Module 5: Reynolds Transport Theorem

Lecture 12: Conservation of mass and examples

- Conservation of mass

Previous Next

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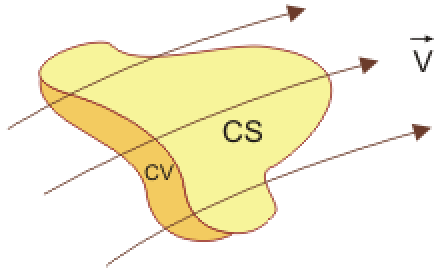
Conservation of mass

N = property =M; Therefore, $\eta = 1$

Apply Reynold's Transport Theorem:



(Fig. 12a)

$$\boxed{\frac{\partial}{\partial t}\iiint 1.\rho\, d\forall = \frac{D}{Dt}\iiint_{c\forall} 1.\rho\, d\forall - \oiint_{CS} 1.\left(\rho\, \vec{V}\,.dA\right)}$$

(Particle-mass cannot be created or destroyed)

The 1st term of the right-hand-side is zero: particle-mass cannot be created or destroyed.

Therefore, $\boxed{\frac{\partial}{\partial t}\iiint_{C\forall} \rho\, d\forall = -\iint_{cs}(\rho\vec{v}.\, dA)}$: conservation of mass equation

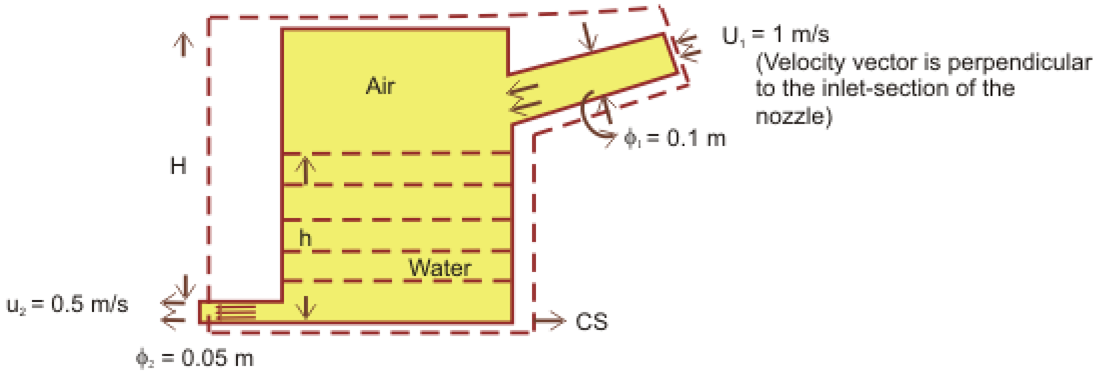
- Note that ρ of a fluid in a CV may change with time, for example, in a tank filled with air and water. If water is drained out, ρ of the mixture (air + water) will vary with time. Message: Be careful while evaluating the integral, whether 'ρ' under evaluation refers to the density of a pure fluid or fluids-mixture.

Previous Next

Module 5: Reynolds Transport Theorem

Lecture 12: Conservation of mass and examples

Example 1: Consider a closed tank partially filled with water. In such case air is trapped at the top. Tank is filled with water at the rate of 1m/s through a nozzle of 10 cm inside-diameter and drained from the bottom at the rate of 0.5m/s through a nozzle of 5 cm diameter. Determine the rate of change in the height of water. The diameter of the tank is 5 m.



(Fig. 12b)

Ans: Consider the CV drawn above. Note that CV is chosen in such a way that the CS is perpendicular to the inlet and outlet nozzles of the tank.

The LHS term of the Reynolds transport theorem or conservation of mass:

$$= \frac{\partial}{\partial t} \iiint_{CV} \rho \, d\forall = \frac{d}{dt}\left(\rho_w A . h_w\right) + \frac{d}{dt} \rho_{air} A \, h_{air}$$

$$\left(\text{where}, h_{air} = H - h_w\right)$$

$$= \frac{d}{dt}\left(\rho_w A h_w\right) \begin{cases} 2^{nd} \text{ term} \\ \text{is zero because} \\ \text{total mass of air} \\ \text{does not change, as the air is trapped.} \end{cases}$$

$$\text{RHS.} \equiv - \iint_{cs} \left(\rho \vec{V} . dA\right)$$

$$= -\rho_{water} \left(-V_1 A_1 + V_2 A_2\right)$$

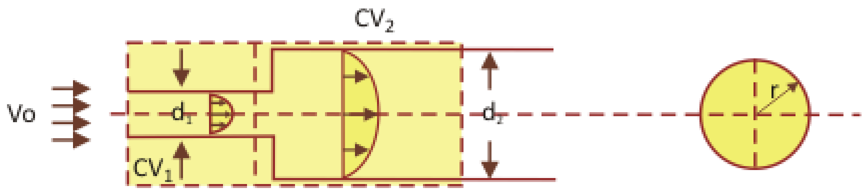
It is assumed that the water velocity is uniform over the inlet- and outlet ports. Also, note that the velocity vector of incoming water to the inlet of the tank is opposite to the area- vector.

Simplifying, $\rho_w A \frac{dh}{dt} = \rho_w (+V_1A_1 - V_2A_2)$

$$\frac{dh}{dt} = \frac{V_1A_1 - V_2A_2}{A} = \frac{1\times(0.1)^2 - 0.5\times(0.05)^2}{5^2}$$

$$= \frac{.01 - .00125}{25} = \frac{.00875}{25} = 0.00035\ m/s$$

Example 2: Consider the steady-state flow of water in a circular tube of variable area:



(Fig. 12c)

The velocity at the inlet to the tube is uniform at V_o. The velocity profile in both sections of the tube is parabolic.

$$v(r) = V_{max}\left(1 - \frac{r^2}{R^2}\right), \text{where } R = d_1/2 \text{ or } d_2/2$$

Apply the conservation of mass equation to calculate maximum velocity in the tube (both sections) in terms of known quantities.

Consider CV_1:

$$\frac{\partial}{\partial t}\iiint \rho\ d\forall = -\iint_{cs_1}(\rho\vec{V}.dA)$$

(1^{st} term is zero for steady-state condition)

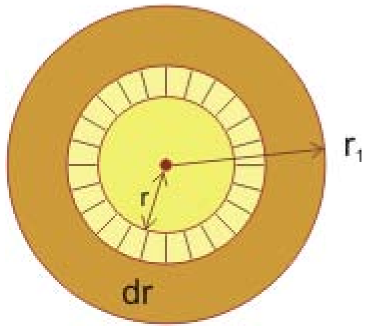
Therefore, $\iint_{cs}\vec{V}.dA = 0$

: $V_oA_1 = \int_0^{r_1} v(r)2\pi\, rdr$, where $\begin{cases} A_1 = \pi/4\, d_1{}^2 \\ r_1 = d_1/2 \end{cases}$

Module 5: Reynolds Transport Theorem

Lecture 12: Conservation of mass and examples

The RHS of the above equation is the integral (over the entire cross-section) of the differential volumetric flow rate:



(Fig. 12d)

$$V_0 \frac{\pi}{4} d_1^{\,2} = \int_0^{r_1} v_{max}\left(1-\frac{r^2}{r_1^{\,2}}\right) 2\pi\, r dr \quad \left\{\begin{array}{c}\text{differential volumetric}\\ \text{flow rate, dq is}\\ v(r) 2\pi\, r dr\end{array}\right.$$

$$V_0 \pi\, r_1^{\,2} = 2\pi\, v_{max} \int_0^{r_1} (1 - r^2/r_1^{\,2}) r dr = 2\pi\, v_{max} \left(\frac{r^2}{2} - \frac{r^4}{4 r_1^{\,2}}\right)\Bigg|_0^{r_1}$$

$$= \pi\, v_{max}\, r_1^{\,2}/2$$

Or $\boxed{v_{max} = 2\, V_0}$ Ans.

Apply the conservation law again over larger CV_2 to show that,

$$V_0 \pi\, r_1^{\,2} = \pi\, v_{max} \frac{r_2^{\,2}}{2}$$

$\boxed{v_{max} = 2\left(\frac{r_1}{r_2}\right)^2 V_0}$ Ans.

Further, v_{max} (for the 2nd section) $= v_{max}$ (for the 1st section) $\times \left(\frac{r_1}{r_2}\right)^2$

or, $(V_{max} A)_1 = (V_{max} A)_2$

Re-visit, $\iint\limits_{cs} \vec{V}.\,dA = 0 \Rightarrow \iint\limits_{cs} dq = 0$

or, $Q_1 = Q_2$ (volumetric flow rate)

or, $(\overline{V}A)_1 = (\overline{V}A)_2$, where $\overline{V} =$ average velocity in the section

Compare to $(V_{max} A)_1 = (V_{max} A)_2$ to show that $V_{max} = 2\overline{V}$.