Module 3: "Thin Film Hydrodynamics"

Lecture 12: ""

The Lecture Contains:

- Linear Stability Analysis
- Some well known instabilities

Linear Stability Analysis

This analysis helps us in understanding the physics of the problem without solving the complex differential equation governing it.
In this analysis we start with a system which is in equilibrium state. A small perturbation is given to the system and we see whether the system returns to its equilibrium state.

The small perturbation can be written as

$$h = h_0 + \varepsilon f(x,t)$$

where

$$f(x,t) \sim O\,(1)$$

and

$$\varepsilon f(x,t) \ll h_0$$

Therefore, we can neglect the higher order terms in ε and discard all the non-linear terms in ε. So,

$$h^3 h_x \approx h_0^3 f_x ; h^3 h_x \frac{\partial \Phi}{\partial h} \approx h_0^3 f_x \frac{\partial \Phi}{\partial h_0}$$

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This linearization around h0, leads to the following expression,

$$f_t - \frac{1}{3\mu}[\Delta\rho g h_0^3 f_x - h_0^3 \gamma f_{xxx} + h_0^3 \Phi_{h0} f_x]_x = 0 \tag{3.11}$$

Note that the above differential equation involves only even-ordered derivatives. Thus, it represents a stationary wave, and the solution to this equation is of the form

$$f = e^{\omega t} \sin Kx \tag{3.12}$$

This solution is periodic in position and exponential in time.

One can verify this solution by substituting it back in the differential equation. This gives

$$\omega - \frac{1}{3\mu}\left[-\Delta\rho g h_0^3 K^2 - h_0^3 \gamma K^4 - h_0^3 \Phi_{h0} K^2\right] = 0 \tag{3.13}$$

The above relation is known as dispersion relation or characteristic equation or *compatibility condition*.

Here,

ω is called the growth coefficient, and

$K = \frac{2\pi}{\lambda}$ is called the wave number.

ε is the amplitude of the initial perturbation.

If $\omega > 0$, the perturbation would grow. So, it causes instability.

If $\omega < 0$, the perturbation would decay. So, it would move towards stability.

If $\omega = 0$, it is the critical boundary between instability and stability.

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Analysis of the terms in the dispersion relation

Consider the terms inside the braces in Eq. (3.13).

I st Term:

It represents the effect of gravity. If $\Delta\rho < 0$, this term would be positive and have a destabilizing effect (***Rayleigh-Taylor instability***). This situation could occur when, for example, there is a heavier liquid on top of a lighter one on a surface.

II Term:

It represents the effect of surface tension. For any positive interfacial tension this term would be negative and thus would have a stabilizing effect.

III Term:

If $\Phi_{ho} < 0$, it would have a destabilizing effect.

$$\Phi = \frac{A}{6\pi h^3} \qquad (3.14)$$

Φ has the units of energy per unit volume. And A is known as Hamaker constant and depends on the properties of the material.

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$$p + \Phi = p(h) + \Phi\,(h)$$

$p + \Phi$ is independent of z. So, for convenience, we evaluate it at the free surface.

$$\Phi_{ho}^{LW} = -\frac{A}{2\pi h_0^4}$$

If $A < 0$, there would be instability due to the LW forces.

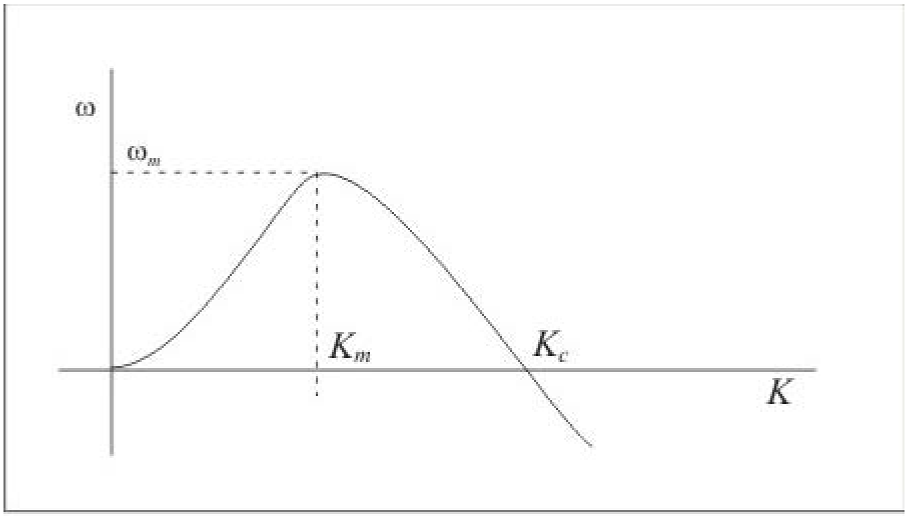
A is the ratio of interaction among the molecules of the liquid to that between the liquid molecules and the surface.

If $A > 0$, it would cause beading. But on the other hand, if $A < 0$, then the surface would be wettable.

In the present problem, $A > 0$ and $\Phi_{ho} < 0$. So the dispersion relation can be written as

$$\omega = cK^2(a - \gamma K^2) \qquad (3.15)$$

The plot has been shown in the figure below (Fig. 3.6).



h (x,t)
Figure 3.6. Growth coefficient versus wave number

Observations

For $K > K_c$, ω becomes negative. This qualitatively means that any perturbation with a very small wavelength would die down.

Sources of disturbance could be mechanical perturbations or thermal perturbations. In the present case, all the modes corresponding to values of K between 0 and K_c lead to instability.

K_m is called the dominant mode as it grows the fastest.

In order to find K_m, we put $\dfrac{\partial \omega}{\partial k} = 0$

$K_m^2 = a/2\gamma$

$\omega_m = ca/4\gamma$

And to find K_c, we put $\omega = 0$.

Number of droplets arising due to instability $= \lambda_m^{-2}$

If $A > 0$, the fluid partially wets the surface, which means the fluid layer breaks up into droplets.

This solution is valid only very close to equilibrium states.

Linear stability analysis is a powerful tool. It helps us analyze the stability of the system around an equilibrium state. But it has its limitations too. It does not give us information about the system over a long time scale. It only gives us the length scales of the deformation. For example, it does not tell us how long it would take before the film ruptures.

An estimate for the time at which the film rupture takes place can be arrived at as follows-

When the film ruptures, $h = 0$ locally, i.e., at some x, h would become zero.

$$0 = h_0 + \varepsilon e^{\omega t} \sin kx$$

$$= h_0 - \varepsilon e^{\omega t} \qquad (\sin kx = -1 \text{ at } x \text{ where rupture takes place})$$

$$\Rightarrow \tau = \frac{1}{\omega} \ln\left(\frac{h_0}{\varepsilon}\right) \qquad (3.16)$$

Maximum value of ω (i.e. corresponding to K_m) would lead us to minimum τ needed for rupture,

$$\tau_{min} = \frac{1}{\omega_m} \ln\left(\frac{h_0}{\varepsilon}\right) \qquad (3.16)$$

The number of holes formed per unit length can also be estimated.

For each $\lambda_m \times \lambda_m$ area, there is one hole.

Number of holes per unit area= $N_H \propto \lambda_m^{-2}$ (3.17)

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Some well known instabilities

1. Falling film

This instability occurs when the Reynolds number is of the order of 1-10 and it is caused by inertial effects. This is the reason behind the ripples observed on the surface of the falling film. The mass/heat transfer rates we estimate can have significant errors.

2. Natural convection

If there are two plates close to each other, and one is cooled and the other is heated, circulation patterns are observed. There is a density variation with variation in temperature.Patterns form starting from a uniform state. This is an example of "self-organization" where the system knows how it wants to evolve.

A point to note here is that this phenomenon is not observable for all values of ΔT. It is only at some critical value of ΔT that this pattern formation takes place. If ΔT were very small in magnitude, there would be no such phenomenon. In fact, in that case, pure conduction would be occurring. Another important point is that the process of evolution should not be confused with instability.

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3. Marangoni flow

Margoni flow arises when we have a free surface but the surface tension is not equal everywhere on the surface. Margoni flow goes from a point of lower surface tension to a point of higher surface tension.

For example, suppose we have a liquid with some powder sprinkled over it. We blow on the surface of the liquid thereby creating a hole in the powder layer. The surface tension at the point where there is no powder left will be higher than at the points where there is powder. So the liquid(with powder) will flow back to close the hole. This also shows us how surfactants tend to stabilize a flow.

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4. Spinodal Phase Separation (Spinodal decomposition)

This is analogous to LW dewetting.
Phase separation occurs by two mechanisms-

1. Nucleation & Growth (An energy barrier has to be overcome)
2. Spinodal Mechanism (There is no energy barrier to overcome)

5. Turbulence

Turbulence can be viewed as instability in the laminar flow profile. This means that the basic solution of the Navier-Stokes Equation is unstable.
There are also other instabilities such as Helmholtz instability, a continuous stirred tank reactor (CSTR) being operated non-isothermally, Rayleigh instability, Rayleigh-Taylor instabilities and a thin film on a non-wettable surface

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