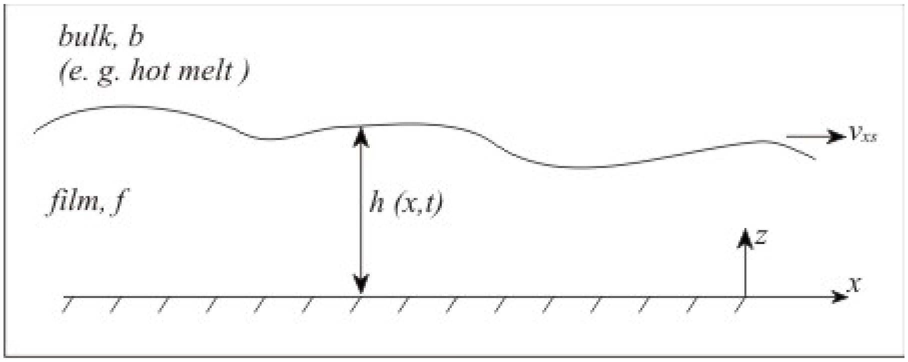

Figure 3.5. Thin film being coated on a surface

Weak flow in the bounding fluid means that pressure distribution is hydrostatic.
Also note that the mean film thickness is constant because of conservation of volume.

$$P_b + \Phi(h) = P_b gL + P_b g(h_0 - h) + P_f gh$$

$$P = (P_b gL + P_b gh_0) + gh(P_f - P_b) - \gamma h_{xx}$$

$(P_b gL + P_b gh_0)$ is a constant and $(P_f - P_b) = \Delta\rho$

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If the film thickness is very small $(< 100\ nm)$, the potential energy due to interaction between the substrate and film needs to be taken into account. Electrostatic interactions can also be important.

$$P_x = g(\Delta\rho)h_x - \gamma h_{xxx} + \Phi_x$$

where Φ accounts for the other interaction potentials. It is a function of h and hence, of x

$$h_t - \frac{1}{3\mu}\left[\left(\Delta\rho g h_x - \gamma h_{xxx} + \frac{\partial\Phi}{\partial h}h_x\right)h^3\right]_x = 0 \qquad (3.9)$$

This is a non-linear PDE of 4th order, which can generally be solved only using numerical techniques.

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