Module 5: Reynolds Transport Theorem

Lecture 13: Continuity, Momentum theorem

- Differential form of mass-conservation
- B. Momentum - conservation equation

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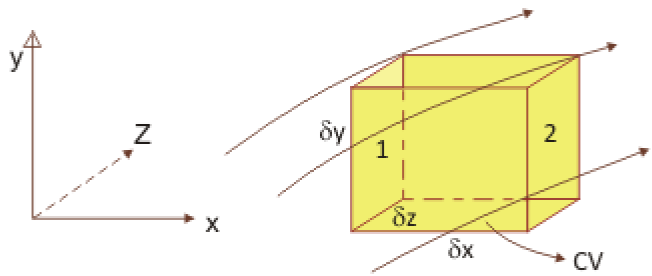
Differential form of mass-conservation

In the previous lecture, we considered the conservation of total mass over a CV, by applying the general Reynolds Transport theorem. Now, we obtain the differential form of the equation, also known as the continuity equation. Consider CV of

$$(\delta x\, \delta y\, \delta z) \qquad \left\{ \begin{array}{l} \text{under limiting} \\ \text{consideration:} \\ \delta x, \delta y, \delta z \to 0 \end{array} \right.$$



(Fig. 13a)

- $\vec{V}(t, x, y, z) = iV_x + jV_y + kV_z$
- CV is bound by six CSs.
- ρ $\equiv$fluid- density

Consider CS marked 1:

Rate of mass-in: $\rho\, V_x(\delta z\, \delta y)$

(It is equivalent to to $' - \rho V.dA'$ of the Reynold's transport theorem)

CS marked 2:

Rate of mass-out: $\left(\rho\, V_x + \dfrac{\partial}{\partial x}(\rho V_x)\delta x\right)\delta z\, \delta y$

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- Net rate of change in mass in x-direction

$$\left(\frac{\partial}{\partial x}(\rho V_x)\right)\delta x\ \delta y \delta z$$
$$=\left(\frac{\partial \rho V_x}{\partial x}\right)\delta\forall$$

Similar expressions can be obtained for the other directions, for examples, $\left(\frac{\partial \rho V_z}{\partial_z}\right)\delta\forall;\ \left(\frac{\partial \rho V_y}{\partial_y}\right)\delta\forall$

- Net rate of accumulation of mass in CV

$$=\frac{\partial}{\partial t}(\rho\ \delta x\ \delta y\ \delta z)=\frac{\partial \rho}{\partial t}\delta\forall$$

Therefore,

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho V_x)+\frac{\partial}{\partial y}(\rho V_y)+\frac{\partial}{\partial z}(\rho V_z)=0$$

The significance of this expression is the same as that for the total mass-conservation rule applied to a CV in the earlier lecture

Re- arrange: $\boxed{\frac{\partial \rho}{\partial t}+\nabla.(\rho\vec{V})=0}$

This is the vector-form of continuity-equation and readily applied for all co-ordinate systems. In this course we will focus mostly on Cartesian and cylindrical 1D or 2D geometry. To this end, the above conservation equation may also be written as:

$$\frac{\partial \rho}{\partial t}+\rho\ \nabla.V+(V.\nabla)\rho=0$$

$$\boxed{\frac{D\rho}{Dt}+\rho\ \nabla.V=0}$$

Re-call the difference between $\frac{\partial \rho}{\partial t}$ and $\frac{D\rho}{Dt}$

- if the flow is steady $\frac{\partial \rho}{\partial t}=0\Rightarrow\nabla.\rho\vec{V}=0$
- if the fluid is incompressible, $\frac{D\rho}{Dt}=0\Rightarrow\nabla.\vec{V}=0$

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In other words, local density my change with time $\left(\frac{\partial \rho}{\partial t} \neq 0\right)$,as the fluid enters and leaves a 'CV' at different rates resulting in the accumulation or depletion of mass of the matter per unit volume of the CV. This is referred as the unsteady-state. However, if the fluid is incompressible, its density will be constant and the material rate of change of density will be zero.

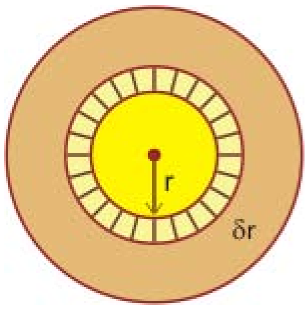
$\nabla.\vec{V} = 0 \text{ for } \rho = C$ (incompressible fluid) can be written for 2D geometries as :

$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$: Cartesian

$\frac{\partial V_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rV_r) = 0$: Cylindrical

Pay special attention to the last term of the above-equation. For 1D radial flow,

$rV_r = \text{const or } \rho(2\,\pi r)\, V_r\, \delta r = \text{const}$.



(Fig. 13b)

$\boxed{\dot{W} = V_r\,(2\pi r\,\delta r)\rho}$ = const under steady-state.

To sum-up, total mass flow rate (kg/s) is constant at all locations in the r-direction. However, mass flux $(kg/m^2 - s)$ changes because the differential area changes in the r-direction

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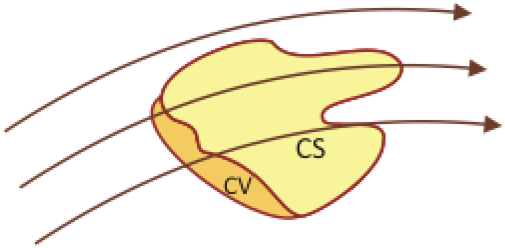
B. Momentum - conservation equation



(Fig. 13c)

Reynolds Transport Theorem:

Property, $N = m\vec{V}. \text{Therefore}, \eta = \vec{V}$

$$\frac{\partial}{\partial t}\iiint_{cv} \vec{V}\rho\, d\forall + \iint_{cs} \vec{V}\,(\rho\vec{V}.d\vec{A}) = \frac{D(m\vec{V})}{Dt}$$

1st Term: local rate of change of momentum in CV

2nd Term: Net rate of momentum in and out of CV through CS.

3rd Term: material or partial –rate of change of momentum, and it may be seen as the external body-force or the source term for the generation of momentum by the external forces.

Mathematically, $\frac{D(m\vec{V})}{Dt} = m\frac{D\vec{V}}{Dt} = F$

Re-write

$$\boxed{F = \frac{\partial}{\partial t}\iiint_{cv} \vec{V}\rho d\forall + \iint_{cs} \vec{V}\,(\rho\vec{V}.d\vec{A})}$$

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