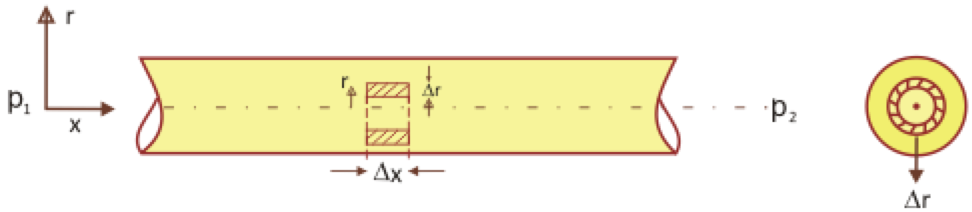
- Steady-state, laminar flow through a horizontal circular pipe

Steady-state, laminar flow through a horizontal circular pipe



(Fig. 17a)

Assumptions: $\rho =$ constant; SS: $\frac{\partial}{\partial t}(.) = 0$; 2D flow: $V_\theta = 0$ (no swirling, circulation)

We are interested in solving $\left.\begin{matrix} V_x\ (x,r) = ? \\ V_r\ (x,r) = ? \end{matrix}\right\}$

Continuity: $\Delta.\vec{V} = 0$

$$\frac{\partial V_x}{\partial X} + \frac{1}{r}\frac{\partial}{\delta r}(rVr) = 0$$

Fully developed flow : $(V_x \neq V_x(x))$

Therefore, $r\ v_r = C$

But $v_r = 0$ at $r = R$ (No- slip condition)

$v_r = 0$ everywhere.

So, the problem is reduced to solving $v_x(r)$.

Previous Next

NS (X-direction):

$$\rho\left(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_r\frac{\partial V_r}{\partial r}\right)=-\frac{\partial p}{\partial X}+\rho g_X(=0)+\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_X}{\partial r}\right)$$

From the formulation of the problem, 1st, 2nd, and 3rd terms on the LHS are zero.

$$0=-\frac{\partial p}{\partial X}+\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dV_X}{dr}\right)$$

r-direction: $0=-\frac{\partial p}{\partial r}-\rho g\sin\theta$

O-direction: $0=-\frac{1}{r}\frac{\partial p}{\partial \theta}-\rho g\cos\theta$

Note: $p=p\,(x,r,\theta)$

Re-look at last two equations.

- $\frac{\partial p}{\partial r}$ and $\frac{\partial p}{\partial \theta}$ depend on gravity. There is no viscous effects
- $\frac{\partial p}{\partial x}$ depends on viscous forces only. There are no gravitational effects.

Therefore, define non-gravitational pressure, so that

$p(x)=p(x,r,\theta)+\rho gr\sin\theta$

or $p(x,r,\theta)=p(x)-\rho gr\sin\theta$

Check: this definition of $p(x)$ or $p(x,r,\theta)$ satisfies the above momentum-balance equation.

Therefore, $\frac{\partial p}{\partial x}=\frac{dp}{dX}$

Substitute,

$$0=-\frac{dp}{dx}+\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dV_x}{dr}\right)$$

Say, $\frac{dP}{dX}=P^1$ (const)

integrate twice to obtain

$$V_X=\frac{r^2p^1}{4\mu}+C_1\ln r+C_2$$

Apply BC1: $r_1 = 0: \dfrac{dV_x}{dr} = 0$ (symmetric BC)

$r = R, \quad V_x = 0$ (No slip BC)

$$\boxed{V_X(r) = -\frac{P^1 R^2}{4\mu}\left(1 - \frac{r^2}{R^2}\right) = V_{max}\left(1 - \frac{r^2}{R^2}\right)}$$

where, $V_{max} = -\dfrac{P^1 R^2}{4\mu} = \dfrac{\Delta P R^2}{4\mu L}; \ \Delta P = P_1 - P_2$

Note that velocity profile is parabolic in the tube.

Calculate,

$Q \equiv$ volumetric flow rate

$$= \overline{V}\,\pi\,R^2 = \int_0^R V_X(r) 2\overline{\pi}\,r\,dr$$

You should be able to show that

$V_{max} = 2\,\overline{V}$

Or, $\overline{V} = \dfrac{V_{max}}{2} = \dfrac{\Delta P R^2}{8\mu L} = \dfrac{(P_1 - P_2)R^2}{8\mu L}$

Re–arrange,

$$\boxed{\frac{\Delta\rho}{L} = \frac{8\mu\overline{V}}{R^2} = \frac{8\mu Q}{\overline{\pi} R^4} = \frac{32\mu\overline{V}}{D^2}}$$

This equation to calculate pressure-drop in a horizontal pipe of Length L and inside-diameter D, for a viscous incompressible fluid flowing under steady-state fully developed laminar condition, is known as Hagen-Poiseuille equation

Previous Next