- NS equation and example
- Ex.1 Couette–flow:

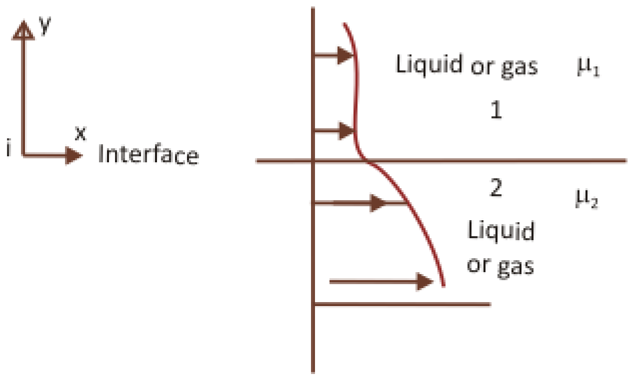
Previous Next

NS equation and examples

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + (V.\nabla)\vec{V}\right] = -\nabla p + \rho\vec{g} + \mu\nabla^2\vec{V}$$

Before we apply the NS equation, let us make a note that the NS equation is a 2nd order PDE. Therefore, 2 BCs are required for $\vec{V}$ along each direction to solve for the velocity-field. We should note the common boundary conditions:

1. No slip condition at the solid surface: it is an experimental observation that the relative velocity at the surface is zero. So, $V_x = V_y = V_z = 0$ at a stationary solid surface
2. At the interface of liquid-gas or liquid–liquid (immiscible fluids), there is no jump in the velocity, shear stress and pressure

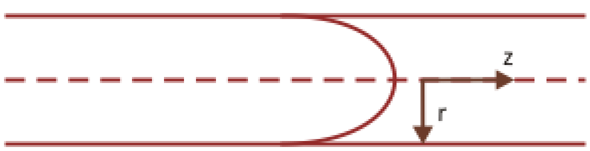


(Fig. 16a)

(gradients maybe different) $\mu_1 \left.\frac{\partial V_x}{\partial y}\right|_{i^-} = \mu_2 \left.\frac{\partial V_x}{\partial y}\right|_{i^+}$

3. Symmetric BC: $\left(\tau_{yx}\big|_{i^-} = \tau_{yx}\big|_{i^+}\right)$

At the center- line of tube or rectangular channel, velocity- gradient is zero.



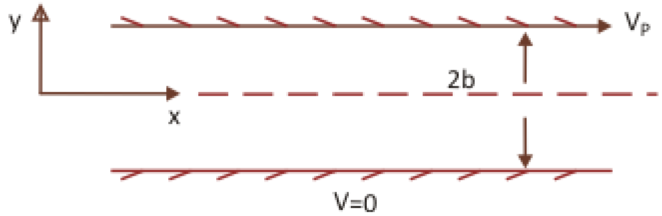
$$\frac{\partial V_z}{\partial r} = 0$$

(Fig. 16b)

Previous Next

Ex.1 Couette–flow

Consider the steady-state 2D-flow of an incompressible Newtonian fluid in a long horizontal rectangular channel. The bottom surface is stationary, whereas the top surface is moved horizontally at the constant velocity, V_p. Determine the velocity field in the channel. Assume fully developed flow.



(Fig. 16c)

Assumption: $\rho = c$, and Newtonian fluid $\Rightarrow$ NS is applicable

2D flow $\Rightarrow$ $V_z = 0$ and $\dfrac{\partial}{\partial Z} = 0$

Fully developed flow: $\dfrac{\partial v_x}{\partial x} = 0$

(Note: this statement is equivalent to saying, neglect the end effects)

Apply continuity: $\nabla.\vec{V} = 0$

$\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0$

For fully developed flow, first term is zero.

Therefore, $\dfrac{\partial v_y}{\partial y} = 0$ $\Rightarrow$ V_y is constant along y direction.

But, $V_y = 0$ at y = ± b (no- slip condition)

So, $V_y = 0$ everywhere. $V_x(y) \neq 0$

Previous Next

Apply NS in x-direction:

$$\rho\left(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}\right)$$

1st term is zero for SS; $\frac{\partial V_x}{\partial x}=0$ for fully developed flow; $V_y=0$. Horizontal $\Rightarrow g_x=0$. Therefore, the equation is simplified as

$$0=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 V_x}{\partial y^2}$$

No external pressure was applied and in addition, it is a long plate $\Rightarrow \frac{\partial p}{\partial x}=0$

$$\mu\frac{\partial^2 V_x}{\partial y^2}=0$$

Integrate twice, $\mu V_x=C_1 y+C_2$

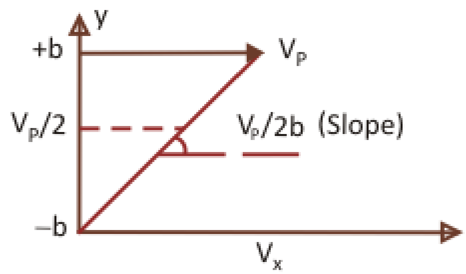
Apply BC (1) $y=-b, V_x=0$ (no slip condition)

BC (2) $y=b, V_x=V_P$

Solve for c_1 and c_2 to obtain

$\boxed{V_x=\frac{V_P}{2}\left(1+\frac{y}{b}\right)}$ $\Rightarrow$ This is the expression for velocity profiles in a Couette flow

(Note: at $y=0, \frac{\partial V_x}{\partial y}=\frac{V_P}{2b}\neq 0$).



(Fig. 16d)

Previous Next

Example 2: Suppose that an external pressure-drop along x-direction is applied across the channel, in addition to the conditions mentioned in example 1. Now, calculate the velocity profiles in the channel.

- X- momentum balance is now modified to

  $0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v_x}{\partial y^2}$

Y – momentum balance:

$$0 = -\frac{\partial p}{\partial y} \quad (V_y = 0)$$

- $p \neq f(y)$ and $\frac{\partial p}{\partial x} = \frac{dp}{dx} \quad (p = p\ (x) \text{only})$

  But , V_x is a function of y only. This implies that $\frac{dp}{dx} = \mu \frac{\partial^2 v_x}{\partial y^2} = p'$ (constant)

Integrating twice, $V_x = \frac{p'}{\mu} \frac{y^2}{2} + c_1\, y + c_2$

Apply BCs to obtain

$$\boxed{V_x = -\frac{p'\, b^2}{2\mu}\left(1 - \frac{y^2}{b^2}\right) + \frac{V_p}{2}\left(1 + \frac{y}{b}\right)}$$

You should check the two BCs.

Special case

1. $p' = 0$ (Same as in example 1)
2. $V_p = 0$

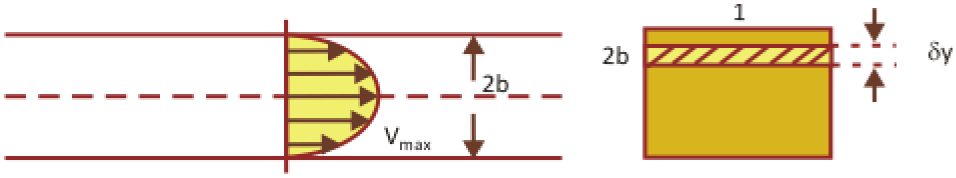
$$V_x = -\frac{p'\, b^2}{2\mu}\left(1 - \frac{y^2}{b^2}\right) = \frac{\Delta p}{L} \frac{b^2}{2\mu}\left(1 - \frac{y^2}{b^2}\right)$$

$V_{max} = \frac{\Delta p}{L} \frac{b^2}{2\mu} (y = 0)$, where, $p' = -\frac{p_1 - p_2}{2} = \frac{\Delta p}{L}$

$$\boxed{V_x\,(y) = V_{max}\left(1 - \frac{y^2}{b^2}\right)}\ ;\ V_{max} = \frac{\Delta p}{L} \frac{b^2}{2\mu}$$

Previous Next

- This is called Poiseuille-flow (velocity profile is parabolic)



(Fig. 16e)

- Determine the average velocity in the channel

$$\overline{V} = {}^{Q}/_{A} = {}^{Q}/_{2b}$$

$$Q = \int_{-b}^{+b} V_x(y)\ .\underbrace{\delta y.1}_{\delta A} = \int_{-b}^{+b} V_{max}\left(1 - \frac{y^2}{b^2}\right)\delta y$$

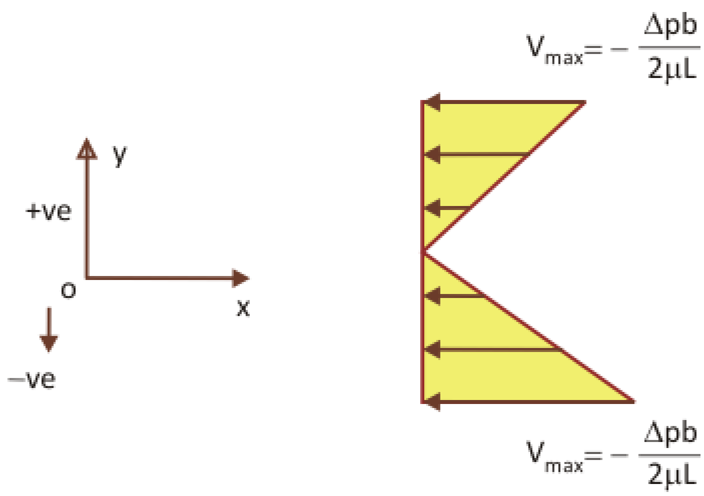
Simplify to show that $\overline{V} = {}^{2}/_{3}\ V_{max}$

or, $V_{max} = 1.5\,\overline{V}$

Next, plot τ_{yx} for Poiseuille and Couette flow

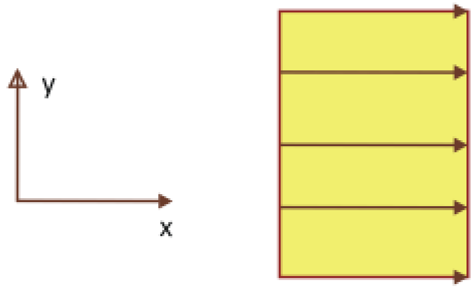
Previous Next

1. $\tau_{yx} = \mu \dfrac{dV_x}{dy} = -2\, V_{max}\, \mu \dfrac{y}{b^2} = -\dfrac{\Delta p}{L} y$: Poiseuille flow



(Fig. 16f)

2. $\tau_{yx} = \dfrac{V_P}{2b}$



(Fig. 16g)

$$\tau_{yx} = \text{constant} = \frac{V_P}{2b}$$

Previous Next