Module 5: Reynolds Transport Theorem

Lecture 15: Equation of motion

- Equation of motion
- Time-Smoothening of The Equation of The Change

Previous Next

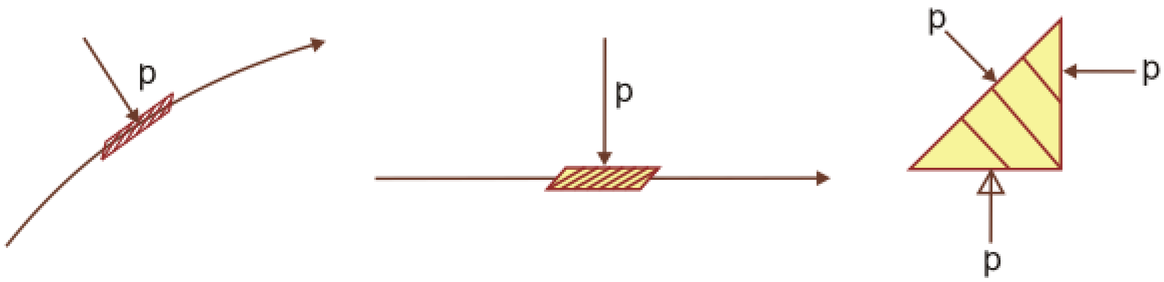
Equation of motion

(Differential form of momentum conservation rate).

Before we apply Reynolds Transport theorem, it is important to understand the forces acting on a fluid-element.

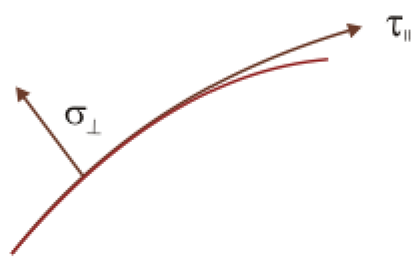
We have earlier noted that the only force acting on a static fluid is the normal force or the pressure acting normal to the surface of the fluid element.

Static fluid:



(Fig. 15a)

In flowing fluid, there is a tangential force or shear force acting parallel to the surface of the fluid-element. Thus, there are two types of stress developed in a flowing fluid: one is the normal stress, σ, and the other is the shear stress, τ :

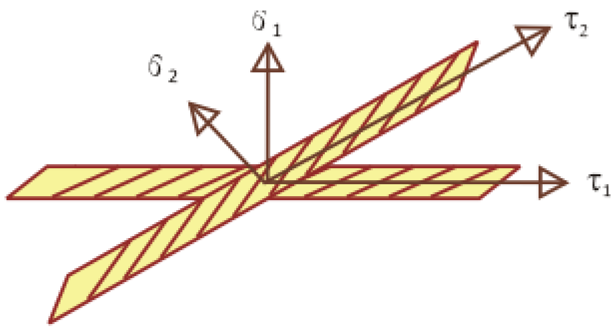


(Fig. 15b)

Previous Next

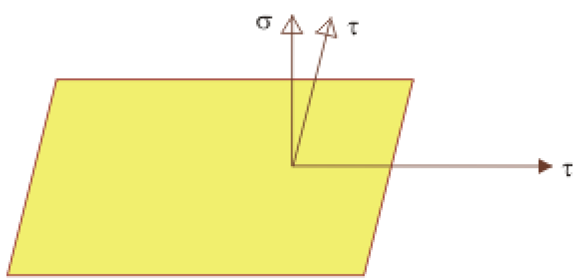
It is shown that, $\sigma, \tau \equiv f(\nabla V)$ where ∇V is the strain rate.

- In the case of a static fluid, $p_x = p_y = p_z$ at any location in the fluid. In a moving fluid, however, $\sigma_1 \neq \sigma_2$ or $\tau_1 \neq \tau_2$:



(Fig. 15c)

- If fluid is inviscid, $\tau = 0$
  And, $\sigma = -p$, as in the case of a static fluid
- For a 3D flow, there are two τ_s and one σ on a plane, as shown below



(Fig. 15d)

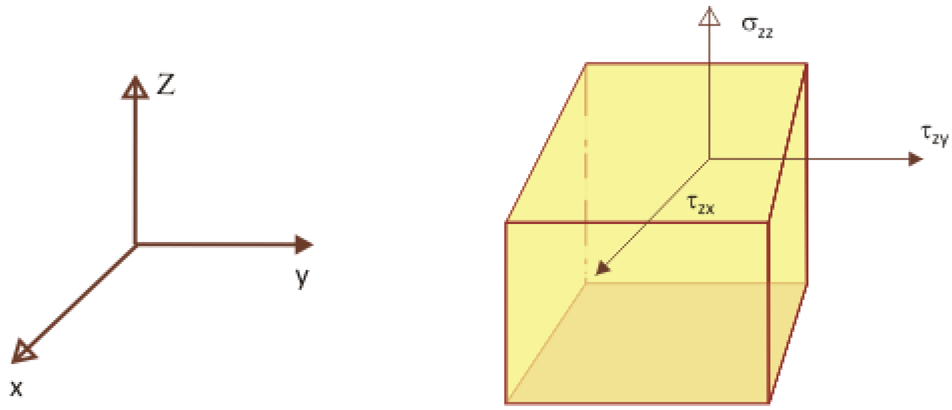
There are sign conventions: τ_{xy}

x: direction of normal to the plane or the direction of the area

y: direction of shear stress

Previous Next

Therefore, for a 3D fluid-element, there are six shear stresses and 3 normal stresses



(Fig. 15e)

- The stresses are represented as

$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

It can be shown that shear stresses are symmetric: $\tau_{xy} = \tau_{yx}$, etc, at any location in the flow-field

- The Newton's law of viscosity relates shear-stress to velocity gradient via the viscosity of a fluid. For 1D flow of a Newtonian fluid,

$$\tau_{yx} = \mu\left(\frac{dV_x}{dy}\right)$$

$\left(\frac{dV_x}{dy}\right)$ is the velocity gradient and can also be represented as the strain rate. In general, when shear stress is applied on a fluid-volume, there are translation, deformation, dilation, rotation and distortion, mathematically expressed by strain rates. Analogous to the representation of shear stresses, strain rates are represented as the combination of the following velocity gradients:

$$\begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} & \frac{\partial V_x}{\partial z} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} & \frac{\partial V_y}{\partial z} \\ \frac{\partial V_z}{\partial x} & \frac{\partial V_z}{\partial y} & \frac{\partial V_z}{\partial z} \end{pmatrix}$$

The Newton's extended law of viscosity relates τ_s and σ_s to ∇V. For example,

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) : 2D$$

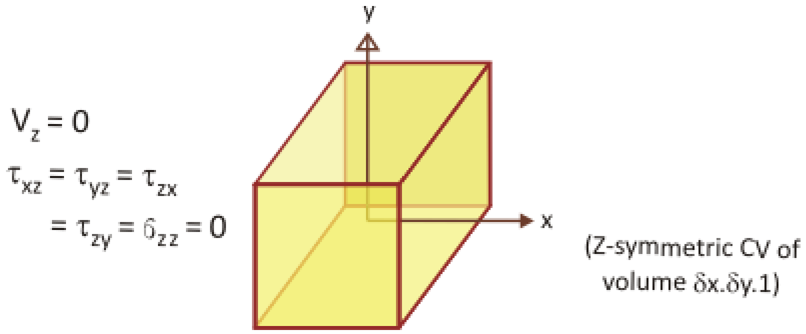
$$\sigma_{xx} = -p + 2\mu \frac{\partial V_x}{\partial x} - \frac{2}{3}\mu\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right] : 2D$$

Again, it is obvious that if a fluid is stationary or inviscid $(\mu = 0)$, $\tau_s = 0$ and $\sigma_s = -p$

Previous Next

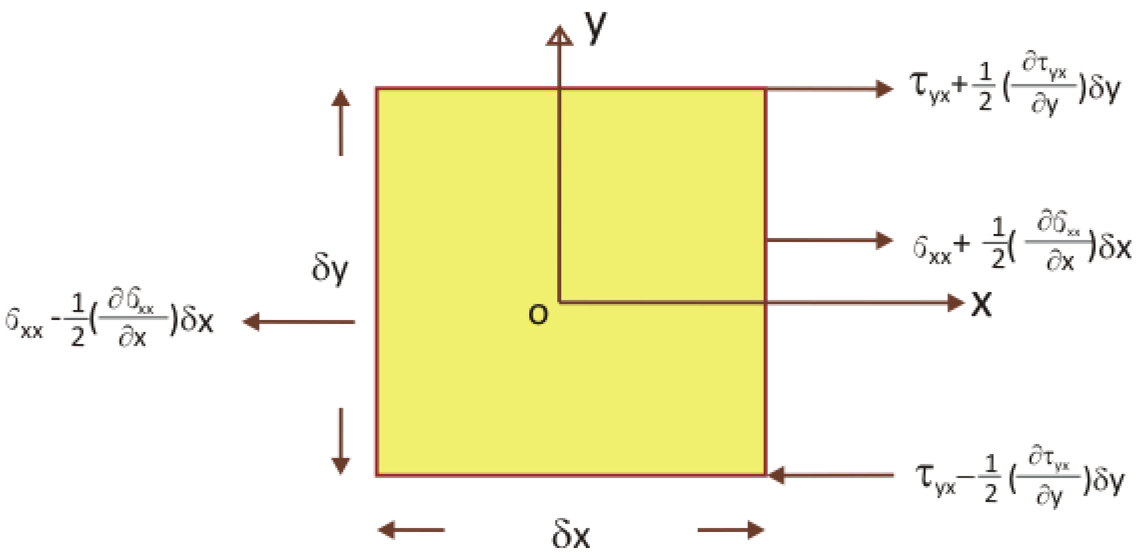
Equation of motion (conservation of momentum in differential form)

Consider $2D$ $(x-y)$ flow of an incompressible fluid $(\rho = c)$



(Fig. 15f)

X-momentum balance:



(Fig. 15g)

See the figure above.

δF_x (due to flow) = Differential force acting on $CV(\delta x\ \delta y 1)$

$$= \text{shear-stress } x(\delta x.1) + \text{normal stress } x(\ \delta y.1)$$

$$= \left(\tau_{yx} + \frac{1}{2}\left(\frac{\partial \tau_{yx}}{\partial x}\right)\delta y - \tau_{yx} + \frac{1}{2}\left(\frac{\partial \tau_{yx}}{\partial y}\right)\delta y\right)\delta x$$

$$+ \left(\sigma_{xx} + \frac{1}{2}\left(\frac{\partial \sigma_{xx}}{\partial x}\right)\delta x - \sigma_{xx} + \frac{1}{2}\left(\frac{\partial \sigma_{xx}}{\partial x}\right)\delta x\right)\delta y$$

$$= \left(\frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \sigma_{xx}}{\partial x}\right).\delta x\ \delta y$$

δF body force = $\rho g_x\ (\delta x\ \delta y)$ (due to gravity)

Newton's 2nd law of motion can be applied on CV

$$F_x = m\ \frac{D\vec{V}}{Dt}$$

or, $\rho\ (\delta x\ \delta y)\ \dfrac{DV_x}{Dt} = \rho g_x\ (\delta x\ \delta y) + \left(\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y}\right)\delta x \delta y$

or, $\rho\ \left(\dfrac{\partial V_x}{\partial t} + V_x\ \dfrac{\partial V_x}{\partial x} + V_y\ \dfrac{\partial V_x}{\partial y}\right) = \rho g_x + \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y}$

Insert the expressions for τ_{yx} and σ_{xx} and simplify for an incompressible fluid

$$\left(\nabla.V = 0 \text{ or } \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0\right)$$

$$= \rho g_x - \frac{\partial p}{\partial x} + 2\mu\ \frac{\partial^2 V_x}{\partial x^2} + \mu\ \left(\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial y\,\partial x}\right)$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu\ \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right) + \mu\frac{\partial}{\partial x}\ \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}\right)$$

$$\boxed{\rho\ \left(\frac{\partial V_x}{\partial t} + V_x\ \frac{\partial V_x}{\partial x} + V_y\ \frac{\partial V_x}{\partial y}\right) = -\ \frac{\partial p}{\partial x} + \mu\ \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right) + \rho g_x}$$

Similarly, one can write (or develop) y-momentum balance equation

$$\rho\left(\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}\right)+\rho g_y$$

In vector form

$$\boxed{\rho\left[\frac{\partial \vec{V}}{\partial t}+(\vec{V}.\nabla)\vec{V}\right]=-\nabla p+\mu\,\nabla^2\vec{V}.+\rho\vec{g}}$$

1st term: Accumulation or transient or unsteady state

2nd term: inertial (convective momentum flux)

3rd term: pressure force

4th term: viscous forces

5th term: External body-force

We call this equation Navier-Stokes (NS) equation

Assumptions: (1) $\rho = C$ (incompressible fluid)

(2) Newtonian fluid

(3) Laminar flow

Previous Next