- Dimensional analysis and similitude
- Buckingham Pi-theorem

Previous Next

Dimensional analysis and similitude

In many engineering applications, scale-up or scale-down of a chemical process or equipment is frequently required. It is not practical to conduct experiments for all conditions of a process to predict the data. The dimensional analysis is a commonly employed technique to scale-up or down a process, and also, predict the results for different conditions.

As an example, consider drag on a sphere falling in stagnant water. If Reynolds number based on the particle size is less than 1, drag can be theoretically calculated as $(3\pi\mu_f d_p v_\infty)$

Alternatively, we can write a general symbolic equation based on our experience:

$F = f\left(\mu_{f,}\rho_{f,}d_{p,}v_\infty\right)$. A few experiments may also be conducted to gain insight into numerical values for F. It is easy to show that $F/\rho v^2 d^2$ is a dimensionless quantity, which can be interpreted as the force per unit kinetic energy per unit cross-sectional area. Realizing that Reynolds number has a mechanistic role to play on the drag, one can write

$$\boxed{\underbrace{\frac{F}{\rho v^2 D^2}}_{\substack{\uparrow \\ \text{dimensionless} \\ \text{number}}} = f\left(\underbrace{\frac{\rho V D}{\mu}}_{\substack{\uparrow \\ \text{dimensionless} \\ \text{number}}}\right)}$$

This is the basis of the dimensional analysis. We now introduce Buckingham Pi-theorem, a very popular technique to obtain a mathematical expression for a complex problem:

Previous Next

Buckingham Pi-theorem

Consider ‘n' number of independent variables for a physical option:

$f\,(q_1, q_2 .. q_n) = 0$

Or

$q_1 \equiv g(q_2 .. q_n)$

The theorem may be interpreted to state that one can form $(n-3)$ independent dimensionless groups of $q_1 .. q_n$ variables so that $h\,(\Pi_1, \Pi_2 .. \Pi_{n-3}) = 0$, where Π_s are the dimensionless groups, and M (mass), L (length) and T (time) are the primary dimensions used to describe the system. For some systems, angle (θ) may also be taken as a primary group, for which one can have (n-4) independent dimensionless groups. We explain the utility of this method in the following examples:

1. Reconsider the previous example of drag on a sphere immersed in a flowing fluid. From the physics of the problem, the independent variables that govern the drag are identified as d_p, v, μ, ρ. Therefore,

$f(F_D, d_p, v, \mu, \rho) = 0$

As per the Buckingham Pi-theorem, the number of dimensionless groups that can be formed is $5 - 3 = 2$.

Therefore, $g\,(\Pi_1, \Pi_2) = 0$

Or $\qquad \Pi_1 = h\,(\Pi_2)$

Choose, $\Pi_1 = f_1\,(d_p, v, \rho, F)$

$\Pi_2 = f_2\,(d_p, v, \rho, \mu)$

Note that there are there repeat variables $(d_p, v, \rho,)$ and two non–repeat variables $\left(\mu_{f,} \rho_{f,}\right)$.

Choice of selecting repeat variables is often arbitrary. Therefore,

$$\Pi_1 = d_p{}^a u^b \rho^c F$$

$$\Pi_2 = d_p{}^a u^b \rho^c \mu$$

$$d_p \equiv [L]; u \equiv \left[\frac{L}{T}\right]; \rho \equiv \left[\frac{M}{L^3}\right]; F \equiv \left[\frac{ML}{T^2}\right] \mu = \left[\frac{M}{LT}\right]$$

Substitute and equate dimensions of M, L, T:

For $\Pi_1 \Rightarrow 0 = c + 1 : M$

$0 = a + b - 3c + 1 : L$

$0 = -b - 2 : T$

Solve to obtain $b = -2, c = -1, a = -2$

Therefore, $\Pi_1 = \left(\frac{F}{\rho v^2 d_p{}^2}\right)$

Similarly, for $\Pi_2 \Rightarrow 0 = c + 1 : M$

$0 = a + b - 3c - 1 : L$

$0 = -b - 1 : T$

Solve to obtain $b = -1, c = -1, a = -1$

Therefore, $\Pi_2 = \left(\frac{\mu}{v d_p \rho}\right)$

$$\left(\frac{F}{\rho v^2 d_p{}^2}\right) = h\left(\frac{\mu}{v d_p \rho}\right)$$

Or $\boxed{F = \rho v^2 d_p{}^2 h\left({}^1/_{R_e}\right)}$ where $R_e = \frac{v d_p \rho}{\mu}$

The exact form of h is found by experiment

Previous Next